MONOGRAFIEEN OVER ASTRONOMIE EN ASTROFYSICA
UITGEGEVEN DOOR
VOLKSSTERRENWACHT URANIA V.Z.W. EN VERENIGING VOOR STERRENKUNDE V.Z.W.
VOL. 4

ASTRONOMICAL FORMULAE FOR CALCULATORS

JEAN MEEUS

DERDE DRUK

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o.l.v. G. BODIFEE en E. WOJCIULEWITSCH

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PREFACE

With the spectacular rise of the pocket calculating machines and the even spectacular fall of their prices in recent years, these wonderful machines are now within reach of everyone. The number of amateur astronomers possessing such calculating machines nearly equals now the number of amateur astronomers themselves. The number of the latter who own a programmable calculating machine is already impressive too and always growing. It is mainly for this last category of interested persons that this book is intended.

Anyone who endeavours to make astronomical calculations has to be very familiar with the essential astronomical conceptions and rules and he must have sufficient knowledge of elementary mathematical techniques. As a matter of course he must have a perfect command of his calculating machine, knowing all possibilities it offers the competent user. However, all these necessities don't suffice. Creating useful, successful and beautiful programs requires much practice. Experience is the mother of all science. This general truth is certainly valid for the art of programming. Only by experience and practice one can learn the innumerable tricks and dodges that are so useful and often essential in a good program.

Astronomical Formulae for Calculators intends to be a guide for the amateur astronomer who wants to do calculations. Before I specify briefly the aims and contents of the book, let me outline first what it is not.

This book is not a general textbook on Astronomy. Elementary astronomical knowledge is taken for granted. For instance, no definitions are given of right ascension and declination, ecliptic, precession, magnitude, etc., but all these notions are used continually throughout the book. Only exceptionally a definition will be given. Nor is this book a textbook on mathematics or a manual for programmable pocket calculators. As I said, the reader is assumed to be able to use his machine appropriately.

What this book intends is to lend a helping hand to every amateur astronomer with mathematical interests and to give him much practical information, advice and examples. About forty topics in the field of calendar problems, celestial phenomena and celestial mechanics are dealt with, and also a few astronomy oriented mathematical techniques, as interpolation and linear regression. For all these cases there is an outline of the problem, its meaning and its signification. The formulae describing the problem in mathematical terms are given and treated at some lenght so as to enable the reader to use them for making his own programs. Many numerical examples are then offered to illustrate the subject and the applications of the formulae.

No programs are given. The reasons are clear. A program is useful only for one type of calculating machine. For instance, a program for a HP-67 cannot be used on a TI-59, and even not on a HP-65. Every calculator thus must learn to create his own programs. There is the added circumstance that the precise contents of a program usually depend on the specific goals of the computation, that are impossible to anticipate always by the author.

The writing of a program to solve some astronomical problem sometimes will require a study of more than one chapter of this book. For instance, in order to create a program for the calculation of the Sun's altitude for a given time of a given date at a given place, one must first convert the date and time to Julian Date (Chapter 3), then calculate the Sun's longitude for that time (Chapter 18), its right ascension (Chapter 8), the sidereal time (Chapter 7), and finally the required altitude of the Sun (Chapter 8).

It is clear that not all topics of mathematical astronomy could have been dealt with in this book. So nothing is said about orbit determination, occultations of stars by the Moon, the calculation of the longitude of the central meridian of Mars and Jupiter for a given instant, meteor astronomy, eclipsing binaries, etc. However, a hasty look on the Table of Contents convinces there is enough fascinating material in this fourth monograph on astronomy and astrophysics edited by *Urania* and *VVS*, to keep every amateur busy for years to come.

G. Bodifée

CONTENTS

SOME SYMBOLS AND ABBREVIATIONS

e	Eccentricity (of an orbit)
h	Altitude above the horizon
r	Radius vector, or distance of a body to the Sun, in AU
v	True anomaly
A	Azimuth
H	Hour angle
M	Mean anomaly
R	Distance from Sun to Earth, in AU
α	Right ascension
δ	Declination
ε	Obliquity of the ecliptic
θ	Sidereal time
θ_0	Sidereal time at Greenwich
π	Parallax
ϕ	Geographical latitude
ϕ'	Geocentric latitude
Δ	Distance to the Earth, in AU
ΔT	Difference ET - UT
$\Delta\varepsilon$	Nutation in obliquity
$\Delta\psi$	Nutation in longitude
$\odot$	Geocentric longitude of the Sun
AU	Astronomical unit
ET	Ephemeris Time
UT	Universal Time
JD	Julian Day
INT	Integer part of
A.E.	Astronomical Ephemeris
IAUC	International Astronomical Union Circular

Following a general astronomical practice (see for instance the *A.E.*), small superior symbols are placed immediately above the decimal point, not after the last decimal. For instance, $28\overset{\circ}{.}5793$ means 28.5793 degrees.

Moreover, note carefully the difference between hours with decimals, and hours - minutes - seconds. For example, $1\overset{h}{.}30$ is *not* 1 hour and 30 minutes, but 1.30 hours, that is 1 hour and 30 hundredths of an hour, or 1^h18^m.

The author wishes to express his gratitude to Mr. G. Bodifée, for his valuable advice and assistance.

1

HINTS AND TIPS

To explain how to calculate or to program on a calculating machine is out of the scope of this book. The reader should, instead, study carefully his instructions manual. However, even then writing good programs cannot be learned in the lapse of one day. It is an art which can be acquired only progressively. Only by practice one can learn to write better and shorter programs.

In this first Chapter, we will give some practical hints and tips, which may be of general interest.

Accuracy

The accuracy of a calculation depends on its aims. If one only wants to know whether an occultation by the Moon will be visible in some countries, an accuracy of 100 kilometers in the northern or southern limit of the region of visibility is probably sufficient ; however, if one wants to organize an expedition to observe a grazing occultation by the Moon, the limit has to be calculated with an accuracy better than 1 kilometer.

If one wants to calculate the position of a planet with the goal of obtaining the moments of rise or setting, an accuracy of 0.01 degree is sufficient. But if the position of the planet is needed to calculate the occultation of a star by the planet, an accuracy of better than 1″ will be necessary because of the small size of the planet's disk.

To obtain a better accuracy it is sometimes necessary to use another method of calculation, not just to keep more decimals in the result of an approximate calculation. For example, if one has to know the position of Mars with an accuracy of 0.1 degree, it suffices to use an unperturbed elliptical orbit (Keplerian motion) although secular perturbations of the orbit are to be taken into account eventually. However, if the position of Mars is to be known with a precision of 10″ or better, perturbations due to the other planets have to be calculated and the program will be a much longer one.

So the calculator, who knows his formulae and the desired accuracy in a given problem, must himself considerate which terms, if any, may be omitted in order to keep the program handsome and as short as possible. For instance, the geometric mean longitude of the Sun, referred to the mean equinox of the date, is given by

$$L = 279°41'48''.04 + 129\,602\,768''.13\,T + 1''.089\,T^2$$

where T is in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 ET. In this expression the last term (secular acceleration of the Sun) is smaller than 1″ if $|T| < 0.96$, that is between the years 1804 and 1996. If an accuracy of 1″ is sufficient, the term in T^2 may thus be dropped for any instant in that period. But for the year −100 we have $T = -20$, so that the last term becomes 436″, which is larger than 0.1 degree.

Rounding

Rounding should be made where it is necessary. Do not retain meaningless decimals in your result. Some "feeling" and sufficient astronomical knowledge are necessary here. For instance, it would be completely irrelevant to give the illuminated fraction of the Moon's disk with an accuracy of 0.000 000 001.

If one calculates by hand and not with a program, the rounding should be performed *after* the whole calculation has been made.

Example : Calculate 1.4 + 1.4 to the nearest integer. If we first round the given numbers, we obtain 1 + 1 = 2. In fact, 1.4 + 1.4 = 2.8, which is to be rounded to 3.

Rounding should be made to the nearest value. For instance, 15.88 is to be rounded to 15.9 or to 16, not to 15. However, calendar dates and years are exceptions. For example, March 15.88 denotes an instant belonging to March 15 ; thus, if we read that an event occurs on March 15.88, it takes place on March 15, not on March 16. Similarly, 1977.69 denotes an event occurring in the year 1977, not 1978.

Trigonometric functions of large angles

Large angles frequently appear in astronomical calculations. In Example 18.a we find that on 1978 November 12.0 the Sun's mean longitude is 28670.77554 degrees. Even larger angles are found for rapidly moving objects, such as the Moon or the bright satellites of Jupiter.

According to the type of the machine, it may be necessary or desirable to reduce the angles to the range 0 − 360 degrees. Some calculating machines give incorrect values for the trigonometric functions of large angles. For instance,

the HP-55 gives sin 360000030° = 0.499 481 3556
TI-52 0.499 998 1862
Casio fx 2200 Error

while the HP-67 gives the correct value 0.500 000 0000.

Angle modes

The calculating machines do not calculate directly the trigonometric functions of an angle which is given in degrees, minutes and seconds. Before performing the trigonometric function, the angle should be converted to degrees and *decimals*. Thus, to calculate the cosine of 23°26′49″, first convert the angle to 23.44694444 degrees, and *then* press the key COS.

Similarly, angles should be converted from degrees, minutes and seconds to degrees and decimals, before they can be interpolated. For instance, it is impossible to apply an interpolation formula directly to the values

5°03′45″
5°34′22″
6°17′09″

Right ascensions

Right ascensions are generally expressed in hours, minutes and seconds. If the trigonometric function of a right ascension must be calculated, it is thus necessary to convert that right ascension to degrees. Remember that one hour corresponds to 15 degrees.

Example 1.a : Calculate $\tan\alpha$, where $\alpha = 9^h14^m55\overset{s}{.}8$.

We first convert α to hours and decimals :

$$9^h14^m55\overset{s}{.}8 = 9.248\,833\,333 \text{ hours.}$$

Then, by multiplying by 15,

$$\alpha = 138\overset{\circ}{.}73250$$

whence $\tan\alpha = -0.877\,517$.

The correct quadrant

When the sine, the cosine or the tangent of an angle is known, the angle itself can be obtained by pressing the corresponding key : arc sin, arc cos, or arc tan, sometimes written as $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$. — The latter are, in fact, incorrect designations, for x^{-1} is the same as $1/x$. But $\cos^{-1} x$ is (incorrectly) used to designate the inverse function, and *not* $1/\cos x$.

In this case, on most pocket calculating machines, arc sin and arc tan give an angle lying between −90 and +90 degrees, while arc cos gives a value between 0 and +180 degrees.

In some cases, the result obtained in this way may not be in the correct quadrant. Each problem must be examined separately. For instance, formulae (8.4) and (25.15) give the sine of the declination. The instruction arc sin will then give the declination always in the correct quadrant, because all declinations lie between −90 and +90 degrees.

This is also the case for the angular separation whose cosine is given by formula (9.1). Indeed, any angular separation lies between the values 0° and 180°, and this is precisely what the operation arc cos gives.

When the tangent of an angle is given, for example by means of formulae (8.1), (8.3) and (18.3), the angle may be obtained directly in the correct quadrant by using a trick : the rectangular/polar transformation applied to the numerator and the denominator of the fraction in the right-hand member of the formula, as explained in Chapter 8 and at some other places in this book.

Powers of time

Some quantities are calculated by means of a formula containing powers of the time (T, T^2, T^3, ...). It is important to note that such polynomial expressions are valid only for values of T which are not too large. For instance, the formula

$$e = 0.016\,751\,04 - 0.000\,0418\,T - 0.000\,000\,126\,T^2$$

given in Chapter 18 for the eccentricity of the Earth's orbit, is valid only for several centuries before and after the year 1900, and *not* for millions of years ! For instance, for $T = 1000$, the above-mentioned formula gives $e = -0.151 < 0$, an absurd result.

The same is true for instance for formula (18.4), which would give the completely invalid results $\varepsilon = 0°$ for $T = -383$, and $\varepsilon = 90°$ for $T = +527$.

One should further carefully note the difference between periodic terms, which remain small throughout the centuries, and secular terms (terms in T^2, T^3, ...) which rapidly increase with time.

In formula (32.1), for instance, the last term is a periodic one which always lies between -0.00033 and $+0.00033$. On the other hand, the term $+0.000\,1178\,T^2$, which is very small when T is very small, becomes increasingly important for larger values of $|T|$. For $T = \pm 10$, that term takes the value $+0.01178$, which is large in comparison to the above-mentioned periodic term. Thus, for large values of T it is meaningless to take into account small periodic terms if secular terms are dropped.

To shorten a program

To make a program as short as possible is not always an art for art's sake, but sometimes a necessity as long as the memory capacities of the calculating machine have their limits.

There exist many tricks to make programs shorter, even for simple calculations. For instance, if one wants to calculate the polynomial

$$Ax^4 + Bx^3 + Cx^2 + Dx + E$$

with A, B, C, D and E constants, and x a variable. Now, one may program the machine directly to calculate this polynomial term after term and adding all terms, so that for each given x the machine obtains the value of the polynomial. However, instead of calculating all the powers of x, it appears to be wiser to write the polynomial as follows :

$$\left[((Ax+B)x + C)x + D\right]x + E$$

In this expression all power functions have disappeared and only additions and multiplications are to be performed. The program will be shorter now. If we use for instance a HP-67 machine and store the constants A to E in the registers 1 to 5, the programs for the calculation will in each case be as follows.

First version	*Second version*
STO A	STO A
4	RCL 1
y^x	×
RCL 1	RCL 2
×	+
RCL A	RCL A
3	×
y^x	RCL 3
RCL 2	+
×	RCL A
+	×
RCL A	RCL 4
x^2	+
RCL 3	RCL A
×	×
+	RCL 5
RCL A	+
RCL 4	
×	
+	
RCL 5	
+	

Thus, by using this simple trick, one has saved five steps, a gain of 23 % in this short program !

2

INTERPOLATION

The astronomical almanacs or other publications contain numerical tables giving some quantities y for *equidistant* values of an argument x. For example, y is the right ascension of the Sun, and the values x are the different days of the year at 0^h ET.

Interpolation is the process of finding values for instants, quantities, etc., intermediate to those given in a table.

In this Chapter we will consider two cases : interpolation from three or from five tabular values. In both cases we will also show how an extremum or a zero of the function can be found. The case of only two tabular values will not be considered here, for in that case the interpolation can but be linear, and this will give no difficulty at all.

Three tabular values

Three tabular values y_1, y_2, y_3 of the function y are given, corresponding to the values x_1, x_2, x_3 of the argument x. Let us form the table of differences

$$\begin{array}{llll} x_1 & y_1 & & \\ & & a & \\ x_2 & y_2 & & c \\ & & b & \\ x_3 & y_3 & & \end{array} \qquad (2.1)$$

where $a = y_2 - y_1$ and $b = y_3 - y_2$ are called the *first differences*. The *second* difference c is equal to $b - a$, that is

$$c = y_1 + y_3 - 2y_2$$

Generally, the differences of the successive orders are gradually smaller. Interpolation from three tabular values is permitted when the second differences are almost constant in that part of the table, that is when the third differences are almost zero. Let us consider, for instance, the distance of Mars to the Earth from 4 to 8 August 1969, at 0^h ET. The values are given in astronomical units, and the differences are in units of the sixth decimal :

August 4	0.659441			
		+5090		
5	0.664531		+30	
		+5120		-1
6	0.669651		+29	
		+5149		0
7	0.674800		+29	
		+5178		
8	0.679978			

Since the third differences are almost zero, we may interpolate from only three tabular values.

The central value y_2 must be choosen in such a manner that it is that value of y that is closest to the required value.

For example, if from the table above we must deduce the value of the function for August 6 at 22^h14^m, then y_2 is the value for August 7.00. In that case, we should consider the tabular values for August 6, 7 and 8, namely the table

$$\begin{array}{rl} \text{August } 6 & y_1 = 0.669651 \\ 7 & y_2 = 0.674800 \\ 8 & y_3 = 0.679978 \end{array} \tag{2.2}$$

and the differences are

$$\begin{array}{ll} a = +0.005149 & \\ & c = +0.000029 \\ b = +0.005178 & \end{array}$$

Let n be the interpolation interval. That is, if the value y of the function is required for the value x of the argument, we have $n = x - x_2$ in units of the tabular interval. The number n is positive if $x > x_2$, that is for a value "later" than x_2 , or from x_2 towards the bottom of the table. If x precedes x_2 , then $n < 0$.

If y_2 has been correctly choosen, then n will be between -0.5 and $+0.5$, although the following formulae will also give correct results for all values of n between -1 and $+1$.

The interpolation formula is

$$y = y_2 + \frac{n}{2}(a + b + nc) \qquad (2.3)$$

Example 2.a : From the table (2.2), calculate the distance of Mars to the Earth on 1969 August 7 at 4^h21^m ET.

We have $4^h21^m = 4.35$ hours and, since the tabular interval is 1 day or 24 hours, we have $n = 4.35/24 = 0.18125$.

Formula (2.3) then gives $y = 0.675\,736$, the required value.

If the tabulated function reaches an *extremum* (that is, a maximum or a minimum value), this extremum can be found as follows. Let us again form the difference table (2.1) for the appropriate part of the ephemeris. The extreme value of the function then is

$$y_m = y_2 - \frac{(a + b)^2}{8c}$$

and the corresponding value of the argument x is given by

$$n_m = -\frac{a + b}{2c}$$

in units of the tabular interval, and again measured from the central value x_2.

Example 2.b : Calculate the time of passage of Mars through the perihelion of its orbit in January 1966, and the value of Mars' radius vector at that instant.

From the *Astronomical Ephemeris* we take the following values for the distance Sun - Mars :

1966 January	11.0	1.381 701
	15.0	1.381 502
	19.0	1.381 535

The differences are $a = -0.000199$ $\quad c = +0.000232$
$b = +0.000033$

from which we deduce

$$y_m = 1.381\,487 \qquad \text{and} \qquad n_m = +0.35776$$

The least distance from Mars to the Sun was thus 1.381 487 AU. The corresponding time is found by multiplying 4 days (the tabular interval) by +0.35776. This gives 1.43104 day, or 1 day and 10 hours later than the central time, or 1966 January 16 at 10^h.

The value of the argument x for which the function y becomes zero can be found by again forming the difference table (2.1) for the appropriate part of the ephemeris. The interpolation interval corresponding to a zero of the function is then given by

$$n_o = \frac{-2y_2}{a + b + cn_o} \tag{2.4}$$

Equation (2.4) can be solved by first putting $n_o = 0$ in the second member. Then the formula gives an approximate value for n_o. This value is then used to calculate the right hand side again, which gives a still better value for n_o. This process, called *iteration* (Latin : *iterare* = to repeat), can be continued until the value found for n_o does not longer vary, to the precision of the calculating machine.

Example 2.c : The *A.E.* gives the following values for the declination of Mercury :

1973 February 26.0	−0° 28′ 13″.4
27.0	+0 06 46.3
28.0	+0 38 23.2

Calculate when the planet's declination was zero.

We firstly convert the tabulated values into seconds of a degree, and then form the differences :

$$\begin{array}{lll} y_1 = -1693.4 & & \\ & a = +2099.7 & \\ y_2 = +\ 406.3 & & c = -202.8 \\ & b = +1896.9 & \\ y_3 = +2303.2 & & \end{array}$$

Formula (2.4) then becomes

$$n_o = \frac{-812.6}{+3996.6 - 202.8\, n_o}$$

Putting $n_o = 0$ in the second member, we find $n_o = -0.20332$. Repeating the calculation, we find successively −0.20125 and −0.20127. Thus n_o = −0.20127 and therefore, the tabular interval being one day, Mercury crossed the celestial equator on

$$\begin{aligned} \text{1973 February 27.0} - 0.20127 &= \text{February 26.79873} \\ &= \text{February 26 at } 19^h 10^m \text{ ET.} \end{aligned}$$

Five tabular values

When the third differences may not be neglected, more than three tabular values must be used. Taking five consecutive tabular values, y_1 to y_5 , we form, as before, the difference table

$$\begin{array}{cccccc}
 & y_1 & & & & \\
 & & A & & & \\
 & y_2 & & E & & \\
 & & B & & H & \\
n \downarrow & y_3 & & F & & K \\
 & & C & & J & \\
 & y_4 & & G & & \\
 & & D & & & \\
 & y_5 & & & &
\end{array}$$

where $A = y_2 - y_1$, $H = F - E$, etc. If n is the interpolation interval, measured from the central value y_3 towards y_4 in units of the tabular interval, we have the interpolation formula

$$y = y_3 + \frac{n}{2}(B + C) + \frac{n^2}{2}F + \frac{n(n^2 - 1)}{12}(H + J) + \frac{n^2(n^2 - 1)}{24}K$$

which may also be written (2.5)

$$y = y_3 + n\left(\frac{B + C}{2} - \frac{H + J}{12}\right) + n^2\left(\frac{F}{2} - \frac{K}{24}\right) + n^3\left(\frac{H + J}{12}\right) + n^4\left(\frac{K}{24}\right)$$

Example 2.d : The *A.E.* gives the following values for the Moon's horizontal parallax :

1979 December 9.0	54′45″.5099
9.5	54 34.4060
10.0	54 25.6303
10.5	54 19.3253
11.0	54 15.5940

The differences (in ″) are

$A = -11.1039$			
	$E = +2.3282$		
$B = -\ 8.7757$		$H = +0.1425$	
	$F = +2.4707$		$K = -0.0395$
$C = -\ 6.3050$		$J = +0.1030$	
	$G = +2.5737$		
$D = -\ 3.7313$			

We see that the third differences may not be neglected, unless an accuracy of about 0″.1 is sufficient.

Let us now calculate the Moon's parallax on December 10 at 3^h20^m ET. The tabular interval being 12 hours, we find

$$n = +0.277\,7778 .$$

Formula (2.5) then gives

$$y = 54'25''.6303 - 2''.0043 = 54'23''.6260.$$

The interpolation interval n_m corresponding to an extremum of the function may be obtained by solving the equation

$$n_m = \frac{6B + 6C - H - J + 3n_m^2\,(H + J) + 2n_m^3 K}{K - 12F} \tag{2.6}$$

As before, this may be performed by iteration, firstly putting $n_m = 0$ in the second member. Once n_m is found, the corresponding value of the function can be calculated by means of formula (2.5).

Finally, the interpolation interval n_o corresponding to a zero of the function may be found from

$$n_o = \frac{-24y_3 + n_o^2\,(K - 12F) - 2n_o^3\,(H + J) - n_o^4 K}{2\ (6B + 6C - H - J)} \tag{2.7}$$

where, again, n_o can be found by iteration, starting from putting $n_o = 0$ in the second member.

Note that the quantities $(6B + 6C - H - J)$, $(K - 12F)$, and $(H + J)$ appear in both formulae (2.6) and (2.7). Consequently, it may be useful to calculate these quantities in a subroutine which will be used in both cases.

Exercise. – From the following values of the heliocentric latitude of Mercury, find the instant when the latitude is zero, by using formula (2.7).

1979 May	25.0 ET	$-1°16'00''.5$
	26.0	$-0\ 33\ 01.3$
	27.0	$+0\ 11\ 12.0$
	28.0	$+0\ 56\ 03.3$
	29.0	$+1\ 40\ 52.2$

Answer : Mercury reaches the ascending node of its orbit for $n_o = -0.251\ 360$, that is on 1979 May 26 at 17^h58^m ET.

Important remarks

1. Interpolation cannot be performed on complex quantities directly. These quantities should be converted, in advance, into a single suitable unit. For instance, angles expressed in degrees, minutes and seconds should be expressed either in degrees and decimals, or in seconds.

Thus, for instance, $12°44'03''.7$ should be written either as $12°.73436$, or as $45843''.7$.

2. *Interpolating times and right ascensions.* – We draw attention on the fact that the time and the right ascension jump to zero when the value of 24 hours is reached. This should be taken into account when interpolation is performed on tabulated values. Suppose, for example, that we wish to calculate the right ascension of Mercury for the instant 1979 April 16.2743 ET, using three tabulated values. We find in the *A.E.* :

1979 April	15.0	$\alpha = 23^h56^m09^s.20$
	16.0	$23\ 58\ 46.63$
	17.0	$0\ 01\ 36.80$

Not only is it necessary to convert these values to hours and decimals, but the last value should be written as $24^h\,01^m\,36^s.80$, otherwise the machine will consider that, from April 16.0 to 17.0, the value of α *decreases* from 23^h58^m... to 0^h01^m....

We find a similar situation in some other cases. For instance, here is the longitude of the central meridian of the Sun for a few dates :

1979 December	25.0	37°.39
	26.0	24.22
	27.0	11.05
	28.0	357.88

It is evident that the variation is −13.17 degrees per day. Thus, one should *not* interpolate directly between 11.05 and 357.88. Either the first value should be written as 371.05, or the second one should be considered as being −2.12.

3

JULIAN DAY AND CALENDAR DATE

In this Chapter we will give a method for converting a date in the Julian or Gregorian calendars into the corresponding Julian Day number (JD), and vice versa.

General remarks

The Julian Day begins at Greenwich mean *noon*, that is at 12^h Universal Time (or 12^h Ephemeris Time, and in that case the expression Julian Ephemeris Day is generally used). For example, 1977 April 26.4 = JD 2443 259.9.

In the methods described below, the Gregorian calendar reform is taken into account. Thus, the day following 1582 October 4 is 1582 October 15.

The "B.C." years are counted astronomically. Thus, the year before the year +1 is the year zero, and the year preceding the latter is the year −1.

We will indicate by INT(x) the integer part of x, that is the integer which precedes its decimal point. For example :

INT(7/4) = 1
INT(8/4) = 2
INT(5.02) = 5
INT(5.9999) = 5
INT(−4.98) = −4

Calculation of the JD

A date may be entered in the machine as consecutive numbers, for instance the year first, then the month number, and finally the day with decimals. Thus, 1976 August 22.09 can be entered by entering successively the numbers 1976, 8 and 22.09.

However, it may be more interesting to enter a date as one single number, namely as *YYYY.MMDDdd*, where *YYYY* is the year, *MM* the month, and *DDdd* the day of the month with decimals. In that case, the month number should always be written as a two-digit number, and a decimal point must separate *YYYY* from *MM*. For example, 1976 August 22.09 should then be entered as 1976.082209. The program must then start with a procedure separating the numbers *YYYY*, *MM* and *DD.dd* and storing them in suitable registers. For example, for 1976 August 22.09, the number 1976.082209 is given to the machine, which stores $YYYY = 1976$ in one register, $MM = 8$ in a second one, and $DD.dd = 22.09$ in a third register.

In what follows, we will suppose that this separation has been performed.

If *MM* is greater than 2, take

$$y = YYYY \qquad \text{and} \qquad m = MM\ ;$$

if $MM = 1$ or 2, take

$$y = YYYY - 1 \qquad \text{and} \qquad m = MM + 12.$$

If the number *YYYY.MMDDdd* is equal or larger than 1582.1015 (that is, in the Gregorian calendar), calculate

$$A = \text{INT}\left(\frac{y}{100}\right) \qquad\qquad B = 2 - A + \text{INT}\left(\frac{A}{4}\right)$$

If $YYYY.MMDDdd < 1582.1015$, it is not necessary to calculate *A* and *B*.

The required Julian Day is then

$$\text{JD} = \text{INT}\,(365.25\,y) + \text{INT}\,(30.6001\,(m+1)) + DD.dd + 1720\,994.5 \qquad (3.1)$$

and, to this result, add the quantity *B* *if* the date is in the Gregorian calendar.

Example 3.a : Calculate the JD corresponding to 1957 October 4.81, the time of launch of Sputnik 1.

Because $MM = 10$ is greater than 2, we have $y = 1957$ and $m = 10$.

Because $1957.100481 > 1582.1015$, the date is in the Gregorian calendar, and we calculate

$$A = \text{INT}\left(\frac{1957}{100}\right) = \text{INT}\,(19.57) = 19$$

$$B = 2 - 19 + \text{INT}\left(\frac{19}{4}\right) = 2 - 19 + 4 = -13$$

$$\mathrm{JD} = \mathrm{INT}\,(365.25 \times 1957) + \mathrm{INT}\,(30.6001 \times 11) + 4.81 + 1720\,994.5 - 13$$

$$\mathrm{JD} = 2436\,116.31$$

Example 3.b : Calculate the JD corresponding to January 27 at 12^h of the year 333.

Because *MM* = 1, we have

$$y = 333 - 1 = 332 \qquad \text{and} \qquad m = 1 + 12 = 13.$$

The number *YYYY.MMDDdd* = 333.01275 being less than 1582.1015, the date is in the Julian calendar, and the quantities A and B are not needed.

$$\mathrm{JD} = \mathrm{INT}\,(365.25 \times 332) + \mathrm{INT}\,(30.6001 \times 14) + 27.5 + 1720\,994.5$$

$$\mathrm{JD} = 1842\,713.0$$

Note. – Your program will not work for negative years. One reason is that, if you enter the date as *YYYY.MMDDdd* preceded by a minus sign, the machine will read *MM* and *DD.dd* as negative numbers. For example, if May 28.63 of the year -584 is entered as -584.052863, the machine will correctly deduce *YYYY* = -584, but will find *MM* = -5 and *DD.dd* = -28.63 instead of the correct values +5 and +28.63.

You may make your program valid for negative years by correcting it as follows.

1. After *YYYY* has been deduced (with proper sign) from the number *YYYY.MMDDdd*, take the absolute value of *.MMDDdd* before calculating *MM* and *DD.dd* ;
2. If $y < 0$, replace, in formula (3.1),

$$\mathrm{INT}\,(365.25\,y) \qquad \text{by} \qquad \mathrm{INT}\,(365.25\,y - 0.75).$$

As an exercise, try your corrected program on -584 May 28.63. The result should be JD = 1507 900.13. But check whether your program is still valid for positive years !

Calculation of the Calendar Date from the JD

The following method is valid for positive as well as for negative years, but not for negative Julian Day numbers.

Add 0.5 to the JD, and let Z be the integer part, and F the fractional (decimal) part of the result.

If $Z < 2299\,161$, take $A = Z$.

If Z is equal to or larger than $2299\,161$, calculate

$$\alpha = \text{INT}\left(\frac{Z - 1867\,216.25}{36524.25}\right)$$

$$A = Z + 1 + \alpha - \text{INT}\left(\frac{\alpha}{4}\right)$$

Then calculate

$$B = A + 1524$$

$$C = \text{INT}\left(\frac{B - 122.1}{365.25}\right)$$

$$D = \text{INT}\,(365.25\,C)$$

$$E = \text{INT}\left(\frac{B - D}{30.6001}\right)$$

The day of the month (with decimals) is then

$$B - D - \text{INT}\,(30.6001\,E) + F$$

The month number m is

$$\begin{array}{lll} E - 1 & \text{if} & E < 13.5 \\ E - 13 & \text{if} & E > 13.5 \end{array}$$

The year is

$$\begin{array}{lll} C - 4716 & \text{if} & m > 2.5 \\ C - 4715 & \text{if} & m < 2.5 \end{array}$$

Example 3.c : Calculate the calendar date corresponding to JD 2436 116.31.

$2436\,116.31 + 0.5 = 2436\,116.81$,

thus $Z = 2436\,116$ and $F = 0.81$

Because $Z > 2299\,161$, we have

$$\alpha = \text{INT}\left(\frac{2436\,116 - 1867\,216.25}{36524.25}\right) = 15$$

$$A = 2436\,116 + 1 + 15 - \mathrm{INT}\left(\frac{15}{4}\right) = 2436\,129$$

Then we find

$B = 2437\,653$, $C = 6673$, $D = 2437\,313$,
$E = 11$,

day of month = 4.81
month $m = E - 1 = 10$ (because $E < 13.5$)
year $= C - 4716 = 1957$ (because $m > 2.5$)

Thus, the required date is 1957 October 4.81

Exercises : Calculate the calendar dates corresponding to JD = 1842 713.0 and to JD = 1507 900.13 .

(Answers : 333 January 27.5 and -584 May 28.63)

Time interval in days

The number of days between two calendar dates can be found by calculating the difference between their corresponding Julian Days.

Example 3.d : The periodic comet Halley passed through perihelion on 1835 November 16 and on 1910 April 20. What is the time interval between these two passages ?

1835 November 16.0 corresponds to JD 2391 598.5
1910 April 20.0 corresponds to JD 2418 781.5

The difference is 27 183 days.

Exercise : Find the date exactly 10 000 days after 1954 June 30.
(Answer : 1981 November 15)

Day of Week

The day of the week corresponding to a given date can be obtained as follows. Compute the JD for that date at 0^h, add 1.5, and divide the result by 7. The remainder of this division will indicate the weekday, as follows : if the remainder is 0, it is a Sunday, 1 a Monday, 2 a Tuesday, 3 a Wednesday, 4 a Thursday, 5 a Friday, 6 a Saturday.

Example 3.e : Find the weekday of 1954 June 30.

I954 June 30.0 corresponds to JD 2434 923.5
2434 923.5 + 1.5 = 2434 925
The remainder after division by 7 is 3. Thus it was a Wednesday.

Day of the Year

The number N of a day in the year can be computed as follows.

For common years :

$$N = \mathrm{INT}\left(\frac{275\,M}{9}\right) - 2\ \mathrm{INT}\left(\frac{M+9}{12}\right) + D - 30$$

For leap (bissextile) years :

$$N = \mathrm{INT}\left(\frac{275\,M}{9}\right) - \mathrm{INT}\left(\frac{M+9}{12}\right) + D - 30$$

where M is the month number, and D is the day of the month.

N takes integer values, from 1 on January 1 to 365 (or 366 in leap years) on December 31.

Example 3.f : 1978 November 14.

Common year, $M = 11$, $D = 14$.
One finds $N = 318$.

Example 3.g : 1980 April 22.

Leap year, $M = 4$, $D = 22$.
One finds $N = 113$.

4

DATE OF EASTER

The method described below has been given by Spencer Jones in his book *General Astronomy* (pages 73-74 of the edition of 1922). It has been published again in the *Journal of the British Astronomical Association*, Vol. 88, page 91 (December 1977) where it is said that it was devised in 1876 and appeared in Butcher's *Ecclesiastical Calendar*.

Unlike the formula given by Gauss, this method has no exception and is valid for all years in the *Gregorian calendar*, that is from the year 1583 on. The procedure for determining the date of Easter is as follows :

Divide	by	Quotient	Remainder
the year x	19	-	a
the year x	100	b	c
b	4	d	e
$b + 8$	25	f	-
$b - f + 1$	3	g	-
$19a + b - d - g + 15$	30	-	h
c	4	i	k
$32 + 2e + 2i - h - k$	7	-	l
$a + 11h + 22l$	451	m	-
$h + l - 7m + 114$	31	n	p

Then n = number of the month (3 = March, 4 = April),
$p + 1$ = day of that month upon which Easter Sunday falls.

Try to have your result displayed in one of the following formats :

DD.M (day.month), for instance 26.3 = 26 March ;

M.DD (month.day), for instance 3.26 = March 26 ;

YYYY.MMDD (year.month day), for instance 1978.0326 = 1978 March 26.

The month and the day of the month may also be displayed successively as integer numbers, but the formats above have the advantage that the complete date is read at a glance.

The calculation of the remainder of a division must be programmed carefully. Suppose that the remainder of the division of 34 by 30 should be found. On the HP-67 machine, we find

$$34/30 = 1.133\,333\,333$$

the fractional part of which is 0.133 333 333. When multiplied by 30, this gives 3.999 999 990. This result differs from 4, the correct value, and may give a wrong date for Easter at the end of the calculation.

On the HP-67, the correct value of the remainder may be found by using the instructions

```
DSP 0
f RND
```

On other machines, it may be necessary to use another trick.

If you have enough program steps, you might add some tests at the beginning of your program. For instance, write your program in such a way that *"Error"* appears if the year is either less than 1583, or not an integer number.

Try your program on the following years :

1978 → March 26	1954 → April 18
1979 → April 15	2000 → April 23
1980 → April 6	1000 → Error

5

EPHEMERIS TIME AND UNIVERSAL TIME

The Ephemeris Time (ET) is a uniform time based on the planetary motions. The Universal Time (UT), necessary for civil life, is based on the rotation of the Earth.

Because the Earth's rotation is slowing down — and, moreover, with unpredictable irregularities — UT is not a uniform time. Since the astronomers need a uniform time, they use ET for the calculation of their accurate ephemerides.

The exact value of the difference ΔT = ET - UT can be deduced only from observations. Table 5.A gives the value of ΔT for some years.

Table 5.A
Value of ΔT in minutes of time

Year	ΔT	*Year*	ΔT	*Year*	ΔT
1710	−0.2	1870	0.0	1940	+0.4
1730	−0.1	1880	−0.1	1950	+0.5
1750	0.0	1895	−0.1	1965	+0.6
1770	+0.1	1903	0.0	1971	+0.7
1800	+0.1	1912	+0.2	1977	+0.8
1840	0.0	1927	+0.4	1987	+1.0 ?

For epochs outside this time interval, an *approximate* value of ΔT (in minutes) can be calculated from

$$\Delta T = +0.41 + 1.2053\,T + 0.4992\,T^2 \tag{5.1}$$

where T is the time in centuries since 1900. We then have

$$\text{UT} = \text{ET} - \Delta T \qquad \text{or} \qquad \text{ET} = \text{UT} + \Delta T$$

Example 5.a : Suppose that the position of Mercury should be calculated for February 6 at 6^h Universal Time of the year −555.

Here we have

$$T = -24.55, \quad \text{whence} \quad \Delta T = +272 \text{ minutes.}$$

Thus

$$\text{ET} = 6^h + 272 \text{ minutes} = 10^h32^m$$

and the calculations should be performed for −555 February 6 at 10^h32^m ET.

Example 5.b : According to the *Astronomical Ephemeris*, the maximum phase of the lunar eclipse of 1977 April 4 took place at $4^h19^m.0$ Ephemeris Time.

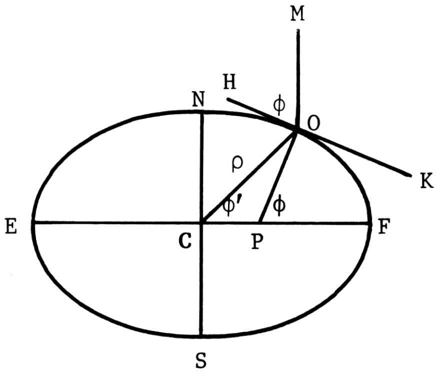
From Table 5.A, ΔT = +0.8 minute in 1977. The corresponding UT was thus

$$4^h19^m.0 - 0^m.8 = 4^h18^m.2.$$

6

GEOCENTRIC RECTANGULAR COORDINATES OF AN OBSERVER

The figure represents a meridian cross-section of the Earth. *C* is the Earth's center, *N* its north pole, *S* its south pole, *EF* the equator, *HK* the horizontal plane of the observer *O*, and *OP* is perpendicular to *HK*. The direction *OM*, parallel to *SN*, makes with *OH* an angle ϕ which is the *geographical latitude* of *O*. The angle *OPF* too is equal to ϕ.



The radius vector *OC*, joining the observer to the center of the Earth, makes with the equator *CF* an angle ϕ' which is the *geocentric latitude* of *O*. We have $\phi = \phi'$ at the poles and at the equator ; for all other latitudes

$$|\phi'| < |\phi|$$

Let f be the Earth's flattening, and b/a the ratio *NC*/*CF* of the polar radius to the equatorial radius. With the value $f = 1/298.257$ now adopted by the International Astronomical Union, we have

$$\frac{b}{a} = 1 - f = 0.996\,647\,19$$

For a place at sea-level, we have

$$\tan \phi' = \frac{b^2}{a^2} \tan \phi$$

If H is the observer's height above sea-level in *meters*, the quantities $\rho \sin \phi'$ and $\rho \cos \phi'$, needed in the calculation of diurnal parallaxes, eclipses and occultations, may be calculated as follows :

$$\tan u = \frac{b}{a} \tan \phi$$

$$\begin{cases} \rho \sin \phi' = \dfrac{b}{a} \sin u + \dfrac{H}{6\,378\,140} \sin \phi \\ \rho \cos \phi' = \cos u + \dfrac{H}{6\,378\,140} \cos \phi \end{cases}$$

$\rho \sin \phi'$ is positive in the northern hemisphere, negative in the southern one, while $\rho \cos \phi'$ is always positive.

The quantity ρ denotes the observer's distance to the center of the Earth (OC in the Figure).

Exercise. - Calculate $\rho \sin \phi'$ and $\rho \cos \phi'$ for the Uccle Observatory, for which $\phi = +50°47'55''$ and $H = 105$ meters.

(Answer : $\rho \sin \phi' = +0.771\,306$ and $\rho \cos \phi' = +0.633\,333$).

7

SIDEREAL TIME AT GREENWICH

The sidereal time at Greenwich at 0^h Universal Time of a given date can be obtained as follows.

Calculate the JD corresponding to that date at 0^h UT (see Chapter 3). Thus, this is a number ending on ·5. Then find T by

$$T = \frac{\text{JD} - 2415\,020.0}{36525} \qquad (7.1)$$

The sidereal time at Greenwich at 0^h UT, expressed in hours and decimals, is then

$$\theta_\circ = 6.646\,0656 + 2400.051\,262\,T + 0.000\,025\,81\,T^2 \qquad (7.2)$$

The result should be reduced to the interval 0 - 24 hours, and then converted to hours, minutes and seconds if necessary.

To reduce $\theta_\circ$ to the interval 0 - 24 hours, it may be easier to divide the numerical values in formula (7.2) by 24 ; this gives

$$\theta_\circ = 0.276\,919\,398 + 100.002\,1359\,T + 0.000\,001\,075\,T^2 \qquad (7.3)$$

This gives the sidereal time in *revolutions*. Multiply the *fractional* part of the result by 24 in order to obtain $\theta_\circ$ in hours.

It is important to note that the formulae (7.2) and (7.3) are valid only for those values of T which correspond to 0^h UT of a given date.

Example 7.a : Find the sidereal time at Greenwich on 1978 November 13 at 0^h Universal Time.

We find

$$\text{JD} = 2443\,825.5 \qquad T = +0.788\,651\,6085$$

and then, by formula (7.3),

$$\begin{aligned}\theta_\circ &= 79.143\,765\,40 \text{ revolution}\\ &= 0.143\,765\,40 \text{ revolution}\\ &= 3.450\,3696 \text{ hours}\\ &= 3^h\,27^m\,01^s\!.331\end{aligned}$$

The *A.E.* gives the same value.

To find the sidereal time *at Greenwich* for any instant UT of a given date, express that instant in hours and decimals, multiply by 1.002 737 908, and add the result to the sidereal time at 0*h* UT.

Example 7.b : Find the sidereal time at Greenwich on 1978 November 13 at $4^h34^m00^s$ UT.

In the preceding example, we have found that the sidereal time at 0^h on that date is 3.450 3696 hours.

$$4^h34^m00^s = 4^h\!.566\,6667$$

$$4^h\!.566\,6667 \times 1.002\,737\,908 = 4^h\!.579\,1698$$

Hence, the required sidereal time is

$$\begin{aligned}\theta_\circ = 3^h\!.450\,3696 + 4^h\!.579\,1698 &= 8^h\!.029\,5394\\ &= 8^h01^m46^s\!.342\end{aligned}$$

The sidereal time obtained by formulae (7.2) or (7.3) is the *mean* sidereal time. The *apparent* sidereal time is obtained by adding the correction $\Delta\psi \cos \varepsilon$, where $\Delta\psi$ is the nutation in longitude (see Chapter 15), and ε the obliquity of the ecliptic. This correction for nutation is called *nutation in right ascension* (or *equation of the equinoxes* in the *A.E.*). The value of ε can here be taken to the nearest $10''$; if $\Delta\psi$ is expressed in seconds of a degree, the correction in seconds of time is

$$\frac{\Delta\psi \cos \varepsilon}{15}$$

Example 7.c : Find the apparent sidereal time at Greenwich on 1978 November 13 at $4^h34^m00^s$ UT.

From Example 7.b, the mean sidereal time at Greenwich for that instant is $8^h01^m46^s\!.342$, while $\Delta\psi = -3''\!.378$ (see Example 15.a). Taking $\varepsilon = 23°26'30''$, the correction to the sidereal time is

$$\frac{-3.378 \times \cos 23°26'30''}{15} = -0.207 \text{ second}$$

and the required apparent sidereal time is

$$8^h01^m46\overset{s}{.}342 - 0\overset{s}{.}207 = 8^h01^m46\overset{s}{.}135$$

8

TRANSFORMATION OF COORDINATES

We will use the following symbols :

α = right ascension. This quantity is generally expressed in hours, minutes and seconds, and thus should firstly be converted into degrees (and decimals) before to be used in a formula. Conversely, if α has been obtained by means of a formula and a calculating machine, it is expressed in degrees ; it may be converted to hours by division by 15, and then, if necessary, be converted into hours, minutes and seconds ;

δ = declination, positive (negative) if north (south) of the celestial equator ;

α_{1950} = right ascension referred to the standard equinox of 1950.0;

δ_{1950} = declination referred to the standard equinox of 1950.0 ;

λ = ecliptical (or celestial) longitude, measured from the vernal equinox along the ecliptic ;

β = ecliptical (or celestial) latitude, positive (negative) if north (south) of the ecliptic ;

l = galactic longitude ;

b = galactic latitude ;

h = altitude, positive (negative) if above (below) the horizon;

A = azimuth, measured westward from the *South*. It should be noted that several authors measure the azimuth from the North. We prefer to count it from the South, because the hour angles too are measured from the South. Thus, a celestial body which is exactly in the southern meridian has $A = H = 0°$;

ε = obliquity of the ecliptic ; this is the angle between the ecliptic and the celestial equator. The mean obliquity of the ecliptic is given by formula (18.4). If, however, the *apparent* right ascension and declination are used (that is, affected by the aberration and the nutation), the true obliquity $\varepsilon + \Delta\varepsilon$ should be used (see Chapter 15). If α and δ are referred to the standard equinox of 1950, then the value of ε for this epoch should be used, namely $\varepsilon_{1950} = 23\overset{\circ}{.}445\,7889$. For the standard equinox of 2000.0, we have $\varepsilon_{2000} = 23°26'21\overset{''}{.}448 = 23\overset{\circ}{.}439\,2911$;

ϕ = the observer's latitude, positive (negative) if in the northern (southern) hemisphere ;

H = the local hour angle, measured westward from the South.

If θ is the local sidereal time, θ_0 the sidereal time at Greenwich, and L the observer's longitude (positive west, negative east from Greenwich), then the local hour angle can be calculated from

$$H = \theta - \alpha \qquad \text{or} \qquad H = \theta_0 - L - \alpha$$

If α is affected by the nutation, then the sidereal time too must be affected by it (see Chapter 7).

For the transformation from equatorial into ecliptical coordinates, the following formulae can be used :

$$\tan \lambda = \frac{\sin \alpha \cos \varepsilon + \tan \delta \sin \varepsilon}{\cos \alpha} \qquad (8.1)$$

$$\sin \beta = \sin \delta \cos \varepsilon - \cos \delta \sin \varepsilon \sin \alpha \qquad (8.2)$$

Transformation from ecliptical into equatorial coordinates :

$$\tan \alpha = \frac{\sin \lambda \cos \varepsilon - \tan \beta \sin \varepsilon}{\cos \lambda} \qquad (8.3)$$

$$\sin \delta = \sin \beta \cos \varepsilon + \cos \beta \sin \varepsilon \sin \lambda \qquad (8.4)$$

Calculation of the local horizontal coordinates :

$$\tan A = \frac{\sin H}{\cos H \sin \phi - \tan \delta \cos \phi} \qquad (8.5)$$

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H \qquad (8.6)$$

Transformation from equatorial coordinates, referred to the standard equinox of 1950.0, into galactic coordinates :

$$\tan x = \frac{\sin (192^\circ\!.25 - \alpha)}{\cos (192^\circ\!.25 - \alpha) \sin 27^\circ\!.4 - \tan \delta \cos 27^\circ\!.4} \qquad (8.7)$$

$$l = 303^\circ - x$$

$$\sin b = \sin \delta \sin 27^\circ\!.4 + \cos \delta \cos 27^\circ\!.4 \cos (192^\circ\!.25 - \alpha) \qquad (8.8)$$

Transformation from galactic coordinates into equatorial coordinates referred to the standard equinox of 1950.0 :

$$\tan y = \frac{\sin (l - 123^\circ)}{\cos (l - 123^\circ) \sin 27^\circ\!.4 - \tan b \cos 27^\circ\!.4} \qquad (8.9)$$

$$\alpha = y + 12^\circ\!.25$$

$$\sin \delta = \sin b \sin 27^\circ\!.4 + \cos b \cos 27^\circ\!.4 \cos (l - 123^\circ) \qquad (8.10)$$

Note the similitude of the formulae (8.1), (8.3), (8.5), (8.7) and (8.9). They can be calculated in one single subroutine. The same remark applies to the formulae (8.2), (8.4), (8.6), (8.8) and (8.10).

The formulae (8.1), (8.3), etc. give tan λ, tan α, etc., and then λ, α, etc. by the function arctan. However, the exact quadrant where the angle is situated in then unknown. It is better *not* to calculate the tangent of the angle by not making the division ; instead, apply the conversion from rectangular into polar coordinates on the numerator and the denominator of the fraction ; this will give the angle λ, α, etc. directly in the correct quadrant.

Example 8.a : Calculate the ecliptical coordinates of Pollux, whose equatorial coordinates are

$$\alpha_{1950} = 7^h42^m15^s\!.525, \qquad \delta_{1950} = +28^\circ08'55''\!.11 .$$

Using the values $\alpha = 115^\circ\!.564\,688$, $\delta = +28^\circ\!.148\,642$, and $\varepsilon = 23^\circ\!.445\,7889$, formulae (8.1) and (8.2) give

$$\tan \lambda = \frac{+1.040\,5017}{-0.431\,5299}, \qquad \text{whence} \quad \lambda = 112^\circ\!.525\,38 \;;$$

$$\beta = +6^\circ\!.68\,058 .$$

Because α and δ are referred to the standard equinox of 1950.0, λ and β too are referred to that equinox.

Exercise. - Using the values of λ and β found in the preceding Example, find α and δ again by means of formulae (8.3) and (8.4).

Example 8.b : Find the azimut and the altitude of Saturn on 1978 November 13 at $4^h34^m00^s$ UT at the Uccle Observatory (longitude $-0^h17^m25^s\!.94$, latitude $+50°47'55''\!.0$); the planet's apparent equatorial coordinates, interpolated from the *A.E.*, being

$$\alpha = 10^h57^m35^s\!.681 \qquad \delta = +8°25'58''\!.10$$

Since these are the *apparent* right ascension and declination, we need the *apparent* sidereal time. The latter has been calculated, for the given instant, in Example 7.c, namely $\theta_o = 8^h01^m46^s\!.135$. We thus have

$$\begin{aligned} H &= \theta_o - L - \alpha \\ &= 8^h01^m46^s\!.135 + 0^h17^m25^s\!.94 - 10^h57^m35^s\!.681 \\ &= -2^h38^m23^s\!.606 = -2^h\!.639\,8906 = -39°\!.598\,358 \end{aligned}$$

Formulae (8.5) and (8.6) then give

$$\tan A = \frac{-0.637\,4019}{+0.503\,4048}, \quad \text{whence} \quad A = -51°\!.6992$$

$$h = +36°\!.5405$$

Exercise. - Find the galactic coordinates of Nova Serpentis 1978, whose equatorial coordinates are

$$\alpha_{1950} = 17^h48^m59^s\!.74, \qquad \delta_{1950} = -14°43'08''\!.2$$

(Answer : $l = 12°\!.9593$, $b = +6°\!.0463$)

Rise or set of a body

The hour angle corresponding to the time of rise or set of a body is obtained by putting $h = 0$ in formula (8.6). This gives

$$\cos H = -\tan \phi \tan \delta$$

However, the instant so obtained refers to the geometric rise or set of the center of the celestial body.

By reason of the atmospheric refraction, the body is actually below the horizon at the instant of its apparent rise or set. The value of 0°34′ is generally adopted for the effect of refraction at the horizon. For the Sun, the calculated times generally refer to the apparent rise or set of the upper limb of the disk ; hence, 0°16′ should be added for the semidiameter. The hour angle H_0 at the time of rise or set should thus be calculated from

$$\cos H_0 = \frac{-0.00\,989 - \sin \phi \sin \delta}{\cos \phi \cos \delta} \qquad \text{for stars and planets;}$$

$$\cos H_0 = \frac{-0.01\,454 - \sin \phi \sin \delta}{\cos \phi \cos \delta} \qquad \text{for the Sun.}$$

In the case of the Moon, the effect of the horizontal parallax also should be taken into account.

The value of $\cos H_0$ being given, there are two possible values for H_0 :

$$-180° < H_0 < 0° \qquad \text{for the rise,}$$

$$0° < H_0 < +180° \qquad \text{for the set.}$$

Pocket calculators generally give the value between 0° and +180° by pressing the key for arcus cosinus (incorrectly labelled $\cos^{-1}$ on many machines). In that case, the sign of H_0 should be changed in the case the time of rise is to be found. This can be performed by the use of a flag, which is set or cleared in the beginning of the program, and which later is interrogated.

The azimut of a star at the time of its *geometric* rise or set is given by

$$\cos A_0 = -\frac{\sin \delta}{\cos \phi}$$

where A_0 should be taken between 180° and 360° (or between −180° and 0°) for the rise, and between 0° and 180° for the set.

Ecliptic and Horizon

If ε = obliquity of the ecliptic,
ϕ = latitude of the observer,
θ = local sidereal time,

then the longitudes of the two points of the ecliptic which are on the horizon are given by

$$\tan \lambda = \frac{-\cos \theta}{\sin \varepsilon \tan \phi + \cos \varepsilon \sin \theta} \qquad (8.11)$$

The angle I between the ecliptic and the horizon is given by

$$\cos I = \cos \varepsilon \sin \phi - \sin \varepsilon \cos \phi \sin \theta \qquad (8.12)$$

Example 8.c : For $\varepsilon = 23^\circ\!.44$, $\phi = +51^\circ$, $\theta = 5^h00^m = 75^\circ$, we find, from formula (8.11),

$\tan \lambda = -0.1879$, whence $\lambda = 169^\circ 21'$ and $\lambda = 349^\circ 21'$.

Formula (8.12) gives $I = 62^\circ$.

Exercices

How does I vary in the course of a sidereal day ?

What happens with formula (8.11) when $\phi = 90^\circ - \varepsilon$ and $\theta = 18^h$? Explain.

9

ANGULAR SEPARATION

The angular distance d between two celestial bodies, whose right ascensions and declinations are known, is given by the formula

$$\cos d = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos (\alpha_1 - \alpha_2) \qquad (9.1)$$

where α_1 and δ_1 are the right ascension and the declination of one body, and α_2 and δ_2 those of the other one.

The same formula may be used when the ecliptical (celestial) longitudes λ and latitudes β of the two bodies are given, provided that α_1, α_2, δ_1 and δ_2 are replaced by λ_1, λ_2, β_1 and β_2.

Formula (9.1) may not be used when d is very near to 0° or 180° because in that case $|\cos d|$ is nearly equal to 1 and varies very slowly with d, so that d cannot be found accurately. For example :

$$\begin{aligned} \cos 0°01'00'' &= 0.999\,999\,958 \\ \cos 0°00'30'' &= 0.999\,999\,989 \\ \cos 0°00'15'' &= 0.999\,999\,997 \\ \cos 0°00'00'' &= 1.000\,000\,000 \end{aligned}$$

If the angular separation is very small, say less than 0°10' or 0°05', this separation should be calculated from

$$d = \sqrt{(\Delta\alpha \,.\, \cos \delta)^2 + (\Delta\delta)^2} \qquad (9.2)$$

where $\Delta\alpha$ is the difference between the right ascensions, $\Delta\delta$ the difference between the declinations, while δ is the declination of any of the two bodies. It should be noted that $\Delta\alpha$ and $\Delta\delta$ should be expressed in the same angular units.

If $\Delta\alpha$ is expressed in hours (and decimals), $\Delta\delta$ in degrees (and decimals), then d expressed in seconds of a degree (″) is given by

$$d = 3600 \sqrt{(15\,\Delta\alpha \,.\, \cos \delta)^2 + (\Delta\delta)^2} \qquad (9.3)$$

If $\Delta\alpha$ is expressed in seconds of time (^{s}), and $\Delta\delta$ in seconds of a degree ($''$), then d expressed in $''$ is given by

$$d = \sqrt{(15\,\Delta\alpha\,.\cos\,\delta)^2 + (\Delta\delta)^2} \tag{9.4}$$

Formulae (9.2), (9.3) and (9.4) may be used only when d is small.

Example 9.a : Calculate the angular distance between Arcturus (α Boo) and Spica (α Vir).

The 1950 coordinates of these stars are

α Boo : $\alpha_1 = 14^h13^m22^s\!.8 = 213^\circ\!.3450$ $\qquad \delta_1 = +19^\circ26'31''$

α Vir : $\alpha_2 = 13^h22^m33^s\!.3 = 200^\circ\!.6388$ $\qquad \delta_2 = -10^\circ54'03''$

Formula (9.1) gives $\cos d = +0.840\,342$,
whence $d = 32^\circ\!.8237 = 32^\circ49'$

Exercise. - Calculate the angular distance between Aldebaran and Antares. (Answer : 169°58′)

One or both bodies may be moving objects. For example : a planet and a star, or two planets. In that case, a program may be written where firstly the quantities δ_1, δ_2 and $(\alpha_1 - \alpha_2)$ are interpolated, after which d is calculated by means of one of the formulae (9.1) or (9.2). Hint : from the interpolated quantities, calculate $\cos d$ by means of formula (9.1). Then, if $\cos d < 0.999\,995$, find d ; if $\cos d > 0.999\,995$, use formula (9.2).

Exercise. - Using the following coordinates, calculate the instant and the value of the least distance between Mercury and Saturn.

1978 *0 h ET*	*Mercury* α_1	*Mercury* δ_1	*Saturn* α_2	*Saturn* δ_2
Sep 12	$10^h23^m17^s\!.65$	$+11^\circ31'46''\!.3$	$10^h33^m01^s\!.23$	$+10^\circ42'53''\!.5$
13	10 29 44.27	+11 02 05.9	10 33 29.64	+10 40 13.2
14	10 36 19.63	+10 29 51.7	10 33 57.97	+10 37 33.4
15	10 43 01.75	+ 9 55 16.7	10 34 26.22	+10 34 53.9
16	10 49 48.85	+ 9 18 34.7	10 34 54.39	+10 32 14.9

Answer : The least angular separation between the two planets is $0^\circ03'44''$, on 1978 September 13 at $15^h06^m\!.5$ ET = 15^h06^m UT.

The same method can be used if one of the bodies is a star. The latter's coordinates are then constant. It is important to note that the α and δ of the star should be referred to the *same equinox* as that of the moving body.

If the moving body is a major planet, whose apparent right ascension and apparent declination referred to the equinox of the date are given (as in the *A.E.*, for instance), then for the star the apparent coordinates too must be used. If one takes the star's position from a catalogue, where they are referred to a standard equinox (for instance that of 1950.0), then the apparent α and δ are found by taking into account the proper motion of the star and the effects of precession, nutation and aberration.

If the α and δ of the moving body are referred to a standard equinox, then the α and δ of the star should be referred to this same standard equinox, the only corrections being those for the proper motion of the star.

10

CONJUNCTION BETWEEN TWO PLANETS

Given three or five ephemeris positions of two planets passing near each other, a program can be written which calculates the time of conjunction *in right ascension*, and the difference in declination between the two bodies at that time. The method consists of calculating the differences $\Delta\alpha$ of the corresponding right ascensions, and then calculating the time when $\Delta\alpha = 0$ by inverse interpolation, using formula (2.4) or (2.7). When that instant is found, direct interpolation of the differences $\Delta\delta$ of the declinations, by means of formula (2.3) or (2.5), yields the required difference in declination at the time of conjunction.

Example 10.a : Calculate the circumstances of the Mercury-Venus conjunction of 1979 November.

The following values, for 0^h ET of the date, are taken from the *Astronomical Ephemeris* :

1979	Mercury		Venus	
	α	δ	α	δ
	$h \quad m \quad s$	$^\circ \quad ' \quad ''$	$h \quad m \quad s$	$^\circ \quad ' \quad ''$
Nov. 7	16 11 38.61	−23 49 45.9	16 04 01.76	−21 07 49.3
8	16 12 55.61	−23 46 54.0	16 09 14.55	−21 24 26.5
9	16 13 40.37	−23 41 19.5	16 14 28.50	−21 40 27.5
10	16 13 50.08	−23 32 50.8	16 19 43.58	−21 55 51.7
11	16 13 22.16	−23 21 16.3	16 24 59.76	−22 10 38.4

We firstly calculate the differences of the right ascensions (in hours and decimals) and of the declinations (in degrees and decimals) :

Nov. 7	$\Delta\alpha = +0.126\,903$	$\Delta\delta = -2.699\,06$
8	$+0.061\,406$	$-2.374\,31$
9	$-0.013\,369$	$-2.014\,44$
10	$-0.098\,194$	$-1.616\,42$
11	$-0.193\,778$	$-1.177\,19$

Applying formula (2.7) to the values of $\Delta\alpha$, we find that $\Delta\alpha$ is zero for the value $n = -0.16960$ of the interpolation interval. Hence, the conjunction in right ascension takes place on 1979 November 8.83040, that is on 1979 November 8 at $19^h55^m.8$ ET, or at 19^h55^m UT.

With the value of n just found, and applying formula (2.5) to the values of $\Delta\delta$, we find $\Delta\delta = -2^\circ.07808$ or $-2°05'$. Thus, at the time of conjunction in right ascension, Mercury is 2°05′ south of Venus.

If the second body is a star, its coordinates may be considered as constant during the time interval considered. We then have

$$\alpha_1' = \alpha_2' = \alpha_3' = \alpha_4' = \alpha_5' \quad \text{and} \quad \delta_1' = \delta_2' = \delta_3' = \delta_4' = \delta_5'$$

The program can be written in such a manner that, if the second object is a star, its coordinates must be entered only once. To achieve this goal, use labels, flags and/or subroutines !

The important remark at the end of Chapter 9 does apply here too : *the coordinates of the star and those of the moving body must be referred to the same equinox.*

As an exercise, calculate the conjunction in right ascension between the minor planet 29 Amphitrite and the star λ Leonis in January 1980. The minor planet's right ascension and declination, referred to the standard equinox of 1950.0, are as follows (from an ephemeris calculated by David W. Dunham) :

0^h ET	α_{1950}	δ_{1950}
1980 January 7	$9^h34^m25^s.279$	+22°06′40″.93
12	9 31 10.656	+22 22 25.44
17	9 27 15.396	+22 39 04.68
22	9 22 45.672	+22 55 48.95
27	9 17 49.742	+23 11 46.00

The star's coordinates for the epoch and equinox of 1950.0 are $\alpha = 9^h28^m52^s.248$ and $\delta = +23°11'22''.21$, and the annual proper motion is $-0^s.0018$ in right ascension and $-0''.042$ in declination. Consequently, the star's position referred to the equinox of 1950.0 but for the epoch 1980.04 is

$$\alpha = 9^h28^m52^s.194\ , \qquad \delta = +23°11'20''.95$$

Now, calculate the conjunction.

(Answer : Amphitrite passes 0°39′ south of λ Leo on 1980 January 15 at 1 h).

11

BODIES IN STRAIGHT LINE

Let (α_1, δ_1), (α_2, δ_2), (α_3, δ_3) be the equatorial coordinates of three heavenly bodies. These three bodies are in "straight line" — that is, they lie on the same great circle of the celestial sphere — if

$$\tan \delta_1 \sin(\alpha_2 - \alpha_3) + \tan \delta_2 \sin(\alpha_3 - \alpha_1) + \tan \delta_3 \sin(\alpha_1 - \alpha_2) = 0 \qquad (11.1)$$

This formula is valid for ecliptical coordinates too, the right ascensions α being replaced by the longitudes λ, and the declinations δ by the latitudes β.

Do not forget that the right ascensions α are generally expressed in hours, minutes and seconds. They should firstly be converted into hours and decimals, and then into degrees by multiplication by 15.

If one or two of the bodies are stars, then once again the important remark at the end of Chapter 9 does apply : *the coordinates of the star(s) must be referred to the same equinox as that of the planet(s).*

Example 11.a : Find the time when Mars is seen in straight line with Pollux and Castor in 1979.

From an ephemeris of Mars and a star atlas, it is easily found that the planet is in straight line with the two stars about 1979 September 21. For this date, the apparent coordinates of the stars are :

Castor (α Gem) : $\alpha_1 = 7^h33^m17\overset{s}{.}0 = 113\overset{\circ}{.}3208$
$\delta_1 = +31°55'54'' = +31\overset{\circ}{.}9317$

Pollux (β Gem) : $\alpha_2 = 7^h44^m03\overset{s}{.}3 = 116\overset{\circ}{.}0138$
$\delta_2 = +28°04'28'' = +28\overset{\circ}{.}0744$

These values have been taken from the Soviet almanac *Astronomicheskii Ezhegodnik* for 1979, pages 360 and 361, but they could have been calculated by means of the method described in Chapter 16. For our problem, these values of α_1 , δ_1 , α_2 and δ_2 may be considered as constants for several days.

The apparent coordinates of Mars (α_3 , δ_3) are variable. Here are the values taken from the *Astronomical Ephemeris*

ET	α_3	δ_3
1979 Sep. 19.0	$7^h54^m33^s.8$ = 118°.6408	+21°43′19″ = +21°.7219
20.0	7 57 08.6 = 119.2858	+21 37 12 = +21.6200
21.0	7 59 42.7 = 119.9279	+21 30 57 = +21.5158
22.0	8 02 16.2 = 120.5675	+21 24 36 = +21.4100
23.0	8 04 49.0 = 121.2042	+21 18 08 = +21.3022

Using all these values, the first member of formula (11.1) takes the following values :

September 19.0	+0.002 1713
20.0	+0.001 2369
21.0	+0.000 3067
22.0	−0.000 6204
23.0	−0.001 5434

Using formula (2.7), we find that the value is zero for

1979 September 21.3304
= 1979 September 21 at 8^h ET (UT).

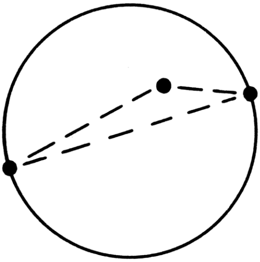
12

SMALLEST CIRCLE CONTAINING THREE CELESTIAL BODIES

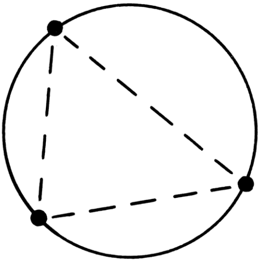
Let A, B, C be three celestial bodies situated not too far from each other on the celestial sphere, say closer than about 6°. We wish to calculate the angular diameter of the smallest circle containing these three bodies. Two cases can occur :

type I : the smallest circle has as diameter the longest side of the triangle ABC ;

type II : the smallest circle is the circle passing through the three points A, B, C.



Type I



Type II

The diameter Δ of the smallest circle can be found as follows. Calculate the lengths of the three sides of the triangle ABC (in degrees) by means of formula (9.1). Formula (9.2) will rarely be required for the present problem.

Let a be the length of the *longest* side of the triangle, and b and c the lengths of the two other sides.

If $a > \sqrt{b^2 + c^2}$, then $\Delta = a$;

if $a < \sqrt{b^2 + c^2}$, then

$$\Delta = \frac{2\,a\,b\,c}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(a+c-b)}} \qquad (12.1)$$

Example 12.a : Calculate the diameter of the smallest circle containing Mercury, Jupiter and Saturn on 1981 September 11 at 0 *h* Ephemeris Time. The positions of these planets at that instant are :

Mercury	$\alpha = 12^h41^m08^s.63$	$\delta = -5°37'54''.2$
Jupiter	12 52 05.21	−4 22 26.2
Saturn	12 39 28.11	−1 50 03.7

The three angular separations are found by means of formula (9.1):

Mercury - Jupiter	3°.00152
Mercury - Saturn	3.82028
Jupiter - Saturn	4.04599 = a

Because $4.04599 < \sqrt{(3.00152)^2 + (3.82028)^2} = 4.85836$, we calculate Δ by means of formula (12.1). The result is

$$\Delta = 4°.26364 = 4°16'$$

This is an example of type II.

Exercise. – Perform the same calculation for the planets Venus, Jupiter and Saturn on 1981 August 29 at 0^h ET, using the following positions :

Venus	$\alpha = 12^h46^m00^s.82$	$\delta = -4°38'59''.7$
Jupiter	12 42 31.51	−3 20 36.0
Saturn	12 34 03.49	−1 14 18.2

Show that this case is of type I, and that $\Delta = 4°32'$.

A program can be written in which firstly the right ascensions and the declinations of the planets are interpolated, after which Δ is calculated. In that case, a test is necessary to compare a

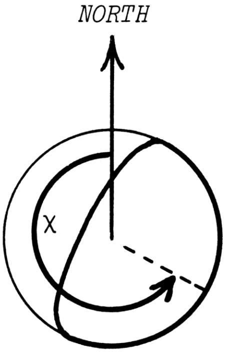
with $\sqrt{b^2 + c^2}$. With such a program, it is possible to calculate (by trial) the minimum value of Δ of a grouping of three planets. Indeed, Δ varies with time, and the method described in this Chapter provides the value of Δ only for a given instant.

Such a program has been used by the author to calculate all the planetary "trios" occurring during the period 1960 - 2005. This list has been published in the French journal *L'Astronomie*, Volume 91, pages 487 - 493 (December 1977).

If one of the bodies is a star, once again the important remark at the end of Chapter 9 does apply : the coordinates of the star should be referred to the same equinox as those of the planets.

13

POSITION ANGLE OF THE MOON'S BRIGHT LIMB



The position angle of the Moon's bright limb is the position angle χ of the *mid-point* of the illuminated limb of the Moon, reckoned eastward from the North Point of the disk (see the Figure).

Let α and δ be the right ascension and the declination of the Sun, α' and δ' the right ascension and the declination of the Moon. Do not forget to express all these quantities in degrees and decimals !

Then the position angle χ of the Moon's bright limb is given by the formula

$$\tan \chi = \frac{\cos \delta \sin (\alpha - \alpha')}{\cos \delta' \sin \delta - \sin \delta' \cos \delta \cos (\alpha - \alpha')}$$

The angle χ is in the vicinity of 270° near First Quarter, near 90° after Full Moon. However, χ is found immediately in the correct quadrant by applying the conversion from rectangular to polar coordinates to the numerator and the denominator of the fraction in the preceding formula.

Example 13.a : Find the position angle of the Moon's bright limb on 1979 February 2 at 21^h ET.

The Moon's equatorial coordinates for that instant are given on page 78 of the *A.E.* for 1979 :

$\alpha' = 1^h54^m18^s\!.175 = 1^h\!.90505 = 28^\circ\!.5757$

$\delta' = +8^\circ01'47''\!.59 = +8^\circ\!.0299$

The coordinates of the Sun are found, by interpolation, from the values given on page 21 of the same publication :

$\alpha = 21\overset{h}{.}05953 = 315\overset{\circ}{.}8930$
$\delta = -16\overset{\circ}{.}7915$

We then find

$$\tan \chi = \frac{-0.91397}{-0.32586}$$

from which $\chi = -109\overset{\circ}{.}62 = 250\overset{\circ}{.}38$.

14

PRECESSION

In this Chapter, we consider the problem of converting the right ascension α and the declination δ of a star, given for an epoch and an equinox, to the corresponding values for another epoch and equinox. Only the *mean* places of a star and the effect of the precession alone are considered here. The problem of finding the apparent place of a star will be considered in Chapter 16.

If no great accuracy is required, and if the two epochs are not widely separated, the following formulae may be used for the annual precessions in right ascension and declination :

$$\Delta\alpha = m + n \sin\alpha \tan\delta \qquad\qquad \Delta\delta = n \cos\alpha \tag{14.1}$$

where m and n are two quantities which vary slowly with time. They are given by

$$m = 3^{\mathrm{s}}.07234 + 0^{\mathrm{s}}.00186\,T\,,$$

$$n = 20''.0468 - 0''.0085\,T\,,$$

T being the time measured in centuries from 1900.0. Here are the values of m and n for some epochs :

Epoch	m	n	n
1700.0	$3^{\mathrm{s}}.069$	$1^{\mathrm{s}}.338$	$20''.06$
1800.0	3.070	1.337	20.06
1900.0	3.072	1.336	20.05
2000.0	3.074	1.336	20.04
2100.0	3.076	1.335	20.03
2200.0	3.078	1.335	20.02

For the calculation of $\Delta\alpha$, the value of n expressed in seconds of time ($^{\mathrm{s}}$) must be used. Remember that 1^{s} corresponds to $15''$.

The effect of the proper motion should be added to the values given by the formulae (14.1).

Example 14.a : The coordinates of Regulus for the epoch and equinox of 1950.0 are

$$\alpha_\circ = 10^h05^m42^s\!.7 \qquad \delta_\circ = +12°12'45''$$

and the annual proper motions are

$-0^s\!.0171$ in right ascension,
$+0''\!.004$ in declination.

Reduce these coordinates to the epoch and the equinox of 1978.0.

Here we have

$$\alpha = 151°\!.428 \qquad \delta = +12°\!.213$$

$$m = 3^s\!.073 \qquad n = 1^s\!.336 = 20''\!.04$$

From the formulae (14.1) we deduce

$$\Delta\alpha = +3^s\!.211, \qquad \Delta\delta = -17''\!.60$$

to which we must add the annual proper motion, giving an annual variation of $+3^s\!.194$ in right ascension, and of $-17''\!.60$ in declination.

Variations during 28 years (from 1950.0 to 1978.0) :

in α : $+3^s\!.194 \times 28 = +89^s\!.4 = +1^m29^s\!.4$

in δ : $-17''\!.60 \times 28 = -493'' = -8'13''$

Required right ascension : $\alpha = \alpha_\circ + 1^m29^s\!.4 = 10^h07^m12^s\!.1$

Required declination : $\delta = \delta_\circ - 8'13'' = +12°04'32''$

The *A.E.* for 1978, page 336, gives $10^h07^m12^s\!.1$ and $+12°04'31''$.

Rigorous method

Newcomb gives the following numerical expressions for the quantities ζ, z and θ which are needed for the accurate reduction of positions from one equinox to another :

$$\begin{aligned} &\text{Initial epoch :} \quad t_0 = 1900.0 + \tau_0 \\ &\text{Final epoch :} \quad t = 1900.0 + \tau_0 + \tau \end{aligned}$$

$$\left.\begin{aligned} \zeta &= (2304''.250 + 1''.396\,\tau_0)\,\tau + 0''.302\,\tau^2 + 0''.018\,\tau^3 \\ z &= \zeta + 0''.791\,\tau^2 + 0''.001\,\tau^3 \\ \theta &= (2004''.682 - 0''.853\,\tau_0)\,\tau - 0''.426\,\tau^2 - 0''.042\,\tau^3 \end{aligned}\right\} \tag{14.2}$$

where τ_0 and τ are measured in *tropical* centuries of 36524.2199 ephemeris days. The fundamental epoch 1900.0 corresponds to JD 2415 020.313. The length of the tropical year is slightly variable with time by about −0.53 second per century, but this very small decrease may be neglected for our purpose.

In other words, if $(\mathrm{JD})_0$ and (JD) are the Julian Days corresponding to the initial and the final epoch, respectively, we have

$$\tau_0 = \frac{(\mathrm{JD})_0 - 2415\,020.313}{36524.2199} \qquad\qquad \tau = \frac{(\mathrm{JD}) - (\mathrm{JD})_0}{36524.2199}$$

For $t_0 = 1950.0 = \mathrm{JD}\ 2433\,282.423$, we have $\tau_0 = +0.5$ and the expressions (14.2) become

$$\left.\begin{aligned} \zeta &= 2304''.948\,\tau + 0''.302\,\tau^2 + 0''.018\,\tau^3 \\ z &= 2304''.948\,\tau + 1''.093\,\tau^2 + 0''.019\,\tau^3 \\ \theta &= 2004''.255\,\tau - 0''.426\,\tau^2 - 0''.042\,\tau^3 \end{aligned}\right\} \tag{14.3}$$

Then, the rigorous formulae for the reduction of the given equatorial coordinates α_0 and δ_0 of the epoch t_0 to the coordinates α and δ of the epoch t are :

$$\begin{aligned} A &= \cos\delta_0 \sin(\alpha_0 + \zeta) \\ B &= \cos\theta \cos\delta_0 \cos(\alpha_0 + \zeta) - \sin\theta \sin\delta_0 \\ C &= \sin\theta \cos\delta_0 \cos(\alpha_0 + \zeta) + \cos\theta \sin\delta_0 \end{aligned}$$

$$\tan(\alpha - z) = \frac{A}{B} \qquad\qquad \sin\delta = C$$

Apply the rectangular/polar coordinates transformation to the quantities A and B. This will give $(\alpha - z)$ directly in the correct quadrant, and also give $\cos\delta = \sqrt{A^2 + B^2}$ which may be used instead of $\sin\delta$ if the star is very close to the pole.

Before making the reduction from $\alpha_\circ$, $\delta_\circ$ to α, δ, calculate the effect of the star's proper motion.

Example 14.b : The star θ Persei has the following mean coordinates for the epoch and equinox of 1950.0 :

$$\alpha_\circ = 2^h40^m46^s.276 \qquad\qquad \delta_\circ = +49^\circ01'06''.45$$

and its annual proper motions referred to that same equinox are

$+0^s.0342$ in right ascension,
$-0''.083$ in declination.

Reduce the coordinates to the epoch and mean equinox of 1978 November 13.19 UT.

The initial epoch is 1950.0 or JD 2433 282.423, and the final one is JD 2443 825.69. Hence, τ = +0.288 665 tropical centuries, or 28.8665 years.

We firstly calculate the effect of the proper motion. The variations over 28.8665 years are

$+0^s.0342 \times 28.8665 = +0^s.987$ in right ascension,
$-0''.083 \times 28.8665 = -2''.40$ in declination.

Thus the star's coordinates, for the mean equinox of 1950.0, but for the epoch 1978 November 13.19, are

$$\alpha_\circ = 2^h40^m46^s.276 + 0^s.987 = 2^h40^m47^s.263 = +40^\circ.196\,929$$

$$\delta_\circ = +49^\circ01'06''.45 - 2''.40 = +49^\circ01'04''.05 = +49^\circ.017\,792$$

Since the initial equinox is that of 1950.0, we can use the formulae (14.3). With the value τ = +0.288 665, we obtain

$\zeta = +665''.383 = +0^\circ.184\,829$
$z = +665''.449 = +0^\circ.184\,847$
$\theta = +578''.522 = +0^\circ.160\,701$

$A = +0.424\,893\,97$
$B = +0.497\,451\,58$
$C = +0.756\,311\,48$

$$\alpha - z = +40^\circ\!.502\,010$$

$$\alpha = +40^\circ\!.686\,857 = 2^h42^m44^s\!.846$$

$$\delta = +49^\circ\!.140\,096 = +49^\circ08'24''\!.35$$

Exercise. - For the same star as in Example 14.b, calculate the equatorial coordinates for the epoch and mean equinox of 1981.0.

Answer : Here, $\tau = +0.31$, and one finds

$$\alpha = 2^h42^m53^s\!.626, \qquad \delta = +49^\circ08'56''\!.58.$$

Exercise. - The equatorial coordinates of α Ursae Minoris, for the epoch and mean equinox of 1950.0, are

$$\alpha = 1^h48^m48^s\!.786, \qquad \delta = +89^\circ01'43''\!.74$$

and the star's annual proper motions for the same equinox are

$+0^s\!.1811$ in right ascension,
$-0''\!.004$ in declination.

Find the coordinates of the star for the epochs and mean equinoxes of 1800.0, 1980.0 and 2100.0.

Answer :

1800.0	$\alpha = 0^h52^m25^s\!.31$	$\delta = +88^\circ14'24''\!.52$
1980.0	2 11 47.60	+89 10 24.41
2100.0	5 53 33.88	+89 32 21.81

It should be noted that the formulae (14.2) are valid only for a limited period of time. If we use them for the year 32 600, for instance, we find for that epoch that α UMi will be at declination -87°, a completely wrong result !

15

NUTATION

The nutation in longitude ($\Delta\psi$) and the nutation in obliquity ($\Delta\varepsilon$) are needed for the calculation of the apparent place of a star and for that of the apparent sidereal time. For a given instant, $\Delta\psi$ and $\Delta\varepsilon$ can be calculated as follows.

Find the time T, measured in Julian centuries from 1900 January 0.5, by means of the formula

$$T = \frac{\text{JD} - 2415\,020.0}{36525} \qquad (15.1)$$

where JD is the Julian Day (see Chapter 3). Then calculate the angles L, L', M, M' and Ω by means of the following formulae, in which the various constants are expressed in degrees and decimals. If T is small or when no high accuracy is required, the terms in T^2 may be neglected.

Sun's mean longitude :

$$L = 279.6967 + 36000.7689\,T + 0.000\,303\,T^2$$

Moon's mean longitude :

$$L' = 270.4342 + 481\,267.8831\,T - 0.001\,133\,T^2$$

Sun's mean anomaly :

$$M = 358.4758 + 35999.0498\,T - 0.000\,150\,T^2$$

Moon's mean anomaly :

$$M' = 296.1046 + 477\,198.8491\,T + 0.009\,192\,T^2$$

Longitude of Moon's ascending node :

$$\Omega = 259.1833 - 1934.1420\,T + 0.002\,078\,T^2$$

We then have, neglecting smaller quantities, and the coefficients being expressed in seconds of a degree (″) :

$$\begin{aligned}
\Delta\psi = &- (17.2327 + 0.01737\,T) \sin \Omega \\
&- (1.2729 + 0.00013\,T) \sin 2L \\
&+ 0.2088 \sin 2\Omega \\
&- 0.2037 \sin 2L' \\
&+ (0.1261 - 0.00031\,T) \sin M \\
&+ 0.0675 \sin M' \\
&- (0.0497 - 0.00012\,T) \sin (2L + M) \\
&- 0.0342 \sin (2L' - \Omega) \\
&- 0.0261 \sin (2L' + M') \\
&+ 0.0214 \sin (2L - M) \\
&- 0.0149 \sin (2L - 2L' + M') \\
&+ 0.0124 \sin (2L - \Omega) \\
&+ 0.0114 \sin (2L' - M')
\end{aligned}$$

$$\begin{aligned}
\Delta\varepsilon = &+ (9.2100 + 0.00091\,T) \cos \Omega \\
&+ (0.5522 - 0.00029\,T) \cos 2L \\
&- 0.0904 \cos 2\Omega \\
&+ 0.0884 \cos 2L' \\
&+ 0.0216 \cos (2L + M) \\
&+ 0.0183 \cos (2L' - \Omega) \\
&+ 0.0113 \cos (2L' + M') \\
&- 0.0093 \cos (2L - M) \\
&- 0.0066 \cos (2L - \Omega)
\end{aligned}$$

If no high accuracy is required, the smaller terms and the terms in T may be neglected. In the expressions for $\Delta\psi$ and $\Delta\varepsilon$, the first term has a period of 6798 days (18.61 years), and the second term a period of 182.62 days.

Example 15.a : Calculate $\Delta\psi$ and $\Delta\varepsilon$ for 1978 November 13 at 4^h35^m Ephemeris Time, that is for 4^h34^m Universal Time.

We find successively :

JD = 2443 825.69
T = +0.788 656 810
L = 28 671°.9485 = 231°.9485
L' = 379 825°.6269 = 25°.6269
M = 28 749°.3715 = 309°.3715

M' = 376 642°.2324 = 82°.2324
Ω = −1266°.1897 = +173°.8103

$\Delta\psi$ = −3″.378
$\Delta\varepsilon$ = −9″.321

According to the *A.E.* the correct values are −3″.383 and −9″.321, respectively.

16

APPARENT PLACE OF A STAR

The *mean place* of a star at any time is its apparent position on the celestial sphere, as it would be seen by an observer at rest on the Sun, and referred to the ecliptic and mean equinox of the date (or to the mean equator and mean equinox of the date).

The *apparent* place of a star at any time is its position on the celestial sphere as it is actually seen from the center of the moving Earth, and referred to the instantaneous equator, ecliptic and equinox. It should be noted that :

- the *mean equinox* is the intersection of the ecliptic of date with the mean equator ;
- the *true equinox* is the intersection of the ecliptic of date with the true equator (that is, the equator affected by the nutation) ;
- there is no "mean" ecliptic, because the ecliptic has a regular motion.

The problem of the reduction of the place of a star from the mean place at one time (for instance of a standard epoch and equinox) to the apparent place of another time, involves the following corrections :

(A) The *proper motion* of the star between the two epochs. We may assume that by its proper motion each star moves on a great circle with an invariable angular speed. Except when the proper motion is an important fraction of the polar distance of the star, not only the proper motion itself, but also its components in right ascension and declination *with respect to a fixed equinox* may be considered as constants during several centuries. Therefore, we start by finding the effect of the proper motion when the axes of reference remain fixed, as in Example 14.b ;

(B) The effect of *precession*. This has been explained in Chapter 14;

(C) The effect of *nutation* (see below) ;

(D) The effect of *annual aberration* (see below) ;

(E) The effect of the *annual parallax*. This correction never exceeds 0".8, and may be neglected in most cases.

The changes in right ascension and in declination due to the *nutation* are

$$\Delta\alpha_1 = (\cos\varepsilon + \sin\varepsilon\sin\alpha\tan\delta)\,\Delta\psi - (\cos\alpha\ \ \tan\delta)\,\Delta\varepsilon$$

$$\Delta\delta_1 = (\sin\varepsilon\cos\alpha)\,\Delta\psi + (\sin\alpha)\,\Delta\varepsilon$$

The quantities $\Delta\psi$ and $\Delta\varepsilon$ may be calculated by means of the method described in Chapter 15, or be taken from the *A.E.*, while ε is the obliquity of the ecliptic, given by formula (18.4).

If $\odot$ is the true longitude of the Sun, which can be calculated by means of the method described in Chapter 18, the changes in right ascension and in declination of a star due to the *annual aberration* are

$$\Delta\alpha_2 = -20''.49\,\frac{\cos\alpha\cos\odot\cos\varepsilon + \sin\alpha\sin\odot}{\cos\delta}$$

$$\Delta\delta_2 = -20''.49\,\Big[\cos\odot\cos\varepsilon\,(\tan\varepsilon\cos\delta - \sin\alpha\sin\delta) + \cos\alpha\sin\delta\sin\odot\Big]$$

where, as above, α and δ are the star's right ascension and declination.

The total corrections to α and δ are therefore $\Delta\alpha_1 + \Delta\alpha_2$ and $\Delta\delta_1 + \Delta\delta_2$, respectively. Calculated from the above formulae, both are expressed in seconds of a degree. Divide the correction to α by 15 in order to obtain it in seconds of time.

Example 16.a : Calculate the apparent place of θ Persei for 1978 November 13.19 UT.

The mean position of this star for that instant, including the effect of proper motion, was found in Example 14.b, namely

$$\alpha = 2^h42^m44^s.846 = 40^\circ.687 \qquad \delta = +49^\circ08'24''.35 = +49^\circ.140$$

The nutations in longitude and in obliquity, for the same instant, were found in Example 15.a :

$$\Delta\psi = -3''\!.378 \qquad \Delta\varepsilon = -9''\!.321$$

The Sun's true longitude, calculated by the method of Chapter 18, is $\odot = 230^\circ\!.45$, while $\varepsilon = 23^\circ\!.44$. (For both values, an accuracy of 0.01 degree is sufficient in this case).

Putting the values of α, δ, ε, $\odot$, $\Delta\psi$ and $\Delta\varepsilon$ in the above-given formulae, one finds

$$\Delta\alpha_1 = +4''\!.059 \qquad \Delta\delta_1 = -7''\!.096$$
$$\Delta\alpha_2 = +29''\!.619 \qquad \Delta\delta_2 = +6''\!.554$$

and the total corrections in right ascension and declination are

$$\Delta\alpha = +4''\!.059 + 29''\!.619 = +33''\!.678 = +2^s\!.245$$
$$\Delta\delta = -7''\!.096 + 6''\!.554 = -0''\!.54$$

Hence, the required apparent coordinates are

$$\alpha = 2^h42^m44^s\!.846 + 2^s\!.245 = 2^h42^m47^s\!.09$$
$$\delta = +49^\circ08'24''\!.35 - 0''\!.54 = +49^\circ08'23''\!.8$$

The values interpolated from the data on page 321 of the *Astronomicheskii Ezhegodnik 1978* are

$$2^h42^m47^s\!.100 \quad \text{and} \quad +49^\circ08'23''\!.86$$

17

REDUCTION OF ECLIPTICAL ELEMENTS FROM ONE EQUINOX TO ANOTHER ONE

For some problems, it may be necessary to reduce orbital elements of a planet, a minor planet or a comet from one equinox to another one. Of course, the semimajor axis a and the eccentricity e do not change when the orbit is referred to another equinox, and thus only the three elements

i = inclination,
ω = argument of perihelion,
Ω = longitude of ascending node

should be taken into consideration here. Let i_0, ω_0, Ω_0 be the known values of these elements at the initial epoch τ_0, and i, ω, Ω their (unknown) values at the final epoch τ. If τ_0 and τ are expressed in *thousands* of tropical years since 1900.0, and if

$$t = \tau - \tau_0$$

calculate the following values :

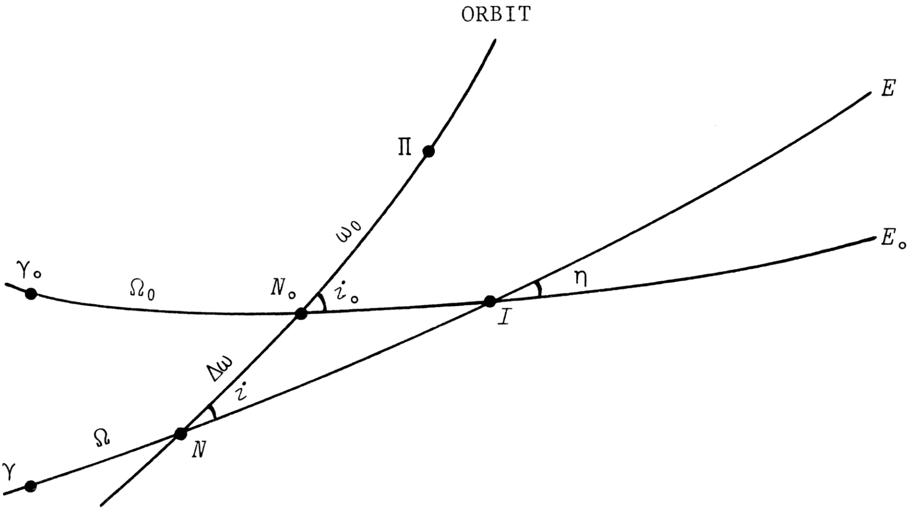
$$\eta = (471''.07 - 6''.75\,\tau_0 + 0''.57\,\tau_0^2)t + (-3''.37 + 0''.57\,\tau_0)t^2 + 0''.05\,t^3$$

$$\theta_0 = 173^\circ.950833 + 32869''\tau_0 + 56''\tau_0^2 - (8694'' + 55''\tau_0)t + 3''t^2$$

$$\theta = \theta_0 + (50256''.41 + 222''.29\,\tau_0 + 0''.26\,\tau_0^2)t + (111''.15 + 0''.26\,\tau_0)t^2 + 0''.1\,t^3$$

In the Figure, E_0 and γ_0 are the ecliptic and the vernal equinox at epoch τ_0, and E and γ the ecliptic and equinox at epoch τ. The angle between the two ecliptics is η. The orbit's perihelion is denoted by Π.

Then the quantities i and $\Omega - \theta$, and thus Ω, can be calculated from



$$\cos i = \cos i_\circ \cos \eta + \sin i_\circ \sin \eta \cos (\Omega_\circ - \theta_\circ) \qquad (17.1)$$

$$\begin{aligned} \sin i \sin (\Omega - \theta) &= \sin i_\circ \sin (\Omega_\circ - \theta_\circ) \\ \sin i \cos (\Omega - \theta) &= -\sin \eta \cos i_\circ + \cos \eta \sin i_\circ \cos (\Omega_\circ - \theta_\circ) \end{aligned} \qquad (17.2)$$

Formula (17.1) should not be used if the inclination is small. Then $\omega = \omega_\circ + \Delta\omega$, where $\Delta\omega$ is found from

$$\begin{aligned} \sin i \sin \Delta\omega &= -\sin \eta \sin (\Omega_\circ - \theta_\circ) \\ \sin i \cos \Delta\omega &= \sin i_\circ \cos \eta - \cos i_\circ \sin \eta \cos (\Omega_\circ - \theta_\circ) \end{aligned} \qquad (17.3)$$

If $i_\circ = 0$, Ω is not determined. Then $i = \eta$ and $\Omega = \theta + 180°$.

Example 17.a : In their *Catalogue Général des Orbites de Comètes de l'an -466 à 1952*, F. Baldet and G. De Obaldia give the following orbital elements for comet Klinkenberg (1744), referred to the mean equinox of 1744.0 :

$$i_\circ = 47\overset{\circ}{.}1220$$
$$\omega_\circ = 151.4486$$
$$\Omega_\circ = 45.7481$$

Reduce these elements to the standard equinox of 1950.0.

We find successively :

$$\tau_\circ = \frac{1744 - 1900}{1000} = -0.156 \qquad \tau = \frac{1950 - 1900}{1000} = +0.050$$

$$t = +0.206$$

$$\eta = +97\overset{\prime\prime}{.}114 = +0\overset{\circ}{.}026\,9761$$

$$\theta_\circ = 173\overset{\circ}{.}950\,833 - 6915\overset{\prime\prime}{.}270 = 172\overset{\circ}{.}029\,925$$

$$\theta = 172\overset{\circ}{.}029\,925 + 10350\overset{\prime\prime}{.}394 = 174\overset{\circ}{.}905\,035$$

Then formulae (17.2) give

$$\sin i \sin (\Omega - \theta) = -0.5907\,2524$$
$$\sin i \cos (\Omega - \theta) = -0.4339\,6271$$

whence, using the key for transformation from rectangular to polar coordinates,

$$\sin i = +0.7329\,9382 \qquad \text{whence} \quad i = 47\overset{\circ}{.}1380$$

$$\Omega - \theta = -126\overset{\circ}{.}3020 \qquad \text{whence} \quad \Omega = 48\overset{\circ}{.}6030$$

Formulae (17.3) give

$$\sin i \sin \Delta\omega = +0.0003\,7954$$
$$\sin i \cos \Delta\omega = +0.7329\,9372$$

whence $\Delta\omega = +0\overset{\circ}{.}0297$, and $\omega = 151\overset{\circ}{.}4783$.

In his *Catalogue of Cometary Orbits* (1975), B.G. Marsden gives the values $i = 47\overset{\circ}{.}1378$, $\omega = 151\overset{\circ}{.}4783$, $\Omega = 48\overset{\circ}{.}6030$.

18

SOLAR COORDINATES

Let JD be the Julian (Ephemeris) Date, which can be calculated by means of the method described in Chapter 3. Then the time T, measured in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 ET, is given by

$$T = \frac{\text{JD} - 2415\,020.0}{36525} \qquad (18.1)$$

This quantity should be calculated with a sufficient number of decimals. For instance, five decimals are not sufficient (unless the Sun's longitude is required with an accuracy not better than one degree) : remember that T is expressed in centuries, so that an error of 0.00001 in T corresponds to an error of 0.37 day in the time.

Then the geometric mean longitude of the Sun, referred to the mean equinox of the date, is given by

$$L = 279^{\circ}.69668 + 36000^{\circ}.76892\,T + 0^{\circ}.000\,3025\,T^2$$

The Sun's mean anomaly is

$$M = 358^{\circ}.47583 + 35999^{\circ}.04975\,T - 0^{\circ}.000\,150\,T^2 - 0^{\circ}.000\,0033\,T^3$$

The eccentricity of the Earth's orbit is

$$e = 0.016\,751\,04 - 0.000\,0418\,T - 0.000\,000\,126\,T^2$$

To find the Sun's true longitude and true anomaly, two different methods can be used.

FIRST METHOD : With the values of M and e, solve Kepler's equation to find the eccentric anomaly E, using one of the methods described in Chapter 22. Then calculate the true anomaly v by means of Formula (25.1).

The Sun's true longitude ☉ is then

$$\odot = L + v - M$$

SECOND METHOD : Calculate the Sun's equation of the center C as follows :

$$\begin{aligned} C = &+ (1^\circ.919\,460 - 0^\circ.004\,789\,T - 0^\circ.000\,014\,T^2) \sin M \\ &+ (0^\circ.020\,094 - 0^\circ.000\,100\,T) \sin 2M \\ &+ 0^\circ.000\,293 \sin 3M \end{aligned}$$

Then the Sun's true longitude is

$$\odot = L + C$$

and its true anomaly is $v = M + C$.

The Sun's radius vector, expressed in astronomical units, can then be obtained by means of one of the following expressions :

$$R = 1.000\,0002\,(1 - e \cos E)$$

$$R = \frac{1.000\,0002\,(1 - e^2)}{1 + e \cos v} \qquad (18.2)$$

In the second formula, the numerator is a quantity which varies slowly with time. It is equal to

0.999 7182	in the year	1800
0.999 7196		1900
0.999 7210		2000
0.999 7224		2100

The Sun's longitude ☉, obtained by the method described above, is the true *geometric* longitude referred to the *mean* equinox of the date. This longitude is the quantity required for instance in the calculation of geocentric planetary positions.

If the *apparent* longitude of the Sun, referred to the *true* equinox of the date, is required, it is necessary to correct ☉ for the nutation and the aberration. Unless high accuracy is required, this can be performed as follows.

$$\Omega = 259^\circ.18 - 1934^\circ.142\,T$$

$$\odot_{app} = \odot - 0^\circ.00569 - 0^\circ.00479 \sin \Omega$$

In some instances, for example in meteor work, it is necessary to have the Sun's longitude referred to the standard equinox of 1950.0. For the 20th century, this can be performed with a sufficient accuracy as follows :

$$\odot_{1950} = \odot - 0^{\circ}.01396\ (\text{year} - 1950)$$

The Sun's latitude is ever less than $1''.2$, and may thus be put equal to zero unless high accuracy is required. In that case, the Sun's right ascension α and declination δ can be calculated from

$$\tan \alpha = \frac{\cos \varepsilon \sin \odot}{\cos \odot} \tag{18.3}$$

$$\sin \delta = \sin \varepsilon \sin \odot$$

where ε, the obliquity of the ecliptic, is given by

$$\begin{aligned} \varepsilon = 23^{\circ}.452\,294 &- 0^{\circ}.013\,0125\,T \\ &- 0^{\circ}.000\,001\,64\,T^2 \\ &+ 0^{\circ}.000\,000\,503\,T^3 \end{aligned} \tag{18.4}$$

If the *apparent* position of the Sun is required, ε should be corrected by

$$+\ 0^{\circ}.00\,256 \cos \Omega \tag{18.5}$$

Formula (18.3) may of course be transformed to

$$\tan \alpha = \cos \varepsilon \tan \odot$$

and then it must be remembered that α must be in the same quadrant as $\odot$. However, for programmable pocket calculators it is better to leave formula (18.2) unchanged, and to apply the rectangular/polar coordinate conversion to the quantities $\cos \varepsilon \sin \odot$ and $\cos \odot$.

The value found for α will be expressed in degrees. Divide the result by 15 in order to express it into hours.

Higher accuracy

A somewhat better accuracy can be obtained as follows. Calculate the angles A, B, C, D, E, H by means of the following expressions, where all numerical values are in degrees and decimals.

$$\begin{aligned} A &= 153.23 + 22518.7541\,T \\ B &= 216.57 + 45037.5082\,T \\ C &= 312.69 + 32964.3577\,T \\ D &= 350.74 + 445\,267.1142\,T - 0.00144\,T^2 \\ E &= 231.19 + 20.20\,T \\ H &= 353.40 + 65928.7155\,T \end{aligned}$$

Then add the following corrections to the Sun's longitude :

$$\begin{aligned} &+ 0\overset{\circ}{.}00134 \cos A \\ &+ 0\overset{\circ}{.}00154 \cos B \\ &+ 0\overset{\circ}{.}00200 \cos C \\ &+ 0\overset{\circ}{.}00179 \sin D \\ &+ 0\overset{\circ}{.}00178 \sin E \end{aligned}$$

and the following corrections to the radius vector :

$$\begin{aligned} &+ 0.000\,005\,43 \sin A \\ &+ 0.000\,015\,75 \sin B \\ &+ 0.000\,016\,27 \sin C \\ &+ 0.000\,030\,76 \cos D \\ &+ 0.000\,009\,27 \sin H \end{aligned}$$

The terms involving A and B are due to the action of Venus, the term with argument C is due to Jupiter, the term with D is due to the Moon, while the term involving E is an inequality of long period.

Example 18.a : Calculate the Sun's position on 1978 November 12 at 0^h ET = JD 2443 824.5.

We find successively :

$$T = +0.788\,624\,230$$
$$L = 28670^\circ\!.77554 = 230^\circ\!.77554$$
$$M = 28748^\circ\!.19863 = 308^\circ\!.19863$$
$$e = 0.016\,718\,00$$

With these values for M and e, the solution of Kepler's equation (see Chapter 22) is $E = 307^\circ\!.43807$. Then we find, using formula (25.1), $v = 306^\circ\!.67358$.

The Sun's true longitude is then

$$\odot = L + v - M = 229^\circ\!.25049 = 229^\circ 15' 02''$$

Using the second method, we find that the equation of the center is

$$C = 1^\circ\!.915\,6746 \sin M + 0^\circ\!.020\,0151 \sin 2M + 0^\circ\!.000\,293 \sin 3M$$
$$= -1^\circ\!.52505$$

whence

$$\odot = L + C = 229^\circ\!.25049,$$ the same result as above.

Then each of the formulae (18.2) gives $R = 0.98984$.

The correct values, according to the *A.E.*, are

$$\odot = 229^\circ 15' 05''\!.85 \quad \text{and} \quad R = 0.989\,8375\ .$$

If the apparent longitude of the Sun is required, we have $\Omega = -1266^\circ\!.13 = +173^\circ\!.87$, whence

$$\odot_{app} = 229^\circ\!.25049 - 0^\circ\!.00569 - 0^\circ\!.00479 \sin 173^\circ\!.87$$
$$= 229^\circ\!.24429 = 229^\circ 14' 39''.$$

According to the *A.E.*, the correct value is 229°14′41″.86.

Using formulae (18.4) and (18.5), we find $\varepsilon = 23^\circ\!.43949$, from which we deduce, using $\odot_{app} = 229^\circ\!.24429$,

$$\alpha = -133^\circ\!.20853 = +226^\circ\!.79147 = 15^h\!.119431 = 15^h 07^m 10^s\!.0$$
$$\delta = -17^\circ\!.53682 = -17^\circ 32' 13''$$

The *A.E.* gives $\alpha = 15^h 07^m 10^s\!.11$ and $\delta = -17^\circ 32' 13''\!.3$

19

RECTANGULAR COORDINATES OF THE SUN

The rectangular geocentric equatorial coordinates X, Y, Z of the Sun are needed for the calculation of an ephemeris of a minor planet or a comet (see Chapters 25 and 26). The origin of these coordinates is the center of the Earth. The X-axis is directed towards the vernal equinox (longitude 0°); the Y-axis lies in the plane of the equator too and is directed towards longitude 90°, while the Z-axis is directed towards the north celestial pole.

The values of X, Y, Z are given for each day at 0^h ET in the *Astronomical Ephemeris* ; they are expressed in astronomical units. If the *A.E.* is not available, or for an instant in the past or in the future, the Sun's rectangular geocentric equatorial coordinates can be calculated from

$$\begin{aligned} X &= R \cos \odot \\ Y &= R \sin \odot \cos \varepsilon \\ Z &= R \sin \odot \sin \varepsilon \end{aligned} \qquad (19.1)$$

where R is the Sun's radius vector expressed in astronomical units, $\odot$ the Sun's true longitude referred to the mean equinox of date, and ε the (mean) obliquity of the ecliptic for that date. The quantities R and $\odot$ can be calculated by the method given in Chapter 18, while ε is given by formula (18.4).

In the formulae (19.1) the latitude of the Sun, which always is very small, has been neglected.

However, the coordinates X, Y, Z, calculated as explained above, are referred to the mean equator and mean equinox of the date. In most cases, it will be necessary to have these coordinates referred to another equator and equinox, for example for the standard equinox of 1950.0. This can be performed in the following way.

If X_0, Y_0, Z_0 are the values at the initial equinox, and X, Y, Z the values at the final equinox, then

$$\begin{aligned} X &= X_x X_0 + Y_x Y_0 + Z_x Z_0 \\ Y &= X_y X_0 + Y_y Y_0 + Z_y Z_0 \\ Z &= X_z X_0 + Y_z Y_0 + Z_z Z_0 \end{aligned} \qquad (19.2)$$

where

$$\begin{aligned} X_x &= \cos\zeta \cos z \cos\theta - \sin\zeta \sin z \\ X_y &= \sin\zeta \cos z + \cos\zeta \sin z \cos\theta \\ X_z &= \cos\zeta \sin\theta \\ Y_x &= -\cos\zeta \sin z - \sin\zeta \cos z \cos\theta \\ Y_y &= \cos\zeta \cos z - \sin\zeta \sin z \cos\theta \\ Y_z &= -\sin\zeta \sin\theta \\ Z_x &= -\cos z \sin\theta \\ Z_y &= -\sin z \sin\theta \\ Z_z &= \cos\theta \end{aligned}$$

the values of ζ, z and θ being found by the formulae (14.2).

It may be interesting to note that we have, approximately,

$$Y_x = -X_y \qquad Z_x = -X_z \qquad Z_y = Y_z$$

Example 19.a : Find the X, Y, Z coordinates of the Sun, referred to the equator and ecliptic of 1950.0, for 1978 November 12 at 0 *h* ET = JD 2443 824.5.

In Example 18.a, we have found the following values for that instant :

$$\odot = 229^\circ\!.25049 \qquad R = 0.98984$$

Formulae (18.1) and (17.4) give

$$T = +0.788\,624\,230 \qquad \varepsilon = 23^\circ\!.442\,031$$

Then formulae (19.1) give

$$\begin{aligned} X &= -0.646\,121 \\ Y &= -0.687\,981 \\ Z &= -0.298\,316 \end{aligned}$$

These values are referred to the equator and ecliptic of the date. They must be reduced to those of 1950.0 by means of formulae (19.2) but firstly we have to calculate ζ, z and θ (Chapter 14). We find

$$\tau_\circ = \frac{2443\,824.5 - 2415\,020.313}{36524.2199} = +0.788\,632\,504$$

$$\tau = \frac{2433\,282.423 - 2443\,824.5}{36524.2199} = -0.288\,632\,503$$

$$\zeta = -665''.374 = -0^\circ.184\,826$$

$$z = -665''.309 = -0^\circ.184\,808$$

$$\theta = -578''.457 = -0^\circ.160\,682$$

Then,

$$\begin{array}{lll} X_x = +0.999\,9753 & Y_x = +0.006\,4513 & Z_x = +0.002\,8044 \\ X_y = -0.006\,4513 & Y_y = +0.999\,9792 & Z_y = -0.000\,0090 \\ X_z = -0.002\,8044 & Y_z = -0.000\,0090 & Z_z = +0.999\,9961 \end{array}$$

and finally, by formulae (19.2),

$$\begin{aligned} X_{1950} &= -0.651\,38 \\ Y_{1950} &= -0.683\,80 \\ Z_{1950} &= -0.296\,50 \end{aligned}$$

According to the *A.E.*, the correct values are

$$\begin{array}{l} -0.651\,3639 \\ -0.683\,8057 \\ -0.296\,5014 \end{array}$$

20

EQUINOXES AND SOLSTICES

The times of the equinoxes and solstices are the instants when the apparent longitude of the Sun is a multiple of 90 degrees. These instants can be calculated as follows.

Firstly, find an approximate time (in Julian Days) by means of the formula

$$\text{JD} = (\text{year} + k/4) \times 365.2422 + 1721\,141.3 \qquad (20.1)$$

where "year" is an integer, and

k = 0 for the March equinox,
1 for the June solstice,
2 for the September equinox,
3 for the December solstice.

For the JD given by formula (20.1), calculate the Sun's apparent longitude $\odot_{app}$ by the method described in Chapter 18. A correction to the JD is then given by

$$+\ 58 \sin (k.90° - \odot_{app}) \text{ days} \qquad (20.2)$$

Using the new value for JD, the calculation should be repeated if necessary, until one finds a correction that is small, say less than 0.001 day.

The final JD can be converted into ordinary calendar date by means of the method described in Chapter 3. The result is expressed in Ephemeris Time.

Example 20.a : Find the instant of the September equinox of the year 1979.

Putting year = +1979, and k = 2, in formula (20.1), we find the approximate value JD = 2444 138.24.

For this instant we find, by the method described in Chapter 18, $\odot_{app} = 28978^\circ\!.144 = 178^\circ\!.144$ and the correction, given by formula (20.2), is then

$$+\,58\ \sin(180^\circ - 178^\circ\!.144) = +\,1.88 \text{ day}.$$

The corrected instant is thus

$$\text{JD} = 2444\,138.24 + 1.88 = 2444\,140.12.$$

With this new value, one finds $\odot_{app} = 179^\circ\!.983$, and the new correction is +0.017 day, giving the new corrected value for the instant JD = 2444 140.137.

Using this latter value again, one finds $\odot_{app} = 180^\circ\!.000$, which shows that the correct instant is indeed JD = 2444 140.137. This corresponds to 1979 September 23 at $15^h17^m17^s$ ET, which must be rounded to 15^h16^m UT.

(In 1979, the difference ET − UT is approximately 50 seconds). The correct value, as given by the *A.E.*, is 15^h17^m UT.

21

EQUATION OF TIME

The equation of time is the difference between the right ascensions of the apparent (true) Sun and the fictitious mean Sun. If the *A.E.* is available, the equation of time E at 0^h UT can be calculated from

$$\begin{aligned} E = 12 \text{ hours } &+ \text{ apparent sidereal time at } 0^h \text{ UT} \\ &- \text{ apparent right ascension of Sun at } 0^h \text{ ET} \\ &- 0.002738\ \Delta T \end{aligned}$$

where ΔT = ET - UT.

Example 21.a : Calculate the equation of time on 1978 January 21 at 0^h Universal Time.

From the *Astronomical Ephemeris* we take the following values :

apparent sidereal time at 0^h UT = $8^h00^m01^s.193$

apparent right ascension of Sun at 0^h ET = $20^h11^m10^s.78$

ΔT = +48.6 seconds

Hence,

$$\begin{aligned} E &= 20^h00^m01^s.193 - 20^h11^m10^s.78 - (0.002738 \times 48^s.6) \\ &= -11^m09^s.72 \end{aligned}$$

If the *A.E.* is not available, the equation of time at any instant can be calculated by means of the following formula given by W.M. Smart (*Text-Book on Spherical Astronomy*, page 149 of the edition of 1956) :

$$\begin{aligned} E = y \sin 2L - 2e \sin M + 4e\, y \sin M \cos 2L \\ - \frac{1}{2} y^2 \sin 4L - \frac{5}{4} e^2 \sin 2M \end{aligned} \qquad (21.1)$$

where $y = \tan^2 \frac{\varepsilon}{2}$, ε being the obliquity of the ecliptic,

L = Sun's mean longitude,
e = eccentricity of the Earth's orbit,
M = Sun's mean anomaly.

The values of ε, L, e and M can be found by means of the formulae given in Chapter 18.

The value of E given by formula (21.1) is expressed in radians. The result may be converted into degrees, and then into hours and decimals by division by 15.

Example 21.b : Calculate the equation of time on 1978 January 21 at 0^h ET = JD 2443 529.5.

We find successively

$$T = +0.780\,547\,5702$$

$$L = 28380^\circ\!.00957 = 300^\circ\!.00957$$

$$M = 28457^\circ\!.44655 = 17^\circ\!.44655$$

$$e = 0.016\,718\,34$$

$$\varepsilon = 23^\circ\!.442\,136$$

$$y = 0.043\,045\,274$$

Formula (21.1) then gives $E = -0.048\,743\,490$ radian

$= -2^\circ\!.792\,7963$

$= -11$ minutes 10.3 seconds

22

EQUATION OF KEPLER

The equation of Kepler is

$$E = M + e \sin E \tag{22.1}$$

where e is the eccentricity of the planet's orbit, M the planet's mean anomaly at a given instant, and E the eccentric anomaly. Generally, e and M are given, and the equation must be solved for E, as in Chapters 18, 25 and 38. The eccentric anomaly E is an auxiliary quantity which is needed to find the true anomaly v.

Equation (22.1) is a transcendental equation in E and cannot be solved directly. We will describe two iteration methods for finding E (iteration = repetition), and finally a formula which gives an approximate result.

FIRST METHOD

It should be noted that in formula (22.1) the angles M and E should be expressed in *radians*. On the calculating machine, the calculations must thus be performed in "radian mode". This can be avoided by multiplying e by $180/\pi$ (conversion from radians into degrees) in equation (22.1). Let e_o be the thus "modified" eccentricity. Kepler's equation is then

$$E = M + e_o \sin E \tag{22.2}$$

and now we can calculate with ordinary degrees.

To solve equation (22.2), give an approximate value to E in the right side of the formula. Then the formula will give a better approximation for E. This is repeated until the required accuracy is obtained ; this process can be performed automatically on a programmable calculator. For the first approximation, use $E = M$.

We thus have

$$\begin{aligned} E_0 &= M \\ E_1 &= M + e \sin E_0 \\ E_2 &= M + e \sin E_1 \\ E_3 &= M + e \sin E_2 \\ &\text{etc.} \end{aligned}$$

E_1 , E_2 , E_3 , etc. are successive and better approximations for the eccentric anomaly E.

Example 22.a : Solve the equation of Kepler for $e = 0.100$ and $M = 5°$, to an accuracy of 0.000 001 degree.

We find

$$e_o = 0.100 \times 180/\pi = 5^\circ\!.729\,577\,95,$$

and the equation of Kepler becomes

$$E = 5 + 5.729\,577\,95 \sin E$$

where all quantities are in degrees. Starting with $E = M = 5°$, we obtain successively :

5.000 000
5.499 366
5.549 093
5.554 042
5.554 535
5.554 584
5.554 589
5.554 589

Hence, the required value is $E = 5^\circ\!.554\,589$.

SECOND METHOD

The first method is very simple, and there will be no problems when e is small. However, the number of required iterations is generally increasing with e. For example, for $e = 0.990$ and $M = 2°$ the successive values of the iteration procedure are as follows :

2.000 000
3.979 598
5.936 635
7.866 758
9.763 644
11.619 294
13.424 417
15.168 909
16.842 404
18.434 883
19.937 269
21.341 978
22.643 349
23.837 929
24.924 579
25.904 408
26.780 556
27.557 863
28.242 483
28.841 471
29.362 399
29.813 009
30.200 940
30.533 515
30.817 592
⋮

After the 50th iteration, the result (32°345 452) still differs from the correct value (32.361 007) by more than 0.01 degree.

When e is larger than 0.4 or 0.5, the convergence may be so slow that a better iteration formula should be used : a better value E_1 for E is

$$E_1 = E_0 + \frac{M + e_o \sin E_0 - E_0}{1 - e \cos E_0} \qquad (22.3)$$

where E_0 is the lastly obtained value for E. In this formula, all quantities are expressed in degrees. It is important to note that the numerator of the fraction contains the "modified" eccentricity e_o defined before, while the denominator contains the ordinary eccentricity e.

Here, again, the process can be repeated as often as is necessary.

Example 22.b : Same problem as in Example 22.a, but now using formula (22.3).

In this case, formula (22.3) takes the following form :

$$E_1 = E_0 + \frac{5 + 5.729\,577\,95 \sin E_0 - E_0}{1 - 0.100 \cos E_0}$$

Starting with $E_0 = M = 5°$, we obtain the following values :

E_0	*correction*	E_1
5.000 000 000	+0.554 616 193	5.554 616 193
5.554 616 193	−0.000 026 939	5.554 589 254
5.554 589 254	−0.000 000 001	5.554 589 253

In this case, an accuracy of 0.000 000 001 degree is obtained after only three iterations.

As an exercise, try the second method on the case mentioned before : $e = 0.99$, $M = 2°$. After only nine or ten iterations, an accuracy of 0.000 0001 degree is reached.

In the first as well as in the second method, a test must be included in the program, because a new iteration should be performed only as long as the required accuracy (for instance 0.000 001 degree) has not been reached. It is important to note a difference in the test for the two methods.

In the first method, formula (22.2) gives directly a new approximation for E. This new value, after being stored, must be compared to the previous one, which thus should be temporarily retained in the machine. Thus, this method requires the use of two registers, one containing the new value of E, and the other containing the previous value.

In the second method, formula (22.3) too gives a new approximation E_1 for the eccentric anomaly, but the fraction in the second member is actually a *correction* to the previous value E_0. On many machines, this correction can be added directly to the value of E_0 contained in a register ("storage register arithmetic", for instance the instruction STO + 0 on the HP-67 machine), after which the absolute value of the correction (which is still displayed) can be tested. This procedure requires only one register for the eccentric anomaly.

THIRD METHOD

The formula

$$\tan E = \frac{\sin M}{\cos M - e} \tag{22.4}$$

gives an *approximate* value for E, and is valid only for small values of the eccentricity.

For the same data as in Example 22.a, the formula (22.4) gives

$$\tan E = \frac{+0.087\,1557}{+0.896\,1947}$$

whence $E = 5^\circ\!.554\,599$, the exact value being $5^\circ\!.554\,589$, an error of only $0''\!.035$. (But for the same eccentricity and $M = 82°$, the error amounts to $35''$).

The greatest error due to the use of formula (22.4) is

$0^\circ\!.0327$ for $e = 0.15$
0.0783 for $e = 0.20$
0.1552 for $e = 0.25$
1.42 for $e = 0.50$
24.7 for $e = 0.99$

For the orbit of the Earth ($e = 0.01674$), the error will be less than $0''\!.2$. In that case, formula (22.4) can safely be used except when very high accuracy is needed.

23

ELEMENTS OF THE PLANETARY ORBITS

The orbital elements of the major planets can be expressed as polynomials of the form

$$a_0 + a_1T + a_2T^2 + a_3T^3$$

where T is the time measured in Julian centuries of 36525 ephemeris days from the epoch 1900 January 0.5 ET = JD 2415 020.0. In other words,

$$T = \frac{\text{JD} - 2415\,020.0}{36525} \qquad (23.1)$$

This quantity is negative before the beginning of the year 1900, positive afterwards. The orbital elements are :

L = mean longitude of the planet ;

a = semimajor axis of the orbit (in fact, this elements is a constant for each planet) ;

e = eccentricity of the orbit ;

i = inclination on the plane of the ecliptic ;

ω = argument of perihelion ;

Ω = longitude of ascending node.

The longitude of the perihelion can be calculated from $\pi = \omega + \Omega$, and the planet's mean anomaly is

$$M = L - \pi = L - \omega - \Omega$$

See also Chapter 25 for the mean anomalies.

The perihelion distance q and the aphelion distance Q are

$$q = a\,(1 - e) \qquad\qquad Q = a\,(1 + e)$$

We have $q + Q = 2a$.

The quantities L and π are measured in two different planes, namely from the vernal equinox along the ecliptic to the orbit's ascending node, and then from this node along the orbit.

Table 23.A gives the coefficients a_i for the orbital elements of the planets Mercury to Neptune. The values for Mercury and Venus are those given by S. Newcomb. The values for Mars are due to F.E. Ross. The elements for Jupiter, Saturn, Uranus and Neptune, due to Gaillot, are *not* affected by the periodic terms of short and long period ; thus they correspond to the purely secular terms.

The elements for the Earth are not given in Table 23.A. Since for this planet we have $i = 0$, the angles ω and Ω are not determined. The Earth's mean anomaly and orbital eccentricity are equal to those of the Sun (see Chapter 18), while the mean longitude and the longitude of the perihelion of the Earth are equal to those of the Sun increased by 180 degrees. Finally, for the Earth we have $a = 1.000\,0002$.

In Table 23.A, the values for the angular quantities L, i, ω and Ω are expressed in degrees and decimals.

Example 23.a : Calculate the orbital elements of Mercury on 1978 June 24.0 ET.

We have (see Chapter 3)

$$1978 \text{ June } 24.0 = \text{JD } 2443\,683.5$$

whence, by formula (23.1),

$$T = +0.784\,763\,8604$$

Consequently, from Table 23.A, we find :

$$\begin{aligned} L &= 178^\circ\!.179\,078 + (149\,474^\circ\!.070\,78 \times 0.784\,763\,8604) \\ &\qquad + (0.000\,3011)\,(0.784\,763\,8604)^2 \\ &= 117\,480^\circ\!.0281 = 120^\circ\!.0281 \end{aligned}$$

$a = 0.387\,0986$

$e = 0.205\,630\,25$

$i = 7^\circ\!.004\,330$

$\omega = 29^\circ\!.044\,410$

$\Omega = 48^\circ\!.076\,160$ $\qquad M = 42^\circ\!.9075$

TABLE 23.A

Elements for the mean equinox of the date

	a_0	a_1	a_2	a_3
MERCURY				
L	178.179 078	+ 149 474.070 78	+ 0.000 3011	
a	0.387 0986			
e	0.205 614 21	+ 0.000 020 46	− 0.000 000 030	
i	7.002 881	+ 0.001 8608	− 0.000 0183	
ω	28.753 753	+ 0.370 2806	+ 0.000 1208	
Ω	47.145 944	+ 1.185 2083	+ 0.000 1739	
VENUS				
L	342.767 053	+ 58 519.211 91	+ 0.000 3097	
a	0.723 3316			
e	0.006 820 69	− 0.000 047 74	+ 0.000 000 091	
i	3.393 631	+ 0.001 0058	− 0.000 0010	
ω	54.384 186	+ 0.508 1861	− 0.001 3864	
Ω	75.779 647	+ 0.899 8500	+ 0.000 4100	
MARS				
L	293.737 334	+ 19 141.695 51	+ 0.000 3107	
a	1.523 6883			
e	0.093 312 90	+ 0.000 092 064	− 0.000 000 077	
i	1.850 333	− 0.000 6750	+ 0.000 0126	
ω	285.431 761	+1.069 7667	+ 0.000 1313	+ 0.000 004 14
Ω	48.786 442	+ 0.770 9917	− 0.000 0014	− 0.000 005 33

TABLE 23.A (continuation)

	a_0	a_1	a_2	a_3
JUPITER				
L	238.049 257	+ 3036.301 986	+ 0.000 3347	− 0.000 001 65
a	5.202 561			
e	0.048 334 75	+ 0.000 164 180	− 0.000 000 4676	− 0.000 000 0017
i	1.308 736	− 0.005 6961	+ 0.000 0039	
ω	273.277 558	+ 0.599 4317	+ 0.000 704 05	+ 0.000 005 08
Ω	99.443 414	+ 1.010 5300	+ 0.000 352 22	− 0.000 008 51
SATURN				
L	266.564 377	+ 1223.509 884	+ 0.000 3245	− 0.000 0058
a	9.554 747			
e	0.055 892 32	− 0.000 345 50	− 0.000 000 728	+ 0.000 000 000 74
i	2.492 519	− 0.003 9189	− 0.000 015 49	+ 0.000 000 04
ω	338.307 800	+ 1.085 2207	+ 0.000 978 54	+ 0.000 009 92
Ω	112.790 414	+ 0.873 1951	− 0.000 152 18	− 0.000 005 31
URANUS				
L	244.197 470	+ 429.863 546	+ 0.000 3160	− 0.000 000 60
a	19.218 14			
e	0.046 3444	− 0.000 026 58	+ 0.000 000 077	
i	0.772 464	+ 0.000 6253	+ 0.000 0395	
ω	98.071 581	+ 0.985 7650	− 0.001 0745	− 0.000 000 61
Ω	73.477 111	+ 0.498 6678	+ 0.001 3117	

TABLE 23.A (end)

	a_0	a_1	a_2	a_3
NEPTUNE				
L	84.457 994	+ 219.885 914	+ 0.000 3205	− 0.000 000 60
a	30.109 57			
e	0.008 997 04	+ 0.000 006 330	− 0.000 000 002	
i	1.779 242	− 0.009 5436	− 0.000 0091	
ω	276.045 975	+ 0.325 6394	+ 0.000 140 95	+ 0.000 004 113
Ω	130.681 389	+ 1.098 9350	+ 0.000 249 87	− 0.000 004 718

The elements calculated by means of the coefficients of Table 23.A are referred to the mean equinox of the date, that is to the ecliptic of the date and to the mean equator of the date. Consequently, those coefficients should be used if one wishes to calculate planetary positions referred to the mean equinox of the date.

In some cases, however, it may be desirable to refer the elements i, ω, Ω to a standard equinox. This is the case, for instance, when one wishes to calculate the least distance between the orbit of a comet and that of a major planet, when the elements of the first orbit are referred to a standard equinox.

By means of the formulae of Chapter 17, it is possible to convert the elements i, ω, Ω from one equinox to another one. However, by means of Tables 23.B and 23.C it is possible to calculate these elements for the major planets directly, referred to the standard equinox of either 1950.0 or 2000.0. The corresponding dates are

1950.0 = 1950 January 0.923 = JD 2433 282.423

2000.0 = 2000 January 1.5 = JD 2451 545.0

It should be noted that, while 1950.0 corresponds to the beginning of the Besselian year 1950 and is 50 *tropical* years later than the epoch 1900.0 = 1900 January 0.813 ET = JD 2415 020.313, the new standard epoch, designated 2000.0, will be exactly 36525 days after the epoch JD 2415 020.0 = 1900 January 0.5.

TABLE 23.B

Elements for equinox 1950.0

	a_0	a_1	a_2	a_3
MERCURY				
i	7.006 790	− 0.005 9671	+ 0.000 000 70	− 0.000 000 036
ω	28.796 761	+ 0.284 3099	+ 0.000 074 64	+ 0.000 000 043
Ω	47.801 352	− 0.125 5041	− 0.000 088 63	− 0.000 000 068
VENUS				
i	3.394 552	− 0.000 8226	− 0.000 032 51	+ 0.000 000 018
ω	54.493 527	+ 0.289 3249	− 0.001 144 35	− 0.000 000 792
Ω	76.368 593	− 0.277 7139	− 0.000 140 39	+ 0.000 000 767
EARTH				
i	−0.006 540	+ 0.013 0855	− 0.000 009 33	+ 0.000 000 014
ω	287.390 758	+ 0.564 7073	+ 0.000 136 10	+ 0.000 003 333
Ω	174.528 170	− 0.241 5735	+ 0.000 007 94	− 0.000 000 028
MARS				
i	1.854 113	− 0.008 1839	− 0.000 023 05	− 0.000 000 045
ω	285.597 172	+ 0.738 5934	+ 0.000 466 47	+ 0.000 006 962
Ω	49.319 212	− 0.294 0497	− 0.000 644 35	− 0.000 008 182
JUPITER				
i	1.307 028	− 0.002 2192	+ 0.000 029 52	+ 0.000 000 125
ω	273.553 214	+ 0.047 5910	− 0.000 210 41	+ 0.000 009 039
Ω	99.865 881	+ 0.166 1852	+ 0.000 958 57	− 0.000 012 500

TABLE 23.B (end)

	a_0	a_1	a_2	a_3
SATURN				
i	2.489 374	+ 0.002 4190	− 0.000 050 22	+ 0.000 000 002
ω	338.439 665	+ 0.821 8494	+ 0.000 706 12	+ 0.000 006 174
Ω	113.356 715	− 0.259 7237	− 0.000 188 62	− 0.000 001 587
URANUS				
i	0.773 723	− 0.001 7599	− 0.000 000 22	+ 0.000 000 121
ω	98.546 561	+ 0.032 5540	− 0.000 501 25	+ 0.000 013 998
Ω	73.700 227	+ 0.055 7505	+ 0.000 429 88	− 0.000 014 630
NEPTUNE				
i	1.774 485	− 0.000 0150	− 0.000 002 27	+ 0.000 000 018
ω	276.190 852	+ 0.036 7891	+ 0.000 038 42	+ 0.000 002 218
Ω	131.234 637	− 0.008 3952	+ 0.000 044 35	− 0.000 002 849

In the case of the Earth, if the inclination is found to be negative, then ω and Ω should *both* be increased or decreased by 180 degrees.

TABLE 23.C

Elements for equinox 2000.0

	a_0	a_1	a_2	a_3
MERCURY				
i	7.010 678	− 0.005 9556	+ 0.000 000 69	− 0.000 000 035
ω	28.839 814	+ 0.284 2765	+ 0.000 074 45	+ 0.000 000 043
Ω	48.456 876	− 0.125 4715	− 0.000 088 44	− 0.000 000 068
VENUS				
i	3.395 459	− 0.000 7913	− 0.000 032 50	+ 0.000 000 018
ω	54.602 827	+ 0.289 2764	− 0.001 144 64	− 0.000 000 794
Ω	76.957 740	− 0.277 6656	− 0.000 140 10	+ 0.000 000 769
EARTH				
i	−0.013 0762	+ 0.013 0855	− 0.000 009 27	+ 0.000 000 014
ω	287.511 505	+ 0.564 7920	+ 0.000 136 10	+ 0.000 003 333
Ω	175.105 679	− 0.241 6582	+0.000 007 94	− 0.000 000 028
MARS				
i	1.857 866	− 0.008 1565	− 0.000 023 04	− 0.000 000 044
ω	285.762 379	+ 0.738 7251	+ 0.000 465 56	+ 0.000 006 939
Ω	49.852 347	− 0.294 1821	− 0.000 643 44	− 0.000 008 159
JUPITER				
i	1.305 288	− 0.002 2374	+ 0.000 029 42	+ 0.000 000 127
ω	273.829 584	+ 0.047 8404	− 0.000 218 57	+ 0.000 008 999
Ω	100.287 838	+ 0.165 9357	+ 0.000 966 72	− 0.000 012 460

TABLE 23.C (end)

	a_0	a_1	a_2	a_3
SATURN				
i	2.486 204	+ 0.002 4449	− 0.000 050 17	+ 0.000 000 002
ω	338.571 353	+ 0.822 0515	+ 0.000 707 47	+ 0.000 006 177
Ω	113.923 406	− 0.259 9254	− 0.000 189 97	− 0.000 001 589
URANUS				
i	0.774 950	− 0.001 7660	− 0.000 000 27	+ 0.000 000 123
ω	99.021 587	+ 0.033 7219	− 0.000 498 12	+ 0.000 013 904
Ω	73.923 501	+ 0.054 5828	+ 0.000 426 74	− 0.000 014 536
NEPTUNE				
i	1.769 715	− 0.000 0144	− 0.000 002 27	+ 0.000 000 018
ω	276.335 328	+ 0.036 8127	+ 0.000 038 49	+ 0.000 002 226
Ω	131.788 486	− 0.008 4187	+ 0.000 044 28	− 0.000 002 858

We see that the inclination of Mercury's orbit on the ecliptic of the date is increasing, but that it is decreasing with respect to the fixed ecliptic of either 1950.0 or 2000.0. The opposite occurs for Saturn.

Between $T = -20$ and $T = +20$, Venus' orbital inclination on the ecliptic of the date is continuously increasing, but with respect to the fixed ecliptic of 1950.0 Venus' inclination reached a maximum about the year +650.

Uranus' inclination on the ecliptic of the date reached a minimum about the year +1110, but with respect to the fixed equinoxes of 1950.0 and 2000.0 its value is continuously decreasing during the time period considered here.

Between $T = -20$ and $T = +20$, Neptune's orbital inclination on the ecliptic of the date is continuously decreasing, but with respect to the fixed ecliptic of 1950.0 Neptune's inclination reached a flat maximum about the year +1550.

The longitudes of the nodes, referred to the equinox of the date, are increasing for all planets. But with respect to the fixed equinoxes of 1950.0 and 2000.0 these longitudes are decreasing except for Jupiter and Uranus.

24

PLANETS : PRINCIPAL PERTURBATIONS

In this Chapter we will mention the most important perturbations in the motion of the planets Mercury, Venus, Mars, Jupiter and Saturn. These periodic terms can be used when a better accuracy is needed than by using the data of Chapter 23 alone. The perturbations in the motions of Jupiter and Saturn are particularly important ; in longitude, they can be larger than 0.3 and 1.0 degree, respectively. For the Earth (Sun), the most important perturbations have been given in Chapter 18.

In the expressions given below, T is the time in Julian centuries from 1900 January 0.5 ET ; see formula (23.1).

M, the Sun's mean anomaly, can be calculated by means of the expression given on the first page of Chapter 18.

The mean anomalies of Mercury, Venus, Mars, Jupiter and Saturn are denoted by M_1 , M_2 , M_4 , M_5 and M_6 , and can be found by means of the formulae given in Chapter 25.

MERCURY

Perturbations in longitude

$+0^\circ\!.00\,204 \times \cos(5M_2 - 2M_1 + 12^\circ\!.220)$
$+0.00\,103 \quad \cos(2M_2 - M_1 - 160^\circ\!.692)$
$+0.00\,091 \quad \cos(2M_5 - M_1 - 37^\circ\!.003)$
$+0.00\,078 \quad \cos(5M_2 - 3M_1 + 10^\circ\!.137)$

Perturbations in radius vector

$+0.000\,007\,525 \times \cos(2M_5 - M_1 + 53^\circ\!.013)$
$+0.000\,006\,802 \quad \cos(5M_2 - 3M_1 - 259^\circ\!.918)$
$+0.000\,005\,457 \quad \cos(2M_2 - 2M_1 - 71^\circ\!.188)$
$+0.000\,003\,569 \quad \cos(5M_2 - M_1 - 77^\circ\!.75)$

VENUS

Term of long period in the *mean* longitude and in the mean anomaly :

$$+\ 0\overset{\circ}{.}00077\ \sin(237\overset{\circ}{.}24 + 150\overset{\circ}{.}27\,T)$$

Perturbations in longitude

$+0\overset{\circ}{.}00\,313 \times \cos(2M - 2M_2 - 148\overset{\circ}{.}225)$
$+0.00\,198 \quad \cos(3M - 3M_2 + 2\overset{\circ}{.}565)$
$+0.00\,136 \quad \cos(M - M_2 - 119\overset{\circ}{.}107)$
$+0.00\,096 \quad \cos(3M - 2M_2 - 135\overset{\circ}{.}912)$
$+0.00\,082 \quad \cos(M_5 - M_2 - 208\overset{\circ}{.}087)$

Perturbations in radius vector

$+0.000\,022\,501 \times \cos(2M - 2M_2 - 58\overset{\circ}{.}208)$
$+0.000\,019\,045 \quad \cos(3M - 3M_2 + 92\overset{\circ}{.}577)$
$+0.000\,006\,887 \quad \cos(M_5 - M_2 - 118\overset{\circ}{.}090)$
$+0.000\,005\,172 \quad \cos(M - M_2 - 29\overset{\circ}{.}110)$
$+0.000\,003\,620 \quad \cos(5M - 4M_2 - 104\overset{\circ}{.}208)$
$+0.000\,003\,283 \quad \cos(4M - 4M_2 + 63\overset{\circ}{.}513)$
$+0.000\,003\,074 \quad \cos(2M_5 - 2M_2 - 55\overset{\circ}{.}167)$

The term of long period (with coefficient $0\overset{\circ}{.}00077$) should be added to both the mean longitude and mean anomaly *before* the equation of Kepler is solved. All other periodic terms must be added to the longitude and to the radius vector obtained *after* solving Kepler's equation.

MARS

Terms of long period in the *mean* longitude and in the mean anomaly :

$$-0\overset{\circ}{.}01\,133\ \sin(3M_5 - 8M_4 + 4M)$$
$$-0\overset{\circ}{.}00\,933\ \cos(3M_5 - 8M_4 + 4M)$$

Perturbations in longitude

$+0\overset{\circ}{.}00\,705 \times \cos(M_5 - M_4 - 48\overset{\circ}{.}958)$
$+0.00\,607 \quad \cos(2M_5 - M_4 - 188\overset{\circ}{.}350)$
$+0.00\,445 \quad \cos(2M_5 - 2M_4 - 191\overset{\circ}{.}897)$
$+0.00\,388 \quad \cos(M - 2M_4 + 20\overset{\circ}{.}495)$
$+0.00\,238 \quad \cos(M - M_4 + 35\overset{\circ}{.}097)$
$+0.00\,204 \quad \cos(2M - 3M_4 + 158\overset{\circ}{.}638)$
$+0.00\,177 \quad \cos(3M_4 - M_2 - 57\overset{\circ}{.}602)$
$+0.00\,136 \quad \cos(2M - 4M_4 + 154\overset{\circ}{.}093)$
$+0.00\,104 \quad \cos(M_5 + 17\overset{\circ}{.}618)$

Perturbations in radius vector

$+0.000\,053\,227 \times \cos(M_5 - M_4 + 41\overset{\circ}{.}1306)$
$+0.000\,050\,989 \quad \cos(2M_5 - 2M_4 - 101\overset{\circ}{.}9847)$
$+0.000\,038\,278 \quad \cos(2M_5 - M_4 - 98\overset{\circ}{.}3292)$
$+0.000\,015\,996 \quad \cos(M - M_4 - 55\overset{\circ}{.}555)$
$+0.000\,014\,764 \quad \cos(2M - 3M_4 + 68\overset{\circ}{.}622)$
$+0.000\,008\,966 \quad \cos(M_5 - 2M_4 + 43\overset{\circ}{.}615)$
$+0.000\,007\,914 \quad \cos(3M_5 - 2M_4 - 139\overset{\circ}{.}737)$
$+0.000\,007\,004 \quad \cos(2M_5 - 3M_4 - 102\overset{\circ}{.}888)$
$+0.000\,006\,620 \quad \cos(M - 2M_4 + 113\overset{\circ}{.}202)$
$+0.000\,004\,930 \quad \cos(3M_5 - 3M_4 - 76\overset{\circ}{.}243)$
$+0.000\,004\,693 \quad \cos(3M - 5M_4 + 190\overset{\circ}{.}603)$
$+0.000\,004\,571 \quad \cos(2M - 4M_4 + 244\overset{\circ}{.}702)$
$+0.000\,004\,409 \quad \cos(3M_5 - M_4 - 115\overset{\circ}{.}828)$

The terms of long period should be added to both the mean longitude and mean anomaly *before* the equation of Kepler is solved. All other periodic terms must be added to the longitude and to the radius vector obtained *after* solving Kepler's equation.

JUPITER

$$
\begin{array}{l}
\upsilon = \dfrac{T}{5} + 0.1 \\
P = 237\overset{\circ}{.}47555 + 3034\overset{\circ}{.}9061\,T \\
Q = 265\overset{\circ}{.}91650 + 1222\overset{\circ}{.}1139\,T \\
S = 243\overset{\circ}{.}51721 + 428\overset{\circ}{.}4677\,T \\
V = 5Q - 2P \\
W = 2P - 6Q + 3S \\
\zeta = Q - P
\end{array}
$$

Perturbations in the mean longitude (A)

$$
\begin{array}{l}
+(0\overset{\circ}{.}331\,364 - 0\overset{\circ}{.}010\,281\,\upsilon - 0\overset{\circ}{.}004\,692\,\upsilon^2)\ \sin V \\
+(0\overset{\circ}{.}003\,228 - 0\overset{\circ}{.}064\,436\,\upsilon + 0\overset{\circ}{.}002\,075\,\upsilon^2)\ \cos V \\
-(0\overset{\circ}{.}003\,083 + 0\overset{\circ}{.}000\,275\,\upsilon - 0\overset{\circ}{.}000\,489\,\upsilon^2)\ \sin 2V \\
+0\overset{\circ}{.}002\,472\ \sin W \\
+0\overset{\circ}{.}013\,619\ \sin \zeta \\
+0\overset{\circ}{.}018\,472\ \sin 2\zeta \\
+0\overset{\circ}{.}006\,717\ \sin 3\zeta \\
+0\overset{\circ}{.}002\,775\ \sin 4\zeta \\
+(0\overset{\circ}{.}007\,275 - 0\overset{\circ}{.}001\,253\,\upsilon)\ \sin \zeta\ \sin Q \\
+0\overset{\circ}{.}006\,417\ \sin 2\zeta\ \sin Q \\
+0\overset{\circ}{.}002\,439\ \sin 3\zeta\ \sin Q \\
-(0\overset{\circ}{.}033\,839 + 0\overset{\circ}{.}001\,125\,\upsilon)\ \cos \zeta\ \sin Q \\
-0\overset{\circ}{.}003\,767\ \cos 2\zeta\ \sin Q \\
-(0\overset{\circ}{.}035\,681 + 0\overset{\circ}{.}001\,208\,\upsilon)\ \sin \zeta\ \cos Q \\
-0\overset{\circ}{.}004\,261\ \sin 2\zeta\ \cos Q \\
+0\overset{\circ}{.}002\,178\ \cos Q \\
+(-0\overset{\circ}{.}006\,333 + 0\overset{\circ}{.}001\,161\,\upsilon)\ \cos \zeta\ \cos Q \\
-0\overset{\circ}{.}006\,675\ \cos 2\zeta\ \cos Q \\
-0\overset{\circ}{.}002\,664\ \cos 3\zeta\ \cos Q \\
-0\overset{\circ}{.}002\,572\ \sin \zeta\ \sin 2Q \\
-0\overset{\circ}{.}003\,567\ \sin 2\zeta\ \sin 2Q \\
+0\overset{\circ}{.}002\,094\ \cos \zeta\ \cos 2Q \\
+0\overset{\circ}{.}003\,342\ \cos 2\zeta\ \cos 2Q
\end{array}
$$

Perturbations in the eccentricity

(The coefficients are given in units of the seventh decimal)

$$
\begin{array}{l}
+(3606 + 130\,\upsilon - 43\,\upsilon^2)\ \sin V \\
+(1289 - 580\,\upsilon)\ \cos V
\end{array}
$$

$-6764 \sin\zeta \sin Q$
$-1110 \sin 2\zeta \sin Q$
$-224 \sin 3\zeta \sin Q$
$-204 \sin Q$
$+(1284 + 116\,\upsilon) \cos\zeta \sin Q$
$+188 \cos 2\zeta \sin Q$
$+(1460 + 130\,\upsilon) \sin\zeta \cos Q$
$+224 \sin 2\zeta \cos Q$
$-817 \cos Q$
$+6074 \cos\zeta \cos Q$
$+992 \cos 2\zeta \cos Q$
$+508 \cos 3\zeta \cos Q$
$+230 \cos 4\zeta \cos Q$
$+108 \cos 5\zeta \cos Q$
$-(956 + 73\,\upsilon) \sin\zeta \sin 2Q$
$+448 \sin 2\zeta \sin 2Q$
$+137 \sin 3\zeta \sin 2Q$
$+(-997 + 108\,\upsilon) \cos\zeta \sin 2Q$
$+480 \cos 2\zeta \sin 2Q$
$+148 \cos 3\zeta \sin 2Q$
$+(-956 + 99\,\upsilon) \sin\zeta \cos 2Q$
$+490 \sin 2\zeta \cos 2Q$
$+158 \sin 3\zeta \cos 2Q$
$+179 \cos 2Q$
$+(1024 + 75\,\upsilon) \cos\zeta \cos 2Q$
$-437 \cos 2\zeta \cos 2Q$
$-132 \cos 3\zeta \cos 2Q$

Perturbations in the perihelion (B)

$+(0\overset{\circ}{.}007\,192 - 0\overset{\circ}{.}003\,147\,\upsilon) \sin V$
$+(-0\overset{\circ}{.}020\,428 - 0\overset{\circ}{.}000\,675\,\upsilon + 0\overset{\circ}{.}000\,197\,\upsilon^2) \cos V$
$+(0\overset{\circ}{.}007\,269 + 0\overset{\circ}{.}000\,672\,\upsilon) \sin\zeta \sin Q$
$-0\overset{\circ}{.}004\,344 \sin Q$
$+0\overset{\circ}{.}034\,036 \cos\zeta \sin Q$
$+0\overset{\circ}{.}005\,614 \cos 2\zeta \sin Q$
$+0\overset{\circ}{.}002\,964 \cos 3\zeta \sin Q$
$+0\overset{\circ}{.}037\,761 \sin\zeta \cos Q$
$+0\overset{\circ}{.}006\,158 \sin 2\zeta \cos Q$
$-0\overset{\circ}{.}006\,603 \cos\zeta \cos Q$
$-0\overset{\circ}{.}005\,356 \sin\zeta \sin 2Q$
$+0\overset{\circ}{.}002\,722 \sin 2\zeta \sin 2Q$
$+0\overset{\circ}{.}004\,483 \cos\zeta \sin 2Q$
$-0\overset{\circ}{.}002\,642 \cos 2\zeta \sin 2Q$
$+0\overset{\circ}{.}004\,403 \sin\zeta \cos 2Q$
$-0\overset{\circ}{.}002\,536 \sin 2\zeta \cos 2Q$
$+0\overset{\circ}{.}005\,547 \cos\zeta \cos 2Q$
$-0\overset{\circ}{.}002\,689 \cos 2\zeta \cos 2Q$

If A is the sum of the perturbations in the mean longitude, B the sum of the perturbations in the perihelion, and e the orbital eccentricity *not* corrected for the perturbations, then the correction to the mean anomaly is

$$A - \frac{B}{e}$$

Perturbations in the semimajor axis
(The coefficients are given in units of the sixth decimal)

$-263 \cos V$
$+205 \cos \zeta$
$+693 \cos 2\zeta$
$+312 \cos 3\zeta$
$+147 \cos 4\zeta$
$+299 \sin \zeta \sin Q$
$+181 \cos 2\zeta \sin Q$
$+204 \sin 2\zeta \cos Q$
$+111 \sin 3\zeta \cos Q$
$-337 \cos \zeta \cos Q$
$-111 \cos 2\zeta \cos Q$

SATURN

Calculate υ, V, W, ζ, etc. as for Jupiter, and moreover $\psi = S - Q$.

Perturbations in the mean longitude (A)

$+(-0\overset{\circ}{.}814\,181 + 0\overset{\circ}{.}018\,150\,\upsilon - 0\overset{\circ}{.}016\,714\,\upsilon^2)\ \sin V$
$+(-0\overset{\circ}{.}010\,497 + 0\overset{\circ}{.}160\,906\,\upsilon - 0\overset{\circ}{.}004\,100\,\upsilon^2)\ \cos V$
$+0\overset{\circ}{.}007\,581\ \sin 2V$
$-0\overset{\circ}{.}007\,986\ \sin W$
$-0\overset{\circ}{.}148\,811\ \sin \zeta$
$-0\overset{\circ}{.}040\,786\ \sin 2\zeta$
$-0\overset{\circ}{.}015\,208\ \sin 3\zeta$
$-0\overset{\circ}{.}006\,339\ \sin 4\zeta$
$-0\overset{\circ}{.}006\,244\ \sin Q$
$+(0\overset{\circ}{.}008\,931 + 0\overset{\circ}{.}002\,728\,\upsilon)\ \sin \zeta\ \sin Q$
$-0\overset{\circ}{.}016\,500\ \sin 2\zeta\ \sin Q$
$-0\overset{\circ}{.}005\,775\ \sin 3\zeta\ \sin Q$
$+(0\overset{\circ}{.}081\,344 + 0\overset{\circ}{.}003\,206\,\upsilon)\ \cos \zeta\ \sin Q$
$+0\overset{\circ}{.}015\,019\ \cos 2\zeta\ \sin Q$
$+(0\overset{\circ}{.}085\,581 + 0\overset{\circ}{.}002\,494\,\upsilon)\ \sin \zeta\ \cos Q$
$+(0\overset{\circ}{.}025\,328 - 0\overset{\circ}{.}003\,117\,\upsilon)\ \cos \zeta\ \cos Q$
$+0\overset{\circ}{.}014\,394\ \cos 2\zeta\ \cos Q$
$+0\overset{\circ}{.}006\,319\ \cos 3\zeta\ \cos Q$
$+0\overset{\circ}{.}006\,369\ \sin \zeta\ \sin 2Q$
$+0\overset{\circ}{.}009\,156\ \sin 2\zeta\ \sin 2Q$
$+0\overset{\circ}{.}007\,525\ \sin 3\psi\ \sin 2Q$
$-0\overset{\circ}{.}005\,236\ \cos \zeta\ \cos 2Q$
$-0\overset{\circ}{.}007\,736\ \cos 2\zeta\ \cos 2Q$
$-0\overset{\circ}{.}007\,528\ \cos 3\psi\ \cos 2Q$

Perturbations in the eccentricity
(The coefficients are given in units of the seventh decimal)

$+(-7927 + 2548\,\upsilon + 91\,\upsilon^2)\ \sin V$
$+(13381 + 1226\,\upsilon - 253\,\upsilon^2)\ \cos V$
$+(248 - 121\,\upsilon)\ \sin 2V$
$-(305 + 91\,\upsilon)\ \cos 2V$
$+412\ \sin 2\zeta$
$+12415\ \sin Q$
$+(390 - 617\,\upsilon)\ \sin \zeta\ \sin Q$
$+(165 - 204\,\upsilon)\ \sin 2\zeta\ \sin Q$
$+26599\ \cos \zeta\ \sin Q$
$-4687\ \cos 2\zeta\ \sin Q$

$-1870 \cos 3\zeta \sin Q$

$-821 \cos 4\zeta \sin Q$

$-377 \cos 5\zeta \sin Q$

$+497 \cos 2\psi \sin Q$

$+(163 - 611\upsilon) \cos Q$

$-12696 \sin \zeta \cos Q$

$-4200 \sin 2\zeta \cos Q$

$-1503 \sin 3\zeta \cos Q$

$-619 \sin 4\zeta \cos Q$

$-268 \sin 5\zeta \cos Q$

$-(282 + 1306\upsilon) \cos \zeta \cos Q$

$+(-86 + 230\upsilon) \cos 2\zeta \cos Q$

$+461 \sin 2\psi \cos Q$

$-350 \sin 2Q$

$+(2211 - 286\upsilon) \sin \zeta \sin 2Q$

$-2208 \sin 2\zeta \sin 2Q$

$-568 \sin 3\zeta \sin 2Q$

$-346 \sin 4\zeta \sin 2Q$

$-(2780 + 222\upsilon) \cos \zeta \sin 2Q$

$+(2022 + 263\upsilon) \cos 2\zeta \sin 2Q$

$+248 \cos 3\zeta \sin 2Q$

$+242 \sin 3\psi \sin 2Q$

$+467 \cos 3\psi \sin 2Q$

$-490 \cos 2Q$

$-(2842 + 279\upsilon) \sin \zeta \cos 2Q$

$+(128 + 226\upsilon) \sin 2\zeta \cos 2Q$

$+224 \sin 3\zeta \cos 2Q$

$+(-1594 + 282\upsilon) \cos \zeta \cos 2Q$

$+(2162 - 207\upsilon) \cos 2\zeta \cos 2Q$

$+561 \cos 3\zeta \cos 2Q$

$+343 \cos 4\zeta \cos 2Q$

$+469 \sin 3\psi \cos 2Q$

$-242 \cos 3\psi \cos 2Q$

$-205 \sin \zeta \sin 3Q$

$+262 \sin 3\zeta \sin 3Q$

$+208 \cos \zeta \cos 3Q$

$-271 \cos 3\zeta \cos 3Q$

$-382 \cos 3\zeta \sin 4Q$

$-376 \sin 3\zeta \cos 4Q$

Perturbations in the perihelion (B)

$+(0^{\circ}\!.077\,108 + 0^{\circ}\!.007\,186\,\upsilon - 0^{\circ}\!.001\,533\,\upsilon^2)\ \sin V$
$+(0^{\circ}\!.045\,803 - 0^{\circ}\!.014\,766\,\upsilon - 0^{\circ}\!.000\,536\,\upsilon^2)\ \cos V$
$-0^{\circ}\!.007\,075\ \sin \zeta$
$-0^{\circ}\!.075\,825\ \sin \zeta\ \sin Q$
$-0^{\circ}\!.024\,839\ \sin 2\zeta\ \sin Q$
$-0^{\circ}\!.008\,631\ \sin 3\zeta\ \sin Q$
$-0^{\circ}\!.072\,586\ \cos Q$
$-0^{\circ}\!.150\,383\ \cos \zeta\ \cos Q$
$+0^{\circ}\!.026\,897\ \cos 2\zeta\ \cos Q$
$+0^{\circ}\!.010\,053\ \cos 3\zeta\ \cos Q$
$-(0^{\circ}\!.013\,597 + 0^{\circ}\!.001\,719\,\upsilon)\ \sin \zeta\ \sin 2Q$
$+(-0^{\circ}\!.007\,742 + 0^{\circ}\!.001\,517\,\upsilon)\ \cos \zeta\ \sin 2Q$
$+(0^{\circ}\!.013\,586 - 0^{\circ}\!.001\,375\,\upsilon)\ \cos 2\zeta\ \sin 2Q$
$+(-0^{\circ}\!.013\,667 + 0^{\circ}\!.001\,239\,\upsilon)\ \sin \zeta\ \cos 2Q$
$+0^{\circ}\!.011\,981\ \sin 2\zeta\ \cos 2Q$
$+(0^{\circ}\!.014\,861 + 0^{\circ}\!.001\,136\,\upsilon)\ \cos \zeta\ \cos 2Q$
$-(0^{\circ}\!.013\,064 + 0^{\circ}\!.001\,628\,\upsilon)\ \cos 2\zeta\ \cos 2Q$

As for Jupiter, the correction to the mean longitude is A, and the correction to the mean anomaly is $A - \frac{B}{e}$.

Perturbations in the semimajor axis

(The coefficients are given in units of the sixth decimal)

$+572\ \sin V$
$+2933\ \cos V$
$+33629\ \cos \zeta$
$-3081\ \cos 2\zeta$
$-1423\ \cos 3\zeta$
$-671\ \cos 4\zeta$
$-320\ \cos 5\zeta$
$+1098\ \sin Q$
$-2812\ \sin \zeta\ \sin Q$
$+688\ \sin 2\zeta\ \sin Q$
$-393\ \sin 3\zeta\ \sin Q$
$-228\ \sin 4\zeta\ \sin Q$
$+2138\ \cos \zeta\ \sin Q$
$-999\ \cos 2\zeta\ \sin Q$
$-642\ \cos 3\zeta\ \sin Q$
$-325\ \cos 4\zeta\ \sin Q$
$-890\ \cos Q$
$+2206\ \sin \zeta\ \cos Q$
$-1590\ \sin 2\zeta\ \cos Q$
$-647\ \sin 3\zeta\ \cos Q$
$-344\ \sin 4\zeta\ \cos Q$
$+2885\ \cos \zeta\ \cos Q$
$+(2172 + 102\,\upsilon)\ \cos 2\zeta\ \cos Q$
$+296\ \cos 3\zeta\ \cos Q$
$-267\ \sin 2\zeta\ \sin 2Q$
$-778\ \cos \zeta\ \sin 2Q$
$+495\ \cos 2\zeta\ \sin 2Q$
$+250\ \cos 3\zeta\ \sin 2Q$
$-856\ \sin \zeta\ \cos 2Q$
$+441\ \sin 2\zeta\ \cos 2Q$
$+296\ \cos 2\zeta\ \cos 2Q$
$+211\ \cos 3\zeta\ \cos 2Q$
$-427\ \sin \zeta\ \sin 3Q$
$+398\ \sin 3\zeta\ \sin 3Q$
$+344\ \cos \zeta\ \cos 3Q$
$-427\ \cos 3\zeta\ \cos 3Q$

Then, after the whole calculation (equation of Kepler, etc.), add the following perturbations to the heliocentric *latitude* :

$$
\begin{aligned}
&+0\overset{\circ}{.}000\,747 \cos \zeta \sin Q\\
&+0\overset{\circ}{.}001\,069 \cos \zeta \cos Q\\
&+0\overset{\circ}{.}002\,108 \sin 2\zeta \sin 2Q\\
&+0\overset{\circ}{.}001\,261 \cos 2\zeta \sin 2Q\\
&+0\overset{\circ}{.}001\,236 \sin 2\zeta \cos 2Q\\
&-0\overset{\circ}{.}002\,075 \cos 2\zeta \cos 2Q
\end{aligned}
$$

25

Elliptic Motion

In this Chapter we will describe two methods for the calculation of a geocentric ephemeris in the case of an elliptic orbit. In the first method, which may be used for the major planets, the geocentric ecliptical longitude and latitude are obtained from the heliocentric ecliptical coordinates of the planet and the geocentric longitude and radius vector of the Sun. In the second method, which is better suited for minor planets and periodic comets, the right ascension and declination of the body, referred to a standard equinox, are obtained directly ; use is made of the geocentric rectangular coordinates of the Sun.

FIRST METHOD

In this method, we use the orbital elements of the planet referred to the *mean equinox of the date*.

From Table 23.A, calculate for the given instant the planet's mean longitude L, semimajor axis a, orbital eccentricity e, inclination i, and longitude of the ascending node Ω.

Calculate the planet's mean anomaly M by means of one of the following formulae :

$$\begin{array}{l|l}
\text{MERCURY} & M_1 = 102\overset{\circ}{.}27\,938 + 149\,472\overset{\circ}{.}51\,529\,T + 0\overset{\circ}{.}000\,007\,T^2 \\
\text{VENUS} & M_2 = 212\overset{\circ}{.}603\,22 + 58\,517\overset{\circ}{.}80\,387\,T + 0\overset{\circ}{.}001\,286\,T^2 \\
\text{MARS} & M_4 = 319\overset{\circ}{.}51\,913 + 19\,139\overset{\circ}{.}85\,475\,T + 0\overset{\circ}{.}000\,181\,T^2 \\
\text{JUPITER} & M_5 = 225\overset{\circ}{.}32\,833 + 3034\overset{\circ}{.}69\,202\,T - 0\overset{\circ}{.}000\,722\,T^2 \\
\text{SATURN} & M_6 = 175\overset{\circ}{.}46\,622 + 1221\overset{\circ}{.}55\,147\,T - 0\overset{\circ}{.}000\,502\,T^2
\end{array}$$

where T is the time in Julian centuries from 1900 January 0.5 ET ;

see formula (23.1). The cases of Uranus and Neptune are not considered here by reason of the large perturbations in their motion.

From the values of e and M, calculate the eccentric anomaly E (Chapter 22), and then the true anomaly v from

$$\tan \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \tag{25.1}$$

If necessary, take into account the principal perturbations (Chapter 24).

The radius vector of the planet can be calculated by means of one of the following two formulae :

$$\begin{aligned} r &= a\,(1 - e \cos E) \\ r &= \frac{a\,(1 - e^2)}{1 + e \cos v} \end{aligned} \tag{25.2}$$

The planet's argument of latitude is

$$u = L + v - M - \Omega \tag{25.3}$$

The ecliptical longitude l can be deduced from $(l - \Omega)$, which is given by

$$\tan (l - \Omega) = \cos i \tan u \tag{25.4}$$

If $i < 90°$, as for the major planets, $(l - \Omega)$ and u must lie in the same quadrant. When a programmable calculator is used, in order to avoid the use of tests, formula (25.4) can better be written as follows :

$$\tan (l - \Omega) = \frac{\cos i \sin u}{\cos u} \tag{25.5}$$

and then the conversion from rectangular to polar coordinates should be applied to the numerator and the denominator of the fraction in the right-hand side. This will give $(l - \Omega)$ directly in the correct quadrant.

The planet's ecliptical latitude b is given by

$$\sin b = \sin u \sin i \tag{25.6}$$

with $-90° < b < +90°$.

We have now obtained the heliocentric ecliptical coordinates

l, b, r of the planet for the given instant. Its geocentric coordinates can be obtained as follows.

Using the method described in Chapter 18, calculate for the given instant the Sun's geometric longitude ⊙ referred to the mean equinox of the date, and its radius vector R. The planet's geocentric longitude λ can be deduced from $(\lambda - \odot)$, which is given by

$$\tan (\lambda - \odot) = \frac{r \cos b \sin (l - \odot)}{r \cos b \cos (l - \odot) + R} = \frac{N}{D} \qquad (25.7)$$

Once again, $(\lambda - \odot)$ can be obtained immediately in the correct quadrant by applying the conversion from rectangular into polar coordinates to the numerator N and the denominator D of the fraction.

The planet's distance Δ to the Earth, in astronomical units, is given by

$$\left.\begin{array}{l} \Delta^2 = N^2 + D^2 + (r \sin b)^2 \\ \text{or} \\ \Delta^2 = R^2 + r^2 + 2rR \cos b \cos (l - \odot) \end{array}\right\} \qquad (25.8)$$

Finally, the planet's geocentric latitude β is given by

$$\sin \beta = \frac{r}{\Delta} \sin b \qquad (25.9)$$

The geocentric coordinates of the planet obtained in this manner are the planet's *geometric* coordinates referred to the mean equinox of the date. If high accuracy is needed, it is necessary to take into account the *effect of light-time* : at time t, the planet is seen in the direction obtained by combining the Earth's (Sun's) position at time t with that of the planet at time $t - \tau$, where τ is the time taken by the light to reach the Earth from the planet. This time is given by

$$\tau = 0.005\,7756\ \Delta \quad \text{day} \qquad (25.10)$$

The *elongation* ψ of the planet, that is its angular distance to the Sun, can be calculated from

$$\cos \psi = \cos \beta \cos (\lambda - \odot) \qquad (25.11)$$

The longitude and latitude of the planet can be converted to right ascension and declination by means of the formulae (8.3) and

(8.4). The equatorial coordinates obtained in this manner are still referred to the *mean* equinox of the date. They may be converted into *apparent* right ascension and declination by correcting for nutation and aberration (see Chapter 16).

Example 25.a : Calculate the heliocentric and geocentric positions of Mercury for 1978 November 12.0 ET.

We obtain successively :

$\mathrm{JD} = 2443\,824.5$

$T = +0.788\,624\,230$

$L = 337^{\circ}.053\,200$

$a = 0.387\,0986$

$e = 0.205\,630\,33$

$i = 7^{\circ}.004\,337$

$\Omega = 48^{\circ}.080\,736$

$M_1 = 259^{\circ}.926\,60$

$E = 248^{\circ}.932\,38$

$v = 238^{\circ}.250\,67$

$r = 0.415\,71$

$u = 267^{\circ}.296\,53$

$l - \Omega = 267^{\circ}.276\,24$

$l = 315^{\circ}.35697 = 315^{\circ}21'25''$

$b = -6^{\circ}.99650 = -6^{\circ}59'47''$

In Example 18.a we have found, for the same instant,

$\odot = 229^{\circ}.25049 \qquad R = 0.98984$

Hence,

$$l - \odot = 86^{\circ}.10648$$

$$\tan(\lambda - \odot) = \frac{+0.411\,6621}{+1.017\,8575}$$

$$\lambda - \odot = 22^{\circ}.02037$$

$$\lambda = 251^{\circ}.27086$$

$$\Delta = 1.09912$$

$$\beta = -2^{\circ}.64058$$

$$\psi = 22^{\circ}.17$$

By means of formula (18.4), we find $\varepsilon = 23^{\circ}.442\,032$. Hence, by means of formulae (8.3) and (8.4),

$$\alpha = 249^{\circ}.31740 = 16^{h}37^{m}16^{s}.2$$

$$\delta = -24^{\circ}.74770 = -24^{\circ}44'52''$$

Let us now compare our results with the values given by the *A.E.* :

		Our result	*A.E.*
l	heliocentric longitude	315°21′25″	315°21′17″
b	heliocentric latitude	−6°59′47″	−6°59′47″
r	radius vector	0.41571	0.41572
α	right ascension	$16^h37^m16^s.2$	$16^h37^m14^s.4$
δ	declination	−24°44′52″	−24°44′39″
Δ	distance to Earth	1.09912	1.09914

The error in l is due to the fact that we neglected the perturbations in the motions of Mercury and the Earth. The errors in α and δ are due partly to this same reason, and partly because we neglected the effects of light-time, nutation and aberration.

SECOND METHOD

Here we use the orbital elements referred to a standard equinox, for instance 1950.0, and the geocentric rectangular equatorial coordinates X, Y, Z of the Sun referred to that *same* equinox. These rectangular coordinates can be taken from the *A.E.*, or calculated by means of the method described in Chapter 19.

The heliocentric longitude and latitude of the planet or comet are not calculated in this method. Instead, we calculate the heliocentric rectangular equatorial coordinates x, y, z of the body, after which the right ascension, declination and other quantities are derived by means of simple formulae.

The following orbital elements are given :

a = semimajor axis, in AU
e = eccentricity
i = inclination
ω = argument of perihelion
Ω = longitude of ascending node
n = mean motion, in degrees/day

where i, ω and Ω are referred to a standard equinox.

If a and n are not given, they can be calculated from

$$a = \frac{q}{1 - e} \qquad n = \frac{0.985\,609}{a\sqrt{a}} \qquad (25.12)$$

where q is the perihelion distance in AU.

All these elements are, strictly speaking, only for one given instant, called the *Epoch*. They vary slowly with time under influence of planetary perturbations. Unless high accuracy is required, the elements may be considered as invariable during several months, for example during the whole apparition of a comet.

Besides the above-mentioned orbital elements, either the value M_0 of the mean anomaly at the epoch, or the time T of passage at the perihelion is given. This allows the calculation of the mean anomaly M at any given instant. The mean anomaly increases by n degrees per day, and is zero at time T.

The orbital elements of a minor planet or of a periodic comet being given, the geocentric position for a given date can be calculated as follows. Firstly, we must calculate the quantities a, b, c and the angles A, B, C, which are constants for a given orbit.

Let ε be the obliquity of the ecliptic. If the orbital elements are referred to the standard equinox of 1950.0, one should use the value

$$\varepsilon_{1950} = 23^\circ\!.445\,7889$$

Then calculate

$$\left.\begin{array}{l} F = \cos\Omega \\ G = \sin\Omega\cos\varepsilon \\ H = \sin\Omega\sin\varepsilon \end{array}\right| \begin{array}{l} P = -\sin\Omega\cos i \\ Q = \cos\Omega\cos i\cos\varepsilon - \sin i\sin\varepsilon \\ R = \cos\Omega\cos i\sin\varepsilon + \sin i\cos\varepsilon \end{array}$$

As a check, we have $F^2 + G^2 + H^2 = 1$, $P^2 + Q^2 + R^2 = 1$, but this calculation is not needed in a program.

Then the quantities a, b, c, A, B, C are given by

$$\left.\begin{array}{ll} \tan A = \dfrac{F}{P} & a = \sqrt{F^2 + P^2} \\[2ex] \tan B = \dfrac{G}{Q} & b = \sqrt{G^2 + Q^2} \\[2ex] \tan C = \dfrac{H}{R} & c = \sqrt{H^2 + R^2} \end{array}\right\} \qquad (25.13)$$

The quantities a, b, c should be taken *positive*, while the angles A, B, C should be placed in the correct quadrant, according to the following rules :

sin A has the same sign as cos Ω ,

sin B and sin C have the same sign as sin Ω.

However, once again it is preferable to apply the conversion from rectangular to polar coordinates to F and P, to G and Q, and to H and R. Not only will this procedure place the angles A, B, C in the correct quadrant, but at the same time it will provide the values of a, b, c, and thus save many program steps.

For each required position, calculate the body's mean anomaly M, then the eccentric anomaly E (see Chapter 22), the true anomaly v by means of formula (25.1), and the radius vector r by means of (25.2). Then the heliocentric rectangular equatorial coordinates of the body are given by

$$\left.\begin{aligned} x &= r\,a \sin(A + \omega + v) \\ y &= r\,b \sin(B + \omega + v) \\ z &= r\,c \sin(C + \omega + v) \end{aligned}\right\} \qquad (25.14)$$

The convenience of these formulae is seen when the rectangular coordinates are required for several positions of the body. The auxiliary quantities a, b, c, A, B, C are functions only of Ω, i and ε, and thus are constants for the whole ephemeris ; for each position only the values of v and r must be calculated. However, it should be noted that Ω, i and ω are constant only if the body is in an unperturbed orbit.

For the same instant, calculate the Sun's rectangular coordinates X, Y, Z (Chapter 19), or take them from the *A.E.*

The geocentric right ascension α and declination δ of the planet or comet are then calculated from

$$\left.\begin{aligned} \tan\alpha &= \frac{Y + y}{X + x} \\ \Delta^2 &= (X + x)^2 + (Y + y)^2 + (Z + z)^2 \\ \sin\delta &= \frac{Z + z}{\Delta} \end{aligned}\right\} \qquad (25.15)$$

where Δ is the distance to the Earth and thus is positive. The correct quadrant of α is indicated by the fact that $\sin \alpha$ has the same sign as $(Y + y)$; however, once more, the transformation from rectangular to polar coordinates, applied to the numerator and the denominator of the fraction, will put α in the correct quadrant without any test.

If α is negative, add 360 degrees. Then transform α from degrees into hours by dividing by 15.

The elongation ψ to the Sun, and the phase angle β (the angle Sun - body - Earth), can be calculated from

$$\cos \psi = \frac{(X + x)X + (Y + y)Y + (Z + z)Z}{R\Delta} = \frac{R^2 + \Delta^2 - r^2}{2R\Delta}$$

$$\cos \beta = \frac{(X + x)x + (Y + y)y + (Z + z)z}{r\Delta} = \frac{r^2 + \Delta^2 - R^2}{2r\Delta}$$

where $R = \sqrt{X^2 + Y^2 + Z^2}$; the angles ψ and β are both between 0 and +180 degrees.

The magnitude is then calculated as follows. In the case of a *comet*, the *total* magnitude is given by

$$m = g + 5 \log \Delta + \kappa \log r \qquad (25.16)$$

where g is the absolute magnitude, and κ a constant which differs from one comet to another. In general, κ is a number between 5 and 15.

In the case of a *minor planet*, we have

$$m = g + 5 \log r\Delta + k\beta$$

where β is the phase angle in degrees, and k is the phase coefficient. Generally, the value $k = 0.023$ is used for minor planets, although for some objects larger values have been found, for instance 0.049 for Ceres.

Example 25.b : Calculate the geocentric position of 433 Eros for 1975 February 11.0 ET, using the following orbital elements (IAUC 2722) :

$$\begin{aligned}
\text{Epoch} &= \text{1975 January 28.0 ET}\\
T &= \text{1975 January 24.70450 ET}\\
a &= 1.457\,9641 \text{ AU}\\
e &= 0.222\,7021
\end{aligned}$$

$$\left.\begin{aligned}
i &= 10^\circ\!.82772\\
\omega &= 178^\circ\!.44991\\
\Omega &= 303^\circ\!.83085
\end{aligned}\right\}\begin{array}{l}\text{ecliptic and equinox}\\ \quad\text{1950.0}\end{array}$$

$$\begin{aligned}
n &= 0.559\,865\,65 \text{ degree/day}\\
g &= 12.4 \text{ (photographic)}
\end{aligned}$$

We first calculate the auxiliary constants of the orbit :

$$\begin{aligned}
F &= +0.556\,742\,97 & P &= +0.815\,895\,71\\
G &= -0.762\,100\,94 & Q &= +0.426\,938\,36\\
H &= -0.330\,513\,88 & R &= +0.389\,920\,29
\end{aligned}$$

whence, by the formulae (25.13),

$$\begin{aligned}
A &= +34^\circ\!.30847 & a &= 0.987\,749\,23\\
B &= -60^\circ\!.74191 & b &= 0.873\,541\,19\\
C &= -40^\circ\!.28610 & c &= 0.511\,152\,87
\end{aligned}$$

For the given date (1975 February 11.0), the time from perihelion is +17.29550 days. Thus the mean anomaly is

$$M = 17.29550 \times 0^\circ\!.559\,865\,65 = +9^\circ\!.683\,156$$

We then find

$$\begin{aligned}
E &= 12^\circ\!.429\,591 & x &= -0.841\,5580\\
v &= 15^\circ\!.554\,375 & y &= +0.725\,7529\\
r &= 1.140\,8828 & z &= +0.258\,2179
\end{aligned}$$

The Sun's geocentric rectangular equatorial coordinates for the date, referred to the same standard equinox (1950.0), are taken from the *Astronomical Ephemeris* :

$$X = +0.770\,0006 \qquad Y = -0.566\,4014 \qquad Z = -0.245\,6064$$

We then obtain further :

$$\begin{aligned}
X + x &= -0.071\,5574\\
Y + y &= +0.159\,3515\\
Z + z &= +0.012\,6115
\end{aligned}$$

$$R = 0.986\,9316$$

$$\Delta = 0.175\,1354$$

$$\begin{aligned}
\alpha_{1950} &= 114\overset{\circ}{.}182\,647 = 7^h36^m44^s \\
\delta_{1950} &= +4^\circ 07\overset{\prime}{.}8 \\
\psi &= 149\overset{\circ}{.}19 \\
\beta &= 26\overset{\circ}{.}30 \\
\text{magnitude} &= 9.5
\end{aligned}$$

As an exercise, calculate an ephemeris for the minor planet 234 Barbara, using the following orbital elements :

$$\begin{aligned}
\text{Epoch} &= \text{1979 November 23.0 ET} \\
M_0 &= 34\overset{\circ}{.}88670 \\
a &= 2.384\,8264 \\
e &= 0.245\,6180 \\
&\left.\begin{aligned}
i &= 15\overset{\circ}{.}38354 \\
\omega &= 191\overset{\circ}{.}11341 \\
\Omega &= 144\overset{\circ}{.}17952
\end{aligned}\right\}\ \begin{array}{l}\text{ecliptic and} \\ \text{equinox 1950.0}\end{array} \\
n &= 0\overset{\circ}{.}267\,620\,22
\end{aligned}$$

Compare your results with the following ephemeris, published in the *Ephemerides of Minor Planets for 1979* (Leningrad, 1978) :

0^h ET	α_{1950}	δ_{1950}
1979 Sept. 4	$1^h24\overset{m}{.}8$	$-\ 9^\circ19'$
14	1 24.6	−12 14
24	1 21.0	−15 04
Oct. 4	1 15.2	−17 30
14	1 08.4	−19 15
24	1 02.2	−20 11
Nov. 3	0 57.9	−20 17
13	0 56.2	−19 39

26

PARABOLIC MOTION

In this Chapter we will give formulae for the calculation of positions of a comet which moves around the Sun in a parabolic orbit. We will assume that the elements of this orbit are invariable (no planetary perturbations) and that they are referred to a standard equinox (for example 1950.0).

The following orbital elements are given :

T = time of passage in perihelion
q = perihelion distance, in AU
i = inclination
ω = argument of perihelion
Ω = longitude of ascending node

Firstly, calculate the auxiliary constants a, b, c, A, B, C as for an elliptic orbit : see formulae (25.13). Then, for each required position of the comet, proceed as follows.

Let $t - T$ be the time since perihelion, in days. This quantity is negative for an instant before the time of perihelion. Calculate

$$W = \frac{0.036\,491\,1624}{q\ \sqrt{q}}\,(t - T) \tag{26.1}$$

Then the true anomaly v and the radius vector r of the comet are given by

$$\tan\frac{v}{2} = s \qquad\qquad r = q\,(1 + s^2) \tag{26.2}$$

where s is the root of the equation

$$s^3 + 3s - W = 0 \tag{26.3}$$

This equation can easily be solved by iteration. One may start from *any* value ; a good choice is $s = 0$. A better value for s is then given by

$$\frac{2s^3 + W}{3(s^2 + 1)} \qquad (26.4)$$

This calculation is repeated until the correct value of s is obtained. It should be noted that in formula (26.4) the cube of s must be calculated ; if s is negative, this operation is not possible on some calculating machines ; when this is the case, calculate $s^2 \times s$ instead of s^3.

Instead of solving equation (26.3) by iteration, s can be obtained directly as follows (J. Bauschinger, *Tafeln zur Theoretischen Astronomie*, page 9 ; Leipzig, 1934) :

$$\begin{aligned} \tan\beta &= \frac{2}{W} = 54.807\,791\ \frac{q\sqrt{q}}{t - T} \\ \tan\gamma &= \sqrt[3]{\tan\frac{\beta}{2}} \qquad (26.5) \\ s &= \frac{2}{\tan 2\gamma} \end{aligned}$$

For the calculation of $\tan\gamma$, one must take the cube root of a quantity which may be negative. When this is the case, the operation is impossible on most calculating machines. This difficulty can be avoided by using a test, a flag, or any other trick. For instance, here are two procedures for calculating the cube root of *any* number on the HP-67 calculating machine :

First method

f $x < 0$
h SF 2
h ABS
3
h $1/x$
h y^x
h F? 2
CHS

Second method

h ABS
h LST x
:
h LST x
h ABS
3
h $1/x$
h y^x
×

However, the author's preference is the iteration formula (26.4) which works without any difficulty.

When s is obtained, v and r can be found by means of (26.2), after which the calculation continues as for the elliptic motion : formulae (25.14) and (25.15).

It should be noted that s has the same sign as $t - T$, and thus is negative before perihelion, positive after perihelion.

In the parabolic motion, $e = 1$ while a and the period of revolution are infinite ; the mean daily motion is zero, and therefore the mean and eccentric anomalies do not exist (in fact, they are zero).

Example 26.a : Calculate the geocentric position of comet Kohler (1977*m*) for 1977 September 29.0 ET, using the following parabolic elements (IAUC 3137) :

$$\begin{aligned} T &= 1977 \text{ November } 10.5659 \text{ ET} \\ q &= 0.990\,662 \\ \left.\begin{aligned} i &= 48^\circ\!.7196 \\ \omega &= 163.4799 \\ \Omega &= 181.8175 \end{aligned}\right\} & \ 1950.0 \end{aligned}$$

$$\text{magnitude} = 6.0 + 5 \log \Delta + 10 \log r$$

We first calculate the auxiliary constants of the orbit :

$F = -0.999\,496\,92$ $P = +0.020\,924\,49$
$G = -0.029\,097\,47$ $Q = -0.903\,973\,29$
$H = -0.012\,619\,22$ $R = +0.427\,076\,64$

whence, by the formulae (25.13),

$A = -\ 88^\circ\!.800\,69$ $a = 0.999\,715\,92$
$B = -178^\circ\!.156\,38$ $b = 0.904\,441\,47$
$C = -\ \ 1^\circ\!.692\,48$ $c = 0.427\,263\,04$

For the given date (1977 September 29.0), the time from perihelion is $t - T = -42.5659$ days. Hence, by formula (26.1),

$$W = -1.575\,2927$$

Starting from the value $s = 0$, we obtain the following successive approximations by means of the iteration formula (26.4) :

$$\begin{array}{r} 0.000\,0000 \\ -0.525\,0976 \\ -0.487\,2672 \\ -0.486\,6745 \\ -0.486\,6743 \end{array}$$

Hence, $s = -0.486\,6743$, and consequently

$$v = -51\overset{\circ}{.}90199 \qquad r = 1.225\,3022$$

If, instead of the iteration procedure, formulae (26.5) are preferred, we obtain successively :

$$\tan\beta = -1.269\,6053$$

$$\beta = -51\overset{\circ}{.}774\,3927$$

$$\tan\gamma = \sqrt[3]{-0.485\,2978} = -0.785\,8436$$

$$\gamma = -38\overset{\circ}{.}161\,8063$$

$s = -0.486\,6743$, as before.

We then find, by means of formulae (25.14),

$$x = +0.474\,2398$$

$$y = -1.016\,9032$$

$$z = +0.492\,3109$$

The Sun's geocentric rectangular equatorial coordinates for the given date, referred to the same standard equinox (1950.0), are taken from the *Astronomical Ephemeris* :

$$X = -0.997\,3057$$

$$Y = -0.085\,7667$$

$$Z = -0.037\,1837$$

We then obtain further :

$$X + x = -0.523\,0659$$

$$Y + y = -1.102\,6699 \qquad R = 1.001\,6772$$

$$Z + z = +0.455\,1272 \qquad \Delta = 1.302\,5435$$

$$\alpha_{1950} = -115\overset{\circ}{.}377\,936 = 16^h18^m29^s$$

$$\delta_{1950} = +20°27\overset{\prime}{.}1$$

$$\psi = 62\overset{\circ}{.}66$$

magnitude = 7.5

27

PLANETS IN PERIHELION AND APHELION

The Julian Day corresponding to the time when a planet is in perihelion or aphelion can be found by means of the following formulae :

Mercury	$JD = 2414\,995.007 + 87.969\,349\,97\,k$
Venus	$JD = 2415\,112.001 + 224.700\,8454\,k - 0.000\,000\,0304\,k^2$
Earth	$JD = 2415\,021.546 + 365.259\,6413\,k + 0.000\,000\,0152\,k^2$
Mars	$JD = 2415\,097.251 + 686.995\,8091\,k - 0.000\,000\,1221\,k^2$
Jupiter	$JD = 2416\,640.884 + 4332.894\,375\,k + 0.000\,1222\,k^2$
Saturn	$JD = 2409\,773.47 + 10\,764.180\,10\,k + 0.001\,3033\,k^2$

where k is an integer for perihelion, and an integer increased by exactly 0.5 for aphelion.

Any other value for k will give a meaningless result !

A positive (negative) value of k will give a date after (before) the beginning of the year 1900.

For example, $k = +14$ and $k = -222$ will give passages through perihelion, while $k = +27.5$ and $k = -119.5$ give passages through aphelion.

An *approximate* value for k can be found as follows, where the "year" should be taken with decimals, if necessary :

Mercury	$k \simeq 4.15201\,(\text{year} - 1900)$
Venus	$k \simeq 1.62549\,(\text{year} - 1900)$
Earth	$k \simeq 0.99997\,(\text{year} - 1900)$
Mars	$k \simeq 0.53166\,(\text{year} - 1900)$

Jupiter	$k \simeq 0.08430$ (year - 1900)
Saturn	$k \simeq 0.03393$ (year - 1900)

Example 27.a : Find the time of passage of Venus at the perihelion nearest to 1978 October 15, that is 1978.79.

An approximate value of k is given by

$$k \simeq 1.62549\,(1978.79 - 1900) = 128.07$$

and, since k must be an integer (perihelion !), we take $k = 128$. Putting this value in the formula for Venus, we find

$$\text{JD} = 2443\,873.709\ ,$$

which corresponds to 1978 December 31.209 = 1978 December 31 at 5^h ET.

Example 27.b : Find the time of passage of Mars through aphelion in 1978.

Taking year = 1978, we find $k \simeq 41.47$. Since k must be an integer increased by 0.5 (aphelion !), we take $k = 41.5$.
Using the formula for Mars, this gives JD = 2443 607.577, which corresponds to 1978 April 9.077 or 1978 April 9 at 2^h ET.

It is important to note that the formula given for the Earth is actually valid for the *barycenter* of the Earth-Moon system. Due to the action of the Moon, the time of least or greatest distance between the centers of Sun and Earth may differ from that for the barycenter by more than one day. For instance, $k = 78$ in the formula for the Earth yields JD = 2443 511.80, which corresponds to 1978 January 3.30, while the correct instant for the Earth is January 1 at 23^h.

Due to mutual planetary perturbations, the instants for Jupiter, calculated by the method described here, may be up to half a month in error. For Saturn, the error may be larger than one month.

For instance, putting $k = 6.5$ in the formula for Jupiter gives 1981 July 19 as the date of an aphelion passage, while the correct date is 1981 July 28. For Saturn, $k = 2$ gives 1944 July 30, while the planet actually reached perihelion on 1944 September 8.

The error would be even larger for Uranus and Neptune. For this reason, no formula is given for these planets. Uranus reached the perihelion on 1966 May 19, and will be in aphelion on 2009 Feb 27.

28

PASSAGES THROUGH THE NODES

Given the orbital elements of a planet or comet, the times t of passages of that body through the nodes of its orbit can easily be calculated as follows.

We have

at the ascending node : $v = -\omega$ or $360° - \omega$

at the descending node : $v = 180° - \omega$

where, as before, v is the true anomaly, and ω the argument of the perihelion. Then, with these values of v, proceed as follows :

Case of an elliptic orbit

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{v}{2}$$

$$M = E - e_o \sin E \qquad (28.1)$$

$$t = T + \frac{M}{n} \text{ days} \qquad (28.2)$$

where e is the orbital eccentricity, while e_o is e converted from radians into degrees, that is

$$e_o = e \times 57\overset{\circ}{.}295\,779\,51$$

In formula (28.1), E should be expressed in degrees. In formula (28.2), T is the time of perihelion passage, M is expressed in degrees, while n is the mean motion in degrees/day.

The corresponding value of the radius vector r can be calculated from

$$r = a\,(1 - e \cos E)$$

where a is the semimajor axis, expressed in astronomical units.

If a and n are not given, they can be calculated from

$$a = \frac{q}{1 - e} \qquad\qquad n = \frac{0.985\,609}{a\sqrt{a}}$$

where q is the perihelion distance in astronomical units.

Case of a parabolic orbit

$$s = \tan\frac{v}{2}$$

$$t = T + 27.403\,896\,(s^3 + 3s)\,q\sqrt{q} \quad \text{days}$$

where the perihelion distance q is expressed in AU. The corresponding value of the radius vector is

$$r = q\,(1 + s^2)$$

Note. — The nodes refer to the ecliptic of the same epoch as that of the equinox used for the orbital elements. For example, if the orbital elements are referred to the standard equinox of 1950.0, the above-mentioned formulae give the passages through the nodes on the ecliptic of 1950.0, *not* on the ecliptic of date. The difference may generally be neglected, except when the inclination is very small.

Example 28.a : We use the same orbital elements for the minor planet Eros as in Example 25.b :

T = 1975 January 24.70450 ET
$\omega = 178\overset{\circ}{.}44991$
$e = 0.222\,7021$
$n = 0\overset{\circ}{.}559\,865\,65$ per day
$a = 1.457\,9641$ AU

For the passage at the descending node, we have

$$v = 180^\circ - \omega = 1\overset{\circ}{.}55009$$

$$\tan\frac{E}{2} = 0.797\,3214 \times 0.013\,5279 = 0.010\,7861$$

$$E = 1\overset{\circ}{.}235\,9474$$

$$M = 1\overset{\circ}{.}235\,9474 - (0.222\,7021 \times 57\overset{\circ}{.}295\,779\,51)\sin 1\overset{\circ}{.}2359474$$
$$= 0\overset{\circ}{.}960\,7206$$

$$t = T + \frac{0.960\,7206}{0.559\,865\,65} = T + 1.71598 \text{ days}$$
$$= 1975 \text{ January } 26.4205$$

$r = 1.13335$ AU

For the ascending node we find similarly :

$v = -\omega = -178^{\circ}.44991$
$E = -178^{\circ}.05595$
$M = -177^{\circ}.62308$
$t = T - 317.26019$ days = 1974 March 13.4443
$r = 1.78247$ AU

Example 28.b : Comet Kohler (1977*m*). We use the same orbital elements as in Example 26.a :

$T =$ 1977 November 10.5659 ET
$q = 0.990\,662$ AU
$\omega = 163^{\circ}.4799$

For the ascending node, we have

$v = -\omega = -163^{\circ}.4799$
$s = -6.888\,371$
$t = T - 9390.2$ days
= 1952 February 25
$r = 47.997$ AU

At the descending node we have

$v = 180^{\circ} - \omega = 16^{\circ}.5201$
$s = +0.145\,1722$
$t = T + 11.8507$ days
= 1977 November 22.4166 ET
$r = 1.0115$ AU

Example 28.c : Calculate the time of passage of Venus at the ascending node nearest to the epoch 1979.0.

We use the elements given in Table 23.A. There we find

$a = 0.723\,3316$, whence $n = 1.602\,133$
$e = 0.006\,820\,69 - 0.000\,047\,74\,T + 0.000\,000\,091\,T^2$
$\omega = 54^{\circ}.384\,186 + 0^{\circ}.508\,1861\,T - 0.001\,3864\,T^2$

The elements e and ω vary with time. We calculate their values for the epoch 1979.0, that is for $T = +0.79$. We find

$e = 0.006\,783\,03$ $\qquad \omega = 54^{\circ}.784\,788$

We then find successively

$v = -\omega = -54^{\circ}.784\,788$ $\qquad M = -54^{\circ}.151\,620$
$E = -54^{\circ}.467\,890$ $\qquad t = T - 33.7997$ days

In Example 27.a, we have found T = 1978 December 31.209 for the time of passage of Venus in the perihelion. Therefore, we have
t = 1978 November 27.409 or 1978 November 27 at 10^h ET.

29

CORRECTION FOR PARALLAX

We wish to calculate the topocentric coordinates of a body (Moon, Sun, planet, comet) when its geocentric coordinates are known. *Geocentric* = as seen from the center of the Earth ; *topocentric* = as seen from the observer's place (Greek : *topos* = place ; compare with Topology).

In other words, we wish to find the corrections $\Delta\alpha$ and $\Delta\delta$ (the parallaxes in right ascension and declination), in order to obtain the topocentric right ascension $\alpha' = \alpha + \Delta\alpha$ and the topocentric declination $\delta' = \delta + \Delta\delta$, when the geocentric values α and δ are known.

Let ρ be the geocentric radius and ϕ' the geocentric latitude of the observer. The expressions $\rho \sin \phi'$ and $\rho \cos \phi'$ can be calculated by the method described in Chapter 6.

Let π be the equatorial horizontal parallax of the body. For the Sun, planets and comets, it is frequently more convenient to use the distance Δ (in astronomical units) to the Earth instead of the parallax. We then have

$$\sin \pi = \frac{\sin 8''.794}{\Delta}$$

or, with sufficient accuracy,

$$\pi = \frac{8''.794}{\Delta} \qquad (29.1)$$

Then, if H is the geocentric hour angle, the rigorous formulae are :

$$\tan \Delta\alpha = \frac{-\rho \cos \phi' \sin \pi \sin H}{\cos \delta - \rho \cos \phi' \sin \pi \cos H} \qquad (29.2)$$

In the case of the declination we may, instead of computing $\Delta\delta$, calculate δ' directly from

$$\tan \delta' = \frac{(\sin \delta - \rho \sin \phi' \sin \pi) \cos \Delta\alpha}{\cos \delta - \rho \cos \phi' \sin \pi \cos H} \qquad (29.3)$$

Except for the Moon, the following non-rigorous formulae may often be used instead of (29.2) and (29.3) :

$$\Delta\alpha = \frac{-\pi \rho \cos \phi' \sin H}{\cos \delta} \qquad (29.4)$$

$$\Delta\delta = -\pi (\rho \sin \phi' \cos \delta - \rho \cos \phi' \cos H \sin \delta) \qquad (29.5)$$

If π is expressed in seconds of a degree (″), then $\Delta\alpha$ and $\Delta\delta$ too are expressed in this unit. To express $\Delta\alpha$ in seconds of time, divide the result by 15.

Example 29.a : Calculate the topocentric coordinates of Mars on 1971 August 12 at $2^h34^m00^s$ UT at the Uccle Observatory, for which

$$\sin \phi' = +0.771\ 306$$
$$\cos \phi' = +0.633\ 333$$
$$L = \text{longitude} = -0^h17^m26^s$$

Mars' geocentric equatorial coordinates for the given instant, interpolated from the *Astronomical Ephemeris*, are

$$\alpha = 21^h24^m46^s\!.85, \qquad \delta = -22°24'09''\!.9$$

The planet's distance at that time is 0.3757 AU and thus, by formula (29.1), its equatorial horizontal parallax is $\pi = 23''\!.41$.

We still need the geocentric hour angle, which is equal to $H = \theta_\circ - L - \alpha$, where $\theta_\circ$, the sidereal time at Greenwich, can be found as indicated in Chapter 7. For the given instant we find $\theta_\circ = 23^h53^m36^s$. Thus

$$H = 23^h53^m36^s + 0^h17^m26^s - 21^h24^m47^s$$
$$= +2^h46^m15^s = +41°\!.562$$

Formula (29.2) then gives

$$\tan \Delta\alpha = \frac{-0.000\ 047\ 687}{+0.924\ 474}$$

whence $\Delta\alpha = -0\overset{\circ}{.}002\,9555 = -0\overset{s}{.}71$

$\alpha' = \alpha + \Delta\alpha = 21^h24^m46\overset{s}{.}14$

Formula (29.3) gives

$$\tan\delta' = \frac{-0.381\,202\,29}{+0.924\,473\,96}$$

whence $\delta' = -22°24'30\overset{''}{.}8$

If, instead of (29.2) and (29.3), we choose the non-rigorous formulae (29.4) and (29.5), we find

$\Delta\alpha = -10\overset{''}{.}64 = -0\overset{s}{.}71$, as above ;

$\Delta\delta = -20\overset{''}{.}9$, whence $\delta' = \delta - 20\overset{''}{.}9 = -22°24'30\overset{''}{.}8$, as above.

As an exercise, perform the calculation for the Moon, again for the Uccle Observatory, using fictive values, for example

$\alpha = 1^h00^m00\overset{s}{.}00 = 15\overset{\circ}{.}000\,000$
$\delta = +5\overset{\circ}{.}000\,000$
$H = +4^h00^m00\overset{s}{.}00 = +60\overset{\circ}{.}000\,000$
$\pi = 0°59'00''$

Compare the results of the rigorous formulae with those of the non-rigorous ones.

We can consider the opposite problem : from the observed topocentric coordinates α' and δ' , deduce the geocentric values α and δ. In the case of a planet or comet, the corrections $\Delta\alpha$ and $\Delta\delta$ are so small, that the formulae (29.4) and (29.5) can be used also for the reduction from topocentric to geocentric coordinates.

Parallax in ecliptical coordinates

It is possible to calculate the topocentric coordinates of a celestial body, from its geocentric values, directly in ecliptical coordinates. The following formulae were given by Joseph Johann von Littrow (*Theoretische und Practische Astronomie*, Vol. I, p. 91 ; Wien, 1821).

Let λ = geocentric ecliptical longitude of the body (Moon, planet, comet),
β = its geocentric ecliptical latitude,
s = its geocentric semidiameter,
λ', β', s' = the required topocentric values of the same quantities,
ϕ = the observer's latitude,
ε = the obliquity of the ecliptic,
θ = the local sidereal time.

$$\phi' = \phi - 0\overset{\circ}{.}193 \sin 2\phi$$

$$N = \cos\lambda \cos\beta - \sin\pi \cos\phi' \cos\theta$$

$$\tan\lambda' = \frac{\sin\lambda \cos\beta - \sin\pi\,(\sin\phi' \sin\varepsilon + \cos\phi' \cos\varepsilon \sin\theta)}{N}$$

$$\tan\beta' = \frac{\cos\lambda' \bigl(\sin\beta - \sin\pi\,(\sin\phi' \cos\varepsilon - \cos\phi' \sin\varepsilon \sin\theta)\bigr)}{N}$$

$$\sin s' = \frac{\cos\lambda' \cos\beta' \sin s}{N}$$

As an exercise, calculate λ', β', s' from the following data :

$\lambda = 181^\circ46'22\overset{''}{.}5$
$\beta = +2^\circ17'26\overset{''}{.}2$
$\pi = 0^\circ59'27\overset{''}{.}7$
$s = 0^\circ16'15\overset{''}{.}5\,0$

$\varepsilon = 23^\circ28'00\overset{''}{.}8$
$\theta = 209^\circ46'07\overset{''}{.}9$
$\phi = +50^\circ05'07\overset{''}{.}8$

Answer :
$\lambda' = 181^\circ48'05\overset{''}{.}2$
$\beta' = +1^\circ29'01\overset{''}{.}3$
$s' = 0^\circ16'25\overset{''}{.}5$

30

POSITION OF THE MOON

In order to calculate an accurate position of the Moon, it is necessary to take into account *hundreds* of periodic terms in the Moon's longitude, latitude and parallax. For this reason, we will limit ourselves to the most important periodic terms, and be satisfied with an accuracy of about $10''$ in the longitude of the Moon, $3''$ in its latitude, and $0''.2$ in its parallax.

Using the method described below, one obtains the geocentric longitude λ and the geocentric latitude β of the center of the Moon, referred to the mean longitude of the date. If necessary, λ and β can be converted to α and δ using formulae (8.3) and (8.4). The equatorial horizontal parallax π of the Moon too is obtained.

When the parallax π is known, the distance between the centers of Earth and Moon can be found from

$$D = \frac{1}{\sin \pi} \quad \text{equatorial radii of the Earth}$$

$$\text{or} \qquad D = \frac{6378.14}{\sin \pi} \quad \text{kilometers}$$

For the given instant (ET !), calculate the JD (see Chapter 3), and then T by means of formula (15.1). Remember that T is expressed in centuries, and thus should be taken with a sufficient number of decimals (at least nine, since during 0.000 000 001 century the Moon moves over an arc of $1''.7$).

Then calculate the angles L', M, M', D, F and Ω by means of the following formulae, in which the various constants are expressed in degrees and decimals.

Moon's mean longitude :

$$L' = 270.434\,164 + 481\,267.8831\,T - 0.001\,133\,T^2 + 0.000\,0019\,T^3$$

Sun's mean anomaly :

$$M = 358.475\,833 + 35\,999.0498\,T - 0.000\,150\,T^2 - 0.000\,0033\,T^3$$

Moon's mean anomaly :

$$M' = 296.104\,608 + 477\,198.8491\,T + 0.009\,192\,T^2 + 0.000\,0144\,T^3$$

Moon's mean elongation :

$$D = 350.737\,486 + 445\,267.1142\,T - 0.001\,436\,T^2 + 0.000\,0019\,T^3$$

Mean distance of Moon from its ascending node :

$$F = 11.250\,889 + 483\,202.0251\,T - 0.003\,211\,T^2 - 0.000\,0003\,T^3$$

Longitude of Moon's ascending node :

$$\Omega = 259.183\,275 - 1934.1420\,T + 0.002\,078\,T^2 + 0.000\,0022\,T^3$$

To the mean values of these arguments must be added some periodic variations, called "additive terms" :

Additive to	*Term*
L'	$+0\overset{\circ}{.}000\,233 \sin(51\overset{\circ}{.}2 + 20\overset{\circ}{.}2\,T)$
M	$-0\overset{\circ}{.}001\,778 \sin(51\overset{\circ}{.}2 + 20\overset{\circ}{.}2\,T)$
M'	$+0\overset{\circ}{.}000\,817 \sin(51\overset{\circ}{.}2 + 20\overset{\circ}{.}2\,T)$
D	$+0\overset{\circ}{.}002\,011 \sin(51\overset{\circ}{.}2 + 20\overset{\circ}{.}2\,T)$
L', M', D, F	$+0\overset{\circ}{.}003\,964 \sin(346\overset{\circ}{.}560 + 132\overset{\circ}{.}870\,T - 0\overset{\circ}{.}009\,1731\,T^2)$
L'	$+0\overset{\circ}{.}001\,964 \sin\Omega$
M'	$+0\overset{\circ}{.}002\,541 \sin\Omega$
D	$+0\overset{\circ}{.}001\,964 \sin\Omega$
F	$-0\overset{\circ}{.}024\,691 \sin\Omega$
F	$-0\overset{\circ}{.}004\,328 \sin(\Omega + 275\overset{\circ}{.}05 - 2\overset{\circ}{.}30\,T)$

The first four terms have a period of 1782 years. The fifth term, with coefficient $0\overset{\circ}{.}003\,964$, is the "Great Venus Term" ; its period is 271 years.

With the values of L', M, M', D and F, corrected for the additive terms, λ, β and π can be obtained by means of the following expressions where, again, all the coefficients are given in degrees and decimals. The terms indicated by (e) or (e^2) should be multiplied by e or e^2, where

$$e = 1 - 0.002\,495\,T - 0.000\,007\,52\,T^2$$

$$
\begin{array}{lrl}
 & \lambda = L' & + 6.288\,750 \sin M' \\
 & & + 1.274\,018 \sin (2D - M') \\
 & & + 0.658\,309 \sin 2D \\
 & & + 0.213\,616 \sin 2M' \\
(e) & & - 0.185\,596 \sin M \\
 & & - 0.114\,336 \sin 2F \\
 & & + 0.058\,793 \sin (2D - 2M') \\
(e) & & + 0.057\,212 \sin (2D - M - M') \\
 & & + 0.053\,320 \sin (2D + M') \\
(e) & & + 0.045\,874 \sin (2D - M) \\
(e) & & + 0.041\,024 \sin (M' - M) \\
 & & - 0.034\,718 \sin D \\
(e) & & - 0.030\,465 \sin (M + M') \\
 & & + 0.015\,326 \sin (2D - 2F) \\
 & & - 0.012\,528 \sin (2F + M') \\
 & & - 0.010\,980 \sin (2F - M') \\
 & & + 0.010\,674 \sin (4D - M') \\
 & & + 0.010\,034 \sin 3M' \\
 & & + 0.008\,548 \sin (4D - 2M') \\
(e) & & - 0.007\,910 \sin (M - M' + 2D) \\
(e) & & - 0.006\,783 \sin (2D + M) \\
 & & + 0.005\,162 \sin (M' - D) \\
(e) & & + 0.005\,000 \sin (M + D) \\
(e) & & + 0.004\,049 \sin (M' - M + 2D) \\
 & & + 0.003\,996 \sin (2M' + 2D) \\
 & & + 0.003\,862 \sin 4D \\
 & & + 0.003\,665 \sin (2D - 3M') \\
(e) & & + 0.002\,695 \sin (2M' - M) \\
 & & + 0.002\,602 \sin (M' - 2F - 2D) \\
(e) & & + 0.002\,396 \sin (2D - M - 2M') \\
 & & - 0.002\,349 \sin (M' + D) \\
(e^2) & & + 0.002\,249 \sin (2D - 2M) \\
(e) & & - 0.002\,125 \sin (2M' + M) \\
(e^2) & & - 0.002\,079 \sin 2M \\
(e^2) & & + 0.002\,059 \sin (2D - M' - 2M) \\
 & & - 0.001\,773 \sin (M' + 2D - 2F) \\
 & & - 0.001\,595 \sin (2F + 2D)
\end{array}
$$

(e) $+ 0.001\,220 \sin (4D - M - M')$
$- 0.001\,110 \sin (2M' + 2F)$
$+ 0.000\,892 \sin (M' - 3D)$
(e) $- 0.000\,811 \sin (M + M' + 2D)$
(e) $+ 0.000\,761 \sin (4D - M - 2M')$
(e^2) $+ 0.000\,717 \sin (M' - 2M)$
(e^2) $+ 0.000\,704 \sin (M' - 2M - 2D)$
(e) $+ 0.000\,693 \sin (M - 2M' + 2D)$
(e) $+ 0.000\,598 \sin (2D - M - 2F)$
$+ 0.000\,550 \sin (M' + 4D)$
$+ 0.000\,538 \sin 4M'$
(e) $+ 0.000\,521 \sin (4D - M)$
$+ 0.000\,486 \sin (2M' - D)$

$B = + 5.128\,189 \sin F$
$+ 0.280\,606 \sin (M' + F)$
$+ 0.277\,693 \sin (M' - F)$
$+ 0.173\,238 \sin (2D - F)$
$+ 0.055\,413 \sin (2D + F - M')$
$+ 0.046\,272 \sin (2D - F - M')$
$+ 0.032\,573 \sin (2D + F)$
$+ 0.017\,198 \sin (2M' + F)$
$+ 0.009\,267 \sin (2D + M' - F)$
$+ 0.008\,823 \sin (2M' - F)$
(e) $+ 0.008\,247 \sin (2D - M - F)$
$+ 0.004\,323 \sin (2D - F - 2M')$
$+ 0.004\,200 \sin (2D + F + M')$
(e) $+ 0.003\,372 \sin (F - M - 2D)$
(e) $+ 0.002\,472 \sin (2D + F - M - M')$
(e) $+ 0.002\,222 \sin (2D + F - M)$
(e) $+ 0.002\,072 \sin (2D - F - M - M')$
(e) $+ 0.001\,877 \sin (F - M + M')$
$+ 0.001\,828 \sin (4D - F - M')$
(e) $- 0.001\,803 \sin (F + M)$
$- 0.001\,750 \sin 3F$
(e) $+ 0.001\,570 \sin (M' - M - F)$
$- 0.001\,487 \sin (F + D)$
(e) $- 0.001\,481 \sin (F + M + M')$
(e) $+ 0.001\,417 \sin (F - M - M')$
(e) $+ 0.001\,350 \sin (F - M)$
$+ 0.001\,330 \sin (F - D)$
$+ 0.001\,106 \sin (F + 3M')$
$+ 0.001\,020 \sin (4D - F)$
$+ 0.000\,833 \sin (F + 4D - M')$

$+ 0.000\,781 \sin (M' - 3F)$
$+ 0.000\,670 \sin (F + 4D - 2M')$
$+ 0.000\,606 \sin (2D - 3F)$
$+ 0.000\,597 \sin (2D + 2M' - F)$
(e) $+ 0.000\,492 \sin (2D + M' - M - F)$
$+ 0.000\,450 \sin (2M' - F - 2D)$
$+ 0.000\,439 \sin (3M' - F)$
$+ 0.000\,423 \sin (F + 2D + 2M')$
$+ 0.000\,422 \sin (2D - F - 3M')$
(e) $- 0.000\,367 \sin (M + F + 2D - M')$
(e) $- 0.000\,353 \sin (M + F + 2D)$
$+ 0.000\,331 \sin (F + 4D)$
(e) $+ 0.000\,317 \sin (2D + F - M + M')$
(e^2) $+ 0.000\,306 \sin (2D - 2M - F)$
$- 0.000\,283 \sin (M' + 3F)$

$$\omega_1 = 0.000\,4664 \cos \Omega$$

$$\omega_2 = 0.000\,0754 \cos (\Omega + 275\overset{\circ}{.}05 - 2\overset{\circ}{.}30\, T)$$

$$\beta = B \times (1 - \omega_1 - \omega_2)$$

$\pi = 0.950\,724$
$+ 0.051\,818 \cos M'$
$+ 0.009\,531 \cos (2D - M')$
$+ 0.007\,843 \cos 2D$
$+ 0.002\,824 \cos 2M'$
$+ 0.000\,857 \cos (2D + M')$
(e) $+ 0.000\,533 \cos (2D - M)$
(e) $+ 0.000\,401 \cos (2D - M - M')$
(e) $+ 0.000\,320 \cos (M' - M)$
$- 0.000\,271 \cos D$
(e) $- 0.000\,264 \cos (M + M')$
$- 0.000\,198 \cos (2F - M')$
$+ 0.000\,173 \cos 3M'$
$+ 0.000\,167 \cos (4D - M')$
(e) $- 0.000\,111 \cos M$
$+ 0.000\,103 \cos (4D - 2M')$
$- 0.000\,084 \cos (2M' - 2D)$
(e) $- 0.000\,083 \cos (2D + M)$
$+ 0.000\,079 \cos (2D + 2M')$
$+ 0.000\,072 \cos 4D$

(e)	$+ 0.000\,064 \cos (2D - M + M')$
(e)	$- 0.000\,063 \cos (2D + M - M')$
(e)	$+ 0.000\,041 \cos (M + D)$
(e)	$+ 0.000\,035 \cos (2M' - M)$
	$- 0.000\,033 \cos (3M' - 2D)$
	$- 0.000\,030 \cos (M' + D)$
	$- 0.000\,029 \cos (2F - 2D)$
(e)	$- 0.000\,029 \cos (2M' + M)$
(e^2)	$+ 0.000\,026 \cos (2D - 2M)$
	$- 0.000\,023 \cos (2F - 2D + M')$
(e)	$+ 0.000\,019 \cos (4D - M - M')$

Example 30.a : Calculate the geocentric longitude, latitude and equatorial horizontal parallax of the Moon on 1979 Dec. 7.0 ET.

We find successively :

JD = 2444 214.5
T = +0.799 301 8480

L' = 108°.7418 M = 332°.5828 M' = 122°.0324
D = 213.5638 F = 315.5204 Ω = 153.2213

With additive terms :

L' = 108°.7469 M = 332°.5812 M' = 122°.0383
D = 213.5705 F = 315.5093

e = 0.998 001

Then the Moon's longitude is equal to the sum of the following quantities :

108°.7469
+5.330 934
−1.042 303
+0.606 608
−0.192 122
+0.085 294
+0.114 318
−0.003 143
−0.026 346
−0.008 506
+0.045 637

+0.020 806
+0.019 198
−0.030 305
+0.006 204
−0.006 834
−0.005 658
+0.002 264
+0.001 069
−0.008 043
+0.007 823
−0.004 326

−0.005 160
−0.000 535
−0.002 409
−0.003 006
+0.002 765
+0.003 206
−0.002 689
+0.001 534
−0.001 213
+0.000 970
+0.001 900

+0.001 266
+0.001 693
−0.000 002
+0.001 755
+0.000 593
+0.000 777
−0.000 467
−0.000 324
−0.000 253
−0.000 753
+0.000 039

+0.000 660
+0.000 285
−0.000 037
−0.000 534
+0.000 423
+0.000 163
+0.000 247

Hence, λ = 113°.6604 = 113°39′37″

In the same manner, we find :

$B = -3\overset{\circ}{.}163\,037$
$\omega_1 = -0.000\,4164$
$\omega_2 = +0.000\,0301$

$\beta = -3\overset{\circ}{.}163\,037 \times 1.000\,3863 = -3\overset{\circ}{.}164\,259 = -3°09'51''$

$\pi = +0\overset{\circ}{.}930\,249 = 55'48\overset{''}{.}9$

The *Astronomical Ephemeris* gives the following values :

$\lambda = 113°39'28\overset{''}{.}27$
$\beta = -3°09'49\overset{''}{.}22$
$\pi = 55'48\overset{''}{.}985$

Lower accuracy. - Of course, when no high accuracy is required, the calculation may be considerably simplified :

- unless T is large, the terms in T^2 and T^3 in the formulae for L', M, M', D and F may be dropped ;
- Ω is not needed ;
- drop the additive terms to L', M, M', D and F ;
- use only a limited number of periodic terms in the expressions for λ, B and π ;
- put $\beta = B$.

As an exercise, calculate the coordinates of the Moon for 1979 December 7 at 0^h ET, with the above-mentioned simplifications. Compare your results with those of Example 30.a.

31

ILLUMINATED FRACTION OF THE MOON'S DISK

The illuminated fraction k of the Moon's disk, as seen from the center of the Earth, can be calculated from

$$k = \frac{1 + \cos i}{2} \qquad (31.1)$$

where i is the Moon's phase angle, that is the angular distance Sun - Earth as seen from the Moon.

The phase angle i can be found as follows. Find the Sun's longitude $\odot$ (Chapter 18), and the Moon's longitude λ and latitude β (Chapter 30). For the Moon, it is sufficient to take into account only a small number of periodic terms. For its latitude, for instance, it is sufficient to calculate

$$\begin{aligned} \beta = &+5^{\circ}\!.1282 \sin F \\ &+0^{\circ}\!.2806 \sin (F + M') \\ &+0^{\circ}\!.2777 \sin (M' - F) \\ &+0^{\circ}\!.1732 \sin (2D - F) \end{aligned}$$

Then, calculate d from

$$\cos d = \cos (\lambda - \odot) \cos \beta \qquad (31.2)$$

d being between 0 and 180 degrees. Then we have, with sufficient accuracy,

$$i = 180^{\circ} - d - 0^{\circ}\!.1468 \frac{1 - 0.0549 \sin M'}{1 - 0.0167 \sin M} \sin d \qquad (31.3)$$

where M and M' are, as before, the mean anomalies of Sun and Moon, respectively.

Example 31.a : Calculate the illuminated fraction of the Moon's disk on 1979 December 25 at 0^h ET.

Instead of calculating the coordinates of Sun and Moon ourselves, we take them from the *A.E.* :

$$\odot = 272°35'23''$$
$$\lambda = 346°39'01''$$
$$\beta = -1°22'54''$$

whence, by means of formula (31.2), $d = 74\overset{\circ}{.}065$.
We have further

$$\text{JD} = 2444\,232.5 \qquad T = +0.799\,794\,6612$$

whence, from the expressions given in Chapter 30,

$$M = 350\overset{\circ}{.}32 \qquad M' = 357\overset{\circ}{.}20$$

and then, by formulae (31.3) and (31.1),

$$i = 180° - 74\overset{\circ}{.}065 - \left(0\overset{\circ}{.}1468 \times \frac{1.0027}{1.0028} \times \sin 74\overset{\circ}{.}065\right)$$

$$i = 180° - 74\overset{\circ}{.}065 - 0\overset{\circ}{.}141 = 105\overset{\circ}{.}794$$

$k = 0.3639$, which should be rounded to 0.36.

Lower accuracy, though still a good result, is obtained by neglecting the Moon's latitude and by calculating an approximate value of i as follows :

$$\begin{aligned} i = 180° - D &- 6\overset{\circ}{.}289 \sin M' \\ &+ 2\overset{\circ}{.}100 \sin M \\ &- 1\overset{\circ}{.}274 \sin (2D - M') \\ &- 0\overset{\circ}{.}658 \sin 2D \\ &- 0\overset{\circ}{.}214 \sin 2M' \\ &- 0\overset{\circ}{.}112 \sin D \end{aligned} \qquad (31.4)$$

Example 31.b : Calculate again the illuminated fraction of the Moon's disk for 1979 December 25.0, but now using formula (31.4).

We have

$$\text{JD} = 2444\,232.5 \qquad T = +0.799\,794\,6612$$

whence, from the expressions given in Chapter 30,

$$M = 350\overset{\circ}{.}324 \qquad M' = 357\overset{\circ}{.}202 \qquad D = 72\overset{\circ}{.}997$$

Then, by formula (31.4), $i = 105\overset{\circ}{.}843$

and, by formula (31.1), $k = 0.3635$, which again rounds to 0.36.

As an exercise, calculate the illuminated fraction of the Moon's disk for 0^h ET of the following dates, and compare your result with the value given in the *Astronomical Ephemeris* :

	A.E.
1978 October 24	0.50
1978 December 13	0.98
1979 April 1	0.18
1979 December 9	0.73

For 1979 December 9.0, the Soviet almanac *Astronomicheskii Ezhegodnik* gives $k = 0.74$ instead of 0.73. Who is correct ?

32

PHASES OF THE MOON

The times of the *mean* phases of the Moon, already affected by the Sun's aberration, are given by

$$\begin{aligned}\mathrm{JD} = 2415\,020.759\,33 &+ 29.530\,588\,68\,k \\ &+ 0.000\,1178\,T^2 \\ &- 0.000\,000\,155\,T^3 \\ &+ 0.000\,33\,\sin(166^{\circ}\!.56 + 132^{\circ}\!.87\,T - 0^{\circ}\!.009\,173\,T^2)\end{aligned} \qquad (32.1)$$

These instants are expressed in Ephemeris Time (Julian Ephemeris Days). In the formula above, an integer value of k gives a New Moon, an integer value increased by

0.25 gives a First Quarter,
0.50 gives a Full Moon,
0.75 gives a Last Quarter

Any other value for k will give meaningless results !

A negative value of k gives a lunar phase before the year 1900, while k is positive after the beginning of the year 1900. Thus, for example,

+479.00 and −2793.00 correspond to a New Moon,
+479.25 and −2792.75 correspond to a First Quarter,
+479.50 and −2792.50 correspond to a Full Moon,
+479.75 and −2792.25 correspond to a Last Quarter.

An approximate value of k is given by

$$k \simeq (\text{year} - 1900) \times 12.3685 \qquad (32.2)$$

where the "year" should be taken with decimals, for example 1977.25 for the end of March 1977.

Finally, in formula (32.1) T is the time in Julian centuries from 1900 January 0.5. Once the correct value of k has been found,

T can be calculated with a sufficient accuracy from

$$T = \frac{k}{1236.85} \qquad (32.3)$$

Then calculate the following angles, which are expressed in degrees and decimals and may be reduced to the interval 0 – 360 degrees before calculating further.

Sun's mean anomaly at time JD :

$$\begin{aligned} M = 359.2242 &+ 29.105\,356\,08\,k \\ &- 0.000\,0333\,T^2 \\ &- 0.000\,003\,47\,T^3 \end{aligned}$$

Moon's mean anomaly :

$$\begin{aligned} M' = 306.0253 &+ 385.816\,918\,06\,k \\ &+ 0.010\,7306\,T^2 \\ &+ 0.000\,012\,36\,T^3 \end{aligned}$$

Moon's argument of latitude :

$$\begin{aligned} F = 21.2964 &+ 390.670\,506\,46\,k \\ &- 0.001\,6528\,T^2 \\ &- 0.000\,002\,39\,T^3 \end{aligned}$$

To obtain the time of the *true* phase, the following corrections should be added to the time of the mean phase given by (32.1). The following coefficients are given in decimals of a day, and smaller quantities have been neglected.

For New and Full Moon :

$$\begin{aligned} &+ (0.1734 - 0.000\,393\,T) \sin M \\ &+ 0.0021 \sin 2M \\ &- 0.4068 \sin M' \\ &+ 0.0161 \sin 2M' \\ &- 0.0004 \sin 3M' \\ &+ 0.0104 \sin 2F \\ &- 0.0051 \sin (M + M') \\ &- 0.0074 \sin (M - M') \\ &+ 0.0004 \sin (2F + M) \\ &- 0.0004 \sin (2F - M) \\ &- 0.0006 \sin (2F + M') \\ &+ 0.0010 \sin (2F - M') \\ &+ 0.0005 \sin (M + 2M') \end{aligned} \qquad (32.4)$$

For First and Last Quarter :

$$
\begin{aligned}
&+ (0.1721 - 0.0004\,T) \sin M \\
&+ 0.0021 \sin 2M \\
&- 0.6280 \sin M' \\
&+ 0.0089 \sin 2M' \\
&- 0.0004 \sin 3M' \\
&+ 0.0079 \sin 2F \\
&- 0.0119 \sin (M + M') \\
&- 0.0047 \sin (M - M') \\
&+ 0.0003 \sin (2F + M) \\
&- 0.0004 \sin (2F - M) \\
&- 0.0006 \sin (2F + M') \\
&+ 0.0021 \sin (2F - M') \\
&+ 0.0003 \sin (M + 2M') \\
&+ 0.0004 \sin (M - 2M') \\
&- 0.0003 \sin (2M + M')
\end{aligned}
\qquad (32.5)
$$

and, in addition :

for First Quarter : $+ 0.0028 - 0.0004 \cos M + 0.0003 \cos M'$

for Last Quarter : $- 0.0028 + 0.0004 \cos M - 0.0003 \cos M'$

Example 32.a : Calculate the instant of the New Moon occurring in February 1977.

Mid-February 1977 being equal to 1977.13, we find by means of formula (32.2)

$$k \simeq (1977.13 - 1900) \times 12.3685 = 953.982$$

whence $k = 954$, since k should be an integer for the New Moon phase. Then, by formula (32.3), $T = +0.77131$, and then formula (32.1) gives

$$\text{JD} = 2443\,192.9407$$

With $k = 954$ and $T = +0.77131$, we further find

$$
\begin{aligned}
M &= 28125^{\circ}.7339 = 45^{\circ}.7339 \\
M' &= 368375^{\circ}.3715 = 95^{\circ}.3715 \\
F &= 372720^{\circ}.9585 = 120^{\circ}.9585
\end{aligned}
$$

The correcting terms given by (32.4) are then, writing extra decimals :

+0.123956
+0.002099
−0.405014
−0.003001
+0.000384
−0.009176
−0.003393

+0.005639
−0.000381
+0.000111
+0.000232
+0.000551
−0.000417

the sum of which is −0.2884 day. Thus, the time of the true New Moon is

$$\text{JD} = 2443\,192.9407 - 0.2884 = 2443\,192.6523$$

which corresponds to 1977 February 18.1523
= 1977 February 18 at $3^h39^m.3$ ET.

The correct value, deduced from the data of the *A.E.*, is $3^h37^m.6$ ET.

Example 32.b : Calculate the time of the Last Quarter of November 1952.

Using the value year = 1952.88, formula (32.2) gives $k \simeq 654.05$, and thus we take $k = 653.75$. We then find

$T = +0.52856$ JD = 2434 326.3814

$$M = 306^\circ.8507$$
$$M' = 173^\circ.8385$$
$$F = 182^\circ.1395$$

and the total correction given by (32.5) is −0.2261 day, whence

$$\text{JD} = 2434\,326.3814 - 0.2261 = 2434\,326.1553$$

which corresponds to 1952 November 9.6553
= 1952 November 9 at $15^h43^m.6$ ET

or 15^h43^m UT, because in 1952 the difference ET − UT was +0.5 minute (see Chapter 5).

The correct value is indeed 15^h43^m UT.

Using the method described in this Chapter, the author has calculated all lunar phases of the years 1971 - 1975. It was found that no instant was more than 2 minutes in error. In 3/4 of the cases, the error was even less than 1.0 minute.

If an accuracy of half an hour is sufficient, one may drop the last term of formula (32.1), and the terms with coefficients less than 0.0030 in formulae (32.4) and (32.5).

33

ECLIPSES

Without too much calculation, it is possible to obtain with good accuracy the principal characteristics of an eclipse of Sun or Moon. For a solar eclipse, the situation is complicated by the fact that the phases of the event are different for different observers at the Earth's surface, while in the case of a lunar eclipse all observers see the same phase at the same instant.

For this reason, we will not condider here the calculation of the local circumstances of a solar eclipse. The interested reader may calculate these circumstances from the Besselian Elements published yearly in the *Astronomical Ephemeris*, where he will find the necessary formulae. More formulae, together with numerical examples, are given in the *Explanatory Supplement to the A.E.*, which may be obtained from the publisher : Her Majesty's Stationery Office, 49 High Holborn, London WC1V 6HB, England.

Firstly, calculate the time (JD) of the *mean* New or Full Moon, using formulae (32.1) to (32.3) of the preceding Chapter. Remember that k must be an integer for a New Moon (solar eclipse), and an integer increased by 0.5 for a Full Moon (lunar eclipse).

Then, calculate the values of M, M' and F for that instant, using the expressions given after formula (32.3).

The value of F will give a first information about the occurrence of a solar or lunar eclipse. If F differs from the nearest multiple of 180° by less than $13^\circ.9$, then there is certainly an eclipse ; if the difference is larger than $21^\circ.0$, there is no eclipse ; between these two values, the eclipse is uncertain and the case must be examined further. On a programmable calculating machine, use can be made of the following rule : there is no eclipse if $|\sin F| > 0.36$.

Note that, after one lunation, F increases by $30^\circ.6705$.

If F is near 0° or 360°, the eclipse occurs near the Moon's ascending node. If F is near 180°, the eclipse takes place near the descending node of the Moon's orbit.

To obtain the *time of maximum eclipse* (for the Earth generally in the case of a solar eclipse), the following corrections should be added to the time of mean conjunction given by (32.1). The following coefficients are given in decimals of a day, and smaller quantities have been neglected.

$$\begin{aligned} &+ (0.1734 - 0.000\,393\,T) \sin M \\ &+ 0.0021 \sin 2M \\ &- 0.4068 \sin M' \\ &+ 0.0161 \sin 2M' \\ &- 0.0051 \sin (M + M') \\ &- 0.0074 \sin (M - M') \\ &- 0.0104 \sin 2F \end{aligned} \tag{33.1}$$

Note that the coefficient of sin $2F$ is negative here, while it was positive in (32.4); the reason is that we calculate here the time of greatest eclipse, not the time of conjunction in longitude.

Then calculate further :

$$\begin{aligned} S = 5.19\,595 & \\ - 0.0048 & \cos M \\ + 0.0020 & \cos 2M \\ - 0.3283 & \cos M' \\ - 0.0060 & \cos (M + M') \\ + 0.0041 & \cos (M - M') \end{aligned} \qquad \begin{aligned} C = &+ 0.2070 \sin M \\ &+ 0.0024 \sin 2M \\ &- 0.0390 \sin M' \\ &+ 0.0115 \sin 2M' \\ &- 0.0073 \sin (M + M') \\ &- 0.0067 \sin (M - M') \\ &+ 0.0117 \sin 2F \end{aligned}$$

$$\gamma = S \sin F + C \cos F$$

$$\begin{aligned} u = 0.0059 & \\ + 0.0046 & \cos M \\ - 0.0182 & \cos M' \\ + 0.0004 & \cos 2\,M' \\ - 0.0005 & \cos (M + M') \end{aligned}$$

SOLAR ECLIPSES

In the case of a solar eclipse, γ represents the least distance from the axis of the Moon's shadow to the center of the Earth, in units of the equatorial radius of the Earth. The quantity γ is positive or negative depending upon the axis of the shadow passing north or south of the Earth's center. When γ is between +0.9972 and −0.9972, the solar eclipse is central : there exists a line of central eclipse on the Earth's surface.

The quantity u denotes the radius of the Moon's *umbral* cone in the fundamental plane, again in units of the Earth's equatorial radius. (The fundamental plane is the plane through the center of the Earth and perpendicular to the axis of the Moon's shadow). The radius of the *penumbral* cone in the fundamental plane is

$$u + 0.5460$$

If $|\gamma|$ is between 0.9972 and $1.5432 + u$, the eclipse is not central. In most cases, it is then a partial one. However, when $|\gamma|$ is between 0.9972 and 1.0260, a part of the umbral cone may touch the surface of the Earth (within the polar regions), but the axis of the cone does *not* touch the Earth. These non-central total or annular eclipses occur when $0.9972 < |\gamma| < 0.9972 + |u|$. Between the years 1950 and 2100, there are seven eclipses of this type :

1950 March 18	annular, not central
1957 April 30	annular, not central
1957 October 23	total, not central
1967 November 2	total, not central
2014 April 29	annular, not central
2043 April 9	total, not central
2043 October 3	annular, not central

If $|\gamma| > 1.5432 + u$, no eclipse is visible from the Earth's surface.

In the case of a *central* eclipse, the type of the eclipse may be determined by the following rules :

if $u < 0$, the eclipse is total ;

if $u > +0.0047$, the eclipse is annular ;

if u is between 0 and +0.0047, the eclipse is either annular or annular-total.

In the last case, the ambiguity is removed as follows. Calculate

$$\omega = 0.00464 \cos W, \qquad \text{where} \qquad \sin W = \gamma$$

Then, if $u < \omega$, the eclipse is annular-total ; otherwise it is an annular one.

In the case of a *partial* solar eclipse, the greatest magnitude is attained at the point of the surface of the Earth which comes closest to the axis of shadow. The magnitude of the eclipse at that point is

$$\frac{1.5432 + u - |\gamma|}{0.5460 + 2u} \qquad (33.2)$$

LUNAR ECLIPSES

In the case of a lunar eclipse, γ represents the least distance from the center of the Moon to the axis of the Earth's shadow, in units of the Earth's equatorial radius. The quantity γ is positive or negative depending upon the Moon's center passing north or south of the axis of shadow.

The radii at the distance of the Moon are :

for the penumbra : $\rho = 1.2847 + u$

for the umbra : $\sigma = 0.7404 - u$

while the magnitude of the eclipse may be found as follows :

for penumbral eclipses : $$\frac{1.5572 + u - |\gamma|}{0.5450} \qquad (33.3)$$

for umbral eclipses : $$\frac{1.0129 - u - |\gamma|}{0.5450} \qquad (33.4)$$

If the magnitude is negative, this indicates that there is no eclipse.

The *semidurations* of the partial and total phases in the *umbra* can be found as follows. Calculate

$$P = 1.0129 - u$$
$$T = 0.4679 - u$$

$$n = 0.5458 + 0.0400 \cos M'$$

Then the semidurations in *minutes* are :

partial phase : $\frac{60}{n}\sqrt{P^2 - \gamma^2}$ total phase : $\frac{60}{n}\sqrt{T^2 - \gamma^2}$

Example 33.a : Solar eclipse of 1978 October 2.

Since October 2 is the 275th day of the year, the given date corresponds to 1978.75. Formula (32.2) then gives

$$k \simeq 974.02, \qquad \text{whence} \quad k = 974.$$

Then, by means of formulae (32.3) and (32.1),

$$\text{JD} = 2443\,783.5524$$

We find further

$$M = 267.^{\circ}8410 \qquad M' = 251.^{\circ}7102 \qquad F = 14.^{\circ}3687$$

Because F is between 13°9 and 21°0, the eclipse is uncertain. We find further :

$$S = 5.3067 \qquad C = -0.1616 \qquad \gamma = +1.1604 \qquad u = +0.0116$$

Because $|\gamma|$ is between 0.9972 and $1.5432 + u$, the eclipse is a partial one. Using formula (33.2), we find that the maximum magnitude is

$$\frac{1.5432 + 0.0116 - 1.1604}{0.5460 + 0.0232} = 0.693$$

Because F is near 0°, the eclipse occurs at the Moon's ascending node. Because γ is positive, the eclipse is visible in the northern hemisphere of the Earth.

To obtain the time of maximum eclipse, we add to JD the terms given by formula (33.1). This gives

$$\begin{aligned} \text{JD} = 2443\,783&.5524 \\ -\,0&.1730 \\ +\,0&.0002 \\ +\,0&.3862 \\ +\,0&.0096 \\ -\,0&.0018 \\ -\,0&.0021 \\ -\,0&.0050 \end{aligned}$$

which gives JD = 2443 783.767, corresponding to 1978 October 2 at 6^h24^m ET.

The correct values, given in the *A.E.*, are $6^h28^m.7$ ET and a maximum magnitude of 0.691.

Example 33.b : Solar eclipse of 1980 February 16.

As in the preceding Example, we find :

k = 991
JD = 2444 285.572
M = $42.6321°$
M' = $330.5979°$
F = $175.7671°$

Corrected JD = 2444 285.871 = 1980 February 16 at 8^h54^m ET

$S = +4.9020$ $C = +0.1421$ $\gamma = +0.2201$ $u = -0.0069$

Because $|\gamma| < 0.9972$, the eclipse is a central one. Because u is negative, the eclipse is a total one. Because $|\gamma|$ is small, the eclipse is visible from the equatorial regions of the Earth. The eclipse takes place near the descending node of the Moon's orbit, because $F \simeq 180°$.

Example 33.c : Lunar eclipse of June 1973.

We find successively :

k = 908.5
JD = 2441 849.299
M = $161.4402°$
M' = $180.7011°$
F = $345.4506°$

Corrected JD = 2441 849.367 = 1973 June 15 at 20^h48^m ET

$S = +5.5285$ $C = +0.0640$ $\gamma = -1.3269$ $u = +0.0197$

The eclipse took place near the Moon's ascending node ($F \simeq 360°$) and the Moon's center passed south of the center of the Earth's umbra ($\gamma < 0$).

According to formula (33.4), the magnitude in the umbra was -0.612. Since this is negative, there was no eclipse in the umbra. Using formula (33.3), we find that the magnitude in the penumbra was 0.459. Thus the eclipse was a penumbral one.

According to the *Connaissance des Temps*, maximum eclipse took place at $20h50.7^m$ ET, and the magnitude was 0.469.

Example 33.d : Find the first lunar eclipse after 1978 July 1.

For 1978.5, formula (32.2) gives $k \simeq 970.93$. Thus we must try the value $k = 971.5$. This gives $F = 117^\circ.6924$, which differs more than 21 degrees from the nearest multiple of 180°, and thus gives no eclipse.

The next Full Moon, $k = 972.5$, gives $F = 148^\circ.3629$, hence again no eclipse. But it is evident that the next Full Moon will give $F \simeq 179^\circ$ and thus give rise to an eclipse. We then find, as before :

$k = 973.5$
JD = 2443 768.787
$M = 253^\circ.2883$
$M' = 58^\circ.8017$
$F = 179^\circ.0334$

Corrected JD = 2443 768.295 = 1978 September 16 at 19^h05^m ET
= 19^h04^m UT

$S = +5.0176$ $C = -0.2134$ $\gamma = +0.2980$ $u = -0.0054$

Formula (33.4) then yields a magnitude of 1.332. Thus the eclipse is a total one in the umbra. We find further :

$P = 1.0183$ $T = 0.4733$ $n = 0.5665$

Semiduration of partial phase :

$$\frac{60}{0.5665}\sqrt{(1.0183)^2 - (0.2980)^2} = 103 \text{ minutes}$$

Semiduration of total phase :

$$\frac{60}{0.5665}\sqrt{(0.4733)^2 - (0.2980)^2} = 39 \text{ minutes}$$

Hence, in Universal Time :

beginning of partial phase : $19^h04^m - 103^m = 17^h21^m$
beginning of total phase : $19^h04^m - 39^m =$ 18 25
maximum of the eclipse : 19 04
end of total phase : $19^h04^m + 39^m =$ 19 43
end of partial phase : $19^h04^m + 103^m =$ 20 47

Exercices

Find the first solar eclipse of the year 1979, and show that it is a total one visible from the northern hemisphere.

Is the solar eclipse of April 1977 a total or an annular one ?

Show that there was no eclipse of the Sun in July 1947.

Show that there will be four solar eclipses in the year 2000, and that all four will be partial eclipses.

Show that there was no lunar eclipse in January 1971.

Show that there will be three total eclipses of the Moon in 1982.

Find the first lunar eclipse of the year 1234. (Answer : the partial lunar eclipse of 1234 March 17).

34

ILLUMINATED FRACTION OF THE DISK OF A PLANET

As for the Moon (see Chapter 31), the illuminated fraction k of the disk of a planet, as seen from the Earth, can be calculated from

$$k = \frac{1 + \cos i}{2}$$

where i is the phase angle. In the case of a planet, this angle can be found from

$$\cos i = \frac{r^2 + \Delta^2 - R^2}{2 r \Delta}$$

r being the planet's distance to the Sun, Δ its distance to the Earth, and R the distance Sun - Earth, all in astronomical units. Combining these two formulae, we find

$$k = \frac{(r + \Delta)^2 - R^2}{4 r \Delta} \qquad (34.1)$$

If the planet's position has been obtained by the first method of Chapter 25, we can find k as follows :

$$k = \frac{r + \Delta + R \cos b \cos (l - \odot)}{2 \Delta}$$

Example 34.a : Find the illuminated fraction of the disks of Mercury, Venus and Mars on 1979 April 17 at 0^h ET.

We will use formula (34.1), and take the values of r, Δ and R from the *Astronomical Ephemeris*.

	Mercury	*Venus*	*Mars*
r	0.466 674	0.728 149	1.387 513
Δ	0.785 473	1.300 500	2.300 530
R	1.003 712	1.003 712	1.003 712
k	0.382	0.821	0.986

For Mercury and Venus, k can take all values between 0 and 1. For Mars, k can never be less than approximately 0.838. In the case of Jupiter, i is never less than 12°, whence k can vary only between 0.989 and 1.

In the case of *Venus*, an *approximate* value of k can be found as follows.

Calculate T by means of formula (18.1). Then,

$$\begin{aligned}
V &= 63\overset{\circ}{.}07 + 22518\overset{\circ}{.}443\,T \\
M &= 178\overset{\circ}{.}48 + 35999\overset{\circ}{.}050\,T \\
M' &= 212\overset{\circ}{.}60 + 58517\overset{\circ}{.}804\,T \\
W &= V + 1\overset{\circ}{.}92 \sin M + 0\overset{\circ}{.}78 \sin M' \\
\Delta^2 &= 1.523\,209 + 1.446\,664 \cos W \qquad (\Delta > 0) \\
k &= \frac{(0.723\,332 + \Delta)^2 - 1}{2.893\,329\,\Delta}
\end{aligned} \tag{34.2}$$

Example 34.b : Find the illuminated fraction of the disk of Venus on 1979 April 17.0 ET, using the approximate method described above.

We find successively :

$\mathrm{JD} = 2443\,980.5$

$T = +0.792\,895\,277$

$V = 17917\overset{\circ}{.}84 = 277\overset{\circ}{.}84$

$M = 28721\overset{\circ}{.}96 = 281\overset{\circ}{.}96$

$M' = 46611\overset{\circ}{.}09 = 171\overset{\circ}{.}09$

$W = V - 1\overset{\circ}{.}88 + 0\overset{\circ}{.}12 = 276\overset{\circ}{.}08$

$\Delta^2 = 1.676\,435$

$\Delta = 1.294\,772$

$k = 0.820$

The correct value, found in the preceding example, is 0.821.

The *elongation* ψ of a planet can be calculated from formula (25.11). If the distances R, r and Δ are known, it can also be found from

$$\cos\psi = \frac{R^2 + \Delta^2 - r^2}{2R\Delta} \tag{34.3}$$

In the case of *Venus*, an *approximate* value of ψ can be found by first calculating Δ from (34.2), and then

$$\cos\psi = \frac{\Delta^2 + 0.4768}{2\Delta} \tag{34.4}$$

Taking for Venus the values given in Example 34.a, we find, by formula (34.3), $\cos\psi = 0.830\,649$, whence $\psi = 33°50'$.

Taking the approximate value of Δ found in Example 34.b, formula (34.4) gives $\cos\psi = 0.8315$, whence $\psi = 33°45'$.

35

POSITIONS OF THE SATELLITES OF JUPITER

The following method enables the configuration of the four great satellites of Jupiter to be computed at any given instant. The accuracy is moderately good ; the results may not be used for accurate calculations. Moreover, the latitudes of the satellites are not considered here : only their apparent distances X, east or west of Jupiter, are calculated ; the Y coordinates are not found.

First, if necessary, convert the date and the instant (UT) into the Julian Day, using the method described in Chapter 3. With the JD, calculate

$$d = \mathrm{JD} - 2415\,020 \, ,$$

where d is the time measured in mean solar days from 1900 January 0, 12^h UT.

Then, calculate the angles M (the Earth's mean anomaly), N (Jupiter's mean anomaly) and J (the difference between the heliocentric mean longitudes of Earth and Jupiter) by means of the following formulae, in which the numerical values are degrees and decimals. If necessary, the results must be reduced to the interval 0 - 360 degrees.

$$M = 358.476 + 0.985\,6003\,d$$

$$N = 225.328 + 0.083\,0853\,d$$

$$J = 221.647 + 0.902\,5179\,d$$

Then calculate

$A = 1\overset{\circ}{.}92 \sin M + 0\overset{\circ}{.}02 \sin 2M$

$B = 5\overset{\circ}{.}537 \sin N + 0\overset{\circ}{.}167 \sin 2N$

$K = J + A - B$

$\Delta = \sqrt{28.07 - 10.406 \cos K} > 0$

$\sin \psi = \frac{\sin K}{\Delta}$ (ψ to be taken between -90° and +90°)

$$u_1 = 84\overset{\circ}{.}5506 + 203\overset{\circ}{.}405\,8630\,(d - \frac{\Delta}{173}) + \psi - B$$

$$u_2 = 41\overset{\circ}{.}5015 + 101\overset{\circ}{.}291\,6323\,(d - \frac{\Delta}{173}) + \psi - B$$

$$u_3 = 109\overset{\circ}{.}9770 + 50\overset{\circ}{.}234\,5169\,(d - \frac{\Delta}{173}) + \psi - B$$

$$u_4 = 176\overset{\circ}{.}3586 + 21\overset{\circ}{.}487\,9802\,(d - \frac{\Delta}{173}) + \psi - B$$

The required X-coordinates of the satellites I to IV are then, respectively :

$$X_1 = 5.906 \sin u_1$$
$$X_2 = 9.397 \sin u_2$$
$$X_3 = 14.989 \sin u_3$$
$$X_4 = 26.364 \sin u_4$$

These X-coordinates are expressed in units of Jupiter's equatorial radius, and they are positive (negative) if the satellite is west (east) of the planet.

Example 35.a : Calculate the configuration of the satellites of Jupiter on 1978 January 31 at 20^h30^m Universal Time.

1978 January 31, 20^h30^m = 1978 January 31.854
= JD 2443 540.354

One finds successively :

$d = 28520.354$

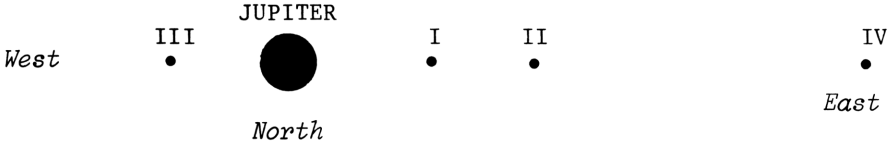
$M = 28468\overset{\circ}{.}145 = 28\overset{\circ}{.}145$
$N = 2594\overset{\circ}{.}950 = 74\overset{\circ}{.}950$
$J = 25961\overset{\circ}{.}777 = 41\overset{\circ}{.}777$

$A = +0\overset{\circ}{.}922$
$B = +5\overset{\circ}{.}431$
$K = +37\overset{\circ}{.}268$

$$\Delta = 4.448 \qquad \sin\psi = +0.13614 \qquad \psi = +7\overset{\circ}{.}824 \qquad d - \frac{\Delta}{173} = 28520.328$$

$$u_1 = 5\,801\,288\overset{\circ}{.}93 \qquad X_1 = -\ 5.51$$
$$u_2 = 2\,888\,914\overset{\circ}{.}50 \qquad X_2 = -\ 9.37$$
$$u_3 = 1\,432\,817\overset{\circ}{.}28 \qquad X_3 = +\ 4.45$$
$$u_4 = 613\,023.00 \qquad X_4 = -22.11$$

From the values of the X-coordinates, we deduce that the configuration of the satellites was as follows :



Example 35.b : Calculate the time of the conjunction between the satellites II and III on the evening of 1979 February 26.

Two satellites are in conjunction when their X-coordinates are equal. We thus calculate X_2 and X_3 for several instants. The results, written with an extra decimal, are as follows.

		X_2	X_3	$X_2 - X_3$
1979 Feb 26	19^h00^m UT	−9.388	−8.610	−0.778
	20 00	−9.334	−9.047	−0.287
	21 00	−9.228	−9.482	+0.254
	22 00	−9.070	−9.905	+0.835
	23 00	−8.868	−10.304	+1.436

By interpolation, we find that $X_2 = X_3$ at 20^h32^m UT.

According to the calculations by Chr. Steyaert (*VVS Hemelkalender 1979*, page 73) the correct time is 20^h47^m UT.

A satellite is in inferior conjunction with Jupiter when its X-coordinate is zero and passing from negative to positive ; or, what is the same, when the corresponding angle u (reduced to the interval 0° – 360°) is 0° or 360°.

Similarly, a satellite is in superior conjunction with Jupiter when its X-coordinate, passing from positive to negative, becomes zero, or when u = 180°.

Exercise. - On 1978 April 16, the satellites I and III were almost simultaneously in conjunction with Jupiter. Confirm that with your program, and compare your results with the following data taken from the *Astronomical Ephemeris* :

17^h27^m UT	III in superior conjunction
17 30	I in inferior conjunction

36

SEMIDIAMETERS OF SUN, MOON AND PLANETS

Sun and Planets

When not available directly from almanac data, the semidiameters s of the Sun and planets can be computed from

$$s = \frac{s_o}{\Delta}$$

where s_o is the body's semidiameter at unit distance (1 AU),
Δ is the body's distance to the Earth, in AU.

The following values of s_o should be used :

Sun	959″.63
Mercury	3.34
Venus	8.41
Mars	4.68
Jupiter :	
equatorial	98.47
polar	91.91
Saturn :	
equatorial	83″.33
polar	74.57
Uranus	34.28
Neptune	36.56

Moon

When not available directly from almanac data, the semidiameter s' of the Moon, expressed in seconds of a degree (″), can be computed from

$$s' = \frac{56\,204.92}{D} = \frac{358\,482\,800}{D'} = 0.272\,476\ \pi'$$

where D = geocentric distance of the Moon in units of the equatorial radius of the Earth ;
D' = geocentric distance of the Moon in kilometers ;
π' = the horizonal equatorial parallax of the Moon in seconds of a degree (″).

Computed in this way, the semidiameter is geocentric, that is it applies to a fictitious observer located at the center of the Earth. The observed semidiameter of the Moon will be slightly larger than the geocentric diameter. It can be obtained, with sufficient accuracy for many purposes, by multiplying the geocentric value by

$$1 + \frac{\sin h}{D}$$

where h is the altitude of the Moon above the observer's horizon,
D is, as above, the geocentric distance of the Moon in units of the Earth's equatorial radius.

The increase in the Moon's semidiameter, due to the fact that the observer is not geocentric, is zero when the Moon is on the horizon, and a maximum (between 14″ and 18″) when the Moon is at the zenith.

37

STELLAR MAGNITUDES

Adding stellar magnitudes

If two stars are of magnitudes m_1 and m_2 , respectively, their combined magnitude m can be calculated as follows :

$$x = 0.4\ (m_2 - m_1)$$

$$m = m_2 - 2.5 \log\ (10^x + 1)$$

where the logarithm is in base 10.

Example 37.a : The magnitudes of the components of Castor (α Gem) are 1.96 and 2.89. Calculate the combined magnitude.

One finds

$$x = 0.4\ (2.89 - 1.96) = 0.372$$

$$m = 2.89 - 2.5 \log\ (10^{0.372} + 1) = 1.58$$

Brightness ratio

If two stars are of magnitudes m_1 and m_2 , respectively, the ratio I_1/I_2 of their apparent luminosities can be calculated as follows :

$$x = 0.4\ (m_2 - m_1) \qquad\qquad \frac{I_1}{I_2} = 10^x$$

If the brightness ratio I_1/I_2 is given, the corresponding magnitude difference $\Delta m = m_2 - m_1$ can be calculated from

$$\Delta m = 2.5 \log \frac{I_1}{I_2}$$

Example 37.b : How many times is Vega (magnitude 0.14) brighter than Polaris (magnitude 2.12) ?

$$x = 0.4\ (2.12 - 0.14) = 0.792$$

$$10^{x} = 6.19$$

Thus, Vega is 6.19 times as bright as the Pole Star.

Example 37.c : A star is 500 times as bright as another one.

The corresponding magnitude difference is

$$\Delta m = 2.5 \log 500 = 6.75$$

Distance and Absolute Magnitude

If π is a star's parallax expressed in seconds of a degree ($''$), this star's distance to us is equal to

$$\frac{1}{\pi}\ \text{parsecs} \qquad \text{or} \qquad \frac{3.2616}{\pi}\ \text{light-years}$$

If π is a star's parallax expressed in seconds of a degree ($''$), and m is the apparent magnitude of this star, its absolute magnitude M can be calculated from

$$M = m + 5 + 5 \log \pi$$

where, again, the logarithm is in base 10.

If d is the star's distance in parsecs, we have

$$M = m + 5 - 5 \log d$$

38

BINARY STARS

The orbital elements of a binary star are the following ones :

P = the period of revolution expressed in mean solar years ;

T = the time of periastron passage, generally given as a year and decimals (for instance, 1945.62);

e = the eccentricity of the true orbit ;

a = the semimajor axis expressed in seconds of a degree (″) ;

i = the inclination of the plane of the true orbit to the plane at right angles to the line of sight. For direct motion in the apparent orbit, i ranges from 0° to 90° ; for retrograde motion, i is between 90 and 180 degrees. When i is 90°, the apparent orbit is a straight line passing through the primary star ;

Ω = the position angle of the ascending node ;

ω = the longitude of periastron ; this is the angle in the plane of the true orbit measured from the ascending node to the periastron, taken always in the direction of motion.

When these orbital elements are known, the apparent position angle θ and the angular distance ρ can be calculated for any given time t, as follows.

$$n = \frac{360^\circ}{P} \qquad\qquad M = n\ (t - T)$$

n is the mean annual motion of the companion, expressed in degrees and decimals, and is always positive. M is the companion's mean anomaly for the given time t.

Then solve Kepler's equation

$$E = M + e \sin E$$

by one of the methods described in Chapter 22, and then calculate the radius vector r and the true anomaly v from

$$r = a\,(1 - e \cos E)$$

$$\tan \frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan \frac{E}{2}$$

Then find $(\theta - \Omega)$ from

$$\tan\,(\theta - \Omega) = \frac{\sin\,(v + \omega)\cos i}{\cos\,(v + \omega)} \tag{38.1}$$

Of course, this equation can be written

$$\tan\,(\theta - \Omega) = \tan\,(v + \omega)\cos i$$

but in this case the correct quadrant for $(\theta - \Omega)$ is not determined. As in previous cases, it is better to leave equation (38.1) unchanged, and to apply to the numerator and the denominator of the fraction the conversion from rectangular to polar coordinates. This procedure will place the angle $(\theta - \Omega)$ at once in the correct quadrant.

When $(\theta - \Omega)$ is found, add Ω to obtain θ. If necessary, reduce the result to the interval 0°– 360°.

The angular separation ρ is found from

$$\rho = \frac{r \cos\,(v + \omega)}{\cos\,(\theta - \Omega)}$$

Example 38.a : According to E. Silbernagel (1929), the orbital elements for η Coronae Borealis are :

P = 41.623 years
T = 1934.008
e = 0.2763
a = 0″.907
i = 59°.025
Ω = 23°.717
ω = 219°.907

Calculate θ and ρ for the epoch 1980.0.

We find successively :

$n = 8.64906$
$t - T = 1980.0 - 1934.008 = 45.992$
$M = 397^\circ.788 = 37^\circ.788$
$E = 49^\circ.897$
$r = 0''.74557$

$$v = 63\overset{\circ}{.}416$$

$$\tan(\theta - \Omega) = \frac{-0.500\,813}{+0.230\,440}$$

$$\theta - \Omega = -65\overset{\circ}{.}291$$

$$\theta = -41\overset{\circ}{.}574 = 318\overset{\circ}{.}4$$

$$\rho = 0\overset{\prime\prime}{.}411$$

It is possible to write a program for obtaining a complete ephemeris of a binary for many years. For the HP-67 machine, the author has written such a program (150 steps). It contains as a subroutine the resolution of Kepler's equation, which can be used as an independent program. After entering the orbital elements, one enters the starting year of the required ephemeris, and the ephemeris interval in years. Then, without any push on a key, the machine displays successively, during pauses,

the year,
the position angle in degrees,
the separation in ″,

then the next year, and so on.

Try to write a similar program for your machine. As an exercise, calculate an ephemeris for γ Virginis, using the following elements (K. Strand, 1937) :

$P = 171.37$ years $i = 146\overset{\circ}{.}05$
$T = 1836.433$ $\Omega = 31\overset{\circ}{.}78$
$e = 0.8808$ $\omega = 252\overset{\circ}{.}88$
$a = 3\overset{\prime\prime}{.}746$

Answer. - Here is an ephemeris with an interval of four years, starting with 1980. The position angle decreases with time, since i is between 90 and 180 degrees.

year = 1980.0	$\theta = 296\overset{\circ}{.}86$	$\rho = 3\overset{\prime\prime}{.}90$
1984.0	293.55	3.58
1988.0	289.55	3.23
1992.0	284.49	2.83
1996.0	277.62	2.37
2000.0	267.10	1.84
2004.0	246.26	1.18
2008.0	125.94	0.38
2012.0	24.65	1.35

Eccentricity of the apparent orbit

The apparent orbit of a binary star is an ellipse whose eccentricity e' is generally different from the eccentricity e of the true orbit. It may be interesting to know e', although this apparent eccentricity has no astrophysical significance. As far as we know, the following formulae have never been published before. (They will be published in an article, by the author, in the *Journal* of the British Astronomical Association).

$$A = (1 - e^2 \cos^2\omega) \cos^2 i$$

$$B = e^2 \sin\omega \cos\omega \cos i$$

$$C = 1 - e^2 \sin^2\omega$$

$$D = (A - C)^2 + 4B^2$$

$$e'^2 = \frac{2\sqrt{D}}{A + C + \sqrt{D}}$$

It should be noted that e' is independent of the orbital elements a and Ω, and that is can be smaller as well as larger than the true eccentricity e.

Example 38.b : Find the eccentricity of the apparent orbit of η Coronae Borealis. The orbital elements are given in Example 38.a.

We find

$$A = 0.25298$$
$$B = 0.01934$$
$$C = 0.96858$$
$$D = 0.51358$$

$$e' = 0.860$$

Hence, for this binary the apparent orbit is much more elongated than the true orbit.

39

LINEAR REGRESSION ; CORRELATION

In many cases, the result of a large number of observations is a series of points in a graph, each point being defined by a x-value and a y-value. It may be necessary to draw through the points the "best" fitting curve.

Several curves can be fitted through a series of points : a straight line, an exponential, a logarithmic curve, a polynomial, etc. We will consider here only the case of a straight line, a problem called *linear regression*.

We wish to calculate the coefficients of the linear equation

$$y = ax + b \tag{39.1}$$

using the least squares method. The slope a and the y-intercept b can be calculated by means of the formulae

$$\begin{aligned} a &= \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2} \\ b &= \frac{\Sigma y\,\Sigma x^2 - \Sigma x\,\Sigma xy}{n\,\Sigma x^2 - (\Sigma x)^2} \end{aligned} \tag{39.2}$$

where n is the number of points. The sign Σ indicates the summation. Thus, Σx signifies the sum of all the x-values, Σy the sum of all the y-values, Σx^2 the sum of the squares of all the x-values, Σxy the sum of the products xy of all the couples of values, etc. It should be noted that Σxy is not the same as $\Sigma x \times \Sigma y$ (the sum of the products is not the same as the product of the sums), and that $(\Sigma x)^2$ is not the same as Σx^2 (the square of the sum is not the same as the sum of the squares) !

An interesting astronomical application is to find the relation between the intrinsic brightness of a comet and its distance to the Sun. The apparent magnitude m of a comet can generally be represented by a formula of the form

$$m = g + 5 \log \Delta + \kappa \log r$$

which we already mentioned in Chapter 25. Here, Δ and r are the distance in astronomical units of the comet to the Earth and to the Sun, respectively. The absolute magnitude g and the coefficient κ must be deduced from the observations. This can be performed when the magnitude m has been measured during a sufficiently long period. For each value of m, the values of Δ and r must be deduced from the ephemeris, or calculated from the orbital elements.

In that case, the unknowns are g and κ. The above formula can be written

$$m - 5 \log \Delta = \kappa \log r + g$$

which is of the form (39.1), when we write $y = m - 5 \log \Delta$, and $x = \log r$. The quantity y may be called the "heliocentric" magnitude, because the effect of the variable distance to the Earth has been removed.

Example 39.a : Table 39.A contains visual magnitude estimates m of the periodic comet Wild 2 (1978*b*), made by John Bortle. The corresponding values of r and Δ have been calculated from the orbital elements (IAUC 3177).

The quantities y and x are now used to calculate Σx, Σy, Σx^2, and Σxy. On a HP-67 and on some other machines, there exists a key (Σ+) whose use stores the different sums Σx, Σy^2, etc. in different registers. This key should be pressed each time after a couple of y and x values has been entered.

With the values of the table, we find

$$n = 19 \qquad \Sigma x = 4.2805 \qquad \Sigma x^2 = 1.0031$$
$$\Sigma y = 192.0400 \qquad \Sigma xy = 43.7943$$

whence, by formulae (39.2),

$$a = 13.67 \qquad b = 7.03$$

Consequently, the "best" line fitting the observations is

$$y = 13.67\,x + 7.03$$

$$\text{or} \quad m - 5 \log \Delta = 13.67 \log r + 7.03$$

TABLE 39.A

1978, UT		m	r	Δ	$y =$ $m - 5 \log \Delta$	$x =$ $\log r$
Febr.	4.01	11.4	1.987	1.249	10.92	0.2982
	5.00	11.5	1.981	1.252	11.01	0.2969
	9.02	11.5	1.958	1.266	10.99	0.2918
	10.02	11.3	1.952	1.270	10.78	0.2905
	25.03	11.5	1.865	1.335	10.87	0.2707
March	7.07	11.5	1.809	1.382	10.80	0.2574
	14.03	11.5	1.772	1.415	10.75	0.2485
	30.05	11.0	1.693	1.487	10.14	0.2287
April	3.05	11.1	1.674	1.504	10.21	0.2238
	10.06	10.9	1.643	1.532	9.97	0.2156
	26.07	10.7	1.582	1.592	9.69	0.1992
May	1.08	10.6	1.566	1.610	9.57	0.1948
	3.07	10.7	1.560	1.617	9.66	0.1931
	8.07	10.7	1.545	1.634	9.63	0.1889
	26.09	10.8	1.507	1.696	9.65	0.1781
	28.09	10.6	1.504	1.703	9.44	0.1772
	29.09	10.6	1.503	1.707	9.44	0.1770
June	2.10	10.5	1.498	1.721	9.32	0.1755
	6.09	10.4	1.495	1.736	9.20	0.1746

Thus we have, for the periodic comet Wild 2 in 1978,

$$m = 7.03 + 5 \log \Delta + 13.67 \log r$$

Coefficient of Correlation

A correlation coefficient is a statistical measure of the degree to which two variables are related to each other. In the case of a linear equation, the coefficient of correlation is

$$r = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{\sqrt{n\,\Sigma x^2 - (\Sigma x)^2}\;\sqrt{n\,\Sigma y^2 - (\Sigma y)^2}} \qquad (39.3)$$

This coefficient is always between +1 and -1. A value of +1 or -1 would indicate that the two variables were totally correlated ; it would denote a perfect functional relationship, all the points

Figure 1

Perfect functional relationship ; positive correlation

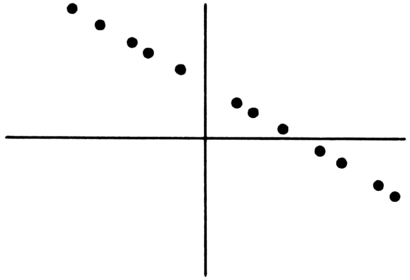


Figure 2

Perfect functional relationship ; negative correlation

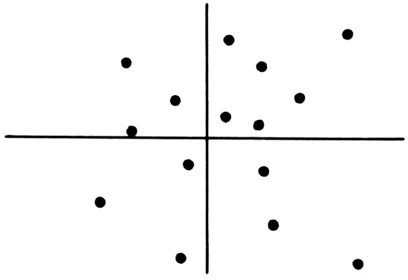


Figure 3

No correlation

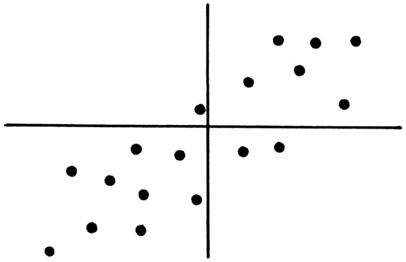


Figure 4

Some correlation

representing paired values of x and y falling on the straight line representing this relationship. If $r = +1$, an increase of x corresponds to an increase of y (Figure 1). If $r = -1$, there is again a perfect functional relationship, but y decreases when x increases (Figure 2).

When $r = 0$, there is no relationship between x and y (Figure 3). In practice, however, when there is no relationship, one may find that r is not exactly zero, due to hazardous coincidences that generally occur except for an infinity of points.

When $|r|$ is between 0 and 1, there is a trend between x and y, although there is no strict relationship (Figure 4). Here, again, it should be noted that, *if* there is actually a strict relationship between the two variables, the calculation may give a value of r not exactly equal to +1 or −1, by reason of inaccuracies inherent to all measures.

Example 39.b : On page 10 of the Belgian journal *Heelal* of September 1978, the following table (Table 39.B) is given. For each of the twenty sunspot maxima which have occurred from 1761 to 1969, y is the height of the maximum (highest smoothed monthly mean), and x is the time interval, in months, since the previous sunspot minimum.

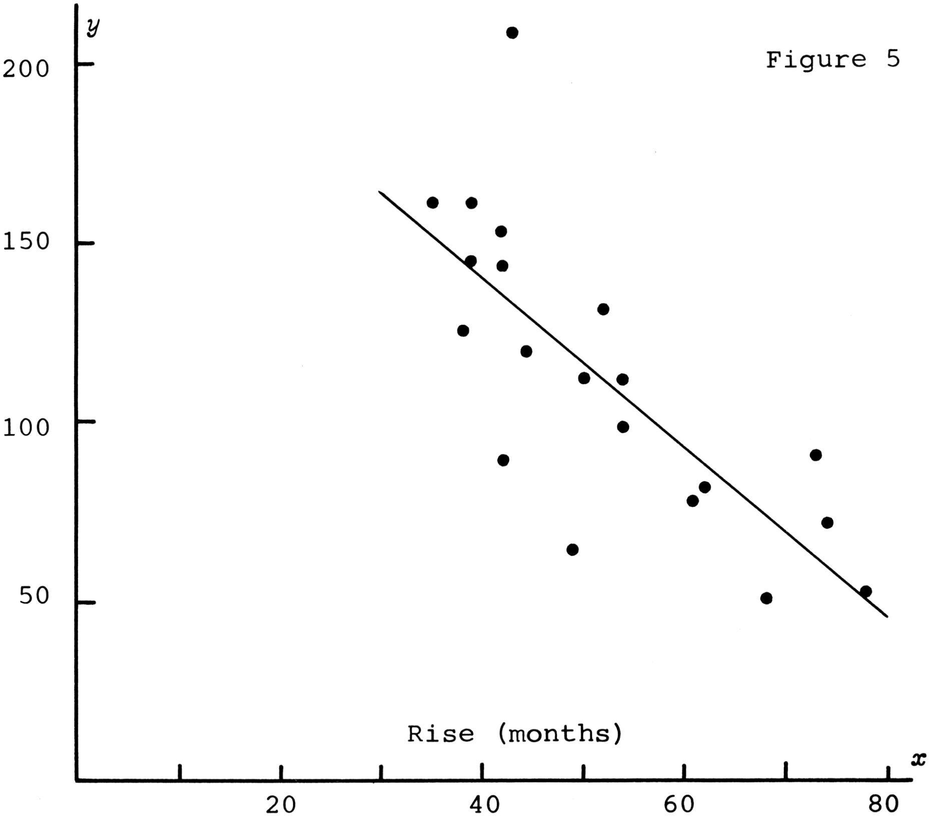
In this case, we find

$\Sigma x = 1039$ $\Sigma x^2 = 57303$ $\Sigma xy = 108\ 987.0$

$\Sigma y = 2249.7$ $\Sigma y^2 = 286\ 027.09$ $n = 20$

TABLE 39.B

epoch of maximum	y	x	*epoch of maximum*	y	x
1761 June	90.4	73	1870 July	144.8	39
1769 Oct.	125.3	38	1884 Jan.	78.1	61
1778 May	161.8	35	1893 Aug.	89.5	42
1787 Nov.	143.4	42	1905 Oct.	63.9	49
1804 Dec.	52.5	78	1917 Aug.	112.1	50
1816 March	50.8	68	1928 June	82.0	62
1829 June	71.5	74	1937 May	119.8	44
1837 Febr.	152.8	42	1947 July	161.2	39
1847 Nov.	131.3	52	1957 Nov.	208.4	43
1860 July	98.5	54	1969 Febr.	111.6	54



and then, by formulae (39.2) and (39.1),

$$y = 235.61 - 2.37\,x$$

This is the equation of the best straight line fitting the given twenty points. These points and the line are shown in Figure 5.

From formula (39.3) we find $r = -0.753$. This shows that there exists an evident trend to connexion, and the negative sign of r indicates that the correlation between x and y is negative : the *longer* the duration of the rise from a minimum to the next maximum of the sunspot activity, the *lower* this maximum generally is.

Example 39.c : As an example of two variables between which we know there *cannot* exist any correlation, we consider the absolute photographic magnitude (x) of a minor planet and the number of the letters (y) in its name. We take the absolute magnitudes from the Leningrad *Ephemerides of Minor Planets* for 1979, and consider only the minor planets 51 to 100, which is a sufficient large sample (Table 39.C).

In this case, we have

$$\Sigma x = 460.8 \qquad \Sigma x^2 = 4278.50 \qquad \Sigma xy = 2869.1$$

$$\Sigma y = 312 \qquad \Sigma y^2 = 2078 \qquad n = 50$$

whence, by formula (39.3), $r = -0.097$. The coefficient of correlation is small, but not zero for the reason mentioned before.

TABLE 39.C

	x	y		x	y
51 Nemausa	8.7	7	76 Freia	9.0	5
52 Europa	7.6	6	77 Frigga	9.6	6
53 Kalypso	9.8	7	78 Diana	9.1	5
54 Alexandra	8.8	9	79 Eurynome	9.3	8
55 Pandora	9.1	7	80 Sappho	9.3	6
56 Melete	9.6	6	81 Terpsichore	9.7	11
57 Mnemosyne	8.4	9	82 Alkmene	9.4	7
58 Concordia	9.9	9	83 Beatrix	9.8	7
59 Elpis	8.7	5	84 Klio	10.3	4
60 Echo	10.0	4	85 Io	8.9	2
61 Danaë	8.8	5	86 Semele	9.8	6
62 Erato	9.8	5	87 Sylvia	8.3	6
63 Ausonia	8.2	7	88 Thisbe	8.2	6
64 Angelina	8.8	8	89 Julia	8.2	5
65 Cybele	7.9	6	90 Antiope	9.3	7
66 Maja	10.6	4	91 Aegina	9.7	6
67 Asia	9.9	4	92 Undina	8.0	6
68 Leto	8.3	4	93 Minerva	8.8	7
69 Hesperia	8.3	8	94 Aurora	8.8	6
70 Panopaea	9.2	8	95 Arethusa	8.9	8
71 Niobe	8.5	5	96 Aegle	9.1	5
72 Feronia	10.3	7	97 Klotho	8.7	6
73 Klytia	10.3	6	98 Ianthe	10.4	6
74 Galatea	10.1	7	99 Dike	11.5	4
75 Eurydike	10.0	8	100 Hekate	9.1	6

It should be noted that here, as in all other statistical studies, the sample must be sufficiently large in order to give a meaningful result. A correlation coefficient close to +1 or to -1 has no physical meaning if it is based on a too small number of cases. When we consider only the eight minor planets 71 to 78 in the example above, we find the high coefficient of correlation +0.785 between x and y, and even the still higher value r = +0.932 for the five minor planets 78 to 82.

This demonstrates that with too few cases the correlation coefficient can be accidently quite large.

As an exercice, show that there is no correlation between the rainfall at the Uccle Observatory and the sunspot activity, using the data of Table 39.D, where

y = total annual rainfall at Uccle, in millimeters,
x = yearly mean of the definitive sunspot numbers.

(Answer : the correlation coefficient is r = -0.090, which shows that there is no significant correlation between x and y.)

TABLE 39.D

year	y	x	*year*	y	x	*year*	y	x
1901	700	2.7	1927	837	69.0	1953	557	13.9
1902	762	5.0	1928	882	77.8	1954	741	4.4
1903	854	24.4	1929	688	64.9	1955	616	38.0
1904	663	42.0	1930	953	35.7	1956	795	141.7
1905	912	63.5	1931	858	21.2	1957	801	190.2
1906	821	53.8	1932	858	11.1	1958	834	184.8
1907	622	62.0	1933	738	5.7	1959	560	159.0
1908	678	48.5	1934	707	8.7	1960	962	112.3
1909	842	43.9	1935	916	36.1	1961	903	53.9
1910	990	18.6	1936	763	79.7	1962	862	37.5
1911	741	5.7	1937	900	114.4	1963	713	27.9
1912	941	3.6	1938	711	109.6	1964	785	10.2
1913	801	1.4	1939	928	88.8	1965	1073	15.1
1914	877	9.6	1940	837	67.8	1966	1054	47.0
1915	910	47.4	1941	744	47.5	1967	707	93.8
1916	1054	57.1	1942	841	30.6	1968	776	105.9
1917	851	103.9	1943	738	16.3	1969	776	105.5
1918	848	80.6	1944	766	9.6	1970	727	104.5
1919	980	63.6	1945	745	33.2	1971	691	66.6
1920	760	37.6	1946	861	92.6	1972	710	68.9
1921	417	26.1	1947	640	151.6	1973	690	38.0
1922	938	14.2	1948	792	136.3	1974	1039	34.5
1923	917	5.8	1949	521	134.7	1975	734	15.5
1924	849	16.7	1950	951	83.9	1976	541	12.6
1925	1075	44.3	1951	878	69.4	1977	855	27.5
1926	896	63.9	1952	926	31.5			