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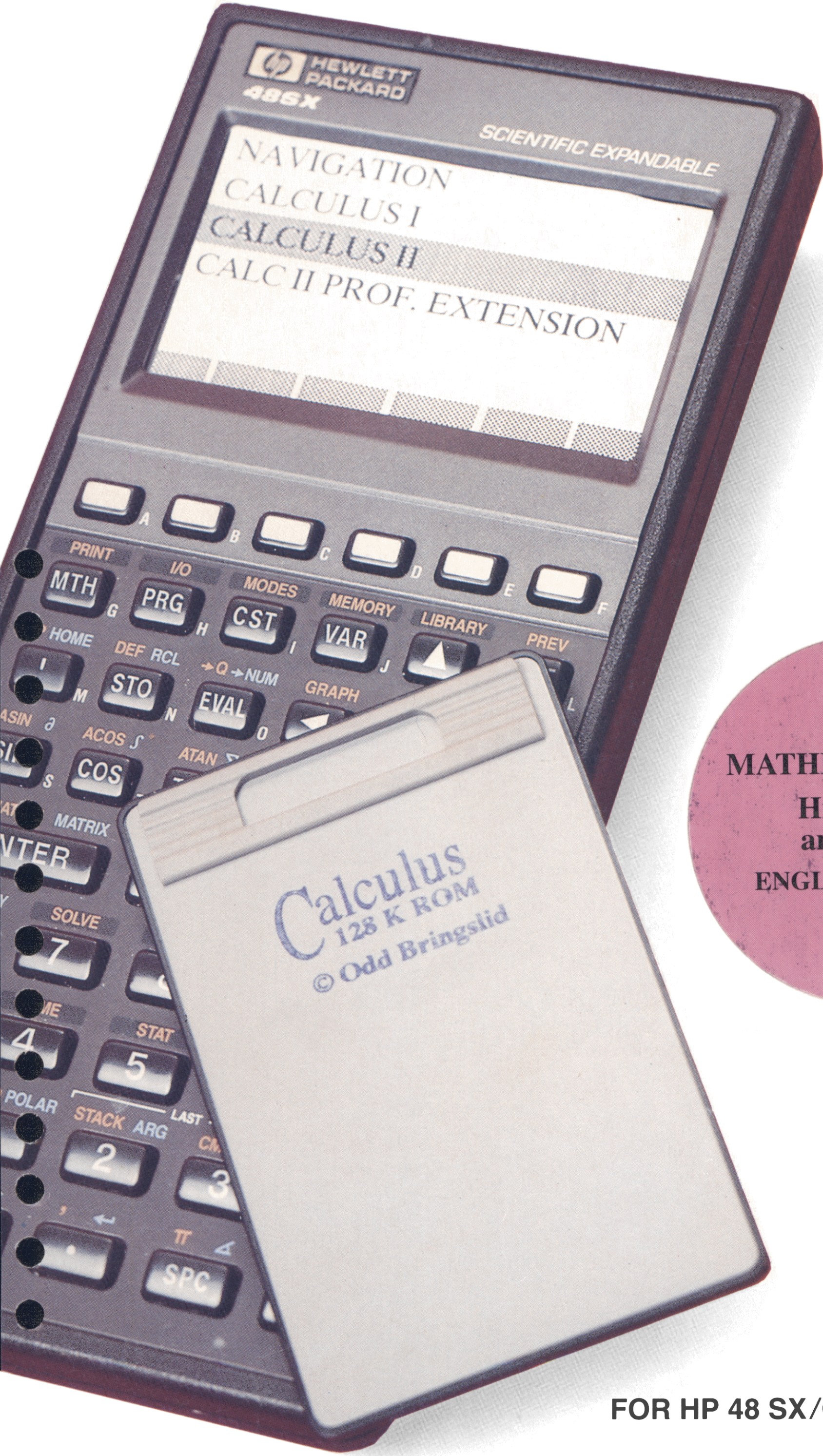
APPLICATION SOFTWARE



NEW MATHEMATIC Pac.1

High School and College

ENGLISH VERSION

FOR HP 48 SX/GX

BAS Calculus
New Mathematics I

VER 3.0

A mathematical program for calculators (HP 48 SX/GX)

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1

In general

Congratulations with BAS CALCULUS!

The development of CALCULUS has served two aims:

- Give students and others a possibility to get things calculated without remembering tricky details
- Give students educational assistance in the way that intermediate results shows up and indicate the solving strategy.

The last point is important and implies that different strategies for solving a problem can be chosen in a menu system.

Hardware requirements

CALCULUS runs under the calculator HP 48SX/GX and will need free RAM capacity of 10 KB. The program card may be inserted into either of the two ports.

Starting up

The LIBRARY menu will show up CALC. Pushing the CALC key will lead you into the CALCULUS main menu and then you simply push the START key. You may also put CALCULUS under the USR interface (see user manual for the HP 48 SX/GX).

User routines

Independent of CALCULUS a set of routines may be accessed from the main menu or the routines may be used in your own programs.

These include routines for sorting, simplifying, viewing text, paging and substituting.

- QSRT sorting, In: {2,4,1,6,3,8}, a list of numbers. Out: sorted list {1,2,3,4,6,8}
- EXCO, simplifying, In: an algebraic , Out: the simplified algebraic.
- VIEW, routine for showing text and GROBs. In: text ("   ") or GROBs in a list, Out: a nice view of indata.
- PAGER, routine for paging in connection with VIEW, In: Text, Out: List of strings.
- WH, routine for substituting. Stacklevel 3: 'x^2' Stacklevel 2: x Stacklevel 1: 1.4. Out: 1.4^2.

User interface

The CALCULUS menu system is easy to use. Using the arrow keys allows you to move the dark bar and select by pushing ENTER.

In the following example you will enter the submenu for FUNCTIONS and select Graphing.

ALGEBRA
DIFFEQUATIONS
FUNCTIONS
FUNC. SEV. VARIABLES
INTEGRATION

Sign of function
Zeroes
Graphing
Extremum points
Function table

RAD PRG
{HOME }

Graphing {f1(x) f2....}
between x = x1 x = x2
: Indp var: x
: x1 x2: 0 2

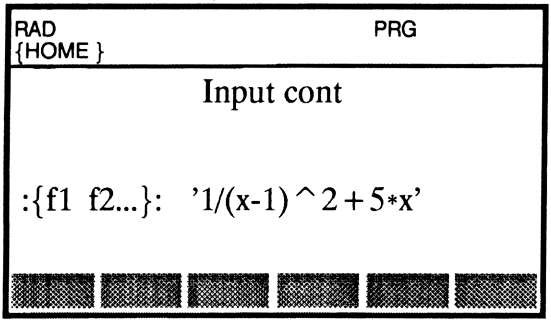
The Editor

You will now enter the editor for input data (input screen). Here the input data can be modified and deleted and you can move around by using the arrow keys.

The cursor is placed right behind :Indp var: and here you enter the independent variable, fx x. The arrow down key is used to get right behind :x1 x2: . Here you enter the interval of x, two x-values separated by a blank

If you want to use another symbol for the independent variable you can use the arrow keys to get right behind the x, delete it and enter t.

Now you push the ENTER key to go further. If the syntax is wrong, fx forgetting the ’ ’ in an algebraic, then you have to correct it before going further.



The function to be drawn is now entered. Remember the ' ' for algebraics.

The input screen will tell you the format of the input data and if several expressions or values are going to be placed in the same line, they must be separated by blanks.

Echoing from the stack

If an expression or a value laying on the stack is going to be used, then the EDIT/→STK/ECHO/ENTER keys will load data in to the input screen. Be sure to place the cursor correctly.

Calculation finished

When a calculation is finished CALCULUS will either show up intermediate results by using the VIEW routine or return directly to the menu. In the last case you will need to use the →STK key to see the result laying on the stack.

Moving up and down in the menu

You can move downwards in the menu system by scrolling the dark bar and pressing ENTER. If you need to move upwards the UPDIR key will help. At any time you can HALT CALCULUS and use the calculator independent of CALCULUS by pressing the →STK key. CONT will get you back to the CALCULUS menu system.

To move faster to the end or to the beginning of the menu you may use the shift right (blue) key together with the down/up arrow key.

Leaving CALCULUS

Pushing the EXIT key will leave CALCULUS.

Intermediate results

In the input screen you may choose PartAns Y/N. Choosing N the result will be laying on the stack and you have to use →STK to see the answer. Choosing Y, different pages of intermediate results will show up.

The degree of details in the partial answers is somewhat different, but some of the results covers "the whole answer". In every case this will give the user a good help.

Different parts of the answer can be found on different pages and the page number can bee seen.

Since partial answers are text, the splitting of long lines is not done taking algebraic considerations. This may simetimes give odd results, like SIN(x) beeing split up like SI
N(x).

This makes it necessary to write the answer down on a sheet of paper to see the answer properly.

When PartAns Y(es) is chosen, all irrational numbers will show up with two figures behind comma. If a more accurate answer is necessary, you may use →STK to look on the stack.

> *Rem.: PartAns N puts the answer only on stack and you must use →STK to look at the answer.*

Mixed calculations

Sometimes it's necessary to use more than one of the CALCULUS routines to solve a problem. A typical example is the calculation of a definite integral. First the indefinite integral has to be calculated.

Example: Calculate $$\int_2^3 (x+1)/(x^3-x^2+x-1)dx$$

First the indefinite integral is calculated by partial fractions.

The answer is transfered into the routine for definite integrals by using EDIT/ ↑STK/ECHO/ENTER. Remember to place the cursor correctly.

Time of computation

HP 48 SX/GX has a 4 bit special processor and the speed of a 386 PC can be 200-300 times the speed of the calculator. The time elapsed when handling complex symbolic expressions may be long.

Strongly dependent of free RAM capacity

If you want to optimize on computation time you should have a lot of free RAM capacity. Using a RAM card as MERGED MEMORY full of data may slow down the calculation considerably.

Pedagogical point

The solution strategies chosen i different cases, are the same as the strategies being used in the mathematical curriculum at university level. This is not necesserily the fastest and smartest algorithms and this may slow CALCULUS down.

Checking expressions and rational numbers

CALCULUS uses general input, that is the expressions are written in standard mathematical form. CALCULUS then has to check what kind of object you have put in (polynomial, rational expression etc.) and this is time consuming.

A specialized input, for instance to write a polynomial as a coefficient vector, could be used.. This will reduce the computation time, but the user interface is more complex.

When an expression has been simplified, CALCULUS has to check for rational numbers in the expression. This can be a rather time consuming process, if the object to be checked is a complex symbolic expression.

For instance to transform $(0.33x^2\text{-}2ax)/(x+5)$ to $(1/3x^2\text{-}2ax)/(x+5)$ will take some time. Not the transformation itself but the checking of the validity of the transformation.

Routines for calculation and formula

Many of the menu choices has an option for calculation and one for viewing a formula/info. Choosing "Go" will start the calculation and choosing "Formula" will let you see the formula the calculation is based on. When the choice is "Info" some furher information about the calculation is coming up.

A full solution to a mathematical problem will include the formula the calculation is based on.

Some of the kind of information mentioned above may sometimes be seen in the input screen.

Inexact arithmetic

CALCULUS is using inexact arithmetic generaly. This means that rational and irrational numbers are approximated by decimal numbers to the internal precision of the calculator.

Within certain limitations rational numbers and multiples of π will be transformed to an exact number.

These limitations lie in the intrinsic routine $\rightarrow Q\pi$. It operates with a precision n FIX (se user manual for HP48SX/GX).

> *Rem.: An irrational number will never be approximated by a rational number, but a rational number may be approximated by a decimal number.*

Flag status and CST menu

The flag status and the CST menu you have before going into CALCULUS will be recovered when you leave CALCULUS (beep off inside).

Warning!

User objects with names like A,B,C.... will be deleted by CALCULUS to avoid a conflict with global CALCULUS objects.

2

Integration

The integration package will let you choose between different integration technics. An integral is being exactly calculated only for those integrands that are implemented in the HP 48 integration routine.

The intention is of a pedagogical kind, it is important to be aware of the method of integration in each case.

Direct integration

The routine for direct integration uses the intrinsic HP 48 routine, but is adapted for indefinite integrals and the constant C is omitted.

Interface:

```
RAD                          PRG
{HOME }
        Direct integration
:       Indp var:   t
:       Integrand:  'ATAN(t/2)'
```

The example calculates $\int \text{Atan}(t/2)dt$

The following functions and linear combinations of them are implemented for exact integration:

- $a*(b*x+c)^m$, no restrictions on a,b,c and m
- $a*\text{Sin}(b*x+c)$, $a*\text{Asin}(b*x+c)$
- $a*\text{Cos}(b*x+c)$, $a*\text{Acos}(b*x+c)$
- $a*\text{Tan}(b*x+c)$, $a*\text{Atan}(b*x+c)$
- $a*\text{Exp}(b*x+c)$, $a*\text{Ln}(b*x+c)$
- Derivated of these functions and some others

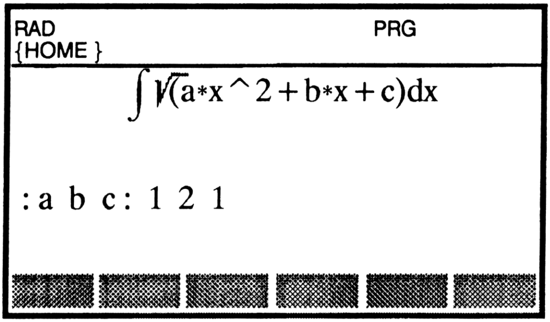
> *Rem.: If no exact answer is found the following will appear:* $\int(c,x,f(x),x)$ *(Indef. int.)*

Integration table

The integrals are already calculated and by inserting parameters the answer is given. The parameters are called a,b,c and four types of functions may be integrated: algebraic, exponential, logarithmic and trigonometric.

Algebraic.

Interface:



In this example the function $f(x) = (x^2 + 2x + 1)^{1/2}$ is integrated. We notice that the integrand equals the absolute value of $(x + 1)$. For other types of integrands the input is quite analogous, under logarithmic you may integrate $\ln(2x^2 + 3x-2)$ by choosing $a = 2$, $b = 3$ and $c = -2$ in the first menu line.

Definite integral

The calculation of a definite integral requires that the indefinite integral has been calculated already.

Interface:

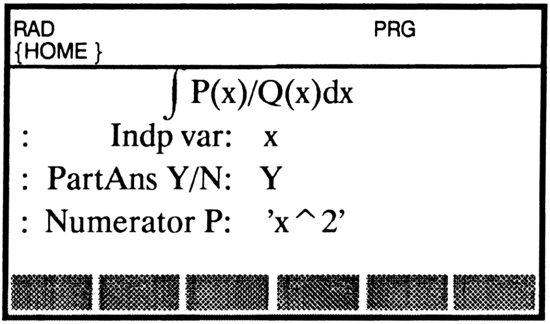
```
RAD                          PRG
{HOME }
         Limits a and b
:            a  b:   1 2
:        Indp var:   x
:      Indef. int.:  '1/2*x^2'
```

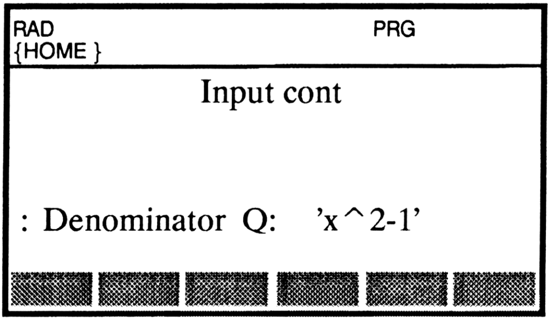
The example calculates $[\ 1/2*x^2]_0^1 = 1/2*2^2\text{-}1/2*1^2=3/2$

Integration by partial fractions

The routine for partial fractions integrates all rational integrands where the denominator is of 4th degree or lower. The way the splitting up is done initialy will show up when Part-Ans Y is selected.

Interface:





The example calculates $\int x^2/(x^2-1)dx$

PartAns Y is selected and three pages of information will tell you the steps through the calculation. The first page tells how the splitting up is done, the second page the values of the coefficients and the third page the final answer.

The routine for partial fractions integrates all rational functions with a denominator of 2nd, 3rd or 4th degree.

$$\frac{P(x)}{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}$$

One of the a_2, a_3 or a_4 must be different from 0. If the denominator is of 1st degree , direct integration may be used, eventualy in connection with polynomial division.

The integrand can not include parameters (only numeric coefficients)

If the roots of the denominator Q(x) is irrational or the integrand includes irrational coefficients, the answer will be given with two decimals. The answer is also laying on the stack and here the number of decimals can be altered by n FIX.

Rem.: The calculation of the coefficients can give numeric instability (rounding errors). A test will check if the answer is not good enough

If the whole answer cannot be seen in text modus (PartAns Y) you may use →STK to see the result on the stack.

Substitution

This integration routine solves integrals of the type

$$\int k*f(g(x))*g'(x)dx$$

f((g(x)) is a function with kernel g(x) and k is a constant. The substitution u = g(x) transforms the integral to the form

$$\int k*f(u)du$$

Other types of substutions are not included.

Interface:

RAD PRG
{HOME}

$\int k*f(g(x))*g'(x)dx$

: Indp var: x

: PartAns Y/N: Y

: Integrand: '3*x*SIN(x^2)'

RAD PRG
{HOME }

Input cont

: Subst. u=g(x): 'x^2'

The example solves $\int 3x*Sin(x^2)dx$

In the formula k=3/2 (3/2*2x=3x), g '(x)=2x, g(x)=x^2, and f(g(x))=Sin(x^2).

Rem.: Use of u as independent variable is prohibited and will automaticly be converted to x

The selection PartAns Y will give the new differential in terms of the initial independent variable, the integrand in terms of the new variable and finaly the answer in the initial independent variable.

Integration by parts

CALCULUS can use integration by parts repeatedly several times and, if the integral cannot be calculated directly, give an answer in terms of a new integral. The last situation will require the use of the integration package if possible or the integrand is simplified in connection with numeric integration.

Interface:

```
RAD                                   PRG
{HOME }
           ∫ u(x)*v(x)dx
:       Indp var:  x
:  PartAns Y/N:  Y
:              u :  'x^2'
```

```
RAD                                   PRG
{HOME }
             Input cont


: v :    'SIN(x)'
```

The example calculates

$$\int x^2 \mathrm{Sin}(x)dx$$

Integration by parts is being used two times here and this may be seen when PartAns Y is selected.

Rem. : Repeted integration is limited to 3 times

The routine for integration by parts integrates functions of the type

$$u*v$$

The integrand is composed of two factors u and v with separte input.

If an exact answer is to be found, all intergrations have to be solved directly (see direct integration). If not, two types of error messages will appear.

If v(x)*dx is exact integrated to V(x), but the integral of V(x)*du/dx*dx cannot be found, a partial integration in the true meaning of the word is done. The answer is given, but it includes a new integral eventualy to be solved by other methods. If v(x)*dx cannot be integrated by direct integration, only an error message is given (no answer).

Be aware that problems with integration by parts may be solved by changing the factors u and v.

Integrals of the type repeated integrands (the initial integrand is coming up), are not implemented in CALCULUS.

> *Rem. If an integrand only consists of one factor (fx. u), the other factor v may be set to 1. Ln(x) = Ln(x)*1.*

Moment of inertia (Applications)

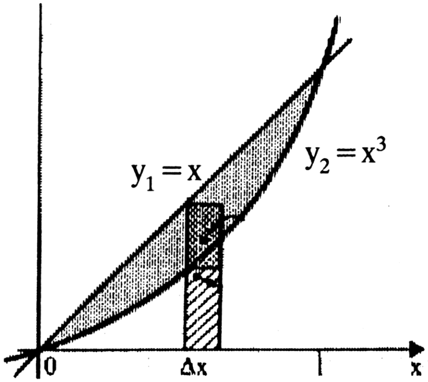
This routine calculates the moment of inertia of a plane region about the axis 'Y = 0' (x-axis) or an axis parallel to the y-axis ('X = A').

Interface:

```
RAD                                PRG
{HOME }
          Region bounded by
          f1,   f2,  x=x1, x=x2
: f1:        'x^3'
: f2:        x
```

```
RAD                                PRG
{HOME }
               Input cont
:          x1   x2:  0 1
:             Axes: 'X=-2'
:        Indp var:   x
```

```
RAD                                PRG
{HOME }
               Input cont



: PartAns Y/N:  Y
```



The example calculates the moment of inertia of the hatched region in the figure about the axis X = -2 (parallel to the y-axis). The region is bounded by two curves and two x-values. If the intersection between the curves is not known, it must be calculated in advance (ALGEBRA Solve equation).

By selecting PartAns Y, you will see the integral that determines the moment of inertia, the calculated indefinite integral and the expression for the definite integral, and the final answer.

Rem.: The antiderivative with limits: $(x1,x2,F(x),x) = [F(x)]_{x_1}^{x_2} = F(x2)-F(x1)$

If CALCULUS cannot integrate the expression directly, a numeric answer is given (**the integrand may not include symbolic paramters**) and you are told eventualy to proceed with another integration method. The integral to be calculated is given.

The formulas for the moment of inertia are given under the menu option Formula and a complete answer to a problem includes the formula.

Center of mass (Applications)

The calculation of center of mass requires that the static moment and the area of the plane region or the arc length of the curve, is calculated in advance.

Interface, plane region:

```
RAD                              PRG
{HOME }
            Static moments
          Mx and My, area A
          Xt=My/A  Yt=Mx/A
: Mx  My  A: '1/19'  '3/26'  '4/31'
```

In the example the center of mass for a plane region is calculated. The static moment about the x-axis is 1/19, about the y-axis 3/26 and the area of the region is 4/31. The answer is given in the form {'Xt= ' 'Yt= '}.

The input for the center of mass of a curve is similar, the static moments and the arc length must be calculated first.

Static moment (Applications)

The static moment (the moment) may be calculated about an axis parallel to either the x-axis or the y-axis. The axis are given as 'X=A' or 'Y=B'.

The interface for the moment of a plane region is identical to the interface for moment of inertia. The input is the same and the answer is coming up in the same manner.

Another option is the moment of a curve. The formula is given under the menu option Formula, and the formula includes the arc differential ds given under the menu selection Arc length.

Interface:

```
RAD                              PRG
{HOME }
      St. mom.  f(x)  x1 to x2


:         f:     'EXP(x)'
:   x1  x2:      0   2
```

```
RAD                              PRG
{HOME }
            Input cont
:          Axes:  'X=0'
:      Indp var:  x
: PartAns Y/N:  Y
```

In the example the moment of the arc of e^x between $x=0$ and $x=2$ is being calculated.

PartAns Y is selected and as you can see direct integration didn't work. The numeric value and the error bound is given. In this case no antiderivative can be found and the numeric value is the only possible answer.

Area (Applications)

The routine for Area calculates the area of a plane region bounded by given curves and x-values.

Interface:

```
RAD                                PRG
{HOME }
        Area bounded by f1(x),
          f2(x), x=x1, x=x2
: f1:      'x^3'
: f2:      x
```

```
RAD                                PRG
{HOME }
              Input cont
:         x1   x2:  0 1
        Indp var:   x
: PartAns Y/N:   N
```

The example calculates the area hatched on the figure under Moment of inertia. The answer is put directly on the stack.

Solid of revolution (Applications)

Selecting this menu option you may calculate the volume or the surface of a solid of revolution.

The interface for the volume of the solid of revolution is the same as the interface for moment of inertia. The example then calculates the volume when the area is rotated about the axis $x = -2$.

A surface of revolution is generated by an arc that is rotated about an axis. The interface is similar to the interface of static moment of curves. The example calculates the surface when the arc of e^x from $x = 0$ to $x = 2$ is rotated about the y-axis.

Arc length (Applications)

The interface is similar to the interface of static moment of a curve exept for the axis. A similar input gives the arc length of the arc of e^x from $x = 0$ to $x = 2$. The integrand has no primitive and a numeric answer is given.

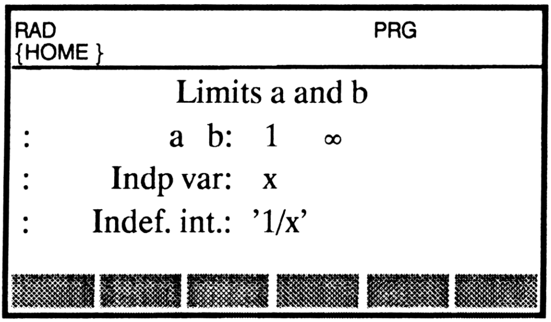
Improper integral

An improper integral is a definite integral where the limits are infinite or the integrand is discontinous in the interval of integration.

Such integrals have to be calculated as a limit. Only integrals where the limits are infinte are implemented in CALCULUS. Both limits may be infinite ($\pm\infty$).

The interface is similar to definite integrals, the indefinite integral has to calculated in advance.

Interface:



RAD PRG
{HOME }

Input cont

: PartAns Y/N : Y

The example calculates $\int_1^\infty (-1/x^2)dx = [1/x]_1^\infty =$

$\lim x \rightarrow \infty \quad 1/x - 1 = -1.$

Rem.: If the limit does not exist an error message is given and the integral is said to diverge

3

Algebra

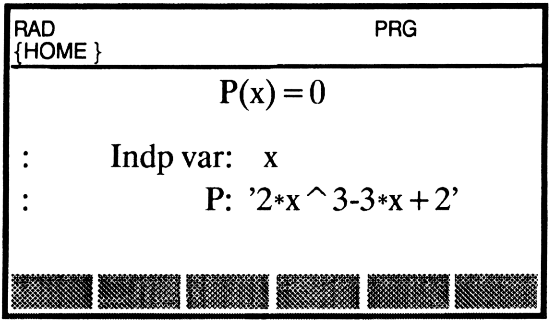
Under ALGEBRA you may simplify expressions, factorize, split up into partial fractions, solve equations and do polynomial division.

The use of parameters (other symbols than the independent variable) is limited, neither the routines for solving equations of 3rd or 4th degree nor partial fractions will allow this.

Roots of a polynomial

This routine finds the roots of a polynomial of 2nd, 3rd or 4th degree. For a polynomial of 2nd degree parameters may be included (symbolic coefficients)

Interface:



The example solves $P_3(x) = 2x^3 - 3x + 2 = 0$

Partial fractions

Here a rational function is split up into partial fractions with denominator of 1st degree or 2nd degree for complex roots. For multiple roots the denominator may be og a higher degree.

If the numerator is of same or higher degree than the denominator, polynomial division is executed first.

By selecting PartAns Y the initial splitting up i shown, the values of the coefficients and the final answer (method of undermined coefficients).

Interface:

```
RAD                                PRG
{HOME }
        P(x)/Q(x) = A/(x-x1) + ....

:        Indp var:  x
: PartAns Y/N:  Y
: Numerator P:  '2*x^3-3*x+2'
```

```
RAD                                PRG
{HOME }
              Input cont



: Denominator Q:  'x^3-x'
```

The example will split up $(2x^3-3x+2)/(x^3-x)$.

Factoring

The routine for factoring will factorize polynomials of 2nd, 3rd or 4th degree. Factors with complex roots will be given as completed squares, fx $(x^2+2*x+2)=(x+1)^2+1$.

Interface:

```
RAD                         PRG
{HOME }
        P(x) = (x-x1)*(x-x2)....

:Indp var:    x
:        P:   'x^3-3*x^2+x-3'
```

The example is factoring $P(x) = x^3-3x^2+x-3$.

Polynomial division

For rational functions where the numerator is of same or higher degree than the numerator, polynomial division will be executed. If this is not the case or the function is not a rational function an error message will be given.

Interface:

```
RAD                         PRG
{HOME }
     P(x)/Q(x) = f + R(x)/Q(x)....
:         P:  '2*x^3-3*x+2'
:         Q:    '(x^2-1)'
: Indp var:      x
```

In the example above $2x^3-3x+2 : x^2-1$ is executed.

Simplify

Interface:

```
RAD                         PRG
{HOME }
           Simplifying expr.

: Indp var:          x
:    Expr.:'(x^3-2)/(x-1/2)+2/x'
```

The example simplifies $(x^3-2)/(x-1/2)+2/x$.

This routine simplifies expressions by drawing together and/or factoring. The same types of expressions as under FUNCTIONS are handled, but the expression don't need to be rational, fx Sin(x)/x-2 = (Sin(x)-2x)/x.

Solve equation

The equation is written as RS = LS, where RS is the right side of the equation and LS the left side. The equation is transformed into RS-LS = 0 and the routine for zeros under FUNCTIONS is used.

Interface:

```
RAD                               PRG
{HOME }
          Equation LS = RS

:     Unknown:  x
:       LS = RS:  'x/(x-1) = a + 2'
```

In the example x/(x-1) = a + 2, constant a, is solved.

4

Functions

Under FUNCTIONS you may examine functions in one variable: sign of function, zeros, max/min- points, points of inflection, graphing, limits etc.

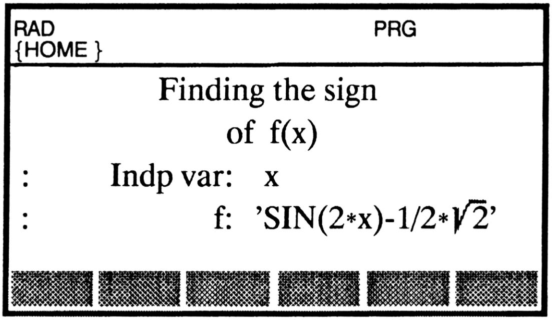
The types of functions that can be handled are:

- Rational functions, $f(x) = P(x)/Q(x)$, fx $(x^2-3)/(x^4+2*x-6)$, degree of $Q(x)$ and $P(x)$ must be less than 5.
- Binomials of rational functions $f_1(x) + f_2(x)$, fx $(x^3-2*x+7)/x + (x-1)/(x^2+3*x-1)$
- All types of functions where the unknown has a single occurrence, fx $Sin(2*x)-0.5$, $Ln(x)^2-3$, etc. Products and ratios of such functions are included: $(Sin(2*x)-0.5)/(Ln(x)^2-3)$

Sign of function

Here the sign of a function in different intervals will be examined. The zeros of the denominator and the numerator in rational functions and the zeros of the function in other cases, will be listed and labeled actual zeros.

Interface:



The example finds the sign of the function:

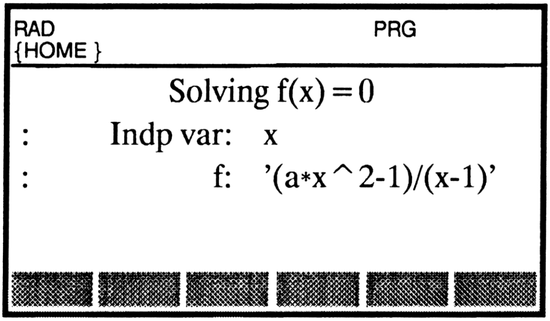
$$f(x) = \mathrm{Sin}(2*x)-1/2\sqrt{2}$$

The zeros are put in a matrix on the stack. You may have to adjust for the interval of definition.

Zeros

This routine finds the zeros for f(x), i.e. solves the equation f(x) = 0.

Interface:



The example solves the equation $(ax^2-1)/(x-1) = 0$.

Graphing

Draws the graph of a function. Graphing uses the intrinsic HP 48 graph routine, and to get a good picture you will sometimes have to use the ZOOM option for scaling. There is full access to the FCN menu for numeric calculations.

Interface:

```
RAD                              PRG
{HOME }
        Graphing {f1(x)  f2....}
         between x=x1 x=x2
:       Indp var:   x
:         x1  x2:    0 2
```

```
RAD                              PRG
{HOME }
             Input cont


:{f1  f2....}: '1/(x-1)^2+5*x'
```

The example draws the graph of $f(x) = 1/(x-1)^2 + 5x$. When the graph of only one function is going to be drawn you will not need the { }.

The graph has an asymptote for x = 1 and the first picture is of little information. We will use the ZOOM option to scale the axis, 10 in the y-direction and 4 in the x-direction.

Extremum points

This routine decides where the function is decreasing and where it is increasing and finds max/min points where the sign of the derivative is changing.

Interface:

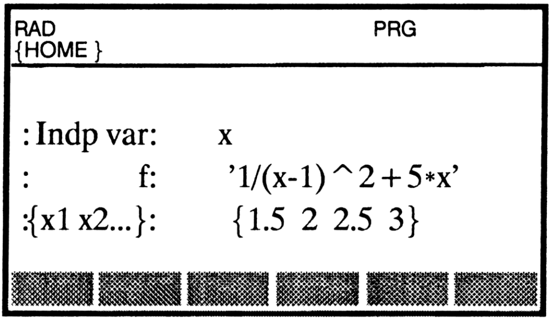
```
RAD                                PRG
{HOME }
         Increasing/decreasing
              function  f(x)
:        Indp var:  x
:               f:  '1/(x-1)^2+5*x'
```

The example examines the function $1/(x-1)^2 + 5x$ for max/min points and decreasing/increasing function. The calculation in this case is somewhat complicated and the time consumed is relative long. The derivative of f(x) is indicated. The extremum points are put on the stack in a matrix.

Function table

The routine will give you a possibility to calculate a function at given points (x-values). The answer on the stack is a two column matrix and can be viewed in the matrix writer by pushing the arrow key (down).

Interface:



The example calculates the function $1/(x-1)^2+5x$ at the points $x=1.5$, $x=2$, $x=2.5$ and $x=3$ and the answer is a matrix with x-values in the left column and y-values in the right column.

Limits

Limits are calculated with the use of L'Hospitals rule where the numerator and the denominator of a ratio is differentiated separately. This CALCULUS routine includes some complicated routines for simplifying expressions and the time consumed might be long.

The calculation is terminated after two rounds of differentiation and the message "Can't find the limit" will show up. If the limit does not exist, the message will be "Limit does not exist"

The following types of indeterminate forms are included:

- $0/0$
- ∞/∞
- $\infty - \infty$
- 0^0
- $0 * \infty$
- 1^∞
- ∞^0

Interface:

```
RAD                                 PRG
{HOME }

              lim  x→a    f(x)
:   Indp var   a :   x 0
:  PartAns Y/N:   Y
:                f:   'x^x'
```

The example calculates lim x→0 x^x

The selection Y in PartAns Y/N shows that you first have to take LN to the expression and then construct a ratio for L'Hopitals rule. The answer for the ratio is 0 and the final answer is $e^0 = 1$.

> *Rem.: In this version of CALCULUS you cannot find the limit when the expression includes parameters (other symbols than the indp var)*

Differentiation

The routine for differentiation finds dy/dx for $y = f(x)$ and for $F(x,y) = 0$ (implicit differentiation). You can choose if you want to find the expression for the derivative or the value at some point.

For ratios the fundamental differentiation rule is used explicit. The use of the intrinsic HP routine will give "an ugly" expression, not suitable for further calculations. The responding time is then longer.

Interface, explicit differentiation:

```
RAD                                 PRG
{HOME }
              y = f(x)
           calculates f '(xo)
: Indp var  xo:   x  x
: PartAns Y/N:     Y
```

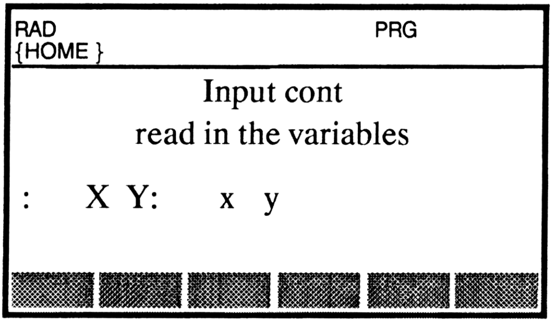
```
RAD                      PRG
{HOME }

        Input cont

: f:     '2*x/(x-1)^2'
```

The example differentiates $f(x) = 2x/(x-1)^2$ and intermediate results are shown when PartAns Y is selected.

When the expression for f(x) is going to be found, x0 is set to x, otherwise the number or symbol for x0.

Interface, implicit differentiation:

```
RAD                      PRG
{HOME }

     F(x,y) = 0,  dy/dx =
       (−∂F/∂x)/∂F/∂y
:  Point x0  y0:   x  y
:        F(x,y):  'x-3*y^2+3*x*y'
```



The example calculates dy/dx from $F(x,y) = x - 3y^2 + 3xy = 0$

When we want to calculate the expression for dy/dx, the input for x0 y0 is x y, otherwise the numbers or symbols for x0 and y0.

Point of inflection

Under this menu option the curvature is examined, for concave and convex curvature. When the curvature is changing there is a point of inflection.

Interface:

```
RAD                              PRG
{HOME }
         Points of inflection
                 f(x)
:        Indp var:  x
:               f:     'x/(x-2)^2'
```

The example calculates the points of inflection for $f(x) = x/(x-2)^2$ and indicates where the curvature is concave and convex. The double derivative and its zeros are also indicated. Points of inflection are put in a matrix on the stack.

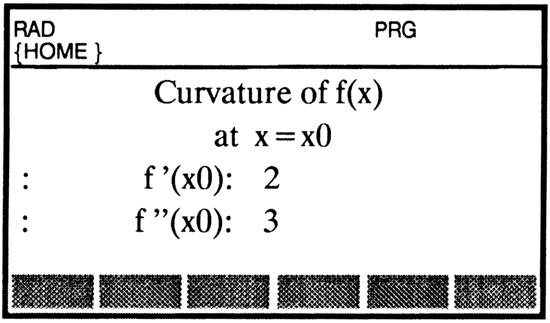
Curvature

The curvature indicates the "degree of sudden turns" in the graph of f(x). A small value indicates a slack curve and the sign determines whether the curve is convex or concave.

The routine for curvature requires that f '(x0) and f ''(x0) are calculated in advance: First find the symbolic derivative, DUP the expression and find f '(x) at the given point using

WH. Then load f '(x) (ECHO from stack) and find f "(x) at the given point using the differentiation routine once more.

Interface:



The example calculates the curvature at a point on the curve where f ' = 2 and f " = 3. The formula for curvature can be found under the menu option Formula.

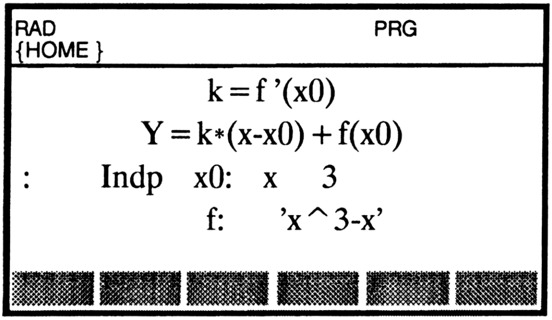
As a curiosity exact values are implemented (exact arithmetic is generaly not implemented).

> *Rem.: The radius of curvature is the inverse of curvature*

Tangent line

The equation for the tangent line at a given point is calculated. The formula can be seen in the input screen.

Interface:



The example calculates the tangent line to the function x^3-x at the point x=3.

5

Linear equations

Two options are avaiable: solving a general system of equations and solving an ordered system with numeric coefficients (numeric coefficient matrix). In an ordered system the coefficient matrix is known:

$$
\begin{aligned}
2x+3y-z&=5\\
x-3y+z&=2\\
3x-3y-2z&=1
\end{aligned}
$$

The coefficient matrix (A) is here [[2 3 -1][1 -3 1][3 -3 -2]] and the right side (B) is [5 2 1]. The system will be singular if there is no solution or an infinte number of solutions and this depends on the coefficient matrix A.

A general, nonordered system looks like this:

$$
\begin{aligned}
2x+3y-z &= 5-2x+y\\
x-3y+z &= 2+x\\
3x-3y-2z &= 1-y
\end{aligned}
$$

Ordered systems will be solved using the intrinsic HP 48 routine if the matrix A is pure numeric. For nonordered systems and systems with parameters, the routine for a general system will be used.

General system

Here the equations may be given in arbitrary form including symbolic parameters. A general elimination technic is used and an error message is given for singular systems.

Singular systems include the case where no solution is possible (selfcontradictory system) or there are an infinte number of solutions (indefinite system).

Interface:

```
RAD                                PRG
{HOME }
        Equations {L1  L2.....}
         Unknown  {x  y. ...}.

:        {x y...}:    {x  y}
:   {L1 L2...}: {'x-y=3' 'x+y=4'}
```

The following system is solved:

$$
\begin{aligned}
x - y &= 3 \\
x + y &= 4
\end{aligned}
$$

This is in fact an ordered system but shows the input environment. Variants of the type $x+y=a$, $x-2y=3y-x+5$ etc. are possible. Remember blanks between the equations in {L1 L2....}.

Ordered system

The coefficient matrix and the right side now has to be known and only numeric elements are possible. This routine is much faster than the general routine. If the system do not include symbolic parameters an idea might be to order the system by hand.

Interface:

```
RAD                              PRG
{HOME }
            [[A]]*X=[B]
:     [B]:     [3  4]
:     {X}:     { x  y}
:   [[A]]:     [[1 -1][1  1]]
```

The example solves the system given under general system. If the system is singular an error message will be given.

As opposed to the use of the intrinsic matrix division routine directly from the keyboard, CALCULUS will transform rational solutions to fractions and in addition label the solution.

6

2D Curves

Curves are not necessarily the graph of a function. This routine handles this kind of curves and is given either as a parametric equation or in polar coordinates.

Parametric equation

Here two variables (say x and y) are given in terms of a third variable, a parameter (t). The direct connection between x and y (parameter eliminated) is often not easy to find.

Area (Parametric equation)

The area bounded by the curve or the curve and the x-axis between two t-values is calculated.

Interface:

```
RAD                                PRG
{HOME }
        Area bounded by  (X,Y),
           between t1 and t2
: X(t):     't^3'
: Y(t):  'SIN(t)'
```

```
RAD                                PRG
{HOME }
               Input cont
:             t1  t2: 0   1
:         Indp var:   t
: PartAns Y/N:   Y
```

The example calculates the area between the curve $X(t) = t^3$, $Y(t) = \text{Sin}(t)$ between $t = 0$ and $t = 1$ and the x-axis.

t-x-y table (Parametric equation)

Mutual dependent values for x and y are calculated for given values of the parameter (t). The table is placed on the stack as a matrix with 3 columns (t-x-y).

Interface:

```
RAD                              PRG
{HOME }
:      X(t):      't^2'
:      Y(t):     'COS(t)'
: Indp var:       t
:{t1 t2...}:     {1 2 3 4}
```

The example calculates the table for $X(t) = t^2$, $Y(t) = Cos(t)$ with parameter values $t = 1,2,3,4$.

Geometry (Parametric equation)

Under this option you may calculate the curvature of a curve and the tangent line at a given point.

Curvature

Here X'(t0), Y'(t0), X"(t0) and Y"(t0) must be calculated in advance under FUNCTIONS/Differentiation.

Interface:

```
RAD                                  PRG
{HOME }
           Curvature of
        X(t)  Y(t)  at t=t0
:        X '(t0):  2
:        X "(t0):  3
```

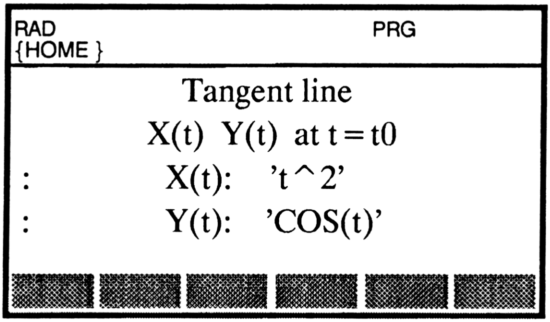
```
RAD                                  PRG
{HOME }
            Input cont

:         Y'(t0):  1
:         Y"(t0):  3
```

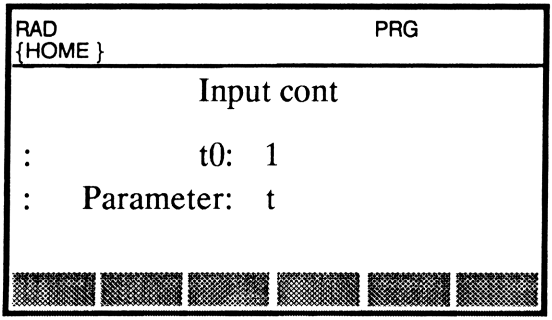
The example calculates the curvature of X(t), Y(t) at the point t=t0 and where X'(t0)=2, X"(t0)=3, Y'(t0)=1, Y"(t0)=3.

Tangent line

Here the tangent line relative to a cartesian coordinate system is calulated at a given point.

Interface:





The example calculates the tangent line to the curve $X(t) = t^2$, $Y(t) = \text{Cos}(t)$ at $t = 1$.

Extremum points (Parametric equation)

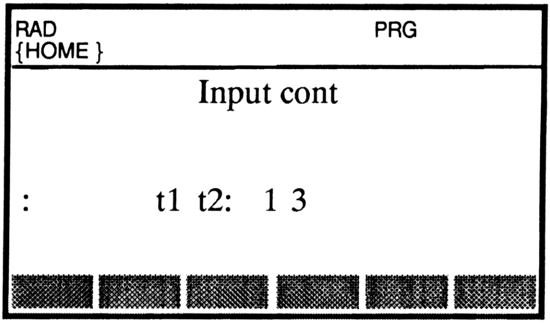
The information is given that extremum points have to be calculated under FUNCTIONS/Extremum points. Be aware of the possibility of singular points (break in the curve).

Graphing (Parametric equation)

Here the HP 48 intrinsic parametric option is used to draw a parametric curve, '(X(t),Y(t))'. Be aware of the use of ' '.

Interface:

```
RAD                                   PRG
{HOME }
          Graphing  ' (X,Y)'
        between t = t1 and t = t2
:        '(X,Y)':       '(t^3,SIN(t))'
:     Parameter:        t
```



The example draws the curve $X(t) = t^3$, $Y(t) = Sin(t)$ for t-values from $t1 = 1$ to $t2 = 3$.

Polar coordinates

In polar coordinates the curve is given as $r = f(\Theta)$. Θ is the angle between the polar axis and the radius vector r. Polar coordinates may easily be transformed into a parametric equation where $X = r*Cos\Theta$, $Y = r*Sin\Theta$. Θ is the parameter.

Area (Polar coordinates)

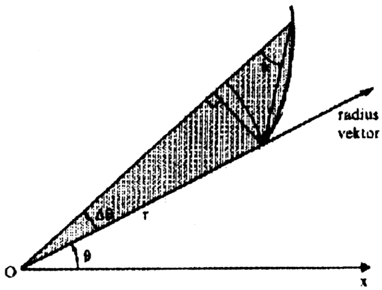
Radius vector "sweeps out" an area as Θ varies. The area may be calculated between two Θ-values.

Interface:

```
RAD                                PRG
{HOME }
      Area bounded by r(Θ),
       between Θ1 and Θ2

: r(Θ):    'Θ^2'
```

```
RAD                                PRG
{HOME }
           Input cont
:          Θ1 Θ2: 0.5 0.7
:        Indp var: Θ
: PartAns Y/N: Y
```

Calculating the area of $r(\Theta) = \Theta^2$ from $\Theta = 0.5$ to 0.7.



Θ r(Θ) table (Polar coordinates)

Mutual dependent values of r and Θ for specific Θ-values (in radians) are calculated.

Interface:

```
RAD                          PRG
{HOME }

:          r(Θ):  'COS(Θ)'
:      Indp var:  Θ
:    {Θ1 Θ2..}:   {1 2 3 4}
```

The example calculates the table for $r(\Theta)=\mathrm{Cos}(\Theta)$ for $\Theta=1,2,3,4$.

Extremum points (Polar coordinates)

Extremum values for f may be found under FUNCTIONS/ extremum points.

Graphing (Polar coordinates)

The intrinsic HP 48 polar graphing routine is used to draw the graph of the curve.

Interface:

RAD PRG
{HOME }

Graphing r(Θ)

between Θ = Θ1 and Θ = Θ2

: r(Θ): 'Θ^2'

: Indp var (Θ): Θ

RAD PRG
{HOME }

Input cont

: Θ1 Θ2: 1 2

The example draws the curve $r(\Theta) = \Theta^2$ between $\Theta = 1$ and $\Theta = 2$ (RAD).

> *Rem.: The symbol for the free variable need not to be* Θ.

Cartesian coordinates (Polar coordinates)

Information about the transformation from polar to cartesian coordinates (i.e. parametric equations).

7

Series

Here intervals of convergence, the terms of the series and sums and ratios of geometric series are calculated. Linear combinations of geometric series are also included to some extent.

Maclaurin series

The Maclaurin series of a function may be terminated after a number of terms and will then give a good approximation to the function around $x = 0$.

The routine for Maclaurin series determines a specific number of terms. Under the menu option Convergence/Ratio test the interval of convergence will be calculated.

Interface:

```
RAD                                PRG
{HOME}
         Macl.series for f(x)
            n'th power in x
:    Indp var  n:    x 3
:                f: '1/(1-x)'
```

In the example four terms in the series for f(x) = 1/(1-x) are calculated (to the third power in x).

> *Rem.: $f^{(n)}(0)$ must exist (n-th derivative included f(0))*

Taylor series

This is corresponding to the Maclaurin series but will give a good approximation about some point x = a (a = 0 will give the Maclaurin series).

Interface:

```
RAD                                PRG
{HOME}
        Taylorseries for f(x)
        about x=a n'th power
:  Indp var  a n:    x 3  3
:               f: '1/(1-x)'
```

The example calculates the Taylor series for f(x) = 1/(x-1) about x = 3 to the third power in x.

> *Rem.: $f^{(n)}(a)$ must exist (n-th derivative included f(a))*

Binomial series

This routine calculates the Maclaurin series for a binomial. A binomial is an addition of two terms. In this case the formulas for the Maclaurin series may be simplified and the general term does not include the derivate of the function. The binomial must be given as $f(x) = (k1 + (k2*x)^n)^m = (a + b)^m$ where k1 and k2 is independent of x, n is a positive integer.

Interface:

```
RAD                              PRG
{HOME }
        Binomial (a+b)^m
        n  is num. of terms
:          n:     3
: (a+b)^m: '(2-x^2/2)^(1/2)'
```

The example calculates the Maclaurin series for the binomial $f(x) = (2-x^2/2)^{1/2}$ ($a = 2$, $b = -x^2/2$ and $m = 1/2$).

The reason for not using the general Maclaurin routine lies in the fact that for binomials there is a simple formula for the terms of the series not including the derivative of the binomial.

> *Rem.: Remember to write square roots etc as a power with fractional exponent m.*

Geometric series

The sum and the ratio of the series are calculated. If the series is infinte the ratio has to be less than 1 in absolute value. The series might be a sum of two series. If the series is not geometric, an error message is given (" Not geometric") and if the general term is too complex (fx the sum of three separate geometric series) the error message "Can't Find" will appear.

Sum n terms

The sum of a finite geometric series is calculated. The general term of the series is given and by selecting PartAns Y both the ratio and the sum will show up in the answer.

Interface:

```
RAD                               PRG
{HOME }
      Series ∑An, k = An + 1/An
       Sm = Ao*(k ^ m-1)/(k-1)
: Num.terms m:  3
:           An:   '3*(2 ^ n-3 ^ n)'
```

```
RAD                          PRG
{HOME }
            Input cont
:   Sumindex n:   n
:      n start:   0
: PartAns Y/N:    Y
```

The example calculates the sum of the series:

$$\sum_{n=0}^{3} 3*(2^n-3^n)$$

> *Rem.: n indicates the sumindex. Another symbol fx k could be used with $3*(2^k-3^k)$ as Ak, the general term.*

Sum ∞

The sum of an infinite series is calculated. The ratio k must be less than 1 in absolute value: $k^2 < 1$, otherwise the series diverges.

Interface:

```
RAD                                PRG
{HOME }
        Series ∑An, k=An+1/An
          S∞ =Ao/(1-k), k^2<1

: An:     '(2^n-3^n)/4^n'
```

```
RAD                                PRG
{HOME }
               Input cont
:   Sumindex n:   n
:        n start:  0
: PartAns Y/N:   Y
```

The example calculates the sum

$$\sum_{n=0}^{\infty} (2^n-3^n)/4^n$$

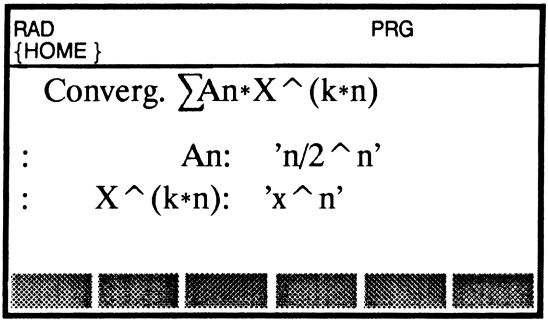
If $k^2 \geq 1$, the error message "Divergent" will appear.

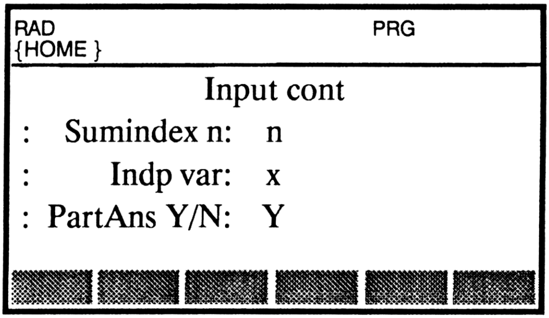
Convergence

Ratio test

The ratio test is frequently used for examination of convergence. A limit has to be calculated. See menu option Info.

Interface:





The example examines convergence for the series

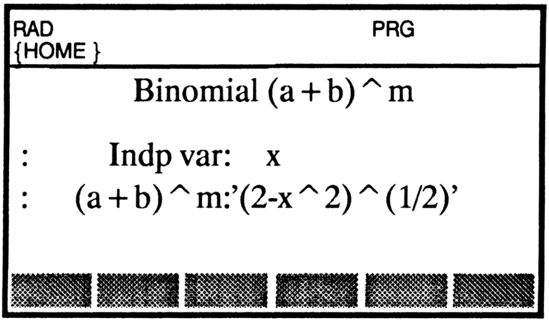
$$\sum_{n=0}^{\infty} n/2^n * x^n$$

> *Rem.: n is summation index, if k is used, Ak is given as $k/2^k$*

Binomial series

Convergence of a binomial series may be evaluated genera-ly using the ratio test and the result may be used to test for convergence in specific cases. In the binomial $(a+b)^m$, b/a need to be less than one in absolute value.

Interface:

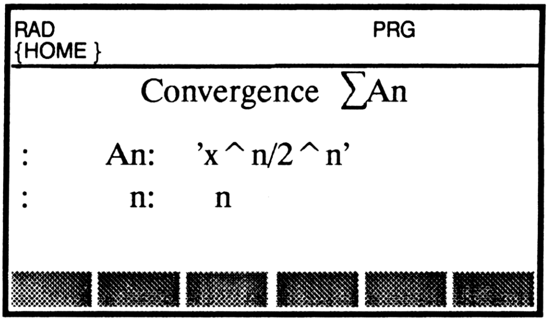


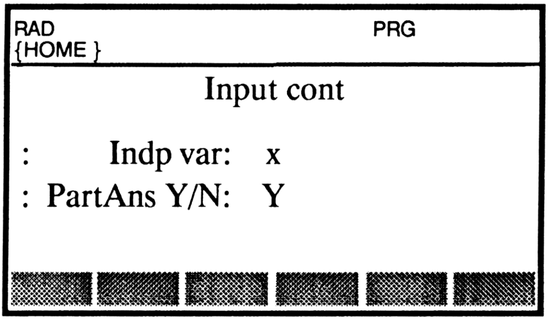
The example evaluates convergence for the Maclaurin series for $f(x) = (2-x^2)^{1/2}$.

Geometric series

A geometric series converges when the ratio k is less than 1 in absolute value, $k^2 < 1$.

Interface:





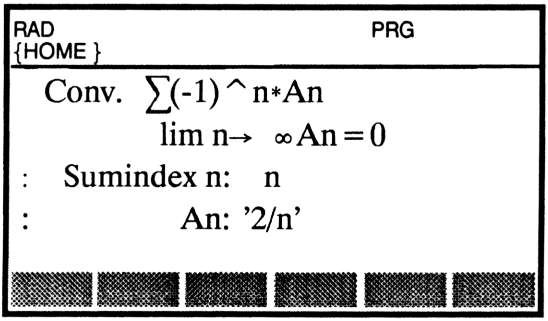
The example evaluates convergence for the series

$$\sum_{n=0}^{\infty} x^n/2^n$$

Leibnitz test

Leibnitz test may be used for alternating series, and is especialy useful at the endpoints of the convergence interval for power series. The ratio test will not give any answer for the endponts.

Interface:



The example evaluates the convergence for

$$\sum_{n=1}^{\infty} (-1)^n * 2/n$$

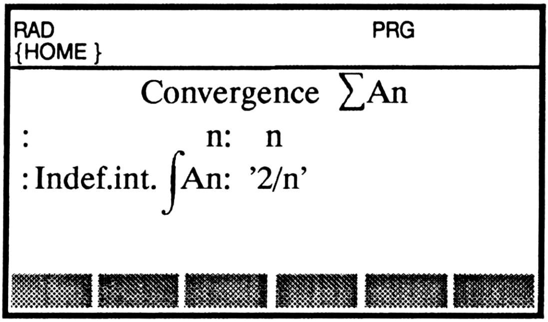
Integral test

Here we will look at A_n as a continous function in n:

$$\int_1^{\infty} An * dn.$$

A_n is the genral term of the series. If this integral converges than the series will converge and similary for divergence.

Interface:



The example evaluates convergence for

$$\sum_{n=1}^{\infty} (-2/n^2)$$

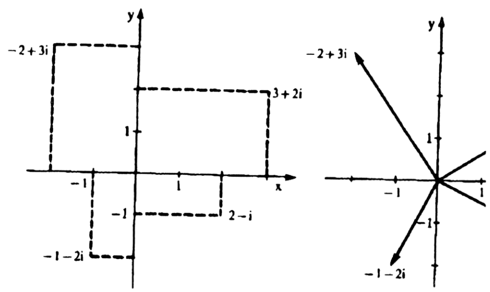
8

Complex numbers

The routines for complex numbers are the intrinsic HP 48 routines, but they are put into a menu system and some important formulas appear in the input screen.

Rem.: Only numeric complex numbers are implemented

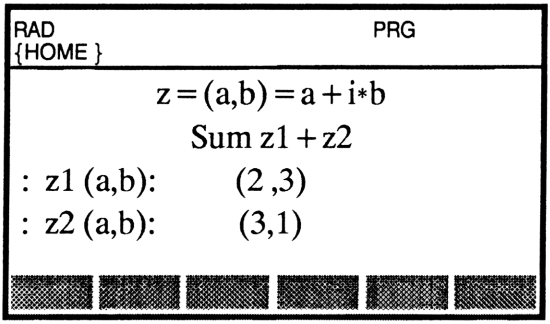
The form a + i*b (Rectangular coordinates)



Sum

Finding the sum of two complex numbers means to find the sum of the real and the imaginary parts separately.

Interface:



The example calculates (2 + 3i) + (3 + i) = 5 + 4i

Product

The product of two compelx numbers is found by multiplying (a1 + i*b1)*(a2 + i*b2) as two ordinary binomials. This gives (a1*a2-b1*b2) + i*(b1*a2 + a1*b2). The interface is similar to the interface above.

Fraction

The ratio between two complex numbers is found by first making the denominator of the fraction real. The numerator will then be a product of two complex numbers.

Interface:

```
RAD                         PRG
{HOME }
     z=(a,b)=a+i*b  z1/z2=
       (a1,b1)*(a2,b2)/
        (a2^2+b2^2)

: (a1,b1) (a2,b2) :   (3,1) (2,4)
```

The example calculates $(3+i)/(2+4i)$

Absolute value

The absolute value of a complex number corresponds to the length of radius vector.

Interface:

```
RAD                         PRG
{HOME }
        z=(a,b)=a+i*b
          ABS(z)=
      (a^2+b^2)^(1/2)

: z (a,b) :      (3,1)
```

The example calculates the absolute value of $3+i$.

Euler form

This routine writes a complex number using the exponential function. This gives a great advantage when finding products, ratios and powers of complex numbers.

Interface:

```
RAD                                PRG
{HOME }
            z=(a,b)=a+i*b
           z=r*EXP(iΘ)=
        r*(COS(Θ)+i*SIN(Θ))

: z (a,b) :        (3,1)
```

The example writes $3+i$ in the form $r*e^{i\Theta}$.

Polar form

A complex number in rectangular form is written in polar form. Polar form is much like the Euler form in that the argument (Θ) and the absolute value (r) is used explicitly in the expression.

Polar form is widely used in the electronic sciences: the absolute value indicates the value of the answer and the argument the phase shift (time shift).

Interface:

```
RAD                               PRG
{HOME}
          z=(a,b)=a+i*b
             z=(r,∠Θ)
      r=ABS(z) Θ=ATAN(b/a)

: z (a,b) :        (3,1)
```

The example transforms 3 + i into polar form.

Powers

This routine calculates $z1^{z2}$, the exponent may be complex. If z2 is a rational number (fraction) square roots etc will be calculated.

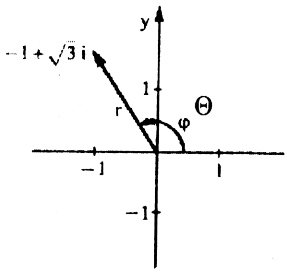
Interface:

```
RAD                               PRG
{HOME}
          z=(a,b)=a+i*b
             z=z1^z2
: z1 (a,b):       (2 ,3)
: z2 (a,b):         3
```

The example calculates $(3+i)^3$.

Polar form

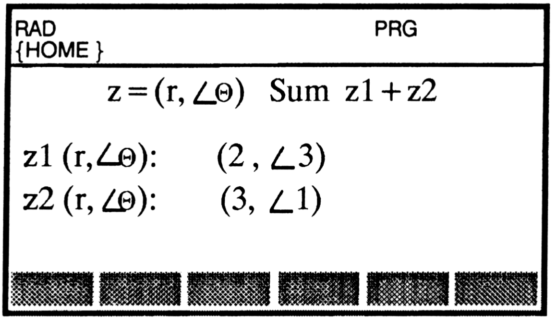
Polar form uses the angle Θ between the polar axis and radius vector r to determine a complex number written $(r, \angle\Theta)$.



Sum

The calculation of the sum by hand is rather complicated but HP 48 makes it an easy task.

Interface:

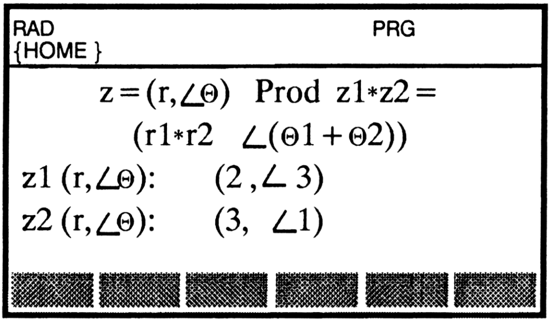


The example calculates $(2, \angle 3) + (3, \angle 1)$.

Product

The product is very easy to find using polar form: the absolute values are multiplied and the arguments are added.

Interface:

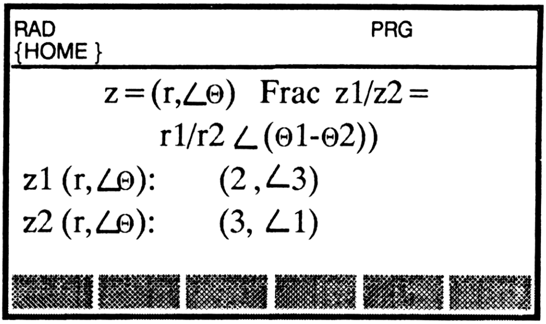


The example calculates $(2, \angle 3)*(3, \angle 1)$.

Fraction

Finding a fraction between two complex numbers in polar form is simply done by dividing the absolute values and subtracting the arguments.

Interface:



The example calculates (2, ∠3)/(3, ∠1).

Absolute value

The absolute vaule of a complex number in polar form is the length of the radius vector r, ABS(z) = r.

Euler form

Euler form writes a complex numer in polar form by the use of the exponential function, $z = re^{i\Theta}$. z has to be a complex number, otherwise an error message will be given.

a + ib form

A complex number in polar form is written i rectangular form.

Powers

Finding powers of complex numbers in polar form is simply done by raising the absolute value and multiplying the argument, both with the exponent of the power.

Interface:

```
RAD                                PRG
{HOME }
        z=(r,∠Θ)  z=z1^n=
             r1^n∠(n*Θ1))
z1 (r, ∠Θ):     (2,∠3)
 :         n:        3
```

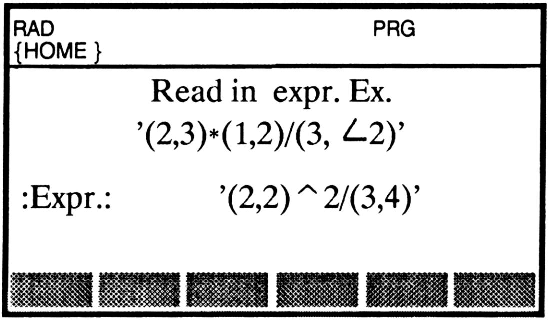
The example calculates $(2, \angle 3)^3$.

> *Rem.: If n is not an integer only the principal value is found.*

Expression

Here expressions are calculated and the components of the expression may be given in mixed polar and rectangular form. Be aware the use of '( )'.

Interface:



The example calculates $(2,3)^2/(3,4)$.

9

Functions of several variables

Here partial derivates are calculated and some applications of partial derivates are included: total differentials and increment estimation (error estimation).

When the differential of a variable is replaced by the increment we will get an expression for the total change in the function if the increment is small. The expression will include the increments of all the free variables of the function.

This may be used for calculating absolute and relative errors for an object described by a function when the function depends on several variables.

Partial differentiation

Partial derivatives may be calculated at some point (x0,y0....) or generaly at the point (x,y...). In the last case the expression for the partial derivative is calculated.

Interface:

```
RAD                                PRG
{HOME }
          f(x,y...)  calculates
          ∂f/∂var   at {xo,yo...}
:    {xo,yo...}: {x,y}
:              f:'x^2+x*y'
```

```
RAD                                PRG
{HOME }
            Input cont
: PartAns Y/N:  Y
:         Var:  x
:     {x,y...}: {x,y}
```

The example calculates $\partial f/\partial x$ for $f(x,y) = x^2 + xy$. Here the formula for the partial derivate is calculated because {x,y} is used for {xo,yo...}. Substituting figures fx {1,2} will give the value of the derivate at the point $x = 1$, $y = 2$.

Total differential

The input is similar to the input under partial derivate and PartAns Y may be selected. Then both the formula for the differential and the value at some point will show up in the answer.

Increment estimation

Here the differentials are replaced by the increment in each variable and this will give a good approximation for the total change of an experssion in several variables when the increments are small.

Interface:

```
RAD                                PRG
{HOME }
        Δf = ∂f/∂x*Δx + ∂f/∂y*Δy
          + ... point {xo yo...}
:     {xo,yo...}:   {1 2}
:                f:'x^2+x*y'
```

```
RAD                                PRG
{HOME }
              Input cont
: PartAns Y/N:   Y
:           {x,y...}: {x,y}
:     {Δx  Δy...}:  0.1  0.2
```

The example calculates the increment Δf in $f(x,y) = x^2 + xy$, when the increment in x is $\Delta x = 0.1$ and in y, $\Delta y = 0.2$ at the ponit $x = 1$ and $y = 2$. If the general expression for the incremnet is wanted you will have to write {Δx,Δy} for {Δx,Δy...}.

Absolute error

By taking the absolute value in the formula for increment estimation, the absloute error will be estimated. This is the maximum error we can get when the errors in x,y,... are known (all errors are in the same direction).

Interface:

```
RAD                                PRG
{HOME}
        Δf=ABS(∂f/∂x*Δx)+...
            point {xo yo...}
:      {xo,yo...}:   {1 2}
:               f:'x^2+x*y'
```

```
RAD                                PRG
{HOME}
              Input cont
: PartAns Y/N:  Y
:        {x,y...}: {x,y}
:     {Δx Δy...}: 0.1 0.2
```

The example calculates the maximum absolute error for $f(x,y) = x^2 + xy$ for error in x, $\Delta x = 0.1$ and error in y, $\Delta y = 0.2$.

Relative error

Maximum relative error is given by the absolute error divided by the value at the given point.

Interface:

```
RAD                                PRG
{HOME}
      Δf/f=ABS(∂f/∂x*Δx)/f+...
           point {xo yo...}
:     {xo,yo...}:   {1 2}
:              f:'x^2+x*y'
```

```
RAD                                PRG
{HOME}
              Input cont
: PartAns Y/N:   Y
:         {x,y...}: {x,y}
:    {Δx  Δy...}:  0.1  0.2
```

The example calculates $\Delta f/f$ for $f(x,y) = x^2 + xy$ at the point $x = 1$, $y = 2$, and the absolute value of all contributions are taken.

Rate of change

The rate of change is the same as the total derivative at a given point.

Interface:

```
RAD                         PRG
{HOME }
      df/dt = ∂f/∂x*(dx/dt) + ...
      at {xo...} and {Dxo...}
:     {xo,yo...}:  {1 2}
:              f:'x^2+x*y'
```

```
RAD                         PRG
{HOME }
            Input cont

: PartAns Y/N :  Y
:          {x,y...}: {x,y}
: {Dxo  Dyo...}:{2 4}
```

The example calculates the total derivative (rate of change) at the point (1,2) for $f(x,y) = x^2 + xy$ where dx/dt at the point is 2 and dy/dt at the point is 4.

If we want to express the total derivative by dx/dt, dy/dt ..., {Dx,Dy...} is written for {Dxo, Dyo...}.

10

Diffequations

Under diffequations 1st order separable, 1st order linear and 2nd order linear equations with constant coefficients are treated. For 2nd order equations two methods are implemented: method of undetermined coefficients and Lagrange method.

Linear 1st order

A linear equation of 1st order is of the form $y' + P(x)*y = Q(x)$. A closed form solution is always possible, but the an exact solution will require that the integration goes well.

General solution
Here the general solution with an integration constant is found. The constant may take any value.

Interface:

```
RAD                         PRG
{HOME }
        y' + P(x)*y = Q(x)
:     Indp var:  x
:            P:  2
:            Q:  x
```

The solution of y' + 2y = x is found. CALCULUS could not find the integral $e^{2x}*x$ directly and the integration package will have to be used.

Initial value problem

The general solution and the initial value y(x0) have to be known.

Interface:

```
RAD                         PRG
{HOME }
     General solution  Y(X)
          Y(X0) = Y0
:          X0:   0
:           Y:   'C*Exp(2*X)'
```

```
RAD                                   PRG
{HOME }
              Input cont
:          Y0:  2
:    Indp var:  x
: PartAns Y/N:  Y
```

The example adapts $y(x) = C*e^{2x}$ to the condition $y(0) = 2$.

Rem.: The constant must have the symbol C

Separable

This routine is solving 1st order separable equations. The equation need not to be linear but it must be possible to put in the form

$$f(y)*dy = g(x)*dx$$

Initialy the equation might be given in different forms but you need to bring in the form $F(x,y,y') = 0$. The equation $y'/(2y^2) = x$ is then written $y'/(2y^2)-x = 0$.

> *Rem.: y' is written Dy*

Interface:

```
RAD                              PRG
{HOME }
            F(x,y,y') = 0
       y' must be written Dy

:           x y:   x y
:             F:     'Dy/(2*y^2)-x'
```

```
RAD                              PRG
{HOME }
            Input cont

: PartAns Y/N:   Y
```

The example solves $y'/(2y^2)-x = 0$.

Rem.: Initial value problems are solved using the menu option 1st order linear/intial value

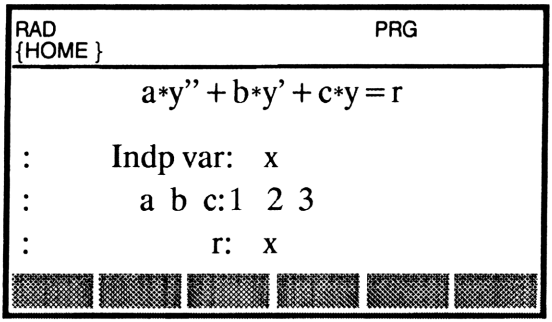
Linear 2nd order

Two methods are available. Lagrange method is the most general, variable coefficients are allowed here, but an "ugly" integral may appear.

The other method is the method of undetermined coefficients. Here the coefficients must be constant and there are certain conditions layed on the right side of the equation: only polynomials, exponentials and trigonometrics are allowed.

Undetermined coefficients

Interface:



```
RAD                     PRG
{HOME}
        Input cont

: PartAns Y/N:  Y
```

The example solves y" + 2y' + 3y = x. By selecting PartAns Y the choice for the particular solution and the homogenous solution will show up as well as the complete answer.

Initial value

Initial value problems requires that the general solution has been found in advance, and further that the value of the function and its derivative are known at the starting point.

Interface:

```
RAD                     PRG
{HOME}
      Gen. solution  Y(X)
   Y(X0) = Y0, y'(X0) = DY0
:Indp var:      x
:       Y: 'A*Sin(x) + B*Cos(x)'
```

```
RAD                                PRG
{HOME }
              Input cont
:              x0:  0
:         Y0 DY0:  1 0
: PartAns Y/N:  Y
```

The example adapts A and B in ASin(x) + BCos(x) to the initial condition y(0) = 1 and y'(0) = 0.

> *Rem.: The constants must have the symbols A and B*

Lagrange method

The coefficients of the equation may be variables and the right side an arbitrary function.

However, a complicated integral is likely to appear, and the method of undertermined coefficients is better where it can be used.

Interface:

```
RAD                        PRG
{HOME }
        a*y" + b*y' + c*y = r

:     Indp var:  x
:      a b c:1  2 -3
:           r:  x
```

```
RAD                        PRG
{HOME }
           Input cont


: PartAns Y/N:  Y
```

The example solves the equation y" + 2y'-3y = x. CALCULUS could not integrate $x/4*e^{-x}$ or $-x/4*e^{3x}$ directly, and the integration package has to be used (integration by parts).

Applications

Here different mathematical models are considered, and we restrict ourselves to 1st order problems. The notion "Rate of change" is central. Applications of 2nd order equations are not considered because it requires much more of knowledge from other fields than mathematics.

Rate of change

Here the rate of change or the "speed of change" is calculated (the derivative with respect to time dY/dt). The dependence on Y of time t may be implicit.

If fx the volume of a sphere is changing with time t, the radius R of the sphere may not be known as a function of time and the answer will then include dR/dt.

> *Rem.: If there is only one variable and the depence of t is explicit, the routine for derivation under FUNCTIONS has to be used.*

Interface:

```
RAD                                  PRG
{HOME }
            V=f(R(t))
          dV/dt=df/dR*dR/dt
         R=R0, dR/dt=Dt0

:V=f(R):'V=4*π/3*R^3'
```

```
RAD                                  PRG
{HOME }
             Input cont

: PartAns Y/N:  Y
:      Var (R):  R
:      R0 Dt0:  R   Dt
```

The example calculates dV/dt in terms of dR/dt where $V=4\pi R^3/3$. If Dt0 is a number fx 4, we will get the rate of change where dR/dt=4. If R0 is a number, fx 2 you put R0=2.

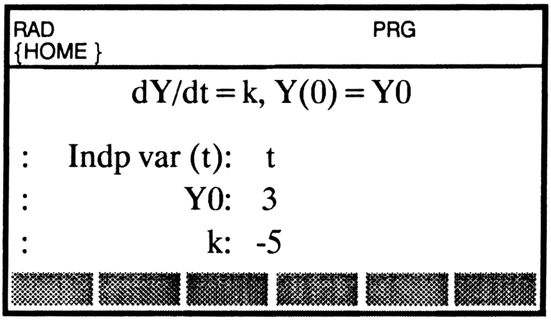
Mathematical models (Applications)

Here some common models where 1st order diffequations are used are calculated.

Linear modell

Mathematical model with constant rate of change gives $dY/dt = k$ and $Y = Y0 + kt$, where Y0 is Y(0).

Interface:



Solves the equation $y' = -5$, $y(0) = 3$.

Exponential model

Here the relative rate of change is constant, 1/Y∗dY/dt = k. This gives $Y = Y0*e^{kt}$ where Y0 = Y(0).

A variant of the exponential model is a model where the rate of change is proportional to "free capacity", that is what is left before reaching a max or a min value. This gives dY/dt = k∗(p-Y), where p is the max or min value, solution $Y = p + (Y0-p)e^{-kt}$.

Interface:

```
RAD                                    PRG
{HOME }
        dY/dt = k(p-Y), Y(0) = Y0

:   Indp var (t):   t
:             Y0:   3
:            p k:   2    -2
```

The example solves the equation y ' = 2y-4

> *Rem. p = 0 gives a pure exponential model*

Logistic model

In this model the relative rate of change is proportional to free capacity $1/N*dN/dt = k(B-N)$, B is the max value (full capacity) eventualy (depends on N0) min value. The solution will be $y = B/(1 + Ce^{-kBt})$, where $C = B/N0-1$, $N0 = N(0)$.

Interface:

```
RAD                              PRG
{HOME }
   1/N*dN/dt = k(B-N), N(0) = N0

:  Indp var (t):  t
:            N0:  3
:           B k:  2   -2
```

The example solves the equation $N' = -4N + 2N^2$ (not linear but separable).

Allometric model

Here relative rate of change for two variables X and Y are proportional, $1/Y*dY/dt = k*(1/X*dX/dt)$. This gives:

$$dY/Y = k*dX/X, \; Y = Y0/X0^k*X^k$$

Interface:

```
RAD                              PRG
{HOME }
        1/Y*dY/dt = k*1/X*dX/dt
          Y(0) = Y0 X(0) = X0
:         X0  Y0:   3 2
:         X  Y  k:  X    Y    1.5
```

The example solves the diffequation Y '/Y = 1.5*X '/X, X(0) = 3, Y(0) = 2.

11

Numerical methods

Here some simple numerical methods are treated, Newton's method for finding roots of an equation and numercal integration using the trapezoidal rule, the rectangular rule or Simpsons rule

Newton's method

Newton's method solves equations of the type $f(x) = 0$. A starting point is required and the method has a fast convergence. However, the convergence is somewhat sensitive to the choice of starting point. The method is said to diverge if a solution with a prescribed accuracy has not been found after 30 iterations. Then a new starting point may be used.

Selecting PartAns Y a table is put on the stack giving different x_n values and $f(x_n)$ values with the format $[x_n\ f(x_n)]$.

Interface:

```
RAD                          PRG
{HOME }
        f(x) = 0, Start:x0
: Indp var:   x
:       x0:   1
:        f:   'x-COS(x)'
```

```
RAD                          PRG
{HOME }
            Input cont
       n is number of accur.
       figures behind comma
: n  PartAns Y/N:3 Y
```

The example solves the equation x-Cos(x) = 0 with starting point 1 and with 3 accurate decimals (correct rounded). We will se that $f(x_3) \approx 1E\text{-}10$.

Numeric integration

Trapezoidal rule

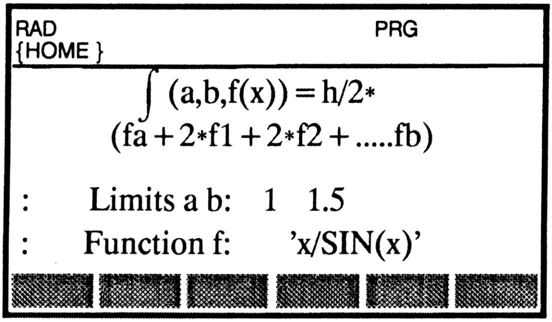
The interval of integration is divided into a certain number of smaller intervals and a number of trapezes are constructed. The sum of the areas for theese trapezes are calculated.

Formula (integration interval [a b]):

$$I \approx (b\text{-}a)/(2n) * (f_0 + 2f_1 + 2f_2 + ...f_n)$$

By selecting PartAns Y the numbers f0, 2f1... will show up on the stack in tabular form.

Interface:



```
RAD                         PRG
{HOME}
          Input cont
: Num. intervals  n:   5
:      Indp var:     x
: PartAns Y/N:     Y
```

The example calculates the integral of Sin(x)/x from x = 1 to 1.5 with 5 subintervals.

Rectangular rule

In rectangular rule the midpoint of each subinterval is used as argument, $f(x_m)$.

Formula (integration intervall [a b]):

$$I \approx (b-a)/n*(f_{m1} + f_{m2} + ...f_{mn})$$

Interface:

```
RAD                          PRG
{HOME }
        ∫ (a,b,f(x)) = h*
   (fm1 + fm2 + fm2 + .....fmn)

:    Limits a b:   1   1.5
:    Function f:  'x/SIN(x)'
```

```
RAD                          PRG
{HOME }
          Input cont
: Num. intervals n:     5
:        Indp var:     x
: PartAns Y/N:      Y
```

The example calculates the integral of x/Sin(x) from x = 1 to x = 1.5 with 5 subintervals.

Simpson's rule

This is normaly the most accurate rule of the three mentioned, but the number of subintervals must be even.

Formula (integration interval [a b]):

$$I \approx (b-a)/(3n) * (f_a + 4f_1 + 2f_2 + 4f_3 + ...f_b)$$

Interface:

```
RAD                                PRG
{HOME }
        ∫(a,b,f(x)) = h/3*
      (fa + 4*f1 + 2*f2 + .....fb)

:     Limits a b:   1   1.5
:     Function f:       'x/SIN(x)'
```

```
RAD                                PRG
{HOME }
             Input cont
: Num. intervals n:     4
:        Indp var:      x
: PartAns Y/N:      Y
```

The example calculates the integral of x/Sin(x) from x = 1 to x = 1.5 with 4 subintervals. An error message is given if the number of subintervals is odd.

12

Solved problems

Graphing functions

When you are going to graph a function you may use the Graphing option under the menu FUNCTIONS. We will specially look at using the intrinsic ZOOM/BOXZ option of the HP 48. When you try to graph a function the first picture may be difficult to understand.

Example 12.1
Plot the graph of the function

$$f(x) = 3*x^5-2*x^2+x-2$$

Initialy there is no interval for x, but Calculus will require that. We need to experiment a bit to get a good picture.

Calculus interface:

```
RAD                          PRG
{HOME }
        Graphing {f1(x)  f2....}
         between x=x1 x=x2
:       Indp var:   x
:         x1  x2:   -2 2
```

```
RAD                          PRG
{HOME }
              Input cont

:{f1  f2....}: '3*x^5-2*x^2+x-2'
```

The choice of an interval for x is [-2,2]. We se that the graph has a "horizontal region" in interval [0,1] and we want to take a further look at this region (it may hide extremum points). We move the cursor to about (-0.12,-2.4) (use the (x,y) key), the - key and the ZOOM/BOXZ keys. Fit a box to the desired region by using arrow keys and use ZOOM.

We will see that the region contains a maximum and a minimum point. We may examine this numericaly by using the FCN menu.

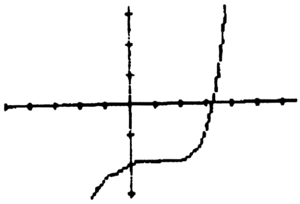


Fig.12.1 3*x^5-2*x^2+x-2

Move the cursor near the maximum point and use EXTR. The maximum point is (0.27, -1.87). In the same way we find a minimum point (0.52,-1.91).

Example 12.2
Given

$$f(x) = \frac{x^2-2*x+1}{x+2}$$

We choose the interval for x to be [-4,4] and then we include the singularity at $x=-2$. In the first picture we only get one branch of the graph. We may ZOOM out to get a wider range for x and try ZOOM/ZFACT/H-FACT 3 /

V-FACT 1/ZOUT. We still don't see the other branch and the reason has to be too small y values and we ZOOM out in the y-direction: ZOOM/ZFACT/V-FACT 10 /H-FACT 1/ZOUT.

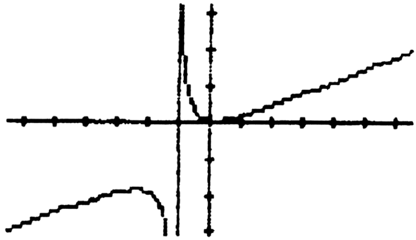


Fig.12.2 (2*x^2-x+1)/(x+2)

The extremum points may be found numericaly by using the FCN menu.

Analyzing functions numericaly

Analyzing a function means to find:

- Roots (crossing of x-axis)
- Extremum points
- Points of inflection
- Graphing

Example 12.3

We will look at the function

$$f(x) = \frac{1/3*x^6-3}{x+2}$$

We will use the graph and the FCN menu. We first graph the function and choose x in the interval [-2,2]. A first glance at the picture indicates two real roots, one minimum point and two points of inflection. The FCN menu gives the roots $x = -1.44$ and $x = 1.44$ and local minimum at (-1.06,-2.69).

To find the points of inflection we may analyze the derivative. By pushing F' (NXT) we get both f(x) and f '(x) in the same picture. The extremum points of f ' give the points of inflection of f. They are (-0.74,1.50) and (0.56,0.50).

But there is another branch. We move the picture to the right by moving the cursor to the left and ZOOM/CNTR. We see that the other branch looks like a straight, almost vertical line, and this suggests that an eventual max/min has a big absolute value. We use ZOOM/ZFACT/H-FACT 1 /V-FACT 200/ZOUT. We see that the derivative has a zero and f a maximum at (-2.38,-151.52).

To find the maximum of f you have to use NXEQ to activate the given function and then push the EXTR key. The cursor has to be in the neighbourhood of the maximum point. To look for a point of inflection for x < -2 we may look for an extremum point for f '. Using NXEQ/EXTR gives an apparent extremum for at x = -2.85. This is however a point of inflection for the derivative with a horisontal tangent line (f '' = 0). So there is no point of inflection for f for x < -2.

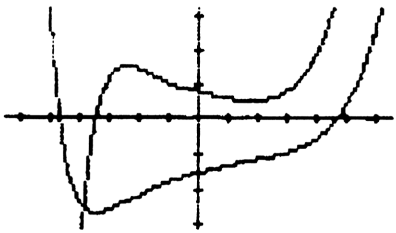


Fig.12.3 (1/3x ^ 6-3)/(x + 2) (x > -2)

Analyzing functions symbolicaly

The degree of any polynomials must the not exeed 4 except for cases like (x-1) ^ 5.

Example 12.4
Given

$$f(x) = \frac{(2x^4-3)}{(x-1)}$$

We first find the roots by using FUNCTIONS/Zeroes.

Calculus interface:

```
RAD                                   PRG
{HOME }
            Solving f(x)=0
:Indp var:      x
:  f       :  '(2*x^4-3)/(x-1)'
```

The zeroes will be {1.11,-1.11, (0,1.11),(0,-1.11)}. These are numerical values but calculated by using formulaes and not numerical iteration.

We may use the option "Sign of function" to find out where the function is positive and where it is negative.

CALCULUS interface:

```
RAD                          PRG
{HOME }
        Finding the sign
            of f(x)
:Indp var:      x
:          f:  '(2*x^4-3)/(x-1)'
```

The answer includes the actual zeroes (zeroes of the function and points of discontinuity/singularity). They are {-1.11, 1, 1.11}.

Sign	Interval
-	x<-1.11
+	-1.11<x<1.00
-	1.00<x<1.11
+	x>1.11

The values of the zeroes are laying on the stack in a matrix

We may now look for the extremum points by using FUNCTIONS/Extremum points. We see that there is no extremum point and the function is increasing for all x.

CALCULUS interface:

```
RAD                          PRG
{HOME }
     Increasing/decreasing
          function f(x)
:Indp var:      x
:          f:   '(2*x^4-3)/(x-1)'
```

The answer includes the expression for the derivative and the zeroes and singularities of the derivative.

Finding the sign of:
$(3\text{-}8*x^3+6x^4)/(x\text{-}1)^2$ (the derivative)
Actual zeroes:
{1.00}

Inc/Dec	Interval
Incr	$x<1.00$
Incr	$x>1.00$

Points of inflection and curvature may be found by using FUNCTIONS/Points of inflection. Points of inflection are

{-0.39,1.45} and the x and y values are put on the stack in a matrix.

CALCULUS interface:

```
RAD                                PRG
{HOME }
            Points of inflection
                    f(x)
:Indp var:      x
:          f:   '(2*x^4-3)/(x-1)'
```

The second derivative and the zeroes are calculated.

Finding the sign of:
$(-6+24x^2-32x^3+12x^4)/(x-1)^3$ (the second derivative)
Actual zeroes:
{-0.39 1.00 1.45}

Curv.	Interval
Convex	$x < -0.39$
Concave	$-0.39 < x < 1.00$
Convex	$1.00 < x < 1.45$
Concave	$x > 1.45$

At last we graph the function. We choose x between -3 and 3. .To get a wider range for y we ZOOM y out by a factor 10 (ZOOM/ZFACT/V-FACT 10/H-FACT 1/ZOUT).

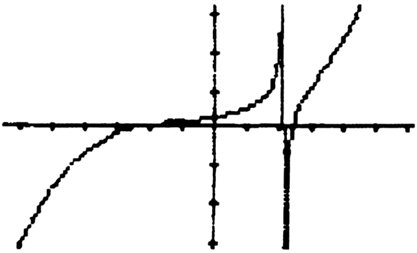


Fig.12.4 (2x^4-3)/(x-1)

Example 12.5
Analyze

$$f(x) = \frac{2}{x} + x - 1$$

We find zeros by using FUNCTION/zeros and remember to write (2/x) + (x-1). Solution is {(0.5, -1.32), (0.5,1.32)} which means that there is no crossing of the x-axis (complex roots).

By using "Extremum points" we may find the maximum and minimum and where the function is decreasing and increasing. The derivative is given as $(x^2-2)/x^2$, and we see that the function is decreasing between x = -1.41 and 1.41 and increasing elsewhere. There is a maximum at x = -1.41 and a minimum at x = 1.41. The x and y values are put on the stack in a matrix.

When investigating the curvature we find no points of inflection and the curve is concave all over.

We now want to graph the function and choose x between say -2 and 2. We cannot see the left branch so we try to get a wider range for y, ZOOM/ZFACT/V-FACT = 3/ZOUT. The curve is somewhat "horisontal" in some regions and then we try ZOOM/ZFACT H-FACT = 3/ZOUT to get a wider range for x-values.

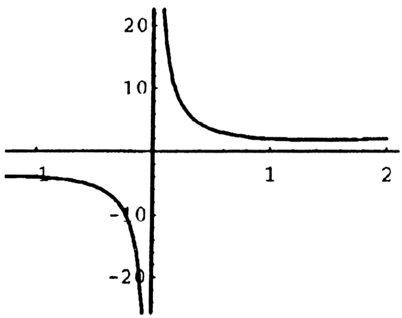


Fig.12.5 2/x + x-1

Example 12.6
In this example we will look at a trancendental function. Generally CALCULUS cannot handle this kind of functions unless the independent variable has a single occurence.

However we will look at some tricks. Given the function

$$f(x) = 2e^{-x} + 1/2*e^{x}$$

We may substitute $e^{x} = u$ and then $e^{-x} = 1/u$. This will transform the function into a rational function.

$$f(u) = 2/u + u/2, \; u = e^{x}$$

We observe that u is always positive (as a real variable). The zeros may be found by using FUNCTIONS/Zeros and this gives u = (0,2) and u = (0,-2). Since they are complex there is no crossing of the x-axis.

To find the extremum points we use FUNCTIONS/Extremum points. We get a maximum at u = -2 but since u is positive there is no maximum. The minimum is for u = 2 and this gives x = ln2 = 0.69. The matrix on the stack gives the value of the function to be 2.

To graph the function we of course use the original expression $f(x) = 2e^{-x} + e^{x}/2$. We choose x between -2 and 2. We moove the picture downwards by mooving the cursor upwards and use CNTR.

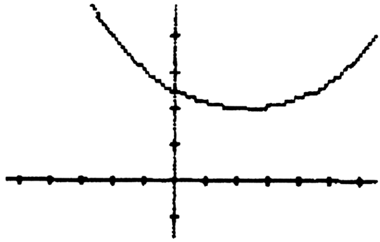


Fig.12.6 2*Exp(-x) + 1/2*Exp(x)

Trancedental functions which cannot be transformed to rational functions has to be examined numericaly. An example would be $f(x) = (x)^{1/2} * e^{-x}$. However if we are going to find extremum points or points of inflection, the derivatives may be rational and CALCULUS will work.

Example 12.7
Analyze

$$f(x) = 1 + x^2/4 - \ln(x)/2$$

The zeros must be found numericaly and we then first graph the function and choose x between $x_1 = 0$ and $x_2 = 3$. We will use a more narrow picture and ZOOM/ZFACT/

H-FACT 2/V-FACT 1. It does not seem to be any zeros and this will be confirmed when we look for extremum points: there is only one minimum point near x = 1.

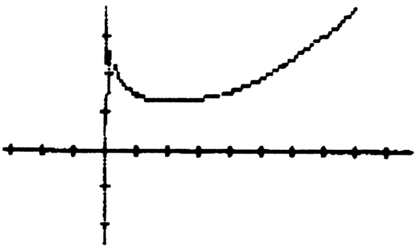


Fig.12.7 1 + x ^ 2/4-Ln(x)/2

We find the extremum points to be: minimum at x = 1 with the value f(1) = 1.25. There is no point of inflection as we can see using the option FUNCTIONS/points of inflection (always concave).

Example 12.8
In the case $f(x) = x^{1/2} * e^{-x}$ we may help CALCULUS a little bit The zero point is apparently x = 0. To look for the extremum points we first find the derivative by using FUNCTIONS/Derivatives.

CALCULUS interface:

```
RAD                                PRG
{HOME }
              Y = f(x)
          calculates f '(xo)
:   Indp var xo:  x   x

:  PartAns Y/N :  N
```

```
RAD                                PRG
{HOME }
             Input cont



:  f:        'x^(1/2)*Exp(-x)'
```

We have to find the sign of the function

$$-e^{-x}*x^{1/2} + 1/2*e^{-x}*x^{-1/2} = e^{-x}*x^{1/2}*(1/(2*x)-1)$$

The last paranthesis is a rational expression and may be handled by CALCULUS. We will find the sign of this expression in different intervals for x and we then use the option FUNCTION/Sign of function. The expression is changing from positive to negative for $x = 1/2$ and this will then be a maximum. In this case the value of the derivative (0) will be put on the stack and the value for f has to be found by using WH in the LIBRARY menu.

When dealing with more complicated rational functions, you might have to rewrite the experession a bit. The rational functions have to be written in the form $f(x)/g(x) + p(x)/h(x)$. This means that the function $f(x) = (x^2-1)/(x-4) + x-5$ has to be written $f(x) = (x^2-1)/(x-4) + (x-5)$. The final paranthesis will be important.

Example 12.9

$$f(x) = (x^2-1)/(x-4) + (x-5)$$

First we try to find the zeros. We will see that the zeros are complex and so there is no crossing of the x-axis.

The extremum points are a maximum at $x = 1.26$ and a minimum at 6.74 with function values given on the stack.

There is no point of inflection, but the curve is convex for $x < 4$ and concave for $x > 4$.

When graphing the function the above analyzing suggests that the range for x-values must be 0 to 8. The first picture is meaningless, but since we saw that the minimum point had a y-value about 17 we try to ZOOM/ZFACT/ V-FACT10/ZOUT. Then we narrow the picture in the x-direction to see the curve at the near flat areas by ZOOM/ZFACT/H-FACT2/ZOUT.

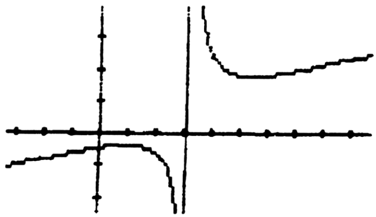


Fig.12.8 (x ^ 2-1)/(x-4) + x-5

Solving equations symbolicaly means not to solve them by numerical iteration (FCN menu). Equations that can be solved are rational equations (polynoms and fractions), and all other equations where the unknown has a single occurence.

In CALCULUS two possible options are available, ALGRBRA/Solve equation and FUNCTIONS/Zeros.

Example 12.10
If you are going to solve $(x^2-1)/(x-4)-x^2 = x-2$, the equation has to be written $(x^2-1)/(x-4) = x^2 + x-2$ if we are using the option ALGEBRA/Solve equation. The the solutions are $\{-1.85, 1, 4.85\}$.

CALCULUS interface:

```
RAD                      PRG
{HOME}
        Equation LS=RS

: Unknown:   x
:LS=RS'(x^2-1)/(x-4)=x^2+x-2'
```

Simplifying expressions (ALGEBRA)

The option Simplify lets you simplify expressions so that fractions are being put over a common denominator and the numerator is factorized. The expression $(x^2-1)/(x-4)-x^2-x+2$ is written

$$(x^2-1)/(x-4)-(x^2+x-2)$$

and simplified into -(x-1)(x + 1.85)(x-4.85)/(x-4).

Integration of rational functions

Any rational function where the degree of a polynimial is less than 5, may be integrated. If the degree of a numerator is equal to or greater than the degree of the denominator, polynomial division will take place. Use the menu option INTEGRATION/partial fractions.

Example 12.11
Integration of (2*x-1)^2/(x^2-1) will then first execute a polynomial division. If the option partial answer Y(es) is selected the details of splitting up into partial fractions are given.

CALCULUS interface:

```
RAD                         PRG
{HOME}
          P(x)/Q(x)dx
: Indep var:    x
: PartAns Y/N:  Y
:   Numerator:  '(2*x-1)^2'
```

RAD PRG
{HOME }

Input cont

:Denominator?(x-1)^2'

Since PartAns Y is selected intermediate results will show up:

Polynomial division
and splitting up:

(2x-1)^2/(x^2-1) =

A/(x-1) + B/(x + 1) + 4 (Next page)

Coefficients:

A = 1/2

B = -9/2 (Next page)

Solution:

-9/2*LN(x + 1) + 1/2*LN(x-1) + 4*x

Example 12.12

If the denominator is of degree 1, polyniomial division together with direct integration will solve the problem. Let us look at the integration of (2*x-1)^2/(x-1). First use polynomial divison under menu ALGEBRA. The result is 4*x + 1/(x-1). Then use direct integration on this expression (imported from stack). To import an expression from the stack to the input screen, push EDIT and use ↑STK/ECHO/ENTER. The answer is $\ln(x-1) + 2x^2$.

Example 12.13
Integrate

$$f(x) = \frac{1}{2x^3-x^2-8x+4}$$

The answer is 1/20*ln(x + 2)-2/15*ln(x + 1/2) + 1/12*ln(x-2).

Example 12.14
Integrate

$$f(x) = \frac{x}{x^4-1}$$

The answer is -1/4*ln(x^2 + 1) + 1/4*ln(x-1) + 1/4*ln(x + 1)

Example 12.15
Integrate

$$f(x) = \frac{x}{x^4+1}$$

The answer is -0.5*Atan(1 + 1.41x) + 0.5*Atan(-1 + 1.41x) which differs from the "obvious" answer 1/2*Atan(x^2) by a constant. Generally irrational numbers will be approximated by decimal numbers. If PartAns Y is choosen the number of digits in the coefficients of partial fractions will be 2, but the answer is laying on the stack where the number of digits may be altered by n FIX.

Example 12.16
Integrate

$$f(z) = \frac{(z^4-5z^3+7z^2-3z-4)}{(z-1)^2(z-3)}$$

Polynomial division is necessary (select PartAns Y) and the answer is

Polynomial division
and splitting up:
$(z^4-5z^3+7z^2-3z-4)/((z-1)^2*(z-3)) =$

$$A/(z-1)^2 + B/(z-1) + C/(z-3) + z$$

Coefficients:

$A = 2$

$B = 1$

$C = -1$

Solution:

$$-2/(z-1) + LN(z-1)-LN(z-3) + 1/2*z^2$$

Integration of algebraic and trancendental functions

In this case you have to use substitution, partial integration, the integration table or direct integration (intrinsic HP function).

Example 12.17
Integrate COS(x)/(2*SIN(x)-3). We see that the derivative of the denominator is proportional to the numerator and we may use substitution u = 2sin(x)-3.

CALCULUS interface:

```
RAD                              PRG
{HOME }
          k*f(g(x))*g'(x)dx
:      Indp var : x
: PartAns Y/N: Y
:    Integrand: 'COS(x)/(2*SIN(X)-3)'
```

RAD PRG
{HOME }

Input cont

:Subst. u = g(x): '2*SIN(x)-3'

The answer will be:

Substitution:

u = 2*SIN(x)-3

du = 2*COS(x)dX

This gives:

1/2udu =

1/2*LN(u)

Solution:

1/2*LN(2*SIN(x)-3)

Example 12.18

Integrate (x-ATAN(x))/(x^2 + 1). We have to separate the integrand into x/(x^2 + 1)-ATAN(x)/(x^2 + 1) and make use of two different substitutions. We then have to run the program two times. First substitution u = x^2 + 1, second

$u = ATAN(x)$. The answer will be $1/2*LN(x^2+1)$-
$1/2*ATAN(x)^2$.

Example 12.19
Integrate x^3*e^{2x}. This is of the form u*v, and we use integration by parts.

CALCULUS interface:

```
RAD                       PRG
{HOME }
          u(x)*v(x)dx
:      Indp var :  x
: PartAns Y/N:  Y
:             u:  'x^3'
```

```
RAD                       PRG
{HOME }
          Input cont


:  v: 'EXP(2*x)'
```

If PartAns Y(es) is selected the answer will look like:

3 times int. by parts

```
   x^3*EXP(2*x)dx =
  x^3*(1/2*EXP(2*x))-
   1/2*EXP(2*x)*3*x^2)dx =
  x^3*(1/2*EXP(2*x))-3*x^2*1/4*EXP(2*x) +
    1/4*EXP(2*x)*6*xdx =
  x^3*1/2*EXP(2*x))-3*x^2*1/4*EXP(2*x)) +
  6*x*1/8*EXP(2*x)-   1/8*EXP(2*x)*6dx =
  x^3*1/2*EXP(2*x)-3*x^2*1/4*EXP(2*x) +
  6*x*1/8*EXP(2*x)-6*1/16*EXP(2*x)
```

We see that 3 times of integration by parts were necessary.

Example 12.20

Integrate LN(x)^2. Here we use integration by parts with the factor v = 1 and u = LN(x)^2. The answer will be LN(x)^2*x-(2*LN(x)*x-x).

Example 12.21

Integrate (x^2-3*x + 4)$^{1/2}$. Here we will use the integration table, Algebraic. We choose the option (a*x^2 + b*x + c) with a = 1, b = -3 and c = 4. The answer will be -0.75*x/2*(x^2-3*x + 4) + 0.88*LN(-3 + 2*x + 2*(x^2-3*x + 4)).

Example 12.22

Integrate[$1/(x^2\text{-}3x\text{-}1)^{1/2} = (x^2\text{-}3x\text{-}1)^{-1/2}$. We again use the integration table, Algebraic. We choose the option 1/ (a*x^2 + b*x + c) with a = 1, b = -3 and c = -1. The answer will be LN(x-1.5 + (x^2-3*x-1)).

Applications of integration and differentiation

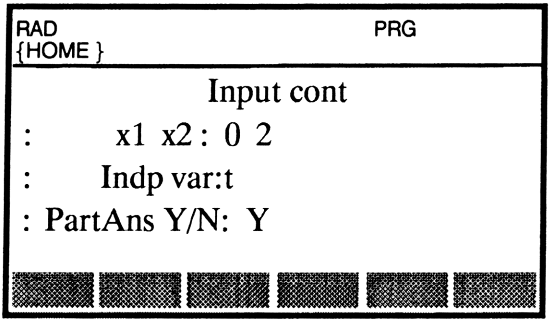
Example 12.23

Calculate the area bounded by the curve $z = (4*t + 1)^{1/2}$, t-axis and the vertical lines t = 0 and t = 2.

We will use the menu option INTEGRATION/Applications/Area. The function f2 on the input screen will here be f2 = 0 (area between z and z = 0, t-axis).

CALCULUS interface:

```
RAD                                PRG
{HOME }
        Area bounded by f1(x),
          f2(x), x = x1, x = x2
:f1: '(4*t + 1)^(1/2)'
:f2: 0
```



When PartAns Y is selected the formula that determins the area is given, the integrated function and the final answer is 13/3.

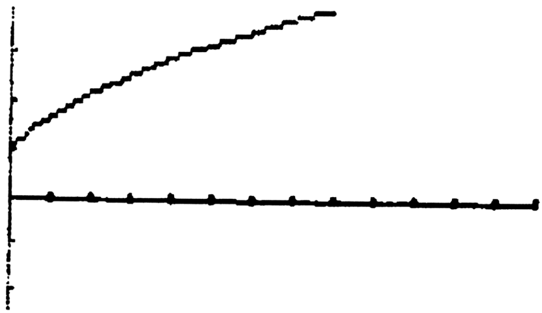


Fig.12.9 (4*t + 1) ^ (1/2)

Example 12.24
Find the area bounded by the curves $y_1 = x^3$, $y_2 = 2x$ and $y_3 = x$ for $x > 0$ and $y > 0$.

The area has to be calculated in two steps, the area between x and 2x from $x=0$ to $x=1$, and the area between 2x and x^3 from $x=1$ to $x=2^{1/2}$. Both answers are laying on the stack and may be added by pushing the + key. The answer is 3/4.

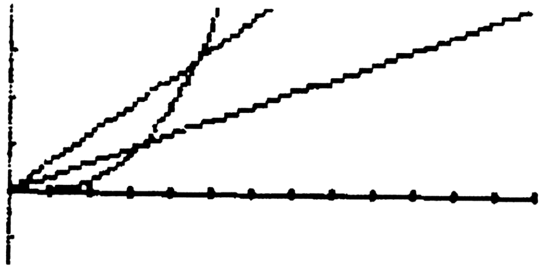


Fig.12.10 y=x^3, y=2*x, y=x

Example 12.25

Calculate the arc length along the curve $y=1/3*(x^2+2)^{3/2}$ from $x=0$ to $x=3$. This may be calculated exactly by hand, but then you have to transform $(x^2(x^2+2)+1)$ into $(x^2+1)^2$. CALCULUS will not be able to do this and a numeric solution is given (= 12).

Example 12.26

Calculate the area of the surface of revolution of the function $y = 2/3*x^{3/2}$, rotation of curve about $x = -2$ from $x = 0$ to $x = 3$.

Interface:

```
RAD                                PRG
{HOME}
              Surface of  f(x),
              x=x1, x=x2
:     f: '2/3*x^(3/2)'
:x1  x2: 0  3
```

```
RAD                                PRG
{HOME}
              Input cont
:        Axes :  x=-2
:       Indp var:    x
: PartAns Y/N:  Y
```

The answer is not exact, 107.23. The integral may be calculated exactly by integration of parts.

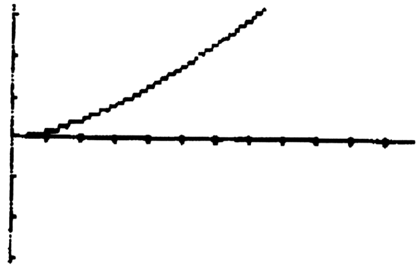


Fig.12.11 2/3x^(3/2)

Example 12.27
Calculate the area of surface of revolution of the function $y = x^2$ about the y-axis from $x = 0$ to $x = 2$. The axis is now given as $X = 0$ and the answer will be 36.177 and may be calculated exactly by the substitution $u = 4x^2 + 1$.

Example 12.28
Calculate the volume of the solid of revolution of the area between the curve $y = 2x^2$ and the x-axis about the Y-axis between $x = 0$ and $x = 5$.

Interface:

```
RAD                                PRG
{HOME }
          Region bounded by,
          f1, f2, x=x1, x=x2
:    f1:'2*x^2'
:     f2:  0
```

```
RAD                                PRG
{HOME }
              Input cont
:        x1  x2 :   0  5
:          Axes:  'X=0'
:      Indp var:  x
```

```
RAD                                PRG
{HOME }
              Input cont
:
:
:     PartAns Y/N:  Y
```

The exact answer will be 625π, and the integral to be evaluated is given.

Example 12.29
Calculate the volume of the solid of revolution of the area between $y = 4-x^2$ and $y = 0$ (x-axis) about the line $Y = 6$ between $x = 0$ and $x = 2$.

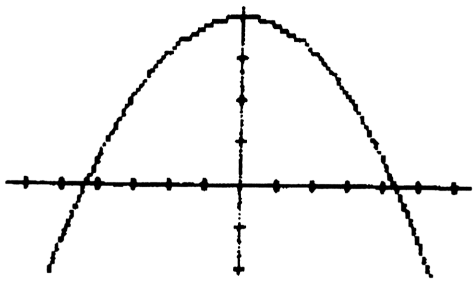


Fig.12.12 4-x^2

The answer is given as 147.45 although the integral is calculated exactly. This has to do with the conversion from decimal numbers to fractions.

Example 12.30
Calculate the center of mass of the area given by $y = 4-x^2$, $x = 0$ and $y = 0$ (betwen $x = 0$ and $x = 2$).

First, the static moments have to be calculated.

Interface:

```
RAD                                  PRG
{HOME }
          Region bounded by,
          f1, f2, x=x1, x=x2
:    f1: '4-x^2'
:     f2:  0
```

```
RAD                                  PRG
{HOME }
              Input cont
:       x1  x2 :   0  2
:         Axes:  'X=0'
:      Indp var:  x
```

```
RAD                                  PRG
{HOME }
              Input cont
:
:
:      PartAns Y/N:  Y
```

Here, the static moment about the y-axis, My is calculated to be 4. In the same way Mx, the moment about the x-axis is calculated to be 128/15.

Now the area of the region has to be calculated. We use the option Area and get the answer 16/3. The center of mass is finaly calculated.

Interface:

```
RAD                                  PRG
{HOME }
            Static moments
          Mx and My, area A
        Xt = My/A  Yt = Mx/A
: Mx  My  A:   '128/15'  4  '16/3'
```

The values of Mx, My and A are laying on the stack and may be inserted by using EDIT/↑STK/ECHO/ENTER. The answer is Xc = 3/4, Yc = 8/5.

Parametric curves

Example 12.31
Given the parametic curve

$$x = 2 + 4\sin t,\ y = 3^{1/2} - 2\cos t,\ t \in [0, 2\pi >$$

Find

- Crossing of x and y-axis
- Singular points
- Extremum values for x and y
- Tangent line at $t = 11\pi/6$
- Area of the region bounded by the curve
- Graph the curve

First we solve $x=0$ and $y=0$. We use ALGEBRA/Solve equation.

Interface, x = 0:

```
RAD                         PRG
{HOME }
         Equation LS = RS

:     Unknown:  x
:      LS = RS:  '2 + 4Sin(t) = 0'
```

The answer is t = 7π/6 and 11π/6. In the same way y = 0 gives t = π/6 and 11π/6.

To find singular points and extrema we use the option FUNCTION/Extremum points for x and y seperately. We see that x has a maxmum point for t = π/2 (= 6) and a minimum point for t = 3π/2 (= -2). y has a maximum for t = π (= 3.73) and minimum for t = 0 (= -0.27). We see that dy/dt and dx/dt dont have zeros at the same time and therefore no singular points.

To find the area we integrate from t = 0 to t = 2π using Area under the menu 2D curves/parametric and the answer is 25.13.

The tangent line is found by using the option Geometry/tangent line under 2D curves/parametric. $t0 = 11\pi/6 \approx 5.76$ and the answer is $y = -0.29x$.

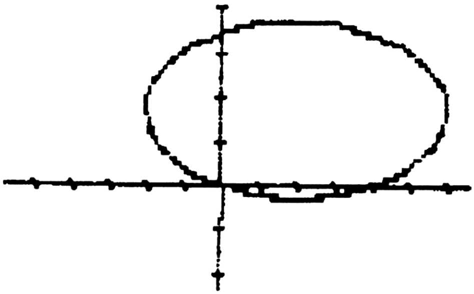


Fig.12.13 x=2+4*Sin(t),y=3^(1/2)-2*Cos(t)

Example 12.32

Graph the curve and calculate the area between the two loops of the curve of Pascal, $r = 4*Cos(t) + 2$, $t \in [-\pi, \pi>$ (Polar curve).

The option 2D curves/Polar coordinates will give the possibility to solve the problems.

The maximum values of abs(r) will be for $t = 0$ and $t = \pi$ (use the option FUNCTION/Extremum) . To find the desired area we have to calculate the area of the outer loop and subtract the area of the inner. The range of t for the outer loop is from -2π/3 to 0 for the lower part and from 0 to 2π/3 for the upper, totaly from -2π/3 to 2π/3. The range of t for the inner loop is from -π to -2π/3 for the upper part and from 2π/3 to π for the lower.

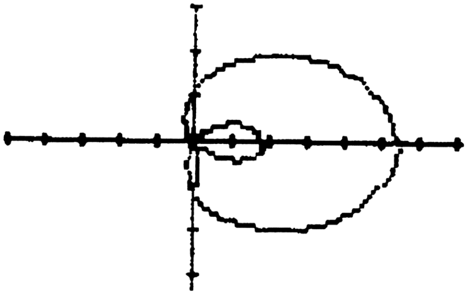
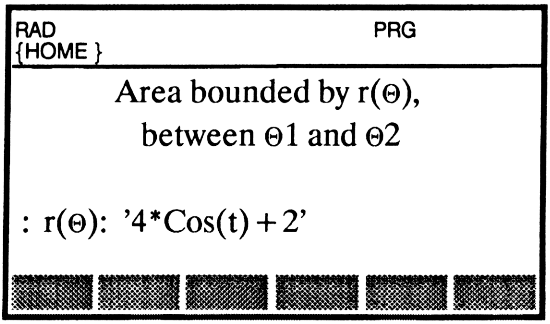


Fig.12.14 r(t) = 4*Cos(t) + 2

Interface (outer area):



```
RAD                              PRG
{HOME }
            Input cont
:        Θ1 Θ2:  '-2*π/3'  0   :
:      Indp var:  t
: PartAns Y/N:  Y
```

Here we have to multiply the answer by 2. In the same way we calculate the inner area and then subtract. The answer is 33.35.

Series

The series will be stated as $\sum_{n=0}^{\infty} A_n * x^{kn}$ (power series) or as $\sum_{n=0}^{\infty} A_n$.

Example 12.33

Find the interval of convergence of the series where $A_n x^{kn} = 1/n * x^n$. We use the ratio rest under SERIES/ Convergence.

Interface:

```
RAD                         PRG
{HOME}
  Converg.    ΣAn*X^(k*n)

:            An:    '1/n'
:      X^(k*n):   'x^n'
```

```
RAD                         PRG
{HOME}
          Input cont
:  Sumindex n:   n
:      Indp var:   x
: PartAns Y/N:   Y
```

The answer is given as:

Ratio test:

an + 1/an =

INV(1 + n)*Xn

lim n→ ∞ an + 1/an =

x (Next page)

Convergence:
[lim n→ ∞
(an + 1/an)] ^ 2 < 1

	$x^2-1<0$	(Next page)
Case	Interval	
Diverg.	$x<-1$	
Conv.	$-1<x<1$	
Diverg.	$x>1$	

Example 12.34
The ratio test does not give any answer at the ends of the interval. Test for convergence at the ends, $x = 1$ and $x = -1$. We then have the series $An = 1/n$ and $An = (-1)^n/n$ (last example). The first will be tested using the integral test (positive series) and the second using Leibnitz test (alternating series).

In the integral test we need to know the indefinite integral of A_n, Ln(n). We see that the series $A_n = 1/n$ diverges and the series $A_n = (-1)^n/n$ converges, the interval of convergence will then be $-1 \leq x < 1$.

Example 12.35
Find the interval of convergence of the series $A_n x^n = n/5^n x^n$

Case	Interval
Diverg.	$x < -5$
Conv.	$-5 < x < 5$
Diverg.	$x > 5$

Example 12.36
The integral test at the end for $x = 5$ gives convergence and the same will Leibntiz test give for $x = -5$. The indefinite integral of A_n may be found by using integration of parts: 5 ^ (-x)*(0.62x-0.39), $x/5^x$ is written $x*5^{-x}$. The interval of convergence will then be $-5 \leq x \leq 5$.

Example 12.37
Find the interval of convergence for the series $(-1)^n*(x-1)^n/n^2$. We first test for absolute convergence of the series $(x-1)^n/n^2$ and get the interval $< 0,2 >$ for x

Example 12.38
Leibnitz test gives convergence also for $x = 2$ in the last example. For $x = 0$ we have the series $An = (-1)^n*(-1)^n/n^2 = 1/n^2$. The integral test will then give convergence (see chapter 7). Interval of convergence: [0,2].

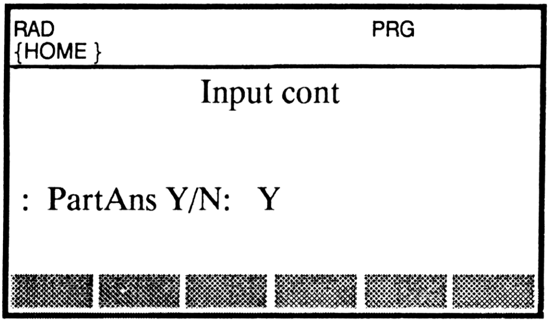
Example 12.39
Test for convergence the binomial series of $f(x) = (16 + x)^{-1/4}$. The interval of convergence will be $<-16,16>$.

Differential equations

Example 12.40
Solve the differential equation y*dy/dx + x = 0. This eqau-tion is separable and need to be written in the form F(x,y,y') = 0. y' = dy/dx is written Dy. We then solve y*Dy + x = 0.

Interface:

```
RAD                                  PRG
{HOME }
              F(x,y,y') = 0
          y' must be written Dy

:           x y:   x y
:             F:     'y*Dy + x'
```



The solution includes the separation and integration since PartAns Y is selected.

Separation gives:

$$Y'*dY=-X*dX$$

Integration gives:

$$1/2*Y\verb|^|2=-1/2*x\verb|^|2+C$$

Solution:

$$Y=s1*(-1/2*x\verb|^|2+C)/(1/2)$$

s1 is here ± 1.

Example 12.41

Solve the differential equation $du/dv=4*v^3*e^{-u}$. The equation is written Du-4*v^3*e^(-u)=0 (separable) and the solution is $u=Ln(v^4+C)$.

Example 12.42

Solve the differential equation $e^{z-x}*dz/dx-x=0$. This is separable and the solution is $e^z =$ (c,x,e^x*x,x) + C. The integral has to be solved by integration by parts.

Example 12.43

Find the particular integral in example 12.41 with the initial condition u(0) = 1. We use Linear 1.st order/Initial value. We get the solution $u = Ln(v^4 + 2.72)$.

Example 12.44

Solve the first order, linear equation $dy/dx + 2y = e^{3x}$. We use the menu option 1st order linear and here $P(x) = 2$ and $Q(x) = e^{3x}$. The solution is $e^{-2x}*C + 1/5*e^{3x}$.

Example 12.45

Solve the second order differential equation y''-4y' + 3y = 2x + Cos(2x). We use the menu option Linear 2nd order/undetermined coeff. with a = 1, b = -4, c = 3 and r(x) = 2x + Cos(2x). Selecting PartAns Y The soution is:

Characteristic eqn:

$$\lambda 2\text{-}4\lambda + 3 = 0$$

$$\lambda 1 = 3 \; \lambda 2 = 1$$

Yh = A*EXP(3x) + B*EXP(x) (nest page)

Choosing Yp:

K + L*x + M*Sin(2x) + N*Cos(2x) (next page)

Yp = 8/9 + 2/3x-8/65Sin(2x)-1/65Cos(2x)

Y = Yp + Yh

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Z

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