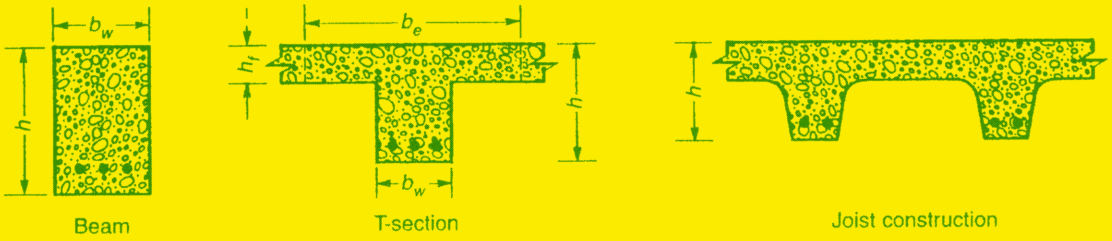
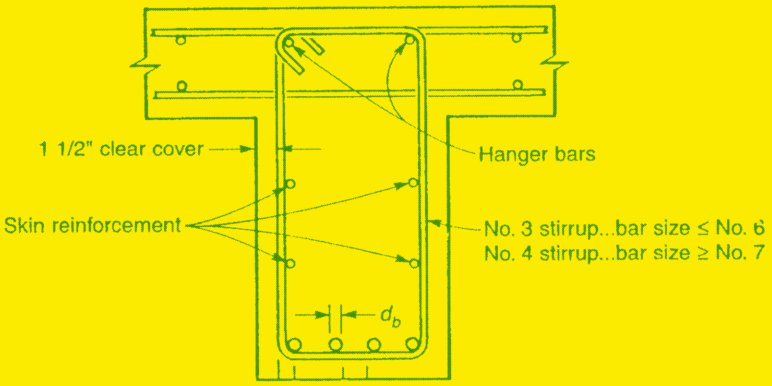
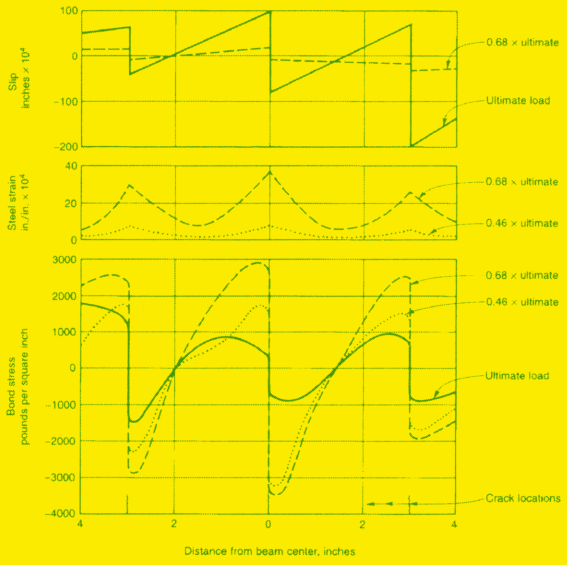


Design of Reinforced Concrete Structures





Alan Williams, Ph.D, S.E., C. Eng.
Registered Structural Engineer

DESIGN OF REINFORCED CONCRETE STRUCTURES

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FOR CIVIL AND STRUCTURAL ENGINEERS

ALAN WILLIAMS, Ph.D., S.E., C.ENG.

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INTRODUCTION

The purpose of this textbook is to provide engineers and students with a comprehensive reference for the design of reinforced concrete structures. It is intended to serve either as a text for a course in reinforced concrete design or as a detailed guide for self study of the subject. The material presented is derived from a four-unit, one-semester course in reinforced concrete design given at the University of California, Irvine, and from a number of years of professional design experience.

The foremost aim of the text is to develop a fundamental appreciation and a clear understanding of the basic factors involved in the analysis and design of reinforced concrete components. By deriving and explaining Code provisions and current design techniques, the concepts of the physical behavior of a reinforced concrete structure under load are illustrated. The theoretical principles and experimental data necessary to develop the design procedures are fully emphasized to enable the reader to apply these principles to the design of complex reinforced concrete elements. The solutions of practical design examples are presented in an orderly and systematic manner to enable readers to consolidate their theoretical knowledge and develop proficiency in recent design procedures for use in the design office.

The design applications are based on the 1995 edition of the *Building Code Requirements and Commentary for Structural Concrete (ACI 318-95)*, published by the American Concrete Institute, which has been adopted by reference in general building codes. This new edition of the Code introduces several revised provisions, particularly in the treatment of bond and anchorage, two-way slab systems, torsion, and slender column design, and reflects the continuing changes occurring in reinforced concrete design practice. The Commentary, published with the Code, gives relevant background information on the development of the Code but does not provide detailed guidance on its interpretation or on its application to design examples. This text provides comprehensive clarification and interpretation of the ACI Code and all designs are according to the provisions of the Code. As such, it provides all the necessary material for candidates taking the Professional Engineer and Structural Registration examinations of the National Council of Examiners for Engineering and Surveying (NCEES), and the individual State Registration examinations.

The text is organized into ten separate sections. Chapter 1 provides a thorough review of the properties of materials involved. The remaining chapters cover the various elements in reinforced concrete structures. In addition, an appendix contains calculator programs, written for the HP-28S calculator, to facilitate the solution of frequently occurring calculation procedures. These are easily applied to give a rapid solution to the problem while also presenting all the intermediate stages in the calculation.

The notation adopted in the text is that used in the Code and conforms to current usage.

Occasional duplication of the meaning of a particular symbol has been unavoidable, but the symbols are explained in the text as they occur and ambiguities should not arise.

The author wishes to express his gratitude to the California State Board of Registration for Professional Engineers and Land Surveyors for their kind permission to reproduce material from past examination papers.

In a book of this type, errors may be expected to occur and the author will appreciate being informed of any errors that may be found in the text. Any suggestions or comments may be sent to the author at the publisher's address.

1

Materials for reinforced concrete

1.1 BASIC CONCEPTS

1.1.1 Materials for concrete

Concrete is a mixture of cement, water, fine aggregate, and coarse aggregate with the addition, in some instances, of an admixture. The cement and water form a paste which hardens as a chemical reaction occurs and binds together the aggregates. For the chemical reaction, or hydration, to be completed, satisfactory curing conditions must be provided. These consist of supplying moisture and a favorable temperature for the duration of the hydration period.

Cement consists of calcium silicate, calcium aluminate, and calcium sulphate and is available in the following varieties[1]:

- Normal Portland cement which is the general cement used in normal building structures.
- Sulfate resistant Portland cement which is used in those parts of a structure exposed to ground water contaminated with sulfates. It has a slower initial rate of hardening than normal Portland cement and also generates a lower heat of hydration.
- High early strength Portland cement which is used when a structure is required for use sooner than would be possible with normal Portland cement.
- Low heat of hydration Portland cement which is used for large volumes of concrete in dam walls and foundations. The slow rate of hydration reduces the heat generated in the interior of the concrete which tends to produce cracking on the surface of the concrete because of thermal stresses.
- Air-entraining Portland cement which has an improved resistance to frost action and to deicing chemicals. Compared with normal Portland cement, air-entraining Portland

cement has a slightly reduced strength, reduced water demand, and improved workability.

- Expansive cement which produces a concrete which expands after setting. This may be used to reduce drying shrinkage stresses and cracking, for grouting purposes, and for repair applications.
- White Portland cement which is manufactured from selected materials containing only small amounts of iron oxides. This may be used for specific architectural effects.

Typical rate of strength development of comparable mixes of normal Portland cement and high early strength Portland cement is shown in Figure 1–1.

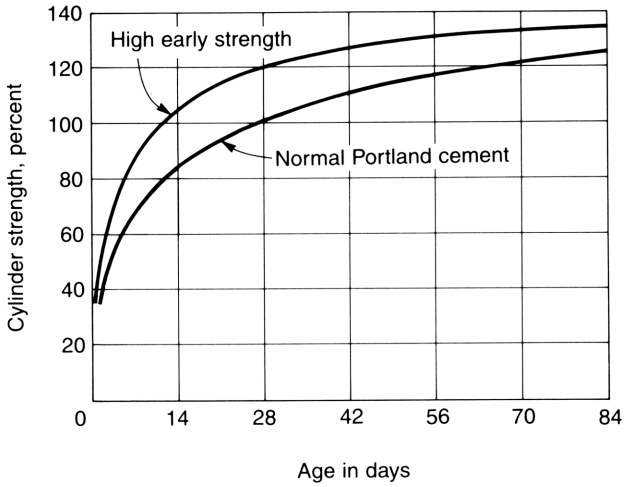

Figure 1–1 Rate of compressive strength development

Water used in the concrete mix should be clean and free from harmful impurities. Sea water is unsuitable for reinforced concrete because of the risk of corrosion of the reinforcement and because of the increased possibility of alkali–silica reaction causing disruption of the concrete. Water containing organic impurities is also unsuitable as is acidic water from peaty areas.

Aggregates, which should be clean, hard, resistant to abrasion, and free from organic material or impurities, are subdivided into two categories:

- Fine aggregate, or sand, which is small enough to pass through the openings of a No. 4 (4.75 mm) wire mesh sieve[2].
- Coarse aggregate, gravel, or stone, which is large enough to be retained on a No. 4 (4.75 mm) wire mesh sieve[2].

The maximum size of coarse aggregate should be as large as possible but is limited by ACI Section 3.3.2, to ensure adequate consolidation, to a maximum of the lesser of

- One fifth the width of the member.
- One third the depth of a slab.
- Three quarters the clear distance between reinforcement.

The grading, or particle–size distribution, of the combined fine and coarse aggregates must be

such as to produce a workable mix which may be placed without segregation and with the least amount of water. In this respect, aggregate proportions which produce a smooth and continuous grading curve without deficiencies or excesses of any size are generally most satisfactory[3]. The aim is to fill the voids between the particles of aggregate with smaller particles so as to reduce the amount of cement paste required. A deficiency in fine aggregate produces harshness and lack of workability in the mix. Similarly an excess of intermediate sizes is accompanied by harshness and lack of workability in the mix. An excess of fine material results in an excess of water and reduced strength. The volume of voids in coarse aggregate may amount to 45 percent and a suitable volumetric ratio of fine to coarse aggregate for this condition is 1:2.

Mix proportions are affected by the amount of moisture contained in the aggregate. Aggregate in a **saturated surface-dry** condition has absorbed its full complement of interior moisture without any free moisture appearing on the surface of the aggregate. The moisture content required to produce a saturated surface-dry condition depends on the properties of the aggregate and may be as much as two percent for fine aggregate and four percent for coarse aggregate. An increase in moisture content in excess of the saturated surface-dry amount results in free moisture being contained between the aggregate particles. A decrease in moisture content below the saturated surface-dry amount results in added moisture being fully absorbed by the aggregate particles until the saturated surface-dry condition is reached. In order to control the water-cement ratio of the mix, continuous checking of the moisture content in the aggregate is required during the mixing process.

In addition, **bulking** is produced in sand as the water content is increased. Bulking is caused by capillary forces which hold a film of water around the sand particles, forcing them apart, and results in damp sand occupying a larger volume than it would occupy if dry. Bulking is more pronounced in fine sand than in coarse sand and the volume may increase by as much as 40 percent at a moisture content of four to six percent. When batching of the mix is by volumetric ratios, close control of the percentage bulking must be maintained. Typical bulking curves for coarse grained and fine grained sands are shown in Figure 1-2.

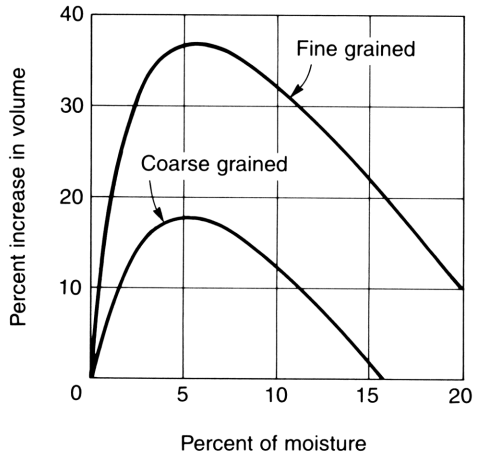

Figure 1-2 Bulking of sand

The average saturated surface-dry bulk density of normal weight aggregate is 95 to 100 pounds per cubic foot. **Lightweight aggregate** has a surface-dry bulk density of 70 pounds per cubic foot or less. Structural lightweight concrete is subdivided, by ACI Section 2.1 into the two categories:

- Sand-lightweight concrete in which the coarse aggregate is lightweight and the fine aggregate is natural sand. In accordance with ACI Section 2.1 the unit weight may not exceed 115 pounds per cubic foot.
- All-lightweight concrete is made entirely with lightweight aggregate and contains no natural sand.

Lightweight aggregates consist of naturally occurring pumice, scoria, and volcanic cinders, or manufactured expanded clay and shale and pelletized fly ash[2].

Admixtures are ingredients other than cement, aggregate, or water which are added to the mix to produce one of the following effects:

- Air-entraining admixtures consist of sulfonated hydrocarbons, resins, and fatty acids. These admixtures cause small air bubbles to be dispersed throughout the mix, improving workability of the mix and the resistance to freezing of the concrete.
- Accelerators consist of calcium chloride. This admixture reduces the setting time and accelerates the development of concrete strength which is advantageous in cold weather and allows the earlier removal and reuse of forms. An excessive amount of calcium chloride may result in corrosion of reinforcement, increase in drying shrinkage, increased susceptibility to sulphate attack, and an increase in alkali-silica reaction.
- Retarders consist of sucrose and lignite which retard the setting time of the mix. This may be advantageous during hot weather, when placing the concrete is likely to be delayed, or to facilitate the production of exposed aggregate surfaces.
- Superplasticizers consist of sulfonated melamine formaldehyde which enable the water-cement ratio to be reduced so as to increase concrete strength. At the same time, the workability of the mix may be improved thus assisting placing the mix in congested forms and allowing pumping of the concrete to be facilitated.
- Fly ash consists of the finely divided residue produced in coal-fired electric power generating plants. It has the advantages of improving the workability of the mix, reducing segregation and bleeding, and lowering the heat of hydration. In the hardened concrete, fly ash reduces permeability and alkali-aggregate reaction and improves resistance to sulfate attack. Disadvantages of fly ash include retarding the setting time of the mix and reducing the strength gain of the concrete.

Aggregates must be checked to ensure that harmful impurities are absent. These consist of:

- A clay film over the surface of the aggregate because of inadequate washing. This adversely affects the strength of the concrete by preventing adhesion between the cement paste and the aggregate.
- Fine dust, produced during the crushing process, and silt, which may similarly coat the aggregates and reduce the concrete strength.
- Organic impurities which may react adversely with the hydration process and result in inferior concrete.

- The presence of reactive silica or carbonate in the aggregate which may combine with alkali produced in the hydration process.. The expansive stresses produced by this reaction may cause extensive cracking and disruption of the surface.

1.1.2 Properties of the concrete mix

The amount of **water** added to the mix affects both the properties of the mix and the hardened concrete. Increasing water content increases the fluidity of the mix, enabling the mix to be placed more easily in the forms. Reducing water content increases the strength of the hardened concrete and reduces permeability and shrinkage. The amount of water in the mix may be expressed as either gallons per sack of cement or in terms of water-cement ratio by weight[2]. One sack of cement contains one cubic foot of cement and weighs 94 pounds. The relationship between 28 day compressive strength and water-cement ratio for comparable mixes of normal Portland cement and high early strength Portland cement is shown in Figure 1-3.

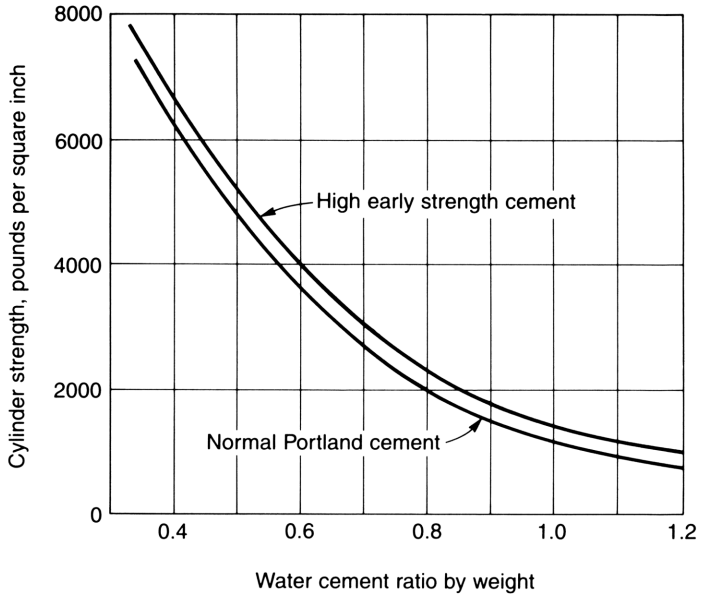

Figure 1-3 Compressive strength and water-cement ratio

Bleeding is caused by the solid particles of the mix settling to the bottom of the form due to gravity and excessive vibration[4]. The resulting concentration of water near the top surface increases the water-cement ratio and produces a weak layer in the concrete. Bleeding may be reduced by using an air-entraining admixture or by adding fly ash to the mix.

Consistency is a measure of the ease with which the freshly mixed concrete flows in the mold. The slump test[5] is used as a measure of consistency. A low-slump, or stiff, mix has a firm consistency. A high-slump, or wet, mix has a fluid consistency. Suitable slump values[6] for vibrated concrete for different types of construction are shown in Table 1-1. The slump test is a means of checking, in the field, to ensure that a uniform consistency is being maintained from different batches of a given mix. A change in slump indicates that some variation has occurred in the mix constituents or properties. With strict control of the concrete materials, slump increases proportionally with the water content of a given mix and is inversely related to concrete strength.

Type of construction	Slump in inches	
	Maximum	Minimum
Mass concrete	2	1
Foundations and footings	3	1
Beams, columns, and walls	4	1
Slabs	3	1

Table 1–1 Required slump

Plasticity is a measure of the ease with which a mix may be molded. A plastic mix is one in which the aggregate particles are completely surrounded by cement paste and in which segregation does not occur during handling and placing and the mix remains homogeneous.

Segregation is the separation of the mix into its separate constituents. Segregation may be avoided by correctly designing the mix and placing it in its final position in the mold.

Setting time of the mix controls the time available for placing the concrete. Setting time is reduced in hot weather and by the use of accelerating admixtures. Initial setting time occurs approximately two hours after adding mixing water to the mix.

Workability is a measure of the ease with which a mix may be placed and compacted in the form. Where congested reinforcement occurs, a highly workable mix is required.

Yield of the mix is the volume of concrete produced from one sack of cement.

1.1.3 Mixing and placing

Batching is the procedure of measuring out the individual constituents of the mix and placing them in the mixer[7]. Normally the proportions of the constituents are specified by weight and are measured to an accuracy of one percent.

The materials are placed in the mixer and thoroughly mixed by rotating blades to produce a uniform consistency. Concrete may be mixed at the jobsite in a stationary mixer or may be delivered, ready mixed, by trucks from a central mixing plant[7].

The concrete may be placed in the forms by conveyor belt, chute, crane skip, or by pumping[7]. The method adopted depends on the location of the member being concreted and the amount of concrete to be deposited.

Consolidation of the deposited concrete is required in order to produce a well compacted mass without honeycombs or entrapped air pockets. Consolidation may be provided either by manual tamping or by mechanical vibrators[8]. The reduced water–cement ratio that may be used with mechanical vibrators results in concrete of increased strength.

Curing the concrete consists of controlling the temperature and moisture content of the concrete, for a specified period, so that its design strength may be attained[9]. Curing of the concrete should start immediately after compaction and be designed to protect the concrete from:

- Premature drying out caused by wind or solar radiation.
- Leaching out of the cement paste by rain.
- Rapid cooling during the first few days after placing.
- Low temperature or frost.
- Excessively high temperature.

Moisture is necessary for the hydration of the cement, and curing for a minimum period of seven days is usually specified. Premature drying out of the concrete arrests the hydration process and results in a loss of strength. Curing may be effected by:

- Leaving the forms in place for the duration of the curing period.
- Covering the surface of the concrete with a waterproof sheet which is sealed to prevent evaporation of moisture from the concrete.
- Continuous fog spraying or sprinkling with water, after sufficient hardening of the concrete has occurred to prevent leaching out of the cement paste or erosion of the surface. In addition, care must be exercised to prevent alternate cycles of wetting and drying occurring, as this may cause surface cracking. This technique is unsuitable if there is any danger of a frost occurring.
- Spraying the surface with a membrane-forming curing compound. These consist of waxes, resins, and chlorinated rubber which form an impervious film on the concrete surface. At the time of application of the compound, the surface of the concrete should be moist but not damp with free water.
- Covering the concrete with a damp absorbent material such as hessian, straw, or sand, which is maintained continuously damp.

Temperature during the curing period should be maintained between 50°F and 70°F. Lower temperatures retard the strength development of the concrete and freezing temperatures may cause permanent damage. The following precautions are necessary or may be adopted during mixing and placing of the fresh concrete during cold weather:

- The aggregates and mixing water must be free from snow, ice, and frost.
- The mixing water and aggregate may be heated to 70°F.
- High early strength Portland cement may be used.
- An accelerating admixture may be added to the mix.
- Surfaces with which the fresh concrete will come into contact must be free from snow, ice, and frost.

The following precautions may be necessary during curing, in cold weather, to ensure that a minimum temperature of 50°F is maintained:

- Covering the concrete surface with insulating materials such as straw or vinyl foam.
- Warming the concrete with electrically heated pads to maintain a minimum temperature of 50°F.
- Using steam curing.

High temperatures during mixing, placing, and curing of the concrete may produce excessive loss of moisture and rapid stiffening of the concrete causing difficulties in compaction and an increased tendency to surface cracking. The following precautions may be necessary during mixing and placing of the fresh concrete during hot weather:

- The temperature of the aggregates may be lowered by refrigeration, sprinkling with cold water, or shading from the sun.
- The temperature of the mixing water may be lowered by adding crushed ice. Care is necessary to ensure that all the ice has melted before mixing is complete.
- A retarding admixture may be added to the mix to delay the initial set of the concrete.
- Surfaces with which the fresh concrete will come into contact may be cooled by shading from the sun and by covering the forms with damp hessian.

The following precautions may be necessary during curing in hot weather:

- Protect the concrete surface from the drying effects of sun and wind by covering with damp hessian.
- Protect the concrete from thermal shock on sudden cooling, due to low overnight temperatures, by covering the concrete surface with insulating materials.

Figure 1–4 shows how the strength of concrete is affected by curing temperature.

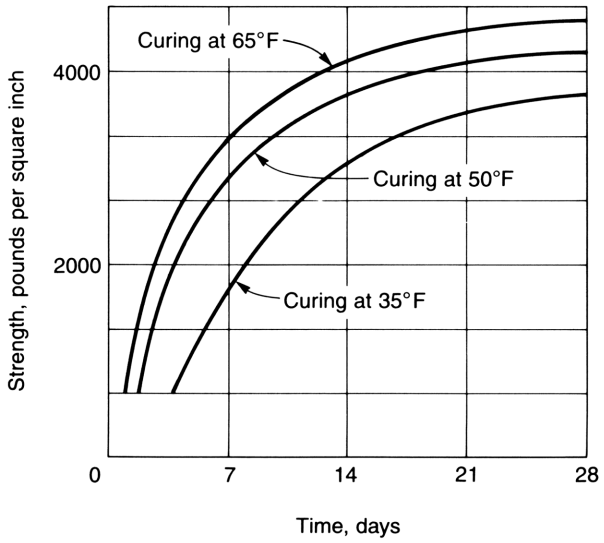

Figure 1–4 Effect of curing temperature on strength

ACI Section 5.11.1 requires that normal Portland cement concrete is maintained above 50°F in a moist condition for a minimum of seven days after placement. High early strength concrete is required by ACI Section 5.11.2 to be similarly cured for a minimum of three days. In accordance with ACI Section 5.11.3 curing times may be accelerated by using steam curing techniques.

During periods of cold or freezing weather, ACI Section 5.12 requires the concrete to be heated and the mix materials to be protected from the cold. During hot weather, ACI Section 5.13 requires precautions to be taken to prevent excessive concrete temperatures and loss of water by evaporation.

1.2 CONCRETE TECHNOLOGY

1.2.1 Durability of concrete

The mix proportions selected for a specific structure must provide not only for the required strength of the structure but also for an appropriate degree of durability of the structure in its working environment. The main characteristic affecting the durability of concrete is its permeability to the ingress of water or other harmful substances[10].

The **permeability** of normal-weight concrete decreases as the quantity of cement paste in the mix increases and as the water-cement ratio decreases. The quantity of cement in the mix may be indirectly controlled by specifying a maximum water-cement ratio, as an adequate cement content must then be provided to achieve satisfactory workability and consistency in the mix. ACI Section 4.2.2 specifies maximum water-cement ratios for various degrees of exposure and these limiting values take precedence over higher values which may be adequate for strength requirements. Fly ash, pozzolan, and ground granulated slag also reduce permeability and the maximum water-cement ratios, inclusive of these additives, specified by ACI Section 4.2.2 are:

0.50 For concrete exposed to water when a low permeability is required.
0.45 For concrete exposed to freezing and thawing when in a moist condition.
0.40 For concrete exposed to deicing salts, brackish water, or seawater.

In the case of lightweight aggregate concrete, the calculation of water-cement ratio is unreliable because of the uncertain water absorption of the aggregate. For both normal weight and lightweight aggregate concrete, ACI Section 4.2.2 specifies the following minimum levels of compressive strength:

4000 For concrete exposed to water when a low permeability is required.
4500 For concrete exposed to freezing and thawing when in a moist condition.
5000 For concrete exposed to deicing salts, brackish water, or seawater.

Improved resistance of the concrete to frost damage may be obtained by the use of air-entraining admixtures. Water expelled from the aggregate during freezing is accommodated in the air bubbles, relieving the internal pressures, and preventing the disruption of the concrete. As the maximum size of the aggregate increases, the mix requires a reduced air content. For severe exposure, in which the concrete is subjected to freeze-thaw cycles under wet conditions or in the presence of deicing chemicals, ACI Section 4.2.1 specifies a percentage air content for the total mix of:

5.5 to 7.0 For a nominal maximum aggregate size of 1.5 to 0.5 inches.

In addition, for severe exposure conditions, ACI Section 4.2.3 specifies the maximum percentage of fly ash, pozzolan, and slag which may be included in the concrete.

For moderate exposure, in which the concrete is subjected to freeze-thaw cycles without prior wetting and is not exposed to deicing chemicals, ACI Section 4.2.1 specifies a percentage air content for the total mix of:

4.5 to 5.5 For a nominal maximum aggregate size of 1.5 to 0.5 inches.

When the concrete is exposed to **sulphate-containing chemicals,** additional precautions are necessary. For very severe exposure, with a sulfate (SO_4) concentration in the water exceeding 10,000 parts per million, ACI Section 4.3.1 requires the use of a high sulfate resistant Portland cement with the addition of pozzolanic material to reduce the cement content. The maximum water-cement ratio is limited to 0.45, in the case of normal weight aggregate concrete, and the minimum compressive strength is specified as 4500 pounds per square inch for both normal weight and lightweight aggregate concrete.

For exposure to sea water, or water containing a sulfate concentration between 150 and 1500 parts per million, the maximum water-cement ratio is limited to 0.50, in the case of normal weight aggregate concrete, and the minimum compressive strength is specified as 4000 pounds per square inch for both normal weight and lightweight aggregate concrete. A sulphate resistant or blended cement is required.

The presence of **chlorides in concrete** may increase the risk of corrosion of the reinforcement. Chlorides may be present in all the constituents of the mix and cause corrosion by reacting with the protective oxide film on the reinforcement. ACI Section 4.4.1 limits the water soluble chloride ion (Cl^-) content in concrete to the following percentages by weight of cement:

0.15	Reinforced concrete exposed to chlorides in service.
0.30	Reinforced concrete that will be damp in service.
1.00	Reinforced concrete that will be dry in service.

Other factors affecting the durability of concrete are:

- The size of the element considered.
- Cover provided to the reinforcement, the requirements for which are specified in ACI Section 7.7.
- The type of aggregates used.
- The compaction of the concrete.
- The curing of the concrete.

1.2.2 Mix Proportions

Mix proportions may be specified either by weight or by volume in terms of the ratios of fine aggregate and coarse aggregate to cement[11]. In the case of a volumetric ratio, a 1:3:5 concrete mix consists of three parts by volume of fine aggregate and five parts by volume of coarse aggregate for each one part by volume of cement used. In the case of mix proportions specified by weight, a 1:3:5 concrete mix consists of three parts by weight of fine aggregate and five parts by weight of coarse aggregate for each one part by weight of cement used.

The water-cement ratio of the mix may be specified either in terms of weight of water to weight of cement or by gallons of water per sack of cement. One sack of cement has a volume of one cubic foot and a weight of 94 pounds. One gallon of water has a weight of 8.34 pounds and one cubic foot of water contains 7.48 gallons. The water-cement ratio is based on the use of saturated surface-dry aggregates and any deficiency or excess of moisture, from this value, must be allowed for. Water content in the aggregate, in excess of the saturated surface-dry water

content, is known as the free moisture content and this reduces the amount of the mixing water required. Water content in the aggregate, less than the saturated surface-dry water content, is known as moisture affinity and this increases the amount of the mixing water required. The free moisture content of the aggregate, expressed as a percentage of the saturated surface-dry weight is given by

$$m = 100(\gamma - \gamma_s)/\gamma_s$$

where γ = wet bulk density
= total weight of the constituent, including contained water, divided by the bulk volume
γ_s = saturated surface-dry bulk density
= total weight of the saturated surface-dry constituent divided by the bulk volume

and $\gamma = \gamma_S(1 + m/100)$

If G = specific gravity of the solid particles of the constituent

and γ_w = unit weight, or density of water

then $\gamma_w G$ = solid density
= weight of the solid constituent divided by the solid, or absolute, volume

Typical properties of the materials used are:

Material	Specific gravity	Bulk density	Solid density
Cement	3.1	94	193
Sand	2.6	100	162
Stone	2.6	95	162
Water	1.0	62.4	62.4

Table 1–2 Properties of concrete constituents

The yield of a mix is the volume of concrete produced from one sack of cement and is calculated assuming that all the voids in the sand and stone are filled with cement, water and entrained air. The ratio of entrained air to the total volume of air plus solids is

$$e = (V_e - V_s)/V_e$$

where V_s = volume of all solids

V_e = total volume of solids plus entrained air

and $V_e = V_s/(1 - e)$

The yield of a mix without voids (i.e., the solid volume) is given by the absolute, or solid, volume method as

$$V_s = \Sigma(\gamma_s V/G\gamma_w)$$

where V = bulk volume of the saturated surface-dry constituent

$\gamma_s V$ = total weight of the saturated surface-dry constituent

G = specific gravity, or relative density, of the constituent

$\gamma_w G$ = solid density

The yield of a mix with entrained air is given by

$$V_e = V_s/(1 - e)$$

where e = ratio of entrained air to the total yield of the mix

The cement factor is defined as the number of sacks of cement required to produce one cubic yard of concrete.

Example 1–1 (Yield of a mix proportioned by weight)

The required proportions of a concrete mix are 1:2.5:3.5 by weight with 5.5 gallons of water per sack and 5 percent air entrainment. The sand contains 4 percent excess moisture and the stone 2 percent deficiency. Using the specific gravities given in Table 1–2, determine the yield of the mix, the cement factor, and the quantities of water and aggregate required to provide one cubic yard of concrete.

Solution

The solid volumes of the constituents for a one sack mix are

cement	$= 94 \times 1/(3.1 \times 62.4)$	$= 0.486$ cubic feet
sand	$= 94 \times 2.5/(2.6 \times 62.4)$	$= 1.448$ cubic feet
stone	$= 94 \times 3.5/(2.6 \times 62.4)$	$= 2.028$ cubic feet
water	$= 5.5 \times 8.34/(1 \times 62.4)$	$= 0.734$ cubic feet

The total solid volume of all constituents is

$$V_s = 0.486 + 1.448 + 2.028 + 0.734$$
$$= 4.696 \text{ cubic feet}$$

Using 5 percent air entrainment, the yield is

$$V_e = V_s/(1 - e)$$
$$= 4.696/(1 - 0.05)$$
$$= 4.943 \text{ cubic feet per sack}$$

The cement factor is given by

$$c = 1 \times 27/4.943$$
$$= 5.46 \text{ sacks per cubic yard}$$

The weight of saturated surface-dry sand required for each cubic yard of concrete is

$$W_s = 94 \times 2.5c$$
$$= 94 \times 2.5 \times 5.46$$
$$= 1283 \text{ pounds}$$

Allowing for the 4 percent of excess moisture, the actual weight of sand required is

$$W = W_s(1 + m)$$
$$= 1283(1 + 0.04)$$
$$= 1334 \text{ pounds}$$

The weight of excess water in the sand is given by

$$\begin{aligned} W_f &= W - W_s \\ &= 1334 - 1283 \\ &= 51 \text{ pounds} \end{aligned}$$

The weight of saturated surface-dry stone required for each cubic yard of concrete is

$$\begin{aligned} W_s &= 94 \times 3.5c \\ &= 94 \times 3.5 \times 5.46 \\ &= 1796 \text{ pounds} \end{aligned}$$

Allowing for the 2 percent moisture deficiency, the actual weight of stone required is

$$\begin{aligned} W &= W_s(1 - m) \\ &= 1796(1 - 0.02) \\ &= 1760 \text{ pounds} \end{aligned}$$

The water deficiency in the stone is

$$\begin{aligned} W_c &= W_s - W \\ &= 1796 - 1760 \\ &= 36 \text{ pounds} \end{aligned}$$

Hence, the total amount of water needed for each cubic yard of concrete is

$$\begin{aligned} W_w &= 5.5c + (W_c - W_f)/8.34 \\ &= 5.5 \times 5.46 + (36 - 51)/8.34 \\ &= 28.23 \text{ gallons} \end{aligned}$$

Thus, to produce one cubic yard of concrete, the constituents required are 5.46 sacks of cement, 1334 pounds of sand, 1760 pounds of stone, and 28.23 gallons of water.

Example 1–2 (Yield of a mix proportioned by volume)

The required proportions of a concrete mix are 1:2:3 by volume with 5 gallons of water per sack. Using the specific gravities given in Table 1–2, determine the yield of the mix, the cement factor, and the quantities of water and aggregate required to provide one cubic yard of concrete.

Solution

The solid volumes of the constituents for a one sack mix are

cement	$= 94 \times 1/(3.1 \times 62.4)$	$= 0.486$ cubic feet
sand	$= 100 \times 2/(2.6 \times 62.4)$	$= 1.233$ cubic feet
stone	$= 95 \times 3/(2.6 \times 62.4)$	$= 1.757$ cubic feet
water	$= 5/7.48$	$= 0.668$ cubic feet

The total solid volume of all constituents is

$$\begin{aligned} V_s &= 0.486 + 1.233 + 1.757 + 0.668 \\ &= 4.144 \text{ cubic feet} \end{aligned}$$

The cement factor is given by

$$c = 1 \times 27/4.144$$
$$= 6.52 \text{ sacks per cubic yard}$$

The volume of sand required for each cubic yard of concrete is

$$V = 2c$$
$$= 13.04 \text{ cubic feet}$$

The volume of stone required for each cubic yard of concrete is

$$V = 3c$$
$$= 19.56 \text{ cubic feet}$$

The total amount of water needed for each cubic yard of concrete is

$$W_w = 5c$$
$$= 5 \times 6.52$$
$$= 32.60 \text{ gallons}$$

Thus, to produce one cubic yard of concrete, the constituents required are 6.52 sacks of cement, 13.04 cubic feet of sand, 19.56 cubic feet of stone, and 32.60 gallons of water. ═══

1.3 MIX DESIGN

1.3.1 Control of concrete quality

To allow for variation in quality control at the concrete plant, the concrete mix must be designed for a target strength in excess of the works specified design strength. A works strength test, in accordance with ACI Section 5.6.1.4, consists of the average of the strengths of two cylinders made from the same sample of concrete and tested at 28 days. ACI Section 5.6.1.1 requires the minimum sampling frequency for each class of concrete to be not less than:

- Once each day.
- Once for each 150 cubic yards of concrete placed each day.
- Once for each 5000 square feet of slab or wall surface area placed each day.

A minimum of five samples from five randomly selected batches is desirable, or samples should be taken from each batch if fewer than five batches are made.

The concrete strength is considered satisfactory provided that:

- The average of any three consecutive strength tests is not less than the specified design strength.
- No individual strength test falls below the specified design strength by more than 500 pounds per square inch.

If either of the above criteria is not obtained, the mix shall be redesigned using a higher target strength.

1.3.2 Mix design criteria

Design of a concrete mix requires the selection of the correct proportions of cement, admixtures, fine aggregate, coarse aggregate, and water to produce concrete with the necessary properties. These properties include:

- The **workability** of the mix which depends on the water content of the mix. The water content must be adjusted to produce a mix which can be placed and compacted in the forms with the equipment available on site.
- The **durability** requirements of ACI Sections 4.2, 4.3, and 4.4, which specify the maximum water–cement ratio, the minimum cement content, the minimum compressive strength, the type of cement, and the type of admixture.
- The **compressive strength** of the hardened concrete which depends on the water–cement ratio of the mix. The minimum compressive strength is governed by the structural design requirements for the element.

Because of the variation of concrete strengths, the concrete mix must be designed to have a target mean strength f'_{cr} much greater than the required design strength f'_c. The variation in concrete strength is attributable to:

- Variation in the quality of the cement supplied at different times or from different manufacturers.
- Variation in the grading or type of aggregates used, which may necessitate a change in the water–cement ratio to maintain the required workability.
- Variation in the mix proportions caused by inaccuracies in the batching equipment or by human error.
- Variation in the making, curing, and testing of the test cylinders.

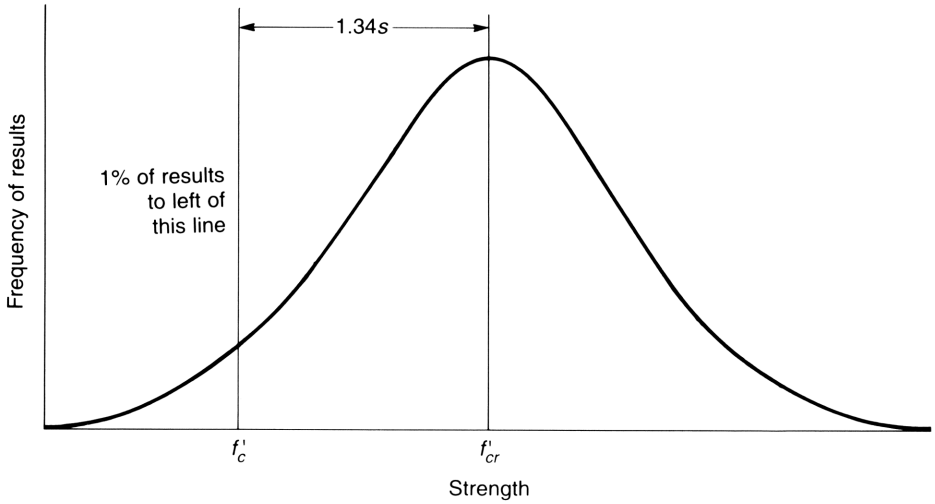

Figure 1–5 Normal distribution of concrete strengths

When a sufficient number of compressive strength test results are available, the distribution of test results are symmetrical about a mean value f'_{cr} and follow a normal Gaussian distribution curve as shown in Figure 1–5. The area beneath the curve represents the total number of test results and the mean value represents the target strength.

The difference between the target strength and the required design strength is determined statistically to ensure that the number of defective test results do not exceed an acceptable number. The margin between the target strength and the design strength is based on the variability of previous test results obtained from a similar concrete mix, where these are available, and is determined from the standard deviation which is obtained from the expression

$$s = [\Sigma(X_i - \overline{X})^2/(n - 1)]^{0.5}$$

where,
- X_i = an individual test result (the average of two cylinder tests)
- n = the number of consecutive tests (minimum of 30 cylinder tests)
- $\overline{X}$ = the mean of n results

The standard deviation used to calculate the margin must be determined from statistical data obtained from concrete mixes using essentially the same plant, ingredients, proportions, and supervision, and producing a concrete with a similar compressive strength. When suitable records are not available, the ACI Code permits the mix to be designed by either of the two alternative methods:

- Using a prescribed margin which increases as the required design strength is increased.
- Using a prescribed water–cement ratio to produce the required design strength.

1.3.3 Mix design based on statistical data

ACI Section 5.3 provides a statistical method of control based on the past test records of the concrete plant. When the records of at least 30 consecutive strength tests are available, based on similar materials and for a specified strength within 1000 pounds per square inch of the works specified strength, the required target mean strength is given by the larger value obtained from the expressions

$$f'_{cr} = f'_c + 1.34s$$
$$f'_{cr} = f'_c + 2.33s - 500$$

where
- f'_{cr} = required target mean strength
- f'_c = specified compressive strength
- s = standard deviation of the strength tests
- $1.34s$ = required margin

The first expression provides a probability that 1–in–100 of the averages of three consecutive tests will be below f'_c. The second expression provides a probability that 1–in–100 of the individual tests will be more than 500 pounds per square inch below f'_c. When the results of 30 consecutive tests are not available, it is acceptable to use two groups of consecutive tests totaling at least 30 tests. The applicable standard deviation is the statistical average of the values calculated for each group and is given by

$$\overline{s} = \{[(n_1 - 1)(s_1)^2 + (n_2 - 1)(s_2)^2]/(n_1 + n_2 - 2)\}^{0.5}$$

where
- n_1 = number of tests in group 1
- n_2 = number of tests in group 2
- s_1 = standard deviation of the tests in group 1
- s_2 = standard deviation of the tests in group 2

The margin specified by the code is based on the standard deviation calculated from test results obtained using the same equipment, materials, and supervision as will be used in the proposed work. This implies that the standard deviation and the required margin remain sensibly constant irrespective of the specified strength, and the target mean strength is obtained by adding a multiple of the standard deviation to the specified strength. However, since the standard deviation may vary with the strength level, ACI Section 5.3.1 requires the test data to be obtained from a concrete with a compressive strength within 1000 pounds per square inch of that specified for the proposed work.

The standard deviation calculated from 30 test results is a reasonable estimate of the standard deviation of the total population. When fewer than 30 results are available, the smaller sample may underestimate the true standard deviation. When the records of 15 to 29 consecutive strength tests only are available, the calculated standard deviation, in accordance with ACI Section 5.3.1.2, must be increased by the modification factor obtained from Table 1.3. The strength tests must span a period of not less than 45 days.

Number of tests	Modification factor
15	1.16
20	1.08
25	1.03
30 or more	1.00

Table 1–3 Modification factor for standard deviation

When the average compressive strength of the test records is not less than the required target mean strength, the same mix proportions may be used for the concrete in the proposed work, provided that the special exposure requirements of ACI Section 4.2, 4.3, and 4.4 are satisfied. When the average compressive strength of the test records is less than the required target mean strength, the mix proportions for the concrete in the proposed work are determined by the trial mix method. Three trial mixes are made using similar materials and conditions with three different water–cement ratios and three cylinders prepared and tested from each mix. From the results of the cylinder tests, a curve is plotted, similar to Figure 1–3, showing the relationship between the water–cement ratio and the compressive strength. The required water–cement ratio for the concrete in the proposed work may be interpolated from the curve.

Example 1–3 (Standard deviation and target mean strength)
A concrete plant is to provide a concrete mix with a specified design strength of 3000 pounds per square inch. The records of 15 consecutive strength tests are: 3050, 3100, 3150, 3250, 3300, 3350, 3480, 3500, 3520, 3650, 3700, 3750, 3850, 3900, 3950. What is the required target strength of the mix?

Solution

The mean value of the strength tests is

$$\begin{aligned}\overline{X} &= \Sigma f'_c/n \\ &= 52{,}500/15 \\ &= 3500 \text{ pounds per square inch}\end{aligned}$$

The standard deviation of the strength tests is

$$\begin{aligned}s &= [\Sigma(X_i - \overline{X})^2/(n-1)]^{0.5} \\ &= (1{,}220{,}800/14)^{0.5} \\ &= 295 \text{ pounds per square inch}\end{aligned}$$

From Table 1–3, the modification factor for 15 records is 1.16. Applying ACI Equation (5–2), the required target strength is given by

$$\begin{aligned}f'_{cr} &= f'_c + 2.33 \times 1.16s - 500 \\ &= 3000 + 2.33 \times 1.16 \times 295 - 500 \\ &= 3297 \text{ pounds per square inch.}\end{aligned}$$

Applying ACI Equation (5–1), the required target strength is given by

$$\begin{aligned}f'_{cr} &= f'_c + 1.34 \times 1.16s \\ &= 3000 + 1.34 \times 1.16 \times 295 \\ &= 3459 \text{ pounds per square inch ... governs.}\end{aligned}$$

1.3.4 Mix design based on a prescribed margin

When a suitable record of test results is not available, in accordance with ACI Section 5.3.2.2 the margin used to determine the target mean strength depends on the required design strength and may be obtained from Table 1–4.

Specified strength f'_c pounds per square inch	Required margin pounds per square inch
Less than 3000	1000
3000 to 5000	1200
over 5000	1400

Table 1–4 Prescribed margin

Using the specified value of the margin, the required target mean strength is calculated and trial mixes used to determine the water–cement ratio.

Example 1–4 (Prescribed margin and target mean strength)
A concrete mix is required with a specified design strength of 3000 pounds per square inch. No records of strength tests are available. What is the required target strength of the mix?

Solution
The prescribed margin is obtained from Table 1–4 and for 3000 pounds per square inch concrete is

$$k = 1200 \text{ pounds per square inch}$$

The required target mean strength is

$$f'_{cr} = f'_c + 1200$$
$$= 3000 + 1200$$
$$= 4200 \text{ pounds per square inch}$$

1.3.5 Mix design without trial mixes

When a suitable record of test results is not available and it is impracticable to make trial mixes, ACI Section 5.4 permits concrete proportions to be based upon other information or experience, provided special permission is obtained from the architect or engineer. Similar concrete mixes used in previous work may be used as a guide. To ensure that all eventualities are covered, the mix selected shall have been previously used with similar materials to those proposed in the current work and have a compressive strength not less than 1200 pounds per square inch greater than that required in the current work. This method is suitable only for small jobs or emergency situations, and may not be used for compressive strengths greater than 4000 pounds per square inch because variations in the mix proportions may cause large variations in the strength of the concrete.

1.4 CONCRETE TESTING

1.4.1 Testing the constituents of the mix

The **grading** of an aggregate is the determination of its particle size distribution by means of a sieve analysis[12]. Standard wire mesh sieves with square openings are used, and a sample of the aggregate is successively sieved through a set of sieves mounted in a vibrating machine. For sand, seven standard sizes are used ranging from ⅜ inch to number 100 (0.15 millimeter). For coarse aggregate, thirteen standard sizes are used ranging from four inches to number 16 (1.18 millimeter). The weight of aggregate retained on each sieve is recorded and the total percentage by weight passing each sieve is calculated and plotted as a grading curve. The abscissae of the graph represent particle size to a logarithmic scale and the ordinates represent percentage by weight finer than the corresponding sieve size. The curves for finer material plot higher and further to the left than those for coarser material. Figure 1–6 shows the sieve analysis[13] of a

natural sand and two continuous gradings obtained by combining this sand with crushed granite aggregate in the proportions by weight of 1:1.9 (combined grading number 1) and 1:1.1 (combined grading number 2).

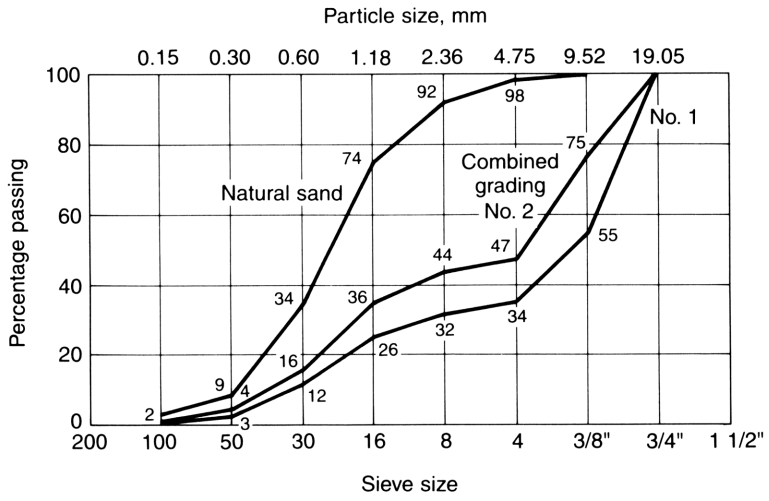

Figure 1–6 Grading curves for natural sand and combined aggregate

The **fineness modulus** of an aggregate indicates its relative fineness, the higher the fineness modulus index, the coarser the aggregate. The fineness modulus is calculated by adding the cumulative percentages by weight of the aggregate retained on each of the standard sieves and dividing the sum by 100. The fineness modulus may be used to select the volume of coarse aggregate required to produce a workable mix[11].

Example 1–6 (Determination of the fineness modulus)
Determine the fineness modulus of the natural sand shown in the grading curve of Figure 1–6.

Sieve size	Percentage passing	Percentage retained	Cumulative percentage retained
⅜ inch	100	0	0
No. 4	98	2	2
No. 8	92	6	8
No. 16	74	18	26
No. 30	34	30	56
No. 50	9	35	91
No. 100	2	7	98
Total	-	98	281

Table 1–5 Fineness modulus

Solution

The calculation proceeds as shown in Table 1–5

The total cumulative percentage retained by weight is

$$\Sigma W = 281 \text{ percent}$$

The fineness modulus is given by

$$FM = \Sigma W/100 = 281/100 = 2.81$$

The **specific gravity** of an aggregate is the ratio of the weight of a given volume of aggregate to the weight of the same volume of water. For **fine aggregate**[14], a sample of the aggregate, weighing approximately 500 grams, is soaked in water for 24 hours and then dried to a saturated surface–dry condition, weighed, and placed in a 500 cubic centimeters flask. The flask is filled with water to the calibrated capacity while stirring the sample to expel any air trapped in the voids. The weight of the flask filled with water plus the sample is recorded. The weight of the flask filled only with water is also recorded. Finally, the sample is oven–dried and weighed. The specific gravity of the aggregate is given by Archimedes' principle as

$$G = W_a/W_w = W_a/(W_s + W_1 - W_2)$$

where W_a = weight of the oven–dried sample
W_w = weight of water displaced by the saturated surface–dry sample
W_1 = weight of flask filled with water
W_2 = weight of flask filled with water plus the saturated surface–dry sample
W_s = weight of the saturated surface–dry sample

For **coarse aggregate**[15], a sample of the aggregate is soaked in water for 24 hours and then dried to a saturated surface–dry condition and weighed. The sample is now placed in a wire basket and weighed while immersed in water. Finally, the sample is oven–dried and weighed. The specific gravity of the aggregate is given by

$$G = W_a/W_w = W_a/(W_s + W_i)$$

where W_i = weight of the immersed sample

The **absorption** of a sample is the ratio, expressed as a percentage, of the weight of water absorbed by a saturated surface–dry sample to the oven–dry weight of the sample and is given by

$$A_b = 100(W_s - W_a)/W_a$$

The **bulk density** or unit weight of an aggregate is the ratio of the weight of a sample of aggregate to its volume[16]. To determine the **dry bulk density**, a container of known volume and weight is filled to the brim in three equal layers with the oven dried sample. A tamping rod is

used to compact each layer and the weight of the sample plus the container is recorded. The bulk density of the oven-dried sample is given by

$$\gamma_d = W_a/V_c = (W_t - W_c)/V_c$$

where W_a = weight of oven dried sample
W_t = weight of oven dried sample plus weight of container
W_c = weight of container
V_c = volume of container

The bulk density of a **saturated surface-dry** sample is given by

$$\gamma_s = \gamma_d(1 + A_b/100)$$

where A_b = absorption of the sample

The **void content** of an aggregate is the ratio, expressed as a percentage, of the volume of voids between the particles of a sample to the total volume of the sample and is given by

$$n = 100(V_n - V_s)/V_n = 100(G\gamma_w - \gamma_d)/G\gamma_w$$

where V_n = total volume of solids plus voids
V_s = volume of solids only
G = specific gravity of the solid particles of the sample
γ_d = bulk density of the oven-dried sample
γ_w = density of water

The **free moisture content** of an aggregate is the ratio, expressed as a percentage, of the weight of surface moisture in a sample of the aggregate to the saturated surface-dry weight of the sample. To obtain the **total moisture content**[17] a sample of the damp aggregate is weighed. The total moisture is now removed by oven drying or by burning alcohol on the sample. The weight of the oven-dried sample is recorded. The total moisture content is given by

$$m_t = 100(W - W_a)/W_a$$

where W = weight of the sample when damp
W_a = weight of the oven-dried sample

The free moisture content is the difference between the total moisture content and the absorbed moisture. The free moisture content is given by

$$m = m_t - A_b$$

The presence of **organic impurities** in fine aggregate is detected by the calorimetric test[18]. The sample of fine aggregate is soaked in a solution of sodium hydroxide for 24 hours and its color compared with a calorimetric standard. The presence of impurities is indicated when the color is darker than the standard.

1.4.2 Testing for consistency of the mix

The consistency of fresh concrete is determined with the **slump test** which utilizes a standard metal cone[5] shown in Figure 1–7. The cone is filled with the sample in three approximately equal layers, each layer being tamped 25 times with a ⅝ inch diameter by 24 inches long steel rod. After the cone is filled, the sample is struck off smooth at the top, and the cone carefully lifted off as the sample subsides downwards. The empty cone is now placed alongside the sample and the amount of slump measured to the nearest ¼ inch. The test must be started within five minutes of taking the sample and be completed in 2½ minutes, as the consistency is affected by delay[13].

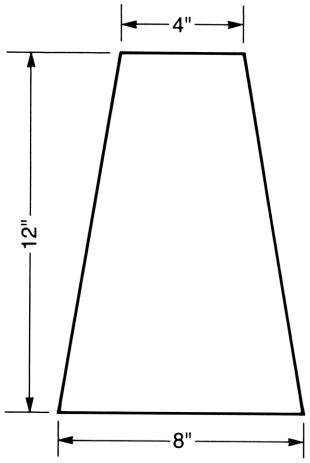

Figure 1–7 Standard slump cone

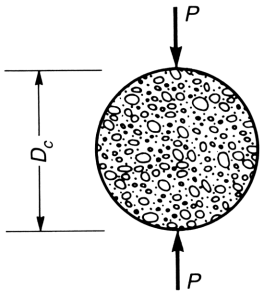

Figure 1–8 Splitting cylinder test

1.4.3 Testing of the concrete

Compressive strength of the concrete is determined from 6 inches diameter by 12 inches high test specimens[19]. The fresh concrete is placed in the mould within 15 minutes of sampling and compacted by either tamping or vibration. The top surface is troweled smooth, covered with a plastic cap, and cured under conditions of controlled temperature and humidity for 28 days. The ends of the cylinders are then capped with a sulfur compound[20] and loaded at a constant rate, in a compression testing machine, to failure[21].

When necessary, specimens of hardened concrete may be taken from the completed structure by cutting cores. The cylinder obtained must have a diameter of not less than three times the maximum size of the coarse aggregate, and a length approximately equal to twice the diameter. For shorter lengths, a correction factor must be applied to the result.

The Schmidt rebound hammer may be used as a non destructive method of measuring in situ concrete strength. The measured rebound of a spring loaded plunger is an indication of the hardness of the concrete surface and the correlation between the rebound and compressive strength may be determined empirically,

1.5 CONCRETE PROPERTIES

1.5.1 Compressive strength

The compressive strength of concrete depends primarily on water–cement ratio, age, and curing conditions. The effect of water–cement ratio on compressive strength is shown in Figure 1–3. The effect of age of the concrete on strength development is shown in Figure 1–1. The effect of curing temperature on compressive strength is shown in Figure 1–4. Design strengths are usually based on the cylinder strength at 28 days.

1.5.2 Tensile strength

The **direct tensile strength** of concrete is determined by means of the split cylinder test[22]. This consists of loading a standard six inches diameter cylinder laterally, as shown in Figure 1–8, until it splits. The value of the tensile strength obtained is used in the design of lightweight concrete. The splitting tensile strength is given by

$$f_{ct} = 2P/\pi \ell D_c \text{ pounds per square inch}$$

where P = maximum applied load, pounds
ℓ = length of cylinder, inches
D_c = diameter of cylinder, inches

ACI Section 9.5.2 specifies the modulus of rupture of lightweight concrete as

$$f_r = 7.5f_{ct}/6.7$$

where f_{ct} = splitting tensile strength of lightweight concrete

ACI Section 11.2, in determining the shear strength of lightweight concrete, specifies the square root of the cylinder strength as

$$(f'_c)^{0.5} = f_{ct}/6.7$$

Similarly ACI Section 12.2, in determining the development length of reinforcement for lightweight concrete, specifies a modification factor of

$$\lambda = 6.7(f'_c)^{0.5}/f_{ct}$$

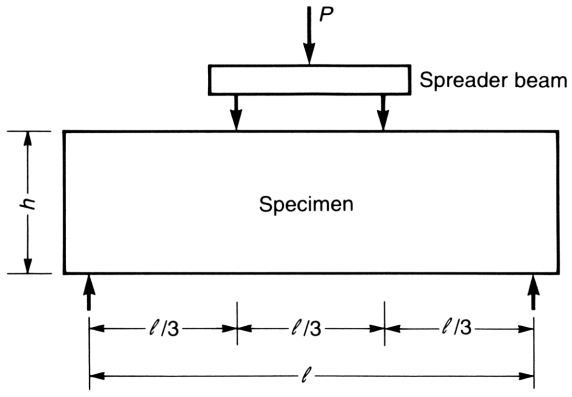

Figure 1–9 Flexural strength test

The **modulus of rupture** of concrete is the tensile resistance of the concrete in flexure. The modulus of rupture may be obtained[23] by testing concrete beams in flexure as shown in Figure 1-9. The modulus of rupture is given by

$$f_r = M/S$$
$$= P\ell/bh^2$$

where M = bending moment at failure
$= P\ell/6$ pound inches
S = section modulus of the beam
$= bh^2/6$
P = maximum applied load, pounds
ℓ = span length, inches
b = specimen width, inches
h = specimen depth, inches

ACI Section 9.5.2, in determining the effective moment of inertia of an element, specifies the modulus of rupture of normal weight concrete as

$$f_r = 7.5(f'_c)^{0.5}$$

1.5.3 Modulus of elasticity

The modulus of elasticity of concrete depends on the rate of loading. The modulus of elasticity obtained from a high-frequency vibration test[24] may be 50 percent higher than that obtained from a static test[25]. For concrete with a density between 90 and 155 pounds per cubic foot, ACI Section 8.5 specifies the modulus of elasticity as

$$E_c = 33(w_c)^{1.5}(f'_c)^{0.5}$$

where w_c = concrete density in pounds per cubic foot

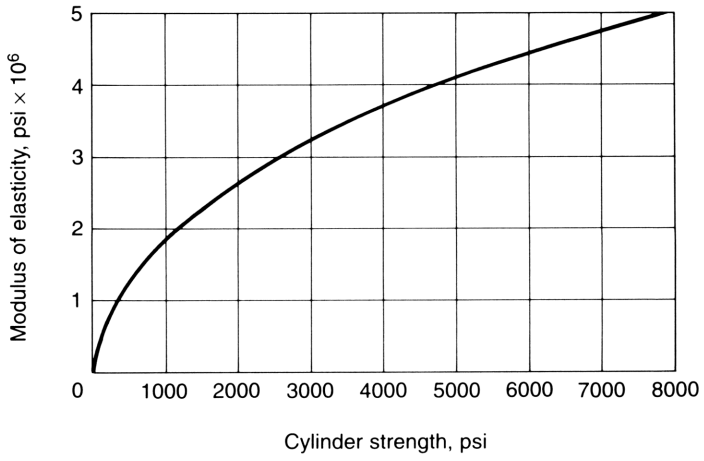

Figure 1-10 Modulus of elasticity of concrete

For normal weight concrete, the modulus of elasticity is given by

$$E_c = 57{,}000(f'_c)^{0.5}$$

and this is plotted in Figure 1–10.

The stress–strain curve for concrete is a non-linear relationship, as shown in Figure 1–11, and the modulus defined by the ACI Code is the secant modulus at a stress of 0.45 times the compressive strength.

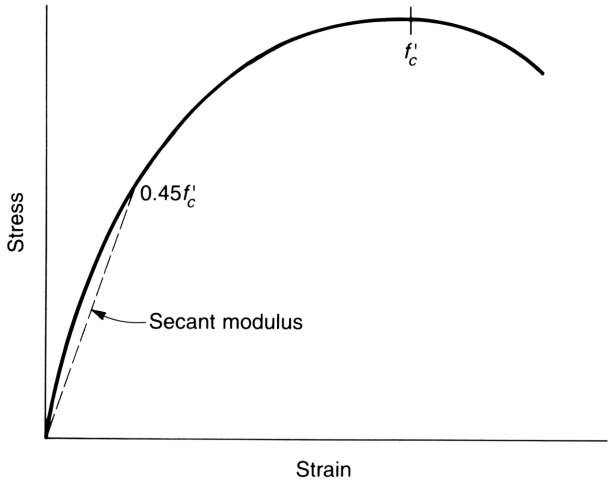

Figure 1–11 Concrete stress–strain relationship

1.5.4 Creep

Concrete, subjected to a sustained compressive force, undergoes an increasing deformation with time which is additional to the initial elastic compressive strain. This additional deformation continues at a decreasing rate as long as the concrete is loaded and is known creep, and this may be obtained from the testing of concrete cylinders[26]. Figure 1–12 shows the relationship between strain and time of loading.

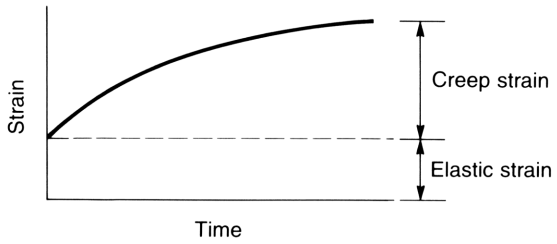

Figure 1–12 Creep in concrete

The magnitude of the creep which occurs is directly proportional to the applied stress, provided the stress does not exceed 40 percent of the compressive strength. The earlier the age of the concrete at which loading is applied, the greater is the creep. Creep reduces as the compressive strength of the concrete increases. Curing conditions also affect the magnitude of creep with steam curing reducing creep compared with air curing.

Creep affects the long-term deflection of reinforced concrete members. ACI Section 9.5.2.5 specifies an increase in deflection due to creep, over a five year period, of twice the initial elastic deflection. The creep at one year is 70 percent of the creep at five years. Additional affects of creep are the loss of prestress in prestressed concrete beams and the redistribution of stress from the concrete to the reinforcement in reinforced concrete members.

1.5.5 Shrinkage

Shrinkage, of an unrestrained concrete member, is due to the progressive removal of moisture from the concrete. Conversely, swelling is produced when concrete is maintained in a damp condition. Shrinkage increases as the water-cement ratio of the concrete mix is increased, as the ratio of surface area to volume of the member increases, and as the ambient relative humidity decreases. The maximum shrinkage strain may approach a value of 0.0008 inches per inch.

When a concrete member is restrained, so that a reduction in length due to shrinkage cannot take place, tensile stresses are caused. This may cause extensive cracking in the concrete unless control joints are provided in long members or reinforcement is provided. ACI Section 7.12 details the shrinkage reinforcement required in structural slabs.

Differential shrinkage of slabs on grade is caused by the upper surface of the slab being warmer and drier than the underside. This causes curling of the slabs and the edges at the joints curl upward. Curling may be reduced by preventing moisture reaching the underside of the slab, by reducing the distance between control joints, and by increasing the thickness of the slab.

1.6 REINFORCEMENT PROPERTIES

1.6.1 Deformed bars

Deformed bars[27,28] are round bars with projecting ribs which are anchored in the concrete and improve the bond resistance. ACI Section 3.5.1 requires all reinforcement to be deformed reinforcement except for spiral ties which may be plain reinforcement. Bar sizes are designated by numbers with Nos. two to eight representing the number of eighths of an inch in the bar diameter. Bar Nos. 9, 10, 11, 14, and 18 have areas equivalent to that of square bars with a side dimension of 1, 1⅛, 1¼, 1½, and 2 inches. Strength of reinforcement is specified by Grade number which is indicative of its minimum yield strength. Thus Grade 40 reinforcement has a yield strength of 40,000 pounds per square inch, Grade 60 reinforcement has a yield strength of 60,000 pounds per square inch, and Grade 75 reinforcement has a yield strength of 75,000

pounds per square inch. ACI Section 3.5.3.2 requires that specified yield strengths exceeding 60,000 pounds per square inch shall be the stress corresponding to a strain of 0.35 percent. This is to ensure that adequate plasticity is available to develop the ultimate strength required. Typical stress–strain curves for reinforcing bars are shown in Figure 1–13.

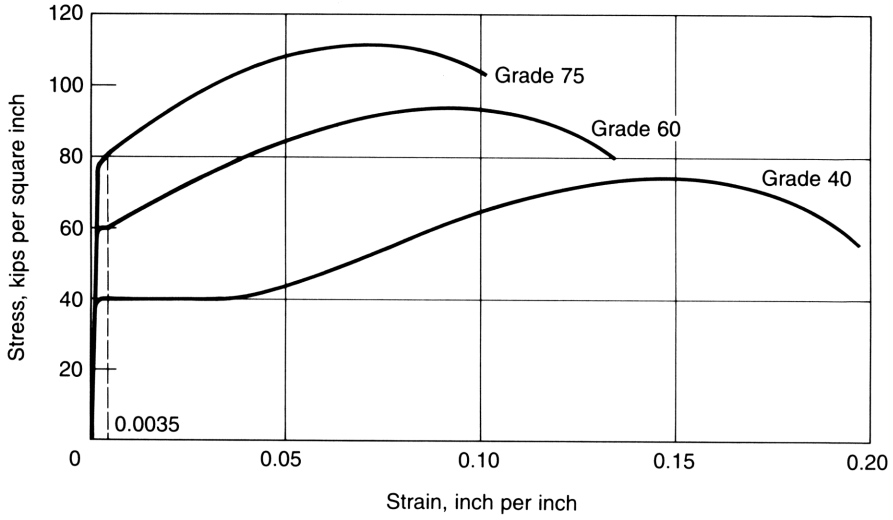

Figure 1–13 Stress–strain curves for reinforcing bars

ACI Section 8.5.2 specifies the modulus of elasticity of steel reinforcement as

$$E_s = 29{,}000{,}000 \text{ pounds per square inch}$$

Details of reinforcing bars corresponding to ASTM A 615 are provided in Table 1–6.

Grade	Minimum yield pounds per square inch	Minimum tensile strength pounds per square inch
40	40,000	70,000
60	60,000	90,000
75	75,000	100,000

Table 1–6 ASTM A 615 reinforcement

1.6.2 Welding reinforcement

In accordance with ACI Section 3.5.2, reinforcement to ASTM A 615 may be welded, provided the chemical composition is known and the correct welding procedure based on the carbon equivalent is specified by the designer. If the carbon content of the steel exceeds 35 percent, special requirements are necessary to ensure satisfactory welding. These include preheating the work and the use of special electrodes. Reinforcement manufactured to ASTM A 706 is a low-alloy steel which is specially formulated for welding and requires no supplementary requirements.

1.6.3 Corrosion protection

To provide additional protection against corrosion, reinforcing bars may be galvanized[29] or epoxy-coated[30], in accordance with ACI Section 3.5.3.7. Epoxy-coated reinforcement has a reduced bond strength as the coating prevents adhesion between the bar and the concrete. ACI Section 12.2.4 provides a special modification factor for determining the development length. Special handling techniques are required during construction to prevent damage to the coating.

Tables 1-7 and 1-8 provide details of reinforcement bar areas.

Bar size	Bar diameter	Number of bars							
		1	2	3	4	5	6	7	8
3	0.38	0.11	0.22	0.33	0.44	0.55	0.66	0.77	0.88
4	0.50	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60
5	0.63	0.31	0.62	0.93	1.24	1.55	1.86	2.17	2.48
6	0.75	0.44	0.88	1.32	1.76	2.20	2.64	3.08	3.52
7	0.88	0.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80
8	1.00	0.79	1.58	2.37	3.16	3.95	4.74	5.53	6.32
9	1.13	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00
10	1.27	1.27	2.54	3.81	5.08	6.35	7.62	8.89	10.16
11	1.41	1.56	3.12	4.68	6.24	7.80	9.36	10.92	12.48

Table 1-7 Areas of bars

Bar size	Bar spacing, inches									
	3	4	5	6	7	8	9	10	11	12
3	0.44	0.33	0.26	0.22	0.19	0.17	0.15	0.13	0.12	0.11
4	0.80	0.60	0.48	0.40	0.34	0.30	0.27	0.24	0.22	0.20
5	1.24	0.93	0.74	0.62	0.53	0.47	0.41	0.37	0.34	0.31
6	1.76	1.32	1.06	0.88	0.75	0.66	0.59	0.53	0.48	0.44
7	2.40	1.80	1.44	1.20	1.03	0.90	0.80	0.72	0.65	0.60
8	3.16	2.37	1.90	1.58	1.35	1.19	1.05	0.95	0.86	0.79
9	4.00	3.00	2.40	2.00	1.71	1.50	1.33	1.20	1.09	1.00
10	5.08	3.81	3.05	2.54	2.18	1.91	1.69	1.52	1.39	1.27
11	6.24	4.68	3.74	3.12	2.67	2.34	2.08	1.87	1.70	1.56

Table 1-8 Areas of bars at various spacings

1.7 SUPPLEMENTARY PROBLEMS

Problem 1–1 (Concrete properties[31])
Answer each part of this problem with a brief discussion.

(i) What are two major benefits of air entrainment in concrete? What is a disadvantage? What is considered to be a normal range of air entrainment?

(ii) What is the primary function of calcium chloride in a concrete mix? What are two disadvantages of its use?

(iii) When welding reinforcing steel, what is meant by preheat, carbon–equivalent, and heat–affected zone?

(iv) What is the main advantage of ASTM A706 reinforcing steel?

(v) Define the terms creep and shrinkage when applied to concrete. List three factors which influence creep and three factors which influence shrinkage.

Problem 1–2 (Concrete properties)
Answer each part of this problem with a brief discussion.

(i) Name three precautions that should be observed when concrete is poured on a hot, windy day.

(ii) Name three precautions that should be observed when concrete is poured during cold weather.

(iii) What are the three principal benefits of using a retarding admixture in concrete?

(iv) What are the two principal benefits of using a superplasticizer admixture in concrete?

(v) What are five benefits of adding fly ash to a concrete mix? What are two disadvantages of its use?

Problem 1–3 (Yield of a mix proportioned by weight)
The required proportions of a concrete mix are 1:2:3 by weight with a water–cement ratio of 0.4. Determine the yield of the mix.

Problem 1–4 (Yield of a mix proportioned by volume)
The required proportions of a concrete mix are 1:2:4 by volume with 6 gallons of water per sack. Determine the yield of the mix and the quantities of water and aggregate required to provide forty cubic yards of concrete.

Problem 1–5 (Target mean strength)
A concrete mix is required with a specified design strength of 4000 pounds per square inch. Records are available of two groups of consecutive strength tests using similar materials and having a specified design strength of 4000 pounds per square inch. Group 1 consists of 14 tests with a standard deviation of 280 pounds per square inch. Group 2 consists of 16 tests with a standard deviation of 320 pounds per square inch. What is the required target strength of the mix?

Problem 1–6 (Target mean strength)
A concrete mix is required with a specified design strength of 5500 pounds per square inch. No records of strength tests are available. What is the required target strength of the mix?

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2

Limit state principles

2.1 GENERAL CONCEPTS

2.1.1 Limit state requirements

The application of limit state principles ensures that a structure remains serviceable throughout its design life. A limit state is that loading condition which, if exceeded, will render the structure unserviceable. A structure is required to satisfy the two criteria of not exceeding either the ultimate limit state or the serviceability limit state. An ultimate limit state occurs at the collapse, or partial collapse, of a structure. A serviceability limit state is produced by excessive deflection, cracking, or vibration.

An ultimate limit state analysis is necessary to determine the maximum load carrying capacity of a structure and a serviceability limit state analysis is necessary to ensure satisfactory behavior under working load conditions.

2.2 ULTIMATE LIMIT STATE PROCEDURE

2.2.1 Strength design requirements

The strength design method, also known as the limit state or load factor design method, utilizes an ultimate limit state analysis to determine the maximum load carrying capacity of the structure. The objective of ultimate limit state design is to ensure a uniform level of reliability for all structures. This is achieved by multiplying the nominal, or unmodified, service loads by load factors to obtain an acceptable probability that the factored loads will be exceeded during the life

of the structure. As shown in Figure 2–1 the shaded portion of the graph denotes the probability of exceeding the factored loads.

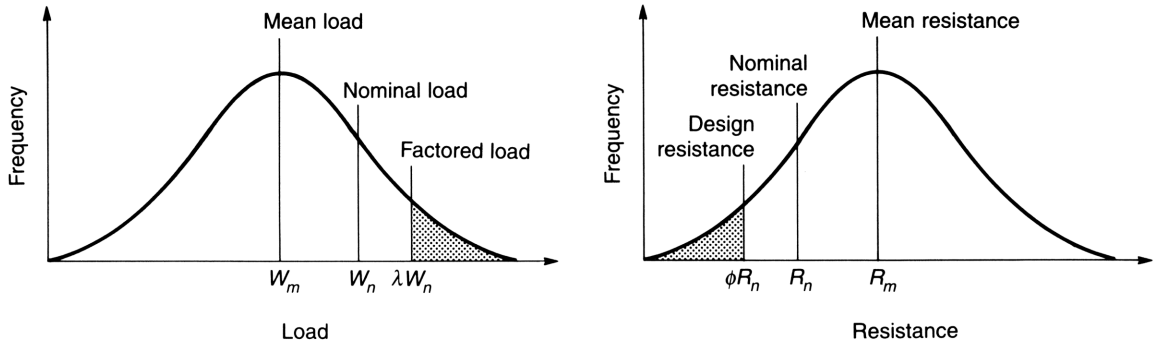

Figure 2–1 Probability curves for loading and resistance capacity

The factored, or ultimate, load is obtained by multiplying the nominal load by a load factor and this may be expressed as

$$W_u = \lambda W$$

where λ = load factor
W_u = ultimate load
W = unmodified service load

The value of the load factor is selected to compensate for the following factors:

- The unfavorable deviation of the loading from the nominal value.
- Uncertainties in methods of analysis.
- Unexpected deviations in structural behavior.
- The reduced probability of the full nominal loads, in a combination of loads, being present simultaneously.

The load factor for dead load is less than the value for live load since the magnitude of the dead load is more accurately known than the magnitude of the live load.

The total effect on the structure of the factored loads shall not exceed the design, or useable, resistance capacity of the structure. If the factored loads acting on a structure produce the forces U, then to ensure that collapse of a member does not occur

$$U = \text{effect produced on a member by the factored loads } W_u \leq \text{design capacity of the member} \leq \phi R_n$$

where ϕR_n = design capacity of the structure
R_n = nominal, or calculated theoretical, resistance capacity of the structure
ϕ = strength reduction factor

The factor ϕ is also referred to as the confidence factor, performance factor, resistance factor and capacity reduction factor. The value of the strength reduction factor is selected to compensate for the following factors:

- The possibility of adverse variations in material strength.
- Inadequate workmanship.
- Dimensional inaccuracies.
- The strength of the material in the structure being less than that of the control specimens.
- The degree of importance of the member to the integrity of the structure.

The strength reduction factor for columns is less than that for beams because of the more serious consequences of column failure and because compressive failure in concrete is sudden and undesirable. As shown in Figure 2–1, the shaded portion of the graph denotes the probability of exceeding the design resistance capacity of the structure.

2.2.2 Design strength and reduction factors

The design strength, or capacity, of a member consists of the theoretical ultimate, or nominal, strength of the member R_n multiplied by the appropriate strength reduction factor ϕ. The design strength must exceed the effect of the factored loads on the member. ACI Section 9.3 defines the reduction factors and these are summarized in Table 2–1. In addition, ACI Section 9.4 specifies the maximum value of the reinforcement yield stress which may be used in design and this is shown in Table 2–1.

Loading condition	Value of ϕ	Upper limit of f_y
Flexure	0.90	80,000 psi
Shear and torsion	0.85	60,000 psi
Columns with spiral reinforcement	0.75	80,000 psi
Columns with lateral ties	0.70	80,000 psi
Bearing on concrete	0.70	–

Table 2–1 Strength reduction factors for the strength design method

2.2.3 Design load and load factors

The basic requirement of strength design is to ensure that the design strength of a member is not less than the required ultimate strength. The required strength consists of the service level loads multiplied by appropriate load factors and this is defined in ACI Equations (9–1), (9–2), (9–3), and (9–4) and summarized in Table 2–2. The dead load is reduced by ten percent when this produces a more critical effect on a member subjected to wind, earthquake, or earth pressure.

Additionally, imposed live load is omitted when this results in an increase in stresses.

Loading combination	Load factors				
	D	L	W	EQ	H
D + L	1.40	1.70	-	-	-
D + L + W	1.05	1.28	1.28	-	-
D + L + EQ	1.05	1.28	-	1.40	-
D + L + H	1.40	1.70	-	-	1.70
D + W	0.90	-	1.30	-	-
D + EQ	0.90	-	-	1.43	-
D + H	0.90	-	-	-	1.70

Table 2–2 Load factors for the strength design method
D = dead load, L = imposed live load, W = wind load, EQ = seismic load, H = earth pressure

2.3 SERVICEABILITY LIMIT STATE PROCEDURE

2.3.1 Working stress design method

The working stress design method which is also known as the service load, or allowable stress, design method is based on the elastic theory. This assumes elastic material properties and a constant modulus of elasticity to predict the material stresses in the structure under the applied working loads. These stresses may not exceed the allowable stresses which are defined as the yield stress of the reinforcement or compressive strength of the concrete multiplied by a reduction factor.

Thus $p_{cb} = \gamma_c f'_c$
and $p_{st} = \gamma_s f_y$
where p_{cb} = permissible compressive stress in concrete in flexure
p_{st} = permissible tensile stress in reinforcement
f'_c = specified compressive stress of concrete
f_y = specified yield strength of reinforcement
γ_c = reduction factor applied to concrete strength
γ_s = reduction factor applied to reinforcement strength

The permissible stresses are limited to values within the elastic range of the materials to ensure that the elastic theory is applicable. The permissible stresses are also limited to provided an adequate margin of safety against failure. The permissible stresses for flexure are given in ACI Section A.3 and the reduction factors are summarized in Table 2.3.

Material	γ
Concrete	0.45
Grade 40 reinforcement	0.50
Grade 60 reinforcement	0.40

Table 2–3 Reduction factors for elastic design

The working stress design method was adequate when material stresses were low and serviceability aspects of design did not require specific consideration. The method models the behavior of a member under service loads satisfactorily but provides an unsatisfactory indication of conditions as loading increases and failure approaches. At this stage, the assumption of a linear relationship between stress and strain is no longer valid.

In working stress design, the forces acting on a structure are normally determined from the actual values of the unmodified service loads. The basic requirement for allowable stress design may be expressed as

$$W_d = D + L$$

where D = dead load
L = live load
W_d = design load

Permanent loads on a structure consist of dead loads, creep, shrinkage, and differential settlement. Transient loads consist of superimposed live loads, wind loads, seismic loads, and temperature. Infrequently occurring transient loads of short duration are accommodated in ACI Section A.2.2 by designing for 75 percent of the applied service load and the design load is given by

$$W_d = 0.75(D + L + W)$$

and

$$W_d = 0.75(D + L + EQ)$$

where W = wind load
and EQ = seismic load

To compensate against inaccuracies in estimating dead load, ACI Section A.2.3 requires a 15 percent reduction in dead load when this produces a more critical effect on the member. In addition, live load is omitted when this results in an increase in stresses. For this condition the design load is given by

$$W_d = 0.75(0.85D + W)$$

and

$$W_d = 0.75(0.85D + EQ)$$

These loading conditions are summarized in Table 2–4.

Loading combination	Load factors			
	D	L	W	EQ
D + L	1.00	1.00	-	-
D + L + W	0.75	0.75	0.75	-
D + L + EQ	0.75	0.75	-	0.75
D + W	0.64	-	0.75	-
D + EQ	0.64	-	-	0.75

Table 2–4 Load factors for the elastic design method
D = dead load, L = imposed live load, W = wind load, EQ = seismic

In general, the working stress design method is an unsatisfactory design method as it does not ensure a constant factor of safety against failure in different types of structures. In particular, the margin of safety provided for an increase in the live load decreases as the ratio of live load to dead load increases. In addition, dead loads may be predicted more accurately than live loads and should therefore have a lower factor of safety.

2.3.2 Serviceability criteria

Serviceability concerns which may affect the correct functioning of a structure are excessive deflections, cracking, and vibration. Following the introduction of reinforcement with yield strengths in excess of 40,000 pounds per square inch, it has become necessary to ensure that cracking and deflection are adequately controlled.

The cracking produced in a member, under working load conditions, must not compromise the purpose for which the member is intended. In normal building construction the cracking in exposed and visible members must be controlled for aesthetic reasons and corrosion protection. The cracking of structures, subjected to aggressive corrosive conditions or where watertightness is required, must be controlled to prevent the structural performance being impaired or adversely affected. For these conditions, addition control measures are required.

Cracking is a random and unpredictable phenomenon and crack widths can not be calculated with any degree of accuracy. Hence, no analytical method is presented in the ACI Code to directly calculate crack widths. To prevent excessive cracking in normal building structures, ACI Section 10.6 requires compliance with simple reinforcement detailing rules.

The deformation of a structure must be controlled to prevent excessive deflections affecting the appearance and efficiency of the structure. The deflection occurring must be compatible with the degree of movement which can be accommodated by supported elements such as finishes, partitions, utilities, glazing, and cladding.

For deflection control, two alternative approaches are provided in the ACI Code:

- Deemed-to-satisfy provisions consisting of limiting span/depth ratios detailed in ACI Sections 9.5.2.1 and 9.5.3.2.
- By analysis and comparison of the deflection computed, in accordance with ACI Section 9.5.2.2, with the allowable values given in ACI Section 9.5.2.6.

Provided that the actual span/depth ratio adopted does not exceed the limiting value specified, a detailed analysis is not required.

Damage to supported elements may be anticipated if the deflection occurring after installation, including the time dependent effects of creep and shrinkage, exceed span/480 for deflection-sensitive elements or span/240 for non-sensitive elements and these are the limits imposed by the ACI Code. To control the flexibility of a member, the immediate deflection under live load is limited to span/180 for roofs and span/360 for floors, provided deflection-sensitive elements are not supported.

Excessive vibration in a structure is undesirable when it causes discomfort or alarm to the occupants, adversely affects the efficiency of the structure, or causes damage to supported elements. Isolation of the source of the vibrations or of portions of the structure may be required.

2.4 LOADING CONDITIONS

2.4.1 Load types

The dead loads acting on a structure consist of the self weight of the structure, finishes, partitions, utilities, glazing, and cladding. Typical values which may be assumed are

- concrete = 150 pounds per cubic foot
- lightweight floor finish = 8 pounds per square foot per inch thickness
- suspended ceiling = 2 pounds per square foot
- plaster ceiling = 8 pounds per square foot per inch thickness
- electrical and mechanical utilities = 2 pounds per square foot
- moveable partitions = 20 pounds per square foot

Imposed live loading may be obtained from the relevant Standards[1] or Codes[2].

2.4.2 Loading combinations

Structures must be designed to withstand the most unfavorable arrangement of design loads. The design ultimate loads specified in ACI Section 9.2 are obtained by multiplying the service loads by the appropriate load factors from Table 2-2 and are given by:

- dead load + imposed load
  1.4D + 1.7L

- adverse dead load + adverse live load + wind load
  1.05D + 1.28L + 1.28W
- beneficial dead load + beneficial live load + wind load
  0.9D + 1.3W
- adverse dead load + adverse live load + seismic load
  1.05D + 1.28L + 1.4EQ
- beneficial dead load + beneficial live load + seismic load
  0.9D + 1.43EQ
- adverse dead load + adverse live load + earth (or water) pressure
  1.4D + 1.7L + 1.7H
- beneficial dead load + beneficial live load + earth (or water) pressure
  0.9D + 1.7H

where D = dead load
L = superimposed live load
W = wind load
EQ = seismic load
H = earth pressure

Wind and earthquake need not be assumed to act simultaneously. Lateral earth or water pressure shall be included in the design when this results in more adverse conditions.

2.4.3 Loading arrangements

ACI Section 8.7 defines the effective span length ℓ for various support conditions and these are shown in Figure 2–2.

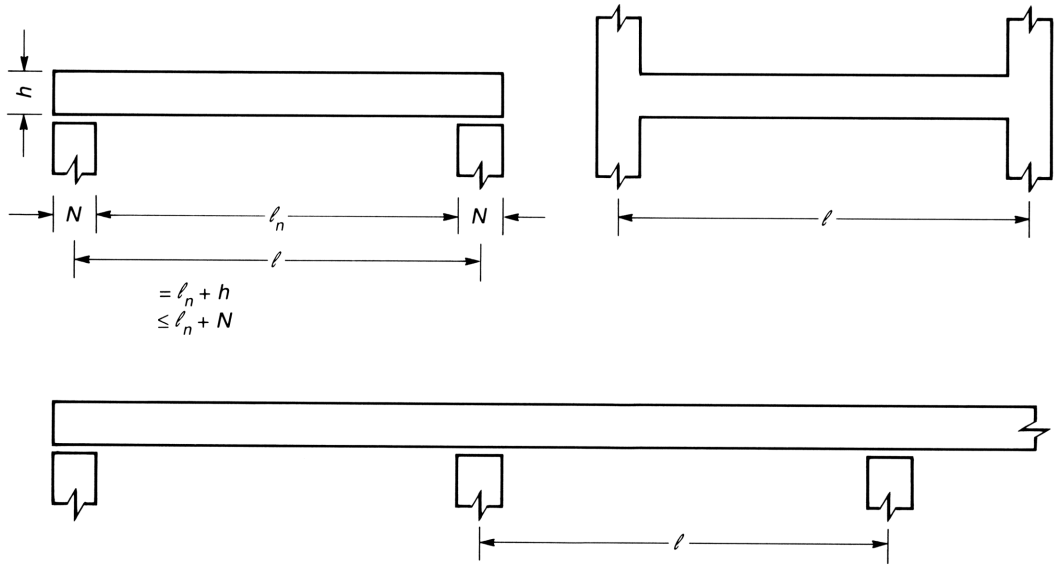

Figure 2–2 Effective span length

When the beam is not integral with its supports, the effective span is considered to be the lesser of the distance between the centers of bearings or the clear span plus the beam depth. For beams forming part of a frame, the effective span is the distance between the centers of columns. When the beam is continuous, the effective span is the distance center-to-center of supports.

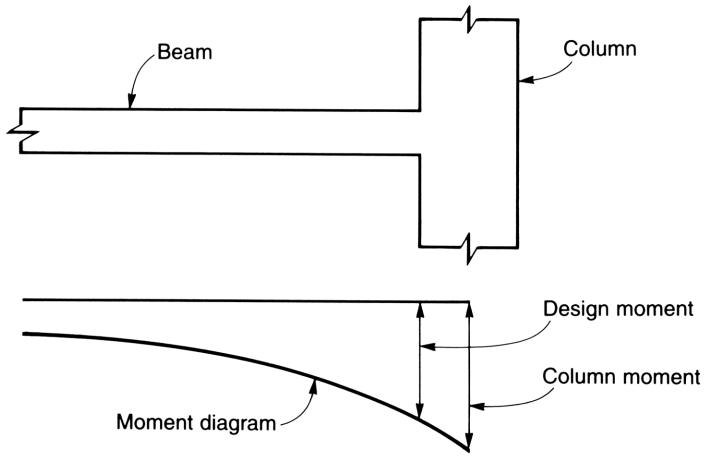

Figure 2–3 Design moment at an integral support

Where beams are integral with supports, the applicable design moment is that occurring at the support face, as shown in Figure 2–3.

In accordance with ACI Section 8.3, the design forces acting on a member shall be determined by elastic analysis using the most adverse distribution of the imposed live load. In theory, for continuous beams, this requires analysis of the loading arrangements shown in Figure 2–4 using skip loading to determine maximum span and support moments.

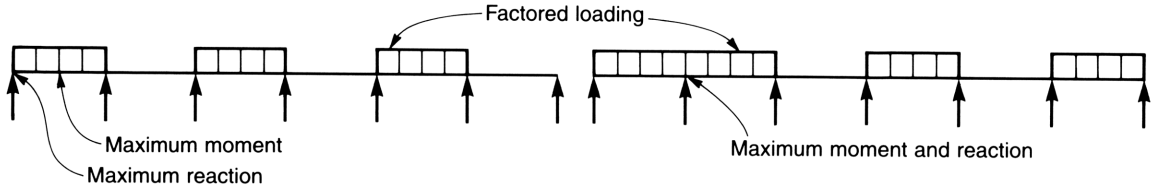

Figure 2–4 Live loading arrangement

To simplify the procedure for support moments, ACI Section 8.9.2 allows consideration of the modified arrangement

- Factored live load on any two adjacent spans with all other spans unloaded.

For maximum span moment, ACI Section 8.9.2 requires

- Factored live load on alternate spans with all other spans unloaded.

Thus, to determine the maximum support moment, the factored live load is applied only to the spans on each side of the support under consideration. For maximum span moment, the factored live load is applied to the span under consideration and all alternate spans.

To determine the design forces acting on the beams in a frame with sway prevented, ACI Section 8.9.1 allows the analysis of a simplified sub-frame. This consists of the beams at one level together with the columns above and below. The ends of the columns remote from the beams are considered to be fixed. This is illustrated in Figure 2–5 and factored live loading is applied to the beam in the same manner as for a continuous beam.

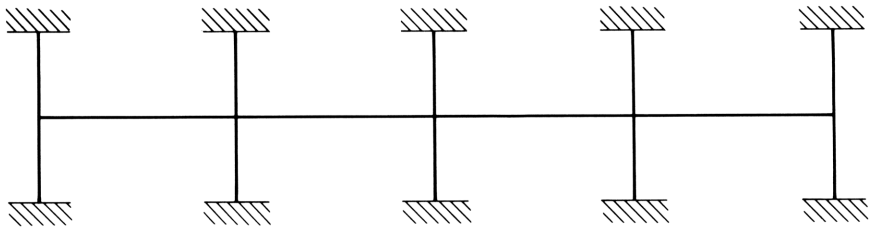

Figure 2–5 Simplified sub-frame

The design axial load and bending moment acting on a column may be determined in accordance with ACI Section 8.8 and as shown in Figure 2–6.

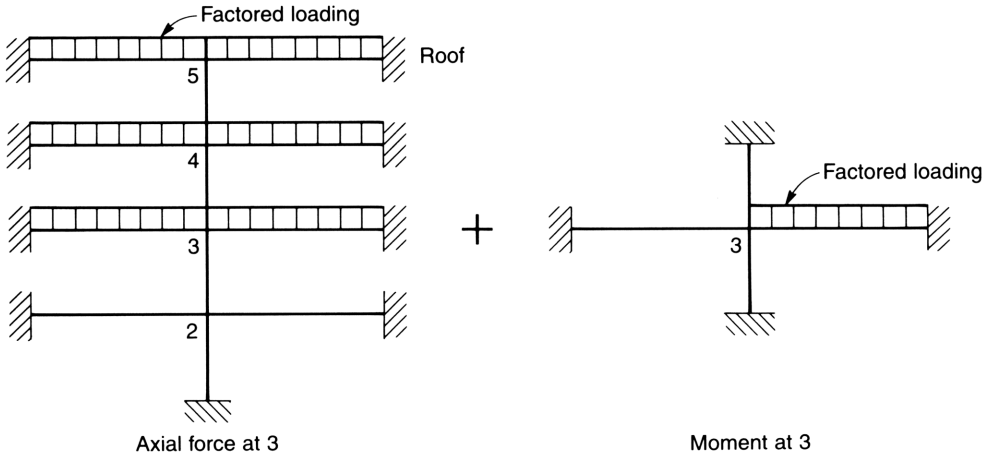

Figure 2–6 Axial force and bending moment at level 3

Example 2–1 (Loading combinations)

Figure 2–7 shows an interior frame of a three story building. The unmodified service loads are:

Roof dead load and floor dead load	= 3 kips per foot
Roof live load	= 0.6 kips per foot
Floor live load	= 1.5 kips per foot
Curtain wall and column weight	= 0.5 kips per foot
Horizontal wind pressure	= 2.2 kips per foot
Vertical wind pressure	= 1.2 kips per foot, upward

Determine the maximum and minimum ultimate design loads acting on the column footings, neglecting any reduction of live loads due to supported area.

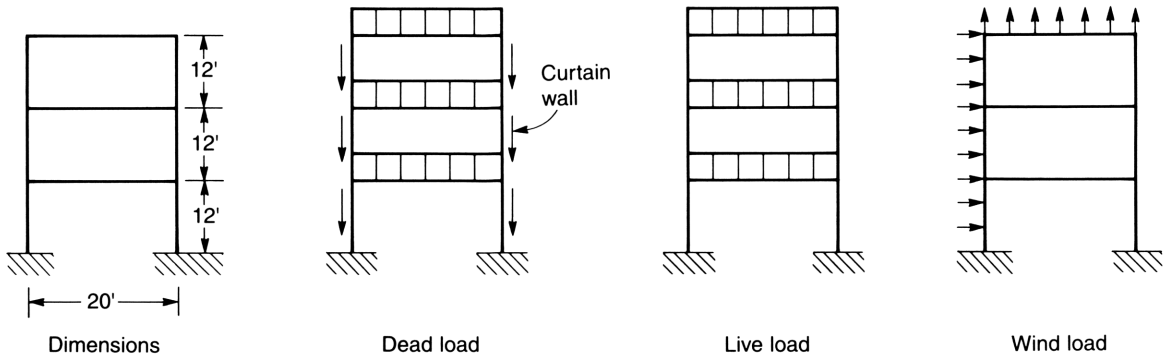

Figure 2–7 Details for Example 2–1

Solution

The unmodified axial load on one column due to roof dead load, floor dead load, curtain wall, and column self weight is

$$\begin{aligned} D &= 3 \times 10 \times 3 + 0.5 \times 12 \times 3 \\ &= 108 \text{ kips} \end{aligned}$$

The unmodified axial load on one column due to roof live load and floor live load is

$$\begin{aligned} L &= 0.6 \times 10 + 1.5 \times 10 \times 2 \\ &= 36 \text{ kips} \end{aligned}$$

The unmodified axial load on one column due to horizontal wind pressure is

$$\begin{aligned} W_h &= \pm 2.2 \times 12 \times 3 \times 18/20 \\ &= \pm 71.28 \text{ kips} \end{aligned}$$

The unmodified axial load on one column due to vertical wind pressure is

$$\begin{aligned} W_v &= -1.2 \times 10 \\ &= -12 \text{ kips} \end{aligned}$$

The controlling load combinations on a column are:

- dead load + imposed load

$$\begin{aligned} P &= 1.4D + 1.7L \\ &= 1.4 \times 108 + 1.7 \times 36 \\ &= 212.40 \text{ kips} \end{aligned}$$

- adverse dead load + adverse live load + wind load

$$\begin{aligned} P &= 1.05D + 1.28L + 1.28(W_h + W_v) \\ &= 1.05 \times 108 + 1.28 \times 36 + 1.28(71.28 - 12) \\ &= 235.36 \text{ kips compression} \end{aligned}$$

- beneficial dead load + beneficial live load + wind load

$$\begin{aligned} P &= 0.9D + 0.0L + 1.3(W_h + W_v) \\ &= 0.9 \times 108 + 1.3(-71.28 - 12) \\ &= 97.20 - 108.26 \\ &= -11.06 \text{ kips tension} \end{aligned}$$

The maximum ultimate design load is 235.36 kips compression.
The minimum ultimate design load is 11.06 kips tension.

2.4.4 Moment redistribution

ACI Section 8.4 allows the support moments determined by elastic analysis to be adjusted by a maximum of 20 percent to reduce the differential between positive and negative moments and relieve the congestion of reinforcement at beam supports. The justification for this is based on the plastic collapse method of design which is illustrated in Figure 2–8 for a fixed–ended beam constructed from an ideal elastic–plastic material.

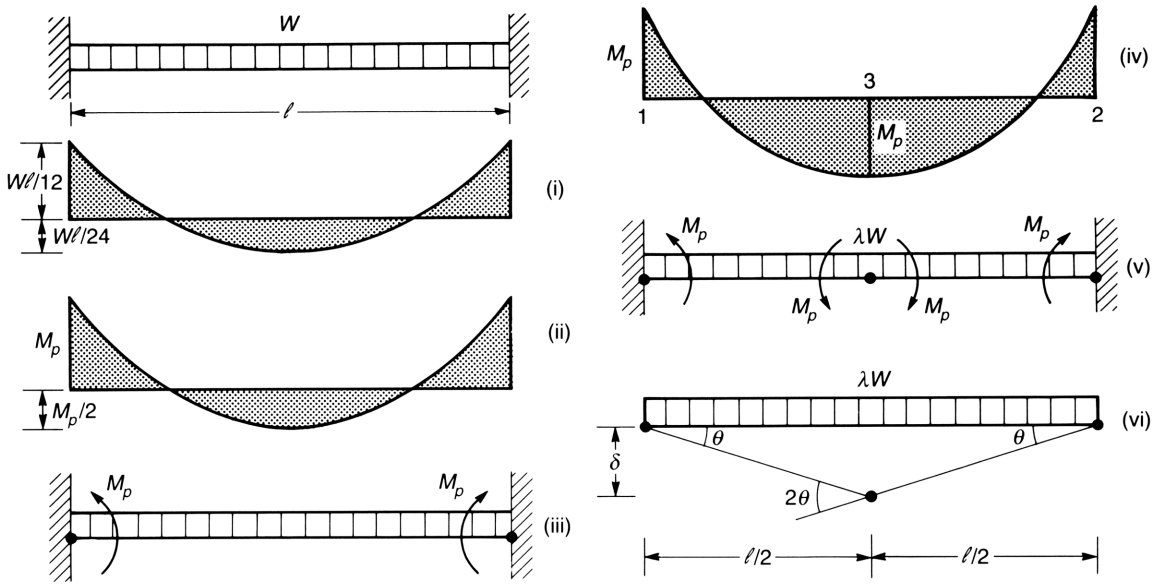

Figure 2–8 Collapse mechanism method of design

(i) The service load W is applied to the beam producing stresses which are within the elastic range and bending moments which may be determined by elastic analysis.

(ii) As the loading is increased, plastic zones form in the outer fibers of the beam, at the positions of maximum moment, which occur at the ends of the beam. Eventually, plastic hinges develop at the ends of the beam with a plastic moment of resistance M_p. The bending moment at mid–span is $0.5M_p$.

(iii) Additional load can still be carried by the beam which acts as a simply supported member subjected to constant applied end moments M_p.

(iv) Further increase in loading results in the formation of a plastic hinge at mid–span with a plastic moment of resistance M_p. The beam has now been converted to an unstable collapse mechanism and supports its ultimate load λW.

(v) Deflection now increases rapidly with additional load and the beam fails as a mechanism. The magnitude of the plastic moment of resistance is obtained from the virtual work equation assuming a rotation θ at the end plastic hinges and a deflection of δ at center–span, as shown at (vi).

Equating internal and external work gives

$$4M_p\theta = \lambda W\delta/2$$
$$= \lambda W\ell\theta/4$$

and $$M_p = \lambda W\ell/16$$

Hence, the beam may be designed for support moments of $\lambda W\ell/16$ compared with the equivalent moment determined by elastic theory of $\lambda W\ell/12$.

The collapse bending moment diagram indicated in Figure 2–8(iv) may be obtained by determining the elastic bending moment diagram due to the collapse loading assuming elastic behavior. The support moments are now reduced while increasing the mid–span moment by an appropriate amount to maintain equilibrium. This operation is termed the redistribution of moments. The procedure may be effected by means of plastic moment distribution[3,4,5] as shown in Table 2–5. The collapse loading is applied to the beam and the bending moments determined assuming elastic behavior. Redistribution and carry–over from each end proceeds as shown.

Joint	1	3	2
Carry–over	0.5 →		← –0.5
Initial elastic moment	$-\lambda W\ell/12$	$\lambda W\ell/24$	$\lambda W\ell/12$
Redistribution and carry–over	$\lambda W\ell/48$	$\lambda W\ell/96$	
Redistribution and carry–over		$\lambda W\ell/96$	$-\lambda W\ell/48$
Final moments	$-\lambda W\ell/16$	$\lambda W\ell/16$	$\lambda W\ell/16$

Table 2–5 Moment redistribution

Use of this technique allows the peak moments, determined by an elastic analysis, to be redistributed without impairing the ultimate load–carrying capacity of the structure.

The concept of a plastic hinge implies ideal ductile behavior in a member with unrestricted rotation at the hinge. Reinforced concrete members exhibit only limited rotational capacity[6,7] and, to prevent premature compression failure, the ACI Code restricts the allowable redistribution to a maximum of 20 percent. In addition, the rotational capacity increases as the tension reinforcement and the neutral axis depth are reduced so that the reinforcement yields long before the concrete fails. The procedure consists of determining the elastic moment diagrams for those load patterns which produce maximum support moments. The negative bending moments at the supports are then reduced as required and the positive mid–span bending moments are increased accordingly to maintain equilibrium. The elastic moment diagrams for the load patterns which produce maximum span moments are now determined. The negative bending moments at the supports are increased as required and the positive mid–span bending moments are reduced accordingly to maintain equilibrium. In this way, the maximum design moments over the entire structure may be minimized without reduction in the overall ultimate strength. The limitations imposed by ACI Section 8.4 are:

- To reduce the required rotation, the maximum allowable increase or decrease in support moment is limited to

$$M_R = 20[1 - (\rho - \rho')/\rho_b] \text{ percent}$$
$$= 20(\rho_b - \rho + \rho')/\rho_b \text{ percent}$$

- To ensure early yielding of the reinforcement, the maximum allowable reinforcement ratio is limited to

$$\rho = 0.50\rho_b + \rho'$$

where ρ = tension reinforcement ratio
$= A_s/bd$
ρ' = compression reinforcement ratio
$= A'_s/bd$
ρ_b = balanced reinforcement ratio defined in ACI Section 10.3.2
$= 0.85 \times 87{,}000\beta_1 f'_c/f_y(87{,}000 + f_y)$
β_1 = compression zone factor given in ACI Section 10.2.7.3

- The redistribution procedure may not be employed when approximate design coefficients are used to determine the bending moments acting on the member.
- The redistribution procedure may not be employed in conjunction with the alternate working stress design method specified in ACI Appendix A.

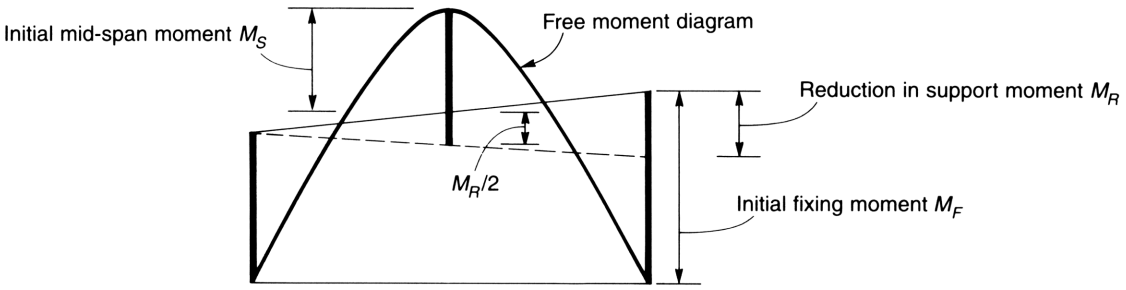

Figure 2–9 Moment redistribution

The effect on the adjacent span moment of a reduction in support bending moment is shown in Figure 2–9. Accepting the slight error introduced by assuming that the maximum span moment occurs at mid–span, the mid–span moment is adjusted by half the amount of the reduction in the support moment.

Example 2–2 (Moment redistribution in a continuous reinforced concrete beam)
A reinforced concrete beam supports a factored dead load, including its self weight, of 3 kips per foot run and a factored imposed live load of 2 kips per foot run and is continuous over two spans of 15 feet. Assuming that the tension reinforcement ratio ρ equals $0.35\rho_b$ determine the required design moments after redistribution.

Solution
The exact bending moments acting on the beam may be determined from continuous beam design

tables[8,9,10]. Figure 2–10 shows the moment coefficients for uniformly distributed load on both spans and the corresponding bending moments, due to dead load only, acting on the beam. The factored dead load moment at the center support is given by

$$\begin{aligned} M_{uD(support)} &= 0.125 w_{uD} \ell^2 \\ &= 0.125 \times 3 \times 15^2 \\ &= 84.38 \text{ kip feet} \end{aligned}$$

The factored dead load moment at the center of the left span is

$$\begin{aligned} M_{uD(span)} &= 0.070 w_{uD} \ell^2 \\ &= 0.070 \times 3 \times 15^2 \\ &= 47.25 \text{ kip feet} \end{aligned}$$

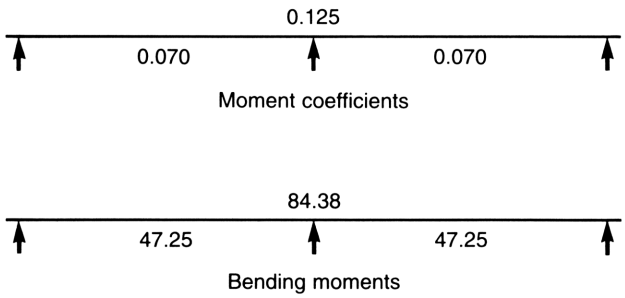

Figure 2–10 Dead load only

Figure 2–11 shows the moment coefficients for uniformly distributed load on both spans and the corresponding bending moments, due to live load only, acting on the beam. The factored live load moment at the center support is given by

$$\begin{aligned} M_{uL(support)} &= 0.125 w_{uL} \ell^2 \\ &= 0.125 \times 2 \times 15^2 \\ &= 56.25 \text{ kip feet} \end{aligned}$$

The factored live load moment at the center of the left span is

$$\begin{aligned} M_{uL(span)} &= 0.070 w_{uL} \ell^2 \\ &= 0.070 \times 2 \times 15^2 \\ &= 31.50 \text{ kip feet} \end{aligned}$$

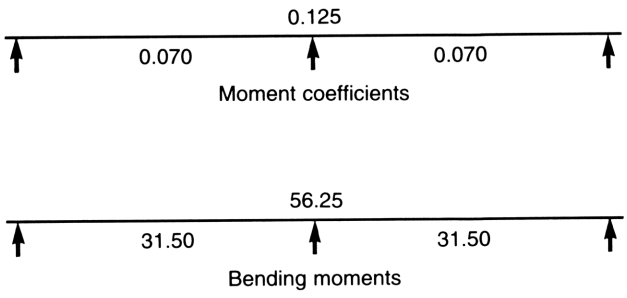

Figure 2–11 Live load on both spans

Figure 2–12 shows the moment coefficients for uniformly distributed load on the left span and the corresponding bending moments, due to live load, acting on the left span. The factored live load moment at the center support is given by

$$\begin{aligned} M_{uL(support)} &= 0.063 w_{uL} \ell^2 \\ &= 0.063 \times 2 \times 15^2 \\ &= 28.35 \text{ kip feet} \end{aligned}$$

The factored live load moment at the center of the left span is

$$\begin{aligned} M_{uL(span)} &= 0.096 w_{uL} \ell^2 \\ &= 0.096 \times 2 \times 15^2 \\ &= 43.20 \text{ kip feet} \end{aligned}$$

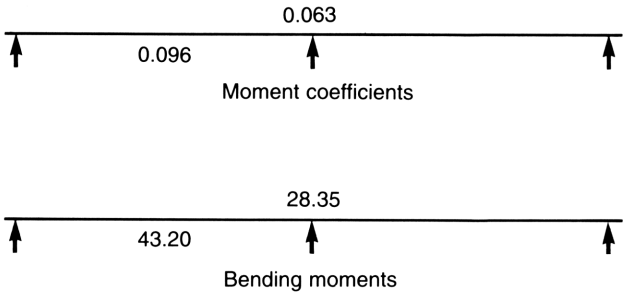

Figure 2–12 Live load on left span only

Combining dead load moments with moments due to live load on both spans results in the values shown in Figure 2–13.

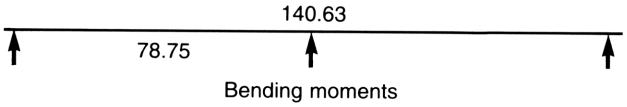

Figure 2–13 Dead load with live load on both spans

The bending moment occurring at the support may be decreased in accordance with ACI Section 8.4 and the maximum allowable decrease in support moment is

$$\begin{aligned} M_R &= 20(\rho_b - \rho + \rho')/\rho_b \text{ percent} \\ &= 20(\rho_b - 0.35\rho_b + 0)/\rho_b \\ &= 20 \times 0.65 \\ &= 13 \text{ percent} \end{aligned}$$

Hence the support moment is reduced by

$$\begin{aligned} M_R &= 0.13 \times 140.63 \\ &= 18.28 \text{ kip feet} \end{aligned}$$

and the span moment is increased by

$$M_R/2 = 18.28/2$$
$$= 9.14 \text{ kip feet}$$

The resultant support moment is

$$M_{u(support)} = 140.63 - 18.28$$
$$= 122.35 \text{ kip feet}$$

The resultant span moment is

$$M_{u(span)} = 78.75 + 9.14$$
$$= 87.89 \text{ kip feet}$$

Combining dead load moments with moments due to live load on the left span results in the values shown in Figure 2–14.

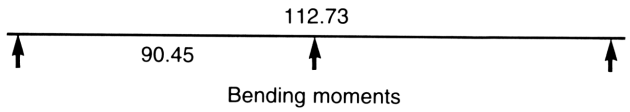

Figure 2–14 Dead load with live load on left span

The bending moment occurring in the left span may be decreased by 2.83 percent, in accordance with ACI Section 8.4, and this reduces the span moment by

$$M_R = 0.0283 \times 90.45$$
$$= 2.56 \text{ kip feet}$$

and the support moment is increased by

$$M_R \times 2 = 2.56 \times 2$$
$$= 5.12 \text{ kip feet}$$

The resultant support moment is

$$M_{u(support)} = 112.73 + 5.12$$
$$= 117.85 \text{ kip feet}$$
$$< 122.35 \text{ kip feet} \ldots \text{ satisfactory}$$

The resultant span moment is

$$M_{u(span)} = 90.45 - 2.56$$
$$= 87.89 \text{ kip feet} \ldots \text{ satisfactory}$$

The final design moments are shown in Figure 2–15.

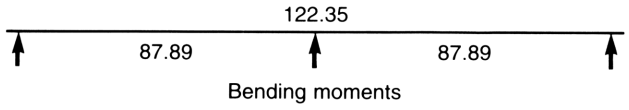

Figure 2–15 Design moments

2.4.5 Approximate design coefficients

In designing continuous beams and one-way slabs, approximate design coefficients may be used instead of an exact analysis provided that the following conditions are observed:

- The loads are uniformly distributed.
- The ratio of live to dead load does not exceed three.
- Adjacent spans do not differ in length by more than 20 percent of the shorter span.

The bending moment coefficients are specified in ACI Section 8.3.3 and are shown in Figure 2-16. The bending moments are obtained from the expression

$$M_u = \tau w_u \ell_n^2$$

where τ = approximate design coefficient
w_u = factored load per unit length
ℓ_n = length of clear span

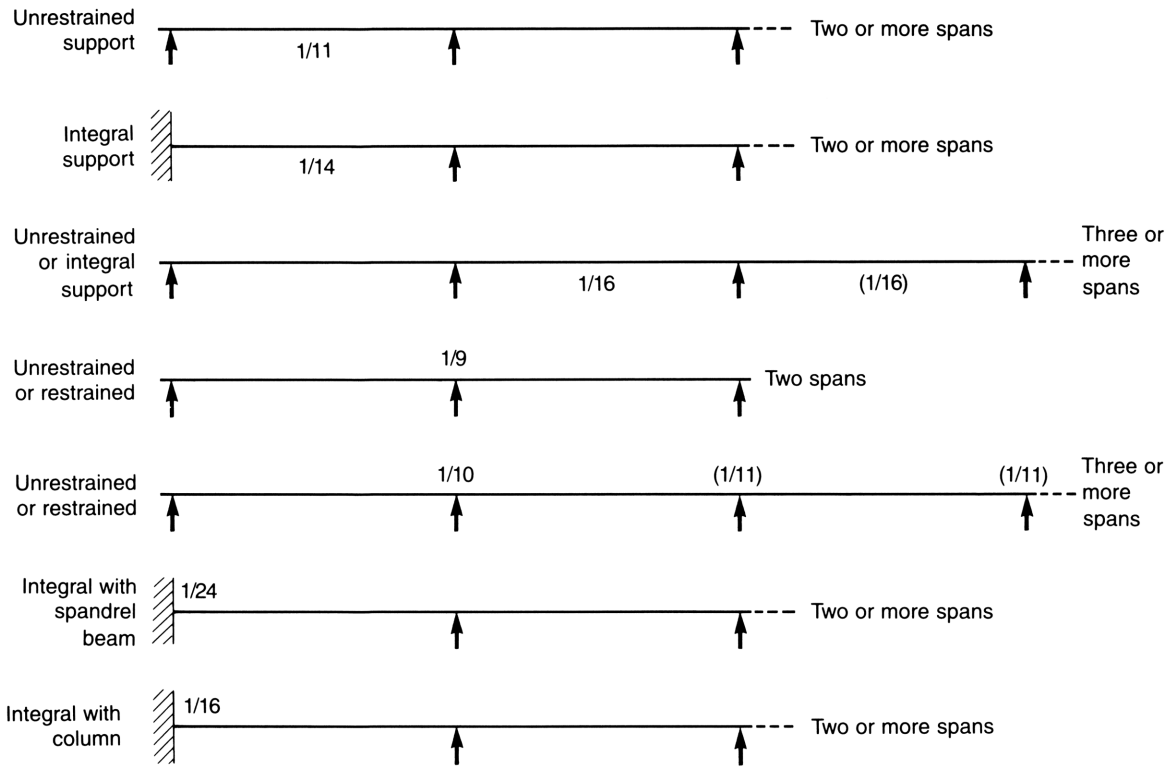

Figure 2-16 Bending moment coefficients

For slabs with a clear span not exceeding ten feet and for beams where the ratio of column stiffnesses to beam stiffnesses exceeds eight at each end of the span, the support moment is given by

$$M_u = -w_u \ell_n^2/12$$

Shear force coefficients are also specified in ACI Section 8.3.3 and are shown in Figure 2-17. The shear force is obtained from the expression

$$V_u = \tau w_u \ell_n$$

where τ = approximate design coefficient
w_u = factored load per unit length
ℓ_n = length of clear span

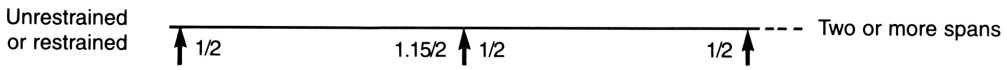

Figure 2–17 Shear force coefficients

Example 2.3 (Approximate design coefficients)
A reinforced concrete beam, continuous over two spans, supports a factored dead load of 3 kips per foot run and a factored live load of 2 kips per foot run. The clear distance between support faces is 15 feet. Determine the design bending moments and shear forces.

Solution
The total factored load is

$$\begin{aligned} w_u &= w_{uD} + w_{uL} \\ &= 3 + 2 \\ &= 5 \text{ kips per foot} \end{aligned}$$

For an unrestrained end support, the mid-span moment is

$$\begin{aligned} M_u &= \tau w_u \ell_n^2 \\ &= 5 \times 15^2 \times 1/11 \\ &= 102.27 \text{ kip feet} \end{aligned}$$

For a two span beam, the bending moment at the center support is

$$\begin{aligned} M_u &= \tau w_u \ell_n^2 \\ &= 5 \times 15^2 \times 1/9 \\ &= 125 \text{ kip feet} \end{aligned}$$

The shear force at the end support is

$$\begin{aligned} V_u &= \tau w_u \ell_n \\ &= 5 \times 15 \times 1/2 \\ &= 37.5 \text{ kips} \end{aligned}$$

The shear force at the center support is

$$\begin{aligned} V_u &= \tau w_u \ell_n \\ &= 5 \times 15 \times 1.15/2 \\ &= 43.13 \text{ kips} \end{aligned}$$

2.5 SUPPLEMENTARY PROBLEMS

Problem 2–1 (Moment redistribution in a continuous reinforced concrete beam)
A reinforced concrete beam supports a factored dead load, including its self weight, of 4 kips per foot run and a factored imposed live load of 4 kips per foot run and is continuous over two spans of 10 feet. Assuming that the tension reinforcement ratio ρ equals $0.45\rho_b$ determine the required design moments after redistribution.

Problem 2–2 (Moment redistribution in a continuous reinforced concrete beam)
A reinforced concrete beam supports a factored dead load, including its self weight, of 2 kips per foot run and a factored imposed live load of 2 kips per foot run and is continuous over three spans of 20 feet. Assuming that the tension reinforcement ratio ρ equals $0.50\rho_b$ determine the required design moments after redistribution.

Problem 2–3 (Approximate design coefficients)
A reinforced concrete one-way slab, continuous over three spans and integral with spandrel beams at the ends, supports a factored load of 200 pounds per square foot. The clear distance between support faces is 10 feet. Determine the design bending moments and shear forces.

Problem 2–4 (Approximate design coefficients)
A reinforced concrete beam, continuous over four spans and integral with columns at the ends, supports a factored load of 7 kips per foot run. The clear distance between support faces is 20 feet. Determine the design bending moments and shear forces.

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3

Flexure of reinforced concrete beams

3.1 GENERAL

3.1.1 Types of members

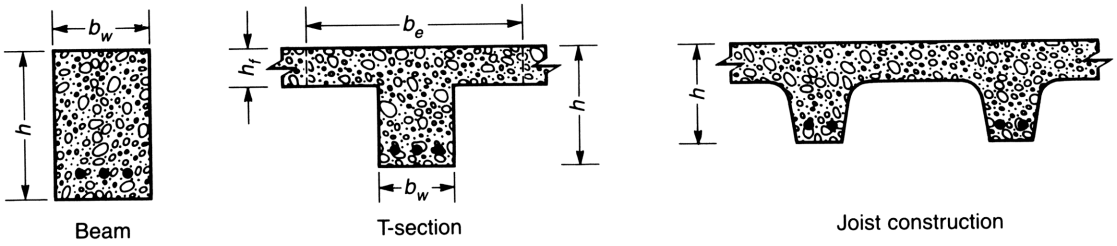

Figure 3–1 Types of beams

The different types of beams are illustrated in Figure 3–1 and consist of:

- **Rectangular beams** which act independently of a floor slab. No limitations are placed on the beam dimensions in the ACI Code. However, to ensure lateral stability, the British Code[1] limits the clear distance between lateral restraints to

$$\ell_r = 60b_w \ldots b_w/d > 0.24$$
$$= 250b_w^2/d \ldots b_w/d \leq 0.24$$

where d = effective depth to tension reinforcement
b_w = breadth of compression face

In practice, the depth of a beam is often dictated by architectural reasons, or from the necessity to limit deflections or provide clearance. The width of a beam is determined from considerations of resistance to the applied shear force, accommodation of tension reinforcement, or from the available width of forms.

- **T–sections** in which the flange and stem are constructed integrally and which are the normal type of construction in buildings. If the flange extends on one side of the stem only, as is the case with a spandrel beam, the flange and stem form an L–section. For both types, for positive applied bending moment which produces tension in the bottom of the section, the slab forms the compression flange of the section. For negative applied bending moment, both types act as a rectangular beam with the stem in compression and the flange ineffective in tension.

  When the stems are widely spaced, the flange mid–way between the stems is only lightly stressed because of shearing deformation in the flange. To allow for this variation in stress, it is convenient to assume a uniform compressive stress acting over a reduced effective width. For a T–section, ACI Section 8.10.2 defines the effective flange width as the lesser of

  $$\begin{aligned} b_e &= \ell/4 \\ &\leq 16h_f + b_w \\ &\leq s + b_w \end{aligned}$$

  where ℓ = beam span
  h_f = flange thickness
  b_w = width of stem
  s = clear distance between stems

  For an L–section, ACI Section 8.10.3 defines the effective flange width as the lesser of

  $$\begin{aligned} b_e &= \ell/12 + b_w \\ &\leq 6h_f + b_w \\ &\leq s/2 + b_w \end{aligned}$$

- **Joist construction** consisting of closely spaced ribs and a top flange. To qualify as joist construction, the maximum clear horizontal spacing between ribs is limited to 30 inches. In addition, the ribs must be at least 4 inches wide and have a depth not exceeding 3.5 times the minimum rib width.

3.1.2 Reinforcement detailing

The minimum spacing of reinforcement must be adequate to allow full consolidation of the concrete around the bars. The minimum clear horizontal spacing between parallel bars in a layer is specified by ACI Sections 7.6.1 and 3.3.2 as the greater of

$$\begin{aligned} s_{(\text{min clear})} &= 1 \text{ inch} \\ &\geq d_b \\ &\geq 1.33 \times \text{maximum aggregate size} \end{aligned}$$

where d_b = diameter of bar

When parallel reinforcement is placed in two or more layers, ACI Section 7.6.2 requires a clear vertical spacing of one inch with the upper bars located directly above the lower bars.

To protect against corrosion of the reinforcement, the ACI Code specifies the concrete cover required for various exposure conditions. The cover is defined as the distance from the face of the beam to the outermost surface of the reinforcement. For **cast in place concrete** cast against and permanently exposed to earth, as is the case with the soffit of grade beams, ACI Section 7.7.1 requires a minimum concrete cover of

c_c = 3 inches

For concrete exposed to earth and weather, the minimum cover is

c_c = 1½ inches ... No. 5 bars or smaller
= 2 inches ... No. 6 bars or larger

For concrete not exposed to the weather or in contact with the ground, as is the case in the interior of a building, the minimum concrete cover for slabs, walls, and joist construction is

c_c = ¾ inch ... No. 11 bars or smaller
= 1½ inches ... No. 14 bars or larger

For concrete not exposed to the weather or in contact with the ground, the minimum concrete cover for beams and columns, including ties, stirrups, and spirals is

c_c = 1½ inches

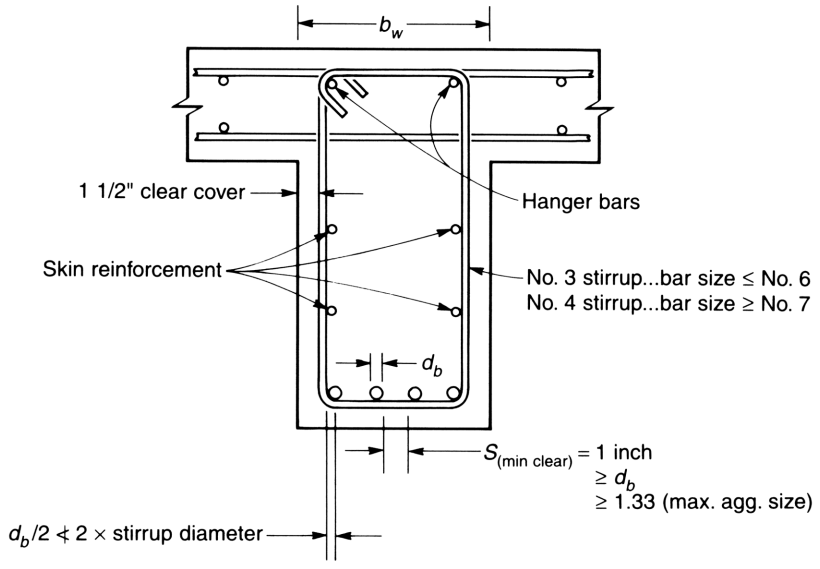

Figure 3–2 Reinforcement detailing requirements

For ¾ inch maximum size aggregate, interior exposure, and using the assumptions regarding size of stirrups shown in Figure 3–2, the number of bars which can be placed in a single layer in beams of varying width is indicated in Table 3–1.

Bar size	Beam width, inches					
	8	10	12	14	16	18
No. 4	2	4	5	6	8	9
No. 5	2	3	5	6	7	8
No. 6	2	3	4	6	7	8
No. 7	2	3	4	5	6	7
No. 8	2	3	4	5	6	7
No. 9	1	2	3	4	5	6
No. 10	1	2	3	4	4	5
No. 11	1	2	3	3	4	5

Table 3–1 Maximum number of bars in a single layer

3.2 STRENGTH DESIGN

3.2.1 Rectangular beam with tension reinforcement only

During the loading to failure of a reinforced concrete beam, the strain distribution over the depth of the beam is assumed to be linear. The stress distribution in the concrete above the neutral axis is essentially parabolic in shape, as shown in Figure 3–3, and this may be derived from a stress–strain curve similar to that given in Figure 1–11.

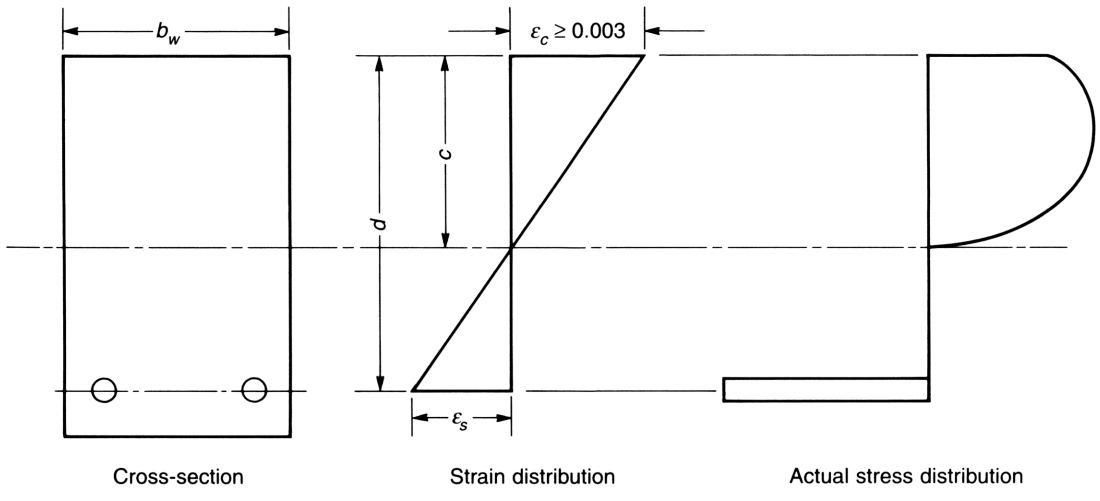

Figure 3–3 Idealized stress distribution

An equivalent rectangular stress distribution has been proposed by Mattock[2] which simplifies calculations with little loss in accuracy. The equivalent rectangular stress block adopted by ACI is specified in ACI Section 10.2 and is shown in Figure 3–4. It is assumed that at failure the tension reinforcement has yielded and that the maximum compressive strain in the concrete is 0.003. In addition, the equivalent uniform compressive stress is defined as $0.85f'_c$ with the tensile strength of the concrete being neglected.

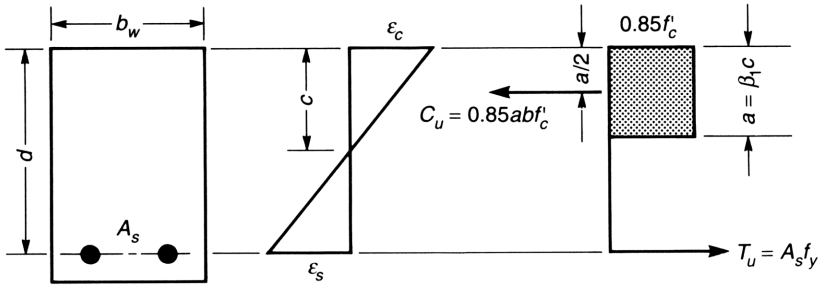

Figure 3–4 Equivalent rectangular stress block

The depth of the stress block is obtained by equating tensile and compressive forces acting on the section and is given by

$$a = A_s f_y / 0.85 f'_c b_w$$
$$= \beta_1 c$$

where c = depth to neutral axis

β_1 = compression zone factor given in ACI Section 10.2.7.3

$= 0.85 \ldots f'_c \leq 4000$ pounds per square inch

$= 0.85 - (f'_c - 4000)/20{,}000 \ldots 4000 < f'_c \leq 8000$ pounds per square inch

$= 0.65$ minimum $\ldots f'_c > 8000$ pounds per square inch

From the Figure, the nominal flexural strength of the member is derived as

$$M_n = A_s f_y (d - a/2)$$
$$= A_s f_y [d - A_s f_y / (2 \times 0.85 f'_c b_w)]$$
$$= A_s f_y d (1 - 0.59 A_s f_y / b_w d f'_c)$$
$$= \rho f_y b_w d^2 (1 - 0.59 \rho f_y / f'_c)$$
$$= \omega f'_c b_w d^2 (1 - 0.59\omega)$$

and $M_n / f'_c b d^2 = \omega(1 - 0.59\omega)$

$M_u / f'_c b d^2 = \omega(0.9 - 0.5294\omega)$

where $\rho = A_s / b_w d$ = tension reinforcement ratio

$\omega = \rho f_y / f'_c$ = tension reinforcement index

$\phi M_n = 0.9 M_n$ = required design strength

$= M_u$ = applied factored moment

$= 1.4 M_D + 1.7 M_L$

and M_D = moment due to dead load

M_L = moment due to live load

Table 3–2 provides a design aid which tabulates ω against $M_u / f'_c b_w d^2$.

ω	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0	0.0000	0.0009	0.0018	0.0027	0.0036	0.0045	0.0054	0.0063	0.0072	0.0081
0.01	0.0089	0.0098	0.0107	0.0116	0.0125	0.0134	0.0143	0.0151	0.0160	0.0169
0.02	0.0178	0.0187	0.0195	0.0204	0.0213	0.0222	0.0230	0.0239	0.0248	0.0257
0.03	0.0265	0.0274	0.0283	0.0291	0.0300	0.0309	0.0317	0.0326	0.0334	0.0343
0.04	0.0352	0.0360	0.0369	0.0377	0.0386	0.0394	0.0403	0.0411	0.0420	0.0428
0.05	0.0437	0.0445	0.0454	0.0462	0.0471	0.0479	0.0487	0.0496	0.0504	0.0513
0.06	0.0521	0.0529	0.0538	0.0546	0.0554	0.0563	0.0571	0.0579	0.0588	0.0596
0.07	0.0604	0.0612	0.0621	0.0629	0.0637	0.0645	0.0653	0.0662	0.0670	0.0678
0.08	0.0686	0.0694	0.0702	0.0711	0.0719	0.0727	0.0735	0.0743	0.0751	0.0759
0.09	0.0767	0.0775	0.0783	0.0791	0.0799	0.0807	0.0815	0.0823	0.0831	0.0839
0.10	0.0847	0.0855	0.0863	0.0871	0.0879	0.0887	0.0895	0.0902	0.0910	0.0918
0.11	0.0926	0.0934	0.0942	0.0949	0.0957	0.0965	0.0973	0.0981	0.0988	0.0996
0.12	0.1004	0.1011	0.1019	0.1027	0.1035	0.1042	0.1050	0.1058	0.1065	0.1073
0.13	0.1081	0.1088	0.1096	0.1103	0.1111	0.1119	0.1126	0.1134	0.1141	0.1149
0.14	0.1156	0.1164	0.1171	0.1179	0.1186	0.1194	0.1201	0.1209	0.1216	0.1223
0.15	0.1231	0.1238	0.1246	0.1253	0.1260	0.1268	0.1275	0.1283	0.1290	0.1297
0.16	0.1304	0.1312	0.1319	0.1326	0.1334	0.1341	0.1348	0.1355	0.1363	0.1370
0.17	0.1377	0.1384	0.1391	0.1399	0.1406	0.1413	0.1420	0.1427	0.1434	0.1441
0.18	0.1448	0.1456	0.1463	0.1470	0.1477	0.1484	0.1491	0.1498	0.1505	0.1512
0.19	0.1519	0.1526	0.1533	0.1540	0.1547	0.1554	0.1561	0.1568	0.1574	0.1581
0.20	0.1588	0.1595	0.1602	0.1609	0.1616	0.1623	0.1629	0.1636	0.1643	0.1650
0.21	0.1657	0.1663	0.1670	0.1677	0.1684	0.1690	0.1697	0.1704	0.1710	0.1717
0.22	0.1724	0.1730	0.1737	0.1744	0.1750	0.1757	0.1764	0.1770	0.1777	0.1783
0.23	0.1790	0.1797	0.1803	0.1810	0.1816	0.1823	0.1829	0.1836	0.1842	0.1849
0.24	0.1855	0.1862	0.1868	0.1874	0.1881	0.1887	0.1894	0.1900	0.1906	0.1913
0.25	0.1919	0.1925	0.1932	0.1938	0.1944	0.1951	0.1957	0.1963	0.1970	0.1976
0.26	0.1982	0.1988	0.1995	0.2001	0.2007	0.2013	0.2019	0.2026	0.2032	0.2038
0.27	0.2044	0.2050	0.2056	0.2062	0.2069	0.2075	0.2081	0.2087	0.2093	0.2099
0.28	0.2105	0.2111	0.2117	0.2123	0.2129	0.2135	0.2141	0.2147	0.2153	0.2159
0.29	0.2165	0.2171	0.2177	0.2183	0.2188	0.2194	0.2200	0.2206	0.2212	0.2218
0.30	0.2224	0.2229	0.2235	0.2241	0.2247	0.2253	0.2258	0.2264	0.2270	0.2276
0.31	0.2281	0.2287	0.2293	0.2298	0.2304	0.2310	0.2315	0.2321	0.2327	0.2332
0.32	0.2338	0.2344	0.2349	0.2355	0.2360	0.2366	0.2371	0.2377	0.2382	0.2388
0.33	0.2393	0.2399	0.2404	0.2410	0.2415	0.2421	0.2426	0.2432	0.2437	0.2443
0.34	0.2448	0.2453	0.2459	0.2464	0.2470	0.2475	0.2480	0.2486	0.2491	0.2496
0.35	0.2501	0.2507	0.2512	0.2517	0.2523	0.2528	0.2533	0.2538	0.2543	0.2549
0.36	0.2554	0.2559	0.2564	0.2569	0.2575	0.2580	0.2585	0.2590	0.2595	0.2600
0.37	0.2605	0.2610	0.2615	0.2620	0.2625	0.2631	0.2636	0.2641	0.2646	0.2651
0.38	0.2656	0.2661	0.2665	0.2670	0.2675	0.2680	0.2685	0.2690	0.2695	0.2700
0.39	0.2705	0.2710	0.2715	0.2719	0.2724	0.2729	0.2734	0.2739	0.2743	0.2748

Table 3–2 Values of $M_u/f'_c b_w d^2$

In addition, the expression for nominal strength may be rearranged to give

$$M_n = M_u/\phi$$
$$= M_u/0.9$$
$$= \rho f_y b_w d^2 - 0.59\rho^2 b_w d^2 f_y^2/f'_c$$

and

$$0 = \rho^2 - \rho(f'_c/0.59f_y) + f'_c M_u/b_w d^2(0.9 \times 0.59f_y^2)$$

The reinforcement ratio required to provide a given factored moment M_u is then

$$\rho = 0.85f'_c[1 - (1 - M_u/0.383f'_c b_w d^2)^{0.5}]/f_y$$
$$\rho = 0.85f'_c[1 - (1 - K_u/0.383f'_c)^{0.5}]/f_y$$

where $K_u = M_u/b_w d^2$ = design moment factor

These expressions may be readily applied using standard calculator programs[3,4] and tables[5,6,7].

For a balanced strain condition, the maximum strain in the concrete and in the tension reinforcement must simultaneously reach the values specified in ACI Section 10.3.2 as

$$\epsilon_c = 0.003 = \text{concrete strain}$$
$$\epsilon_s = f_y/E_s = \text{steel strain} = f_y/(29 \times 10^6)$$

For this condition, the depth of the stress block is given by

$$a_b = A_b f_y/0.85f'_c b_w$$
$$= \rho_b f_y d/0.85f'_c$$
$$= \beta_1 c_b$$
$$= \epsilon_c \beta_1 d/(\epsilon_c + \epsilon_s)$$
$$= 0.003\beta_1 d/(0.003 + f_y/E_s)$$
$$= 87{,}000\beta_1 d/(87{,}000 + f_y)$$

where ρ_b = reinforcement ratio required to produce a balanced strain condition

$A_b = b_w d\rho_b$

and the balanced reinforcement ratio is given by

$$\rho_b = 0.85 \times 87{,}000\beta_1 f'_c/f_y(87{,}000 + f_y)$$

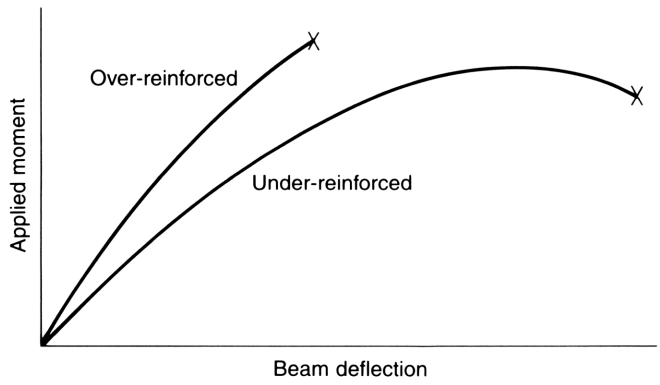

Figure 3–5 Reinforced concrete beam deflections

Reinforced concrete beams may be classified either as **under-reinforced** or **over-reinforced.** During testing of an **under-reinforced** beam, the tension reinforcement yields at a load less than the ultimate load. As more load is applied, the reinforcement continues to yield, the neutral axis rises, and finally the concrete crushes at the ultimate load. The primary cause of failure is due to yielding of the reinforcement which causes the neutral axis to rise and reduces the depth of concrete available for compression. The characteristics of an under-reinforced beam are the development of large tensile strains and full utilization of the tension reinforcement. Wide cracks are produced with large deflections at ultimate load giving adequate warning of impending failure. In an **over-reinforced** beam, the reinforcement remains elastic at the ultimate load and failure is caused by the concrete crushing explosively. Cracks and deflections are small and a sudden, brittle failure occurs with little prior warning. The characteristic behavior of under-reinforced and over-reinforced beams is shown in Figure 3-5. To ensure an under-reinforced beam with ductile behavior and yielding of the tension reinforcement, ACI Section 10.3.3, limits the maximum allowable reinforcement ratio to

$$\rho_{max} = 0.75\rho_b$$

The maximum allowable design strength of a singly reinforced member is then

$$M_{max} = \phi A_{max} f_y d(1 - 0.59\rho_{max} f_y/f'_c)$$
$$= \phi\rho_{max} f_y b_w d^2(1 - 0.59\rho_{max} f_y/f'_c)$$

where $A_{max} = b_w d\rho_{max}$

The maximum allowable design moment factor is

$$K_{u(max)} = M_{max}/bd^2$$

The maximum allowable reinforcement index is

$$\omega_{max} = \rho_{max} f_y/f'_c$$

A minimum reinforcement ratio is required to ensure that the flexural strength of the reinforced section is greater than that of the uncracked concrete section otherwise sudden failure occurs when the modulus of rupture of the concrete is exceeded and the first flexural crack forms. The moment causing flexural cracking in a rectangular beam is

$$M_{cr} = f_r S$$

where f_r = modulus of rupture of the concrete
$= 7.5(f'_c)^{0.5}$ ... from ACI Equation (9-9)
$S = b_w h^2/6$

and $M_{cr} = 1.25 b_w h^2 (f'_c)^{0.5}$
$\approx 1.25 b_w d^2 (f'_c)^{0.5}$

After cracking, the nominal moment of resistance of the section is

$$M_n = A_s f_y jd$$

where jd = lever-arm of the internal resisting moment
$= d - a/2$

and a = depth of the equivalent rectangular stress block

Then $M_n = A_s f_y(d - a/2)$
$\approx A_s f_y d$ ... for a low reinforcement ratio

Applying a factor of safety of 2.4 to the cracking moment gives

$M_n = 2.4M_{cr}$
and $A_s f_y d = 2.4 \times 1.25 b_w d^2 (f'_c)^{0.5}$
$A_s = 3b_w d(f'_c)^{0.5}/f_y$

and the minimum allowable reinforcement ratio given by ACI Section 10.5.1 is derived as

$\rho_{min} = 3(f'_c)^{0.5}/f_y$
$\geq 200/f_y$

In the case of a T-section with the flange in tension, the minimum area of reinforcement required is given by ACI Section 10.5.2 as the smaller value of

$A_{min} = 3b_e d(f'_c)^{0.5}/f_y$
or $A_{min} = 6b_w d(f'_c)^{0.5}/f_y$
where b_e = effective flange width

In accordance with ACI Section 10.5.3, however, the minimum reinforcement provided need not exceed one-third more than required by analysis.

Values of ρ_{max} and ρ_{min} for typical reinforcement and concrete strengths are given in Table 3-3. In addition, values are tabulated of the maximum design moment factor $K_{u(max)}$, with units of pounds per square inch, and of the maximum reinforcement index ω_{max}.

f'_c	β_1	f_y = 40,000				f_y = 60,000			
		ρ_{max}	ρ_{min}	$K_{u(max)}$	ω_{max}	ρ_{max}	ρ_{min}	$K_{u(max)}$	ω_{max}
3000	0.85	0.0278	0.0050	783	0.371	0.0160	0.0033	702	0.321
3500	0.85	0.0325	0.0050	913	0.371	0.0187	0.0033	819	0.321
4000	0.85	0.0371	0.0050	1044	0.371	0.0214	0.0033	936	0.321
4500	0.83	0.0405	0.0050	1149	0.360	0.0233	0.0034	1029	0.311
5000	0.80	0.0437	0.0053	1248	0.349	0.0252	0.0035	1116	0.302

Table 3-3 Moment factors and reinforcement ratios for singly reinforced beams

For reinforced concrete slabs, the minimum reinforcement provided shall be sufficient to control temperature and shrinkage stresses. ACI Section 7.12.2 requires a minimum reinforcement ratio of

$\rho_{min} = 0.0020$ ... Grade 40 and 50 deformed bars
$\rho_{min} = 0.0018$ ... Grade 60 deformed bars

The different methods available for the analysis, or evaluation, of a given reinforced concrete beam are illustrated in Examples 3–1 to 3–4. The different methods available for the design of a reinforced concrete beam are illustrated in Examples 3–5 to 3–8. Example 3–9 illustrates the use of ACI Section 10.5 in determining required minimum reinforcement areas.

Example 3–1 (Analysis of a given reinforced concrete beam using Figure 3–4) ═══
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with 3 square inches of Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the maximum ultimate applied moment which the beam can support.

Solution
The tension reinforcement ratio provided is

$$\begin{aligned}\rho &= A_s/b_w d \\ &= 3/(12 \times 18) \\ &= 0.0139\end{aligned}$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned}\rho_{min} &= 0.0033 \\ &< \rho \ldots \text{satisfactory}\end{aligned}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned}\rho_{max} &= 0.0214 \\ &> \rho \ldots \text{satisfactory}\end{aligned}$$

Hence the reinforcement provided in the section is satisfactory.

Equating the tensile and compressive forces acting on the section gives the depth of the equivalent rectangular stress block as

$$\begin{aligned}a &= A_s f_y/0.85 f'_c b_w \\ &= 3 \times 60{,}000/(0.85 \times 4000 \times 12) \\ &= 4.41 \text{ inches}\end{aligned}$$

The lever-arm of the internal resisting moment is

$$\begin{aligned}jd &= d - a/2 \\ &= 18 - 4.41/2 \\ &= 15.79 \text{ inches}\end{aligned}$$

The nominal moment of resistance is

$$\begin{aligned}M_n &= A_s f_y jd \\ &= 3 \times 60{,}000 \times 15.79/12{,}000 \\ &= 236.9 \text{ kip feet}\end{aligned}$$

The maximum ultimate applied moment that the beam can support is

$$\begin{aligned} M_u &= 0.9M_u \\ &= 213.2 \text{ kip feet} \end{aligned}$$

Example 3–2 (Analysis of a given reinforced concrete beam using basic equations)
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with 3 square inches of Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the maximum ultimate applied moment which the beam can support.

Solution
The tension reinforcement ratio provided is

$$\begin{aligned} \rho &= A_s/b_w d \\ &= 3/(12 \times 18) \\ &= 0.0139 \end{aligned}$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned} \rho_{min} &= 0.0033 \\ &< \rho \ldots \text{ satisfactory} \end{aligned}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned} \rho_{max} &= 0.0214 \\ &> \rho \ldots \text{ satisfactory} \end{aligned}$$

Hence the reinforcement provided in the section is satisfactory.

The nominal flexural strength of the member is

$$\begin{aligned} M_n &= A_s f_y d(1 - 0.59A_s f_y/b_w d f'_c) \\ &= 3 \times 60{,}000 \times 18[1 - 0.59 \times 3 \times 60{,}000/(12 \times 18 \times 3000)]/12{,}000 \\ &= 236.8 \text{ kip feet} \end{aligned}$$

The maximum ultimate applied moment that the beam can support is

$$\begin{aligned} M_u &= 0.9M_u \\ &= 213.1 \text{ kip feet} \end{aligned}$$

Example 3–3 (Analysis of a given reinforced concrete beam using Table 3–2)
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with 3 square inches of Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the maximum ultimate applied moment which the beam can support.

Solution

The tension reinforcement ratio provided is

$$\begin{aligned}\rho &= A_s/b_w d \\ &= 3/(12 \times 18) \\ &= 0.0139\end{aligned}$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned}\rho_{min} &= 0.0033 \\ &< \rho \ldots \text{satisfactory}\end{aligned}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\begin{aligned}\rho_{max} &= 0.0214 \\ &> \rho \ldots \text{satisfactory}\end{aligned}$$

Hence the reinforcement provided in the section is satisfactory.

The tension reinforcement index is

$$\begin{aligned}\omega &= \rho f_y/f'_c \\ &= 0.0139 \times 60{,}000/4000 \\ &= 0.2085\end{aligned}$$

From Table 3–2, the corresponding value of

$$M_u/f'_c b_w d^2 = 0.1646$$

The maximum ultimate applied moment that the beam can support is

$$\begin{aligned}M_u &= 0.1646 f'_c b_w d^2 \\ &= 0.1646 \times 4000 \times 12 \times 18^2/12{,}000 \\ &= 213.3 \text{ kip feet}\end{aligned}$$

Example 3–4 (Analysis of a given reinforced concrete beam using a calculator program) = A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with 3 square inches of Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the maximum ultimate applied moment which the beam can support.

Solution

Using calculator program TRA in the Appendix.
Enter data:

$$\begin{aligned}A_s &= 3 \text{ square inches} \\ b_w &= 12 \text{ inches} \\ d &= 18 \text{ inches} \\ f_y &= 60 \text{ kips per square inch} \\ f'_c &= 4 \text{ kips per square inch}\end{aligned}$$

Display output functions:

ρ = 0.0139
ρ_{max} = 0.0214
ρ_{min} = 0.0033
M_u = 213.1 kip feet

Example 3–5 (Design of a reinforced concrete beam using Figure 3–4)
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 300 kip feet.

Solution
The required nominal moment of resistance is

$$M_n = M_u/0.9 = 300/0.9 = 333.3 \text{ kip feet}$$

Assume the depth of the equivalent rectangular stress block is

$$a = 6.7 \text{ inches}$$

The lever-arm of the internal resisting moment is

$$jd = d - a/2 = 18 - 6.7/2 = 14.65 \text{ inches}$$

The required reinforcement area is given by

$$A_s = M_n/f_y jd = 333.3 \times 12000/(60{,}000 \times 14.65) = 4.55 \text{ square inches}$$

For this area of reinforcement, the depth of the equivalent rectangular stress block is

$$a = A_s f_y/0.85 f'_c b_w = 4.55 \times 60{,}000/(0.85 \times 4000 \times 12) = 6.69 \text{ inches} \approx 6.7 \text{ inches}$$

and the initial estimate of the depth of the stress block is satisfactory.

The reinforcement ratio is

$$\rho = A_s/b_w d = 4.55/(12 \times 18) = 0.0211$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{min} = 0.0033$$
$$< \rho \ldots \text{satisfactory}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{max} = 0.0214$$
$$> \rho \ldots \text{satisfactory}$$

Hence the section size is adequate.

Example 3–6 (Design of a reinforced concrete beam using basic equations)
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 300 kip feet.

Solution
The reinforcement ratio required to provide a given factored moment M_u is

$$\rho = 0.85f'_c[1 - (1 - M_u/0.383f'_c b_w d^2)^{0.5}]/f_y$$
$$= 0.85 \times 4000\{1 - 300 \times 12{,}000/(0.383 \times 4000 \times 12 \times 18^2]^{0.5}\}/60{,}000$$
$$= 0.0210$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{min} = 0.0033$$
$$< \rho \ldots \text{satisfactory}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{max} = 0.0214$$
$$> \rho \ldots \text{satisfactory}$$

Hence the section size is adequate.

The reinforcement area required is

$$A_s = \rho b_w d$$
$$= 0.0210 \times 12 \times 18$$
$$= 4.54 \text{ square inches}$$

Example 3–7 (Design of a reinforced concrete beam using Table 3–2)
A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 300 kip feet.

Solution

The design moment factor is

$$K_u = M_u/b_w d^2$$
$$= 300 \times 12{,}000/(12 \times 18^2)$$
$$= 925.9 \text{ pounds per square inch}$$

and

$$K_u/f'_c = 925.9/4000$$
$$= 0.2315$$

From Table 3–2, the corresponding tension reinforcement index is

$$\omega = 0.316$$

The required tension reinforcement ratio is given by

$$\rho = \omega f'_c/f_y$$
$$= 0.316 \times 4000/60{,}000$$
$$= 0.0211$$

The minimum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{min} = 0.0033$$
$$< \rho \ldots \text{ satisfactory}$$

The maximum allowable reinforcement ratio is obtained from Table 3–3 as

$$\rho_{max} = 0.0214$$
$$> \rho \ldots \text{ satisfactory}$$

Hence the section size is adequate.

The reinforcement area required is

$$A_s = \rho b_w d$$
$$= 0.0211 \times 12 \times 18$$
$$= 4.55 \text{ square inches}$$

Example 3–8 (Design of a reinforced concrete beam using a calculator program)

A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 300 kip feet.

Solution

Using calculator program TRD in the Appendix.

Enter data:

M_u = 300 kip feet

b_w = 12 inches

$d = 18$ inches
$f_y = 60$ kips per square inch
$f'_c = 4$ kips per square inch

Display output functions:

$\rho = 0.0211$
$\rho_{max} = 0.0214$
$\rho_{min} = 0.0033$
$A_s = 4.55$ square inches

Example 3–9 (Minimum reinforcement area)

A reinforced concrete beam, with an effective depth of 18 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 40 kip feet.

Solution

The reinforcement ratio required to provide a given factored moment M_u is

$$\begin{aligned}\rho &= 0.85f'_c[1 - (1 - M_u/0.383f'_c b_w d^2)^{0.5}]/f_y \\ &= 0.85 \times 4000\{1 - 40 \times 12{,}000/(0.383 \times 4000 \times 12 \times 18^2]^{0.5}\}/60{,}000 \\ &= 0.00233\end{aligned}$$

and

$$\begin{aligned}1.33\rho &= 1.33 \times 0.00233 \\ &= 0.0031\end{aligned}$$

The minimum reinforcement ratio in accordance with ACI Section 10.5.1 is

$$\begin{aligned}\rho_{min} &= 200/f_y \\ &= 200/60{,}000 \\ &= 0.0033 \\ &> \rho \\ &> 1.33\rho\end{aligned}$$

Hence, in accordance with ACI Section 10.5.3, the required minimum reinforcement ratio is

$$\begin{aligned}\rho_{rm} &= 1.33\rho \\ &= 0.0031\end{aligned}$$

The required minimum reinforcement area is

$$\begin{aligned}A_s &= \rho_{rm} b_w d \\ &= 0.0031 \times 12 \times 18 \\ &= 0.67 \text{ square inches}\end{aligned}$$

Example 3–10 (Design of a reinforced concrete beam with tension reinforcement only) ═ A reinforced concrete beam, with an effective depth of 16 inches and a width of 12 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. The beam carries a superimposed dead load, including its self weight, of 2.5 kips per foot run and a superimposed live load of 2.7 kips per foot run over on effective span of 15 feet. Determine the area of tension reinforcement required.

Solution

The applied dead load moment is given by

$$\begin{aligned} M_D &= w_D \ell^2/8 \\ &= 2.5 \times 15^2/8 \\ &= 70.31 \text{ kip feet} \end{aligned}$$

The applied live load moment is given by

$$\begin{aligned} M_L &= w_L \ell^2/8 \\ &= 2.7 \times 15^2/8 \\ &= 75.94 \text{ kip feet} \end{aligned}$$

The factored moment at mid span is obtained from ACI Equation (9–1) as

$$\begin{aligned} M_u &= 1.4M_D + 1.7M_L \\ &= 1.4 \times 70.31 + 1.7 \times 75.95 \\ &= 227.53 \text{ kip feet} \end{aligned}$$

The maximum allowable factored moment for a singly reinforced beam is obtained from Table 3.3 as

$$\begin{aligned} M_{max} &= K_{u(max)} b_w d^2 \\ &= 936 \times 12 \times 16^2/12{,}000 \\ &= 239.62 \text{ kip feet} \\ &> M_u \end{aligned}$$

Hence the section size is adequate.

The design moment factor is

$$\begin{aligned} K_u &= M_u/b_w d^2 \\ &= 227.53 \times 12{,}000/(12 \times 16^2) \\ &= 888.79 \text{ pounds per square inch} \end{aligned}$$

and

$$\begin{aligned} K_u/f'_c &= 888.79/4000 \\ &= 0.2222 \end{aligned}$$

From Table 3–2, the corresponding tension reinforcement index is

$$\omega = 0.299$$

The required tension reinforcement ratio is given by

$$\rho = \omega f'_c/f_y$$
$$= 0.299 \times 4000/60{,}000$$
$$= 0.020$$

The required area of tension reinforcement is

$$A_s = \rho b_w d$$
$$= 0.020 \times 12 \times 16$$
$$= 3.83 \text{ square inches}$$

Provide three No. 10 bars which, from Table 1–7, give an area of 3.81 square inches. ═══

Example 3–11 (Analysis of a reinforced concrete slab) ═══════════

A one-way, reinforced concrete slab is continuous over four, equal, clear spans of 15 feet. The slab has an overall depth of 7 inches with a concrete cylinder strength of 3000 pounds per square inch. The reinforcement at the top of the slab at the first interior support consists of No. 4, Grade 60, bars at 11 inches on center. In addition to its own weight, the slab carries a superimposed dead load of 26 pounds per square foot. Determine the maximum uniformly distributed superimposed live load which the slab can support.

Solution

The minimum allowable concrete cover to the reinforcement in the slab, in accordance with ACI Section 7.7.1, is

$$c_c = 0.75 \text{ inches}$$

The effective depth provided is

$$d = h - c_c - d_b/2$$
$$= 7 - 0.75 - 0.5/2$$
$$= 6 \text{ inches}$$

The area of tension reinforcement in a 12 inch width of slab with No. 4 bars at 11 inches on center is obtained from Table 1–8 as

$$A_s = 0.22 \text{ square inches}$$

The tension reinforcement ratio provided is given by

$$\begin{aligned}\rho &= A_s/b_w d \\ &= 0.22/(12 \times 6) \\ &= 0.00306 \\ &> \rho_{min} \ldots (= 0.0018 \text{ from ACI Section } 7.12.2) \\ &< \rho_{max} \ldots (= 0.0160 \text{ from ACI Section } 10.3.3)\end{aligned}$$

Hence the reinforcement provided is satisfactory.

The tension reinforcement index is

$$\begin{aligned}\omega &= \rho f_y/f'_c \\ &= 0.00306 \times 60{,}000/3000 \\ &= 0.0612\end{aligned}$$

From Table 3–2, the corresponding value of

$$K_u/f'_c = 0.0531$$

The allowable factored moment is given by

$$\begin{aligned}M_u &= K_u b_w d^2 \\ &= 0.0531 \times 3000 \times 12 \times 6^2/12{,}000 \\ &= 5.73 \text{ kip feet}\end{aligned}$$

The self weight of the slab is

$$\begin{aligned}w_{SD} &= 7 \times 12 \times 0.15/144 \\ &= 0.0875 \text{ kips per foot}\end{aligned}$$

The total dead load is given by

$$\begin{aligned}w_D &= w_{SD} + w_{AD} \\ &= 0.0875 + 0.026 \\ &= 0.1135 \text{ kips per foot}\end{aligned}$$

Using the approximate moment coefficients given in ACI Section 8.3.3, the negative moment produced by the dead load at the first interior support is

$$\begin{aligned}M_D &= w_D \ell_n^2/10 \\ &= 0.1135 \times 15^2/10 \\ &= 2.55 \text{ kip feet}\end{aligned}$$

The factored dead load moment is obtained from ACI Section 9.2.1 as

$$\begin{aligned}M_{uD} &= 1.4 \times M_D \\ &= 1.4 \times 2.55 \\ &= 3.57 \text{ kip feet}\end{aligned}$$

The allowable factored live load moment is given by

$$\begin{aligned}M_{uL} &= M_u - M_{uD} \\ &= 5.73 - 3.57 \\ &= 2.16 \text{ kip feet}\end{aligned}$$

The allowable non-factored live load moment is obtained from ACI Section 9.2.1 as

$$\begin{aligned} M_L &= M_{uL}/1.7 \\ &= 2.16/1.7 \\ &= 1.27 \text{ kip feet} \end{aligned}$$

The allowable non-factored live load is obtained from ACI Section 8.3.3 as

$$\begin{aligned} w_L &= 10M_L/\ell_n^2 \\ &= 10 \times 1.27 \times 1000/15^2 \\ &= 56.4 \text{ pounds per foot} \end{aligned}$$

The slab can support a uniformly distributed live load of 56.4 pounds per square foot. ═══

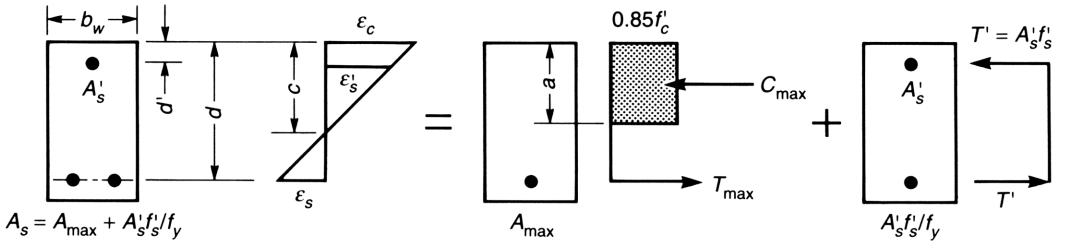

Figure 3–6 Reinforced concrete beam with compression reinforcement

3.2.2 Rectangular beam with compression reinforcement

When the applied factored moment exceeds the maximum design strength of a singly reinforced member with the maximum allowable reinforcement ratio, compression reinforcement and additional tension reinforcement must be provided, as shown in Figure 3–6. The difference between the applied factored moment and the maximum design moment strength of a singly reinforced section is

$$M_r = M_u - M_{max} = \text{residual moment}$$

The required area of compression reinforcement is given by

$$A'_s = M_r/\phi f'_s(d - d')$$

where f'_s = stress in the compression reinforcement

The additional area of tension reinforcement required is

$$\begin{aligned} A_T &= M_r/\phi f_y(d - d') \\ &= A'_s f'_s/f_y \end{aligned}$$

The total required area of tension reinforcement is

$$\begin{aligned} A_s &= A_{max} + A_T \\ &= A_{max} + M_r/\phi f_y(d - d') \end{aligned}$$

The depth of the stress block is obtained by equating horizontal forces and is given by

$$a = f_y A_{max}/0.85 f'_c b_w$$

The neutral axis depth is given by ACI Section 10.2.7.1 as

$$c = a/\beta_1$$

The strain in the compression reinforcement is obtained from the strain diagram in Figure 3–6 and is

$$\epsilon'_s = \epsilon_c(1 - d'/c)$$

where $\epsilon_c = 0.003$

The stress in the compression reinforcement is

$$\begin{aligned} f'_s &= \epsilon_c E_s(1 - d'/c) \\ &= 0.003 \times 29{,}000{,}000(1 - d'/c) \\ &= 87{,}000(1 - d'/c) \\ &\leq f_y \end{aligned}$$

The required area of compression reinforcement is

$$A'_s = A_T f_y / f'_s$$

The maximum allowable tension reinforcement ratio is given by ACI Section 10.3.3 as

$$\begin{aligned} \rho'_{max} &= 0.75\rho_b + A'_s f'_s / b_w d f_y \\ &= \rho_{max} + \rho' f'_s / f_y \end{aligned}$$

where $\rho' = A'_s / b_w d$

In order to analyze a given member with compression reinforcement, an initial estimate of the neutral axis depth is required. Assuming that the compressive strain in the concrete is 0.003 the stress in the compression and tension reinforcement may now be determined. The total compressive force in the concrete and the compression reinforcement is then compared with the tensile force in the tension reinforcement. The initial estimate of the neutral axis depth is then adjusted until these two values are equal. The maximum nominal moment provided by the section is obtained by taking moments of the forces in the concrete and in the compression reinforcement about the centroid of the tension reinforcement.

Example 3–12 (Design of a rectangular beam with compression reinforcement) ═══

A reinforced concrete beam is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. The beam carries a superimposed live load of four kips per foot run over an effective span of thirty feet. The beam has an overall depth of 30 inches, a width of 12 inches, an effective depth to the centroid of the tension reinforcement of 27 inches, and a depth to the centroid of the compression reinforcement of 3 inches. Determine the areas of tension and compression steel required.

Solution

The self weight of the beam is

$$\begin{aligned} w_D &= 30 \times 12 \times 0.15/144 \\ &= 0.375 \text{ kips per foot} \end{aligned}$$

The dead load moment is given by

$$\begin{aligned} M_D &= w_D \ell^2/8 \\ &= 0.375 \times 30^2/8 \\ &= 42 \text{ kip feet} \end{aligned}$$

The live load moment is given by

$$\begin{aligned} M_L &= w_L \ell^2/8 \\ &= 4 \times 30^2/8 \\ &= 450 \text{ kip feet} \end{aligned}$$

The factored moment at mid span is obtained from ACI Equation (9–1) as

$$\begin{aligned} M_u &= 1.4M_D + 1.7M_L \\ &= 1.4 \times 42 + 1.7 \times 450 \\ &= 823.8 \text{ kip feet} \end{aligned}$$

The maximum allowable reinforcement ratio for a singly reinforced beam is obtained from Table 3–3 as

$$\rho_{max} = 0.016$$

The maximum reinforcement area for a singly reinforced beam is

$$\begin{aligned} A_{max} &= b_w d \rho_{max} \\ &= 12 \times 27 \times 0.016 \\ &= 5.18 \text{ square inches} \end{aligned}$$

The maximum allowable design moment factor for a singly reinforced beam is obtained from Table 3–3 as

$$K_{u(max)} = 702 \text{ pounds per square inch}$$

The maximum design moment of a singly reinforced section is

$$\begin{aligned} M_{max} &= K_{u(max)} b_w d^2 \\ &= 702 \times 12 \times 27^2/12{,}000 \\ &= 511.8 \text{ kip feet} \end{aligned}$$

The residual moment is given by

$$\begin{aligned} M_r &= M_u - M_{max} \\ &= 312 \text{ kip feet} \end{aligned}$$

The additional area of tension reinforcement required is

$$\begin{aligned} A_T &= 12M_r/\phi f_y(d - d') \\ &= 12 \times 312/[0.9 \times 60(27 - 3)] \\ &= 2.89 \text{ square inches} \end{aligned}$$

The total required area of tension reinforcement is given by

$$\begin{aligned} A_s &= A_{max} + A_T \\ &= 5.18 + 2.89 \\ &= 8.07 \text{ square inches} \end{aligned}$$

The depth of the stress block is obtained by equating horizontal forces and is

$$\begin{aligned} a &= f_y A_{max}/0.85 f'_c b_w \\ &= 60 \times 5.18/(0.85 \times 3 \times 12) \\ &= 10.16 \text{ inches} \end{aligned}$$

The neutral axis depth is given by ACI Section 10.2.7.1 as

$$\begin{aligned} c &= a/\beta_1 \\ &= 10.16/0.85 \\ &= 11.95 \text{ inches} \end{aligned}$$

The stress in the compression reinforcement is

$$\begin{aligned} f'_s &= \epsilon_c E_s(1 - d'/c) \\ &= 87{,}000(1 - 3/11.95) \\ &= 60{,}000 \text{ pounds per square inch ... maximum} \end{aligned}$$

The required area of compression reinforcement is

$$\begin{aligned} A'_s &= A_T f_y/f'_s \\ &= 2.89 \times 60/60 \\ &= 2.89 \text{ square inches} \end{aligned}$$

The maximum allowable tension reinforcement area is given by ACI Section 10.3.3 as

$$\begin{aligned} A'_{max} &= A_{max} + A'_s f'_s/f_y \\ &= 5.18 + 2.89 \times 60/60 \\ &= 8.07 \text{ square inches} \\ &= A_s \text{ ... satisfactory} \end{aligned}$$

Example 3–13 (Analysis of a rectangular beam with compression reinforcement)
A reinforced concrete beam is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. The beam has a width of 24 inches, an effective depth to the centroid of the tension reinforcement of 27 inches, and a depth to the centroid of the compression reinforcement of 3 inches. The compression reinforcement consists of 4 No. 11 bars and the tension reinforcement consists of 6 No. 11 bars. Determine the design moment strength of the section.

Solution

The area of tension reinforcement provided by 6 No. 11 bars is obtained from Table 1–7 as

$$A_s = 9.36 \text{ square inches}$$

The tension reinforcement ratio provided is given by

$$\begin{aligned} \rho &= A_s/b_w d \\ &= 9.36/(24 \times 27) \\ &= 0.014 \end{aligned}$$

The area of compression reinforcement provided by 4 No. 11 bars is

$$A'_s = 6.24 \text{ square inches}$$

The compression reinforcement ratio provided is given by

$$\begin{aligned}\rho' &= A'_s/b_w d \\ &= 6.24/(24 \times 27) \\ &= 0.00963\end{aligned}$$

Assume that the depth of the neutral axis is

$$c = 4.98 \text{ inches}$$

The stress in the tension reinforcement is obtained from Figure 3–6 as

$$\begin{aligned}f_s &= E_s\epsilon_s \\ &= E_s\epsilon_c(d - c)/c \\ &= 29 \times 10^6 \times 0.003(d - c)/c \\ &= 87{,}000(d/c - 1) \\ &= 87{,}000(27/4.98 - 1) \\ &= 60 \text{ kips per square inch } \ldots \text{ maximum}\end{aligned}$$

The stress in the compression reinforcement is obtained from Figure 3–6 as

$$\begin{aligned}f'_s &= E_s\epsilon'_s \\ &= E_s\epsilon_s(c - d')/c \\ &= 29 \times 10^6 \times 0.003\ (c - d')/c \\ &= 87{,}000(1 - d'/c) \\ &= 87{,}000(1 - 3/4.98) \\ &= 34{,}590 \text{ pounds per square inch} \\ &= 34.59 \text{ kips per square inch}\end{aligned}$$

The maximum allowable tension reinforcement ratio is given by ACI Section 10.3.3 as

$$\begin{aligned}\rho'_{max} &= \rho_{max} + \rho' f'_s/f_y \\ &= 0.0214 + 0.00963 \times 34.59/60 \\ &= 0.0269 \\ &> \rho \ \ldots \text{ satisfactory}\end{aligned}$$

The depth of the stress block is given by ACI Section 10.2.7.1

$$\begin{aligned}a &= \beta_1 c \\ &= 0.85 \times 4.98 \\ &= 4.23 \text{ inches}\end{aligned}$$

The compressive force in the compressive stress block is

$$\begin{aligned}C_c &= 0.85 f'_c ba \\ &= 0.85 \times 4 \times 24 \times 4.23 \\ &= 345.4 \text{ kips}\end{aligned}$$

The compressive force in the compression reinforcement is

$$\begin{aligned}C_s &= f'_s A'_s \\ &= 34.59 \times 6.24 \\ &= 215.8 \text{ kips}\end{aligned}$$

The tensile force in the tension reinforcement is

$$\begin{aligned} T &= f_y A_s \\ &= 60 \times 9.36 \\ &= 561.6 \text{ kips} \end{aligned}$$

Hence $T \approx C_c + C_s$

and the initial estimate for the depth of the neutral axis is correct.

The design moment strength of the section is obtained by taking moments about the centroid of the tension reinforcement and is given by

$$\begin{aligned} \phi M_n &= \phi[C_c(d - a/2) + C_s(d - d')]/12 \\ &= 0.9[345.4(27 - 4.23/2) + 215.8(27 - 3)]/12 \\ &= 1033 \text{ kip feet} \end{aligned}$$

3.2.3 Flanged section with tension reinforcement only

A flanged member, with a flange width b, may be designed as a rectangular beam when the depth of the equivalent stress block is less than the flange thickness. The depth of the stress block is given by

$$\begin{aligned} a &= A_s f_y / 0.85 f'_c b \\ &= 1.18 \omega d \end{aligned}$$

where ω may be determined from Table 3–2 using the calculated value of $M_u/f'_c bd^2$.
The required reinforcement ratio, corresponding to M_u, is given by

$$\rho = \omega f'_c / f_y$$

and the required area of tension reinforcement is

$$A_s = \rho bd$$

When the depth of the equivalent stress block equals the flange thickness, the design moment strength of the section is given by

$$\begin{aligned} \phi M_n &= 0.85 \phi f'_c b h_f (d - h_f/2) \\ &= 0.765 f'_c b h_f (d - h_f/2) \end{aligned}$$

and the corresponding area of tension reinforcement is

$$A_s = 0.85 f'_c b h_f / f_y$$

The conditions at ultimate load in a flanged member, when the depth of the equivalent rectangular stress block exceeds the flange thickness, are shown in Figure 3–7. The area of reinforcement required to balance the compressive force in the outstanding flanges is given by

$$\begin{aligned} A_{sf} &= C_f / f_y \\ &= 0.85 f'_c h_f (b - b_w) / f_y \end{aligned}$$

The corresponding design moment strength of the outstanding flanges is

$$M_f = \phi A_{sf} f_y (d - h_f/2)$$

The residual moment to be developed by the beam web is

$$M_r = M_u - M_f$$

The required reinforcement ratio to provide the residual moment is

$$\rho_w = 0.85 f'_c [1 - (1 - K_w/0.383 f'_c)^{0.5}]/f_y$$

where

$$K_w = M_r/b_w d^2$$

The corresponding reinforcement area required to develop the residual moment is

$$A_{sw} = b_w d \rho_w$$

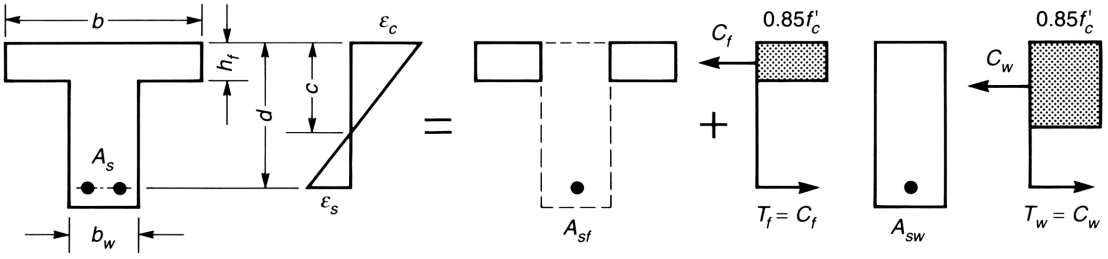

Figure 3–7 Flanged member with tension reinforcement

Alternatively, the value of ω_w may be determined from Table 3–2 using the calculated value of $M_r/f'_c b_w d^2$. The corresponding reinforcement area required to develop the residual moment is

$$A_{sw} = \omega_w b_w d f'_c/f_y$$

The total reinforcement area required is

$$A_s = A_{sf} + A_{sw}$$

The total reinforcement ratio is

$$\rho = A_s/bd$$

The maximum allowable reinforcement ratio is given by Section 10.3.3 as

$$\rho'_{max} = 0.75 b_w(\rho_b + \rho_f)/b$$

where $\rho_f = A_{sf}/b_w d$

and $0.75\rho_b = \rho_{max}$

$$= 0.75 \times 0.85 \times 87{,}000\, \beta_1 f'_c/f_y(87{,}000 + f_y)$$

is the maximum allowable reinforcement ratio for a rectangular section which may be obtained from Table 3–3.

To analyze a flanged member, when the depth of the equivalent rectangular stress block exceeds the flange thickness, the compressive force developed by the outstanding flanges is first determined as

$$C_f = 0.85f'_c h_f(b - b_w)$$

The area of tension reinforcement to balance this compression force is

$$A_{sf} = C_f/f_y$$

The corresponding design moment strength of the outstanding flanges is

$$M_f = \phi A_{sf} f_y(d - h_f/2)$$

The residual reinforcement area is

$$A_{sw} = A_s - A_{sf}$$

where A_s = total tension reinforcement area provided in the section

The residual reinforcement ratio referred to the web is

$$\rho_w = A_{sw}/b_w d$$

The design moment strength of the residual reinforcement is

$$M_w = \phi A_{sw} f_y d(1 - 0.59\rho_w f_y/f'_c)$$

The total design moment strength of the section is

$$M_u = M_f + M_w$$

Alternatively, the value of $M_w/f'_c b_w d^2$ may be determined from Table 3–2 using the calculated value of the reinforcement index

$$\omega_w = \rho_w f_y/f'_c$$

and the value of M_w determined.

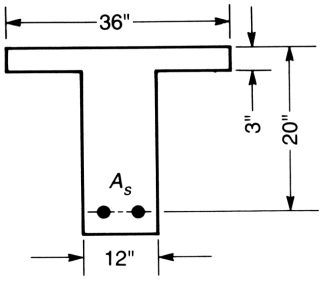

Figure 3–8 Flanged section for Examples 3–14 and 3–15

Example 3–14 (Design of a flanged beam with tension reinforcement)
The reinforced concrete flanged beam, shown in Figure 3–8, is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. Determine the area of tension reinforcement required to support an applied factored moment of 450 kip feet.

Solution
Assuming that the depth of the equivalent rectangular stress block exceeds the flange thickness, the compression force developed by the outstanding flanges is

$$\begin{aligned} C_f &= 0.85f'_c h_f(b - b_w) \\ &= 0.85 \times 3 \times 3(36 - 12) \\ &= 183.6 \text{ kips} \end{aligned}$$

The area of tension reinforcement required to balance the compressive force is

$$\begin{aligned} A_{sf} &= C_f/f_y \\ &= 183.6/60 \\ &= 3.06 \text{ square inches} \end{aligned}$$

The corresponding design moment strength of the outstanding flanges is

$$\begin{aligned} M_f &= \phi A_{sf} f_y(d - h_f/2) \\ &= 0.9 \times 3.06 \times 60(20 - 1.5)/12 \\ &= 254.75 \text{ kip feet} \end{aligned}$$

The residual moment to be developed by the beam web is

$$\begin{aligned} M_r &= M_u - M_f \\ &= 450 - 254.75 \\ &= 195.25 \text{ kip feet} \end{aligned}$$

and

$$\begin{aligned} M_r/f'_c b_w d^2 &= 195.25 \times 12/(3 \times 12 \times 20^2) \\ &= 0.1627 \end{aligned}$$

From Table 3–2, the corresponding tension reinforcement index is

$$\omega_w = 0.2057$$

The reinforcement area required to develop the residual moment is

$$\begin{aligned} A_{sw} &= \omega_w b_w d f'_c/f_y \\ &= 0.2057 \times 12 \times 20 \times 3/60 \\ &= 2.47 \text{ square inches} \end{aligned}$$

The depth of the equivalent rectangular stress block is given by

$$\begin{aligned} a &= A_{sw} f_y/(0.85 \times f'_c \times b_w) \\ &= 2.47 \times 60/(0.85 \times 3 \times 12) \\ &= 4.84 \text{ inches} \\ &> h_f \end{aligned}$$

Hence the depth of the equivalent rectangular stress block does exceed the flange thickness as assumed.

The total reinforcement area required is

$$\begin{aligned} A_s &= A_{sf} + A_{sw} \\ &= 3.06 + 2.47 \\ &= 5.53 \text{ square inches} \end{aligned}$$

The corresponding reinforcement ratio is

$$\begin{aligned} \rho &= A_s/bd \\ &= 5.53/(36 \times 20) \\ &= 0.00768 \end{aligned}$$

The maximum allowable reinforcement ratio for the beam web is obtained from Table 3–3 as

$$\rho_{max} = 0.016$$

The maximum allowable reinforcement ratio for the flanged beam is given by ACI Section 10.3.3 as

$$\begin{aligned} \rho'_{max} &= \rho_{max} b_w/b + 0.75A_{sf}/bd \\ &= 0.016 \times 12/36 + 0.75 \times 3.06/(36 \times 20) \\ &= 0.00852 \\ &> \rho \ldots \text{ satisfactory} \end{aligned}$$

Example 3–15 (Analysis of a flanged beam with tension reinforcement)

The reinforced concrete flanged beam, shown in Figure 3–8, is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. The area of tension reinforcement provided is six square inches. Determine the design moment strength of the section.

Solution

From Example 3–14, it is clear that the depth of the equivalent rectangular stress block exceeds the flange thickness and that the area of tension reinforcement provided is less than the allowable maximum. The compression force developed by the outstanding flanges is

$$\begin{aligned} C_f &= 0.85f'_c h_f(b - b_w) \\ &= 0.85 \times 3 \times 3(36 - 12) \\ &= 183.6 \text{ kips} \end{aligned}$$

The area of tension reinforcement required to balance the compressive force is

$$\begin{aligned} A_{sf} &= C_f/f_y \\ &= 183.6/60 \\ &= 3.06 \text{ square inches} \end{aligned}$$

The corresponding design moment strength of the outstanding flanges is

$$\begin{aligned} M_f &= \phi A_{sf} f_y(d - h_f/2) \\ &= 0.9 \times 3.06 \times 60(20 - 1.5)/12 \\ &= 254.75 \text{ kip feet} \end{aligned}$$

The residual reinforcement area is

$$\begin{aligned} A_{sw} &= A_s - A_{sf} \\ &= 6.00 - 3.06 \\ &= 2.94 \text{ square inches} \end{aligned}$$

The corresponding tension reinforcement index is

$$\begin{aligned} \omega_w &= \rho_w f_y / f'_c \\ &= 2.94 \times 60/(3 \times 12 \times 20) \\ &= 0.245 \end{aligned}$$

From Table 3–2, the corresponding value of

$$M_w/f'_c b_w d^2 = 0.1887$$

The design moment strength of the residual reinforcement is

$$\begin{aligned} M_w &= 0.1887 \times 3 \times 12 \times 20^2/12 \\ &= 226.44 \text{ kip feet} \end{aligned}$$

The total design moment strength of the flanged section is

$$\begin{aligned} M_u &= M_f + M_w \\ &= 254.75 + 226.44 \\ &= 481.19 \text{ kip feet} \end{aligned}$$

3.3 SERVICEABILITY REQUIREMENTS

3.3.1 Crack control

To protect the reinforcement from corrosion, ACI Section 10.6 specifies limitations on reinforcement distribution in order to control flexural cracking. Crack width and reinforcement corrosion increase[8,9,10] as

- reinforcement stress increases
- concrete cover increases
- the area of concrete surrounding each bar increases

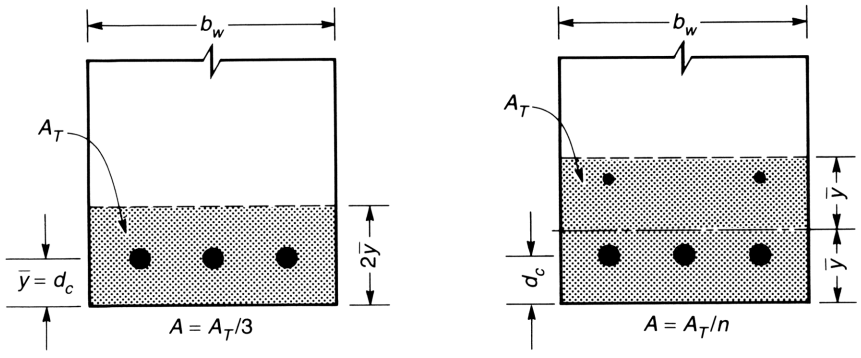

Figure 3–9 Crack control details

It has been determined, from an extensive research program[8], that the maximum crack width at the service load in beams reinforced with deformed bars is given by

$$w = 0.000091 \times f_s(d_c A)^{0.33} \text{ inches}$$

where f_s = stress in the reinforcement at the service load as determined by the straight-line, elastic, theory

d_c = concrete cover measured to center of reinforcement closest to tensile face of concrete

A = effective concrete area in the tension block per equivalent number of bars

Specifying a maximum allowable crack width for exterior exposure of 0.013 inches gives

$$\begin{aligned} w/0.000091 &= 0.013/0.000091 \\ &= 143 \\ &\approx 145 \end{aligned}$$

Specifying a maximum allowable crack width for interior exposure of 0.016 inches gives

$$\begin{aligned} w/0.000091 &= 0.016/0.000091 \\ &= 176 \\ &\approx 175 \end{aligned}$$

Since, in practice, crack widths are extremely variable, the ACI Code adopts a procedure which gives satisfactory compliance with the crack width requirements without the necessity of calculating actual crack widths.

Crack control is achieved, in the ACI Code, by limiting the value of the parameter z to 145 kips per inch for exterior exposure and 175 kips per inch for interior exposure where the parameter z is given by ACI Equation (10–5) as

$$z = f_s(d_c A)^{0.33}$$

where f_s = stress in reinforcement at service load determined by the straight-line, elastic, theory

$\approx 0.6f_y$ kips per square inch

d_c = concrete cover measured to center of reinforcement closest to tensile face of concrete

A = effective concrete area in the tension block per equivalent number of bars

The derivation of the crack control parameters is illustrated in Figure 3–9. The equivalent number of bars is given by

$$n = A_s/A_L$$

where A_s = total tension reinforcement area

A_L = area of the largest bar in the group

$\bar{y}$ = height of the centroid of the tension reinforcement

A_T = total concrete area in the tension block

$= 2\bar{y}b_w$

$A = A_T/n$

From the form of the ACI Equation, it is clear that cracking is more effectively controlled by using many smaller bars at close spacing rather than fewer large bars at wider spacing.

These provisions are adequate for normal exposure conditions. Where particularly corrosive conditions exist, or watertight construction is required, special requirements are necessary[11].

Over the depth of a beam, the crack widths vary from a minimum at the level of the reinforcement to a maximum at a height of one-third the distance to the neutral axis. Crack control over the sides of beams is achieved in the ACI Code by providing a specified area of skin reinforcement at a limited spacing. If the reinforcement provided complies with these requirements, the maximum specified crack width will not be exceeded. To control cracking in the webs of beams with an effective depth exceeding three feet, ACI Section 10.6.7 requires that skin reinforcement be provided in the lower half of each face of the web with an area of

$$A_{sk} \geq 0.012(d - 30) \text{ square inches per foot depth}$$

The maximum spacing of this reinforcement shall not exceed twelve inches or d/6 and the reinforcement shall be uniformly distributed along both sides of the member.

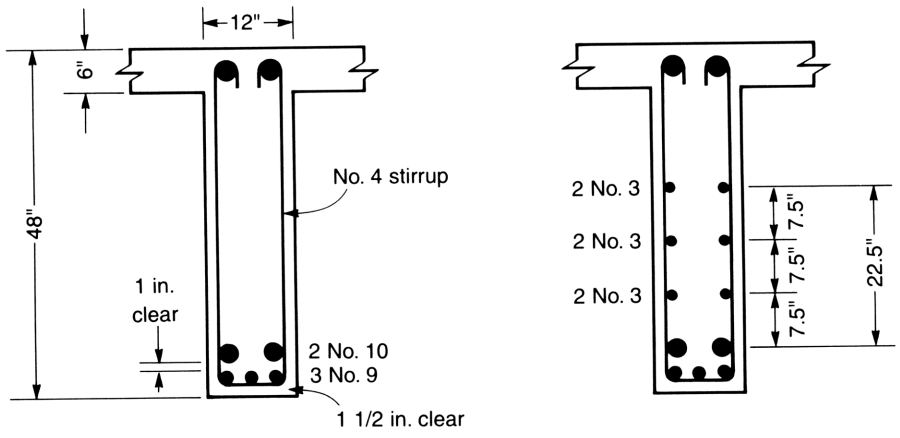

Figure 3–10 Details for Example 3–16

Example 3–16 (Crack control reinforcement)

The reinforced concrete T-beam shown in Figure 3–10 is reinforced with Grade 60 bars at the positions indicated. Determine the value of the parameter z and the required skin reinforcement.

Solution

The total area of tension reinforcement provided is

$$A_s = 3 \times 1 + 2 \times 1.27$$
$$= 5.54 \text{ square inches}$$

Concrete cover to the center of the bottom layer of reinforcement is

$$d_c = 1.5 + 0.5 + 1.128/2$$
$$= 2.56 \text{ inches}$$

Height of the centroid of the top layer of tension reinforcement is

$$y_2 = d_c + 1.128/2 + 1 + 1.27/2$$
$$= 4.76 \text{ inches}$$

Height of the centroid of the total tension reinforcement is

$$\bar{y} = (1 \times 3d_c + 1.27 \times 2y_2)/A_s$$
$$= (1 \times 3 \times 2.56 + 1.27 \times 2 \times 4.76)/5.54$$
$$= 3.57 \text{ inches}$$

The total area of the concrete tension block is

$$A_T = 2\bar{y}b_w$$
$$= 2 \times 3.57 \times 12$$
$$= 85.65 \text{ square inches}$$

Equivalent number of No. 10 bars is

$$n = A_s/1.27$$
$$= 5.54/1.27$$
$$= 4.36 \text{ bars}$$

Effective concrete area in the tension block is

$$A = A_T/n$$
$$= 85.65/4.36$$
$$= 19.64 \text{ square inches per bar}$$

Stress in the reinforcement at service load may be taken as

$$f_s = 0.6f_y$$
$$= 0.6 \times 60$$
$$= 36 \text{ kips per square inch}$$

Hence, the parameter z is given by ACI Equation (10–5) as

$$z = f_s(d_cA)^{0.33}$$
$$= 36\ (2.56 \times 19.64)^{0.33}$$
$$= 131 \text{ kips per inch}$$
$$< 145$$

Hence, the reinforcement distribution is satisfactory for exterior exposure.
The effective depth of the tensile reinforcement is

$$d = h - \bar{y}$$
$$= 48 - 3.57$$
$$= 44.43 \text{ inches}$$

Since the web depth exceeds three feet, skin reinforcement must be provided in each face with an area of

$$A_{sk} = 0.012(d - 30)$$
$$= 0.012(44.43 - 30)$$
$$= 0.173 \text{ square inches per foot}$$

The required bars must be placed within a distance from the tensile reinforcement of

$$d/2 = 22.22 \text{ inches}$$

The maximum allowable spacing is

$$d/6 = 44.43/6$$
$$= 7.41 \text{ inches}$$

Use No. 3 bars at 7.5 inch spacing, as shown in Figure 3–10, to provide a reinforcement area in each face of

$$A'_{sk} = 0.11 \times 12/7.5$$
$$= 0.176 \text{ square inches per foot}$$
$$> A_{sk} \ldots \text{satisfactory}$$

3.3.2 Deflections

The deformation of a structure must be controlled to prevent excessive deflections affecting the appearance and efficiency of the structure. The deflection occurring must be compatible with the degree of movement which can be accommodated by supported elements such as finishes, partitions, utilities, glazing, and cladding.

Acceptable deflections, based on the observed behavior of structures[12], are given in ACI Table 9.5(b). For members not supporting deflection sensitive elements, the allowable, immediate deflections due to live load are

- $\ell/180$ ... flat roofs
- $\ell/360$ ... floors

where ℓ = span length

For roof or floor construction supporting non–sensitive elements, the allowable short–term plus long–term deflection due to loading applied after the construction of the elements is

- $\ell/240$

For roof or floor construction supporting deflection sensitive elements, the corresponding allowable deflection is

- $\ell/480$

For members not supporting deflection sensitive elements ACI Table 9.5(a) provides deemed–to–satisfy provisions consisting of limiting **span/depth ratios**. These ratios are based on past experience and are applicable to normal types of construction. Members conforming to these span/depth ratios are considered to have sufficient stiffness to prevent allowable deflections being exceeded. For normal weight concrete and Grade 60 reinforcement the minimum thicknesses specified are given in Table 3–4. For Grade 40 reinforcement, the tabulated values are multiplied by the factor

$$R_{40} = 0.8$$

For lightweight concrete, the tabulated values are multiplied by the factor

$$R_{LW} = 1.65 - 0.005w_c \geq 1.09$$

where w_c = unit weight of the concrete in pounds per cubic foot

End conditions	Beam	Slab
Simply supported	$\ell/16$	$\ell/20$
Continuous at one end	$\ell/18.5$	$\ell/24$
Continuous at both ends	$\ell/21$	$\ell/28$
Cantilever	$\ell/8$	$\ell/10$

Table 3–4 Minimum thickness of beams and slabs

For two-way reinforced concrete slab systems, minimum depth requirements are specified in ACI Table 9.5(c).

For other situations, where deflections have to be computed, consideration must be given to both short-term and long-term deflections. Factors affecting the magnitude of deflections include:

- An increase in restraint at the support reduces deflections
- An increase in steel stress, under service loads, increases deflections as cracking of the member increases
- Adding compression reinforcement reduces deflections as shrinkage is restrained at the compression face
- Reducing ambient humidity increases deflections by increasing shrinkage
- An increase in the duration of loading increases deflections because of creep effects
- Early loading of a member increases deflection as creep increases when concrete is stressed at an early age

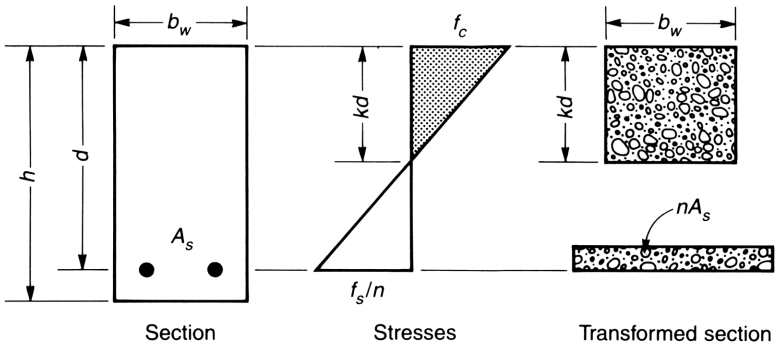

Figure 3–11 Reinforced concrete member at service load

Short-term deflections are calculated in the ACI Code by assuming a constant effective moment of inertia[13] over the whole length of the member. This is not an accurate assumption[14,15] as the inertia varies along the length of the member. At a crack, the stiffness is reduced due to the decrease in the neutral axis depth, and between cracks the stiffness is increased by the uncracked concrete in the tension zone. Short-term deflections may be computed, as shown in Figure 3-11, using the effective moment of inertia given by ACI Equation (9-7) as

$$I_e = (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3] I_{cr}$$

where
- I_g = moment of inertia of gross concrete section, neglecting reinforcement
  = $b_w h^3/12$ ... for a rectangular section
- I_{cr} = moment of inertia of cracked transformed section about the neutral axis
  = $b_w(kd)^3/3 + nA_s(d - kd)^2$ ... for a rectangular section
- k = neutral axis depth ratio at service load for a singly reinforced section
  = $[2\rho n + (\rho n)^2]^{0.5} - \rho n$
- n = modular ratio
  = E_s/E_c
- ρ = reinforcement ratio
  = $A_s/b_w d$
- M_a = maximum moment in the member at the stage that deflection is required
- M_{cr} = cracking moment of the section
  = $2f_r I_g/h$
- f_r = modulus of rupture of normal weight concrete
  = $7.5(f'_c)^{0.5}$ ... from ACI Equation (9-10)

In accordance with ACI Section 9.5.2.4, the effective moment of inertia for prismatic members is calculated at midspan for simple and continuous spans and at the support for cantilevers.

When subjected to **long-term** loads, additional deflection is produced due to the combined effects of creep and shrinkage on the member. The additional deflection is determined from ACI Equation (9-10) by multiplying the short-term deflection by the factor[16]

$$\lambda = \xi/(1 + 50\rho')$$

where
- ρ' = reinforcement ratio for compression reinforcement.
  = $A'_s/b_w d$
- ξ = time-dependent factor for sustained loads given in ACI Section 9.5.2.5
  = 2.0 ... for a 5 year period
  = 1.4 ... for a 12 months period
  = 1.2 ... for a 6 months period
  = 1.0 ... for a 3 months period

The ACI Code multiplier λ represents the additional deflection due to the combined effects of creep and shrinkage on the member. However, only the additional deflection due to creep may be correctly expressed as a multiple of the short-term deflection, the additional deflection due to shrinkage is dependent on the free shrinkage occurring and this is independent of the short-term deflection[15]. In determining the combined total short-term plus long-term deflection, the nonlinear effects due to the cracking must be considered and the following procedure may be adopted[1]:

- Calculate the short term deflection δ_T under the total dead load plus live load using the appropriate values of M_a and I_e
- Calculate the short term deflection δ_S under the permanent load only using the appropriate values of M_a and I_e
- Calculate the additional long-term deflection $\lambda\delta_S$ caused by the permanent load using the appropriate value of λ

The required combined total deflection is given by

$$\begin{aligned}\delta_F &= \delta_T - \delta_S + \delta_S + \lambda\delta_S \\ &= \delta_T + \lambda\delta_S\end{aligned}$$

Example 3–17 (Deflection of a reinforced concrete beam)

The reinforced concrete beam designed in Example 3–10 has an overall depth of 18 inches and a concrete strength of 3000 pounds per square inch. All dead load is in place before the falsework is removed. If 30 percent of the superimposed live load may be considered sustained, determine if the deflection of the beam may be considered satisfactory.

Solution

The unit weight of the concrete is

$$w_c = 150 \text{ pounds per cubic foot}$$

The modulus of elasticity of the concrete is given by ACI Section 8.5.1 as

$$\begin{aligned}E_c &= 33(w_c^3 f'_c)^{0.5} \\ &= 33(150^3 \times 3000)^{0.5} \\ &= 3320 \text{ kips per square inch}\end{aligned}$$

The modulus of elasticity of the reinforcement is given by ACI Section 8.5.2 as

$$E_s = 29{,}000 \text{ kips per square inch}$$

The modular ratio is

$$\begin{aligned}n &= E_s/E_c \\ &= 29/3.32 \\ &= 8.73\end{aligned}$$

The reinforcement ratio is given by

$$\begin{aligned}\rho &= A_s/b_w d \\ &= 3.81/(12 \times 16) \\ &= 0.0198\end{aligned}$$

and

$$\rho n = 0.173$$

From Table 3–5, the corresponding neutral axis depth ratio is obtained as

$$k = 0.4401$$

The moment of inertia of the cracked transformed section about the neutral axis

$$\begin{aligned} I_{cr} &= b_w(kd)^3/3 + nA_s(d - kd)^2 \\ &= 12(0.4401 \times 16)^3/3 \times 8.73 \times 3.81(16 - 0.4401 \times 16)^2 \\ &= 4067 \text{ inches}^4 \end{aligned}$$

The moment of inertia of the gross concrete section, neglecting reinforcement is

$$\begin{aligned} I_g &= b_w h^3/12 \\ &= 12 \times 18^3/12 \\ &= 5832 \text{ inches}^4 \end{aligned}$$

The modulus of rupture of normal weight concrete is obtained from ACI Equation (9–9) as

$$\begin{aligned} f_r &= 7.5(f'_c)^{0.5} \\ &= 7.5(3000)^{0.5} \\ &= 411 \text{ pounds per square inch} \end{aligned}$$

The cracking moment of the section is given by ACI Equation (9–8) as

$$\begin{aligned} M_{cr} &= 2f_r I_g/h \\ &= 2 \times 411 \times 5832/(18 \times 12000) \\ &= 22.18 \text{ kip feet} \end{aligned}$$

For the total dead plus live load, the maximum bending moment on the beam is

$$\begin{aligned} M_a &= M_D + M_L \\ &= 70.31 + 75.94 \\ &= 146.25 \text{ kip feet} \end{aligned}$$

For this bending moment the effective moment of inertia is given by ACI Equation (9–7) as

$$\begin{aligned} I_e &= (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3]I_{cr} \\ &= 0.152^3 \times 5832 + (1 - 0.152^3)4067 \\ &= 4073 \text{ inches}^4 \end{aligned}$$

The short-term deflection produced by the total dead plus live load is

$$\begin{aligned} \delta_T &= 180M_a L^2/(E_c \times I_e) \\ &= 180 \times 146.25 \times 15^2/(3320 \times 4073) \\ &= 0.44 \text{ inches} \end{aligned}$$

For the dead plus the sustained live load, the maximum bending moment on the beam is

$$\begin{aligned} M_a &= M_D + 0.3M_L \\ &= 70.31 + 0.3 \times 75.94 \\ &= 93.09 \text{ kip feet} \end{aligned}$$

For this bending moment the effective moment of inertia is given by ACI Equation (9–7) as

$$\begin{aligned} I_e &= (M_{cr}/M_a)^3 I_g + [1 - (M_{cr}/M_a)^3]I_{cr} \\ &= 0.238^3 \times 5832 + (1 - 0.238^3)4067 \\ &= 4091 \text{ inches}^4 \end{aligned}$$

The short-term deflection produced by the dead load plus the sustained live load is

$$\begin{aligned}\delta_S &= 180M_aL^2/(E_c \times I_e)\\ &= 180 \times 93.09 \times 15^2/(3320 \times 4091)\\ &= 0.28 \text{ inches}\end{aligned}$$

The multiplier for additional long-term deflection is given by ACI Equation (9-10) as

$$\begin{aligned}\lambda &= \xi/(1 + 50\rho')\\ &= 2/(1 + 0)\\ &= 2\end{aligned}$$

The final deflection produced by the immediate deflection of the total loads plus the long-term deflection of the sustained loads is

$$\begin{aligned}\delta_F &= \delta_T + \lambda\delta_S\\ &= 0.44 + 2 \times 0.28\\ &= 1.00 \text{ inches}\end{aligned}$$

The allowable deflection, for non-sensitive elements, in accordance with ACI Section 9.5.2.6 is

$$\begin{aligned}\delta_A &= \ell/240\\ &= 15 \times 12/240\\ &= 0.75 \text{ inches}\\ &< \delta_F \ldots \text{ unsatisfactory}\end{aligned}$$

3.3.3 Working stress design method

The working stress design method is based on the straight-line, elastic theory and is referred to in the ACI Code Appendix A as the Alternate Design Method. Prior to the introduction of the strength design method in the 1956 Code, the working stress design method was the only method approved by the Code. In the working stress design method, the Code stipulates permissible service load stresses which may not be exceeded when the service loads are applied to the member. The basic assumptions adopted by ACI Section A.5 are

- The strain distribution over the depth of the member is linear.
- Stresses in the concrete and in the reinforcement are proportional to the induced strain.
- Tensile stress in the concrete is neglected.

Permissible service load stresses are given in ACI Section A.3 as

$$\begin{aligned}p_{cb} &= \text{maximum permissible stress in the concrete}\\ &= 0.45f'_c\\ p_{st} &= \text{maximum permissible stress in the reinforcement}\\ &= 20 \text{ kips per square inch } \ldots \text{ for Grade 40 reinforcement}\\ &= 24 \text{ kips per square inch } \ldots \text{ for Grade 60 or greater reinforcement}\end{aligned}$$

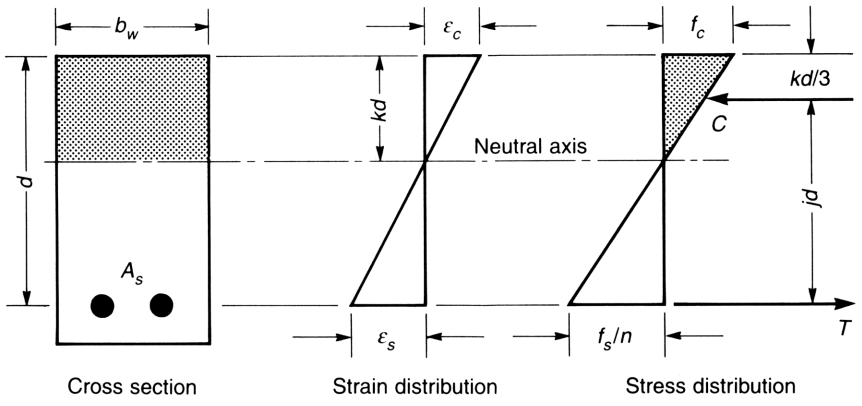

Figure 3–12 Working stress design method

The assumptions adopted by ACI Section A.5 are shown in Figure 3–12. The depth of the neutral axis is obtained by equating tensile and compressive forces acting on the section and is given by

$$kd = 2A_s f_s / b_w f_c$$

The neutral axis depth factor is

$$k = 2\rho f_s / f_c$$

where f_s = steel stress
f_c = maximum concrete stress
ρ = tensile reinforcement ratio
$= A_s / b_w d$

From the strain diagram

$$\epsilon_s / \epsilon_c = (d - kd)/kd$$
$$= (1 - k)/k$$

and $$f_s E_c / f_c E_s = (1 - k)/k$$
$$f_s / f_c = n(1 - k)/k$$

where n = modular ratio
$= E_s / E_c$
E_s = the modulus of elasticity of the reinforcement
= 29,000 kips per square inch ... from ACI Section 8.5.2
E_c = the modulus of elasticity of the concrete
$= 33(w_c^3 f'_c)^{0.5}$ ... from ACI Section 8.5.1
w_c = unit weight of the concrete, pounds per cubic foot

Hence $$k = 2\rho n(1 - k)/k$$
$$= [2\rho n + (\rho n)^2]^{0.5} - \rho n$$

Table 3–5 provides a design aid which tabulates ρn against k and facilitates the analysis of a given section. For a given section, the known values of reinforcement area, concrete strength, and section dimensions enable the determination of ρn and the corresponding value of k.

ρn	0.000	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0	0.0000	0.0437	0.0613	0.0745	0.0855	0.0951	0.1037	0.1115	0.1187	0.1255
0.01	0.1318	0.1377	0.1434	0.1488	0.1539	0.1589	0.1636	0.1682	0.1726	0.1769
0.02	0.1810	0.1850	0.1889	0.1927	0.1964	0.2000	0.2035	0.2069	0.2103	0.2136
0.03	0.2168	0.2199	0.2230	0.2260	0.2290	0.2319	0.2347	0.2375	0.2403	0.2430
0.04	0.2457	0.2483	0.2509	0.2534	0.2559	0.2584	0.2608	0.2632	0.2655	0.2679
0.05	0.2702	0.2724	0.2747	0.2769	0.279	0.2812	0.2833	0.2854	0.2875	0.2895
0.06	0.2916	0.2936	0.2956	0.2975	0.2995	0.3014	0.3033	0.3051	0.307	0.3088
0.07	0.3107	0.3125	0.3142	0.3160	0.3178	0.3195	0.3212	0.3229	0.3246	0.3263
0.08	0.3279	0.3296	0.3312	0.3328	0.3344	0.3360	0.3376	0.3391	0.3407	0.3422
0.09	0.3437	0.3452	0.3467	0.3482	0.3497	0.3511	0.3526	0.3540	0.3554	0.3569
0.10	0.3583	0.3597	0.3610	0.3624	0.3638	0.3651	0.3665	0.3678	0.3691	0.3705
0.11	0.3718	0.3731	0.3744	0.3756	0.3769	0.3782	0.3794	0.3807	0.3819	0.3832
0.12	0.3844	0.3856	0.3868	0.3880	0.3892	0.3904	0.3916	0.3927	0.3939	0.3951
0.13	0.3962	0.3974	0.3985	0.3996	0.4007	0.4019	0.4030	0.4041	0.4052	0.4063
0.14	0.4074	0.4084	0.4095	0.4106	0.4116	0.4127	0.4137	0.4148	0.4158	0.4169
0.15	0.4179	0.4189	0.4199	0.4209	0.4219	0.4229	0.4239	0.4249	0.4259	0.4269
0.16	0.4279	0.4288	0.4298	0.4308	0.4317	0.4327	0.4336	0.4346	0.4355	0.4364
0.17	0.4374	0.4383	0.4392	0.4401	0.4410	0.4419	0.4429	0.4437	0.4446	0.4455
0.18	0.4464	0.4473	0.4482	0.4491	0.4499	0.4508	0.4516	0.4525	0.4534	0.4542
0.19	0.4551	0.4559	0.4567	0.4576	0.4584	0.4592	0.4601	0.4609	0.4617	0.4625
0.20	0.4633	0.4641	0.4649	0.4657	0.4665	0.4673	0.4681	0.4689	0.4697	0.4705
0.21	0.4712	0.4720	0.4728	0.4736	0.4743	0.4751	0.4758	0.4766	0.4774	0.4781
0.22	0.4789	0.4796	0.4803	0.4811	0.4818	0.4825	0.4833	0.4840	0.4847	0.4855
0.23	0.4862	0.4869	0.4876	0.4883	0.4890	0.4897	0.4904	0.4911	0.4918	0.4925
0.24	0.4932	0.4939	0.4946	0.4953	0.4960	0.4966	0.4973	0.4980	0.4987	0.4993
0.25	0.5000	0.5007	0.5013	0.5020	0.5026	0.5033	0.5040	0.5046	0.5053	0.5059
0.26	0.5066	0.5072	0.5078	0.5085	0.5091	0.5097	0.5104	0.5110	0.5116	0.5123
0.27	0.5129	0.5135	0.5141	0.5147	0.5154	0.5160	0.5166	0.5172	0.5178	0.5184
0.28	0.5190	0.5196	0.5202	0.5208	0.5214	0.5220	0.5226	0.5232	0.5238	0.5243
0.29	0.5249	0.5255	0.5261	0.5267	0.5272	0.5278	0.5284	0.5290	0.5295	0.5301
0.30	0.5307	0.5312	0.5318	0.5323	0.5329	0.5335	0.5340	0.5346	0.5351	0.5357
0.31	0.5362	0.5368	0.5373	0.5379	0.5384	0.5389	0.5395	0.5400	0.5406	0.5411
0.32	0.5416	0.5422	0.5427	0.5432	0.5437	0.5443	0.5448	0.5453	0.5458	0.5464
0.33	0.5469	0.5474	0.5479	0.5484	0.5489	0.5494	0.5499	0.5505	0.5510	0.5515
0.34	0.5520	0.5525	0.5530	0.5535	0.5540	0.5545	0.5550	0.5554	0.5559	0.5564
0.35	0.5569	0.5574	0.5579	0.5584	0.5589	0.5593	0.5598	0.5603	0.5608	0.5613
0.36	0.5617	0.5622	0.5627	0.5632	0.5636	0.5641	0.5646	0.5650	0.5655	0.566
0.37	0.5664	0.5669	0.5674	0.5678	0.5683	0.5687	0.5692	0.5696	0.5701	0.5705
0.38	0.5710	0.5714	0.5719	0.5723	0.5728	0.5732	0.5737	0.5741	0.5746	0.575
0.39	0.5755	0.5759	0.5763	0.5768	0.5772	0.5776	0.5781	0.5785	0.5789	0.5794

Table 3–5 Values of the neutral axis factor k

The lever-arm of the internal resisting moment is obtained from Figure 3-12 as

$$jd = d - kd/3$$

and the lever-arm factor is

$$j = 1 - k/3$$

The resisting moment of the concrete is given by

$$M_c = Cjd = f_c jkb_w d^2/2$$

and for a given applied service moment M_w the maximum concrete stress is given by

$$f_c = 2M_w/jkb_w d^2$$

The resisting moment of the reinforcement is given by

$$M_s = Tjd = f_s j\rho b_w d^2$$

and for a given applied service moment M_w the reinforcement stress is given by

$$f_s = M_w/j\rho b_w d^2$$

For a permissible concrete stress of p_{cb} and a permissible reinforcement stress of p_{st} the service moment capacity of the section is the lesser of

$$M_c = p_{cb} jkb_w d^2/2$$

and

$$M_s = p_{st} j\rho b_w d^2$$

In designing a reinforced concrete beam by the working stress method, it is necessary to ensure that the permissible stresses are not exceeded in either the concrete or the reinforcement. By adjusting the amount of reinforcement in the section, a balanced stress condition may be attained in which the maximum stress in the concrete and the stress in the tension reinforcement simultaneously reach the permissible values specified in ACI Section A.3. This condition is termed a balanced design. A balanced design may not necessarily be the most economical design as this depends on the relative cost of the materials rather than permissible stresses.

From the strain diagram in Figure 3-12, and assuming a balanced design in which

$$f_c = p_{cb}$$

and

$$f_s = p_{st}$$

we obtain the expression

$$p_{st}/p_{cb} = n(1 - k)/k$$

or

$$k = np_{cb}/(p_{st} + np_{cb})$$

and the lever-arm factor is

$$j = 1 - k/3$$

The balanced moment is given by

$$M_{bal} = p_{cb}jkb_wd^2/2$$
$$= p_{st}j\rho_{bal}b_wd^2$$
$$= K_{bal}b_wd^2$$

where K_{bal} = balanced moment factor

$$= p_{st}j\rho_{bal}$$
$$= p_{cb}jk/2$$
$$= 1.5p_{cb}j(1 - j)$$

and $\rho_{bal} = K_{bal}/p_{st}j$

or $A_s = M_{bal}/p_{st}jd$

Values of K_{bal}, ρ_{bal}, and j for typical reinforcement and concrete stresses are given in Table 3–6 and facilitate the design of a section.

f'_c	p_{cb}	n	f_y = 40,000 p_{st} = 20,000			f_y = 60,000 p_{st} = 24,000		
			ρ_{bal}	K_{bal}	j	ρ_{bal}	K_{bal}	j
3000	1350	9.19	0.0129	225	0.872	0.0096	204	0.886
3500	1575	8.51	0.0158	274	0.866	0.0118	248	0.880
4000	1800	7.96	0.0188	323	0.861	0.0140	295	0.875
4500	2025	7.50	0.0219	374	0.856	0.0164	342	0.871
5000	2250	7.12	0.0250	426	0.852	0.0188	390	0.867

Table 3–6 Moment factors and reinforcement ratios for working stress design

To design a member with given values of M_w, p_{cb}, p_{st}, and b_w, the moment factor K_{bal} is first obtained from Table 3–6 for the specific values of p_{cb} and p_{st}. The effective depth is then calculated from the expression

$$d = (M_w/b_wK_{bal})^{0.5}$$

The required reinforcement area is given by

$$A_s = M_w/p_{st}jd$$

and the stresses in the concrete and the reinforcement are p_{cb} and p_{st} respectively.

To design a member with given values of M_w, p_{cb}, p_{st}, b_w, and d, the moment factor K_w is first determined from the expression

$$K_w = M_w/b_wd^2$$

The moment factor for balanced design K_{bal} is now obtained from Table 3–6 for the specific values of p_{cb} and p_{st} and compared with K_w.

If $K_w = K_{bal}$
then $f_c = p_{cb}$

and the section is balanced.

If $K_w > K_{bal}$
then $f_c > p_{cb}$

and the section has inadequate capacity and requires an increase in either b_w, d, or f'_c or the addition of compression reinforcement.

If $K_w < K_{bal}$
then $f_c < p_{cb}$

and for this situation the actual value for j is greater than that tabulated in Table 3–6. Hence a conservative estimate for the reinforcement area may be obtained by using the tabulated value of j.

Example 3–18 (Analysis of a reinforced concrete beam under service loads)
Determine the maximum stress in the concrete and the stress in the reinforcement under service loads, for the beam designed in Example 3–10.

Solution
The relevant details are obtained from Example 3–10 and are

f'_c = concrete cylinder strength
= 4000 pounds per square inch
f_y = reinforcement yield strength
= 60,000 pounds per square inch
b_w = width of section
= 12 inches
d = effective depth provided
= 16 inches
A_s = area of tension reinforcement
= 3.81 square inches
ρ = tension reinforcement ratio
= 0.020
M_w = total service moment
$= M_D + M_L$
= 70.31 + 75.94
= 146.25 kip feet

Assume the unit weight of the concrete is

w_c = 145 pounds per cubic foot

The modular ratio is obtained from Table 3–6 and is

$$n = 7.96$$

and

$$\rho n = 0.020 \times 7.96$$
$$= 0.159$$

The corresponding value of the neutral axis factor is obtained from Table 3–5 and is

$$k = 0.427$$

and

$$j = \text{lever-arm factor}$$
$$= 1 - k/3$$
$$= 0.858$$

The maximum concrete stress caused by the service moment is given by

$$f_c = 2M_w/jkb_wd^2$$
$$= 2 \times 146.25 \times 12{,}000/(0.858 \times 0.427 \times 12 \times 16^2)$$
$$= 3119 \text{ pounds per square inch}$$

The reinforcement stress caused by the service moment is given by

$$f_s = M_w/A_sjd$$
$$= 146.25 \times 12{,}000/(3.81 \times .858 \times 16)$$
$$= 33{,}554 \text{ pounds per square inch}$$

Example 3–19 (Design of a reinforced concrete slab for service loads)
A simply supported reinforced concrete slab supports an inclusive load of 200 pounds per square foot over an effective span of 12 feet. The slab is reinforced with No. 4 Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. Determine the required effective depth and the spacing of tension reinforcement required by the working stress design method

Solution

The service moment applied to a 12 inch width of slab is given by

$$M_w = w\ell^2/8$$
$$= 1.5 \times 200 \times 12^2$$
$$= 43{,}200 \text{ pound inches}$$

The moment factor for balanced design is obtained from Table 3–6 and is

$$K_{bal} = 204 \text{ pounds per square inch}$$

and

$$d = \text{required effective depth}$$
$$= (M_w/b_wK_{bal})^{0.5}$$
$$= (43{,}200/12 \times 204)^{0.5}$$
$$= 4.2 \text{ inches}$$

The lever-arm factor for balanced design is obtained from Table 3-6 and is

$$j = 0.886$$

and

$$A_s = \text{required reinforcement area}$$
$$= M_w/p_{st}jd$$
$$= 43{,}200/(24{,}000 \times 0.886 \times 4.2)$$
$$= 0.484 \text{ square inches per foot}$$

From Table 1-8, No. 4 bars at 5 inches on center provide a reinforcement area of

$$A_s = 0.48 \text{ square inches per foot ... satisfactory}$$

3.4 SUPPLEMENTARY PROBLEMS

Problem 3-1 (Analysis of a given reinforced concrete beam using Figure 3-4)
A reinforced concrete beam, with an effective depth of 24 inches and a width of 15 inches, is reinforced with 4 square inches of Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the maximum ultimate applied moment which the beam can support.

Problem 3-2 (Analysis of a given reinforced concrete beam using basic equations)
Using the same data as in Problem 3-1 determine the maximum ultimate applied moment which the beam can support.

Problem 3-3 (Analysis of a given reinforced concrete beam using Table 3-2)
Using the same data as in Problem 3-1 determine the maximum ultimate applied moment which the beam can support.

Problem 3-4 (Analysis of a given reinforced concrete beam using a calculator program)
Using the same data as in Problem 3-1 determine the maximum ultimate applied moment which the beam can support.

Problem 3-5 (Design of a reinforced concrete beam using Figure 3-4)
A reinforced concrete beam, with an effective depth of 24 inches and a width of 15 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required for the beam to support a factored moment of 470 kip feet.

Problem 3-6 (Design of a given reinforced concrete beam using basic equations)
Using the same data as in Problem 3-5 determine the maximum ultimate applied moment which the beam can support.

Problem 3–7 (Design of a given reinforced concrete beam using Table 3–2)
Using the same data as in Problem 3–5 determine the maximum ultimate applied moment which the beam can support.

Problem 3–8 (Design of a given reinforced concrete beam using a calculator program)
Using the same data as in Problem 3–5 determine the maximum ultimate applied moment which the beam can support.

Problem 3–9 (Minimum reinforcement area)
A reinforced concrete beam, with an effective depth of 24 inches and a width of 15 inches, is reinforced with Grade 60 bars and has a concrete cylinder strength of 4000 pounds per square inch. Determine the area of tension reinforcement required to enable the beam to support a factored moment of 80 kip feet.

Problem 3–10 (Design of a rectangular beam with compression reinforcement)
A reinforced concrete beam is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. The beam has an effective depth of 24 inches, a width of 12 inches, and a depth to the centroid of the compression reinforcement of 3 inches. Determine the areas of tension and compression steel required to support a factored moment of 500 kip feet.

Problem 3–11 (Analysis of a rectangular beam with compression reinforcement)
A reinforced concrete beam is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. The beam has an effective depth of 24 inches, a width of 12 inches, and a depth to the centroid of the compression reinforcement of 3 inches. The area of compression reinforcement provided is 0.9 square inches and the area of tension reinforcement provided is 5 square inches. Determine the design moment strength of the section.

Problem 3–12 (Design of a flanged beam with tension reinforcement)
The reinforced concrete flanged beam, shown in Figure 3–8, is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. Determine the area of tension reinforcement required to support an applied factored moment of 500 kip feet.

Problem 3–13 (Analysis of a flanged beam with tension reinforcement)
The reinforced concrete flanged beam, shown in Figure 3–8, is reinforced with Grade 60 bars and has a concrete cylinder strength of 3000 pounds per square inch. The area of tension reinforcement provided is five square inches. Determine the design moment strength of the section.

Problem 3–14 (Crack control reinforcement)
A reinforced concrete beam with a width of 12 inches is reinforced with three, No. 8, Grade 60 bars. Shear reinforcement consists of No. 4 bars with a clear concrete cover of 1½ inches. Determine the value of the crack parameter z.

Problem 3–15 (Short-term deflection of a reinforced concrete beam)
A beam reinforced in tension only with three, No. 9, Grade 60 bars, supports a service bending moment due to live load and dead load of 100 kip feet. The width of the beam is 12 inches, the effective depth is 18 inches, and the overall depth is 21 inches. The concrete strength is 4000 pounds per square inch and the effective span is 17 feet. Determine the short-term deflection.

Problem 3–16 (Long-term deflection of a reinforced concrete beam)
The beam, analyzed in Problem 3–15, has 80 percent of the total applied service load sustained. Determine the total deflection caused by short-term and long-term effects.

Problem 3–17 (Determination of the reinforcement stress in a beam under service loads)
Using the same data as in Problem 3–15, determine the stress in the tension reinforcement under the applied service load.

Problem 3–18 (Design of a reinforced concrete beam for service loads)
A beam reinforced in tension only with Grade 60 bars, supports a service bending moment due to live load and dead load of 120 kip feet. The width of the beam is 12 inches, the effective depth is 18 inches, and the concrete strength is 4000 pounds per square inch. Determine the area of tension reinforcement required.

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4

Shear and torsion of concrete beams

4.1 DESIGN FOR SHEAR

4.1.1 General shear concepts

This chapter covers one-way, or flexural, shear effects in reinforced concrete beams. Two-way, or punching, shear effects are covered in the appropriate chapters on slabs and footings.

In an elastic, homogeneous material, the shear stress in a member at any level in a cross section is given by

$$v = VQ/Ib_w$$

where
- V = vertical shear force acting at the cross section
- Q = first moment of area about the neutral axis of that portion of the cross sectional area above the level considered
- I = moment of inertia of the cross section about the neutral axis
- b_w = width of the cross section at the level considered

For a rectangular section, as shown in Figure 4-1, the maximum shear stress is developed at the neutral axis of the section and has a value of

$$v_{max} = 1.5V/b_w d$$

A small element of the beam, taken at the neutral axis where flexural stresses are zero, has only shear stresses acting as shown in the Figure. These shear stresses produce a principal tensile stress, acting on a diagonal plane, at an angle of 45 degrees. Since a reinforced concrete beam is not a truly homogeneous member, the above expression is not strictly applicable and it is convenient to measure shear by means of the nominal stress given by

$v = V/b_w d$

where d = effective depth of the tension reinforcement

This nominal stress acting on a nominal area ($b_w d$) may be used as an indication of the magnitude of the principal tensile stress acting on the section.

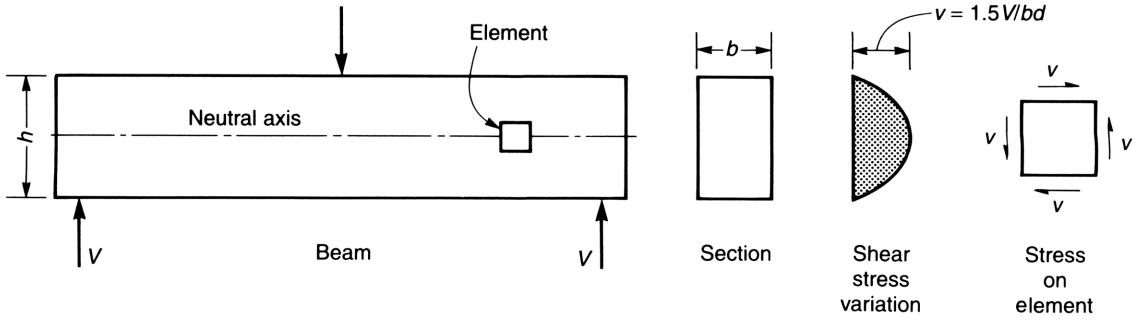


Figure 4–1 Variation of shear stress

The diagonal cracking which occurs in a reinforced concrete beam during loading is caused by the principal tensile stresses. Figure 4–2 shows the tension stress trajectories in a beam and the cracks which are formed perpendicular to the trajectories. Also indicated are the stresses acting above, on, and below the neutral axis. The value of the principal tensile stress is given by

$p_t = f_b/2 + (f_b^2/4 + v^2)^{0.5}$

where f_b = flexural stress, positive when tensile

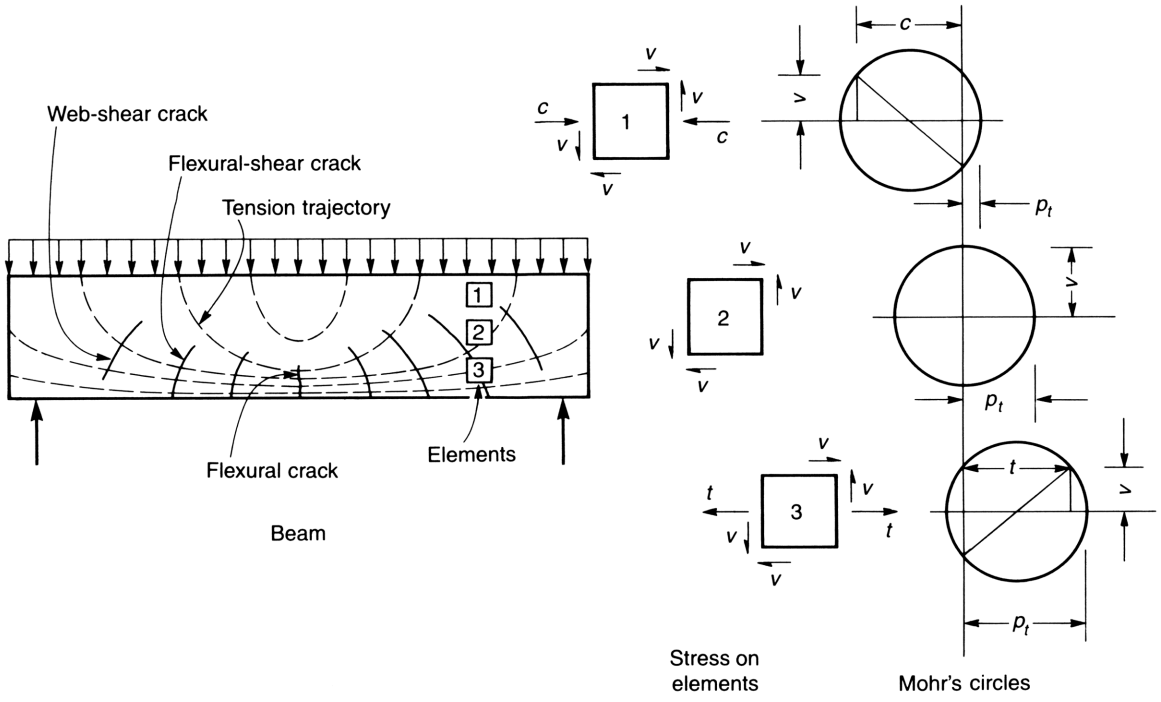


Figure 4–2 Stress trajectories and crack formation

The plane on which the principal tensile stress acts is inclined at an angle θ to the vertical given by

$$\tan(2\theta) = 2v/f_b$$

When the principal tensile stress exceeds the tensile strength of the concrete, diagonal tension cracking is produced in the concrete. The tensile strength of the concrete depends on the water–cement ratio of the mix and on the grading, maximum size, and type of aggregate used, and varies between the limits[1]

$$f_t = 5(f'_c)^{0.5}$$

and

$$f_t = 7.5(f'_c)^{0.5}$$

Diagonal tension cracks occur[2] at a nominal shear stress given by

$$v_{cr} = V_{cr}/b_w d$$
$$= 3.5(f'_c)^{0.5} \text{ ... sections with small moment}$$
$$= 1.9(f'_c)^{0.5} \text{ ... sections with large moment}$$

where V_{cr} = applied shear force at which shear cracks occur

After cracking occurs, the shear resistance of the member is provided[3,4,5], as shown in Figure 4–3, by the three components

- V_c = shear resistance provided by the concrete in the uncracked portion of the beam. This may contribute 20 to 40 percent of the total shear capacity.
- V_a = shear resistance provided by the interlocking of the aggregate particles across the crack. This may contribute 33 to 50 percent of the total shear capacity.
- V_d = shear resistance provided by the dowel action of the tension reinforcement. This may contribute 15 to 25 percent of the total shear capacity.

The relative values of these three components depend on concrete strength, type of aggregate, and steel percentage.

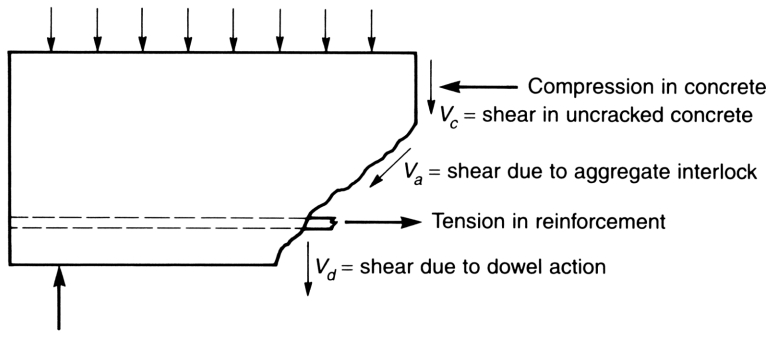


Figure 4–3 Forces acting at a shear crack

The types of shear failure modes[3,4,5,6] which may occur in a reinforced concrete beam are shown in Figure 4–4. The mode of failure is influenced by the shear span to effective depth ratio given by

a/d = shear span to effective depth ratio

where a = shear span
= distance between the support and the nearest concentrated load

For small values of the a/d ratio, an **arch–rib failure** occurs when a diagonal web crack develops in the beam and extends from the support to the adjacent concentrated load. Arch action develops with the flexural reinforcement acting as the tie of a tied–arch and a compression strut forming between the support and the concentrated load. The stress in the reinforcement is approximately constant along the whole length of the beam with high bond stresses developing at the ends. As loading is increased, eventual failure occurs by crushing of the concrete along the compression strut.

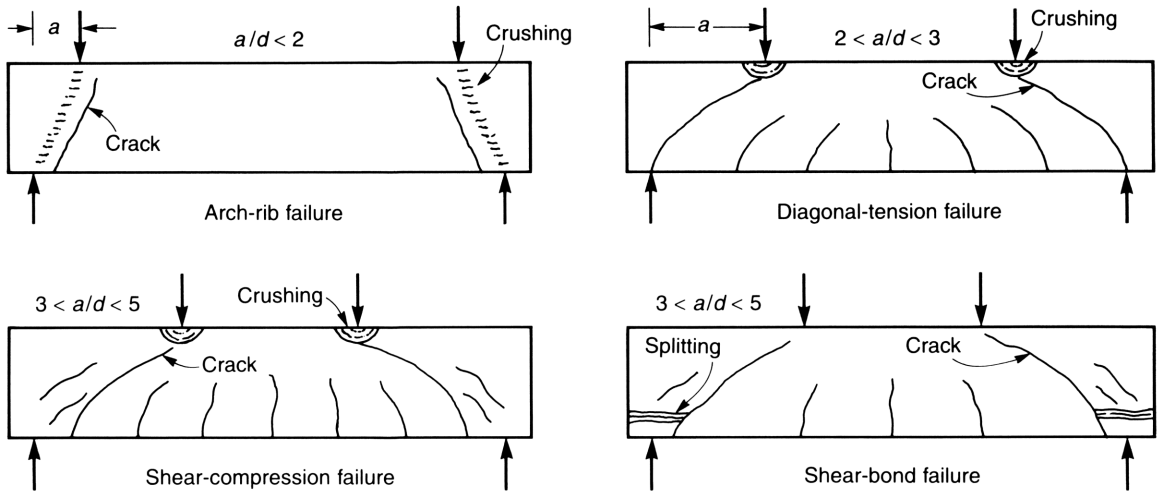


Figure 4–4 Shear failure modes

With larger values of the a/d ratio, a **diagonal–tension failure** may be produced. As the loading on the beam increases, one large web–shear crack develops in the vicinity of the support and turns in the direction of the loading point. The depth of concrete above the crack, which provides the compression zone of the beam, is gradually reduced until compression failure occurs under the load.

Shear–compression failure occurs at an a/d ratio of between three and five. With increasing load, flexural cracks form in the soffit of the beam and propagate into the web at 45 degrees to form flexural–shear cracks. Failure occurs when the concrete in the compression zone above the crack crushes.

Shear–bond failure occurs when the bottom of a flexural–shear crack propagates along the tension reinforcement causing splitting of the concrete and bond failure.

4.1.2 Design of beams without shear reinforcement

The ACI Code uses empirical expressions[2] to determine the shear capacity of reinforced concrete beams. The relationship between the variables affecting the shear capacity of beams without shear reinforcement is given by ACI Equations (11–3) and (11–5). For members subject to shear and flexure only, either ACI Equation (11–3) or (11–5) may be used. These give the nominal shear capacity of the concrete as

$$V_c = 2b_w d(f'_c)^{0.5}$$

or

$$V_c = [1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d \leq 3.5b_w d(f'_c)^{0.5}$$

where $\rho_w = A_s/b_w d$

V_u = factored shear force at the section

M_u = factored moment occurring simultaneously with V_u at the section

and $V_u d/M_u \leq 1.0$ ... to limit V_c at points of contraflexure

$(f'_c)^{0.5} \leq 100$ pounds per square inch

For most designs, it is convenient to assume that in ACI Equation (11–5) the second term

$$2500\rho_w V_u d/M_u = 0.1(f'_c)^{0.5}$$

which gives ACI Equation (11–3)

$$V_c = 2b_w d(f'_c)^{0.5}$$

Research[7] has indicated that ACI Equations (11–3) and (11–5) overestimate the shear capacity for higher strength concretes. Because of this, ACI Section 11.1.2 limits the contribution of concrete strength to the shear capacity by establishing the upper limit of

$$(f'_c)^{0.5} \leq 100 \text{ pounds per square inch}$$

Hence concrete strength in excess of 10,000 pounds per square inch does not contribute to the shear capacity of the member.

The objective of shear design is to prevent shear failure prior to a ductile flexural failure. A reinforced concrete beam, without shear reinforcement, subject to accidental overload may fail without warning on the formation of a shear crack. To prevent this from occurring, ACI Section 11.5.5 requires the provision of a minimum area of shear reinforcement when the factored applied shear force exceeds one half of the design shear strength. Shear reinforcement shall be provided when

$$V_u > \phi V_c/2$$

where V_u = factored applied shear force

ϕ = strength reduction factor

V_c = nominal shear capacity of the section without shear reinforcement

This provision does not apply to reinforced concrete slabs and footings as, for these members, redistribution of the loading can occur.

Example 4–1 (Beam without shear reinforcement)

A reinforced concrete beam, with an effective depth of 20 inches and a width of 12 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.0124. The factored shear force at the critical section is 14 kips and the factored moment occurring simultaneously is 35 kip feet. Determine if shear reinforcement is required.

Solution

The shear strength provided by the concrete is given by ACI Equation (11–3) as

$$\begin{aligned}\phi V_c &= 2\phi b_w d(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 12 \times 20(4000)^{0.5}/1000 \\ &= 25.8 \text{ kips}\end{aligned}$$

The factored applied shear at the critical section is

$$\begin{aligned}V_u &= 14 \text{ kips} \\ &> \phi V_c/2\end{aligned}$$

Hence shear capacity is unsatisfactory according to the simplified Equation and it is necessary to check with the more detailed ACI Equation (11–5). Then, the shear capacity is given by

$$\phi V_c = \phi[1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d$$

where ρ_w = reinforcement ratio
$= 0.0124$

and M_u = factored moment occurring with V_u
$= 35$ kip feet

Hence $V_u d/M_u = 14 \times 20/(12 \times 35)$
$= 0.67$
< 1 ... satisfactory

and $\phi V_c = 0.85(1.9(4000)^{0.5} + 2500 \times 0.0124 \times 0.67)12 \times 20/1000$
$= 28.73$ kips
$> 2V_u$ ... satisfactory
$< 3.5 \times 25.8/2$ ... satisfactory

Hence shear reinforcement is not required.

As shown in Figure 4–2, the presence of compressive stress, acting simultaneously with the shear on a section, reduces the principal tensile stress, and the presence of tensile stress increases the principal tensile stress. For members subject to **axial compression,** the nominal shear capacity of a member has been determined from experimental data[8] and is given by ACI Equation (11–4) as

$$V_c = 2(1 + N_u/2000A_g)b_w d(f'_c)^{0.5}$$

where N_u = factored axial compressive load occurring simultaneously with V_u
A_g = gross area of the section

Alternatively, a more accurate value of the nominal shear capacity is provided by ACI Equations (11–5) and (11–6) as

$$V_c = [1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_m]b_w d$$
$$\leq 3.5 b_w d(f'_c)^{0.5}(1 + N_u/500A_g)^{0.5}$$

where $M_m = M_u - N_u(4h - d)/8$

h = overall depth of member

A negative value of M_m is invalid and, for this condition, the nominal shear capacity is given by

$$V_c = 3.5 b_w d(f'_c)^{0.5}(1 + N_u/500A_g)^{0.5}$$

For members subject to **axial tension** the nominal shear capacity of a member is given by ACI Equation (11–8) as

$$V_c = 2 b_w d(f'_c)^{0.5}(1 + N_u/500A_g)$$
$$\geq 0$$

where N_u = factored axial tensile load occurring simultaneously with V_u

= negative for tension

Example 4–2 (Beam with compressive force)

A reinforced concrete beam, with an effective depth of 20 inches, an overall depth of 22 inches, and a width of 12 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.0124. The factored shear force at the critical section is 14 kips and the factored moment occurring simultaneously is 35 kip feet. Determine the shear capacity of the beam if an axial compressive force of 10 kips is applied to the section.

Solution

The shear capacity of the beam is given by ACI Equation (11–4) as

$$\phi V_c = 2\phi(1 + N_u/2000A_g)b_w d(f'_c)^{0.5}$$
$$= 2 \times 0.85 \times 12 \times 20[1 + 10{,}000/(2000 \times 12 \times 22)](4000)^{0.5}/1000$$
$$= 26.3 \text{ kips}$$

Alternatively the shear capacity may be determined by ACI Equations (11–5) and (11–6) as

$$\phi V_c = \phi[1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_m]b_w d$$

where $M_m = M_u - N_u(4h - d)/8$

$$= (35 \times 12) - 10(4 \times 22 - 20)/8$$
$$= 335 \text{ kip inches}$$

Hence $V_u d/M_m = 14 \times 20/335$

$$= 0.84$$

The limiting value of ϕV_c is given by ACI Equation (11–6) as

$$\phi V_{c(max)} = 3.5\phi b_w d(f'_c)^{0.5}(1 + N_u/500A_g)^{0.5}$$
$$= 3.5 \times 0.85 \times 12 \times 20(4000)^{0.5}[1 + 10{,}000/(500 \times 12 \times 22)]^{0.5}/1000$$
$$= 46.8 \text{ kips}$$

and $\phi V_c = 0.85(1.9(4000)^{0.5} + 2500 \times 0.0124 \times 0.84)12 \times 20/1000$

$$= 29.83 \text{ kips}$$
$$< \phi V_{c(max)} \ldots \text{ satisfactory}$$

Example 4–3 (Beam with tensile force)

A reinforced concrete beam, with an effective depth of 20 inches, an overall depth of 22 inches, and a width of 12 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.0124. The factored shear force at the critical section is 14 kips and the factored moment occurring simultaneously is 35 kip feet. Determine the shear capacity of the beam if an axial tensile force of 10 kips is applied to the section.

Solution

The shear capacity of the beam is given by ACI Equation (11–8) as

$$\begin{aligned}\phi V_c &= 2\phi(1 + N_u/500A_g)b_w d(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 12 \times 20[1 - 10{,}000/(500 \times 12 \times 22)](4000)^{0.5}/1000 \\ &= 23.8 \text{ kips}\end{aligned}$$

Lightweight concrete has a lower shear capacity than normal weight concrete, for a given compressive strength, and ACI Section 11.2.1.2 requires the application of a reduction factor to $(f'_c)^{0.5}$ to compensate. The specified reduction factors are

- replace $(f'_c)^{0.5}$ with $0.75(f'_c)^{0.5}$ ... for all-lightweight concrete
- replace $(f'_c)^{0.5}$ with $0.85(f'_c)^{0.5}$ ... for sand-lightweight concrete

Alternatively, when the splitting tensile strength f_{ct} has been determined

- replace $(f'_c)^{0.5}$ with $f_{ct}/6.7$ ... for sand-lightweight concrete

4.1.3 Rectangular beam with shear reinforcement

When the design shear strength of the concrete is exceeded, it is necessary to provide shear reinforcement. The nominal shear strength of the shear reinforcement is derived from the **truss analogy** illustrated in Figure 4–5.

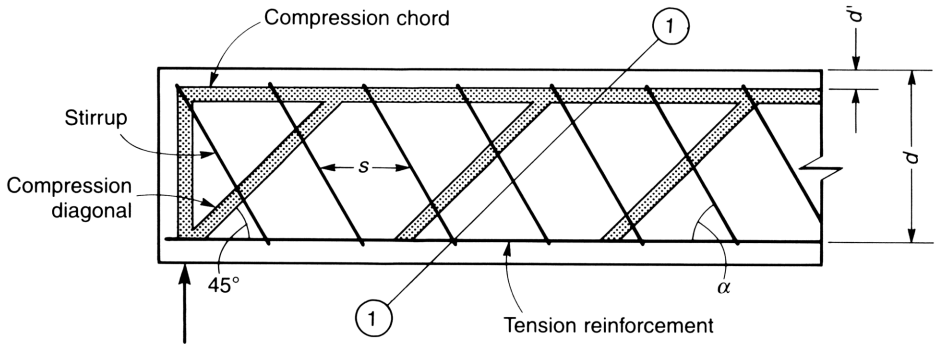


Figure 4–5 Truss analogy

In applying the analogy, it is assumed that diagonal cracking occurs along planes inclined at 45 degrees to the longitudinal axis and that compression diagonals are formed in the concrete between the cracks. Tension reinforcement provides the tension chord of the truss and the compression chord is provided by the top compression reinforcement and the compression zone in the top of the beam. Stirrups provide the tension diagonals and are inclined at an angle of α degrees. The number of stirrups crossing plane 1–1 is given by

$$n = (d - d')(1 + \cot\alpha)/s$$

where α = angle of inclination of the stirrups
s = stirrup spacing
d = effective depth to the tension reinforcement
d' = depth to the compression reinforcement

The vertical component of the forces in the stirrups is

$$V_s = nA_v f_y \sin\alpha$$
$$= A_v f_y(\sin\alpha + \cos\alpha)(d - d')/s$$

where A_v = area of one stirrup

Since $d - d' \approx d$

ACI Equation (11– 16) is derived as

$$V_s = A_v f_y d(\sin\alpha + \cos\alpha)/s$$

When the shear reinforcement is vertical

$$\alpha = 90 \text{ degrees}$$

and ACI Equation (11–15) is given as

$$V_s = A_v f_y d/s$$

To prevent a compression failure of compression diagonals, prior to yielding of the shear reinforcement, the nominal shear strength of the shear reinforcement is limited by ACI Section 11.5.6.8 to

$$V_s = 8b_w d(f'_c)^{0.5}$$

Control of the width of diagonal tension cracks, to ensure the continued effectiveness of aggregate interlock, is provided by ACI Section 11.5.2 which limits the design yield strength of the shear reinforcement to

$$f_y = 60{,}000 \text{ pounds per square inch}$$

A **maximum spacing** of stirrups is specified to prevent a shear failure plane forming between two adjacent stirrups and resulting in no shear reinforcement contributing to the shear capacity of the section. To ensure that a diagonal tension crack, in the lower half of the beam, will be crossed by at least one stirrup ACI Section 11.5.4.2 requires the spacing of vertical stirrups to be limited to

$$s_{(max)} \leq d/2$$

In addition, to reduce diagonal tension crack widths when the required shear strength of the shear reinforcement exceeds $4b_w d(f'_c)^{0.5}$, ACI Section 11.5.4.3 requires a minimum of three stirrups to cross a crack in the lower half of the beam. The spacing of vertical stirrups is then limited to

$$s_{(max)} \leq d/4$$

In order to correspond with current test data[2] ACI Section 11.5.1.2 limits the angle of inclination of stirrups to

$$\alpha \geq 45 \text{ degrees}$$

The truss analogy assumes that the total shear, after the formation of a diagonal tension crack, is carried by the shear reinforcement. However, test data[2] indicates that, provided the angle of inclination of the compression diagonals is assumed to be at 45 degrees, the **total shear** applied to a member is carried by both the concrete and shear reinforcement. Thus, the factored applied shear force acting on a member, in accordance with ACI Equations (11-1) and (11-2), is given by

$$V_u = \phi V_c + \phi V_s$$

where ϕ = strength reduction factor for shear given by ACI Section 9.3.2.3
$= 0.85$

V_c = nominal shear strength of the concrete
$= 2b_w d(f'_c)^{0.5}$ ... ACI Equation (11-3)

or $= [1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d$ ... ACI Equation (11-5)

V_s = nominal shear strength of the shear reinforcement
$= A_v f_y d/s$ ... ACI Equation (11-15)
$= A_v f_y d(\sin\alpha + \cos\alpha)/s$ ... ACI Equation (11-16)

A_v = area of shear reinforcement

s = spacing of shear reinforcement specified in ACI Section 11.5.4 as
$\leq d/2$ or 24 inches for vertical stirrups when $V_s \leq 4b_w d(f'_c)^{0.5}$
$\leq d/4$ or 12 inches when $4b_w d(f'_c)^{0.5} < V_s \leq 8b_w d(f'_c)^{0.5}$

Recent research[9] has shown that the truss analogy is also applicable when the angle of inclination of the compression diagonals is assumed to be less than 45 degrees provided that the shear strength contributed by the concrete is appropriately reduced.

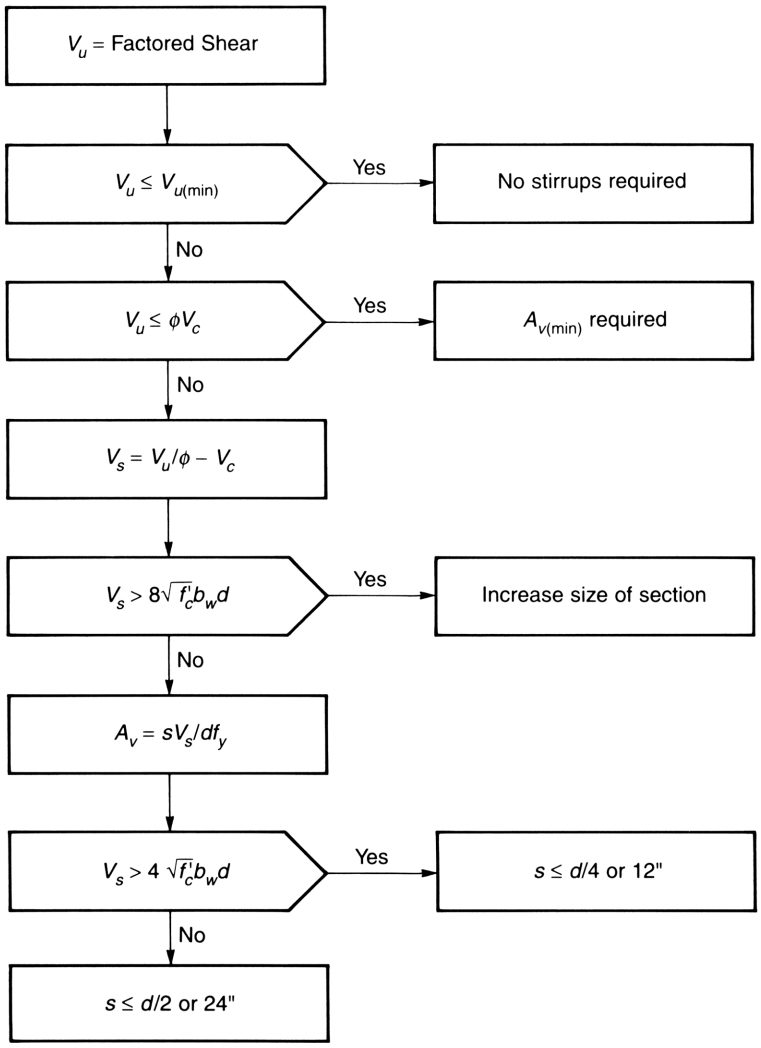
To ensure that sudden failure does not occur on the formation of a diagonal tension crack, caused by an unexpected overload, shear reinforcement is required to raise the shear capacity above the shear cracking load. A **minimum area** of shear reinforcement is specified in ACI Section 11.5.5 when the factored shear force exceeds

$$V_u = \phi V_c/2 = V_{u(min)}$$

and the minimum area of shear reinforcement is given by ACI Equation (11-13) as

$$A_{v(min)} = 50b_w s/f_y$$

Flow Chart 4-1 summarizes the shear provisions of the ACI Code.



Flow Chart 4–1 Design for shear

With **high strength** concrete, the shear force to produce diagonal tension cracking is increased. This requires a greater load to be carried by the combined effects of aggregate interlock, dowel action, uncracked concrete, and shear reinforcement after cracking occurs. As a result, the minimum area of shear reinforcement specified by ACI Equation (11–13) is inadequate to prevent brittle failure[10,11] when diagonal tension cracking occurs in high strength concrete members. For concrete strengths exceeding 10,000 pounds per square inch, ACI Section 11.1.2.1 requires the minimum area of shear reinforcement to be

$$A_{v(min)} = b_w s f'_c/100 f_y$$
$$\leq 150 b_w s/f_y$$

When this increased shear reinforcement is provided it is unnecessary to impose the restriction of ACI Section 11.1.2 by limiting $(f'_c)^{0.5}$ to maximum value of 100 pounds per square inch.

As shown in Figure 4–6, inclined **bent–up bars** may be used as shear reinforcement. In accordance with ACI Section 11.5.1.2, the minimum angle of inclination permitted is 30 degrees and, in accordance with ACI Section 11.5.6.6, only the center three–fourths of the inclined portion of the bar may be considered effective. The shear strength of a series of equally spaced bent–up bars is given by ACI Equation (11–16) as

$$V_s = A_v f_y d(\sin\alpha + \cos\alpha)/s$$

The maximum allowable spacing of the inclined bars is derived from Figure 4–6 as

$$s_{(max)} = 0.375d(1 + \cot\alpha)$$

The shear strength of a single bent–up bar is given by ACI Equation (11–17) as

$$V_s = A_v f_y \sin\alpha \leq 3b_w d(f'_c)^{0.5}$$

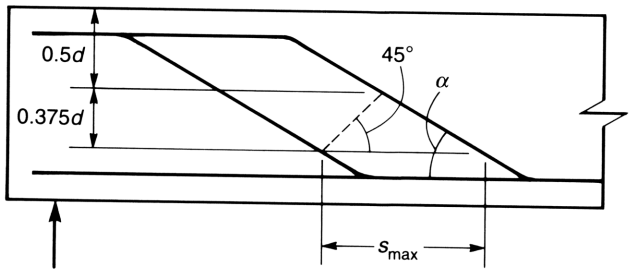


Figure 4–6 Bent–up bars

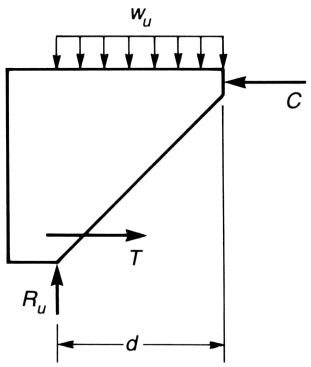


Figure 4–7 Maximum shear

When the support reaction produces a compressive stress in the member, ACI Section 11.1.3 specifies that the **critical factored shear force** is that acting at a distance equal to the effective depth from the face of the support. As shown in Figure 4–7, a diagonal tension crack at the support extends upwards to the compression zone at a distance from the support approximately equal to the effective depth. Loads applied to the top of the beam over this distance are transferred directly to the support by the compression diagonal in the concrete. The maximum factored design shear force is given by

$$V_u = R_u - dw_u$$

and the beam must be designed for this shear force over a distance equal to the effective depth from the face of the support. When the support does not produce compression in the end of the beam, or when a concentrated load is located closer to the support than a distance equal to the effective depth, the section must be designed for the maximum shear at the support.

Figure 4–8 illustrates the design principles involved.

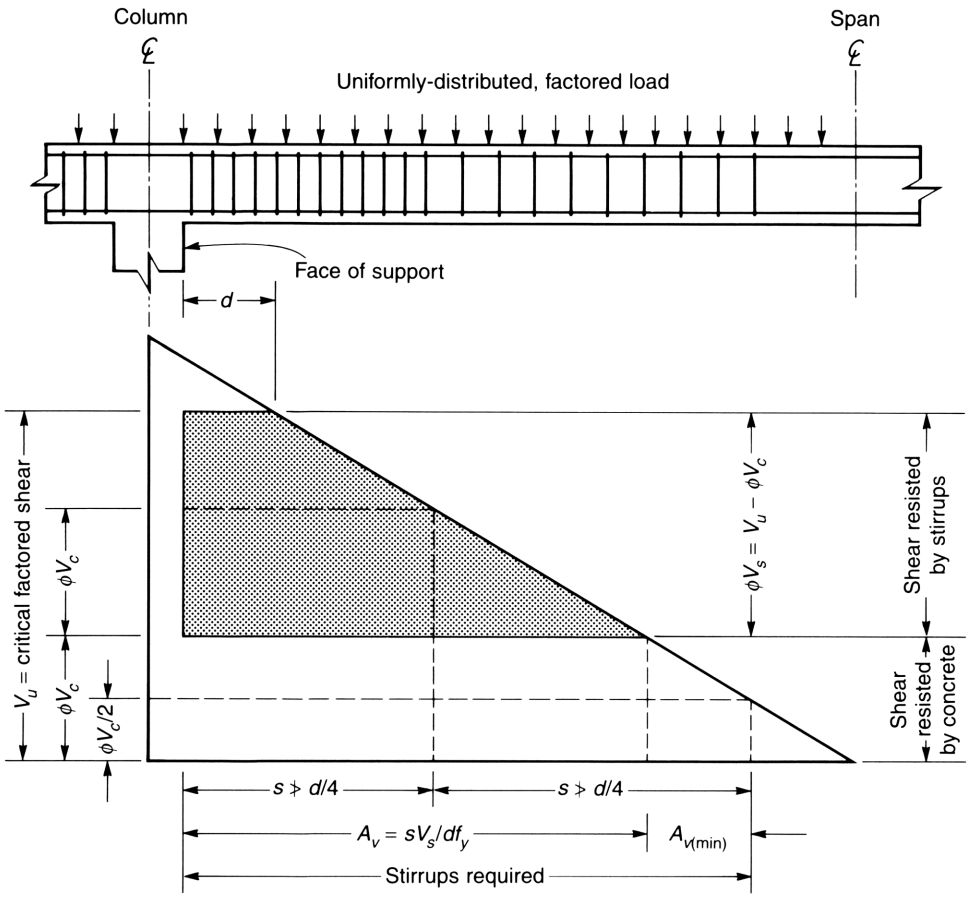


Figure 4–8 Shear in a reinforced beam

Example 4–4 (Vertical stirrup shear reinforcement)

A reinforced concrete beam, simply supported over a clear span of 23 feet, is reinforced with Grade 60 bars and has a compressive strength of 3000 pounds per square inch. The beam has an effective depth of 24 inches and a width of 20 inches. The beam supports a factored load, including its own weight, of 21.2 kips per foot run. Determine the critical design shear and the required spacing, at the critical design shear location, for No.4 stirrups with four vertical legs.

Solution

The support reaction is given by

$$\begin{aligned} R_u &= w_u \ell_n/2 \\ &= 21.2 \times 23/2 \\ &= 244 \text{ kips} \end{aligned}$$

The critical section for shear, in accordance with ACI Section 11.1.3.1, is located a distance from the support equal to the effective depth of the section. The factored shear force at this location is

$$V_u = R_u - w_u d$$
$$= 244 - 21.2 \times 2$$
$$= 202 \text{ kips}$$

The design shear strength provided by the concrete is obtained from ACI Section 9.3 and ACI Equation (11-3) as

$$\phi V_c = 2\phi b_w d(f'_c)^{0.5}$$
$$= 2 \times 0.85 \times 20 \times 24(3000)^{0.5}/1000$$
$$= 44.7 \text{ kips}$$

The design shear strength required from the shear reinforcement is given by ACI Equations (11-1) and (11-2) as

$$\phi V_s = V_u - \phi V_c$$
$$= 202 - 44.7$$
$$= 157.3 \text{ kips}$$

In accordance with ACI Section 11.5.6.8, since

$$\phi V_s < 4 \times \phi V_c$$

the concrete section is adequate. In accordance with ACI Section 11.5.4.3, since

$$\phi V_s > 2 \times \phi V_c$$

The maximum stirrup spacing is given by

$$s = d/4$$
$$= 24/4$$
$$= 6 \text{ inches}$$

The area of shear reinforcement required is given by ACI Equation (11-15) as

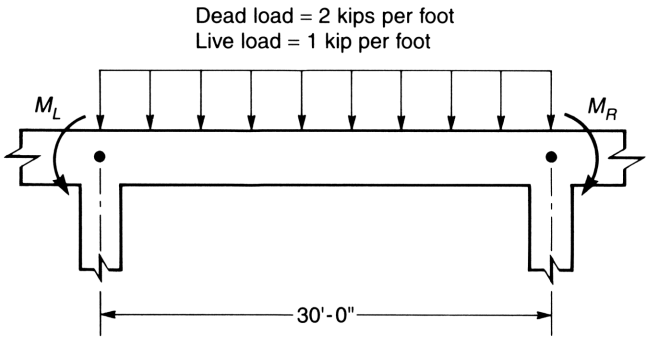
$$A_v/s = \phi V_s/\phi d f_y$$
$$= 157.3 \times 12/(0.85 \times 24 \times 60)$$
$$= 1.54 \text{ inches}^2/\text{foot}$$

Shear reinforcement consisting of four vertical legs of No.4 stirrups at six inch spacing provides a reinforcement area of

$$A_v/s = 4 \times 0.39$$
$$= 1.56 \text{ inches}^2/\text{foot}$$
$$> 1.54 \text{ ...satisfactory}$$

The minimum area of shear reinforcement, which may be provided, is given by ACI Equation (11-13) as

$$A_v/s = 50b_w/f_y$$
$$= 50 \times 12 \times 20/60{,}000$$
$$= 0.20 \text{ inches}^2/\text{foot}$$
$$< 1.54 \text{ ... satisfactory}$$



Unfactored moment

	M_L kip feet	M_R kip feet
Dead load	−150	−150
Live load	−75	−75
Seismic	+/−210	−/+210

Figure 4–9 Example 4–5 loading details

Example 4–5 (Bent-up bar shear reinforcement[12])

The reinforced concrete beam, shown in Figure 4–9, supports the nonfactored loading indicated with the self weight included in the dead loads. The beam is reinforced, as indicated in Figure 4–10, with Grade 40 bars and has a compressive strength of 3000 pounds per square inch. Determine the factored shear force acting at section A–A and the design shear capacity provided.

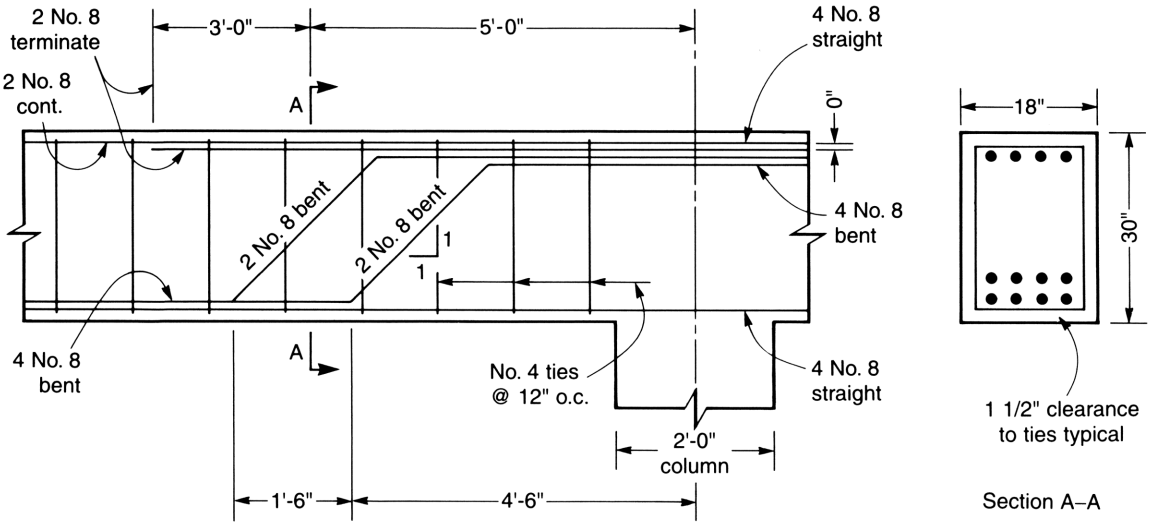


Figure 4–10 Example 4–5 reinforcement details

Solution

The **dead loads** acting on the beam and the resulting shear and moment diagrams are shown in Figure 4–11. The shear acting on section A–A is

$$\begin{aligned} V_D &= R_R - 5w_D \\ &= 30 - 5 \times 2 \\ &= 20 \text{ kips} \end{aligned}$$

The moment acting on section A–A is

$$\begin{aligned} M_D &= 5 \times R_R - w_D \times 5^2/2 + M_R \\ &= 5 \times 30 - 2 \times 12.5 - 150 \\ &= -25 \text{ kip feet} \end{aligned}$$

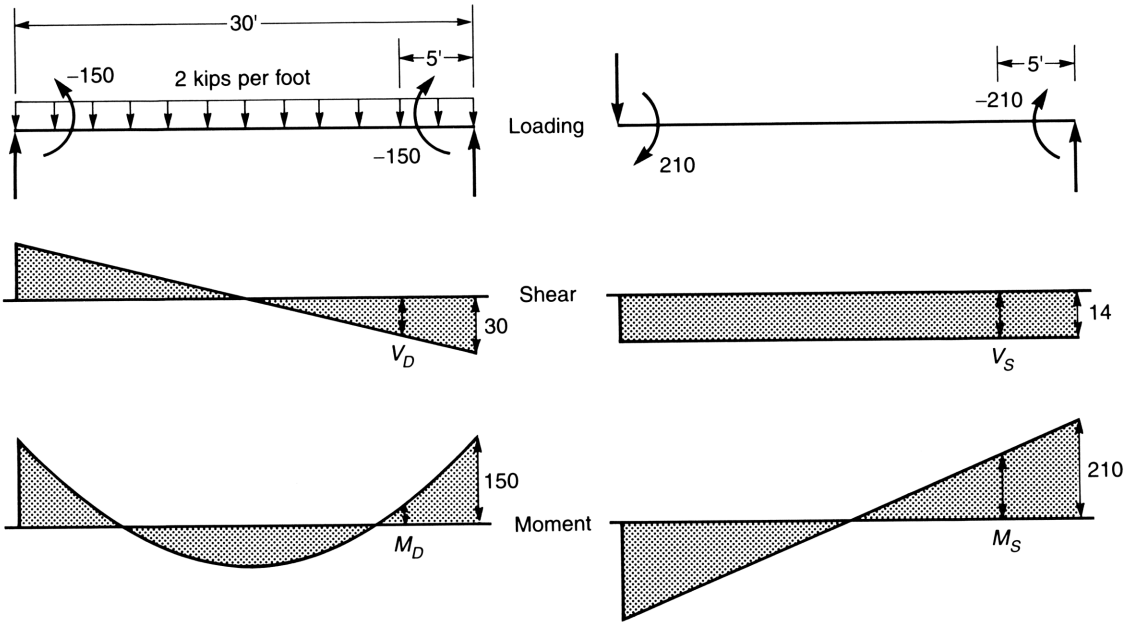


Figure 4–11 Dead load shear and moment

Figure 4–12 Seismic shear and moment

The **live loads** acting on the beam are half the dead load values and the resulting shear and moment acting on section A–A are

$$\begin{aligned} V_L &= 10 \text{ kips} \\ M_D &= -12.5 \text{ kip feet} \end{aligned}$$

The **seismic loads** acting on the beam and the resulting shear and moment diagrams are shown in Figure 4–12. The shear acting on section A–A is

$$\begin{aligned} V_S &= (M_L - M_R)/30 \\ &= (210 + 210)/30 \\ &= 14 \text{ kips} \end{aligned}$$

The moment acting on section A–A is

$$\begin{aligned} M_S &= 5 \times R_R + M_R \\ &= 5 \times 14 - 210 \\ &= -140 \text{ kip feet} \end{aligned}$$

The loading combination dead plus live plus seismic controls, and the **factored shear force and moment** at section A–A are given by ACI Equation (9–2) and ACI Section 9.2.3 as

$$V_U = 1.05V_D + 1.28V_L + 1.40V_S$$
$$= 1.05 \times 20 + 1.28 \times 10 + 1.40 \times 14$$
$$= 53.4 \text{ kips}$$
$$M_U = 1.05M_D + 1.28M_L + 1.40M_S$$
$$= 1.05 \times -25 + 1.28 \times -12.5 + 1.40 \times -140$$
$$= -238.3 \text{ kip feet}$$

At section A–A, the effective depth of the top reinforcement is

$$d = h - d_c$$
$$= 30 - 1.5 - 0.5 - 0.5$$
$$= 27.5 \text{ inches}$$

The **design shear capacity of the concrete section** is given by ACI Equation (11–5) as

$$\phi V_c = \phi[1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d$$

where ρ_w = reinforcement ratio
$= 4 \times 0.79/(18 \times 27.5)$
$= 0.0064$

ϕ = strength reduction factor
$= 0.85$ ... from ACI Section 9.3.2.3

and M_u = factored moment occurring with V_u
$= 238.3$ kip feet

Hence $V_u d/M_u = 53.4 \times 27.5/(12 \times 238.3)$
$= 0.51$
< 1 ... satisfactory

and $\phi V_c = 0.85(1.9(3000)^{0.5} + 2500 \times 0.0064 \times 0.51)18 \times 27.5/1000$
$= 47.2$ kips
$< 3.5b_w d(f'_c)^{0.5}$ ... satisfactory

The **design shear capacity of the vertical ties** is given by ACI Equation (11–15) as

$$\phi V_s = \phi A_v f_y d/s$$

where A_V = area of shear reinforcement
$= 2 \times 0.2$
$= 0.4$

f_y = yield strength of the ties
$= 40$ kips per square inch

s = spacing of ties
$= 12$ inches

and $\phi V_s = 0.85 \times 0.4 \times 40 \times 27.5/12$
$= 31.2$ kips

In accordance with ACI Section 11.5.6.8, since

$$\phi V_s < \phi \times 8b_w d(f'_c)^{0.5}$$

the concrete section is adequate. In accordance with ACI Section 11.5.4.3, since

$$\phi V_s < \phi \times 4 b_w d (f'_c)^{0.5}$$

The maximum stirrup spacing is given by

$$\begin{aligned} s &= d/2 \\ &= 27.5/2 \\ &= 13.8 \text{ inches} \\ &> 12 \text{ inches} \ldots \text{ satisfactory} \end{aligned}$$

The minimum area of shear reinforcement, which may be provided, is given by ACI Equation (11–13) as

$$\begin{aligned} A_v &= 50 b_w s / f_y \\ &= 50 \times 18 \times 12/40{,}000 \\ &= 0.27 \text{ square inches} \\ &< 0.40 \ldots \text{ satisfactory} \end{aligned}$$

In accordance with ACI Sections 11.5.4.2 and 11.5.6.6 the maximum allowable spacing of **bent-up bars** is given by

$$\begin{aligned} s_{(max)} &= 0.375 d(1 + \cot\alpha) \\ &= 0.375 \times 27.5(1 + \cot 45°) \\ &= 20.63 \text{ inches} \\ &> 18 \text{ inches} \end{aligned}$$

Hence the bent-bars may be considered as a series of parallel bars and the design shear capacity of the bars is given by ACI Equation (11–16) as

$$\begin{aligned} \phi V_{sb} &= \phi A_v f_y d(\sin\alpha + \cos\alpha)/s \\ &= 0.85 \times 2 \times 0.79 \times 40 \times 27.5 \times 1.41/18 \\ &= 115.7 \end{aligned}$$

The **total design shear capacity** at section A–A is given by ACI Section 11.5.6.7 as

$$\begin{aligned} \Sigma(\phi V) &= \phi V_c + \phi V_s + \phi V_{sb} \\ &= 47.2 + 31.2 + 115.7 \\ &= 194.1 \text{ kips} \end{aligned}$$

4.1.4 Shear in deep beams

For the purposes of shear design, ACI Section 11.8.1 classifies a deep beam as one in which

$$\ell_n/d < 5$$

where ℓ_n = clear span
d = effective depth

The shear strength of deep beams is greater than that predicted by normal beam theory and requires special consideration. In order to use deep beam design techniques, beams must be loaded along the top edge so that compression struts can develop between the loads and the supports. For deep beams loaded along the sides or bottom, the design techniques for normal

beams shall be employed. Similarly, for continuous deep beams, ACI Section 11.8.3 specifies that the design techniques used for normal beams shall be used, together with minimum horizontal and vertical reinforcement.

The tension stress trajectories in a deep beam tend to be more horizontal than in a normal beam resulting in diagonal shear cracks occurring almost vertically. Because of this, ACI Section 11.8.8 requires the provision of both horizontal and vertical shear reinforcement.

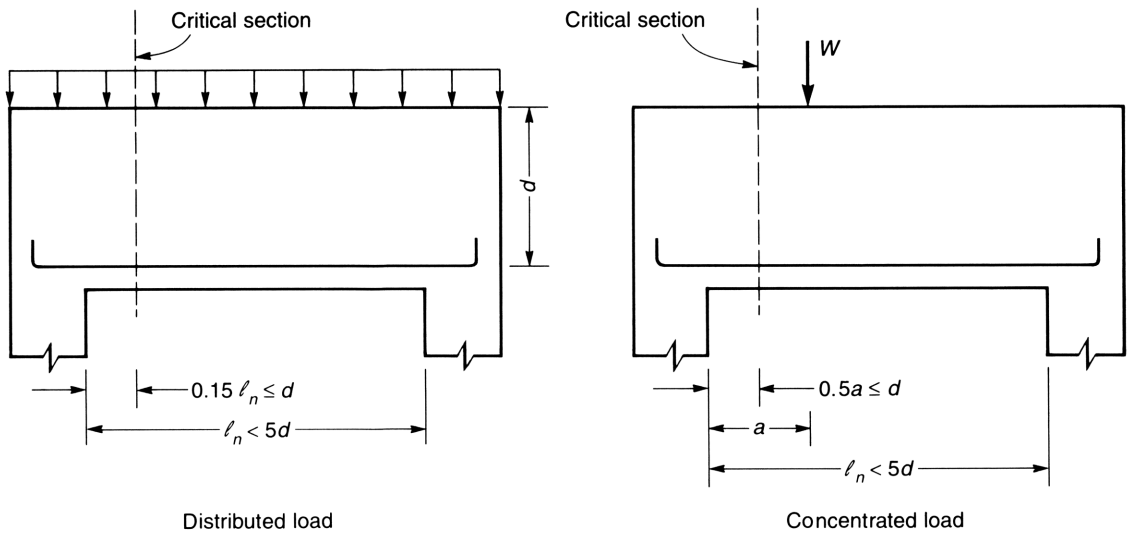


Figure 4–13 Critical sections for deep beams

Figure 4–13 shows the locations of the **critical sections** for shear as specified in ACI Section 11.8.5 and the shear reinforcement required at the critical sections shall be used throughout the span. Hence, the shear force may be calculated only at the critical sections, a determination made as to whether shear reinforcement is required and if so to continue this throughout the member.

Based on experimental data, the nominal **shear capacity of the concrete** is given either by ACI Equation (11–28) or (11–29). These give the nominal shear capacity of the concrete as

$$V_c = 2b_w d(f'_c)^{0.5}$$

or

$$V_c = (3.5 - 2.5M_u/V_u d)[1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d \le 6b_w d(f'_c)^{0.5}$$

where $\rho_w = A_s/b_w d$

V_u = factored shear force at the section

M_u = factored moment occurring simultaneously with V_u at the section

and $2.5 \ge (3.5 - 2.5M_u/V_u d)$

When the design shear strength of the concrete is exceeded, it is necessary to provide shear reinforcement. The nominal **shear strength of the shear reinforcement** is given by ACI

Equation (11–30) as

$$V_s = f_y d[A_v(1 + \ell_n/d)/12s + A_{vh}(11 - \ell_n/d)/12s_2$$

where A_v = area of shear reinforcement perpendicular to tension reinforcement
A_{vh} = area of shear reinforcement parallel to tension reinforcement
s = spacing of vertical reinforcement
s_2 = spacing of horizontal reinforcement

Minimum areas of horizontal and vertical shear reinforcement are specified in ACI Sections 11.8.9 and 11.8.10 and are shown in Figure 4–14. These amounts of shear reinforcement are recommended by ACI Section R11.8.7, even when shear reinforcement is not required, in order to control cracking.

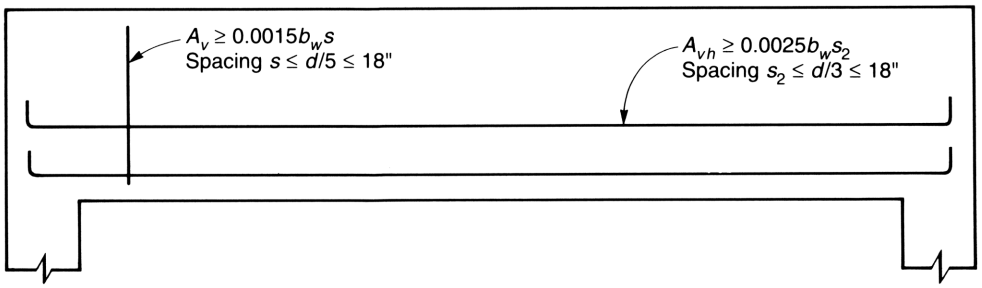


Figure 4–14 Minimum shear reinforcement for deep beams

Substituting the minimum areas of horizontal and vertical shear reinforcement in ACI Equation (11–30) gives

$$V_s = f_y b_w(0.029d - 0.001\ell_n)/12$$

It is assumed, in the ACI Code, that the total shear applied to the member is carried by both the concrete and shear reinforcement. Thus, the factored applied shear force acting on a member, in accordance with ACI Equations (11–1) and (11–2), is given by

$$V_u = \phi V_c + \phi V_s$$
$$= \phi V_n$$

where ϕ = strength reduction factor for shear given by ACI Section 9.3.2.3
$= 0.85$
V_c = nominal shear strength of the concrete
V_s = nominal shear strength of the shear reinforcement
V_n = combined nominal shear strength of concrete and shear reinforcement

The combined nominal shear strength of the concrete and the shear reinforcement is limited by ACI Section 11.8.4. When

$$\ell_n/d < 2$$
$$V_n \leq 8b_w d(f'_c)^{0.5}$$

when $2 \leq \ell_n < 5$

$$V_n = 2b_w d(f'_c)^{0.5}(10 + \ell_n/d)/3$$

Example 4–6 (Shear in a deep beam)

A reinforced concrete beam, with a clear span of 10 feet, an effective depth of 30 inches and a width of 10 inches, has a concrete cylinder strength of 4500 pounds per square inch and a reinforcement ratio of 0.010. The factored shear force at the critical section is 70 kips and the factored moment occurring simultaneously is 175 kip feet. Determine the shear reinforcement required.

Solution

The shear strength provided by the concrete is given by ACI Equation (11–29) as

$$\phi V_c = \phi(3.5 - 2.5M_u/V_u d)[1.9(f'_c)^{0.5} + 2500\rho_w V_u d/M_u]b_w d$$

where $\rho_w = A_s/b_w d$

$= 0.010$

V_u = factored shear force at the section

$= 70$ kips

M_u = factored moment occurring simultaneously with V_u at the section

$= 175$ kip feet

$(3.5 - 2.5M_u/V_u d) = 3.5 - 2.5 \times 175/(70 \times 2.5)$

$= 3.5 - 2.5 \times 1.0$

$= 1.0$

< 2.5 ... satisfactory

and $\phi V_c = 0.85 \times 1.0[1.9(4500)^{0.5} + 2500 \times 0.01 \times 1.0]10 \times 30/1000$

$= 38.88$ kips

$< V_u$ ... shear reinforcement is required

Using the minimum areas of horizontal and vertical shear reinforcement specified in ACI Sections 11.8.9 and 11.8.10 gives the shear strength provided by the shear reinforcement as

$$\phi V_s = \phi f_y b_w(0.029d - 0.001\ell_n)/12$$

$$= 0.85 \times 60 \times 10(0.029 \times 30 - 0.001 \times 120)/12$$

$$= 31.88 \text{ kips}$$

The combined design shear strength of the concrete and the shear reinforcement is given by ACI Section 11.1.1 as

$$\phi V_n = \phi V_c + \phi V_s$$

$$= 38.88 + 31.88$$

$$= 70.76 \text{ kips}$$

$> V_u$ ... satisfactory

The required area of vertical shear reinforcement is given by

$$A_v/s = 0.0015b_w$$

$$= 0.0015 \times 10 \times 12$$

$$= 0.18 \text{ inches}^2/\text{foot}$$

Shear reinforcement consisting of two vertical legs of No.3 stirrups at the maximum permitted spacing of 6 inches provides a reinforcement area of

$$A_v/s = 2 \times 0.22$$
$$= 0.44 \text{ inches}^2/\text{foot}$$
$$> 0.18 \ldots \text{satisfactory}$$

The required area of horizontal shear reinforcement is given by

$$A_v/s_2 = 0.0025b_w$$
$$= 0.0025 \times 10 \times 12$$
$$= 0.30 \text{ inches}^2/\text{foot}$$

Shear reinforcement consisting of two horizontal No.3 bars at a vertical spacing of 9 inches provides a reinforcement area of

$$A_v/s = 2 \times 0.15$$
$$= 0.30 \text{ inches}^2/\text{foot}$$
$$= \text{required value} \ldots \text{satisfactory}$$

4.1.5 Design of corbels

A corbel is a short cantilever bracket supporting a load bearing member. To ensure compliance with the available experimental data[13,14], ACI Section 11.9.1 limits the shear span to a length not exceeding the effective depth and the applied horizontal force to a value not exceeding the vertical force. As shown in Figure 4–15, the forces acting on the corbel interface are:

- A **shear V_u** which is resisted at the interface by shear–friction reinforcement consisting of closed horizontal ties. The design shear strength of this reinforcement is derived by the shear–friction design method specified in ACI Sections 11.7.4.1 and 11.9.3.2.1 as

$$\phi V_n = \phi A_{vf} f_y \mu \text{ kips}$$
$$\leq 0.8\phi b_w d \text{ kips}$$
$$\leq 0.2\phi b_w d f'_c \text{ kips}$$

where ϕ = strength reduction factor
= 0.85 ... from ACI Section 11.9.3.1
A_{vf} = area of shear–friction reinforcement
μ = coefficient of friction at the shear interface
= 1.4λ ... for concrete placed monolithically

and λ = 1.0 ... for normal weight concrete

- A **horizontal force** N_{uc} caused by temperature changes in the supported member. The force transferred to the corbel equals the frictional resistance between the supported member and the bearing plate and ACI Section 11.9.3.4 specifies a minimum horizontal force of 20 percent of the vertical force. The area of reinforcement A_n required to resist the horizontal force is given by

$$N_{uc} = \phi A_n f_y$$

- A **moment M_u** caused by the eccentricity of the vertical load about the interface and the couple caused by the horizontal force and the force in the main reinforcement which is

given by

$$M_u = [V_u a + N_{uc}(h - d)]$$

This moment is resisted by the flexural reinforcement A_f determined by the normal flexural theory of ACI Section 10.3.

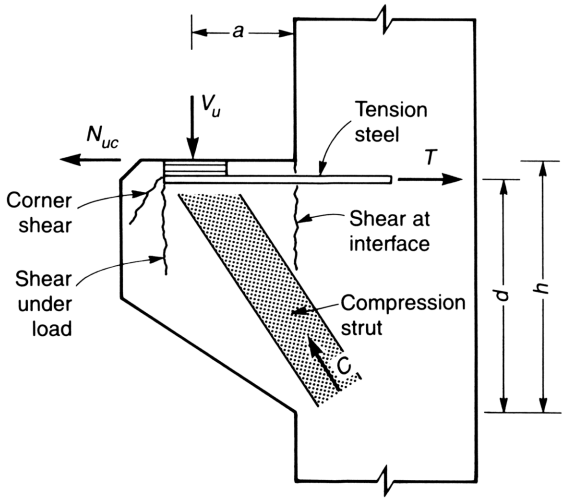


Figure 4–15 Basis of corbel design

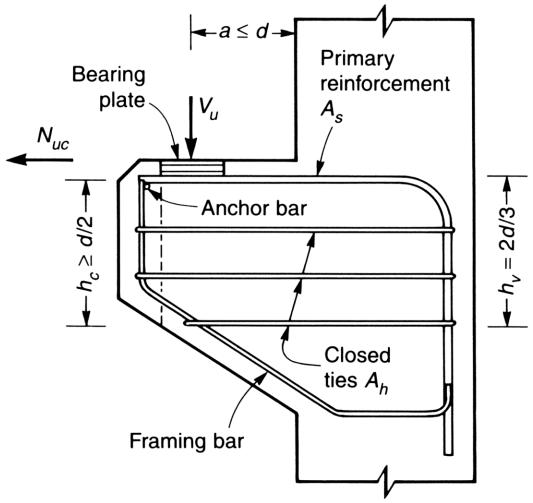


Figure 4–16 Corbel detailing requirements

Corbels may fail in one of the following ways:

- By **yielding or bond failure** of the primary tension reinforcement. Experimental data indicates that the primary reinforcement area must equal the sum of the flexural reinforcement area and the reinforcement area required to resist the horizontal force and is given by ACI Section 11.9.3.5 as

  $$A_s = A_f + A_n$$

  but must not be less than

  $$A_s = 2A_{vf}/3 + A_n$$

  To prevent bond failure, ACI Section 11.9.6 requires the primary tension reinforcement to be welded at the front face of the corbel to a transverse bar of equal size or by bending back the bars to form a horizontal loop. The former method is shown in Figure 4–16. An alternative technique is to weld the reinforcement to an angle located at the front corner of the corbel. The rear end of the tension reinforcement is bent down into the column.

- By the formation of a **corner shear crack** as shown in Figure 4–15. This may be prevented, as specified in ACI Section 11.9.7, by ensuring that the bearing plate does not project beyond the anchor bar or beyond the straight portion of the tension bars.

- By the formation of a **shear crack** under the front edge of the bearing plate. To prevent this, ACI Section 11.9.2 requires a minimum depth to the corbel at the front edge of the

bearing plate of

$$h_c = d/2$$

- By **fracture of the principal tension reinforcement** if the concrete cracks under the action of M_u and N_{uc}. To prevent this ACI Section 11.9.5 requires a minimum ratio of tension reinforcement given by

$$\rho = A_s/bd = 0.04f'_c/f_y$$

- By **shear along the interface** between the corbel and the column. Experimental data indicates that the area of the horizontal closed ties A_h must equal half the flexural reinforcement area. ACI Section 11.9.4 specifies this area as

$$A_h = (A_s - A_n)/2$$

As shown in Figure 4–16, the horizontal ties must be distributed in the upper two-thirds of the effective depth of the corbel.

Example 4–7 (Design of a concrete corbel[15])

A concrete corbel projecting from an 18 inch square concrete column is shown in Figure 4–17. Normal weight concrete is used with a strength of 3000 pounds per square inch and the corbel is reinforced with Grade 60 bars. Determine whether the corbel's dimensions and location of the vertical load V_u are satisfactory. Determine the maximum vertical load and horizontal tensile load N_{uc} that may be applied simultaneously assuming that $N_{uc} = 0.20V_u$. Determine the required size and spacing of horizontal ties.

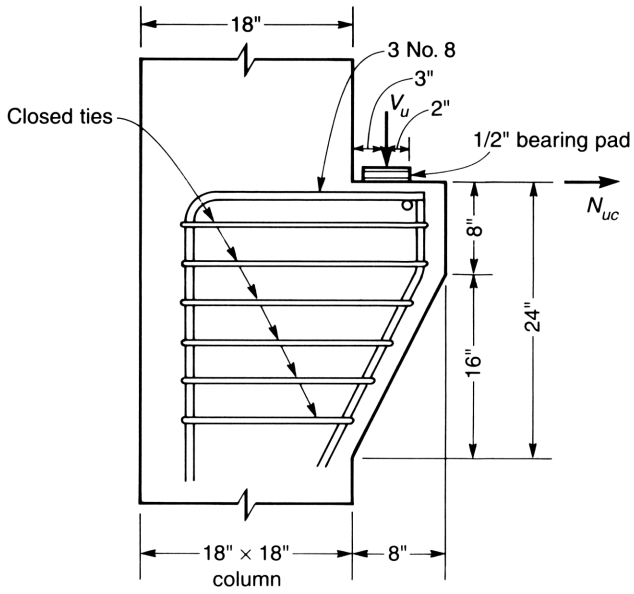


Figure 4–17 Details for Example 4–7

Solution

Assuming the primary tension reinforcement consists of one inch diameter bars with one and a half inch clear concrete cover, the effective depth to the tension reinforcement measured at the face of the column is

$$\begin{aligned} d &= h - d_c \\ &= 24 - 1.5 - 0.5 \\ &= 22 \text{ inches} \end{aligned}$$

The shear span–to–depth ratio is

$$\begin{aligned} a/d &= 3/22 \\ &< 1 \end{aligned}$$

and the requirement of ACI Section 11.9.1 is satisfied.

The corbel depth at the outside edge of the bearing area is

$$\begin{aligned} h_c &= 8 + 3 \times 16/8 \\ &= 14 \text{ inches} \\ &> d/2 \end{aligned}$$

and the requirement of ACI Section 11.9.2 is satisfied.

The maximum design shear strength is given by ACI Section 11.9.3.2.1 as

$$\begin{aligned} \phi V_n &= 0.8\phi b_w d \\ &= 0.8 \times 0.85 \times 18 \times 22 \\ &= 269 \text{ kips} \end{aligned}$$

or

$$\begin{aligned} \phi V_n &= 0.2\phi b_w d f'_c \\ &= 0.2 \times 0.85 \times 18 \times 22 \times 3 \\ &= 202 \text{ kips ... governs} \end{aligned}$$

For normal weight concrete, placed monolithically, the coefficient of friction at the shear interface is obtained from ACI Section 11.7.4.3 as

$$\mu = 1.4\lambda$$

where $\lambda = 1.0$ ... for normal weight concrete

and $\mu = 1.4$

The required area of shear–friction reinforcement is given by ACI Equation (11–25) as

$$\begin{aligned} A_{vf} &= V_u/\phi f_y \mu \\ &= V_u/(0.85 \times 60 \times 1.4) \\ &= 0.01401 V_u \text{ square inches} \end{aligned}$$

The required area of reinforcement to resist the tensile force, N_{uc}, is obtained from ACI Section 11.9.3.4 as

$$\begin{aligned} A_n &= N_{uc}/\phi f_y \\ &= 0.2 V_u/(0.85 \times 60) \\ &= 0.00392 V_u \text{ square inches} \end{aligned}$$

In accordance with ACI Section 11.9.3.5, the required area of primary tension reinforcement shall not be less than

$$A_s = 2A_{vf}/3 + A_n$$
$$= (2 \times 0.01401/3 + 0.00392)V_u$$
$$= 0.0133\ V_u \text{ square inches}$$

The primary tension reinforcement provided consists of three No.8 bars with an area of

$$A_s = 2.37 \text{ square inches}$$

Hence
$$V_u = 2.37/0.0133$$
$$= 179 \text{ kips} \ldots \text{governs}$$
$$< 202 \text{ kips}$$

Using the value $V_u = 179$ kips for the ultimate shear, the applied ultimate moment is given by

$$M_u = V_u a + N_{uc}(h - d)$$
$$= 179[3 + 0.2(24 - 22)]$$
$$= 608 \text{ kip inches}$$

The required area of flexural reinforcement is derived from ACI Section 10.2 and is

$$A_f = 0.85b_w df'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$
$$= 0.52 \text{ square inches}$$

In accordance with ACI Section 11.9.3.5, the required area for the primary tension reinforcement is given by

$$A_s = A_f + A_n$$
$$= 0.52 + 0.00392V_u$$
$$= 0.52 + 0.00392 \times 179$$
$$= 1.22 \text{ square inches}$$
$$< 2.37$$

Hence
$$V_u = 179 \text{ kips}$$
and
$$N_{uc} = 0.2V_u$$
$$= 35.8 \text{ kips}$$

The required area of closed ties is given by ACI Section 11.9.4 as

$$A_h = 0.5(A_s - A_n)$$
$$= 0.5(2.37 - 0.00392 \times 179)$$
$$= 0.83 \text{ square inches}$$

In accordance with ACI Section 11.9.4 the ties must be distributed over a depth of

$$h_v = 2d/3$$
$$= 2 \times 22/3$$
$$= 14.7 \text{ inches}$$

Using four No.3 ties at a spacing of 3.5 inches, provides a reinforcement area of

$$A_h = 4 \times 2 \times 0.11$$
$$= 0.88 \text{ square inches}$$
$$> 0.83 \ldots \text{satisfactory}$$

4.2 DESIGN FOR TORSION

4.2.1 General torsion concepts

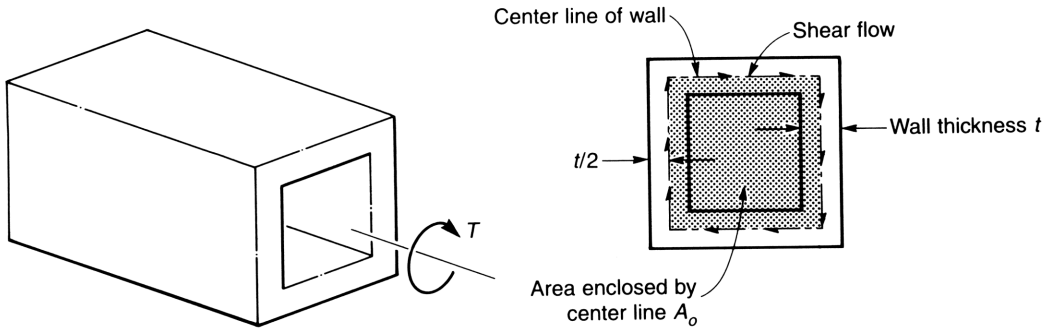


Figure 4–18 Torsional shear flow in a thin walled tube

A hollow torsional member with thin walls is shown in Figure 4–18. The shear stress in the thin walls is uniformly distributed throughout the wall thickness and is tangential to the walls. The shear flow in the walls is given by

$$q = \tau t$$

where τ = shear stress in the walls
t = wall thickness

The applied torsion is resisted by the moment of the shear flow in the walls about the centroid of the section and is given by

$$T = \int qr\,ds$$

where ds = an elemental length of wall
r = distance from the centroid of the section to the midpoint of a wall

and $T = 2A_o q$

where A_0 = area enclosed by the center line of the walls

Hence $\tau = T/2A_o t$

Research data[16] indicates that, after cracking, the concrete in the core of a solid reinforced concrete beam has little effect on the torsional strength of the member and may be neglected. The beam may thus be analyzed as a thin walled tube. In order to maintain a consistent approach, a beam before cracking is also analyzed as a thin walled tube.

It is further assumed, in ACI Section R11.6, that after cracking all torsion is resisted by torsion reinforcement and the concrete is ineffective. In addition, it is assumed that the shear strength of the concrete V_c is unaffected by the torsion. When both shear and torsion reinforcement are required, the sum of the individual areas must be provided in accordance with ACI Section 11.6.3.8. The spacing of the reinforcement is governed by the minimum required spacing of either the shear or torsion reinforcement.

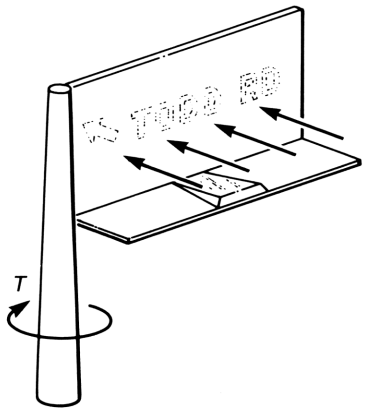


Figure 4–19 Sign structure

In a statically determinate structure subjected to torsional loading, equilibrium is provided by the torsional reactions at the supports, and at any section the torsional forces are readily determined. The torsional stresses may not be reduced by redistribution and this condition is referred to as **equilibrium torsion.** This is illustrated by the sign structure shown in Figure 4–19. The torsional reaction T is necessary to provide equilibrium and the section must be designed with this capacity otherwise collapse may result.

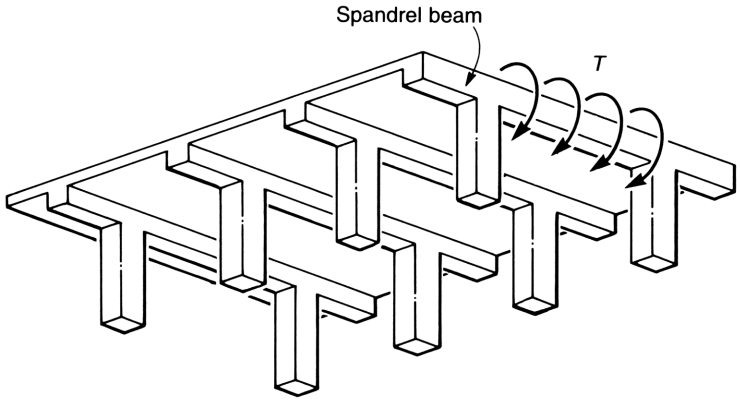


Figure 4–20 One-way slab

In an indeterminate structure, redistribution of internal forces is possible by reducing torsional strength while providing a compensating increase in flexural and shear strength. The spandrel beam shown in Figure 4–20 is subject to torsional stresses from the supported slab. In a member subjected to torsional stresses only, the principal tensile stress equals the torsional stress and cracking occurs when the torsional stress reaches the value

$$\tau = 4(f'_c)^{0.5}$$

At this stage, large rotations can occur in the spandrel beam to maintain compatibility of deformations with adjacent members and allow a redistribution of forces. This condition is referred to as **compatibility torsion.** In accordance with ACI Section 11.6.2.2 the member may be designed for the factored torque causing cracking which is given by

$$T_u = 4\phi(f'_c)^{0.5}(A_{cp}^2/p_{cp})$$

where A_{cp} = area enclosed by the outside perimeter of the concrete cross section
p_{cp} = outside perimeter of the concrete cross section
ϕ = strength reduction factor
= 0.85 ... from ACI Section 9.3.2.3

4.2.2 Design of beams without torsion reinforcement

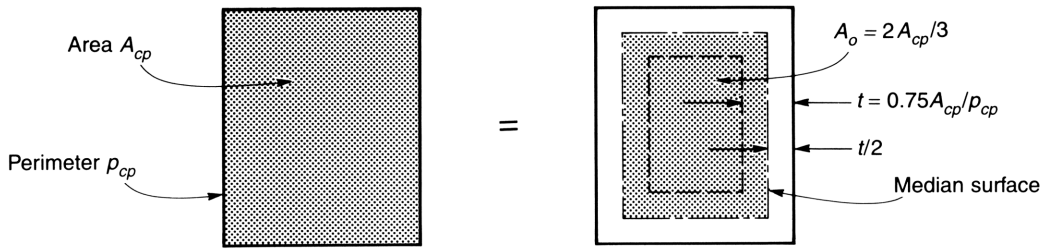


Figure 4–21 Reinforced concrete section before cracking

As specified in ACI Section 11.6.1 and shown in Figure 4–21, prior to cracking of a solid concrete section, the member is idealized as a thin walled tube with

$$t = \text{wall thickness} = 0.75A_{cp}/p_{cp}$$

and A_o = area enclosed by the centerline of the thin walled tube
$= 2A_{cp}/3$

For beams cast monolithically with a slab, the corresponding values of A_{cp} and P_{cp}, including the slab contribution, are defined in ACI Section 11.6.1 and are shown in Figure 4–22.

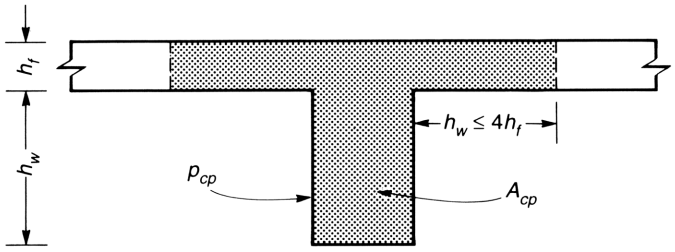


Figure 4–22 Effective flange width

Hence for a member subject to pure torsion, the torsional stress at cracking is given by

$$\tau_{cr} = 4(f'_c)^{0.5}$$
$$= T_{cr}/2A_o t$$
$$= T_{cr}p_{cp}/A_{cp}^2$$

and $$T_{cr} = 4(f'_c)^{0.5}(A_{cp}^2/p_{cp})$$

ACI Section 11.6.1 specifies the nominal torsional capacity of the section as one-quarter of this value and torsional effects may be neglected and closed stirrups are not required when the factored torque is given by

$$T_u \leq \phi(f'_c)^{0.5}(A_{cp}^2/p_{cp})$$

ACI Section 11.6.2.4 specifies that the **critical factored torsion** is that acting at a distance equal to the effective depth from the face of the support. If a concentrated torque occurs within this distance, the critical section shall be taken at the face of the support.

Example 4–8 (Beam without torsion reinforcement)

A reinforced concrete beam, with a depth of 22 inches and a width of 12 inches, has a concrete cylinder strength of 4000 pounds per square inch. The factored torsion at the critical section is 4 kip feet. Determine if torsion reinforcement is required.

Solution

The area enclosed by the outside perimeter of the concrete cross section is

$$A_{cp} = 22 \times 12$$
$$= 264 \text{ square inches}$$

The length of the outside perimeter of the concrete cross section is

$$p_{cp} = 2(22 + 12)$$
$$= 68 \text{ inches}$$

Torsional effects may be neglected and closed stirrups are not required when the factored torque does not exceed

$$T_u = \phi(f'_c)^{0.5}(A_{cp}^2/p_{cp})$$
$$= 0.85(4000)^{0.5}(264^2/68)/12{,}000$$
$$= 4.59 \text{ kip feet}$$
$$> 4 \text{ kip feet}$$

Hence torsional effects may be neglected and closed stirrups are not required.

4.2.3 Rectangular beam with torsion reinforcement

After torsional cracking has occurred, ACI Section R11.6.3.6 specifies that the resistance to torsion is provided mainly by closed stirrups, longitudinal bars, and concrete compression diagonals. The concrete outside the closed stirrups is relatively ineffective and due to the large rotations which occur prior to failure the concrete cover may spall. For this reason, as

illustrated in Figure 4–23, the effective area is defined as

A_{oh} = area enclosed by the centerline of the closed stirrups

and p_h = perimeter of the centerline of the closed stirrups

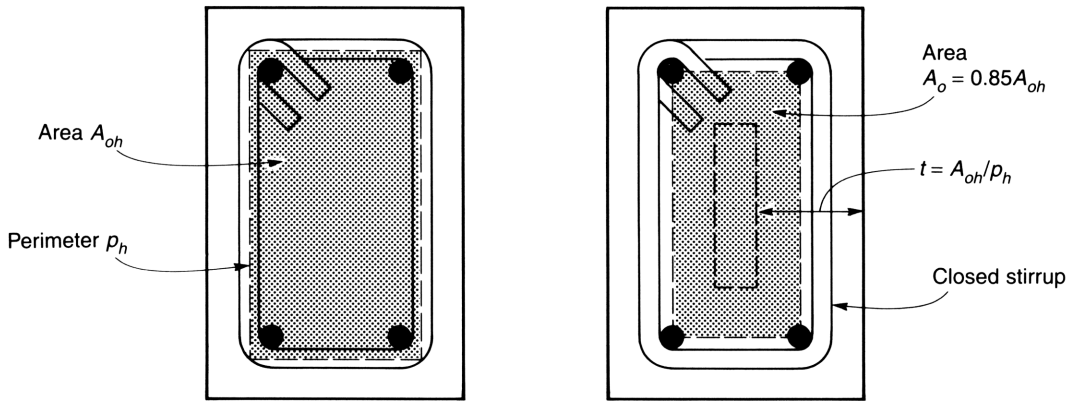


Figure 4–23 Reinforced concrete section after cracking

For beams cast monolithically with a slab, the corresponding values of A_{oh} and p_h, including the slab contribution, are defined in ACI Section R11.6.3.6 and are shown in Figure 4–24.

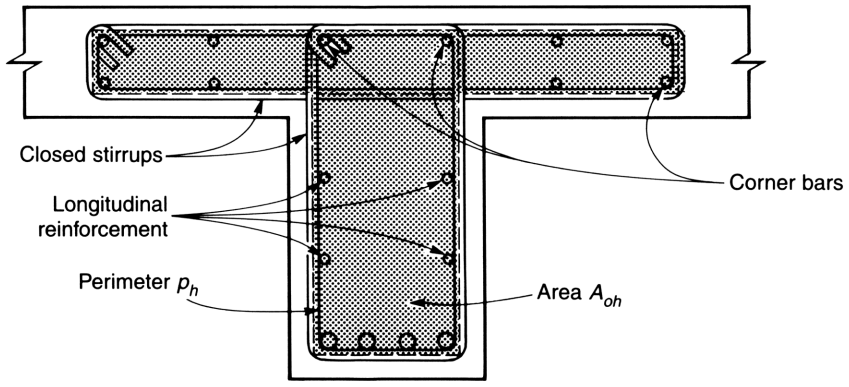


Figure 4–24 Flange contribution after cracking

As specified in ACI Section 11.6.3.6 and shown in Figure 4–23, after cracking of a solid concrete section, the member is idealized as a thin walled tube with the following relationships empirically defined

t = wall thickness
$= A_{oh}/p_h$

and A_o = area enclosed by the centerline of the thin walled tube
$= 0.85A_{oh}$

Torsion cracks are produced diagonally on all four faces of a reinforced concrete beam forming a continuous spiral failure surface around the perimeter as shown in Figure 4–25.

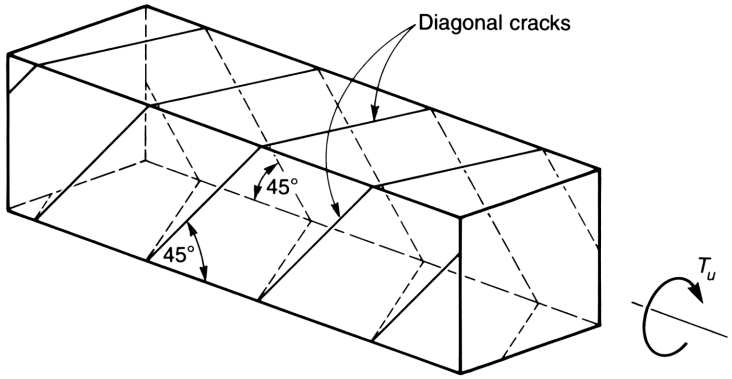


Figure 4–25 Spiral failure surface in a reinforced concrete beam

For a reinforced concrete beam subjected to pure torsion, the principal tensile stresses are produced at an angle of 45 degrees and ACI Section 11.6.3.6 specifies that the angle of inclination of the cracks may be assumed to be 45 degrees. After cracking, the beam is idealized as a tubular space truss consisting of closed stirrups, longitudinal reinforcement, and concrete compression diagonals between the torsion cracks. To control the torsional cracking, it is necessary to provide transverse and longitudinal reinforcement on each face. Closed stirrups are required to resist the vertical component and longitudinal reinforcement is required to resist the horizontal component of the torsional stresses.

As specified in ACI Section 11.6.2.1, reinforcement is required to resist all the applied torsion when the factored torsion exceeds the value

$$T_u = \phi(f'_c)^{0.5}(A^2_{cp}/p_{cp})$$

The area of **closed stirrups** required is specified in ACI Section 11.6.3.6 and is derived by reference to Figure 4–26.

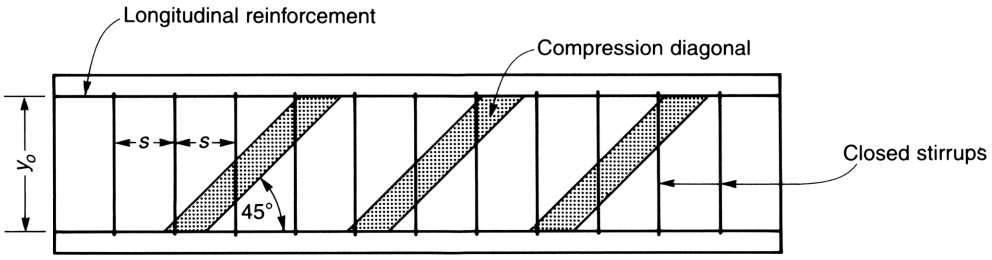


Figure 4–26 Elevation on a vertical face

The number of stirrups crossing a crack inclined at 45 degrees is

$n = y_o/s$

where y_o = distance between centerlines of longitudinal bars on a vertical face

s = spacing of closed stirrups

The force in these stirrups at yield is

$V_y = A_t f_{yv} y_o/s$

where A_t = area of one leg of a closed stirrup

f_{yv} = yield strength of the closed stirrups

Tte shear flow, or shear force per unit length of the perimeter, is

$$q = \tau t = T_n/2A_o$$

where T_n = nominal torsional capacity

The shear force on one vertical face is

$$V_y = qy_o = T_n y_o/2A_o$$

Equating the shear force on one vertical side to the force in the stirrups gives

$$T_n = 2A_o A_t f_{yv}/s$$

The area of **longitudinal reinforcement** required is specified in ACI Section 11.6.3.7 and is derived by reference to Figure 4–26. Since the concrete compression diagonal is inclined at 45 degrees, the shear force on one vertical face is accompanied by a longitudinal force of equal magnitude which is given by

$N_y = V_y$

Similarly, the shear force on one horizontal face is accompanied by a longitudinal force of equal magnitude which is given by

$N_x = V_x$

The total longitudinal force on all four faces is given by

$$\begin{aligned} N &= 2(N_y + N_x) \\ &= 2(V_y + V_x) \\ &= 2(T_n y_o/2A_o + T_n x_o/2A_o) \\ &= T_n(y_o + x_o)/A_o \\ &= 2A_t f_{yv}(y_o + x_o)/s \\ &= A_t p_h f_{yv}/s \end{aligned}$$

where x_o = distance between centerlines of longitudinal bars on a horizontal face

The force in the longitudinal bars at yield is

$N = A_\ell f_{y\ell}$

where A_ℓ = total area of longitudinal reinforcement to resist torsion

$f_{y\ell}$ = yield strength of the longitudinal reinforcement

Equating the total longitudinal force on all four faces to the force in the longitudinal bars gives

$$A_\ell = A_t p_h f_{yv}/f_{y\ell} s$$

Since the resultant longitudinal force acts along the centroid of the analogous space truss, the line of action of the force in the longitudinal bars must also act along this centroid. Hence the bars are distributed around the perimeter of the closed stirrups with a **maximum spacing,** in accordance with ACI Section 11.6.6.2, of 12 inches. To provide anchorage for, and to transfer the force from the compression diagonals to the stirrups, a longitudinal bar is required in each corner of the stirrups. To prevent buckling of the longitudinal bars, the **minimum diameter** specified is

$$d_{b\ell} = s/24$$
$$\geq \text{No.3 bar}$$

To ensure that sudden failure does not occur at torsional cracking, a **minimum area** of torsional reinforcement is specified in ACI Sections 11.6.5.2 and 11.6.5.3 when the factored torsion exceeds

$$T_u = \phi(f'_c)^{0.5}(A^2_{cp}/p_{cp})$$

The minimum combined area of stirrups for shear and torsion is given by ACI Equation (11–23) as

$$(A_v + 2A_t) = 50 b_w s/f_{yv}$$

where A_v = area of two legs of a closed stirrup for shear resistance
A_t = area of one leg of a closed stirrup for torsion resistance

The total **minimum area** of longitudinal reinforcement for torsion is given by ACI Equation (11–24) as

$$A_{\ell(\min)} = 5A_{cp}(f'_c)^{0.5}/f_{y\ell} - A_t p_h f_{yv}/f_{y\ell} s$$

where $A_t/s \geq 25 b_w/f_{yv}$

To control crack widths, ACI Section 11.6.6.1 specifies a **maximum spacing** of the closed stirrups of

$$s = p_h/8$$
$$\leq 12 \text{ inches}$$

To prevent **crushing of the concrete** compression diagonals, the combined shear stress due to the factored torsion and shear forces is limited by ACI Equation (11–18). As shown in Figure 4–27, the stress due to shear acts uniformly over the full width of the section and is given by

$$\tau_v = V_u/b_w d$$

The torsional shear acts in the walls of the analogous tube and is given by

$$\tau_t = T_u/2A_o t$$

where $A_o = 0.85A_{oh}$
$t = A_{oh}/p_h$

and $\tau_t = T_u p_h/(1.7A^2_{oh})$

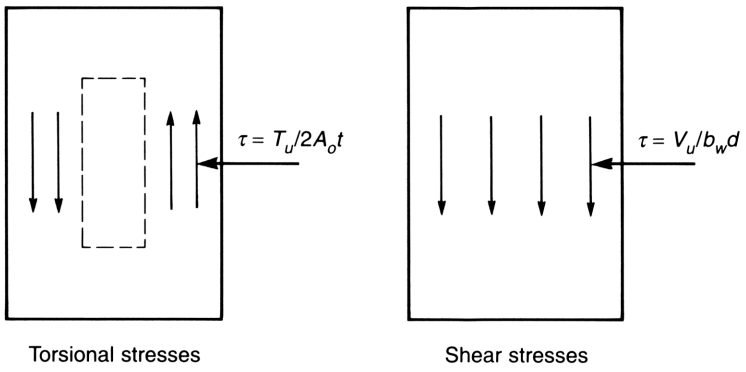


Figure 4–27 Addition of torsion and shear stresses

Since these stresses act on different areas, they are combined using the square root of the sum of the squares rather than by direct addition. The nominal shear strength V_c provided by the concrete is given by ACI Equations (11–3) or (11–5). The maximum allowable nominal shear strength provided by the shear reinforcement is given by ACI Section 11.5.6.8 as

$$V_s = 8b_w d(f'_c)^{0.5}$$

Hence ACI Equation (11–18) is derived as

$$\{(V_u/b_w d)^2 + [T_u p_h/(1.7A_{oh}^2)]^2\}^{0.5} \leq \phi[V_c/b_w d + 8(f'_c)^{0.5}]$$

Since torsional cracking occurs on all faces of a beam, it is necessary to provide closed stirrups to resist torsional forces. In addition, the resultant of the diagonal compression forces in adjacent faces of a beam tend to cause spalling of the corners as shown in Figure 4–28. To prevent this, the spacing and cover to the closed stirrups should be as small as practical. In accordance with ACI Section 11.6.4.2, 135 degree hooks are required to anchor the stirrups as 90 degree hooks fail when the corners spall.

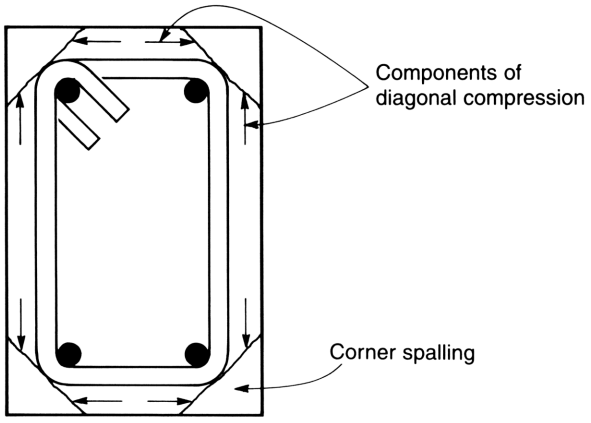


Figure 4–28 Corner spalling

Example 4–9 (Torsion in a continuous reinforced concrete beam[17])

The loading diagram and resulting forces acting on a cast-in-place continuous concrete beam, in which redistribution can occur, are shown in Figure 4–29. The beam is of normal weight concrete with a cylinder strength of 4000 pounds per square inch and reinforcement consists of Grade 60 bars. The longitudinal reinforcement required, for bending only, is 1.38 square inches top and bottom. Determine the combined shear and torsion transverse reinforcement requirements for the given beam. Determine the combined flexural and longitudinal torsion reinforcement requirements for the given beam.

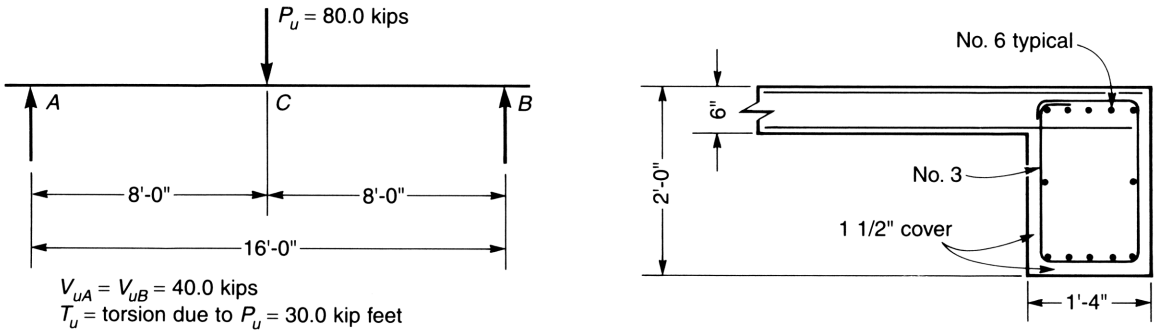


Figure 4–29 Details for Example 4–9

Solution

The critical section for shear and torsion in accordance with ACI Sections 11.1.3.1 and 11.6.2.4, is located a distance from the support equal to the effective depth of the section. At this location, the factored applied shear force and torsional moment are given as

$$V_u = 40 \text{ kips}$$
$$T_u = 30 \text{ kip feet}$$

In accordance with ACI Section 11.6.1.1, the overhanging flange width to be included in the section properties is given by

$$y_{flange} = h_w = 18 \text{ inches}$$

The area enclosed by the outside perimeter of the concrete cross section is

$$A_{cp} = 24 \times 16 + 6 \times 18 = 492 \text{ square inches}$$

The length of the outside perimeter of the concrete cross section is

$$p_{cp} = 2(24 + 16) + 2 \times 18 = 116 \text{ inches}$$

Torsional effects may be neglected and closed stirrups are not required when the factored torque does not exceed

$T_u = \phi(f'_c)^{0.5}(A^2_{cp}/p_{cp})$

where ϕ = strength reduction factor
= 0.85 ... from ACI Section 9.3.2.3

and $T_u = 0.85(4000)^{0.5}(492^2/116)/12{,}000$
= 9.35 kip feet
$< T_u$

Hence torsional effects must be considered

In accordance with ACI Section 11.6.2.2 the factored torque causing cracking is given by

$T_u = 4\phi(f'_c)^{0.5}(A^2_{cp}/p_{cp})$
$= 4 \times 9.35$
= 37.40 kip feet
$> T_u$

Since no redistribution of the torsion occurs the torsional reinforcement must be designed for the full torsional moment

T_u = 30 kip feet

The area of closed stirrups required for torsional effects is given by ACI Equation (11–21) as

$A_t/s = T_u/2A_of_{yv}$

where A_o = area enclosed by the centerline of the thin walled tube
$= 0.85A_{oh}$
$= 0.85x_oy_o$ ... since closed stirrups are not provided in the slab
A_{oh} = area enclosed by the centerline of the closed stirrups
x_o = distance between centerlines of longitudinal bars on a horizontal face
$= b_w - d_b - 2$(clear cover)
$= 16 - 0.375 - 2 \times 1.5$
= 12.625 inches
y_o = distance between centerlines of longitudinal bars on a vertical face
$= h - d_b - 2$(clear cover)
$= 24 - 0.375 - 2 \times 1.5$
= 20.625

Then $A_t/s = 12 \times 30 \times 12/(2 \times 0.85 \times 12.625 \times 20.625 \times 60)$
= 0.163 square inches/foot/leg

The design shear strength provided by the concrete is obtained from ACI Section 9.3 and ACI Equation (11–3) as

$\phi V_c = 2\phi b_w d(f'_c)^{0.5}$

where d = effective depth
$= 24 - 1.5 - 0.375 - 0.75/2$
= 21.75 inches

and $\phi V_c = 2 \times 0.85 \times 16 \times 21.75(4000)^{0.5}/1000$
= 37.42 kips

The design shear strength required from the shear reinforcement is given by ACI Equations (11–1) and (11–2) as

$$\begin{aligned} \phi V_s &= V_u - \phi V_c \\ &= 40 - 37.42 \\ &= 2.58 \text{ kips} \end{aligned}$$

The area of shear reinforcement required is given by ACI Equation (11–15) as

$$\begin{aligned} A_v/s &= \phi V_s/\phi d f_y \\ &= 2.58 \times 12/(0.85 \times 21.75 \times 60) \\ &= 0.0280 \text{ inches}^2/\text{foot} \end{aligned}$$

The total area of shear and torsional stirrups required, in accordance with ACI Section 11.6.3.8, is given by

$$\begin{aligned} A/s &= A_v/s + 2A_t/s \\ &= 0.028 + 2 \times 0.163 \\ &= 0.354 \text{ square inches/foot} \end{aligned}$$

Closed stirrups consisting of No.3 bars at a spacing of seven inches provide an area of

$$A/s = 0.38 \text{ which is satisfactory}$$

The minimum area of torsion and shear reinforcement, which may be provided, is given by ACI Equation (11–23) as

$$\begin{aligned} A/s &= 50 b_w/f_{yv} \\ &= 12 \times 50 \times 16/60{,}000 \\ &= 0.160 \text{ square inches/foot} \\ &< 0.38 \ldots \text{ satisfactory} \end{aligned}$$

In accordance with ACI Section 11.5.4.3 the maximum spacing of shear reinforcement is

$$\begin{aligned} s &= d/2 \\ &= 21.75/2 \\ &= 10.9 \text{ inches} \\ &> 7 \ldots \text{ satisfactory} \end{aligned}$$

The maximum spacing of torsion reinforcement is defined in ACI Section 11.6.6.1 as

$$s = p_h/8$$

where p_h = perimeter of the centerline of the closed stirrups

$$\begin{aligned} &= 2(x_o + y_o) \\ &= 2(12.625 + 20.625) \\ &= 66.50 \text{ inches} \end{aligned}$$

and

$$\begin{aligned} s &= 66.5/8 \\ &= 8.3 \text{ inches} \\ &> 7 \ldots \text{ satisfactory} \end{aligned}$$

Hence closed stirrups consisting of No.3 bars at a spacing of seven inches is satisfactory.

The longitudinal torsional reinforcement required is given by ACI Equation (11–22) as

$$\begin{aligned} A_\ell &= A_t p_h f_{yv}/f_{y\ell} s \\ &= 0.163 \times 66.5/12 \\ &= 0.903 \text{ square inches} \end{aligned}$$

Since $50 b_w s/f_{yv} < 2A_t$

the longitudinal torsional reinforcement shall not be less than the value given by ACI Equation (11–24) as

$$\begin{aligned} A_{\ell(\min)} &= 5A_{cp}(f'_c)^{0.5}/f_{y\ell} - A_t p_h f_{yv}/f_{y\ell} s \\ &= 5 \times 492 \times (4000)^{0.5}/60{,}000 - 0.163 \times 66.5/12 \\ &= 2.59 - 0.90 \\ &= 1.69 \end{aligned}$$

and this value governs

In accordance with ACI Section 11.6.6.2 one bar shall be placed in each corner of the closed stirrup and the remaining bars distributed around the perimeter at a maximum spacing of twelve inches. Using six longitudinal torsional bars, consisting of four corner bars and two side bars, gives a total area of reinforcement required at both the top and bottom face of the section of

$$\begin{aligned} A_t &= A_{st} + 2A_\ell/6 \\ &= 1.38 + 2 \times 1.69/6 \\ &= 1.94 \text{ square inches} \end{aligned}$$

Five No.6 bars provide an area of

$$\begin{aligned} A &= 2.20 \text{ square inches} \\ &> A_t \text{ ... satisfactory} \end{aligned}$$

The area of reinforcement required at each side of the beam is

$$\begin{aligned} A_s &= A_\ell/6 \\ &= 1.69/6 \\ &= 0.28 \text{ square inches} \end{aligned}$$

One No.6 bar provides an area of

$$\begin{aligned} A &= 0.44 \text{ square inches} \\ &> A_s \text{ ... satisfactory} \end{aligned}$$

A check on crushing of the concrete compression diagonals is provided by ACI Equation (11–18). The left-hand-side of ACI Equation (11–18) is

$$\begin{aligned} &\{(V_u/b_w d)^2 + [T_u p_h/(1.7A_{oh}^2)]^2\}^{0.5} \\ &= \{(40/16 \times 21.75)^2 + [30 \times 66.5 \times 12{,}000/(1.7 \times 260.39^2)]^2\}^{0.5} \\ &= (13{,}212 + 43{,}137)^{0.5} \\ &= 237 \text{ pounds per square inch} \end{aligned}$$

The right-hand-side of ACI Equation (11-18) is

$$\begin{aligned}
&\phi[V_c/b_w d + 8(f'_c)^{0.5}] \\
&= 0.85[2(f'_c)^{0.5} + 8(f'_c)^{0.5}] \\
&= 0.85 \times 10 \times (4000)^{0.5} \\
&= 537 \text{ pounds per square inch} \\
&> 237 \ldots \text{ satisfactory}
\end{aligned}$$

4.3 REINFORCEMENT DETAILING

4.3.1 Shear reinforcement

As specified in ACI Section 12.13.2 stirrups must be properly anchored so that the full tensile strength may be developed at mid-depth. To ensure this, ACI Section 12.13.1 requires the stirrups to be extended as close as practical to the tension and compression faces and to be hooked around a longitudinal bar at each face. In addition, ACI Section 7.1.3 specifies standard hooks which have been determined to be satisfactory based on experimental data. These hooks are illustrated in Figure 4-30 and described below.

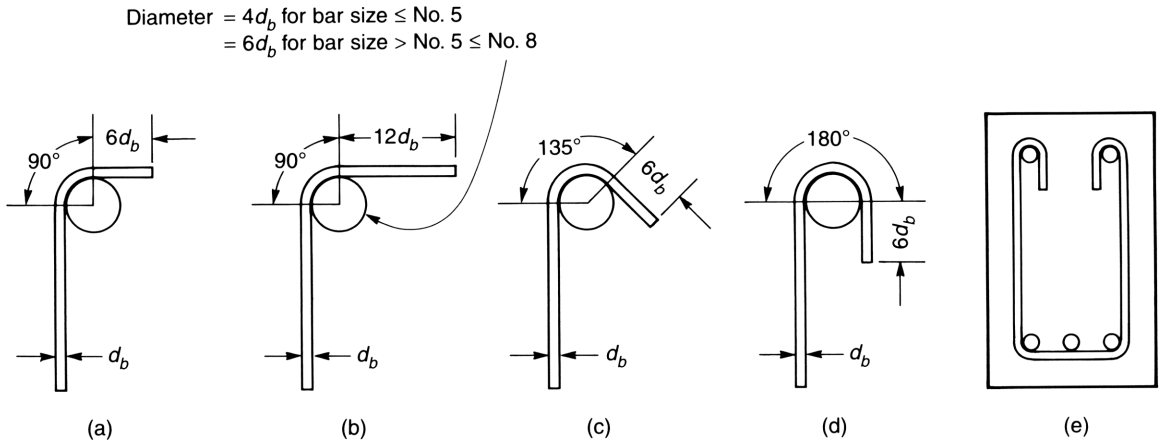


Figure 4-30 Anchorage of stirrups for shear reinforcement

- The 90 degree hook with six diameters extension at the free end, shown at (a), may be used for No.5 bars and smaller with any yield strength. As specified in ACI Sections 12.13.2.1 and R12.13.2.2 this anchorage may also be used for No.6, No.7, and No.8 bars provided that the yield strength does not exceed 40,000 pounds per square inch.
- The 90 degree hook with twelve diameters extension at the free end, shown at (b), may be used for No.6, No.7, and No.8 bars with a yield strength exceeding 40,000 pounds per square inch provided that an embedment is provided between the midheight of the member and the outside end of the hook of

$$\ell_b = 0.014 d_b f_y / (f'_c)^{0.5}$$

- The 135 degree and 180 degree hooks with six diameters extension at the free end, shown at (c) and (d), are the preferred hooks for No.6, No.7, and No.8 bars with a yield strength exceeding 40,000 pounds per square inch. The embedment length ℓ_b is also required.

A typical stirrup in a rectangular beam is shown at (e).

4.3.2 Torsion reinforcement

As specified in ACI Section 11.6.4.2 and shown in Figure 4–31, torsion stirrups in a rectangular beam must be anchored with a 135 degree hook, as 90 degree hooks fail when the corners spall. To reduce the tendency of corner spalling, the spacing and cover of the stirrups should be as small as practical. Since torsional stresses occur on all faces of a member, ACI Section 11.6.4.1 requires all stirrups to be closed. This may be achieved by means of the single piece stirrup shown. A more convenient method utilizes a two piece stirrup consisting of an open stirrup with 135 degree hooks closed by means of a crosstie with 135 degree hooks at both ends. When corner spalling is prevented by an adjacent slab, ACI Section 11.6.4.2 relaxes this requirement and permits the use of closed stirrups or crossties with 90 degree hooks.

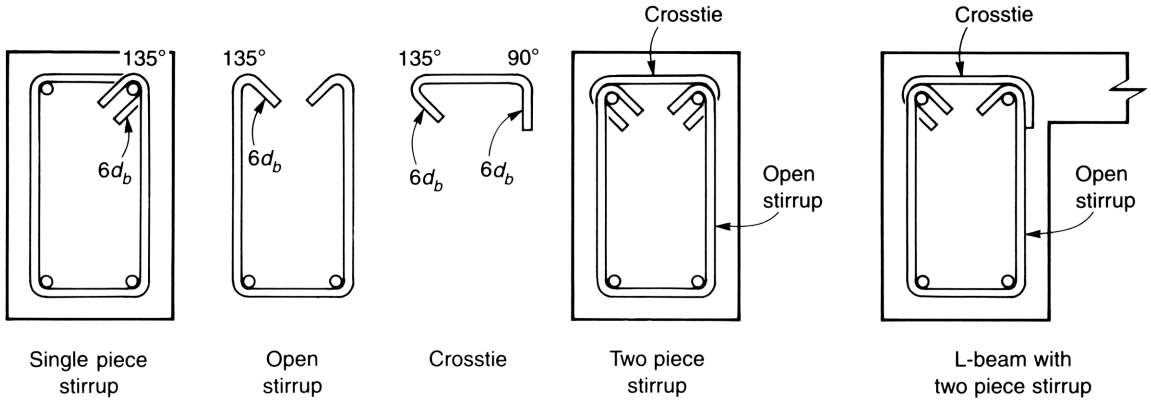


Figure 4–31 Anchorage of stirrups for torsion reinforcement

4.4 SUPPLEMENTARY PROBLEMS

Problem 4–1 (Beam without shear reinforcement)
A reinforced concrete beam, with an effective depth of 30 inches and a width of 15 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.015. The factored shear force at the critical section is 15 kips and the factored moment occurring simultaneously is 40 kip feet. Determine if shear reinforcement is required.

Problem 4–2 (Beam with compressive force)
A reinforced concrete beam, with an effective depth of 30 inches, an overall depth of 32 inches, and a width of 15 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.015. The factored shear force at the critical section is 15 kips and the factored moment occurring simultaneously is 40 kip feet. Determine the shear capacity of the beam if an axial compressive force of 12 kips is applied to the section.

Problem 4–3 (Beam with tensile force)
A reinforced concrete beam, with an effective depth of 30 inches, an overall depth of 32 inches, and a width of 15 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.015. The factored shear force at the critical section is 15 kips and the factored moment occurring simultaneously is 40 kip feet. Determine the shear capacity of the beam if an axial tensile force of 12 kips is applied to the section.

Problem 4–4 (Vertical stirrup shear reinforcement)
A reinforced concrete beam, simply supported over a clear span of 20 feet, is reinforced with Grade 60 bars and has a compressive strength of 3000 pounds per square inch. The beam has an effective depth of 24 inches and a width of 18 inches. The beam supports a factored load, including its own weight, of 22 kips per foot run. Determine the critical design shear and the required spacing, at the critical design shear location, for No.4 stirrups with four vertical legs.

Problem 4–5 (Shear in a deep beam)
A reinforced concrete beam, with a clear span of 12 feet, an effective depth of 42 inches and a width of 12 inches, has a concrete cylinder strength of 4000 pounds per square inch and a reinforcement ratio of 0.015. The factored shear force at the critical section is 90 kips and the factored moment occurring simultaneously is 200 kip feet. Determine the shear reinforcement required.

Problem 4–6 (Beam without torsion reinforcement)
A reinforced concrete beam, with a depth of 32 inches and a width of 15 inches, has a concrete cylinder strength of 4000 pounds per square inch. The factored torsion at the critical section is 5 kip feet. Determine if torsion reinforcement is required.

Problem 4–7 (Beam with torsion reinforcement)
A reinforced concrete beam, with a depth of 32 inches and a width of 15 inches, has a concrete cylinder strength of 4000 pounds per square inch. The required flexural reinforcement area at the top and bottom of the beam is 1.70 square inches, all longitudinal reinforcement is size No.6, and 1.5 inches of concrete cover is provided to the No.3 stirrups. The factored torsion at the critical section is 35 kip feet and the factored shear is 50 kips. Determine the shear and torsion reinforcement required.

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5

Bond and anchorage

5.1 GENERAL

5.1.1 Bond effects in reinforced concrete

The tensile force developed in a reinforcing bar is transferred to the surrounding concrete by means of the bond stress developed between the reinforcement and the concrete at the interface. The embedment length of the reinforcement must be sufficient to develop by bond the maximum tensile force in the reinforcement. The length required to develop the yield strength of the reinforcement is called the development or anchorage length.

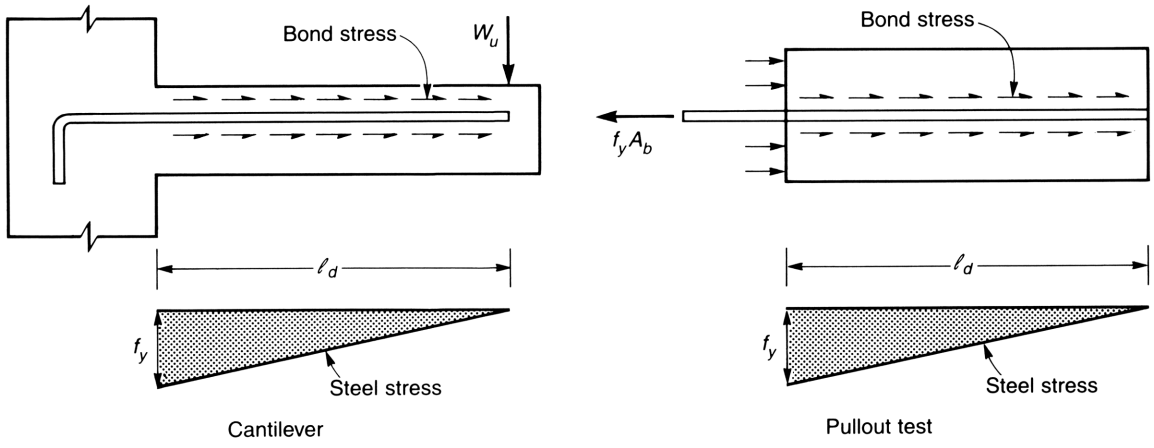


Figure 5–1 Development length

At the ultimate load, the reinforcement in the cantilever shown in Figure 5–1 develops its yield strength when the anchorage length provided is not less than the development length of the reinforcement. A similar situation exists in the pullout test shown in Figure 5–1. The tensile force in the reinforcement is given by

$$A_s f_y = \pi d_b^2 f_y / 4$$

This force is resisted by the bond stress acting on the surface area of the reinforcement. Equating these two forces gives

$$\ell_d \pi d_b f_{ba} = \pi d_d^2 f_y / 4$$

and

$$\ell_d / d_b = f_y / 4 f_{ba}$$

where

ℓ_d = development length
d_b = bar diameter
f_{ba} = average bond stress over the development length

The bond stress developed at the interface depends on the surface condition and diameter of the reinforcement, the cover, spacing, and location of the reinforcement, the provision of confining or transverse reinforcement, concrete strength and type of aggregate used, and whether the reinforcement is in tension or compression.

In a plain round bar, bond at the interface is produced by adhesion, micro-mechanical locking, and friction. In a deformed bar, provided that splitting of the concrete is prevented by confining reinforcement, bond is produced by the wedging effect of the bar ribs in the concrete[1], as shown in Figure 5–2. As the concrete tends to slip over the bar, the bearing pressure between the concrete and the ribs increases. This bearing pressure produces radial cracking around the ribs which tends to split the concrete if not restrained by confining reinforcement.

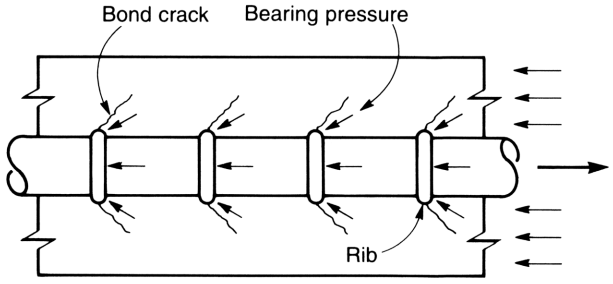


Figure 5–2 Bond resistance in a deformed bar

On the formation of a flexural crack in a beam, the concrete slips over the reinforcement in opposite directions on either side of the crack[2], as shown in Figure 5–3. A break occurs in the slip curve at a crack with slip to the left plotted as positive and slip to the right plotted as negative. The total ordinate of the slip curve at a crack is the width of the crack. Similarly, with the bond stress distribution curves, bond stress opposing positive slip is plotted as positive. The curves of slip and bond stress illustrate how the bond stress builds up as the slip increases and the ribs wedge against the concrete. As slip increases further, the concrete shears at the ribs

and the bond stress reduces rapidly, with the peak values in the bond stress curve moving away from the cracks where the slip is at a maximum.

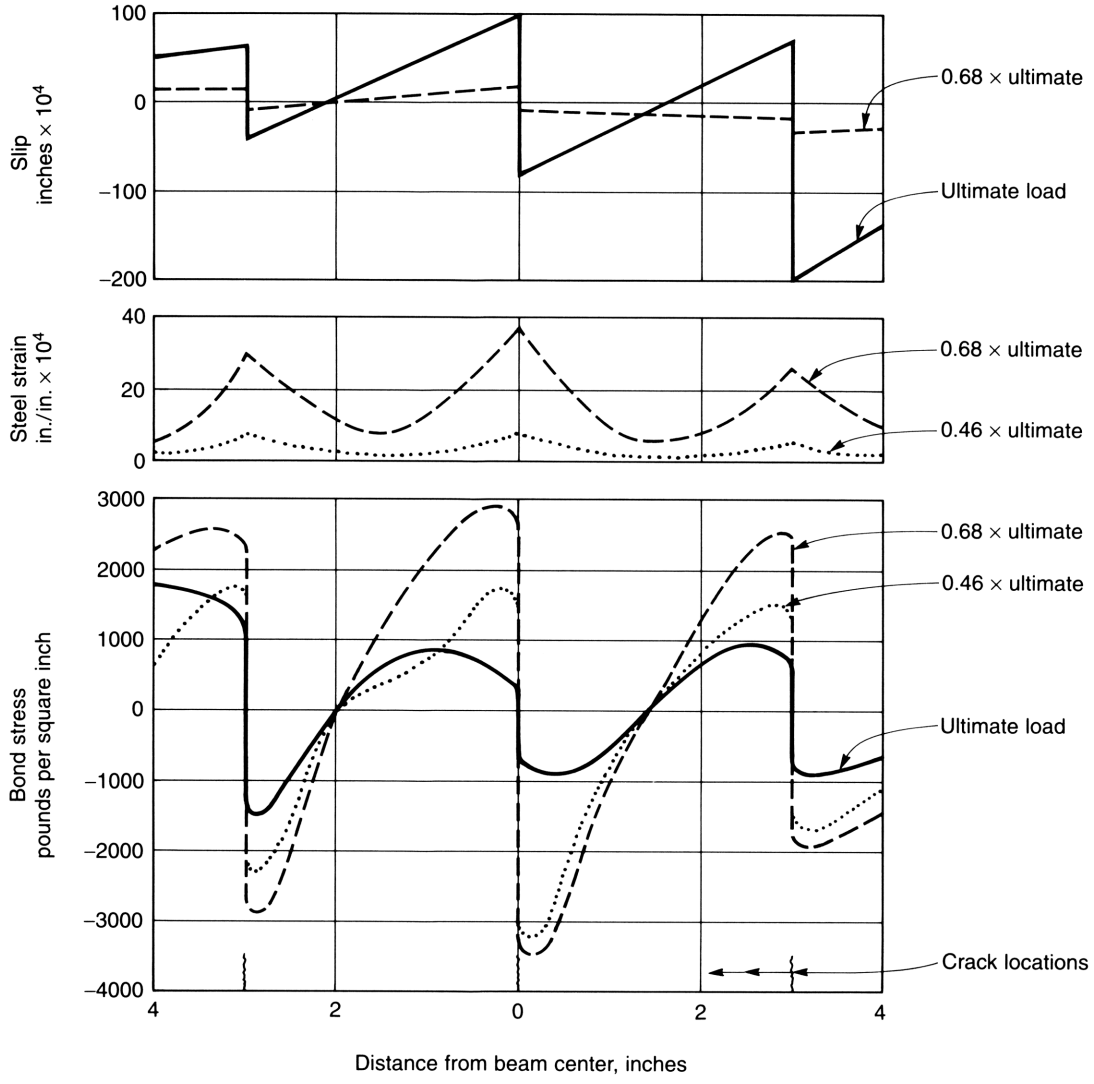


Figure 5–3 Slip, strain, and bond stress relationships

Bond strength increases as bar diameter is reduced[1] and also increases as the relative rib area increases[3], provided adequate confining reinforcement is used. For reinforcing bars complying with ASTM A615 and A706 the rib spacing is limited to a maximum of 70 percent of the bar diameter and the minimum specified rib height to 4 percent for No.4 bars or smaller, 4.5 percent for No.5 bars, and 5 percent for larger bars.

The radial bond cracks produced by the ribs bearing on the surrounding concrete tend to propagate to the concrete surface[4]. As shown in Figure 5–4, the cracks will spread vertically to the beam soffit when the bottom cover to the bars is smaller than either the side cover or one-half the center-to-center spacing of the bars. Otherwise, horizontal splitting will occur.

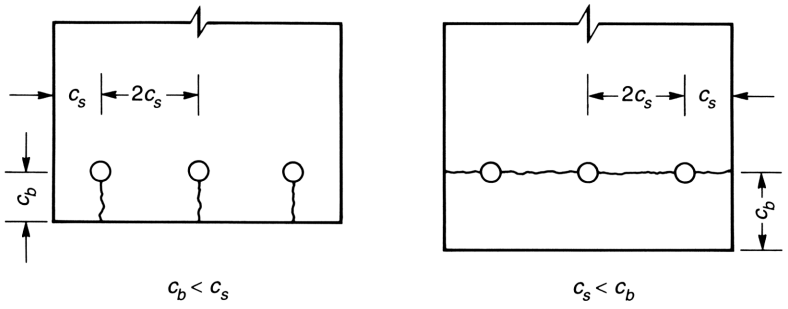


Figure 5–4 Concrete splitting

The loss of confinement around the bars, caused by the spalling of the concrete, produces bond failure and must be prevented either by increasing concrete cover and bar spacing or by providing transverse reinforcement to confine the concrete[5,6]. To be effective, the transverse reinforcement must be adjacent to and on the outside of the bar being developed and must cross the potential splitting planes as shown in Figure 5–5.

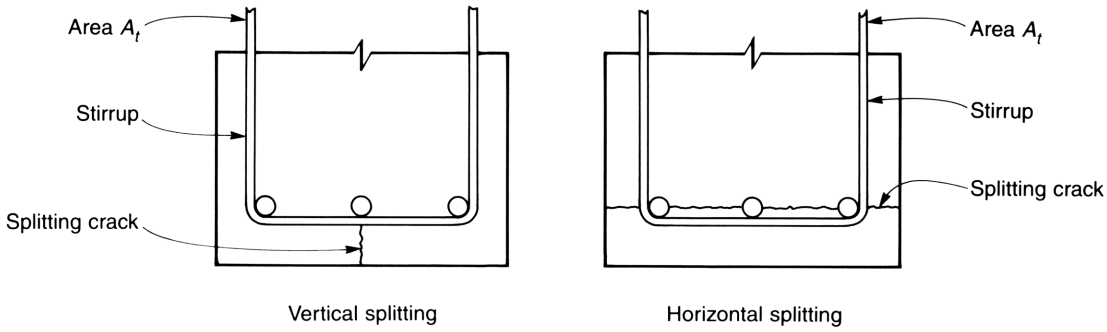


Figure 5–5 Confining reinforcement

The effectiveness of the transverse reinforcement also depends on the area of confining reinforcement provided along the developed bar and is defined by the transverse reinforcement index

$$K_{tr} = A_{tr} f_{yt} / 1500sn$$

where A_{tr} = cross-sectional area of transverse reinforcement crossing a potential plane of splitting
= A_t for the vertical splitting shown in Figure 5–5
= $2A_t$ for the horizontal splitting shown in Figure 5–5

f_{yt} = yield strength of the transverse reinforcement
s = spacing of the transverse reinforcement
n = number of bars being developed along a potential plane of splitting
= 1 for the vertical splitting shown in Figure 5–5
= 3 for the horizontal splitting shown in Figure 5–5
1500 = constant with the unit pounds per square inch

Factors which reduce the bond strength include top-cast bars, lightweight concrete, and epoxy-coated bars. A reinforcing bar with more than 12 inches of concrete cast below it has its bond strength reduced by the excess water and entrapped air which collects underneath. Lightweight concrete has a lower tensile strength than normal weight concrete and this results in a reduced splitting resistance and reduced bond strength. The bond strength of epoxy-coated bars is reduced because the coating reduces the adhesion and friction between the bar and the concrete.

5.2 ANCHORAGE REQUIREMENTS

5.2.1 Development length: straight bars in tension

The development length for tension reinforcement is based on extensive research data and professional experience and is given by ACI Equation (12.1) and ACI Section 12.2.4 as

$$\ell_d/d_b = 0.075 f_y \alpha\beta\gamma\lambda/[(f'_c)^{0.5}(c + K_{tr})/d_b]$$

where ℓ_d = development length
d_b = bar diameter
α = reinforcement location factor
= 1.3 for horizontal bars with more than 12 inches of concrete below
= 1.0 for other bars
β = coating factor
= 1.5 for epoxy-coated bars with cover $< 3d_b$ or clear spacing $< 6d_b$
= 1.2 for all other epoxy-coated bars
= 1.0 for uncoated reinforcement
$\alpha\beta \leq 1.7$
γ = reinforcement size factor
= 0.8 for No.6 and smaller bars
= 1.0 for No.7 and larger bars
λ = lightweight aggregate factor
= 1.3 for lightweight aggregate concrete
= $6.7(f'_c)^{0.5}/f_{ct} \geq 1.0$ when f_{ct} is specified
= 1.0 for normal weight concrete
c = spacing or cover dimension
= distance from center of bar to nearest concrete surface
≤ one-half the center-to-center spacing of the bars being developed
$K_{tr} = A_{tr} f_{yt}/1500sn$

and $(c + K_{tr})/d_b \leq 2.5$

In addition, as specified by ACI Section 12.2.1, the minimum value of the development length is

$$\ell_d = 12 \text{ inches}$$

Experimental data for high strength concrete is not available and for this reason ACI Section 12.1.2 specifies that

$$(f'_c)^{0.5} \leq 100 \text{ pounds per square inch}$$

When excess reinforcement is provided in a member, the tensile stress in the reinforcement is less than the design value and development lengths are correspondingly lower. The development length may then be reduced by the factor

$$\epsilon = A_{s(reqd)}/A_{s(provided)}$$

This reduction factor may not be applied where anchorage is required for the full yield strength of the reinforcement. This occurs for shrinkage and temperature reinforcement specified in ACI Section 7.12.2.3, for integrity reinforcement specified in ACI Section 7.13, for positive moment at a support as specified in ACI Section 12.11.2, and for tension lap splices as specified in ACI Sections 12.15.1 and 13.3.8.5.

The strength reduction factor ϕ is not applied in the determination of development length since the specified lengths already include an allowance for understrength.

For commonly occurring design situations ACI Equation (12–1) may be simplified, as specified in ACI Section 12.2.2, by assuming an estimated value for the factor $(c + K_{tr})/d_b$.

- Using the minimum clear spacing d_b specified in ACI Section 7.6.1, the minimum clear cover d_b to primary reinforcement specified in ACI Section 7.7, and in the absence of stirrups

  $$c = d_b$$
  $$K_{tr} = 0$$
  $$(c + K_{tr})/d_b = 1.0$$

  Then, for No.7 and larger bars $\gamma = 1.0$ and

  $$\ell_d/d_b = 0.075 f_y \alpha\beta\lambda/(f'_c)^{0.5}$$

  For No.6 and smaller bars $\gamma = 0.8$ and

  $$\ell_d/d_b = 0.06 f_y \alpha\beta\lambda/(f'_c)^{0.5}$$

- Using the minimum clear spacing d_b specified in ACI Section 7.6.1, the minimum clear cover d_b to primary reinforcement specified in ACI Section 7.7, and minimum stirrups specified in ACI Equation (11–13)

  $$c = d_b$$
  $$K_{tr} = 50 b_w/1500n \approx 0.5 d_b$$
  $$(c + K_{tr})/d_b = 1.5$$

Then, for No.7 and larger bars $\gamma = 1.0$ and

$$\ell_d/d_b = 0.05f_y\alpha\beta\lambda/(f'_c)^{0.5}$$

For No.6 and smaller bars $\gamma = 0.8$ and

$$\ell_d/d_b = 0.04f_y\alpha\beta\lambda/(f'_c)^{0.5}$$

- Using a minimum clear spacing of $2d_b$, the minimum clear cover d_b to primary reinforcement specified in ACI Section 7.7, and in the absence of stirrups

$$c = 1.5d_b$$
$$K_{tr} = 0$$
$$(c + K_{tr})/d_b = 1.5$$

Then, for No.7 and larger bars $\gamma = 1.0$ and

$$\ell_d/d_b = 0.05f_y\alpha\beta\lambda/(f'_c)^{0.5}$$

For No.6 and smaller bars $\gamma = 0.8$ and

$$\ell_d/d_b = 0.04f_y\alpha\beta\lambda/(f'_c)^{0.5}$$

Making the further assumptions of using uncoated, Grade 60 reinforcement in the bottom of normal weight concrete beams gives

$$\beta = 1.0$$
$$\alpha = 1.0$$
$$\lambda = 1.0$$

and Table 5–1 may be derived.

f'_c	Bar size ≤ No.6		Bar size ≥ No.7	
	Case 1*	Case 2**	Case 1*	Case 2**
3000	44	66	55	82
3500	41	61	51	76
4000	38	57	47	71
4500	36	54	45	67
5000	34	51	42	64

* Clear cover ≥ d_b clear spacing ≥ $2d_b$ OR:
Clear cover ≥ d_b clear spacing ≥ d_b and minimum stirrups
** All other conditions

Table 5–1 Values of ℓ_d/d_b for Grade 60 bars from ACI Section 12.2.2 with $\alpha = \beta = \lambda = 1.0$

Example 5–1 (Development length for a straight bar in tension)
The reinforced concrete cantilever shown in Figure 5-6 is reinforced with Grade 60 bars and has a compressive strength of 3000 pounds per square inch. The beam has an overall depth of 24 inches and a width of 12 inches. The beam supports the factored load indicated, and its own weight may be neglected. Determine if the anchorage length provided for the No.10 bars is satisfactory.

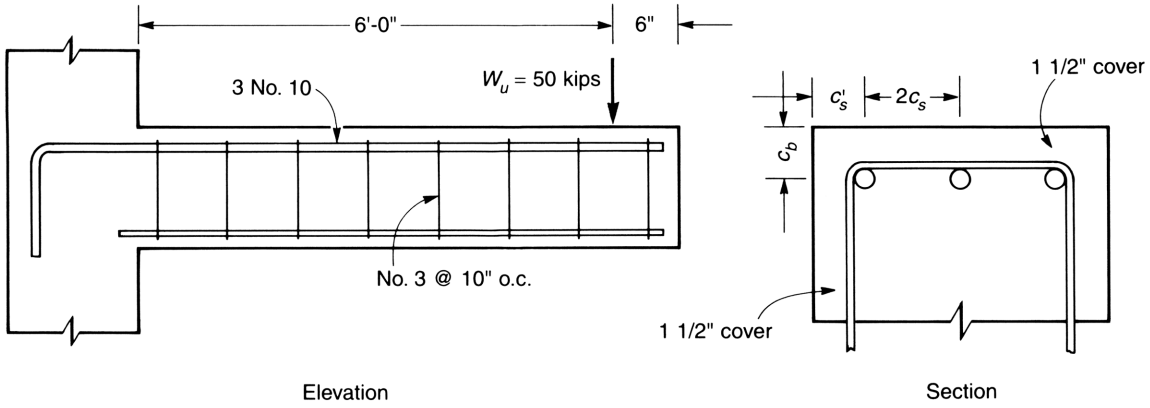


Figure 5–6 Details for Example 5–1

Solution
The factored support reaction is given by

$$R_u = W_u = 50 \text{ kips}$$

The minimum shear reinforcement required is given by ACI Equation (11-13) as

$$A_v/s = 12 \times 50b_w/f_y = 0.12 \text{ inches}^2/\text{foot}$$

Shear reinforcement consisting of two vertical legs of No.3 stirrups at 10 inch spacing provides a reinforcement area of

$$A_v/s = 2 \times 0.13 = 0.26 \text{ inches}^2/\text{foot} \ldots \text{satisfactory}$$

The factored support moment is given by

$$M_u = W_u \times 6 = 300 \text{ kip feet}$$

The effective depth provided is

$$d = 24 - 1.5 - 0.375 - 1.27/2 = 21.49 \text{ inches}$$

The flexural reinforcement required is given by ACI Section 10.2 as

$$A_s = 0.85b_w df'_c[1 - (1 - M_u/0.383f'_c b_w d^2)^{0.5}]/f_y$$
$$= 3.74 \text{ square inches}$$

Tension reinforcement consisting of three No.10 bars provides a reinforcement area of

$$A_s = 3.81 \ldots \text{ satisfactory}$$

Using ACI Section 12.2.2

Since the clear cover and the clear spacing of the bars exceeds d_b and minimum stirrups are provided, the required development may be derived from Table 5–1. The top bar factor is

$$\alpha = 1.3$$

and the excess reinforcement factor is

$$\epsilon = A_{s(reqd)}/A_{s(provided)}$$
$$= 3.74/3.81$$
$$= 0.98$$

The required development length is

$$\ell_d = 55 \times 1.3 \times 0.98d_b$$
$$= 89 \text{ inches} \ldots \text{ unsatisfactory}$$

Using ACI Equation (12–1)

The relevant cover dimensions are obtained from Figure 5–6 as

$$c_b = 1.5 + 0.375 + 1.27/2$$
$$= 2.51 \text{ inches}$$
$$c'_s = c_b$$
$$2c_s = (12 - 2 \times 1.5 - 2 \times 0.375 - 3 \times 1.27)/2 + 1.27$$
$$= 5.71 \text{ inches}$$
$$c_s = 2.86 \text{ inches}$$
$$> c_b$$

Hence $c = 2.51$ inches, and vertical splitting controls

The transverse reinforcement index is

$$K_{tr} = A_{tr}f_{yt}/1500sn$$

where A_{tr} = cross-sectional area of transverse reinforcement crossing the plane of splitting
$= A_t$ for vertical splitting
$= 0.11$ square inches

f_{yt} = yield strength of the transverse reinforcement
$= 60{,}000$ pounds per square inch

s = spacing of the transverse reinforcement
$= 10$ inches

n = number of bars being developed along the potential plane of splitting
$= 1$ for vertical splitting

Hence $K_{tr} = 0.11 \times 60{,}000/(1500 \times 10 \times 1)$
$= 0.44$ inches
and $(c + K_{tr})/d_b = (2.51 + 0.44)/1.27$
$= 2.32$
< 2.5 ... satisfactory
Then $\ell_d/d_b = 0.075 f_y \alpha\beta\gamma\epsilon\lambda/[(f'_c)^{0.5}(c + K_{tr})/d_b]$
where α = reinforcement location factor
= 1.3 for top bars
β = coating factor
= 1.0 for uncoated reinforcement
γ = reinforcement size factor
= 1.0 for No.10 bars
ϵ = excess reinforcement factor
= 0.98
λ = lightweight aggregate factor
= 1.0 for normal weight concrete
$(c + K_{tr})/d_b = 2.32$
and $\ell_d/d_b = 0.075 \times 60{,}000 \times 1.3 \times 1.0 \times 1.0 \times 0.98 \times 1.0/2.32(3000)^{0.5}$
$= 45$
$\ell_d = 45 \times 1.27$
$= 57$ inches

The anchorage length provided is

$\ell_{d(provided)} = 6 \times 12 + 4.5$
$= 76.5$ inches
> 57 ... satisfactory

5.2.2 Development length: straight bars in compression

The requirements for determining the development length of compression reinforcement are given in ACI Section 12.3. Shorter basic development lengths are specified for bars in compression than are specified for bars in tension. The end bearing of the bars and the lack of flexural cracks both reduce the development lengths of compression bars. Based on extensive test data, the basic development length is defined in ACI Section 12.3.2 as

$\ell_{db} = 0.02 d_b f_y/(f'_c)^{0.5}$
$\geq 0.0003 d_b f_y$

The first expression governs when the concrete strength is less than 4444 pounds per square inch. To determine the actual development length, the basic development length is modified by factors to account for excess reinforcement and the use of confining reinforcement. These modifiers are defined in ACI Section 12.3.3 and are given by

ϵ = excess reinforcement factor
$= A_{s(reqd)}/A_{s(provided)}$

ξ = confinement factor
= 0.75

To qualify for the confinement factor, spiral reinforcement must be provided with a diameter of 0.25 inch and a pitch not exceeding 4 inches. Alternatively, column ties may be provided using a minimum of No.4 bars at a spacing not exceeding 4 inches.

Then ℓ_d = development length or anchorage length
$= \epsilon \xi \ell_{db}$
$\geq$ 8 inches

Table 5–2 provides values of the basic development length for Grade 60 bars.

Bar size No.	Concrete strength			
	3000	3500	4000	≥4444
3	8.2	7.6	7.1	6.8
4	11.0	10.1	9.5	9.0
5	13.7	12.7	11.9	11.3
6	16.4	15.2	14.2	13.5
7	19.2	17.8	16.6	15.8
8	21.9	20.3	19.0	18.0
9	24.7	22.9	21.4	20.3
10	27.8	25.8	24.1	22.9
11	30.9	28.6	26.8	25.4

Table 5–2 Values of ℓ_{db} for Grade 60 bars in compression

5.2.3 Bundled bars

Longer development lengths are specified for bundled bars than are specified for individual bars in both tension and compression. The difficulty of compacting concrete around groups of bars in contact, reduces the bond resistance at the interior of the group and necessitates an increased development length. For bundled bars, ACI Section 12.4 specifies that the development length shall be increased 20 percent for a three-bar bundle and 33 percent for a four-bar bundle. No increase is required for a two-bar bundle. To determine the effective diameter d_b of a bundle to use in the development length expressions of ACI Sections 12.2 and 12.3, an equivalent single bar is adopted with an area equal to that of the bundle.

Example 5–2 (Development length for bundled bars in tension)
The reinforced concrete beam shown in Figure 5–7 is reinforced with Grade 60 bars and has a compressive strength of 3000 pounds per square inch. The beam has an overall depth of 24 inches and a width of 12 inches. The flexural reinforcement in the beam is fully stressed under the factored load indicated, and the No.10 bars are bundled, as shown, to provide space for inserts cast into the beam soffit. Determine if the anchorage length provided for the bundled bars is satisfactory.

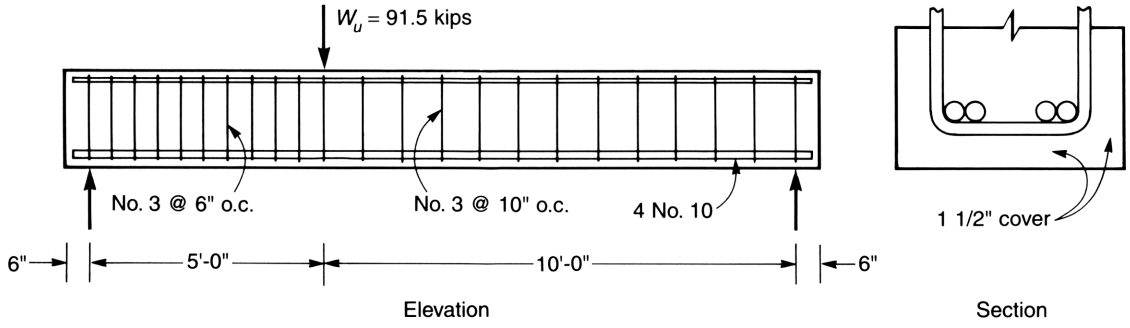


Figure 5–7 Details for Example 5–2

Solution

The combined area of two bundled No.10 bars is

$$A_s = 2.54 \text{ square inches}$$

The effective diameter of the equivalent single bar is

$$d_{b(equiv)} = (4A_s/\pi)^{0.5}$$
$$= 1.80 \text{ square inches}$$

Using ACI Section 12.2.2

The clear cover to the equivalent single bars is

$$c_c = 1.5 + 0.375$$
$$= 1.875 \text{ inches}$$
$$> d_{b(equiv)}$$

The clear spacing of the equivalent single bars is

$$s_c = 12 - 2 \times 1.5 - 2 \times 0.375 - 2 \times 1.80$$
$$= 4.65 \text{ inches}$$
$$> 2d_{b(equiv)}$$

Stirrups are provided and the required development length is obtained from Table 5–1 as

$$\ell_d = 55d_{b(equiv)}$$
$$= 55 \times 1.8$$
$$= 99 \text{ inches}$$

Development length provided is

$$\ell_{d(provided)} = 5 \times 12 + 4.5$$
$$= 64.5 \text{ inches}$$
$$< \ell_d \ldots \text{ unsatisfactory}$$

Using ACI Equation (12–1)

The relevant cover dimensions are

$c_b = 1.5 + 0.375 + 1.80/2$
$= 2.78$ inches
$c_s' = c_b$
$2c_s = 12 - 2 \times 1.5 - 2 \times 0.375 - 1.80$
$= 6.45$ inches
$c_s = 3.23$ inches
$> c_b$

Hence $c = 2.78$ inches, and vertical splitting controls

The transverse reinforcement index is

$K_{tr} = A_{tr}f_{yt}/1500sn$

where A_{tr} = cross-sectional area of transverse reinforcement crossing the plane of splitting
$= A_t$ for vertical splitting
$= 0.11$ square inches

f_{yt} = yield strength of the transverse reinforcement
$= 60{,}000$ pounds per square inch

s = spacing of the transverse reinforcement
$= 6$ inches

n = number of bars being developed along the potential plane of splitting
$= 1$ for vertical splitting

Hence $K_{tr} = 0.11 \times 60{,}000/(1500 \times 6 \times 1)$
$= 0.73$ inches

$(c + K_{tr})/d_{b(equiv)} = (2.78 + 0.73)/1.80$
$= 1.95$
< 2.5 ... satisfactory

Then $\ell_d/d_{b(equiv)} = 0.075f_y\alpha\beta\gamma\epsilon\lambda/[(f_c')^{0.5}(c + K_{tr})/d_b]$

where α = reinforcement location factor
$= 1.0$ for bottom bars

β = coating factor
$= 1.0$ for uncoated reinforcement

γ = reinforcement size factor
$= 1.0$

ϵ = excess reinforcement factor
$= 1.0$

λ = lightweight aggregate factor
$= 1.0$ for normal weight concrete

$(c + K_{tr})/d_{b(equiv)} = 1.95$

and $\ell_d/d_{b(equiv)} = 0.075 \times 60{,}000 \times 1.0 \times 1.0 \times 1.0 \times 1.0 \times 1.0/1.95(3000)^{0.5}$
$= 42$

$\ell_d = 42 \times 1.80$
$= 76$ inches
$> \ell_{d(provided)}$ ... unsatisfactory

5.2.4 Development length: hooked bars in tension

Hooks, or mechanical anchorages, may be provided to the ends of straight bars in order to reduce the tension anchorage length required. Anchorage is developed by a combination of bond resistance along the straight portion of the bar and by bearing stresses on the concrete within the hook. Standard hook dimensions are specified in ACI Sections 7.1 and 7.2 and are shown in Figure 7-8. Both the 90 degree hook and the 180 degree hook have the same anchorage capacity and the development length is measured from the critical section to the outside edge of the hook. Research indicates that the failure of a hooked bar is caused by the high bearing stresses within the hook splitting the concrete cover. Thus, the anchorage capacity may be increased by providing lateral confinement to the hook either by increasing concrete cover or by providing closely spaced ties. As specified in ACI Section 12.5.5 hooks are not effective in developing bars in compression.

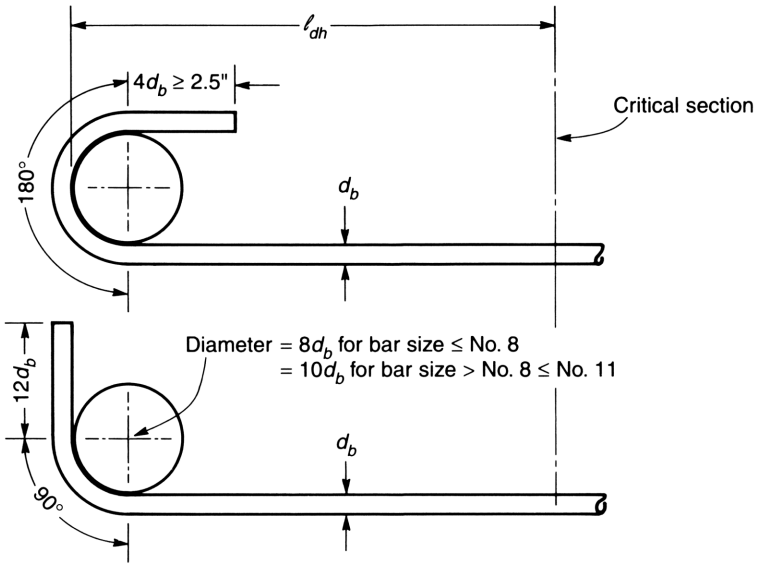


Figure 5-8 Standard hooks

The basic development length for tension reinforcement, with a yield strength of 60,000 pounds per square inch, terminating in a standard hook, is given by ACI Section 12.5.2 as

$$\ell_{hb} = 1200 d_b / (f'_c)^{0.5}$$

Table 5-3 provides values of the basic development length of Grade 60 hooked bars. The actual development or anchorage length is obtained by multiplying the basic development length by the modifiers defined in ACI Sections 12.5.3 and 12.5.4 and summarized in Flow Chart 5-1. The modifiers are cumulative with the following restrictions imposed

$$\ell_{dh} = \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\ell_{hb}$$
$$= \text{development or anchorage length}$$
$$\geq 8d_b$$
$$\geq 6 \text{ inches}$$

Bar size No.	Concrete strength				
	3000	3500	4000	4500	5000
3	8.2	7.6	7.1	6.7	6.4
4	11.0	10.1	9.5	8.9	8.5
5	13.7	12.7	11.9	11.2	10.6
6	16.4	15.2	14.2	13.4	12.7
7	19.2	17.8	16.6	15.7	14.8
8	21.9	20.3	19.0	17.9	17.0
9	24.7	22.9	21.4	20.2	19.1
10	27.8	25.8	24.1	22.7	21.6
11	30.9	28.6	26.8	25.2	23.9

Table 5–3 Values of ℓ_{hb} for Grade 60 hooked bars

The modifiers defined in ACI Sections 12.5.3 and 12.5.4 are given by

η_1 = reinforcement strength factor
= $f_y/60{,}000$

where f_y = yield strength of the reinforcement

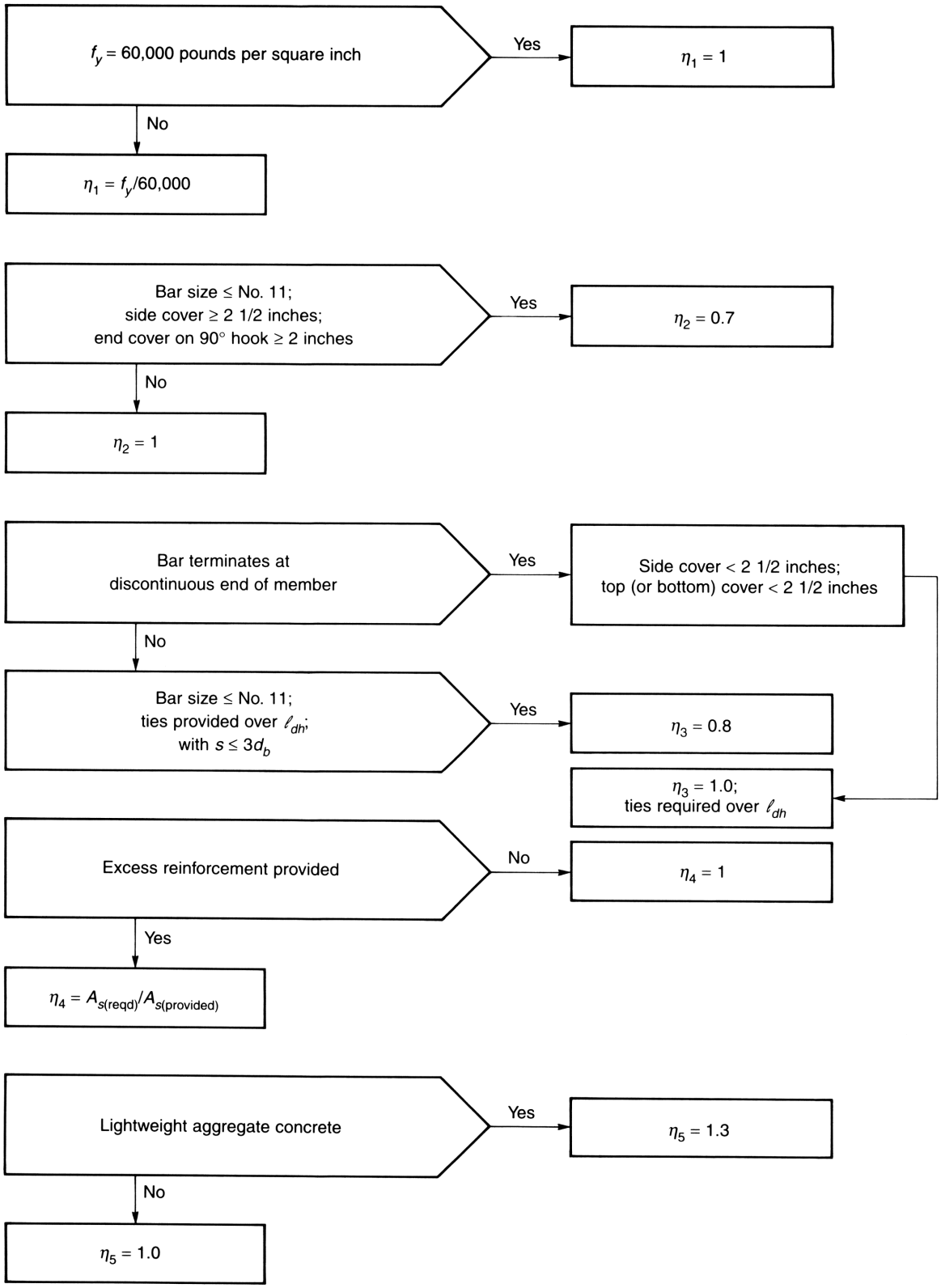
η_2 = cover factor
= 0.7 for bars $\leq$ No.11 in size, with side cover $\geq$ 2.5 inches, and end cover on a 90 degree hook $\geq$ 2 inches
= 1.0 for all other conditions

η_3 = tie factor
= 0.8 for bars $\leq$ No.11 in size, with ties provided over the full development length at a spacing $\leq 3d_b$
= 1.0 at the discontinuous end of members with side cover $<$ 2.5 inches, top (or bottom) cover $<$ 2.5 inches, and with ties provided over the full development length at a spacing $\leq 3d_b$

η_4 = excess reinforcement factor
= $A_{s(reqd)}/A_{s(provided)}$

η_5 = lightweight aggregate factor
= 1.3 for lightweight aggregate concrete
= 1.0 for normal weight concrete

η_6 = coating factor
= 1.2 for epoxy-coated bars
= 1.0 for uncoated reinforcement



Flow Chart 5–1 Modifiers for standard hooked bars

Example 5–3 (Development length for hooked bars)

The reinforced concrete beam for Example 5–2 and shown in Figure 5–7 is reinforced with hooked bars. Determine if the anchorage length provided for the hooked bars is satisfactory.

Solution

Since the bars terminate at the discontinuous end of a member with side cover and bottom cover less than 2.5 inches, ties must be provided over the full development length at a spacing not exceeding $3d_{b(equiv)}$. The spacing required is

$$\begin{aligned} s &= 3d_{b(equiv)} \\ &= 3 \times 1.8 \\ &= 5.4 \text{ inches} \end{aligned}$$

Hence, at the end of the beam, stirrups are required at a spacing of five inches. The basic development length is given by ACI Section 12.5.2 as

$$\begin{aligned} \ell_{hb} &= 1200d_{b(equiv)}/(f'_c)^{0.5} \\ &= 1200 \times 1.8/(3000)^{0.5} \\ &= 39.4 \text{ inches} \end{aligned}$$

All modifiers are unity and the actual development length is given by

$$\begin{aligned} \ell_{dh} &= \ell_{hb} \\ &= 39.4 \text{ inches} \\ &< \ell_{d(provided)} \ldots \text{ satisfactory} \end{aligned}$$

5.3 CURTAILMENT REQUIREMENTS

5.3.1 Curtailment of flexural reinforcement

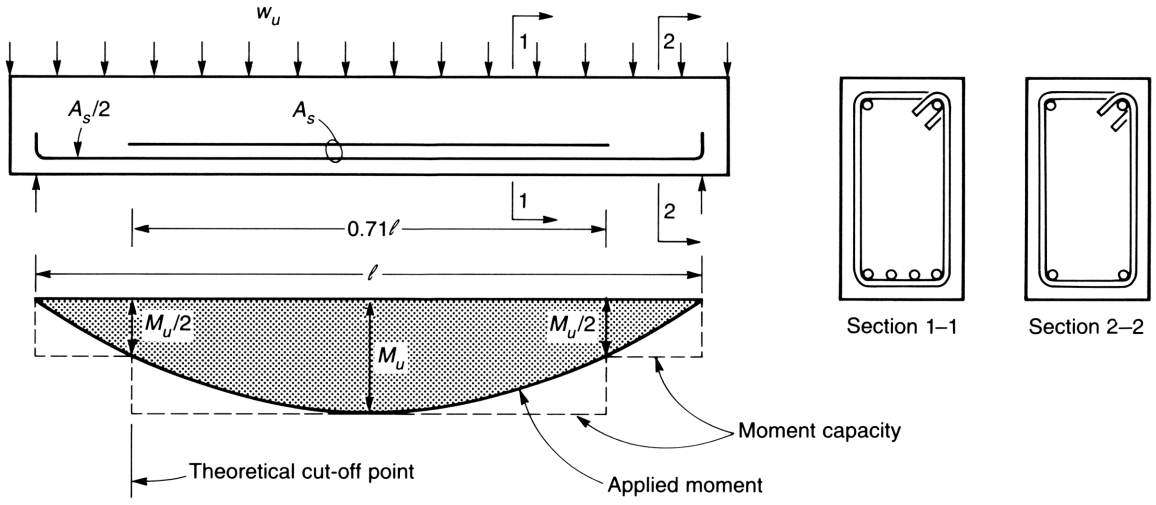


Figure 5–9 Theoretical cut–off points

In the simply supported beam loaded with a uniformly distributed factored load as shown in Figure 5–9, the bending moment diagram is parabolic in shape. The maximum moment occurs at midspan and requires an area of flexural reinforcement given by

$$A_s = M_u/(d - 0.5a)$$

where M_u = factored moment at midspan
d = effective depth
a = depth of equivalent rectangular stress block

As the value of the moment decreases parabolically towards the supports, the required area of reinforcement is correspondingly reduced. When the moment has reduced to half its maximum value, the required area of reinforcement is theoretically given by

$$A_s' \approx 0.5M_u/(d - 0.5a)$$
$$\approx A_s/2$$

As shown in Figure 5–9, it appears that half of the reinforcement may be curtailed at this theoretical cut-off point. In practice, when tension reinforcement is curtailed along the length of a flexural member the actual, or physical, cut-off point may not be less than the appropriate anchorage length from the last point at which it is assumed to be fully stressed. The point at which, in theory, a bar is no longer required to resist flexure is the theoretical cutoff point and is that point where the design strength of the section, considering only the continuing bars, is equal to the required ultimate strength. However, reinforcement must be continued beyond the theoretical cutoff point to account for the following effects

- Inaccuracies in the magnitude of the moments caused by changes in the applied loads, settlement of supports, and lateral loads.
- Possible errors in construction.
- The presence of diagonal shear cracks which cause the calculated tensile stress to occur approximately an effective depth beyond the theoretical cut-off point as shown in Figure 5–10.

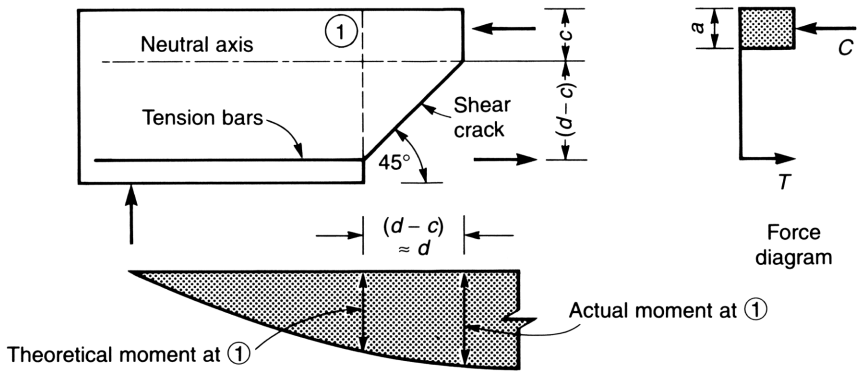


Figure 5–10 Increase in theoretical moment

To account for these effects, ACI Section 12.10.3 specifies that reinforcement shall extend beyond the theoretical cut-off point for a distance at least equal to the greater of the effective depth of the member or twelve times the bar diameter, except at end supports and at the free end of cantilevers. These requirements are illustrated in Figure 5–11.

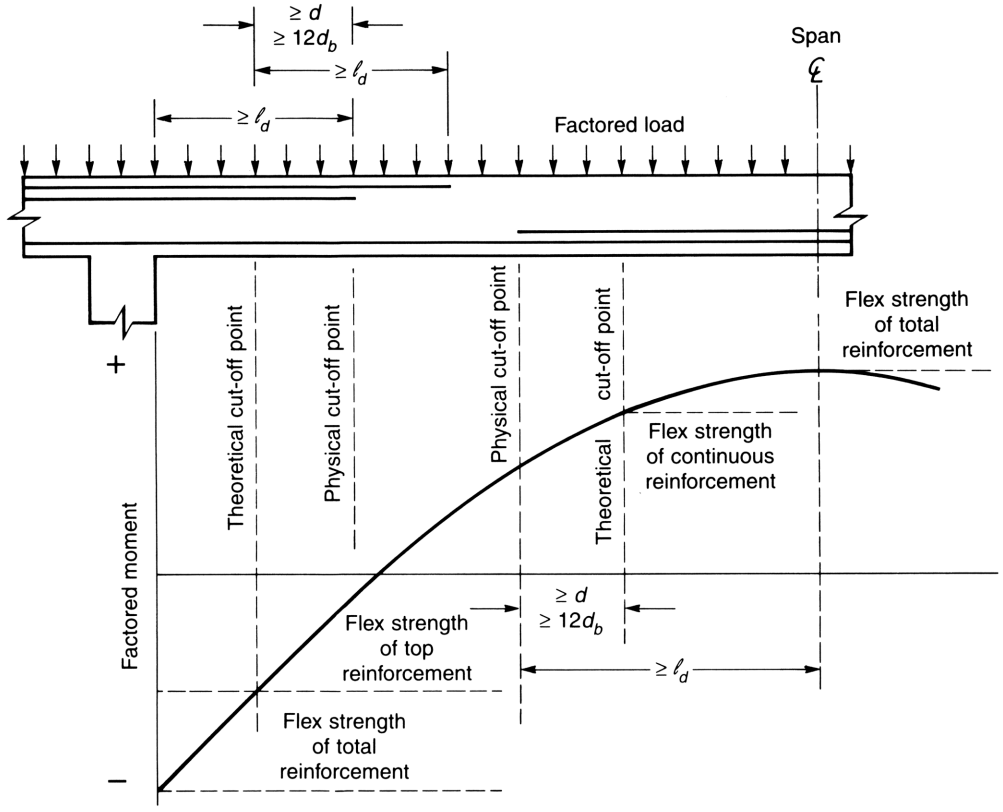


Figure 5–11 Curtailment of flexural reinforcement

To prevent excessive flexural and diagonal shear cracking at cutoff points and a local reduction in shear strength, additional precautions are required for the curtailment of tension reinforcement. In accordance with ACI Section 12.10.5, one of the following conditions must be satisfied at the physical cutoff point

- The applied shear force does not exceed two-thirds the shear capacity of the section.
- Additional stirrups with an area of $60b_ws/f_y$ are provided along the curtailed reinforcement for a distance equal to three-quarters of the effective depth at a maximum spacing of $d/8\beta_b$, where β_b is the ratio of the area of terminated reinforcement to total reinforcement.
- For number 11 bars and smaller, the continuing reinforcement provides twice the flexural capacity required and the applied shear force does not exceed three-quarters the shear capacity of the section.

5.3.2 Positive flexural reinforcement

In order to control local bond stresses, unless anchorage is provided by means of a standard hook, a limitation is placed on the size of the reinforcement by specifying a maximum value for the development length. When the moment distribution in a beam is parabolic, as in Figure 5–9, the reduction in stress from midspan to supports is also approximately parabolic. At midspan the steel stress at ultimate load equals the yield stress f_y and a development length of 0.5ℓ is available to anchor the reinforcement. At a point 0.355ℓ from midspan, the steel stress is $0.5f_y$ and a development length of only 0.145ℓ is available to anchor the reinforcement.

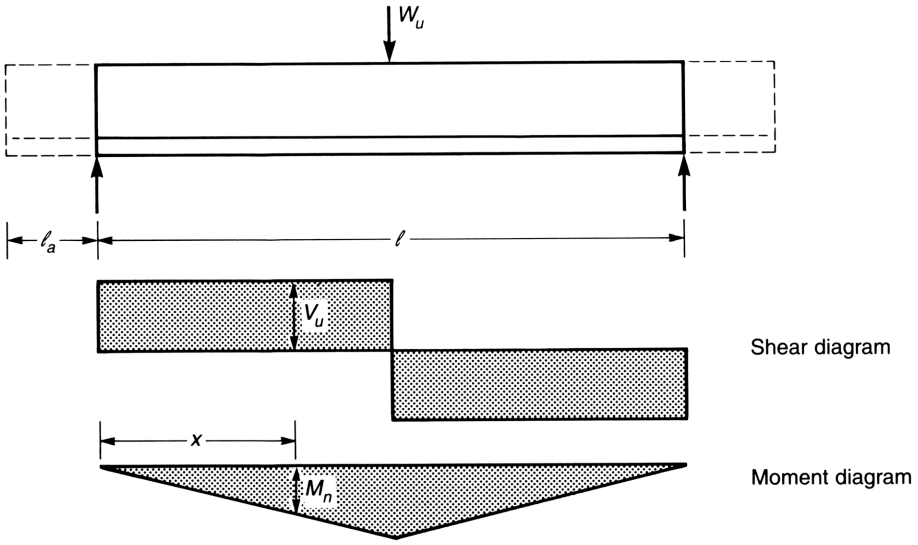


Figure 5–12 Anchorage at supports

The shear and moment diagram for a simply supported beam with a central concentrated load is shown in Figure 5–12. The bending moment at any section equals the area under the shear force diagram and

$$x = M_n/V_u$$

To provide adequate anchorage to the reinforcement

$$x \geq \ell_d$$

and

$$\ell_d \leq M_n/V_u$$

When the reinforcement extends beyond the point of support a distance ℓ_a, this length is also available to provide anchorage and ACI Equation (12–2) is derived as

$$\ell_d \leq M_n/V_u + \ell_a$$

For the case of uniformly distributed loading with a parabolic moment diagram, ACI Equation (12–2) is conservative. At a point of inflection in a beam, a similar limitation is imposed to prevent local bond failure.

ACI Equation (12–2) applies when a beam frames into a girder. An increase of 30 percent in the value of M_n/V_u is allowed when the beam frames into a column, as the ends of the reinforcement are confined by a compression reaction giving the modified expression

$$\ell_d \leq 1.3M_n/V_u + \ell_a$$

This requirement is not imposed at a simple support when the reinforcement terminates with a standard hook beyond the center line of the support.

ACI Sections 12.11 and 7.13 impose requirements on the amount of positive flexural reinforcement which must be carried into the support. To allow for variations in the magnitude of the moments caused by changes in the applied loads, settlement of supports, and lateral loads:

- 33 percent of the reinforcement at midspan in a simply supported beam shall extend into the support a minimum distance of 6 inches.
- 25 percent of the reinforcement at midspan in a continuous member shall extend into the support a minimum distance of 6 inches.

For lateral load resisting systems, reversal of moments may occur at the supports. To allow for this contingency, ACI Section 12.11.2 requires that positive moment reinforcement extended into the support shall develop the full yield strength of the reinforcement in tension. These requirements are illustrated in Figure 5–13.

To provide structural integrity, additional requirements are imposed by ACI Section 7.13.2.2. Perimeter beams must be provided with closed stirrups throughout, with 25 percent of the reinforcement made continuous at supports with a Class A splice. At a discontinuous support, a standard hook must be provided. In other beams, 25 percent of the reinforcement shall be made continuous at supports with a Class A splice unless closed stirrups are provided when the requirement is waived. Similarly at a discontinuous support, a standard hook is required unless closed stirrups are provided.

5.3.3 Negative flexural reinforcement

ACI Sections 12.12 and 7.13 impose requirements on the amount of negative flexural reinforcement which must be carried into the span. To allow for variations in the moments at a support, 33 percent of the reinforcement at the support shall extend beyond the point of inflection a distance of

$$x = d \geq 12d_b \geq \ell_n/16$$

where d = effective depth
d_b = bar diameter
ℓ_n = clear span

These requirements are illustrated in Figure 5–14.

To provide structural integrity, additional requirements are imposed by ACI Section 7.13.2.2. Perimeter beams must be provided with closed stirrups throughout with 16.7 percent of the reinforcement at supports made continuous at midspan with a Class A splice.

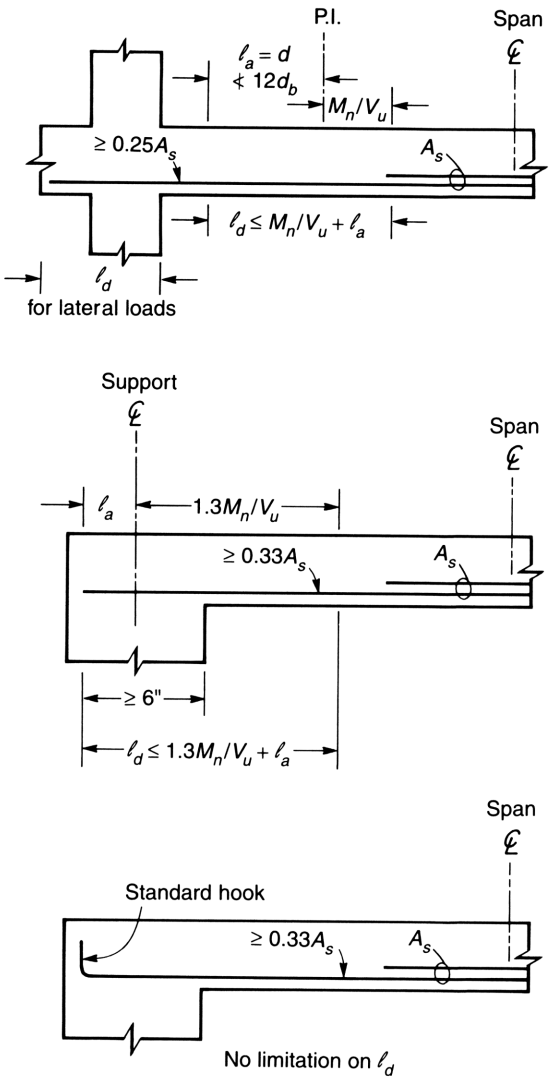


Figure 5–13 Positive flexural reinforcement

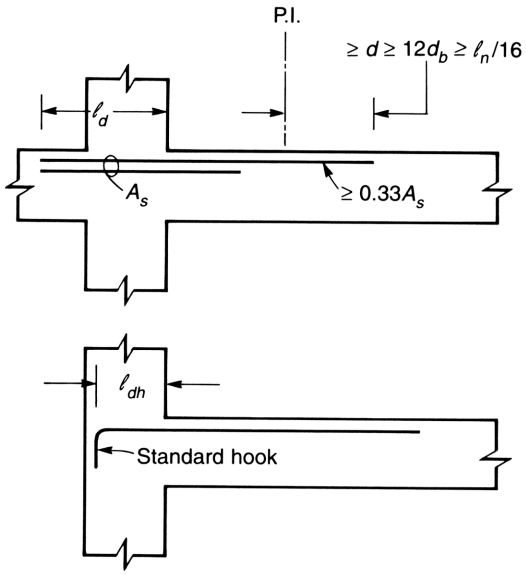


Figure 5–14 Negative flexural reinforcement

Example 5–4 (Curtailment requirements[7])

One span of a continuous reinforced concrete beam subjected to gravity loading is shown in Figure 5–15. An analysis of pattern loading on the beam results in the maximum moment envelope indicated. The factored load w_u is present with all load patterns generating the moment envelope and the factored shear force V_u does not exceed two-thirds the shear capacity of the section everywhere along the span. The concrete strength is 4000 pounds per square inch, all reinforcement is Grade 60, and all bars have a clear spacing of 5 inches and a clear cover of 2 inches. Determine the location of the cut-off points L_3 and L_4 and check the development length requirements at the inflection point L_1.

Solution

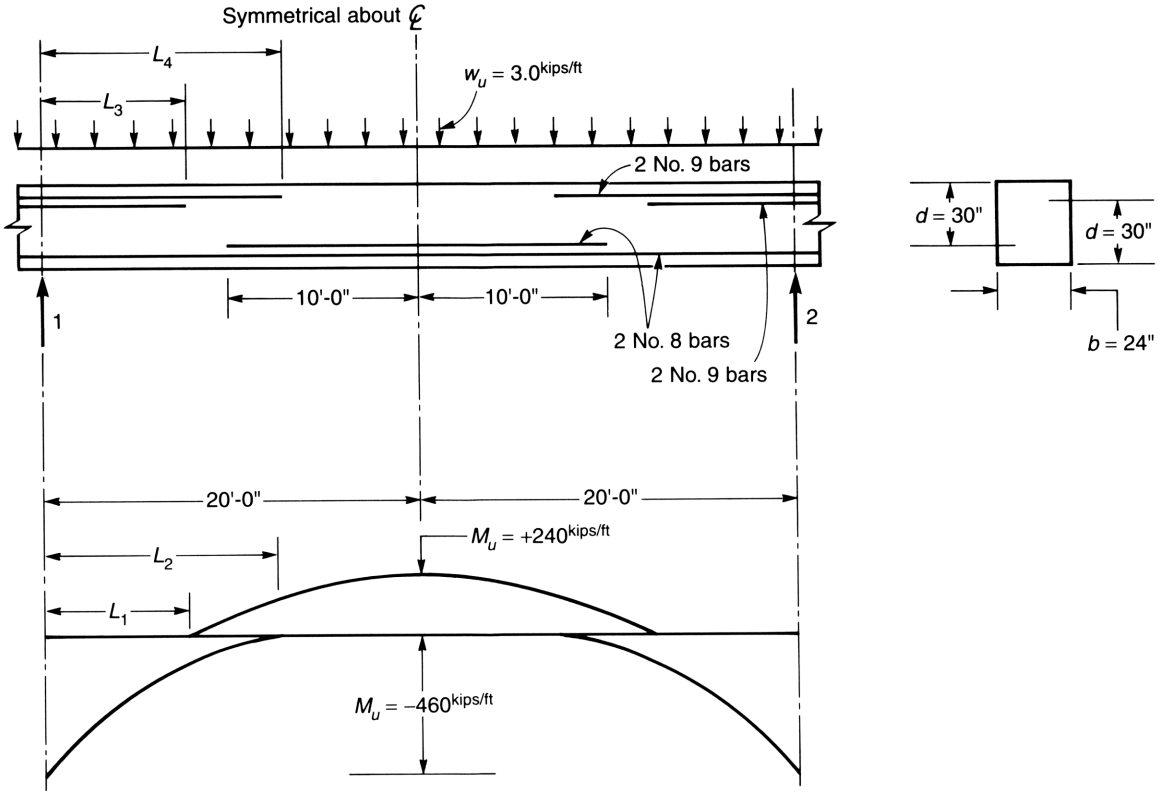


Figure 5–15 Details for Example 5–4

Physical cut–off point for first two top bars, Dimension L_3 :
Two No.9 bars continue beyond the cut–off point giving a reinforcement area of

$$A_s = 2 \text{ square inches}$$

The design flexural strength of the member beyond this point is given by ACI Section 10.2 as

$$\begin{aligned} M_u &= \phi A_s f_y d(1 - 0.59A_s f_y/b_w d f'_c) \\ &= 0.9\times2\times60{,}000\times30[1-0.59\times2\times60{,}000/(24\times30\times3000)]/12{,}000 \\ &= 263 \text{ kip feet} \end{aligned}$$

The theoretical cut–off point at which the applied moment = 263 kip feet is a distance x from support 1 where

$$\begin{aligned} M_u &= M_1 - V_1 x + w_u x^2/2 \\ 263 &= 460 - 60x + 1.5x^2 \\ x &= 3.6 \text{ feet} \end{aligned}$$

The physical cut–off point is located an additional distance beyond the theoretical cut–off point given by the greater of

$12d_b = 12 \times 1.128$
$= 13.5$ inches
or $d = 30$ inches ... governs

The physical cut-off point is a distance L_3 from support 1 where

$L_3 = x + d$
$= 3.6 \times 12 + 30$
$= 73$ inches

In addition, the physical cut-off point may not be less than a development length from the support. Since the clear cover exceeds d_b and the clear spacing exceeds $2d_b$ the development length for a No.9 top bar is obtained from Table 5-1 and ACI Section 12.2.2 as

$\ell_d = 47\alpha\epsilon d_b$
where $\alpha = 1.3$
and $\epsilon = A_{s(reqd)}/A_{s(provided)}$

The flexural reinforcement required is given by ACI Section 10.2 as

$A_{s(reqd)} = 0.85b_w df'_c[1 - (1 - M_u/0.383f'_c b_w d^2)^{0.5}]/f_y$
$= 3.56$ square inches

The tension reinforcement provided consists of four No.9 bars giving a reinforcement area of

$A_{s(provided)} = 4.0$ square inches
and $\epsilon = A_{s(reqd)}/A_{s(provided)}$
$= 3.56/4.0$
$= 0.89$

The required development length is

$\ell_d = 47 \times 1.3 \times 0.89 \times 1.128$
$= 61$ inches
< 73 inches

Hence the physical cut-off point is a distance from support 1 given by

$L_3 = 73$ inches

Physical cut-off point for second two top bars, Dimension L_4 :
The theoretical cut-off point at which the negative moment is zero is a distance L_2 from support 1 where

$0 = M_1 - V_1L_2 + w_uL_2^2/2$
$0 = 460 - 60L_2 + 1.5L_2^2$
$L_2 = 10.3$ feet

The physical cut-off point is located an additional distance beyond the theoretical cut-off point given by the greater of

$12d_b = 12 \times 1.128$
$= 13.5$ inches

$\ell_n/16 < 30$ inches
or $d = 30$ inches ... governs

The physical cut-off point is a distance L_4 from support 1 where

$$L_4 = L_2 + d$$
$$= 10.3 \times 12 + 30$$
$$= 154 \text{ inches}$$

In addition, the physical cut-off point may not be less than a development length from the theoretical cut-off point of the first two curtailed bars. Since the clear cover exceeds d_b and the clear spacing exceeds $2d_b$ the development length for a No.9 top bar is obtained from Table 5-1 and ACI Section 12.2.2 as

$$\ell_d = 47\alpha\epsilon d_b$$
where $\alpha = 1.3$
and $\epsilon = A_{s(reqd)}/A_{s(provided)}$
$$= 1.0$$

The required development length is

$$\ell_d = 47 \times 1.3 \times 1.0 \times 1.128$$
$$= 69 \text{ inches}$$

Hence the physical cut-off point may not be less than a distance from support 1 given by

$$L_4 = x + \ell_d$$
$$= 3.6 \times 12 + 69$$
$$= 112 \text{ inches}$$
$$< 154 \text{ inches}$$

Hence the physical cut-off point is a distance from support 1 given by

$$L_4 = 154 \text{ inches}$$

Local bond requirements at the inflection point L_1

The point of inflection, at which the positive moment is zero, is located a distance y from midspan given by

$$0 = M_u - w_u y^2/2$$
$$0 = 240 - 1.5y^2$$
$$y = 12.65 \text{ feet}$$
$$L_1 = 20 - y$$
$$= 7.35 \text{ feet}$$

Two No.8 bars are located in the section at the point of inflection giving a reinforcement area

$$A_s = 1.58 \text{ square inches}$$

The nominal flexural strength of the member at the point of inflection is given by ACI Section 10.2 as

$$M_n = A_s f_y d(1 - 0.59A_s f_y/b_w d f'_c)$$
$$= 232 \text{ kip feet}$$

The applied shear force at the point of inflection is given by

$$V_u = V_1 y/0.5\ell$$
$$= 60 \times 12.65/20$$
$$= 38 \text{ kips}$$

Since the clear cover exceeds d_b and the clear spacing exceeds $2d_b$ the development length for the No.8 bottom bars is obtained from Table 5–1 and ACI Section 12.2.2 as

$$\ell_d = 47\alpha\epsilon d_b$$

where $\alpha = 1.0$

and $\epsilon = A_{s(reqd)}/A_{s(provided)}$

$$= 1.0$$

and $\ell_d = 47 \times 1.0 \times 1.0 \times 1.0$

$$= 47 \text{ inches}$$

To ensure satisfactory local bond conditions, ACI Section 12.11.3 specifies that

$$\ell_d \leq M_n/V_u + \ell_a$$

where ℓ_a is given by the greater of

$$12d_b = 12 \times 1.0$$
$$= 12 \text{ inches}$$

or $d = 30$ inches ... governs

and $M_n/V_u + \ell_a = 232 \times 12/38 + 12$

$$= 103 \text{ inches}$$
$$\geq \ell_d \text{ ... satisfactory}$$

5.4 REINFORCEMENT SPLICES

5.4.1 Splice lengths for tension reinforcement

Lap splicing of reinforcing bars is used as a means of transferring, by bond, the force from one bar to another. This becomes necessary when commercially available lengths of bar are too short or when short lengths are used for convenience in construction.

As shown in Figure 5–16, the bars may be in contact and lightly wired together, or may be spaced apart. In accordance with ACI Section 12.14.2.3 the bars in noncontact splices shall not be spaced farther apart than one-fifth the lap length, nor six inches, so as not to create an unreinforced section in the member. In either case, the bond strength of a bar in a tension lap splice is equal to that of a single bar provided that the same anchorage length and confinement conditions are provided.

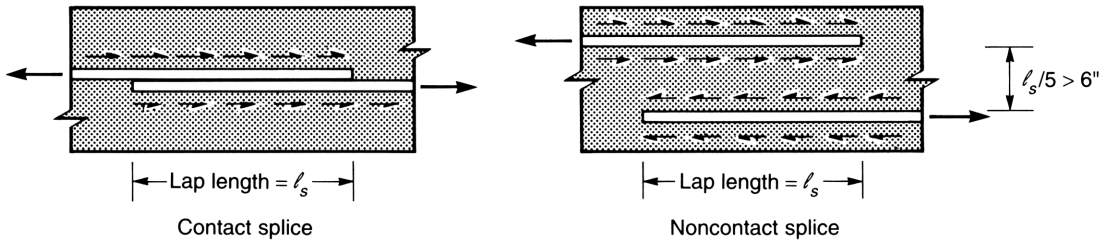


Figure 5–16 Lap splices

The requirements for reinforcement splices are covered in ACI Sections 12.14 to 12.19. In accordance with ACI Section 12.14, lap splices shall not be used for bars larger than No.11. For bundled bars, the lap length required for the individual bars must be increased twenty percent for a three–bar bundle and thirty three percent for a four–bar bundle and the splices must be staggered.

In accordance with Section 12.15, the required length of lap for tension reinforcement, which may not be less than 12 inches, depends on the splice classification and is given by:

- Class A splice, lap length = $1.0\ell_d$
- Class B splice, lap length = $1.3\ell_d$

A Class A splice may only be used when both of the following conditions are satisfied:

- The reinforcement area provided at the splice is not less than twice that required by analysis over the entire length of the splice.
- Not more than half of the total reinforcement may be spliced within the required lap length. This may be achieved by staggering alternate bar splices at least a lap length.

The purpose of the two–level classification is to encourage designers to stagger splices at sections with low applied moments so as to reduce the tendency to cracking.

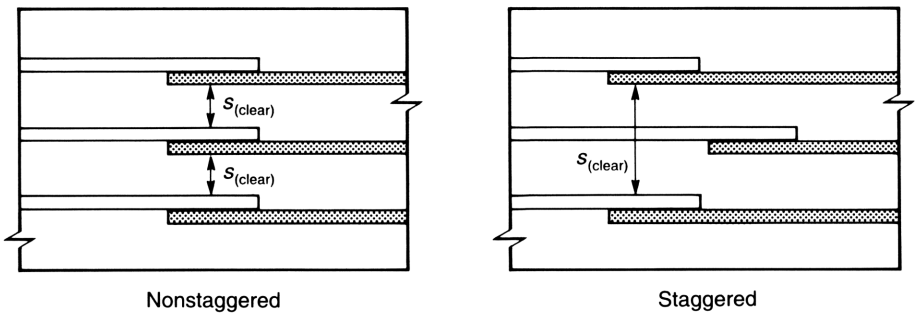


Figure 5–17 Clear spacing of lap splices in slabs

In determining the lap length, all applicable modifiers must be used in calculating the development length, with the exception of the modifier for excess reinforcement since the splice classification already reflects any excess reinforcement at the splice location. The definition of clear spacing given in ACI Section R12.15.1 is illustrated in Figure 5–17 for lap splices in slabs and walls and in Figure 5–18 for lap splices in beams and columns.

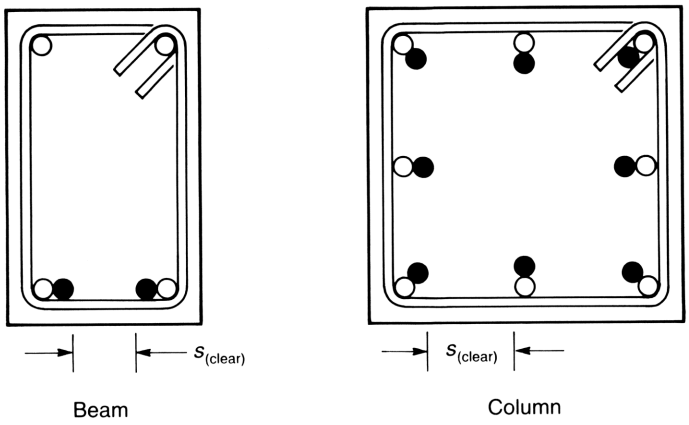


Figure 5–18 Lap splices in beams and columns

Example 5–5 (Lap length and curtailment requirements)
The reinforcement details of a cantilever retaining wall are shown in Figure 5–19. The dowel bars are fully stressed at the base of the wall and the factored moment reduces to half of its maximum value at a height of 5 feet above the base. The factored shear force V_u does not exceed two-thirds the shear capacity of the section everywhere along the wall. The concrete strength is 3000 pounds per square inch and all reinforcement is Grade 60. Determine the location of the cut-off points L_1 and L_2.

Solution
The relevant cover dimensions at the base of the wall are obtained from Figure 5–19 and the distance from center of bar to face of wall is

$$\begin{aligned} c_b &= 3.0 + d_b/2 \\ &= 3.0 + 1.0/2 \\ &= 3.5 \text{ inches} \end{aligned}$$

The center-to-center spacing of the bars being developed is

$$\begin{aligned} 2c_s &= s_{(clear)} + d_b \\ &= 6 - 2d_b + d_b \\ &= 6 - 2 + 1 \\ &= 5 \text{ inches} \\ c_s &= 2.5 \text{ inches} \\ &< c_b \end{aligned}$$

Hence $c = 2.5$ inches, and splitting between the bars controls

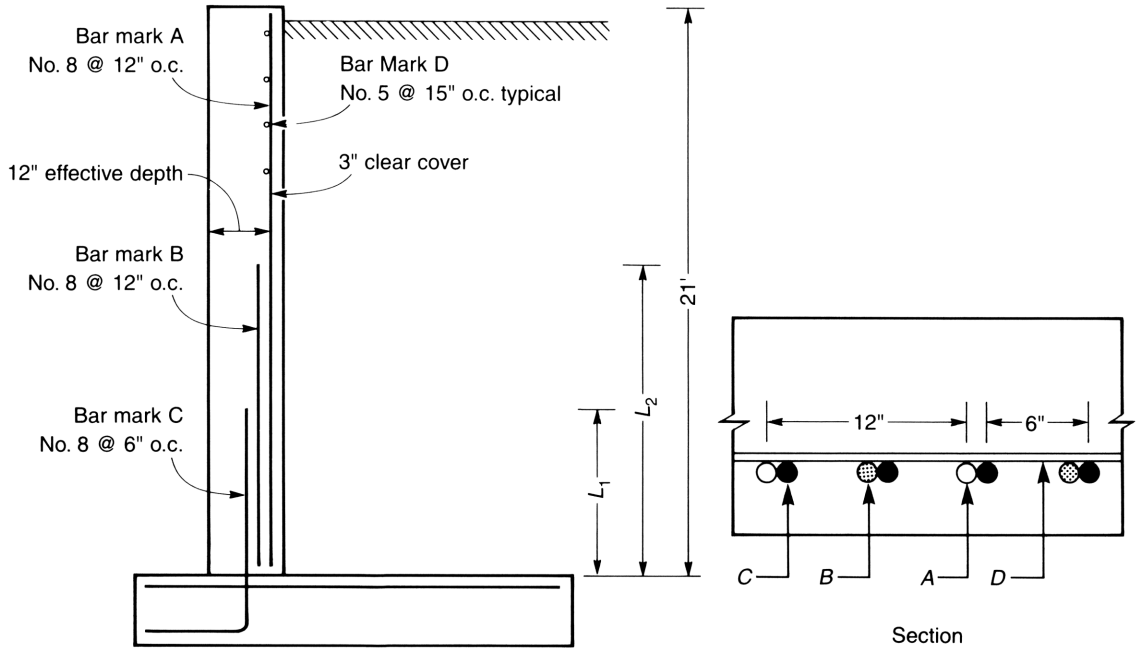


Figure 5–19 Details for Example 5–5

The transverse reinforcement index is

$$K_{tr} = A_{tr}f_{yt}/1500sn$$
$$= 0 \text{ ... since the transverse reinforcement is inside bars being developed}$$

and $(c + K_{tr})/d_b = (2.5 + 0)/1.0$
$$= 2.50 \text{ ... satisfactory}$$

Then $\ell_d/d_b = 0.075f_y\alpha\beta\gamma\epsilon\lambda/[(f'_c)^{0.5}(c + K_{tr})/d_b]$

where α = reinforcement location factor
$= 1.0$

β = coating factor
$= 1.0$ for uncoated reinforcement

γ = reinforcement size factor
$= 1.0$ for No.8 bars

ϵ = excess reinforcement factor
$= 1.0$

λ = lightweight aggregate factor
$= 1.0$ for normal weight concrete

$(c + K_{tr})/d_b = 2.50$

and $\ell_d/d_b = 0.075 \times 60{,}000/2.50(3000)^{0.5}$
$$= 33$$

$\ell_d = 33 \times 1.0$
$= 33$ inches

Since all the reinforcement is spliced at the base of the wall, a class B splice is required and the lap length is given by

$$L_1 = 1.3 \times \ell_d$$
$$= 43 \text{ inches}$$

Five feet above the base, the factored moment is half that at the base and this is the theoretical cut-off point for half the reinforcement. The physical cut-off point is located an additional distance beyond the theoretical cut-off point given by the greater of

$$12d_b = 12 \times 1.0$$
$$= 12 \text{ inches}$$

or

$$d = 12 \text{ inches}$$

Hence, the physical cut-off point is a distance L_2 from the base where

$$L_2 = 5 \times 12 + d$$
$$= 60 + 12$$
$$= 72 \text{ inches}$$

5.4.2 Splice lengths for compression reinforcement

Research has indicated that end bearing as well as length of a splice contributes to the capacity of a compression splice. For this reason, ACI Section 12.16.1 gives two values for the compression lap splice length

$$\ell_s = 0.0005 f_y d_b \ldots f_y \leq 60{,}000 \text{ pounds per square inch}$$

or

$$= (0.0009 f_y - 24) d_b \ldots f_y > 60{,}000 \text{ pounds per square inch}$$

For concrete strengths less than 3000 pounds per square inch, the length of the lap is increased by one-third.

When bars of different sizes are lap spliced, the lap length is given by the larger of

- The development length of the larger bar.
- The splice length of the smaller bar.

The length of the lap splice may be reduced by 17 percent when the bars are enclosed by ties with an effective area of

$$A_e = 0.0015hs$$

where

h = dimension of column perpendicular to the tie legs

s = spacing of ties

The length of the lap splice may be reduced by 25 percent when the bars are enclosed by a spiral conforming to ACI Sections 7.10.4 and 10.9.3. The lap length may not be less than 12 inches.

In columns subject to flexure and axial loads, tensile stresses may occur on one face of the column. When the tensile stress due to factored loads exceeds $0.5f_y$ the lap provided shall be a Class B tension lap splice. When the tensile stress due to factored loads does not exceed $0.5f_y$

a Class B tension lap splice. When the tensile stress due to factored loads does not exceed $0.5f_y$ a class A lap is adequate if half or less of the bars are spliced and alternate bar splices are staggered by ℓ_d otherwise the lap provided shall be a Class B tension lap splice.

Example 5–6 (Splice lengths for compression reinforcement)
The design details of a reinforced concrete column and pile cap are shown in Figure 5–20. The 18 inch square column has a specified compressive strength of 4000 pounds per square inch and is reinforced with 8 No.11, Grade 60 bars which are fully stressed. The 5 foot square pile cap has a specified compressive strength of 3000 pounds per square inch. Four No.6, Grade 60 dowel bars are provided at the interface. Determine the required length of the dowels in the pile cap and in the column.

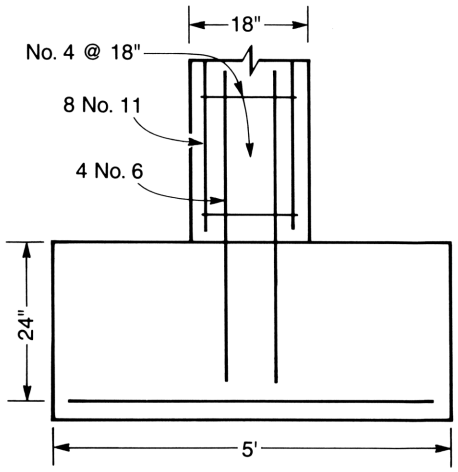


Figure 5–20 Details for Example 5–6

Solution
The basic development length of a deformed bar in compression is given by ACI Section 12.3.2 as

$$\ell_{db} = 0.02d_b f_y/(f'_c)^{0.5}$$
$$\geq 0.0003d_b f_y$$

The governing value for No.6 dowels in 3000 pounds per square inch concrete is obtained from Table 5–2 as

$$\ell_{db} = 16.4 \text{ inches.}$$

The minimum area of dowels is given by ACI Section 15.8.2.1 as

$$A_{(reqd)} = 0.005A_{(col)}$$
$$= 0.005 \times 18 \times 18$$
$$= 1.62 \text{ square inches}$$

The area provided is

$$A_{(provided)} = 1.76 \text{ square inches ... satisfactory}$$

The excess reinforcement modifier is

$$\epsilon = A_{s(reqd)}/A_{s(provided)}$$
$$= 1.62/1.76$$
$$= 0.92$$

No confining reinforcement is provided in the pile cap and the actual development length is

$$\ell_d = 0.92\ell_{db}$$
$$= 0.92 \times 16.4$$
$$= 15 \text{ inches.}$$

In the column, as specified in ACI Section 12.16.2, the lap splice length is computed as the larger of the development length of the larger bar, or the splice length of the smaller bar. For the No.11 column bars in 4000 pounds per square inch concrete, the basic development length is obtained from Table 5–2 as

$$\ell_{db} = 26.8 \text{ inches.}$$

As specified in ACI Section 12.3.3.2, no modification is applicable for the ties provided, and the actual development length is

$$\ell_d = \ell_{db}$$
$$= 27 \text{ inches}$$

The splice length of the No.6 bars may be reduced by 17 percent when the bars are enclosed by ties with an effective area of

$$A_e = 0.0015hs$$

where h = dimension of column perpendicular to the tie legs

s = spacing of ties

and

$$A_e = 0.0015 \times 18 \times 18$$
$$= 0.49 \text{ square inches}$$

The actual area provided is

$$A_{(provided)} = 2 \times 0.44$$
$$= 0.88 \text{ square inches}$$
$$> A_e$$

For the No.6, Grade 60 dowel bars, the required splice length is

$$\ell_s = 0.83 \times 0.0005d_bf_y$$
$$= 0.83 \times 0.0005 \times 0.75 \times 60{,}000$$
$$= 19 \text{ inches}$$
$$< \ell_d$$

Hence, the required length of the dowel bars in the column is

$$\ell_d = 27 \text{ inches}$$

5.4.3 Welded splices and mechanical connections

Welded splices and mechanical connections are permitted in accordance with ACI Sections 12.14.3, 12.15.3, 12.16.3, and 12.17.3. A welded splice or mechanical connection is required to develop at least 125 percent of the specified yield strength of the bar. This requirement is waived for bars not exceeding No.5 in size if twice the required area of reinforcement is provided and the splices are staggered at least 24 inches

Reinforcing bars complying with ASTM A706 are manufactured with a maximum carbon equivalent of 0.55 percent and may be welded without supplementary requirements. Reinforcing bars complying with ASTM A615 have no limitation on carbon equivalent and require additional preheating for the welding process.

5.5 REINFORCEMENT DETAILING

5.5.1 Empirical detailing procedures

By making the simplifying assumptions that a beam is a member of a system of approximately equal spans which are loaded with approximately equal uniform loads, empirical detailing procedures have been established[9,10,11] and are illustrated in Figure 5–21.

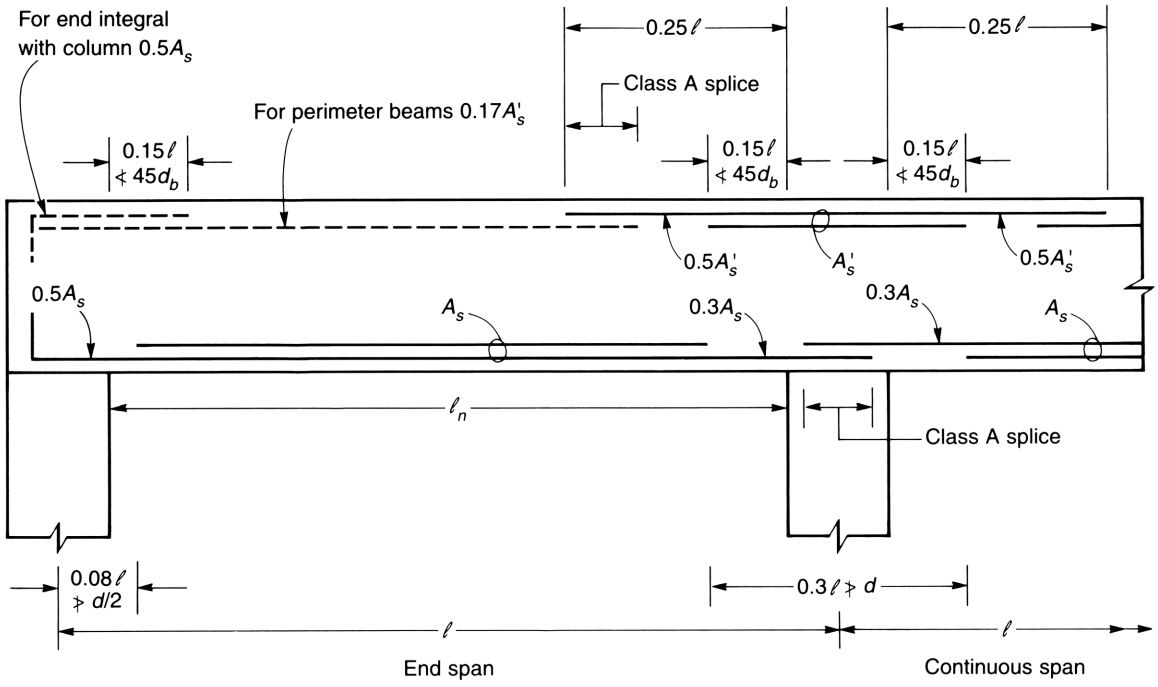


Figure 5–21 Empirical detailing procedure

5.6 SUPPLEMENTARY PROBLEMS

Problem 5–1 (Development length for a straight bar in tension)
The reinforced concrete beam shown in Figure 5–22 is reinforced with Grade 60 bars in normal weight concrete and has a compressive strength of 3000 pounds per square inch. The beam has an overall depth of 20 inches and a width of 10 inches. The beam supports a negative moment which requires 2.1 square inches of tension reinforcement. Determine the required development length of the No.8 bars.

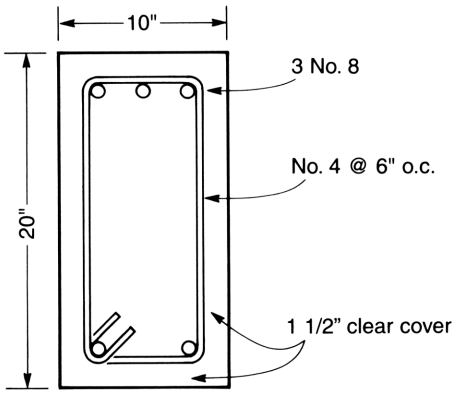


Figure 5–22 Details for Problem 5–1

Problem 5–2 (Development length for a hooked bar in tension)
The reinforced concrete slab shown in Figure 5–23 is reinforced with Grade 60 epoxy-coated bars in normal weight concrete and has a compressive strength of 4000 pounds per square inch. The slab cantilevers from a concrete wall with a compressive strength of 4000 pounds per square inch. The slab reinforcement is fully stressed at the wall interface. Determine the required wall width if a standard hook is provided on the end of the No.6 bars.

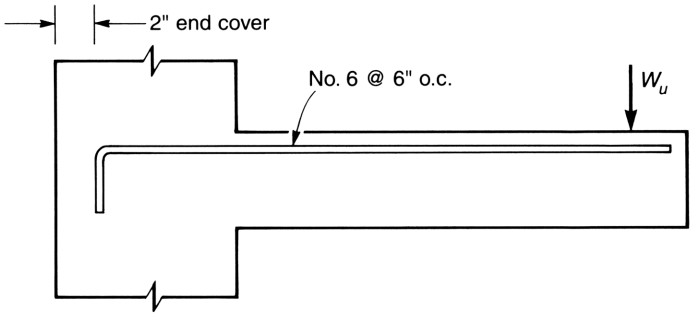


Figure 5–23 Details for problem 5–2

Problem 5–3 (Development length in lightweight aggregate concrete for a straight bar)
An 8 inch thick lightweight aggregate concrete slab is reinforced with Grade 60 bars has a compressive strength of 3000 pounds per square inch. The reinforcement consists of No.6 bars at 8 inch centers with a clear cover of ¾ inch. The required area of reinforcement is 0.6 square inches per foot. Determine the required development length.

Problem 5–4 (Development length for a straight bar in compression)
The reinforced concrete beam shown in Figure 5–24 is reinforced with Grade 60 bars in normal weight concrete and has a compressive strength of 3000 pounds per square inch. The beam has an overall depth of 20 inches and a width of 12 inches. The beam supports a positive moment which requires 1.4 square inches of tension reinforcement. Determine the required development length of the No.8 bars.

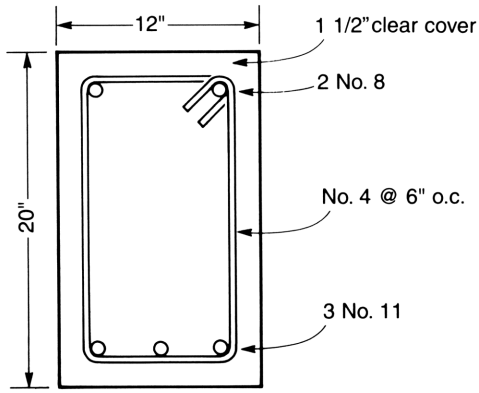


Figure 5–24 Details for Problem 5–4

Problem 5–5 (Development length for a bundled bar in tension)
A reinforced concrete beam is reinforced with bundles of 3 No.7 Grade 60 bars in normal weight concrete with a compressive strength of 3000 pounds per square inch. The bundled bars are fully stressed in tension and are provided with a clear cover of 2 × the equivalent diameter and a clear spacing of 3 × the equivalent diameter. The bars are fully stressed in tension. Determine the required development length of the bundled bars.

Problem 5–6 (Curtailment of a straight bar in tension)
A reinforced concrete beam is reinforced with 6 No.8, Grade 60 bars in normal weight concrete and has a compressive strength of 3000 pounds per square inch. The beam supports a positive moment which requires 4.3 square inches of tension reinforcement at midspan. At a distance of 4 feet from midspan, the bending moment has reduced to a value which fully stresses only four bars. Determine the distance from midspan at which two bars may be curtailed.

Problem 5–7 (Splice length for a straight bar in compression)
The two No.8 bars shown in Figure 5–24 are both lap spliced at the same location. Determine the required splice length of the No.8 bars.

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6

Design of reinforced concrete columns

6.1 GENERAL PRINCIPLES

6.1.1 Column instability

A short column, such as a 6 inches diameter by 12 inches long concrete cylinder, which is loaded axially, fails by crushing at the squash load given by

$$P_u = Af'_c$$

where A = cross sectional area of the column
and f'_c = concrete compressive strength

A long, pin-ended, elastic, column, which is loaded axially, fails by buckling at the Euler critical load given by

$$P_c = \pi^2 EI/\ell^2$$

and the stress in the column is given by

$$\begin{aligned} f_c &= P_c/A \\ &= \pi^2 EI/A\ell^2 \\ &= \pi^2 Er^2/\ell^2 \\ &= \pi^2 E/(\ell/r)^2 \end{aligned}$$

where E = modulus of elasticity
I = moment of inertia
ℓ = length of the column
r = radius of gyration
and ℓ/r = slenderness ratio

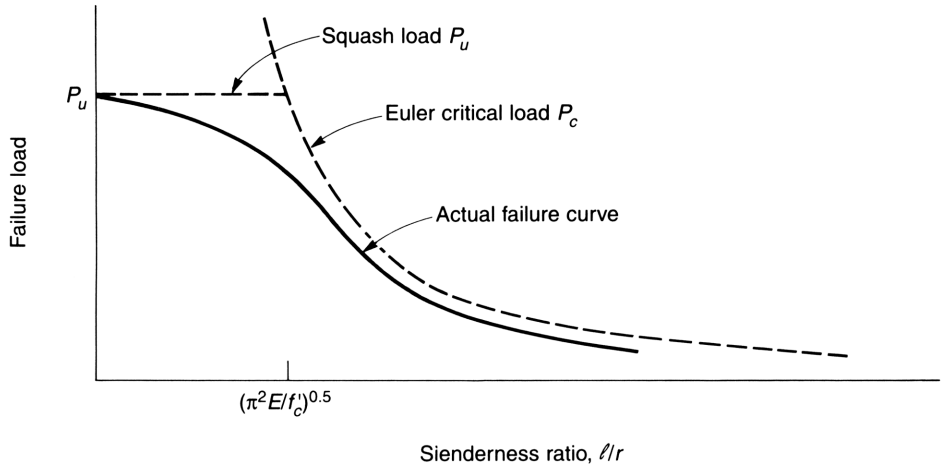


Figure 6–1 Failure load and slenderness ratio

At the theoretical transition point between a crushing failure and a buckling failure a column fails at the stress

$$f'_c = P_u/A = \pi^2 E/(\ell/r)^2$$

and

$$\ell/r = (\pi^2 E/f'_c)^{0.5}$$

For values of the slenderness ratio less than $(\pi^2 E/f'_c)^{0.5}$ the theoretical failure load is the squash load P_u. For values of the slenderness ratio greater than $(\pi^2 E/f'_c)^{0.5}$ the theoretical failure load conforms to the Euler critical load P_c. In practice, a concrete column is non-elastic, has initial imperfections which cause eccentricity of the loading, and exhibits creep effects which produce deviations from the theoretical behavior as shown in Figure 6–1.

6.1.2 Effective column length

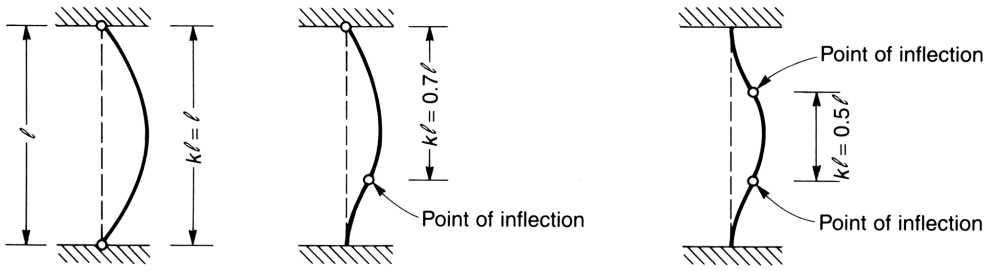


Figure 6–2 Effective lengths for a non-sway column

The Euler expression applies only to ideal pin-ended columns and neglects other end conditions which may influence the column instability. These conditions are covered by introducing the

concept of an effective length factor k which is used to provide an effective length $k\ell$ equivalent to the Euler pin-ended length. For a non-sway column, the effective length values are shown in Figure 6-2 and for a sway column the effective length values are shown in Figure 6-3.

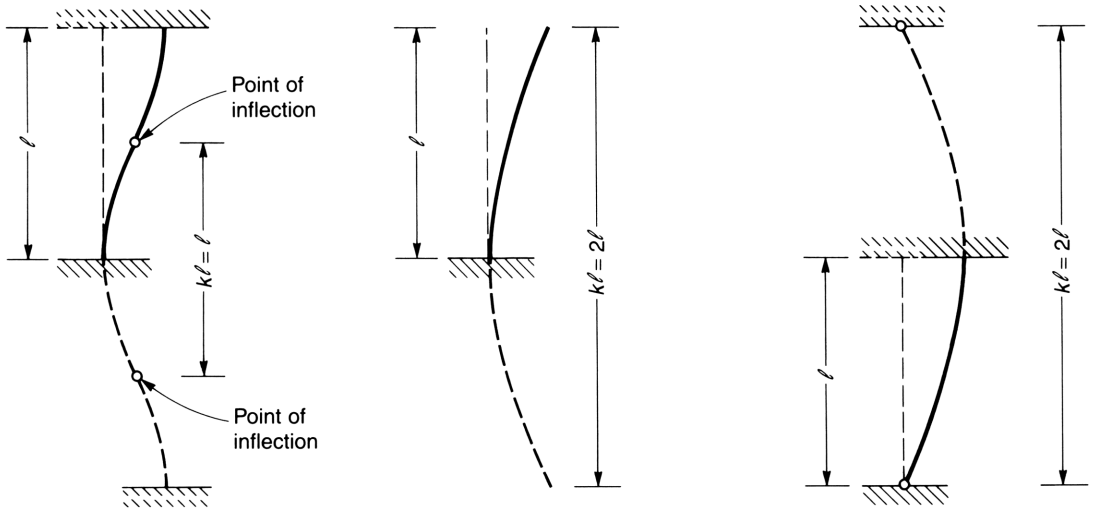


Figure 6-3 Effective lengths for a sway column

For a practical situation in a framed structure, ideal fixed or pinned-end conditions are never realized since the ends of a column frame into beams which are neither completely rigid nor completely flexible. As shown in Figure 6-4, the effective length factor for a column in a non-sway frame is always less than 1.0 and in a sway frame always exceeds 1.0. As specified in ACI Section 10.12.1 the effective length factor for a column in a non-sway frame shall be taken as 1.0 unless analysis shows that a lower value is justified. As specified in ACI Section 10.12.1 the effective length factor for a column in a sway frame shall be greater than 1.0.

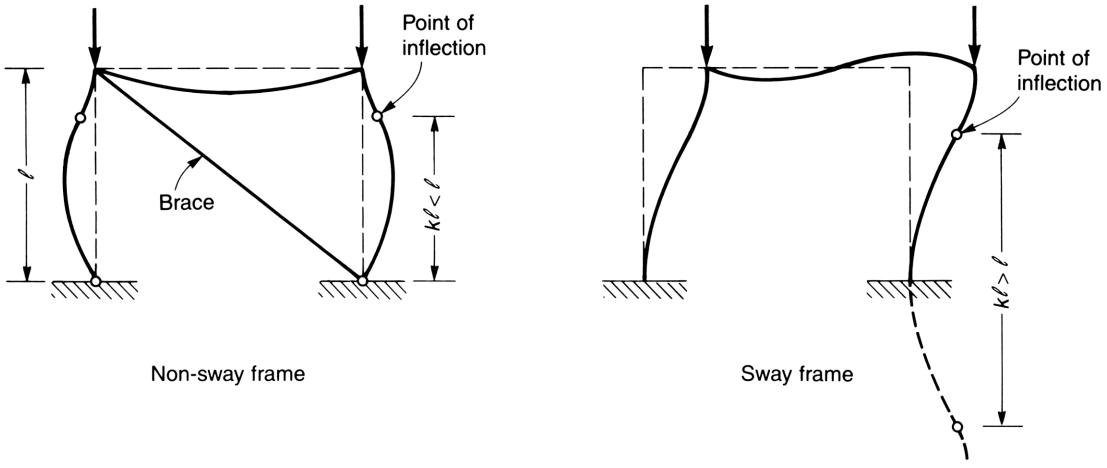


Figure 6-4 Effective lengths in a sway and non-sway frame

6.1.3 Slenderness effects

Reinforced concrete columns may be classified as either short columns or long columns. In the design of long columns, slenderness effects must be considered and the secondary moments, due to the P-delta effects, added to the primary moments. The P-delta effects in a given column are caused by the eccentricity of the gravity load above that column, which produces secondary moments augmenting the primary moments in the story. As shown in Figure 6-5, the primary moment on a column in the sway frame, in the second story, caused by the lateral loads acting on the frame is

$$M_p = F_2 h_{s2}/2$$

and the secondary moment is

$$M_s = P_2 \Delta_2$$

The secondary moment in a non-sway frame, caused by the displacement of the column center line, is given by

$$M_b = P_1 \Delta_1$$

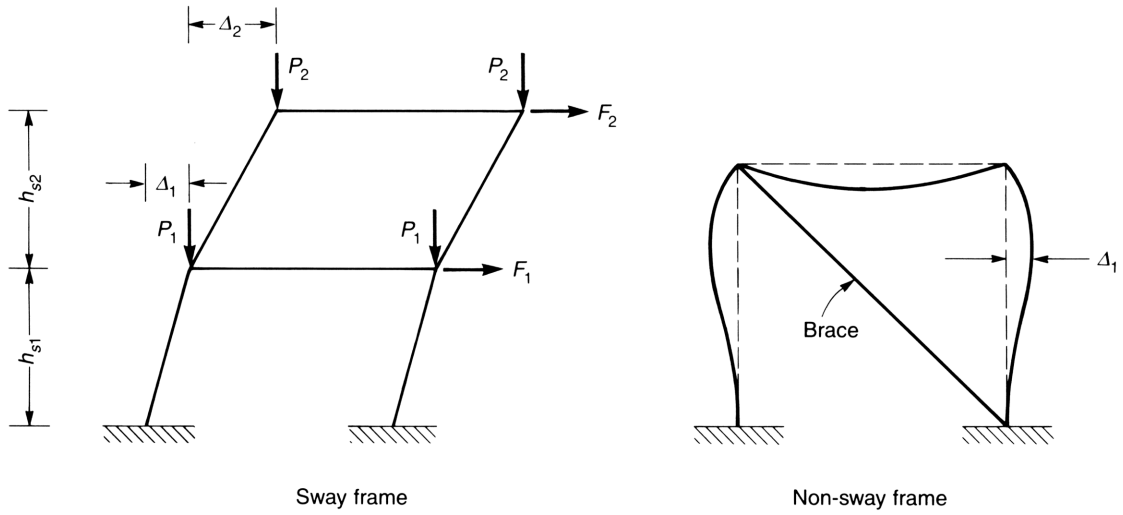


Figure 6-5 P-delta effects

Classification as a short column depends on the slenderness ratio of the column which is given by $k\ell_u/r$ where

k = effective length factor defined in ACI Sections 10.12.1 and 10.13.1
ℓ_u = unsupported column height defined in ACI Section 10.11.3
$k\ell_u$ = effective length of the column
r = radius of gyration of the column, defined in ACI Section 10.11.2 as:
r = 0.3 × rectangular column dimension, in the direction stability is being considered
r = 0.25 × circular column diameter

The unsupported column height is the clear distance between floor slabs, between floor slab and beams, or between floor slab and column capital as shown in Figure 6-6.

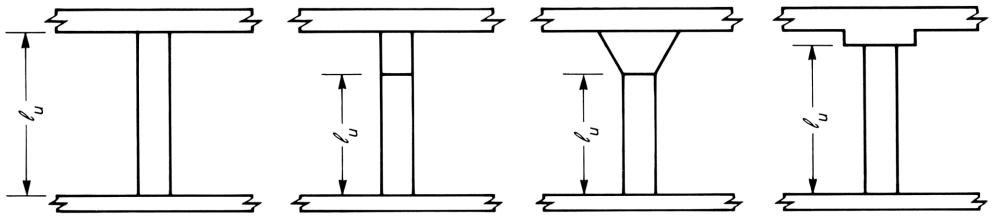


Figure 6-6 Unsupported column height

The value of the effective length factor depends on the restraint conditions at each end of the column[1]. For a column forming part of a frame with rigid joints, ACI Section R10.12.1 presents alignment charts for determining the effective length for the two conditions of sidesway prevented and sidesway permitted, and these are illustrated in Figure 6-7.

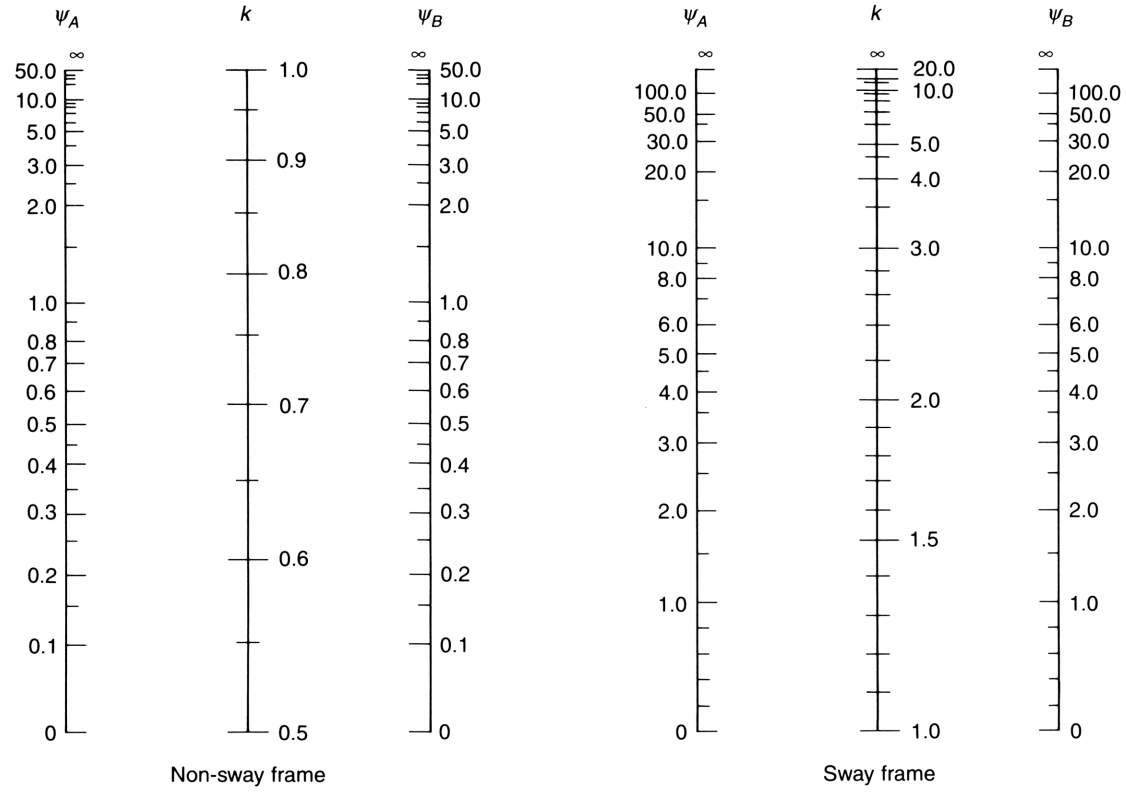


Figure 6-7 Alignment charts for effective length factor

To utilize the alignment charts, the stiffness ratio at the two ends of the column under consideration must be determined and this is defined by

$$\psi = \Sigma(E_c I_c/\ell_c)/\Sigma(E_b I_b/\ell_b)$$

where $\Sigma(E_c I_c/\ell_c)$ = the sum of the EI/ℓ values for all columns meeting at the joint

$\Sigma(E_b I_b/\ell_b)$ = the sum of the EI/ℓ values for all beams meeting at the joint

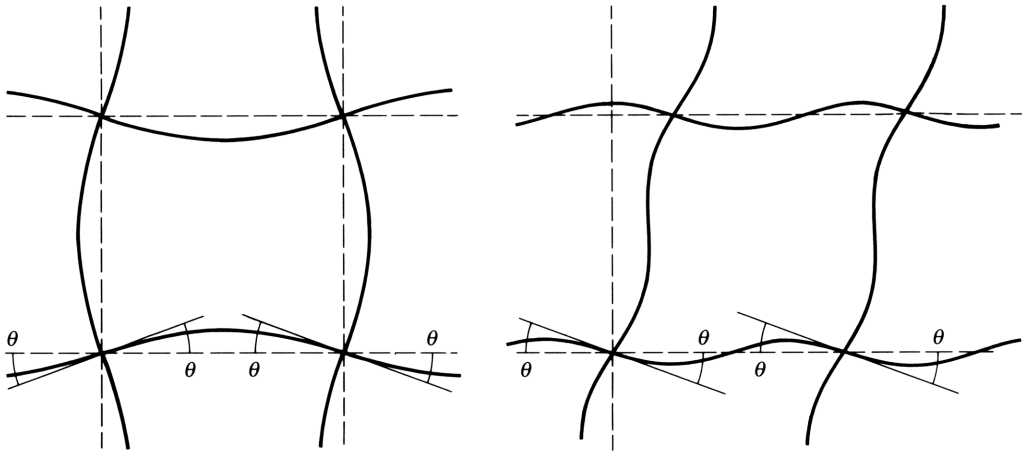


Figure 6–8 Frame displacement modes

The alignment charts are based on the critical displacement modes shown in Figure 6–8 and on the following assumptions

- All joints are rigid.
- All members are prismatic.
- All members act in a linear-elastic manner.
- All columns buckle simultaneously.
- For the non-sway frame, all columns are bent in single curvature and all beams are bent in single curvature and have a stiffness value of 2EI/L as shown in Table 6–1.
- For the sway frame, all columns are bent in double curvature and all beams are bent in double curvature and have a stiffness value of 6EI/L as shown in Table 6–1.

When one end of a beam is pinned, its stiffness value is 3EI/L and when one end of a beam is fixed, its stiffness value is 4EI/L as shown in Table 6–1. Hence, for these two cases, adjustments must be made to the $(E_b I_b/\ell_b)$ values in determining the stiffness ratio ψ since the charts are based on stiffness values for the beams of 2EI/L and 6EI/L for braced and sway frames, respectively. For a non-sway frame, the beam stiffness is multiplied by a factor of 1.5 when its far end is pinned and by a factor of 2 when its far end is fixed. For a sway frame, the beam stiffness is multiplied by a factor of 0.5 when its far end is pinned and by a factor of 0.67 when its far end is fixed.

When one end of a column is pinned, ψ is theoretically infinite but for practical design purposes may be assumed[2] to have a value of 10. When one end of a column is fixed, ψ is theoretically zero but for practical design purposes may be assumed[2] to have a value of 1.0.

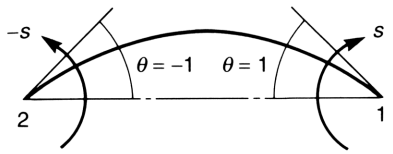
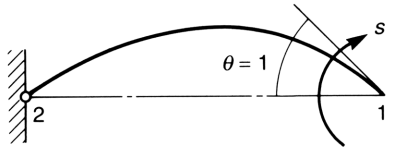
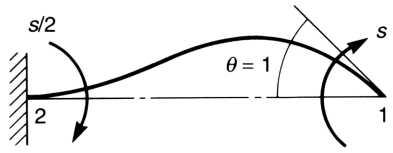
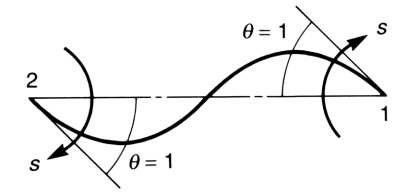
Configuration	Stiffness, s	Description
	2EI/L	Braced frame
	3EI/L	Pinned end
	4EI/L	Fixed end
	6EI/L	Sway frame

Table 6–1 Beam stiffness

Example 6–1 (Effective length determination)

The non-sway frame shown in Figure 6–9 consists of members having identical EI values. Determine the effective lengths of both columns.

Solution

For the pinned connection at joint 5, AISC[2] Section C–C2 specifies a stiffness ratio

$$\psi_1 = 10$$

For the columns connected to joint 2, the sum of the relative stiffness values is

$$\begin{aligned}\Sigma(I_c/L_c) &= I_{25}/L_{25} + I_{24}/L_{24} \\ &= I/L + I/L \\ &= 2I/L\end{aligned}$$

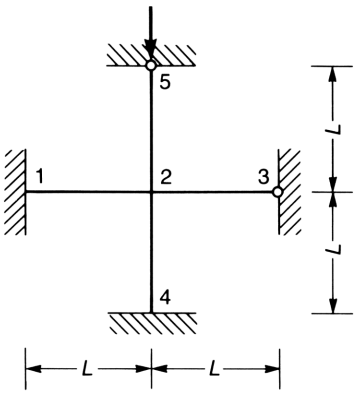


Figure 6–9 Details for Example 6–1

Allowing for the fixed end at joint 1 and the pinned end at joint 3, the sum of the adjusted relative stiffness values for the two beams connected to joint 2 is

$$\begin{aligned}\Sigma(I_b/L_b) &= (4/2)I_{21}/L_{21} + (3/2)I_{23}/L_{23}\\ &= 2I/L + 1.5I/L\\ &= 3.5I/L\end{aligned}$$

Hence, the stiffness ratio at joint 2 is given by

$$\begin{aligned}\psi_2 &= \Sigma(I_c/L_c)/\Sigma I_b/L_b)\\ &= 2/3.5\\ &= 0.57\end{aligned}$$

Hence, from the alignment chart for a non-sway frame, the effective-length factor for column 25 is

$$K_{25} = 0.82$$

For the fixed connection at joint 4, AISC[2] Section C-C2 specifies a stiffness ratio

$$\psi_1 = 1$$

Hence, from the alignment chart for a non-sway frame, the effective-length factor for column 24 is

$$K_{24} = 0.74$$

In determining effective lengths it is necessary to use realistic EI values[3] which allow for the cracking which occurs in flexural members at the factored load. For an overall frame analysis, the EI values adopted should approach the mean values for the individual members. Based on the results of frame tests and analyses, and after incorporating a suitable strength reduction factor, ACI Section R10.11.1 specifies the following values for the moment of inertia:

- For beams and cracked walls

$$I = 0.35I_g$$

- For columns and uncracked walls

$$I = 0.70I_g$$

- For flat plates and flat slabs

$$I = 0.25I_g$$

where I_g = moment of inertia of the gross cross section of a member about its centroidal axis, neglecting reinforcement

The modulus of elasticity of the concrete is given by ACI Section 8.5.1 as

$$E_c = 33(w_c^3 f'_c)^{0.5}$$
$$= 57{,}000(f'_c)^{0.5} \text{ ... for normal weight concrete}$$

where w_c = unit weight of concrete

For the analysis of service load deflections, the moment of inertia is taken as 1.25 times the value assumed for factored loads and the strength reduction factor is taken as 1.0. This gives values for the moment of inertia 1.43 times the above values.

When sustained loads act on a structure, as is the case with earth pressure, creep effects are allowed for by dividing the moments of inertia by $(1 + \beta_d)$ where β_d is the ratio of factored axial dead load to total factored axial load.

In accordance with ACI Section R10.11.4, a column may be assumed braced by inspection when it is located in a story in which the bracing elements prevent deflections which may affect the column strength. A non-sway frame is further defined in ACI Section 10.11.4.1 as one in which the second-order magnification of sway moments is 5 percent or less. This may be determined from ACI Equation (10-7) which requires that

$$Q = \text{stability index for a story}$$
$$= \Sigma P_u \Delta_o / V_u \ell_c$$
$$\leq 0.05$$

where ΣP_u = total vertical load on a story
V_u = story shear
Δ_o = first-order relative lateral deflection of the top and bottom of the story
ℓ_c = length of a column measured center-to-center of the joints

For a sway frame, slenderness effects may be neglected, in accordance with ACI Section 10.13.2, when the slenderness ratio is given by

$$k\ell_u/r < 22$$

For a non-sway frame, slenderness effects may be neglected in accordance with ACI Equation (10-8) when the slenderness ratio is given by

$$k\ell_u/r < 34 - 12M_1/M_2$$

where M_2 = larger factored moment at end of column
M_1 = smaller factored moment at end of column, negative if the column is bent in double curvature, positive if column is bent in single curvature.

and $M_1/M_2 \geq -0.5$

6.1.4 Reinforcement detailing

Transverse reinforcement, consisting of lateral ties or spirals, shall be provided throughout the height of a column to provide confinement and to prevent buckling of the longitudinal bars. The tie spacing requirements in accordance with ACI Section 7.10.5.2 are:

$$
\begin{aligned}
s_{max} &\leq 16 \times \text{longitudinal bar diameters} \\
&\leq 48 \times \text{tie bar diameters} \\
&\leq \text{least dimension of column}
\end{aligned}
$$

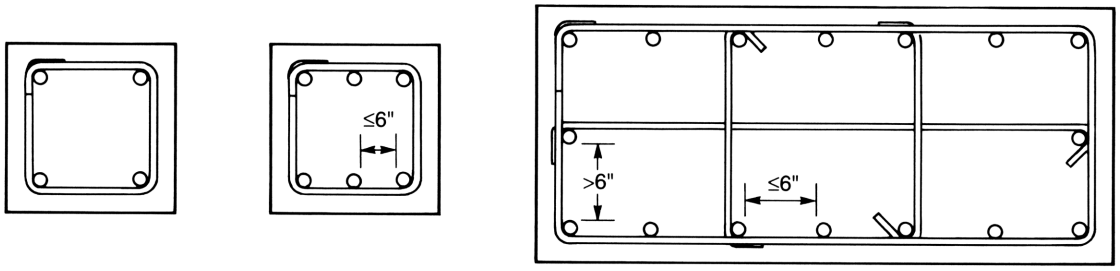


Figure 6–10 Tie details

As specified in ACI Section 7.10.5.3 and shown in Figure 6–10, every corner bar and alternate bar shall have lateral support provided by the corner of a tie or crosstie with no unsupported bar more than 6 inches from a supported bar. At the ends of a column, as shown in Figure 6–11, ties shall be located not more than one-half a tie spacing above the top of footing or slab and not more than one-half a tie spacing below the reinforcement in the slab above. When beams frame into all four faces of a column, the ties may terminate not more than one-half a tie spacing below the bottom reinforcement in the shallowest beam.

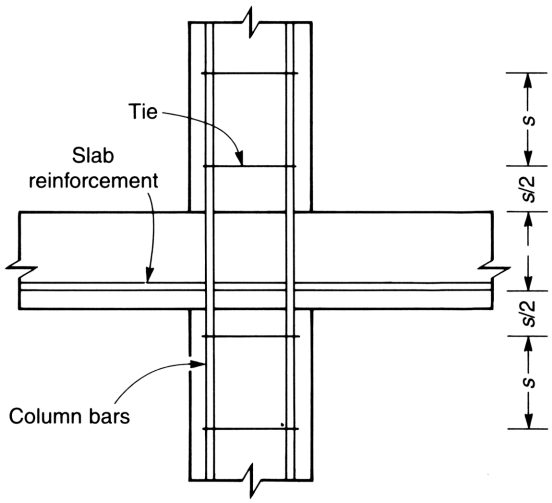


Figure 6–11 Termination of ties

The minimum tie size specified in ACI Section 7.10.5.1 is

No.3 ... longitudinal bars $\leq$ No.10
No.4 ... longitudinal bars $>$ No.10

Longitudinal reinforcement limits are imposed by ACI Section 10.9.1 in order to control creep, reduce steel congestion, and provide for unforeseen flexural stresses. These limits are

$$\rho_g \geq 0.01$$
$$\leq 0.08$$

where ρ_g = ratio of reinforcement area to gross concrete area

At least four longitudinal bars are required for columns with rectangular or circular ties. In accordance with ACI Section 7.6.1, longitudinal bars shall have a clear spacing not less than 1.5 times the bar diameter, but not less than 1.5 inches. The minimum clear cover to tie reinforcement is specified by ACI Section 7.7.1 as 1.5 inches for concrete not exposed to weather or in contact with ground.

The minimum diameter of spiral reinforcement specified in ACI Section 7.10.4 is ⅜ inch for cast-in place construction, and the clear spacing between spirals shall not exceed 3 inches nor be less than 1 inch. Lap splices shall be not less than 48 tie bar diameters with a minimum of 12 inches.

A concrete column with lateral ties fails abruptly when the concrete crushes, causing spalling and buckling of the longitudinal bars between ties. By contrast, a column with adequate spiral reinforcement fails at a higher load, and in a ductile manner, since the spiral confines the concrete core and prevents buckling of the reinforcement. To ensure sufficient spiral reinforcement is provided to produce a ductile failure, ACI Section 10.9.3 specifies a minimum ratio of volume of spiral reinforcement to volume of core (out-to-out of spirals) of

$$\rho_s = 0.45f'_c[(A_g/A_c) - 1]/f_y$$

where A_g = gross area of the section
A_c = area of core measured out-to-out of the spiral

Example 6–2 (Spiral reinforcement)

An 18inch diameter spirally reinforced column has 1½ inches cover to the spiral. The concrete strength is 4000 pounds per square inch and the spiral is Grade 60 reinforcement. Determine the diameter of the spiral reinforcement, and the maximum allowable pitch.

Solution

In accordance with ACI Section 7.10.4.2, the minimum diameter of the spiral reinforcement is

$$d_b = ⅜ \text{ inch.}$$

From ACI Equation (10-6) the minimum spiral reinforcement ratio is

$$\rho_s = 0.45f'_c[(A_g/A_c) - 1]/f_y$$

where A_g = gross area of the section
$= \pi 18^2/4$
$= 254.5$ square inches

A_c = area of core measured out-to-out of the spiral
= $\pi 15^2/4$
= 176.7 square inches

and ρ_s = $0.45 \times 4[(254.5/176.7) - 1]/60$
= 0.0132
= V_s/V_c

where V_s = volume of spiral bar per turn of spiral
= $\pi A_b d_s$
= $\pi \times 0.11(15 - 0.375)$
= 5.05

V_c = volume of concrete core per turn of spiral
= $A_c s$
= 176.7s

and s = $V_s/\rho_s A_c$
= $5.05/(0.0132 \times 176.7)$
= 2.17 inches

Providing a pitch of 2 inches gives a clear spacing of

s_c = $s - d_b$
= 2.00 - 0.375
= 1.63 inches
> 1 inch
< 3 inches ... satisfactory

6.2 Short columns

6.2.1 Short columns with axial load only

For a short column, the axial load carrying capacity is determined from the properties of its cross-section and this is not reduced by second-order effects. The axial load capacity may be determined by reference to Figure 6-12.

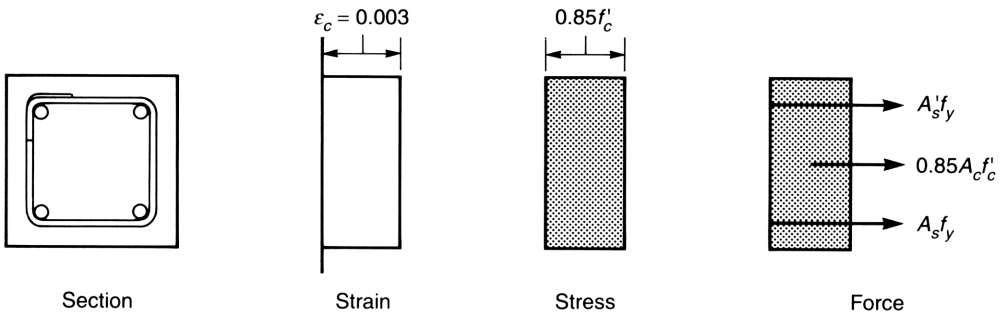


Figure 6-12 Compression in a short column

From experimental data, the concrete fails at a compressive strain of 0.003 after the reinforcement has yielded. Because of creep effects caused by the relative slow rate of loading in an actual structure, the maximum stress in the concrete is limited to 0.85 times the cylinder strength f'_c. The theoretical design axial load strength at zero eccentricity is then given by ACI Section R10.3.5. as

$$\phi P_o = \phi[0.85f'_c(A_g - A_{st}) + A_{st}f_y]$$

where ϕ = strength reduction factor specified in ACI Section 9.3
= 0.75 for compression members with spiral reinforcement
= 0.70 for compression members with lateral ties
f'_c = concrete cylinder strength
f_y = reinforcement yield strength
A_g = gross area of the section
A_{st} = reinforcement area

Spirally reinforced columns are assigned a higher strength reduction factor than tied columns to reflect the improved ductility provided by the spiral.

To account for accidental eccentricity, ACI Section 10.3.5 requires a spirally reinforced column to be designed for a minimum eccentricity of approximately 0.05h which gives a maximum design axial load strength at zero eccentricity, in accordance with ACI Equation (10–1), of

$$\begin{aligned}\phi P_{n(max)} &= 0.85\phi P_o \\ &= 0.85 \times 0.75[0.85f'_c(A_g - A_{st}) + A_{st}f_y]\end{aligned}$$

In the case of a column with lateral ties, a minimum eccentricity of approximately 0.10h is specified which gives a maximum design axial load strength at zero eccentricity, in accordance with ACI Equation (10–2), of

$$\begin{aligned}\phi P_{n(max)} &= 0.80\phi P_o \\ &= 0.80 \times 0.70[0.85f'_c(A_g - A_{st}) + A_{st}f_y]\end{aligned}$$

Example 6–3 (Short column with axial load)

The stub column shown in Figure 6–13 supports a steel column and base plate with the indicated loads. The stub column is 18 inches square and is reinforced with Grade 60 deformed bars. Concrete compressive strength is 3000 pounds per square inch. The stub column is subjected to axial load only and is effectively braced against sidesway by the floor slab. Determine the design axial load strength of the stub column, the minimum allowable reinforcement area, and the required size and spacing of the lateral ties.

Solution

From ACI Section 10.12.1, the effective length factor for a column braced against sidesway is

$$k = 1.0$$

The radius of gyration, in accordance with ACI Section 10.11.2 is

$$r = 0.3h$$

where $h = \text{width of column}$
$= 18 \text{ inches}$
and $r = 0.3 \times 18$
$= 5.4 \text{ inches}$

The slenderness ratio, in accordance with ACI Section 10.11.4 is $k\ell_u/r$

where $\ell_u = \text{unsupported column height}$
$= 30 \text{ inches}$
and $k\ell_u/r = 1.0 \times 30/5.4$
$= 5.6$

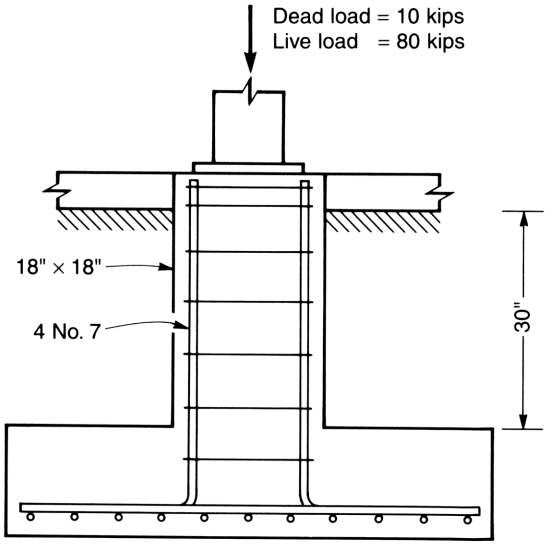


Figure 6–13 Details for Example 6–3

From ACI Section 10.11.4.1, slenderness effects may be neglected when

$$k\ell_u/r < 34 - 12M_1/M_2$$

where $M_2 = \text{larger factored moment at end of column}$
$= 0$
$M_1 = \text{smaller factored moment at end of column, negative if the column is bent in double curvature, positive if the column is bent in single curvature.}$
$= 0$
and $M_1/M_2 \geq -0.5$
then $M_1/M_2 = -0.5 \ldots \text{for double curvature}$
and $34-12M_1/M_2 = 34 + 6$
$= 40$
$> k\ell_u/r$

Hence the column is considered a short column and the design axial load strength, for a column with lateral ties, is given by ACI Equation (10–2) as

$$\phi P_n = 0.80\phi[0.85f'_c(A_g - A_{st}) + f_y A_{st}]$$

where ϕ = strength reduction factor specified in ACI Section 9.3
= 0.70 for compression members with lateral ties
f'_c = concrete cylinder strength
= 3.0 kips per square inch
f_y = reinforcement yield strength
= 60 kips per square inch
A_g = gross area of the section
= 18 × 18
= 324 square inches
A_{st} = reinforcement area
= 2.4 square inches

Then $\phi P_n = 0.8 \times 0.7[0.85 \times 3(324 - 2.4) + 60 \times 2.4]$
= 540 kips

The design axial load strength of the stub column is

$$\phi P_n = 540 \text{ kips}$$

The applied ultimate load is given by ACI Section 9.2.1 as

$$P_u = 1.4P_D + 1.7P_L$$
$$= 1.4 \times 10 + 1.7 \times 80$$
$$= 150 \text{ kips}$$

The ratio of design load strength to applied ultimate load is

$$\phi P_n/P_u = 540/150$$
$$= 3.6$$
$$> 2$$

Hence, from ACI Section 10.8.4 the reduced effective area of the section is

$$A'_g = A_g/2$$
$$= 324/2$$
$$= 162 \text{ square inches}$$

The minimum allowable reinforcement area, in accordance with ACI Section 10.9.1 is

$$A_{s(min)} = 0.01A'_g$$
$$= 0.01 \times 162$$
$$= 1.62 \text{ square inches}$$

The reinforcement area provided is

$$A_s = 2.4 \text{ square inches}$$
$$> A_{s(min)} \ldots \text{ satisfactory}$$

The minimum allowable size of lateral ties for enclosing bars smaller than No.10 is given by ACI Section 7.10.5.1 as No.3.

ACI Section 7.10.5.2 specifies a tie spacing not greater than:

$$\begin{aligned} h &= 18 \text{ inches} \\ 48d_t &= 48 \times 3/8 \\ &= 18 \text{ inches} \\ 16d_b &= 16 \times 7/8 \\ &= 14 \text{ inches} \ldots \text{ governs} \end{aligned}$$

No.3 ties at a pitch of 14 inches are required.

Example 6–4 (Circular column with spiral reinforcement and axial load)

A 20 inch diameter spirally reinforced column with 1½ inches cover to the spiral is reinforced with ten No.9 bars. The concrete strength is 5000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 9 feet, it supports axial load only without end moments, and it is braced against side sway. The ends of the column are effectively pinned ended and the column may be assumed to buckle in single curvature. Determine the design axial load strength of the column, the minimum permitted diameter of the spiral reinforcement, and the maximum allowable pitch of the spiral.

Solution

From ACI Section 10.12.1, the effective length factor of a column braced against sidesway may conservatively be taken as

$$k = 1.0$$

The radius of gyration, in accordance with ACI Section 10.11.2 is

$$r = 0.25h$$

where h = diameter of column
$= 20$ inches

and $r = 0.25 \times 20$
$= 5$ inches

The slenderness ratio is

$$\begin{aligned} k\ell_u/r &= 1 \times 9 \times 12/5 \\ &= 21.6 \end{aligned}$$

From ACI Section 10.11.4.1, slenderness effects in the non-sway column may be neglected when

$$k\ell_u/r < 34 - 12M_1/M_2$$

where M_2 = larger factored moment at end of column
$= 0$

M_1 = smaller factored moment at end of column, positive if the column is bent in single curvature.
$= 0$

and $M_1/M_2 \geq -0.5$

then $M_1/M_2 = 1.0$ ... for single curvature
and $34-12M_1/M_2 = 34 - 12$
$= 22$
$> k\ell_u/r$

Hence the column is considered a short column and the design axial load strength, for a spirally reinforced column, is given by ACI Equation (10–1) as

$$\phi P_n = 0.85\ \phi[0.85f'_c(A_g - A_{st}) + f_yA_{st}]$$
$$= 0.85 \times 0.75[0.85 \times 5(314.2 - 10) + 60 \times 10]$$
$$= 1207 \text{ kips}$$

In accordance with ACI Section 7.10.4.2, the minimum diameter of the spiral reinforcement is

$$d_b = \tfrac{3}{8} \text{ inch.}$$

From ACI Equation (10–6) the minimum spiral reinforcement ratio is

$$\rho_s = 0.45f'_c[(A_g/A_c) - 1]/f_y$$

where A_g = gross area of the section
$= \pi 20^2/4$
$= 314.2$ square inches

A_c = area of core measured out-to-out of the spiral
$= \pi 17^2/4$
$= 227.0$ square inches

and $\rho_s = 0.45 \times 5[(314.2/227.0) - 1]/60$
$= 0.0144$
$= V_s/V_c$

where V_s = volume of spiral bar per turn of spiral
$= \pi A_b d_s$
$= \pi \times 0.11(17 - 0.375)$
$= 5.75$

V_c = volume of concrete core per turn of spiral
$= A_c s$
$= 227.0s$

and $s = V_s/\rho_s A_c$
$= 5.75/(0.0144 \times 227.0)$
$= 1.76$ inches

Providing a pitch of 1.75 inches gives a clear spacing of

$$s_c = s - d_b$$
$$= 1.75 - 0.375$$
$$= 1.38 \text{ inches}$$
$$> 1 \text{ inch}$$
$$< 3 \text{ inches ... satisfactory}$$

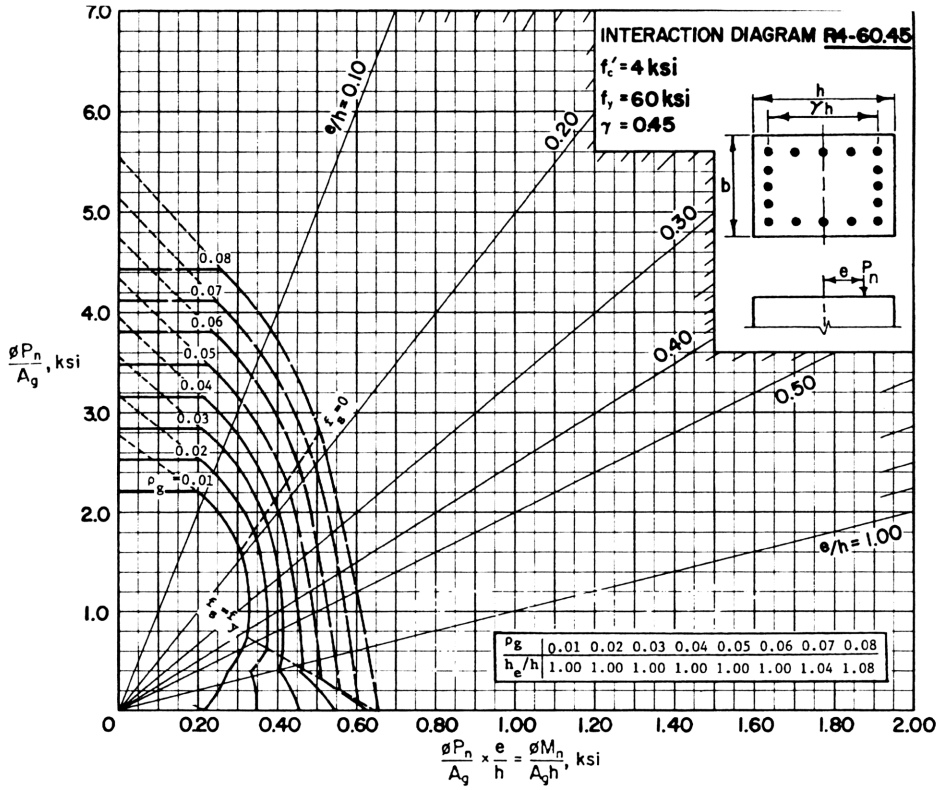


Figure 6–14 Rectangular column reinforced on four sides

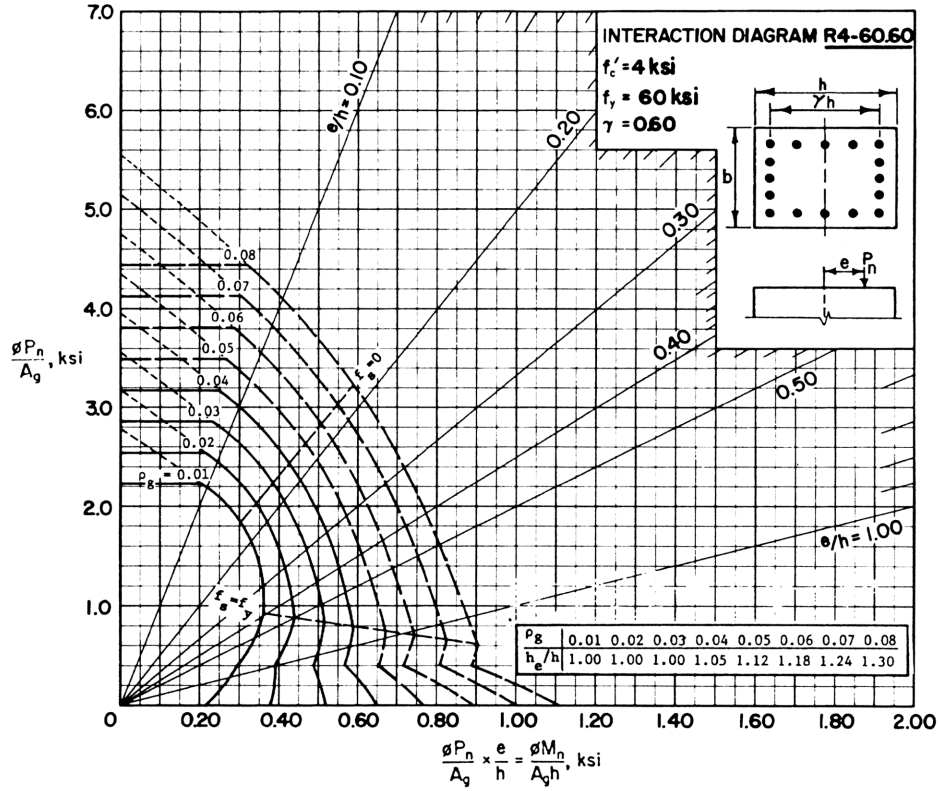


Figure 6–15 Rectangular column reinforced on four sides

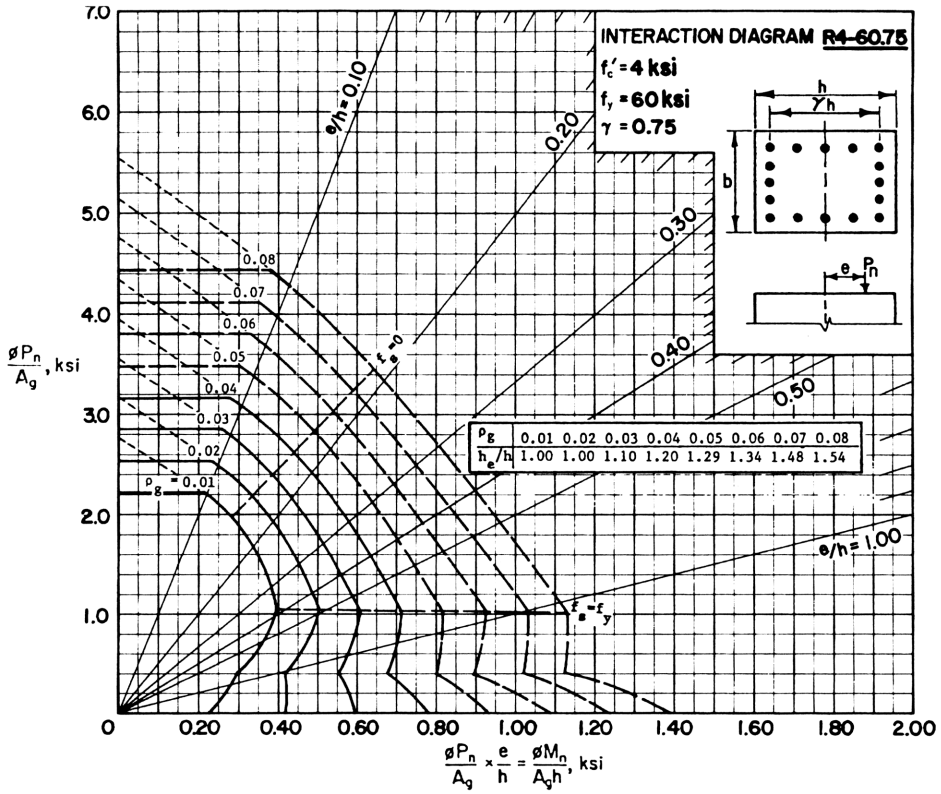


Figure 6–16 Rectangular column reinforced on four sides

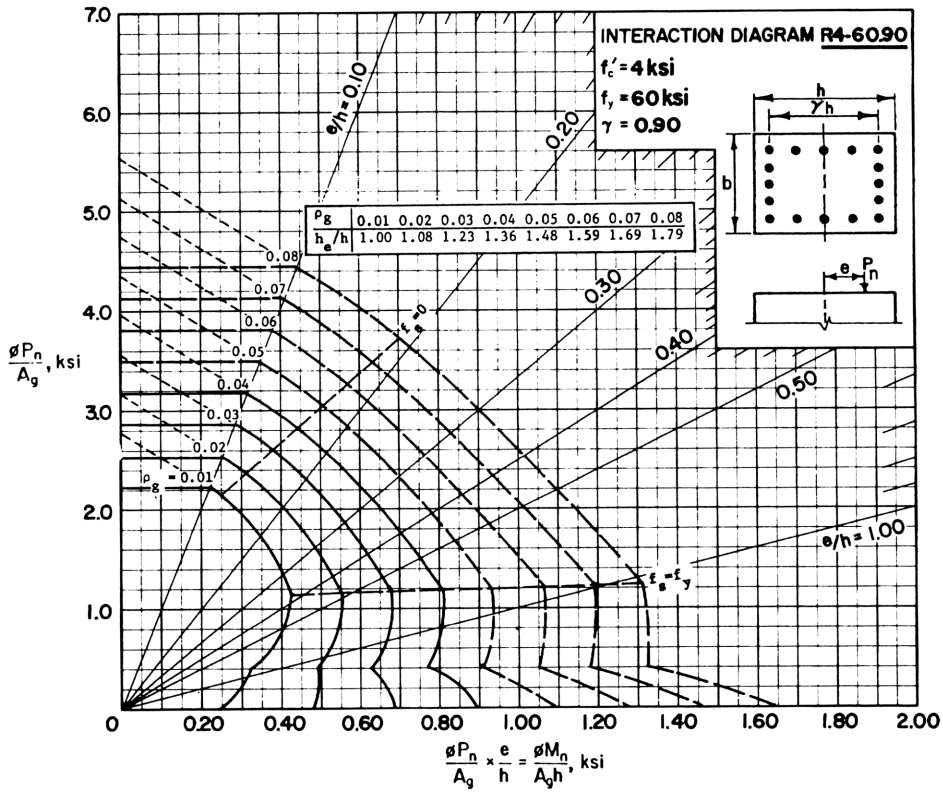


Figure 6–17 Rectangular column reinforced on four sides

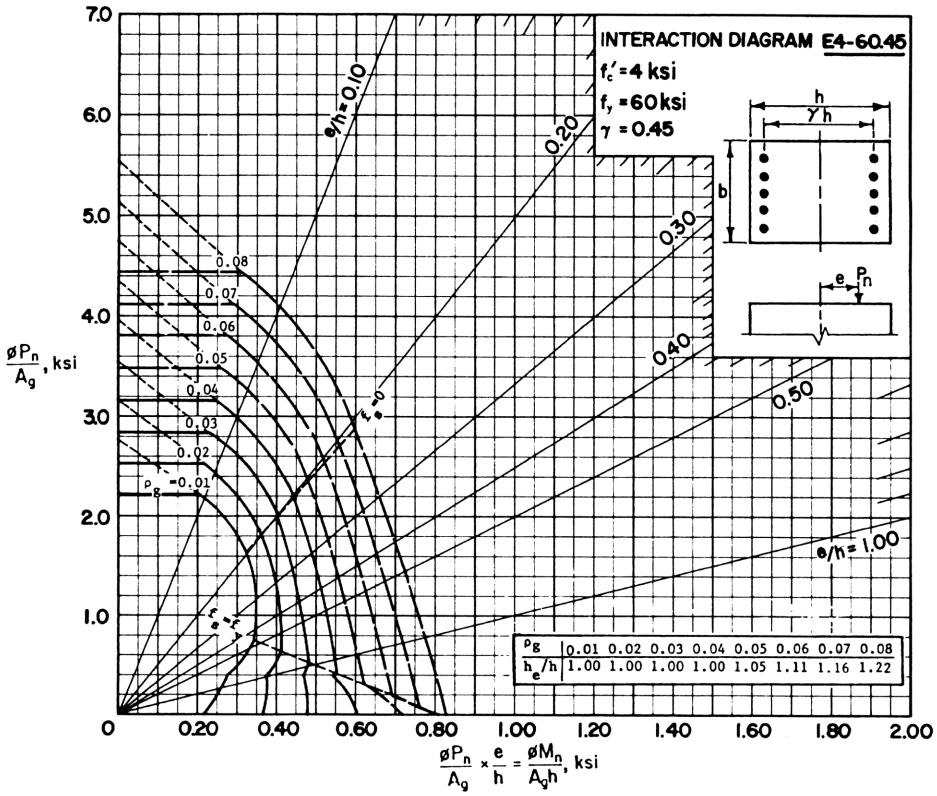


Figure 6–18 Rectangular column reinforced on two sides

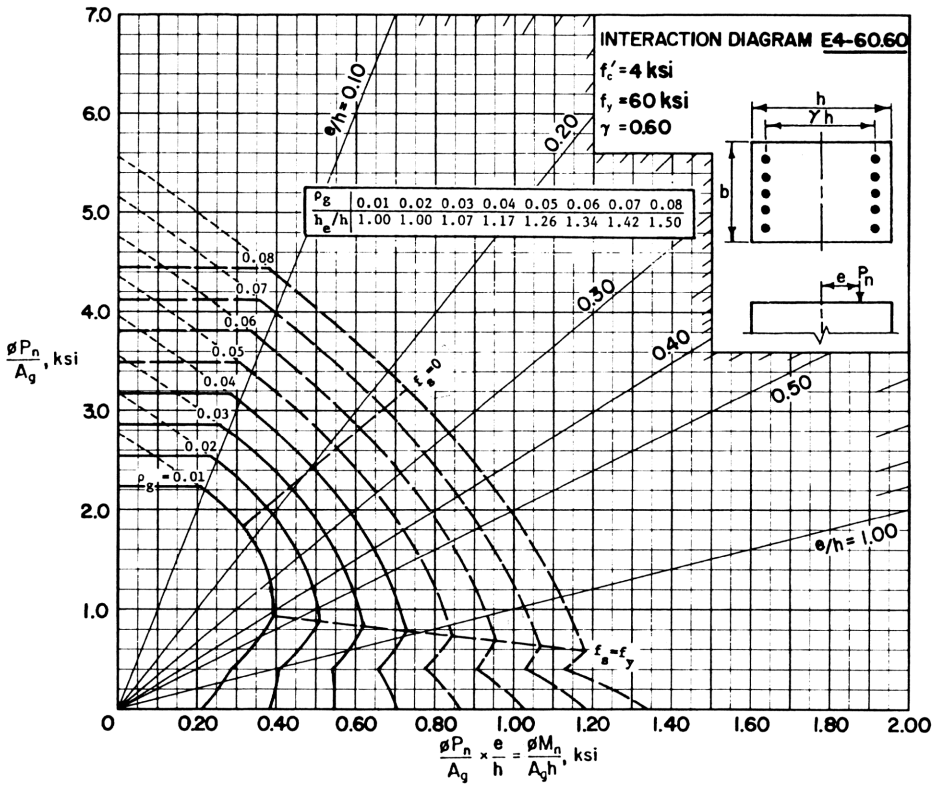


Figure 6–19 Rectangular column reinforced on two sides

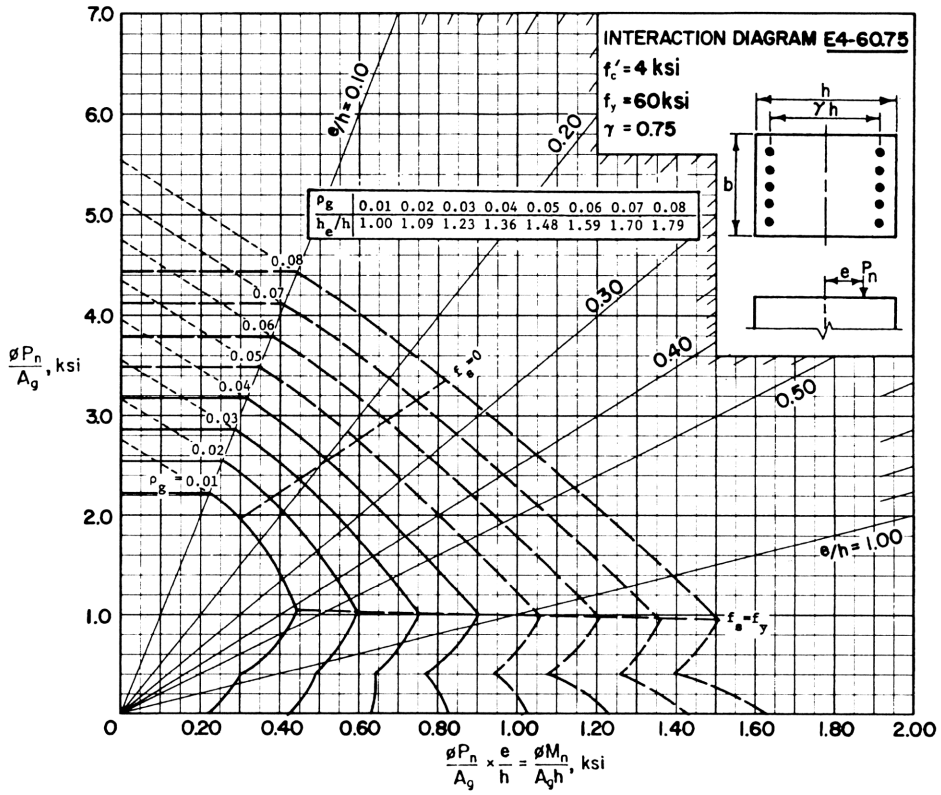


Figure 6–20 Rectangular column reinforced on two sides

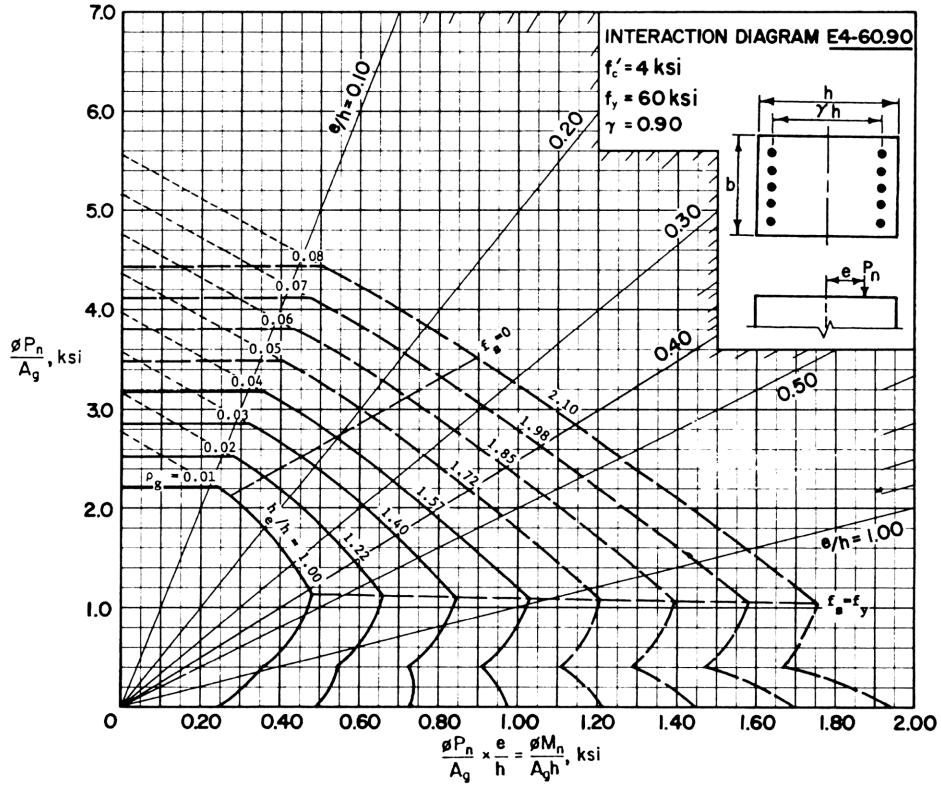


Figure 6–21 Rectangular column reinforced on two sides

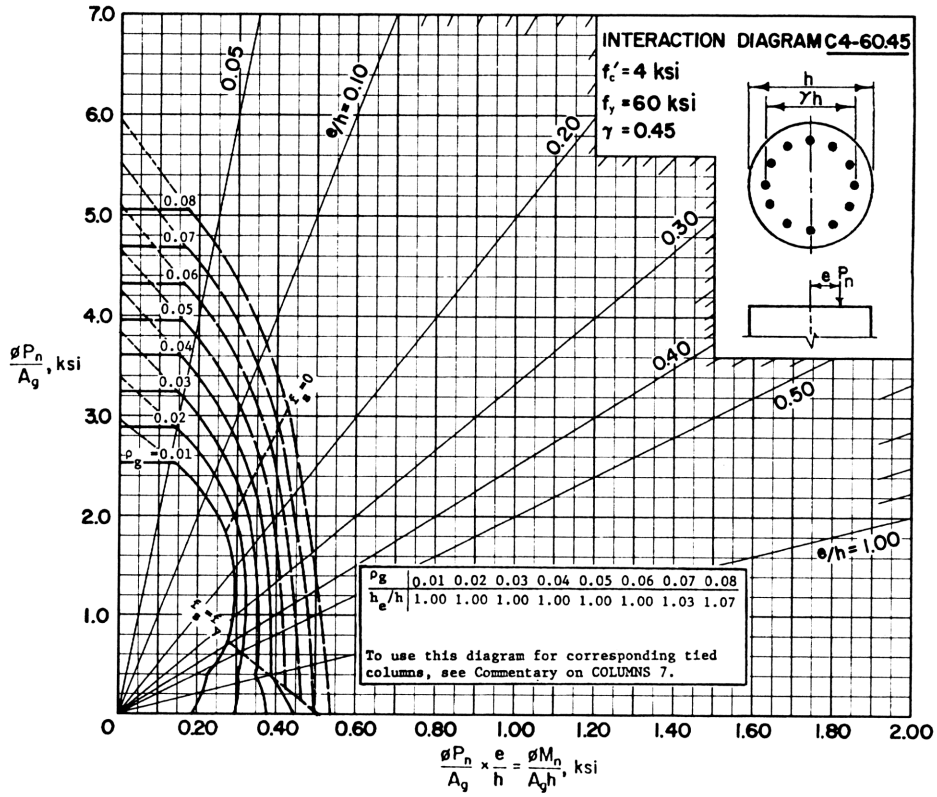


Figure 6–22 Circular column

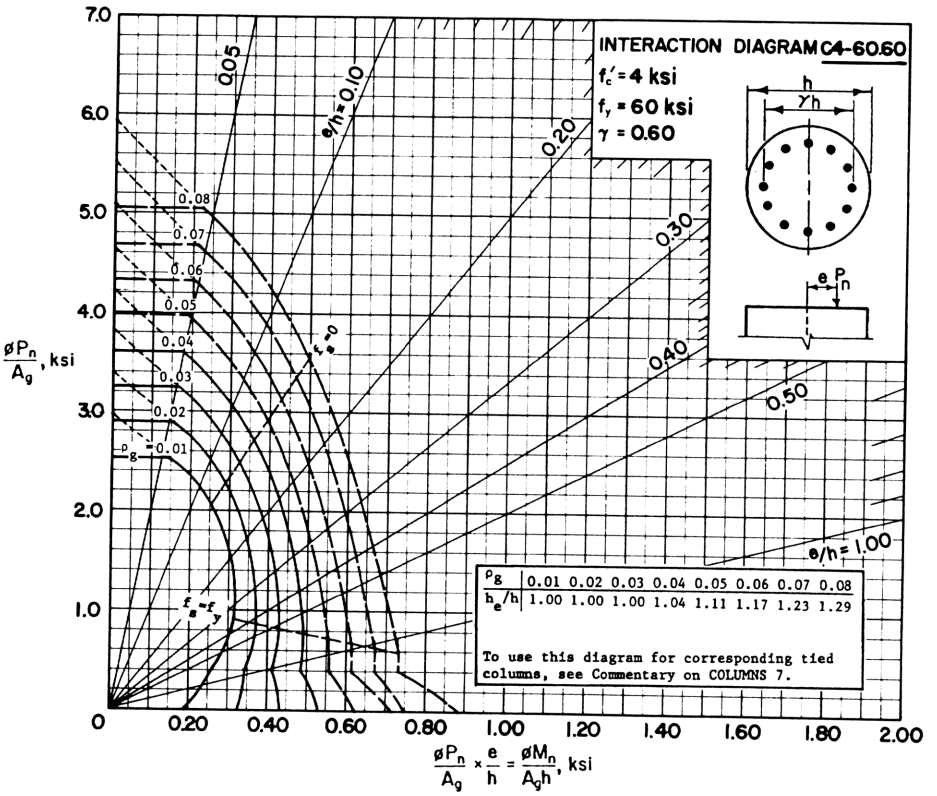


Figure 6–23 Circular column

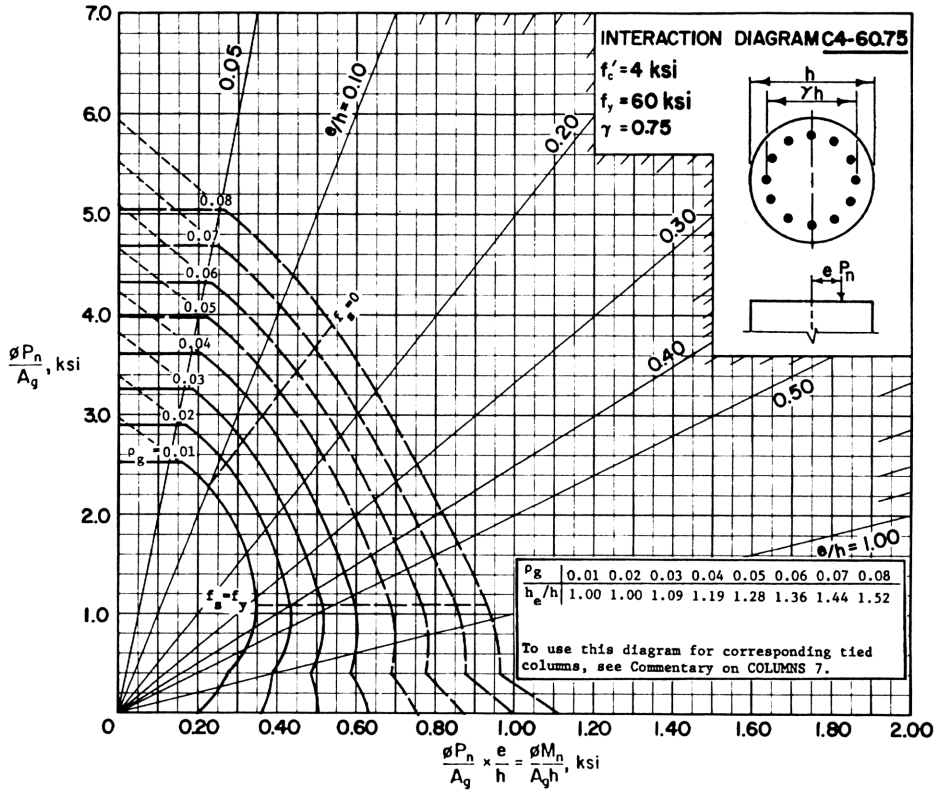


Figure 6–24 Circular column

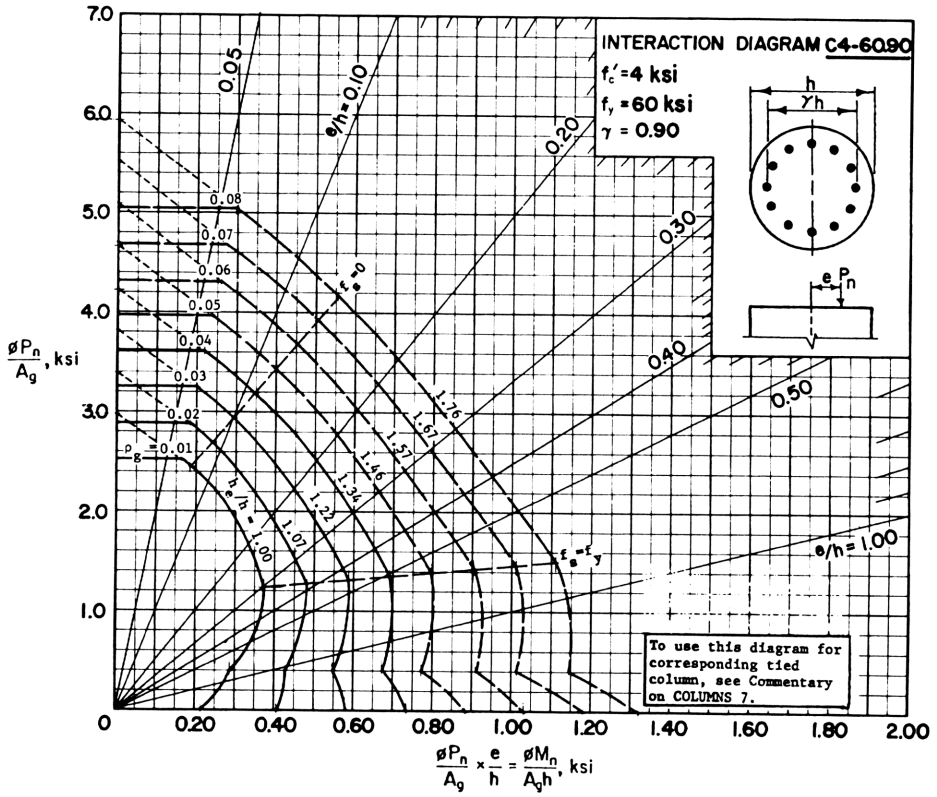


Figure 6–25 Circular column

6.2.2 Short column with applied moment

For a short column, the axial load carrying capacity reduces as the moment applied to the column increases and causes additional stresses in the column. Design may be facilitated by means of tables and charts[4,5,6] and, by courtesy of the American Concrete Institute, a representative selection of the interaction diagrams published by the Institute are reproduced in Figures 6–14 to 6–25. Derivation of an interaction diagram is illustrated in Example 6–5.

Example 6–5 (Interaction diagram for a short column)
Derive the interaction diagram for the column shown in Figure 6–26.

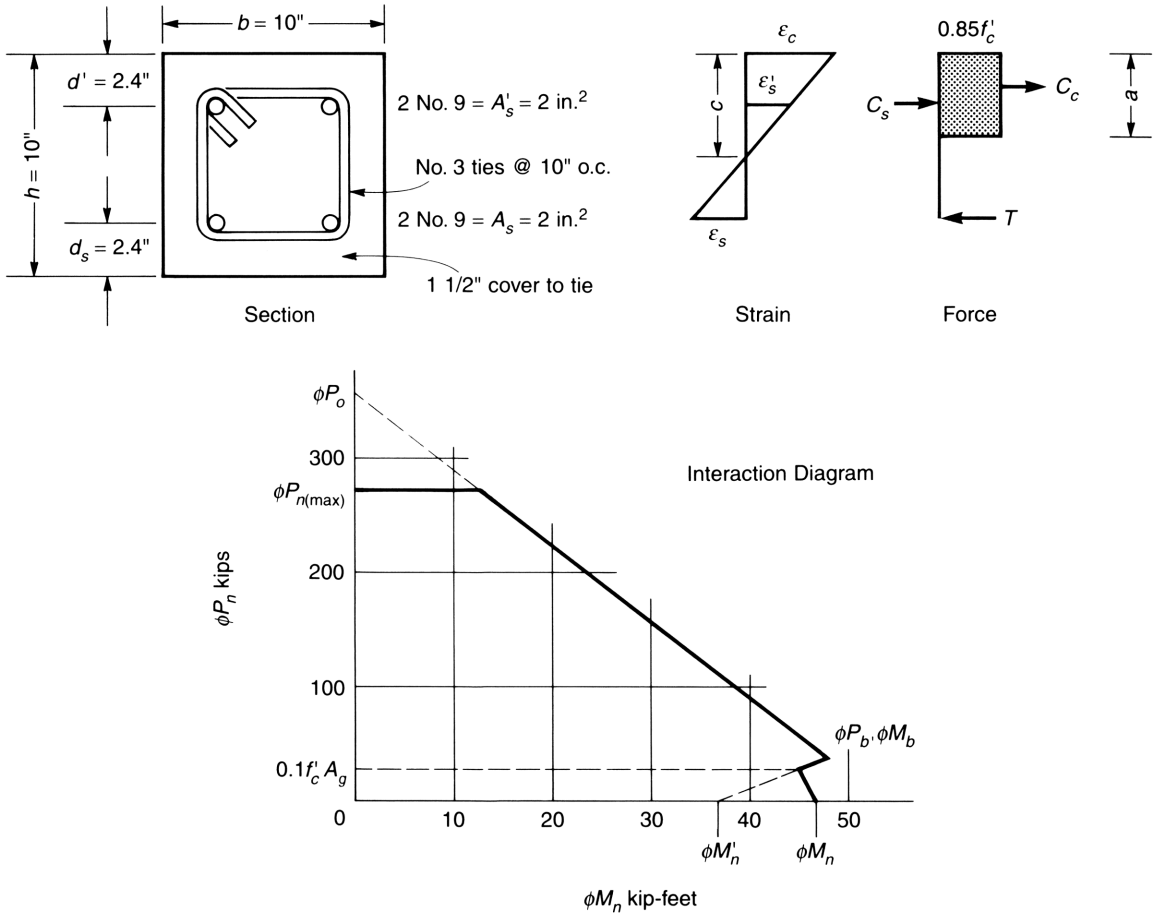


Figure 6–26 Flexure and compression in a short tied column

Solution

The theoretical design axial load strength at zero eccentricity is given by ACI Section 10.3.5. as

$$\phi P_o = \phi[0.85f'_c(A_g - A_{st}) + A_{st}f_y]$$

where ϕ = strength reduction factor specified in ACI Section 9.3
= 0.70 for compression members with lateral ties

f'_c = concrete cylinder strength
= 3.0 kips per square inch
f_y = reinforcement yield strength
= 60 kips per square inch
A_g = gross area of the section
= 100 square inches
A_{st} = reinforcement area
= 4 square inches

Then $\phi P_o = 0.7[0.85 \times 3(100 - 4) + 4 \times 60]$
$= 339$ kips

and this is plotted on the interaction diagram in Figure 6–26.

To account for accidental eccentricity, ACI Section 10.3.5 requires a spirally reinforced column to be designed for a minimum eccentricity of approximately 0.10h which gives a maximum design axial load strength at zero eccentricity, in accordance with ACI Equation (10–2), of

$$\phi P_{n(max)} = 0.80\phi P_o$$
$$= 0.80 \times 339$$
$$= 271 \text{ kips}$$

and this is plotted on the interaction diagram.

As increasing moment is applied to the column, a balanced strain condition is produced when the strain in the concrete and in the tension reinforcement simultaneously reach the values specified in ACI Section 10.3.2 as

$\epsilon_c = 0.003$
$\epsilon_s = f_y/E_s$
$= 60/29{,}000$
$= 0.00207$

From the strain diagram in Figure 6–26, the depth to the neutral axis and the stress in the compression reinforcement may be derived as

$c = d\epsilon_c/(\epsilon_c + \epsilon_s)$
$= 0.003d/(0.003 + \epsilon_s)$
$= 0.003dE_s/(0.003E_s + \epsilon_s E_s)$
$= 87{,}000d/(87{,}000 + f_y)$
$= 87d/(87 + f_y)$
$= 87 \times 7.6/(87 + 60)$
$= 4.5$ inches

and $f'_s = E_s\epsilon_c(c - d')/c$
$= 87(1 - d'/c)$
$= 87(1 - 2.4/4.5)$
$= 40.58$ kips per square inch
$< f_y$

In accordance with ACI Section 10.2.7.1, the depth of the equivalent rectangular stress block is given by

$$a = c\,\beta_1$$

where $\beta_1 = 0.85$ as defined in ACI Section 10.2.7.3

Hence $a = 4.5 \times 0.85$

$= 3.83$ inches

Then, the forces in the concrete, compression reinforcement and tensile reinforcement, are as indicated in the force diagram in Figure 6–26 and are given by

$$C_c = 0.85f'_c(ab - A'_s)$$
$$= 0.85 \times 3(3.83 \times 10 - 2)$$
$$= 92.57 \text{ kips}$$
$$C_s = f'_s A'_s$$
$$= 40.58 \times 2$$
$$= 81.16 \text{ kips}$$
$$T = f_y A_s$$
$$= 60 \times 2$$
$$= 120 \text{ kips}$$

Hence, the nominal axial load capacity at the balanced strain condition is given by

$$P_b = C_c + C_s - T$$
$$= 92.57 + 81.16 - 120$$
$$= 53.73 \text{ kips}$$

The design axial load capacity at the balanced strain condition is given by

$$\phi P_b = 0.7 \times 53.73$$
$$= 37.61 \text{ kips}$$

The nominal moment capacity at the balanced strain condition is obtained by summing moments about the mid-depth of the section and is given by

$$M_b = C_c(h/2 - a/2) + C_s(h/2 - d') + T(h/2 - d_s)$$
$$= 92.57(5 - 1.92) + 81.16(5 - 2.4) + 120(5 - 2.4)$$
$$= 808 \text{ kip inches}$$

The design moment capacity at the balanced strain condition is

$$\phi M_b = 0.7 \times 808/12$$
$$= 47.14 \text{ kip feet}$$

and the point corresponding to the balanced strain condition is plotted on the interaction diagram.

Neglecting compression reinforcement, the design moment strength of the section without applied axial load is obtained from

$$\phi M_n = \phi A_s f_y d(1 - 0.59\rho f_y/f'_c)/12$$
$$= 0.9 \times 2 \times 60 \times 7.6(1 - 0.59 \times 0.0263 \times 60/3)/12$$
$$= 47.16 \text{ kip feet}$$

The ratio of the distance between the compression and tension reinforcement to the overall depth of the section is

$$\begin{aligned}(d - d')/h &= (7.6 - 2.4)/10 \\ &= 0.52 \\ &< 0.7\end{aligned}$$

Also the value of

$$\begin{aligned}0.1f'_cA_g &= 0.1 \times 3 \times 100 \\ &= 30 \text{ kips} \\ &< \phi P_b\end{aligned}$$

Hence, in accordance with ACI Section 9.3.2.2, the strength reduction factor ϕ decreases from a value of 0.9, at zero axial load, to a value of 0.7, at an axial load of $0.1f'_cA_g$. Using a value of 0.7 for the strength reduction factor gives a value for the design moment strength of

$$\begin{aligned}\phi M'_n &= \phi M_n \times 7/9 \\ &= 47.16 \times 7/9 \\ &= 36.68 \text{ kip feet}\end{aligned}$$

The values of ϕM_n, $\phi M'_n$ and $0.1f'_cA_g$ are plotted on the interaction diagram which may be constructed as shown. All combinations of axial load and moment which lie within the interaction diagram are valid.

Example 6–6 (Circular column with spiral reinforcement and end moments)

A 20 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch spiral is reinforced with ten No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 9 feet, it is braced against side sway, and is bent in single curvature with factored end moments of $M_1 = M_2 = 250$ kip feet. Determine the design axial load strength of the column.

Solution

From ACI Section 10.12.1, the effective length factor of a column braced against sidesway may conservatively be taken as

$$k = 1.0$$

The radius of gyration, in accordance with ACI Section 10.11.2 is

$$r = 0.25h$$

where h = diameter of column
= 20 inches

and $r = 0.25 \times 20$
$= 5$ inches

The slenderness ratio is

$$k\ell_u/r = 1 \times 9 \times 12/5$$
$$= 21.6$$

From ACI Section 10.11.4.1, slenderness effects in the non-sway column may be neglected when

$$k\ell_u/r < 34 - 12M_1/M_2$$

where M_2 = larger factored moment at end of column
$= 250$ kip feet

M_1 = smaller factored moment at end of column, positive if the column is bent in single curvature.
$= 250$ kip feet

and $M_1/M_2 \geq -0.5$

then $M_1/M_2 = 1.0$ ... for single curvature

and $34-12M_1/M_2 = 34 - 12$
$= 22$
$> k\ell_u/r$

Hence, slenderness effects need not be considered.

The ratio of the distance between centroids of longitudinal steel to the overall diameter of the column is

$$\gamma = (20 - 4.88)/20$$
$$= 0.76$$
$$\approx 0.75$$

The reinforcement ratio is

$$\rho_g = 10/314.2$$
$$= 0.0318$$

and $M_u/A_gh = 250 \times 12/(314.2 \times 20)$
$= 0.477$ kips per square inch

Utilizing interaction diagrams in Figure 6–24 the allowable design axial stress is given by

$$P_u/A_g = 1.7 \text{ kips per square inch}$$

The allowable design axial load is

$$P_u = 1.7A_g$$
$$= 1.7 \times 314.2$$
$$= 534 \text{ kips}$$

6.3 Long columns

6.3.1 Determination of P Delta effects

For a slender column which does not conform to the criteria given in ACI Sections 10.12.2 and 10.13.2, it is necessary to consider the secondary bending stresses caused by the P–delta effect.

The secondary stresses are produced by the eccentricity of the axial force about the displaced center line of the member. The P–delta effects are automatically determined in a second–order frame analysis and the column cross section may then be designed directly for the calculated axial force and bending moment using the short column design procedure. Alternatively, provided the slenderness ratio of a long column does not exceed 100, ACI Sections 10.12.3 and 10.13.4.3 permit slenderness effects to be allowed for by means of the approximate moment magnifier method. A conventional first–order frame analysis may be utilized and the P–delta effects estimated by amplifying the primary bending moments by a moment magnification factor[7]. The column cross section may then be designed for the calculated axial force and the magnified bending moment using the short column design procedure. Separate procedures are used for sway and non–sway situations.

6.3.2 Magnified moments: non–sway frames

For the non–sway case, the magnified factored moment given by ACI Equation (10–9) is

$M_c = \delta_{ns} M_2$

where
M_2 = larger factored moment due to gravity or transverse loads
δ_{ns} = moment magnification factor for non–sway frames
$= C_m/(1 - P_u/0.75P_c)$ ... from ACI Equation (10–10)
≥ 1.0
C_m = equivalent moment correction factor
$= 0.6 + 0.4M_1/M_2 \geq 0.4$ ... from ACI Equation (10–14) for non–sway frames without transverse loads
$= 1.0$ ... for members with transverse loads between supports
M_1 = smaller factored moment at end of column, negative if the column is bent in double curvature, positive if the column is bent in single curvature.
0.75 = stiffness reduction factor
P_u = factored axial load at a given eccentricity
$\leq \phi P_n$
P_n = nominal axial load capacity at a given eccentricity
P_c = Euler critical load from ACI Equation (10–11)
$= \pi^2 EI/(k\ell_u)^2$
I $\approx 0.25I_g$ ... from ACI Section R10.12.3

The value of the moment magnification factor is applicable to a member bent in single curvature under uniform bending moment. For other conditions, ACI Section 10.12.3.1 specifies the use of a moment correction factor C_m to produce an equivalent uniform bending moment. The value of the moment correction factor reduces to:

C_m = 1.0 ... for a column bent in single curvature with equal end moments
= 0.6 ... for a column bent in single curvature with one end hinged
= 0.4 ... for a column with one end fixed

Typical values of C_m are shown in Figure 6–27. When the calculated moments at the ends of

a non-sway column are very small or zero, ACI Section 10.12.3.2 specifies that $C_m = 1.0$ and the column design is based on a minimum factored moment of

$$M_2 = P_u(0.6 + 0.03h)$$

where h = width or diameter of column

The stages in the analysis of a slender non-sway column may be summarized as:

- Calculate the factored axial force and moments in the column using conventional first-order frame analysis.
- Determine the end restraint conditions and calculate the effective length of the column.
- Verify that the column is slender using ACI Equation (10-8).
- Determine the moment correction factor C_m from ACI Equation (10-14), the moment magnification factor δ_{ns} from ACI Equation (10-10), and the magnified moment from ACI Equation (10-9).
- Calculate the values of M_c/A_gh and P_u/A_g and using the relevant interaction diagram for a short column determine if these values lie inside or outside the interaction diagram.

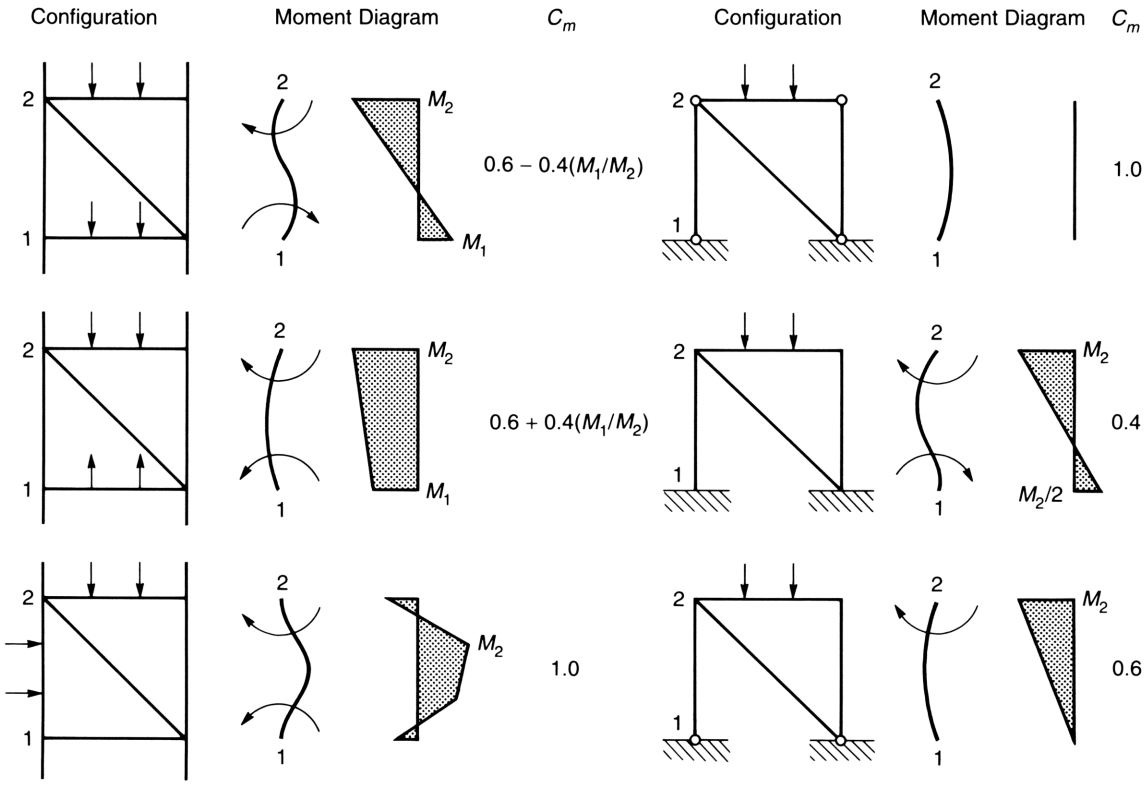


Figure 6-27 Values of the moment correction factor

Example 6–7 (Slender circular column with spiral reinforcement and end moments) ═══
A 20 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch diameter spiral is reinforced with ten No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 12 feet, it is braced against side sway, is bent in single curvature with factored end moments of $M_1 = M_2 = 250$ kip feet, and has an Euler critical load value of $P_c = 3400$ kips. Determine if the column can support a factored axial load of 700 kips.

Solution

From ACI Section 10.12.1, the effective length factor of a column braced against sidesway may conservatively be taken as

$$k = 1.0$$

The radius of gyration, in accordance with ACI Section 10.11.2 is

$$r = 0.25h$$

where h = diameter of column
= 20 inches

and $r = 0.25 \times 20$
$= 5$ inches

The slenderness ratio is

$$k\ell_u/r = 1 \times 9 \times 12/5$$
$$= 21.6$$

From ACI Section 10.11.4.1, slenderness effects in the non-sway column may be neglected when

$$k\ell_u/r < 34 - 12M_1/M_2$$

where M_2 = larger factored moment at end of column
= 250 kip feet

M_1 = smaller factored moment at end of column, positive if the column is bent in single curvature.
= 250 kip feet

and $M_1/M_2 \geq -0.5$

then $M_1/M_2 = 1.0$ ... for single curvature

and $34-12M_1/M_2 = 34 - 12$
$= 22$
$< k\ell_u/r$

Hence the column is considered a long column and slenderness effects must be considered.

From ACI Equation (10–14), the moment correction factor is given by

$$C_m = 0.6 + 0.4M_1/M_2$$
$$= 0.6 + 0.4 \times 250/250$$
$$= 1.0$$

From ACI Equation (10–10), the magnification factor for gravity loads is

$$\begin{aligned}\delta_{ns} &= C_m/(1 - P_u/0.75P_c)\\ &= 1/[1 - 500/(0.75 \times 4000)]\\ &= 1.24\end{aligned}$$

From ACI Equation (10–9) the magnified moment is

$$\begin{aligned}M_c &= \delta_{ns}M_2\\ &= 1.24 \times 250\end{aligned}$$

The ratio of the distance between centroids of longitudinal steel to the overall diameter of the column is

$$\begin{aligned}\gamma &= (20 - 4.88)/20\\ &= 0.76\\ &\approx 0.75\end{aligned}$$

The reinforcement ratio is

$$\begin{aligned}\rho_g &= 10/314.2\\ &= 0.0318\end{aligned}$$

and

$$\begin{aligned}M_u/A_gh &= 1.24 \times 250 \times 12/(314.2 \times 20)\\ &= 0.59 \text{ kips per square inch}\\ P_u/A_g &= 500/314.2\\ &= 1.59 \text{ kips per square inch}\end{aligned}$$

These values lie outside of the relevant interaction diagram shown in Figure 6–24 and the column can not support the axial load of 500 kips.

6.3.3 Magnified moments: sway frames

For the sway case, the magnified factored moments given by ACI Equations (10–16) and (10–17) are

$$M_1 = M_{1ns} + \delta_s M_{1s}$$

$$M_2 = M_{2ns} + \delta_s M_{2s}$$

where M_1 = smaller factored moment at end of column, negative if the column is bent in double curvature, positive if the column is bent in single curvature.

M_2 = larger factored moment at end of column

M_{1ns} = factored moment at the end of the column at which M_1 acts, due to loads which cause no appreciable side sway, calculated using first-order frame analysis

M_{2ns} = factored moment at the end of the column at which M_2 acts, due to loads which cause no appreciable side sway, calculated using first-order frame analysis

M_{1s} = factored moment at the end of the column at which M_1 acts, due to loads which cause appreciable side sway, calculated using first-order frame analysis

M_{2s} = factored moment at the end of the column at which M_2 acts, due to loads which cause appreciable side sway, calculated using first-order frame analysis

δ_s = moment magnification factor for sway frames
= $1/[1 - \Sigma P_u/(0.75\Sigma P_c)]$ ... from ACI Equation (10-19), where the summations extend over all the columns in a story.
≤ 2.5 ... from ACI Section 10.13.6

0.75 = stiffness reduction factor

P_u = factored axial load at a given eccentricity
$\leq \phi P_n$

P_n = nominal axial load capacity at a given eccentricity

P_c = Euler critical load from ACI Equation (10-11)
= $\pi^2 EI/(k\ell_u)^2$

I $\approx 0.25 I_g$ ... from ACI Section R10.12.3

The maximum moments do not occur at the ends of the column, in the case of a slender column with high axial load, when

$$\ell_u/r > 35/(P_u/f'_c A_g)^{0.5}$$

In this case, the column is designed for the magnified moment

$$M_c = \delta_{ns} M_2$$
$$= \delta_{ns}(M_{2ns} + \delta_s M_{2s})$$
$$= C_m(M_{2ns} + \delta_s M_{2s})/(1 - P_u/0.75P_c)$$

Example 6-8 (Slender circular column in a sway frame)

A 20 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch diameter spiral is reinforced with ten No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 12 feet and its effective length factor is 1.5. The maximum factored end moments due to sway and non-sway loading are of M_{2s} = 200 kip feet and M_{2ns} = 10 kip feet, and it has an Euler critical load value of P_c = 3400 kips. In the story in which the column is located, the summation of the total factored column loads is ΣP_u = 1200 kips and the summation of the total column critical loads is ΣP_c = 13,600 kips. Determine if the column can support a factored axial load of 400 kips.

Solution

The radius of gyration, in accordance with ACI Section 10.11.2 is

$r = 0.25h$

where h = diameter of column
= 20 inches

and $r = 0.25 \times 20$
= 5 inches

The slenderness ratio is

$$k\ell_u/r = 1.5 \times 12 \times 12/5$$
$$= 43.2$$

From ACI Section 10.13.2, slenderness effects in the sway column may be neglected when

$$k\ell_u/r < 22$$

Hence the column is considered a long column and slenderness effects must be considered.

From ACI Equation (10–20), the maximum moment occurs at the end of the column when

$$\ell_u/r < 35/(P_u/f'_cA_g)^{0.5}$$
$$< 35/(300/4 \times 314.2)^{0.5}$$
$$< 71.6$$

and

$$\ell_u/r = 12 \times 12/5$$
$$= 28.8$$
$$< 71.6$$

Hence the maximum moment occurs at the end of the column.

From ACI Equation (10–19), the moment magnification factor for sway frames is given by

$$\delta_s = 1/[1 - \Sigma P_u/(0.75\Sigma P_c)]$$
$$\leq 2.5 \ldots \text{ from ACI Section 10.13.6}$$

and

$$\delta_s = 1/[1 - 1200/(0.75 \times 13{,}600)]$$
$$= 1.13$$
$$< 2.5 \ldots \text{ satisfactory}$$

The maximum magnified factored moment is

$$M_2 = M_{2ns} + \delta_s M_{2s}$$
$$= 10 + 1.13 \times 200$$
$$= 237 \text{ kip feet}$$

The ratio of the distance between centroids of longitudinal steel to the overall diameter of the column is

$$\gamma = (20 - 4.88)/20$$
$$= 0.76$$
$$\approx 0.75$$

The reinforcement ratio is

$$\rho_g = 10/314.2$$
$$= 0.0318$$

and

$$M_2/A_gh = 237 \times 12/(314.2 \times 20)$$
$$= 0.45 \text{ kips per square inch}$$
$$P_u/A_g = 400/314.2$$
$$= 1.27 \text{ kips per square inch}$$

These values lie inside of the relevant interaction diagram shown in Figure 6–24 and the column can support the axial load of 400 kips. ═══

6.3.4 Biaxial bending

For the case of biaxial bending, column strengths may be estimated, in accordance with ACI Section R10.3.5, by the interaction equation

$$1/P_{ni} = 1/P_{nx} + 1/P_{ny} - 1/P_o$$

where P_{ni} = nominal axial load capacity with eccentricities of e_x and e_y about the x-axis and y-axis, respectively

P_o = nominal axial load capacity with $e_x = e_y = 0.0$

P_{nx} = nominal axial load capacity with $e_y = 0.0$

P_{ny} = nominal axial load capacity with $e_x = 0.0$

In accordance with ACI Section 10.11.6, when biaxial bending occurs in a slender column, the computed moments about each of the principal axes must be magnified.

6.4 TRANSFER OF FORCE AT BASE OF COLUMN

6.4.1 Transfer by bearing on concrete

The force at the base of a column must be transferred to the footing by bearing on the concrete at the interface or by a combination of bearing on the concrete and force transferred by reinforcement. The design bearing capacity of the column concrete at the interface is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_1$$

where ϕ = strength reduction factor specified in ACI Section 9.3.2.4 for bearing on concrete

$= 0.7$

A_1 = area of column

f'_c = compressive strength of the column concrete

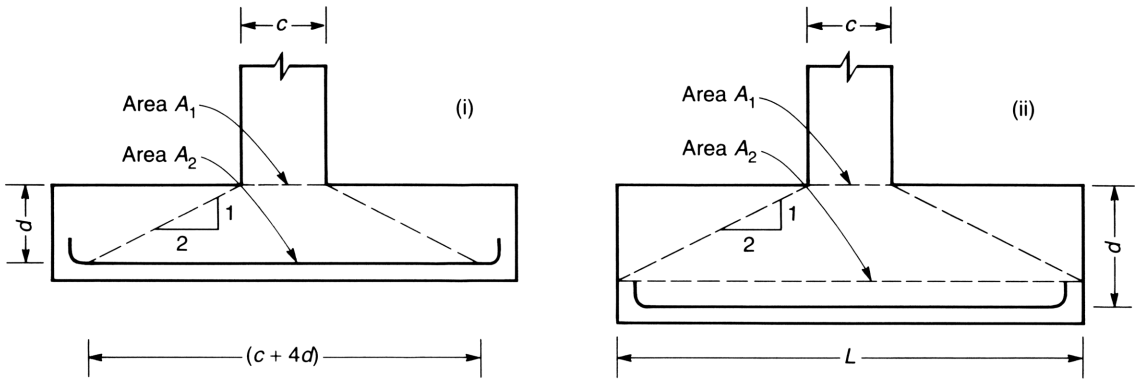


Figure 6–28 Supporting area in a footing

The bearing capacity of the footing concrete at the interface is based on the increased supporting area available, as defined by ACI Section 10.17.1. As shown in Figure 6–28, this is the area of the base of the pyramid, radiating from the column base with side slopes of 1 vertical to 2 horizontal, formed wholly within the footing. The area is given by

$$A_2 = (c + 4d)^2 \leq L^2$$

where c = width of column
d = effective depth of footing
L = width of footing

The design bearing capacity of the footing concrete at the interface is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_1 (A_2/A_1)^{0.5} \leq 2 \times 0.85\phi f'_c A_1$$

6.4.2 Transfer by reinforcement

When the axial load in the column exceeds the bearing capacity, reinforcement must be provided to transfer the excess force. The force which the dowels can transfer is given by

$$\phi P_s = \phi A_s f_y$$

where ϕ = strength reduction factor specified in ACI Section 9.3.2.2 for axial compression in a column
A_s = area of reinforcement
f_y = yield strength of the reinforcement

A minimum area of reinforcement is required at the interface and this is specified in ACI Section 15.8.2.1 as

$$A_{s(min)} = 0.005A_1$$

where A_1 = area of column

Reinforcement across the interface may be furnished either by extending the column reinforcement into the footing or by providing dowel bars. The length of the dowel bars must be sufficient to provide a lap splice with the column reinforcement and to provide a development length in the footing.

Example 6–9 (Transfer of force at base of column)

The design details of a reinforced concrete column and pile cap are shown in Figure 6–29. The 18 inch square column has a specified compressive strength of 4000 pounds per square inch and is reinforced with 8 No.11 Grade 60 bars. The 5 foot square pile cap has a specified compressive strength of 3000 pounds per square inch. Four No.8 Grade 60 dowel bars are provided at the interface. Determine the force which the dowels can transfer across the interface, the bearing capacity of the column and the pile cap, and the total load which may be transferred across the interface.

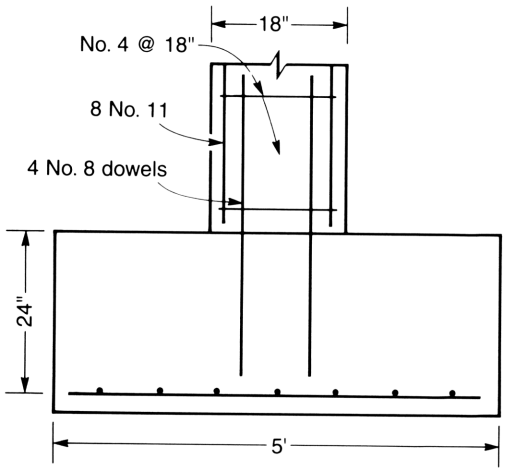


Figure 6–29 Details for Example 6–9

Solution

The force which the dowels can transfer is

$$\phi P_s = \phi A_s f_y$$

where ϕ = strength reduction factor specified in ACI Section 9.3.2.2 for axial compression in members without spiral reinforcement
$= 0.7$

A_s = area of four No.8 bars
$= 3.16$ square inches

f_y = yield strength of the dowels
$= 60{,}000$ pounds per square inch

and $\phi P_s = 0.7 \times 3.16 \times 60$
$= 133$ kips

The minimum required area of dowel reinforcement is specified in ACI Section 15.8.2.1 as

$$A_{s(min)} = 0.005A_1$$

where A_1 = area of column
$= 18 \times 18$
$= 324$ square inches

and $A_{s(min)} = 0.005 \times 324$
$= 1.62$ square inches
$< A_s$ ... satisfactory

Hence the force which the dowels can transfer is

$$\phi P_s = 133 \text{ kips}$$

The design bearing capacity of the column concrete at the interface is given by ACI Section 10.17.1 as

$\phi P_n = 0.85\phi f'_c A_1$

where ϕ = strength reduction factor specified in ACI Section 9.3.2.4 for bearing on concrete
= 0.7

A_1 = area of column
= 18 × 18
= 324 square inches

f'_c = compressive strength of the column concrete
= 4000 pounds per square inch

and $\phi P_n = 0.85 \times 0.7 \times 4 \times 324$
= 771 kips.

Hence the design bearing capacity of the column concrete at the interface is

ϕP_n = 771 kips.

The bearing capacity of the pile cap concrete at the interface is based on the increased supporting area available, as defined by ACI Section 10.17.1. This is the area of the base of the pyramid, radiating from the column base with side slopes of 1 vertical to 2 horizontal, formed wholly within the pile cap. The area is given by

$A_2 = (c + 4d)^2$
$\leq L^2$

where c = width of column
= 18 inches

d = effective depth of pile cap
= 24 inches

L = width of pile cap

and $A_2 = (18 + 4 \times 24)^2$
= 12996 square inches
$\leq L^2$
$= 60^2$
= 3600 square inches ... governs

The design bearing capacity of the pile cap concrete at the interface is given by ACI Section 10.17.1 as

$\phi P_n = 0.85\phi f'_c A_1 (A_2/A_1)^{0.5}$
$\leq 2 \times 0.85\phi f'_c A_1$

where $(A_2/A_1)^{0.5} = (3600/324)^{0.5}$
= 3.33
> 2

and $\phi P_n = 2 \times 0.85\phi f'_c A_1$
$= 2 \times 0.85 \times 0.7 \times 3 \times 324$
= 1157 kips

Hence the design bearing capacity of the pile cap concrete is

ϕP_n = 1157 kips

The total load which may be transferred across the interface is specified in ACI Section 15.8.1.2 as the design strength of the dowel bars plus the minimum bearing capacity of the two elements at the interface. Hence the total load which may be transferred across the interface is

$$\begin{aligned} P_u &= \phi P_s + \phi P_{n(min)} \\ &= 133 + 771 \\ &= 904 \text{ kips} \end{aligned}$$

6.5 SUPPLEMENTARY PROBLEMS

Problem 6–1 (Effective length determination)
The sway frame shown in Figure 6–30 consists of 12 inch square columns and 12 inch wide by 24 inch deep beams. The concrete strength for both beams and columns is 4000 pounds per square inch. Determine the effective length of column 12 and column 23.

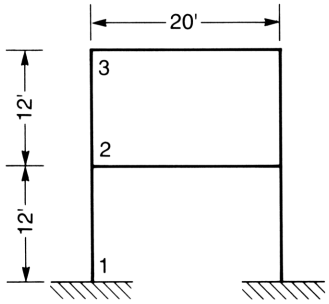


Figure 6–30 Details for Problem 6–1

Problem 6–2 (Spiral reinforcement)
A 22 inch diameter spirally reinforced column has 1½ inches cover to the spiral. The concrete strength is 4000 pounds per square inch and the spiral is Grade 60 reinforcement. Determine the minimum diameter of the spiral reinforcement, and the maximum allowable pitch.

Problem 6–3 (Short column with axial load)
A 20 inches square column is reinforced with eight No.9, Grade 60 deformed bars. Concrete compressive strength is 4000 pounds per square inch. The column is subjected to axial load only and is classified as a short column. Determine the design axial load strength of the column, the minimum allowable reinforcement area, and the required size and spacing of the lateral ties.

Problem 6–4 (Circular column with spiral reinforcement and axial load)
A 22 inch diameter spirally reinforced column with 1½ inches cover to the spiral is reinforced with twelve No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 9 feet, it supports axial load only without end moments, and it is braced against side sway. The ends of the column are effectively pinned ended and the column may be assumed to buckle in single curvature. Determine the design axial load strength of the column, the minimum permitted diameter of the spiral reinforcement, and the maximum allowable pitch of the spiral.

Problem 6–5 (Interaction diagram for a short column)
Derive the interaction diagram for the column shown in Figure 6–31. The concrete strength is 3500 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. Determine if the column can support a factored axial load of 100 kips and a factored bending moment of 40 kip feet.

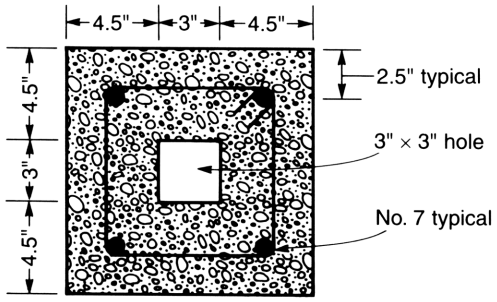


Figure 6–31 Details for Problem 6–5

Problem 6–6 (Circular column with spiral reinforcement and end moments)
A 22 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch spiral is reinforced with twelve No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 9 feet, it is braced against side sway, and is bent in single curvature with factored end moments of $M_1 = M_2 = 200$ kip feet. Determine the design axial load strength of the column.

Problem 6–7 (Slender circular column with spiral reinforcement and end moments)
A 22 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch diameter spiral is reinforced with twelve No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 14 feet, it is braced against side sway, is bent in single curvature with factored end moments of $M_1 = M_2 = 200$ kip feet, and has an Euler critical load value of $P_c = 3600$ kips. Determine if the column can support a factored axial load of 700 kips.

Problem 6–8 (Slender circular column in a sway frame)
A 22 inch diameter spirally reinforced column with 1½ inches cover to the ⅜ inch diameter spiral is reinforced with twelve No.9 bars. The concrete strength is 4000 pounds per square inch and the tensile strength of the reinforcement is 60,000 pounds per square inch. The unsupported length of the column is 14 feet and its effective length factor is 1.5. The maximum factored end moments due to sway and non-sway loading are M_{2s} = 250 kip feet and M_{2ns} = 10 kip feet, and it has an Euler critical load value of P_c = 3600 kips. In the story in which the column is located, the summation of the total factored column loads is ΣP_u = 1500 kips and the summation of the total column critical loads is ΣP_c = 14,400 kips. Determine if the column can support a factored axial load of 500 kips.

Problem 6–9 (Transfer of force at base of column)
The design details of a reinforced concrete column and footing are shown in Figure 6–32. The 12 inch square column has a specified compressive strength of 5000 pounds per square inch and is reinforced with 4 No.9 Grade 60 bars. The 4 foot square footing has a specified compressive strength of 3000 pounds per square inch. Four No.4 Grade 60 dowel bars are provided at the interface. Determine the force which the dowels can transfer across the interface, the bearing capacity of the column and the footing and the total load which may be transferred across the interface.

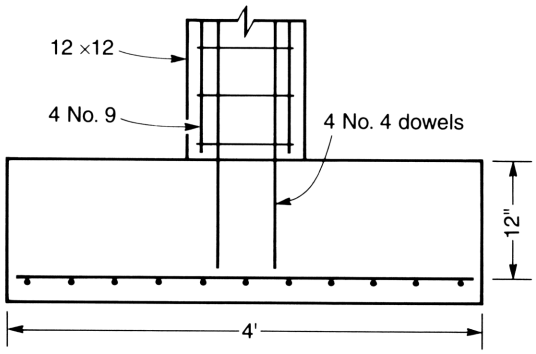


Figure 6–32 Details for Problem 6–9

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7

Design of reinforced concrete slabs

7.1 GENERAL

7.1.1 Types of slabs

Reinforced concrete slabs form horizontal diaphragms in a structural frame, at the floor and roof levels, and may be supported on beams or columns. The different types of slab systems are illustrated in Figure 7-1 and consist of:

- One-way slabs which are supported by beams on two opposite sides. The slab carries its load by spanning between the supporting beams and may be analyzed as a shallow beam either by the approximate methods of ACI Section 8.3.3 or by exact analytical techniques with redistribution of moments.
- Two-way slabs which are supported by beams on all four sides. The slab panel carries its load by spanning in two orthogonal directions, so that two-way flexural action is provided, enabling higher loads to be supported with smaller deflections. However, if the ratio of the long side to the short side of the panel exceeds two, the applied loading is supported predominantly by flexural action in the direction of the shorter span and the panel acts as a one-way slab.
- Flat plates which are slabs supported directly by columns, at their corners, without utilizing beams. They provide a clear, unobstructed soffit, are simple to construct, and provide a minimum depth of construction.
- Flat slabs which are flat plates thickened at the columns with drop panels or column capitals. This increases the resistance of the slab to the high moments and shears at the columns and enables increased loads to be supported over longer spans.
- Two-way grids or waffle slabs which are flat slabs constructed with voids formed in the

soffit with dome shaped fiberglass forms. This produces an orthogonal grillage with a lesser dead load than a flat slab. At the columns, the voids are omitted so as to produce a solid slab equivalent to a drop panel.

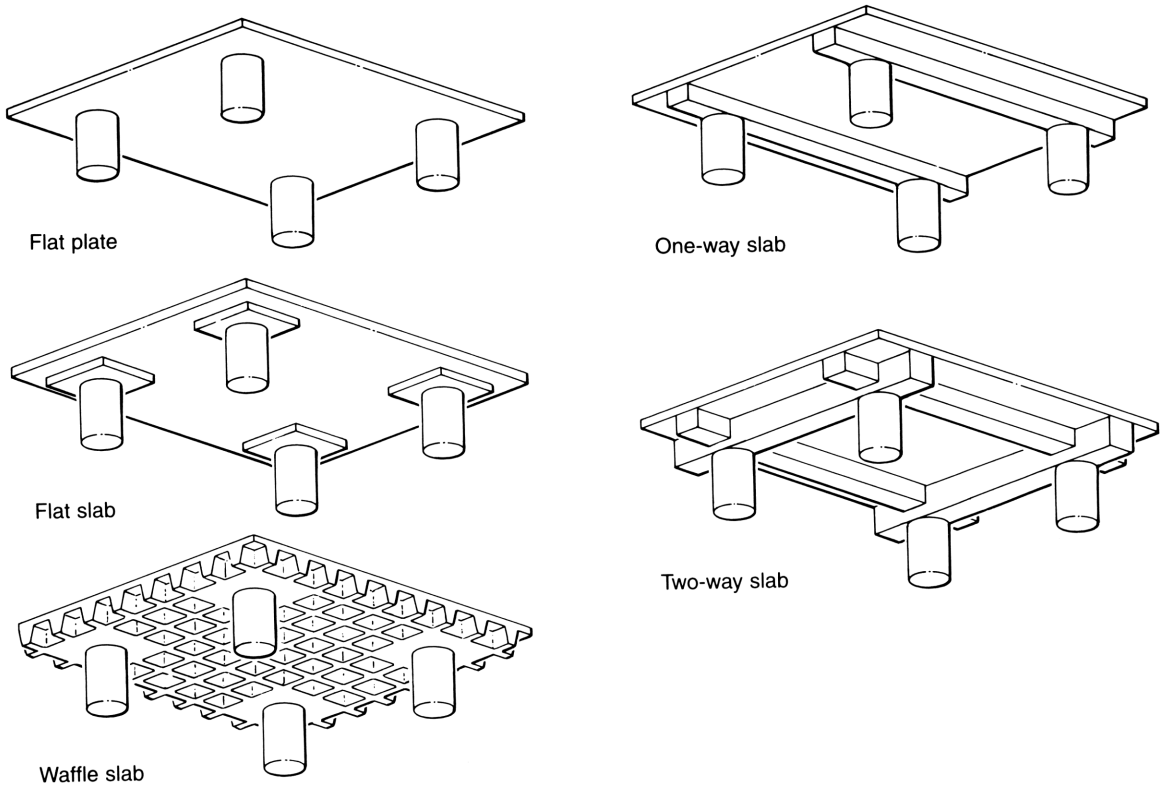

Figure 7–1 Types of structural slabs

7.2 ONE–WAY SLABS

7.2.1 Design of one–way slabs

One–way slabs are designed with the principal reinforcement placed perpendicular to the supporting beams, in the direction of flexure. The area of reinforcement required may be determined from the bending moments calculated from the approximate coefficients of ACI Section 8.3.3 or by exact analytical techniques. For the design, a one foot wide strip of slab is analyzed and the calculated reinforcement used throughout the slab. A concentrated load applied to the slab also produces transverse bending, and reinforcement is required parallel to the supporting beams to resist this bending moment. The area of distribution reinforcement required is determined from empirical detailing rules in ACI Section 7.12.2.1. The location of the principal reinforcement and the distribution reinforcement is shown in Figure 7–2.

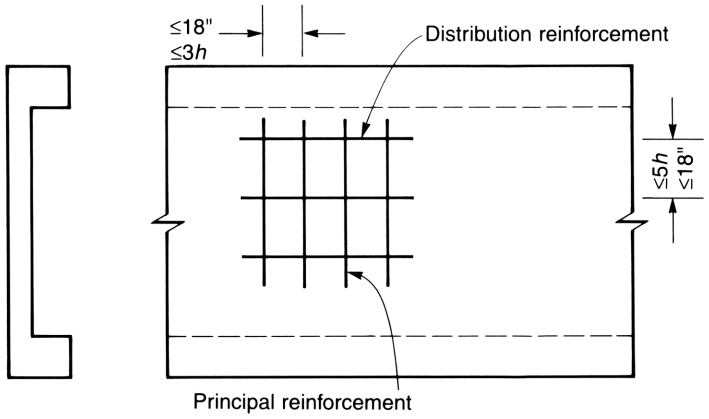

Figure 7–2 One–way slab reinforcement

7.2.2 Deflection criteria

For reinforced concrete one–way slabs not supporting deflection sensitive construction, the allowable deflection may be deemed satisfied if the minimum thickness requirements of ACI Table 9.5(a) are adopted. For normal weight concrete and Grade 60 reinforcement the minimum thicknesses specified are:

h = $\ell/20$ ... simply supported slab
h = $\ell/24$ ... slab continuous at one end
h = $\ell/28$ ... slab continuous at both ends
h = $\ell/10$ ... cantilever slab

When the one–way slab supports deflection sensitive construction, deflections must be calculated in accordance with the requirements of ACI Section 9.5.2 and the maximum allowable short–term plus long–term deflection is given by ACI Table 9.5(b) as:

$\delta = \ell/480$

where ℓ = span length defined in ACI Section 8.7

7.2.3 Reinforcement detailing

The maximum spacing of the principal reinforcement is limited by ACI Section 7.6.5 to

s = 3h
≤ 18 inches

where h = slab thickness

The spacing of distribution reinforcement, which is required to resist the shrinkage and temperature stresses produced by lateral restraints imposed on the slab in addition to the transverse flexural stresses caused by concentrated loads, is limited by ACI Section 7.12.2.2 to

s = 5h
≤ 18 inches

The minimum reinforcement ratio specified for distribution steel in ACI Section 7.12.2.1 is

$A_s/A_g = 0.0020$ ... for Grade 40 and 50 reinforcement
$A_s/A_g = 0.0018$ ... for Grade 60 reinforcement
$A_s/A_g = 0.0018 \times 60{,}000/f_y$ ... for $f_y > 60{,}000$ pounds per square inch
≥ 0.0014

where A_s = area of distribution reinforcement
A_g = gross area of slab cross section

The minimum cover to reinforcement in slabs not exposed to the weather, for No.11 bars and smaller, is specified in ACI Section 7.7.1 as ¾ of an inch.

The curtailment of reinforcement in one-way slabs may be determined in accordance with the requirements of ACI Section 12.10. When loads are uniformly distributed and spans are nominally equal, the simplified detailing rules[1,2,3] shown in Figure 7-3 may be adopted.

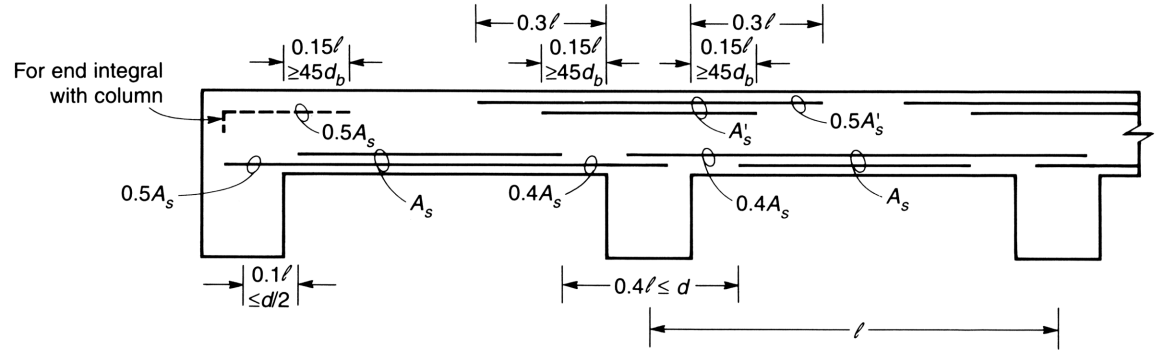

Figure 7-3 Reinforcement detailing in slabs

Example 7-1 (Design of a one-way slab)

A solid, normal weight, reinforced concrete slab, for an assembly area with movable seats, spans between 8 inch wide beams at 15 feet centers over 5 bays. The discontinuous ends of the slab are unrestrained. The superimposed dead load, due to finishes and services is 40 pounds per square foot. The concrete strength is 3000 pounds per square inch and all reinforcement consists of Grade 60 No.4 bars. The slab does not support any deflection sensitive partitions or ceilings.

Determine the minimum permissible thickness of the slab and the maximum allowable effective depth. Determine the required spacing of No.4 bars in the end span and at the penultimate support. Determine the maximum permissible spacing of No.4 bars to provide the transverse reinforcement required and the nominal maximum size of coarse aggregate allowable. What is the maximum diameter of pipe which may be embedded in the slab without prior approval of the structural engineer.

Solution

From ACI Table 9.5(a), the minimum permissible thickness of the end span is

$$h = \ell/24$$

where ℓ = distance center–to–center of supports
= 15 feet

and $h = 15 \times 12/24$
$= 7.5$ inches

From ACI Section 7.7.1, the minimum concrete cover for a slab not exposed to the weather is ¾ of an inch. Hence, the maximum effective depth is

$$d = 7\frac{1}{2} - \frac{3}{4} - \frac{1}{4}$$
$$= 6.5 \text{ inches}$$

Hence, the minimum permissible thickness of the slab and the maximum allowable effective depth are 7.5 and 6.5 inches, respectively.

From UBC[4] Table 16–A or ASCE[5] Table 2, the specified live load for a public assembly area with movable seats is

$$w_L = 100 \text{ pounds per square foot}$$

In accordance with UBC Section 1606 or ASCE Section 4.8.2, no reduction is permitted in the live load.

The dead load is given by

$$w_D = 40 + 150 \times 7.5/12$$
$$= 133.75 \text{ pounds per square foot}$$

The factored loading is specified by ACI Equation (9–1) as

$$w_u = 1.4w_D + 1.7w_L$$
$$= 1.4 \times 133.75 + 1.7 \times 100$$
$$= 357.25 \text{ pounds per square foot}$$

The approximate moment factors given in ACI Section 8.3.3 are applicable and the moment in the end span is given by

$$M_u = w_u\ell_n^2/11$$

where ℓ_n = clear span
= 14.33 feet

and $M_u = 357.25 \times 14.33^2/11{,}000$
$= 6.67$ kip feet

The moment at the penultimate support is

$$M_u = w_u\ell_n^2/10$$
$$= 357.25 \times 14.33^2/10{,}000$$
$$= 7.34 \text{ kip feet}$$

The design moment factor for the end span is given by

$$K_u = M_u/bd^2$$
$$= 12 \times 6.67/(12 \times 6.5^2)$$
$$= 0.158 \text{ kips per square inch.}$$

The required reinforcement area in the end span is given by

$$A_s = 0.85bdf'_c[1 - (1-K_u/0.383f'_c)^{0.5}]/f_y$$

This may be solved by calculator program to give

$$A_s = 0.236 \text{ square inches per foot}$$

No.4 bars at 10 inches on center provide an area of 0.24 square inches per foot.

The design moment factor for the penultimate support is given by

$$K_u = M_u/bd^2$$
$$= 12 \times 7.34/(12 \times 6.5^2)$$
$$= 0.174 \text{ kips per square inch.}$$

and $A_s = 0.261$ square inches per foot

No.4 bars at 9 inches on center provide an area of 0.27 square inches per foot.

Hence, the required spacing of No.4 bars in the end span and at the penultimate support are 10 and 9 inches, respectively.

The minimum area of transverse reinforcement required is specified in ACI Section 7.12.2.1 as

$$A_s = 0.0018A_c$$

where A_c = gross concrete area
$$= 12 \times 7.5$$
$$= 90 \text{ square inches}$$

and $A_s = 0.0018 \times 90$
$$= 0.162 \text{ square inches per foot}$$

The maximum allowable spacing is given by ACI Section 7.12.2.2 as

$$s_{max} = 5h$$
$$= 37.5 \text{ inches}$$
$$\leq 18 \text{ inches ... which governs}$$

Hence, provide No.4 bars at 15 inches on center to give an area of 0.16 square inches per foot.

The nominal maximum size of coarse aggregate is specified by ACI Section 3.3.2 as

$$h/3 = 7.5/3$$
$$= 2.5 \text{ inches}$$

The maximum diameter of pipe allowable is specified in ACI Section 6.3.5.1 as

$$h/3 = 7.5/3$$
$$= 2.5 \text{ inches}$$

7.3 TWO-WAY SLABS

7.3.1 Design of two-way slabs

The following methods may be employed in the design of two-way slabs for gravity loads:

- The direct design method which uses empirical coefficients to proportion the free span moment between a strip of slab over the columns (the column strip) and a strip of slab between the columns (the middle strip).
- The equivalent frame method in which the structure is divided into equivalent frames with a width extending to the mid point between column centers. The equivalent frame may be analyzed by any frame analysis technique.
- The yield line method[6,7] which is based on limit state theory applied to an appropriate collapse mechanism. Yield line analysis is suitable for the design of irregularly shaped slabs and for slabs supporting concentrated loads.

7.3.2 Direct design method

The criteria necessary for the use of the direct design method are specified in ACI Section 13.16.1 and are:

- A minimum of three continuous spans are required in each direction. The method is not applicable to a system with two continuous spans because of the large negative moment produced at the central support of a two span system.
- Panels shall be rectangular with an aspect ratio not exceeding two. Panels with an aspect ratio exceeding two behave essentially as one-way slabs.
- Successive spans shall not differ by more than one third of the longer span. Span differences greater than this produce negative moments beyond the point prescribed for the curtailment of the negative moment reinforcement.
- Columns may be offset not more than ten percent of the span. An offset exceeding this amount produces unacceptable deviations from the prescribed design values.
- The method is applicable to uniformly distributed gravity loads only, with the live load not exceeding two times the dead load. The panel is analyzed for the single load case of maximum design load applied to the panel. The application of lateral loads requires a frame analysis procedure.
- Where beams support panel edges, the ratio of the beam stiffnesses in two perpendicular directions shall be between 0.2 and 5.0. Stiffnesses outside these limits produce unacceptable deviations from the prescribed design values.

The initial procedure consists of subdividing the panel into sections as defined in ACI Section 13.2 and as shown in Figure 7-4. A column strip consists of a strip of slab extending $0.25\ell_{min}$ on each side of a column centerline and includes any beam which is present. The width of a column strip in the direction of both the shorter and longer span is given by

$$w_c = 0.5\ell_{min}$$

The width of a middle strip in the direction of the shorter span is given by

$$w_m = \ell_{max} - 0.5\ell_{min}$$

The width of a middle strip in the direction of the longer span is given by

$$w_m = \ell_{min} - 0.5\ell_{min}$$

where ℓ_{min} = lesser of ℓ_1 and ℓ_2
ℓ_{max} = greater of ℓ_1 and ℓ_2
ℓ_1 = length of span in the direction that moments are being determined, measured center-to-center of supports
ℓ_2 = length of span transverse to ℓ_1

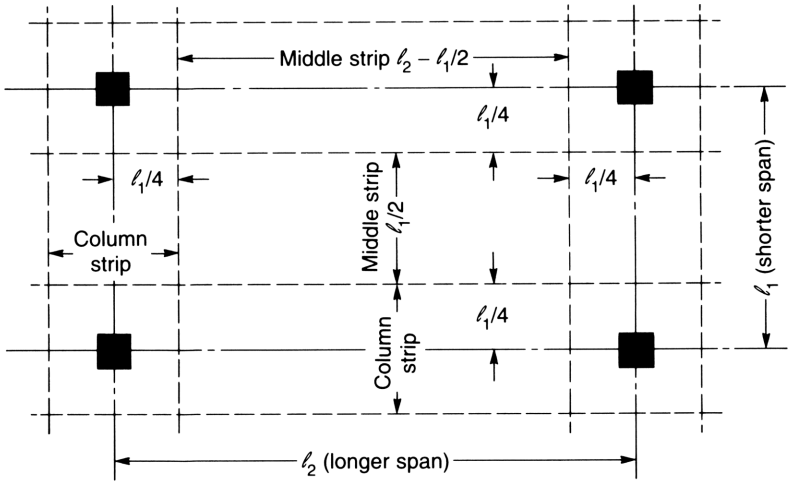

Figure 7–4 Subdivision of panels

The total free or static moment acting in the direction of ℓ_1 is given by ACI Equation (13-3) as

$$M_o = w_u \ell_2 \ell_n^2 / 8$$

where ℓ_n = clear span in the direction of ℓ_1
$= \ell_1 - c_1$
$\geq 0.65\ell_1$
c_1 = column width in the direction of ℓ_1

This moment is distributed, in accordance with ACI Section 13.6.3, to give positive and negative moments across the full width of the panel. At an interior panel, irrespective of support conditions, the total positive moment distributed across the panel at mid-span is

$$M_m = 0.35M_o$$

At an interior panel, irrespective of support conditions, the total negative moment distributed across the panel at column center lines is

$$M_c = 0.65M_o$$

In an end panel, the moments distributed across the full width of the panel depend on the support conditions. These values are specified in ACI Section 13.6.3 and are shown in Figure 7–5.

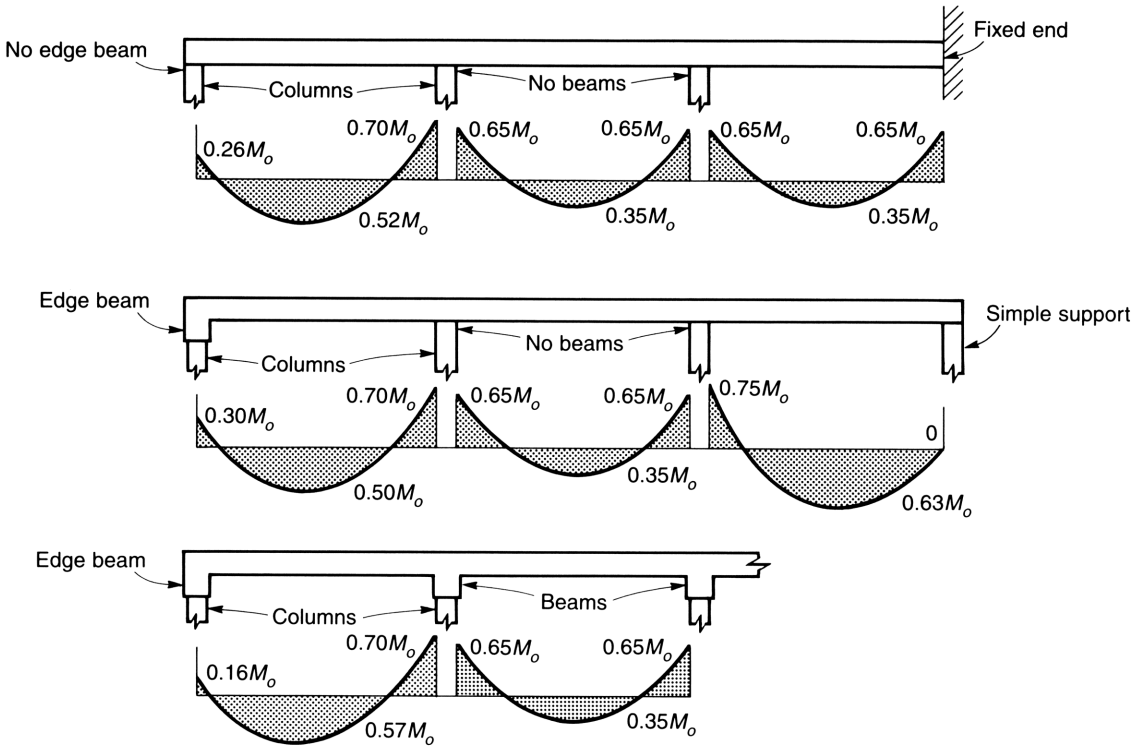

Figure 7–5 Distribution of static moment for various support conditions

At the first interior support, the panel is designed for the larger of the two distributed moments, or the unbalanced moment is distributed to adjoining elements in proportion to their stiffnesses.

The distributed moments across the full width of the panel are further subdivided between column and middle strips as specified in ACI Section 13.6.4. The proportion attributed to the column strip depends on the relative stiffness of the panel and edge beam, if present, where the relative stiffness is defined as

$$\alpha = E_{cb}I_b/E_{cs}I_s$$

where E_{cb} = modulus of elasticity of beam concrete
E_{cs} = modulus of elasticity of panel concrete
I_b = moment of inertia of the equivalent beam defined in ACI Section 13.2.4
I_s = moment of inertia of the panel between column lines

Dimensions of the equivalent beam and slab are shown in Figure 7–6.

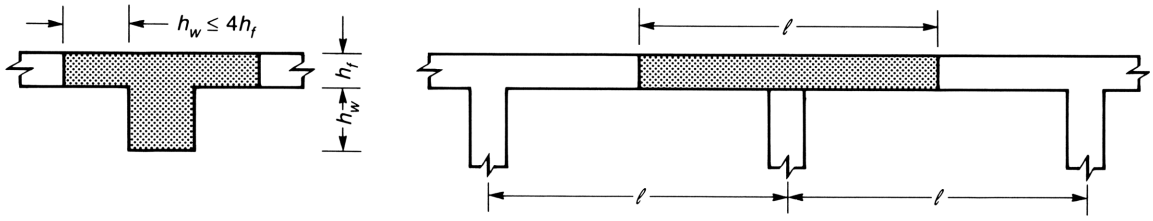

Figure 7–6 Dimensions of equivalent beam and slab

The negative moment at an interior support is distributed to a column strip as shown in Table 7–1.

ℓ_2/ℓ_1	0.5	1	2
$(\alpha_1\ell_2/\ell_1) = 0$	75	75	75
$(\alpha_1\ell_2/\ell_1) \geq 1.0$	90	75	45

Table 7–1 Percentage distribution of total interior negative moment to column strip

At an edge support, allowance must be made for the relative torsional stiffness of the edge beam, if present, and this is defined in ACI Section 13.0 as

$$\beta_t = E_{cb}C/2E_{cs}I_s$$

where C = torsional constant given in ACI Section 13.0

$$= \Sigma(1 - 0.63x/y)x^3y/3$$

x = smaller dimension of rectangular component

y = larger dimension of rectangular component

The negative moment at an exterior support is distributed to a column strip as shown in Table 7–2.

ℓ_2/ℓ_1		0.5	1	2
$(\alpha_1\ell_2/\ell_1) = 0$	$\beta_1 = 0.0$	100	100	100
	$\beta_1 \geq 2.5$	75	75	75
$(\alpha_1\ell_2/\ell_1) \geq 1.0$	$\beta_1 = 0.0$	100	100	100
	$\beta_1 \geq 2.5$	90	75	45

Table 7–2 Percentage distribution of total exterior negative moment to column strip

The positive moment at mid-span is distributed to a column strip as shown in Table 7–3.

ℓ_2/ℓ_1	0.5	1	2
$(\alpha_1\ell_2/\ell_1) = 0$	60	60	60
$(\alpha_1\ell_2/\ell_1) \geq 1.0$	90	75	45

Table 7–3 Percentage distribution of total positive moment to column strip

The proportion of the distributed moment assigned to half middle strips is defined in ACI Section 13.6.6 as that portion of the distributed moment which is not attributed to a column strip.

In accordance with ACI Section 13.6.5 beams at panel edges shall be assigned the percentage

of column strip moment given in Table 7–4. Beams shall be designed, in addition, to support all loads applied directly to the beams.

$\alpha_1 \ell_2/\ell_1$	0	≥ 1.0
% assigned to beam	0	85

Table 7–4 Percentage distribution of column strip moments to beam

Positive and negative moments may be modified by a maximum of ten percent, in accordance with ACI Section 13.6.7, provided that the total static moment is not reduced.

Example 7.2 (Design of a flat plate for flexure)

A reinforced concrete flat plate floor without beams has 18 inches square columns at 20 feet centers in both directions and supports a factored load of 250 pounds per square foot. For an interior bay, compute the positive and negative moments of the column strip and of the middle strip for an end span, interior span, and interior support.

Solution

The clear span in both directions is

$$\begin{aligned} \ell_n &= \ell_1 - c_1 \\ &= 20 - 1.5 \\ &= 18.5 \text{ feet} \\ &> 0.65\ell_1 \ldots \text{ satisfactory} \end{aligned}$$

For an interior bay, the widths of a column strip and of a middle strip in the north–south direction are defined by ACI Section 13.2 as

$$\begin{aligned} w_c &= \ell_2/2 \\ &= 20/2 \\ &= 10 \text{ feet} \end{aligned}$$

and

$$\begin{aligned} w_m &= \ell_2 - w_c \\ &= 20 - 10 \\ &= 10 \text{ feet} \end{aligned}$$

The total factored static moment is given by

$$\begin{aligned} M_o &= w_u \ell_2 \ell_n^2/8 \\ &= 0.25 \times 20 \times 18.5^2/8 \\ &= 214 \text{ kip feet} \end{aligned}$$

$$\alpha_1 = \alpha_2 = \beta_1 = 0.0$$

and

$$\ell_2/\ell_1 = 1.0$$

Hence M_o is distributed, as specified in ACI Section 13.6.3.3, to give positive and negative moments across the full width between columns, and then distributed to the column and middle strips in accordance with ACI Sections 13.6.4 and 13.6.6 as indicated in Table 7–5.

Strip	Section	End span	Interior support	Interior span
Full Width	Distribution Coeff Moment	0.52 111	0.70 150	0.35 75
Column Strip	Distribution Coeff Moment	0.60 67	0.75 112	0.60 45
Middle Strip	Distribution Coeff Moment	0.40 44	0.25 38	0.40 30

Table 7–5 Distribution of moments for Example 7–2

7.3.3 Shear requirements

The treatment of shear at the column supports of flat plates or slabs is covered in ACI Section 11.12. A two-way slab must be designed for both flexural or one-way shear and punching or two-way shear. The critical section for flexural shear is located a distance from the face of the columns equal to the effective depth, as shown in Figure 7–7, and the slab is considered to act as a wide beam spanning between columns.

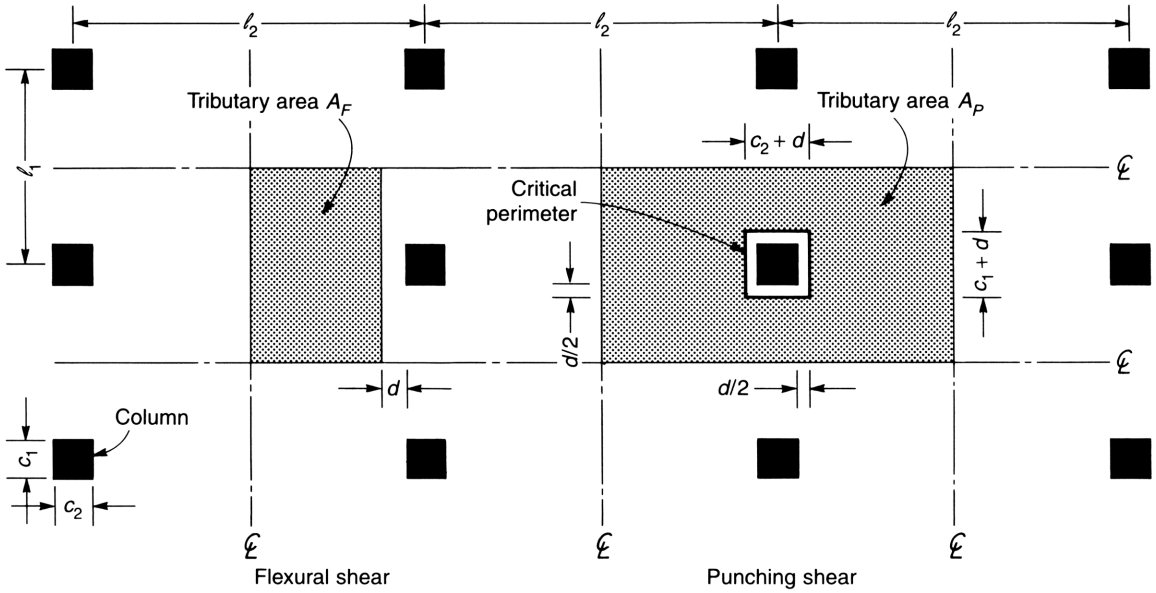

Figure 7–7 Critical sections for shear

The factored shear force acting on the critical section is

$$V_u = w_u A_F$$

where w_u = distributed factored load on the slab

A_F = tributary area contributing to the flexural shear

The flexural shear capacity of the slab is given by ACI Equation (11-3) as

$$\phi V_c = 2\phi d \ell_1 (f'_c)^{0.5}$$

where ϕ = strength reduction factor
= 0.85 ... from ACI Section 9.3.2.3
ℓ_1 = distance between panel center lines

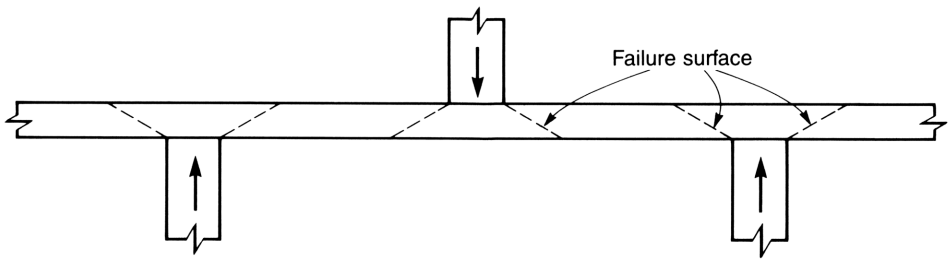

Figure 7–8 Punching shear failure surface

Punching shear[8] is a type of shear failure which occurs in the form of a truncated cone around the periphery of a column or concentrated load as shown in Figure 7-8. Punching shear usually occurs suddenly, without warning of impending failure. The Code recommendations are empirical and are expressed in terms of a nominal shear stress acting on a defined area. The critical perimeter for punching shear, as specified in ACI Section 11.12.1.2, is located a distance from the face of columns or concentrated loads equal to half the effective depth. As shown in Figure 7-7 the length of the critical perimeter is given by

$$b_o = 2(c_1 + c_2) + 4d$$

where c_1 = short side of column
c_2 = long side of column
d = average effective depth of the two layers of slab reinforcement

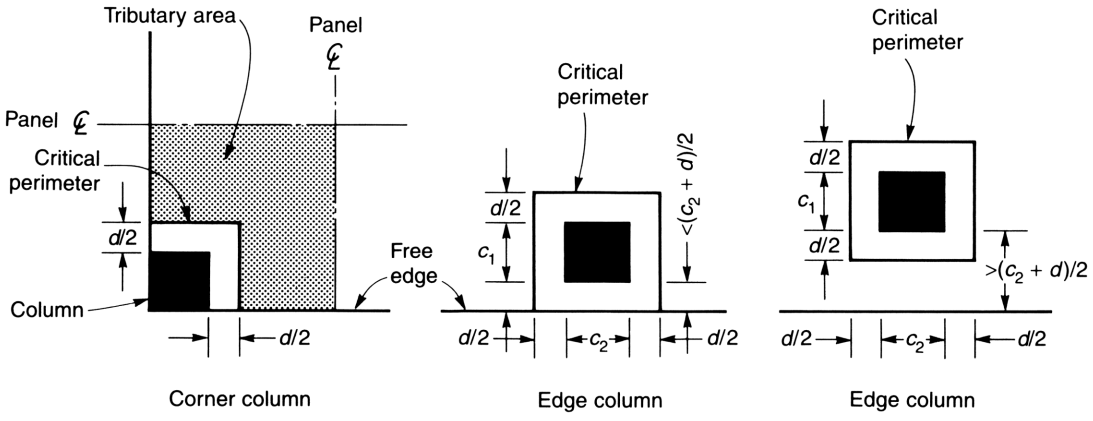

Figure 7–9 Length of critical perimeter

The factored punching shear force acting on the critical perimeter is

$$V_u = w_u A_P$$

where w_u = distributed factored load on the slab
A_P = tributary area contributing to the punching shear

The slab punching shear capacity, for a square column, is given by ACI Equation (11–37) as

$$\phi V_c = 4\phi d b_o (f'_c)^{0.5}$$

For a rectangular column, when the ratio of the lengths of the long and short sides of the column exceeds 2.0, experimental data indicates that this expression is unconservative and the punching shear capacity is given by ACI Equation (11–35) as

$$\phi V_c = \phi d b_o (2 + 4/\beta_c)(f'_c)^{0.5}$$

where β_c = ratio of long side to short side of column
≤ 2

In addition, experimental data indicates that the shear capacity reduces as the ratio of the length of the critical perimeter to the effective depth increases and the punching shear capacity for this condition is given by ACI Equation (11–36) as

$$\phi V_c = \phi d b_o (2 + \alpha_s d/b_o)(f'_c)^{0.5}$$

where α_s = 40 for interior columns
= 30 for edge columns
= 20 for corner columns

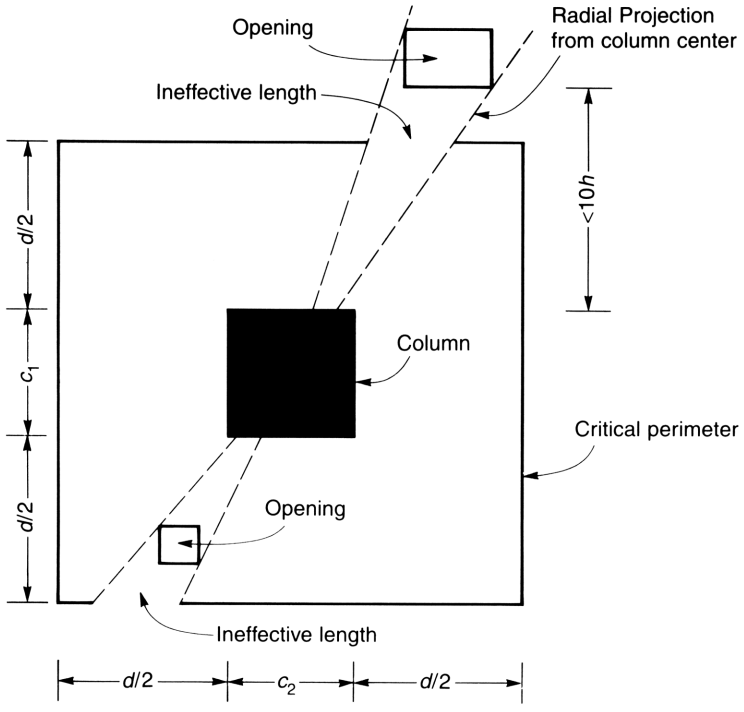

Figure 7–10 Effect of openings on the critical perimeter

Where a column is close to a free edge, the critical perimeter is taken as the lesser of the three sided perimeter or the four sided perimeter, as shown in Figure 7–9. For a corner column, the critical perimeter is two sided as shown.

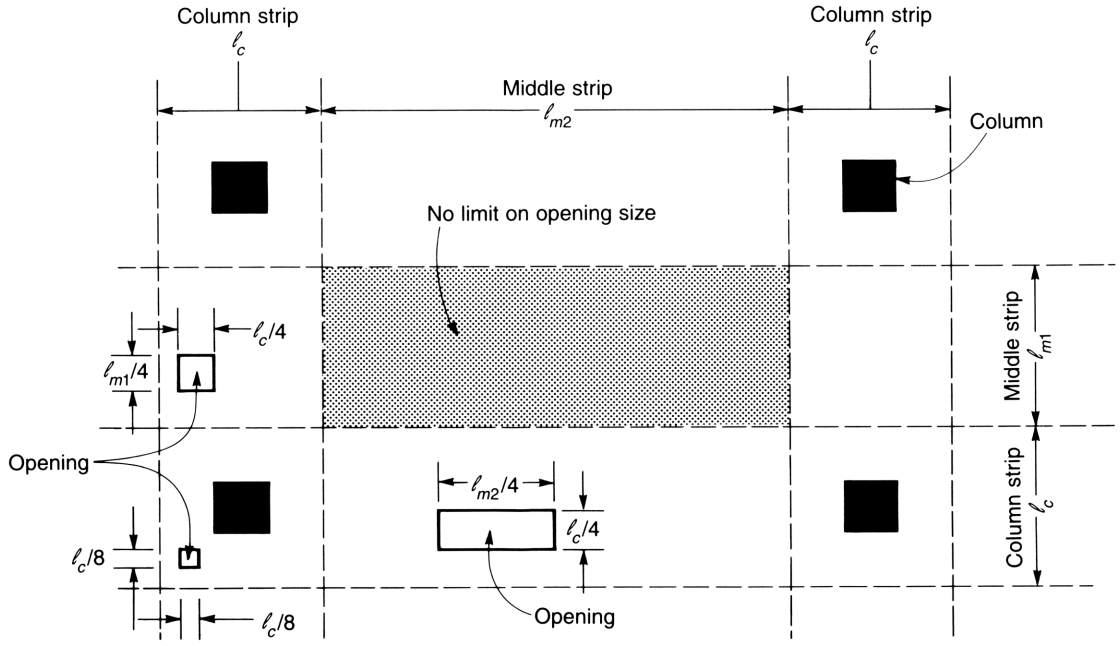

Figure 7–11 Maximum permitted opening size

When openings in the slab are within ten times the thickness of the slab from the edge of a column, that part of the critical perimeter that is enclosed by radial projections from the center of the column to the opening shall be considered ineffective as shown in Figure 7–10. As specified in ACI Section 13.4, and shown in Figure 7–11, openings of any size are permitted in the area common to intersecting middle strips. In the area common to intersecting column strips, the width of the opening shall not exceed one-eighth the width of the column strip. In the area common to one column strip and one middle strip, not more than one-quarter the reinforcement in either strip shall be interrupted by openings. The area of reinforcement interrupted by all openings shall be replaced by an equivalent amount added on the sides of the opening.

Example 7.3 (Design of a flat plate for shear)

The reinforced concrete flat plate floor without beams, shown in Figure 7–12, has 20 inches diameter columns at 20 feet centers in both directions and supports a factored load of 250 pounds per square foot. For an interior column, compute the minimum required effective depth to resist punching shear if the concrete strength is 4000 pounds per square inch.

Solution

The critical section for punching shear in a flat plate, as defined by ACI Section 11.12.1.2, is located on the perimeter of a circle with a diameter of

$$h_o = h + d$$

where h = diameter of column
$= 20$ inches
d = effective depth

and $h_o = 20 + d$

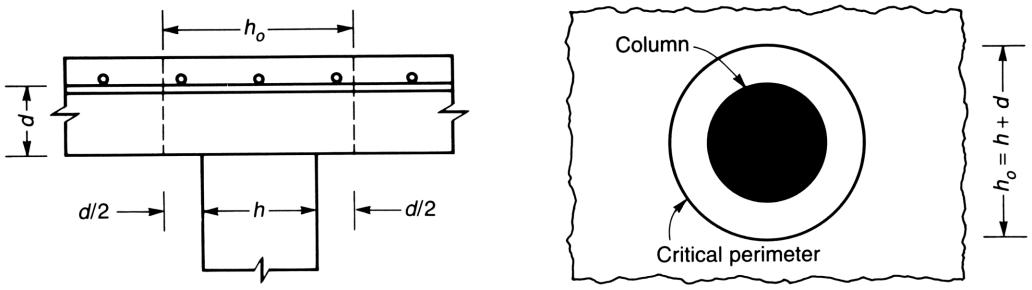

Figure 7–12 Details for Example 7–3

The length of the critical perimeter is

$$b_o = \pi h_o$$
$$= \pi(20 + d)$$

Hence the shear capacity of the slab for two-way action is given by ACI Equation (11–37) as

$$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$$

where ϕ = strength reduction factor
$= 0.85$ from ACI Section 9.3

and $\phi V_c = 4 \times 0.85 \times \pi(20 + d) \times d(4000)^{0.5}$
$= 13512.79d + 675.64d^2$

The factored applied shear at the critical perimeter is

$$V_u = w_u(\ell_1 \times \ell_2 - \pi h_o^2/4)$$
$$= 250(20 \times 20 - \pi(20 + d)^2/(4 \times 144)$$
$$= 99454.51 - 54.55d - 1.36d^2$$
$$= \phi V_c$$

Hence $0 = 677d^2 + 13567.34d - 99454.51$

and $d = 5.7$ inches

When the loading on a flat slab is distributed uniformly around a column, equal moments develop in the slab on either side of the column as shown in Figure 7–13. Shear stress at the critical perimeter is then due entirely to the gravity loading and is uniformly distributed around the critical perimeter. If unequal moments develop in the slab on either side of the column due to non-uniform loading, the unbalanced moment is transmitted from the slab to the column. Experimental data indicates that, for a square column, approximately 60 percent of the unbalanced moment is transmitted by flexure and the remainder by shear, as shown in Figure 7–14. When both gravity loads and unbalanced moments occur, the shear stress due to both conditions are combined to give

$v_u = V_u/db_o + \gamma_v M_u c/J_c$

where

V_u = factored shear force acting on the critical perimeter

d = average effective depth

b_o = length of the critical perimeter
$= b_1 + b_2$

M_u = unbalanced moment at the centroid of the critical perimeter

c = distance from the centroid of the critical perimeter to edge of perimeter

γ_v = fraction of the unbalanced moment transferred by shear
$= 1 - \gamma_f$

γ_f = fraction of the unbalanced moment transferred by flexure
$= 1/[1 + 0.67(b_1/b_2)^{0.5}]$

b_1 = width of the critical perimeter measured in the direction of M_u
$= c_1 + d$

b_2 = width of the critical perimeter measured perpendicular to b_1
$= c_2 + d$

c_1 = width of the column measured in the direction of M_u

c_2 = width of the column measured perpendicular to c_1

J_c = polar moment of inertia of critical perimeter

$J_c/c = [b_1 d(b_1 + 3b_2) + d^3]/3$ ... for an interior column

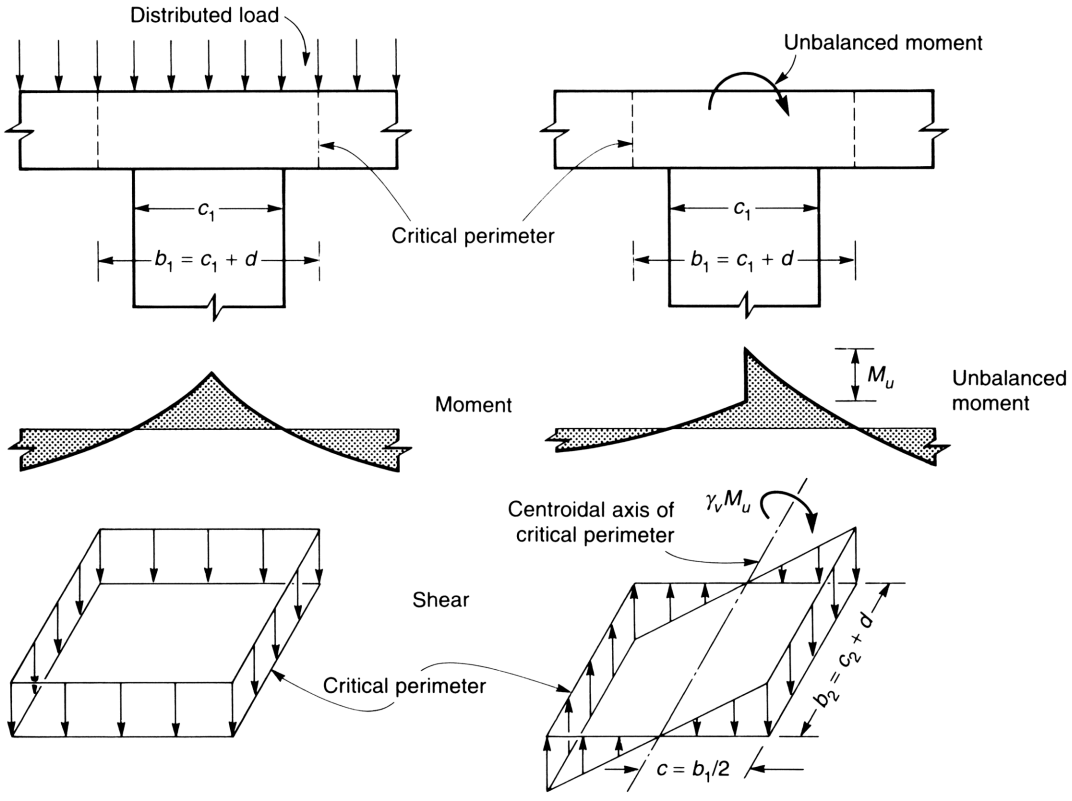

Figure 7–13 Shear due to distributed load

Figure 7–14 Shear due to unbalanced moment

For an edge column, the polar moment of inertia is given by

$$J_c/c = [2b_1^2d(b_1 + 2b_2) + d^3(2b_1 + b_2)]/6b_1 \quad \text{... bending perpendicular to edge}$$

where $c = b_1^2/(2b_1 + b_2)$

For a corner column, the polar moment of inertia is given by

$$J_c/c = [b_1^2d(b_1 + 4b_2) + d^3(b_1 + b_2)]/6b_1$$

where $c = b_1^2/2(b_1 + b_2)$

For an edge column, ACI Section 13.6.3.6 specifies that the unbalanced moment to be transferred by shear is

$$M_u = 0.3M_o$$

where M_o = total static moment acting perpendicular to the edge of slab
$= w_u\ell_2\ell_n^2/8$

For a column, ACI Section 13.5.3.2 specifies that the fraction of the unbalanced moment to be transferred by flexure is $\gamma_f M_u$ and for an edge column the unbalanced moment consists of the total moment in the column strip and is given by

$$M_u = 0.26M_o \quad \text{... from ACI Section 13.6.3.3}$$

where M_o = total static moment acting perpendicular to the edge of slab
$= w_u\ell_2\ell_n^2/8$

The additional reinforcement required to resist the moment $\gamma_f M_u$ is placed within an effective slab width given by ACI Section 13.5.3.2 as

$$b_c = c_2 + 3h$$

where c_2 = width of column measured perpendicular to the direction of M_u
h = overall slab depth

Example 7.4 (Design for shear due to unbalanced moment)

The 18 inches square interior column of a flat plate floor supports a factored shear force of 100 kips and an unbalanced moment of 40 kip feet. The effective depth of the slab is 7 inches and the concrete strength is 4000 pounds per square inch. Determine if the shear capacity of the slab is adequate.

Solution

The critical section for punching shear in a flat plate, as defined by ACI Section 11.12.1.2, is located on the perimeter of a rectangle with sides of length

$$b_1 = c_1 + d$$
$$= 18 + 7$$
$$= 25 \text{ inches}$$
$$= b_2$$

The length of the critical perimeter is

$$b_o = 2(b_1 + b_2)$$
$$= 2(25 + 25)$$
$$= 100 \text{ inches}$$

The polar moment of inertia of the critical perimeter is given by

$$J_c/c = [b_1 d(b_1 + 3b_2) + d^3]/3 \ldots \text{ for an interior column}$$
$$= [25 \times 7(25 + 3 \times 25) + 7^3]/3$$
$$= 5948 \text{ inches}^3$$

The fraction of the unbalanced moment transferred by shear is

$$\gamma_v = 1 - 1/[1 + 0.67(b_1/b_2)^{0.5}]$$
$$= 1 - 1/1.67$$
$$= 0.40$$

The ratio of the long side to the short side of the column is

$$\beta = c_2/c_1$$
$$= 25/25$$
$$= 1.00$$
$$< 2$$

Hence the allowable shear stress for two-way action is given by ACI Equation (11-37) as

$$\phi v_c = 4\phi(f'_c)^{0.5}$$

where ϕ = strength reduction factor
= 0.85 from ACI Section 9.3

and $\phi v_c = 4 \times 0.85(4000)^{0.5}$
= 215 pounds per square inch

The combined shear stress due to the applied shear force and the unbalanced moment is

$$v_u = V_u/db_o + \gamma_v M_u c/J_c$$
$$= 100 \times 1000/(7 \times 100) + 0.4 \times 40 \times 12{,}000/5948$$
$$= 142 + 32$$
$$= 174 \text{ pounds per square inch}$$
$$< \phi v_c$$

Hence the shear capacity of the slab is satisfactory.

Example 7.5 (Design for punching shear and moment[9])

The reinforced concrete flat plate floor, shown in Figure 7-15, with an effective depth of 7.5 inches, has 20 inches square columns at 26 feet centers in the north-south direction and at 28 feet centers in the east-west direction. The concrete strength is 4500 pounds per square inch. The total unfactored dead load (including self weight of slab and partitions) is 125 pounds per square foot. The unfactored floor live load is 30 pounds per square foot and no reduction need be considered. For an interior column, compute the punching shear force and shear capacity of the slab. For an interior bay, compute the positive and negative factored moments in the column strip and middle strip in the east-west direction and the width of the column and middle strips

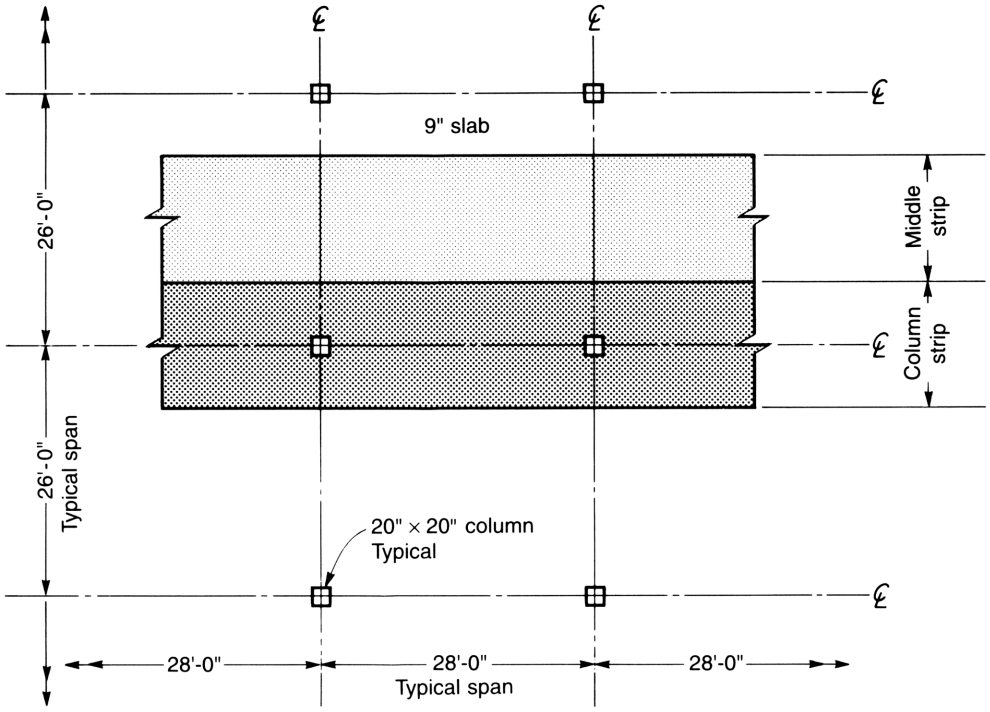

Figure 7–15 Details for Example 7–5

Solution

The factored load is obtained from ACI Section 9.2 Equation (9–1) as

$$\begin{aligned} w_u &= 1.4\, w_D + 1.7 w_L \\ &= 1.4 \times 125 + 1.7 \times 30 \\ &= 226 \text{ pounds per square foot} \end{aligned}$$

The critical section for punching shear in a flat plate, as defined by ACI Section 11.12.1.2, is located on the perimeter of a rectangle with sides of length

$$\begin{aligned} b_1 &= c_1 + d \\ &= 20 + 7.5 \\ &= 27.5 \text{ inches} \\ &= b_2 \end{aligned}$$

The length of the critical perimeter is

$$\begin{aligned} b_o &= 2(b_1 + b_2) \\ &= 2(27.5 + 27.5) \\ &= 110 \text{ inches} \end{aligned}$$

The ratio of the long side to the short side of the column is

$$\begin{aligned}\beta &= c_2/c_1\\ &= 20/20\\ &= 1.17\\ &< 2\end{aligned}$$

Hence the shear capacity of the flat plate for two-way action is given by ACI Equation (11-37) as

$$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$$

where ϕ = strength reduction factor
= 0.85 from ACI Section 9.3

and $\phi V_c = 4 \times 0.85 \times 110 \times 7.5(4500)^{0.5}/1000$
= 188 kips

The factored applied shear at the critical perimeter is

$$\begin{aligned}V_u &= w_u(\ell_1 \times \ell_2 - b_1 \times b_2)\\ &= 0.226(28 \times 26 - 27.5 \times 27.5/144)\\ &= 163 \text{ kips}\\ &< \phi V_c\end{aligned}$$

Hence, the shear capacity of the slab is satisfactory.

The clear span in the east-west direction is

$$\begin{aligned}\ell_n &= \ell_1 - c_1\\ &= 28 - 20/12\\ &= 26.33 \text{ feet}\\ &> 0.65\ell_1\end{aligned}$$

Then in accordance with ACI Section 13.6.2.5 the value to be adopted for the clear span is

$$\ell_n = 26.33 \text{ feet}$$

For an interior bay, the widths of a column strip and of a middle strip in the east-west direction are defined by ACI Section 13.2 as

$$\begin{aligned}w_c &= \ell_2/2\\ &= 26/2\\ &= 13 \text{ feet}\end{aligned}$$

and $w_m = \ell_2 - w_c$
$= 26 - 13$
$= 13$ feet

From ACI Section 13.6.2, the total factored static moment in the east-west direction across the full width of a panel is

$$\begin{aligned}M_o &= w_u\ell_2\ell_n^2/8\\ &= 0.226 \times 26 \times 26.33^2/8\\ &= 509 \text{ kip feet}\end{aligned}$$

From ACI Section 13.6.3, the total positive moment across the full width of a panel is

$$M_m = 0.35M_o$$
$$= 178 \text{ kip feet}$$

From ACI Section 13.6.3, the total negative moment across the full width of a panel is

$$M_c = 0.65M_o$$
$$= 331 \text{ kip feet}$$

From ACI Section 13.6.4, $\alpha_1\ell_2/\ell_1 = 0.0$ and the column strip negative moment at an interior support is

$$M_{cc} = 0.75M_c$$
$$= 248 \text{ kip feet}$$

From ACI Section 13.6.6, the middle strip negative moment at an interior support is

$$M_{mc} = 0.25M_c$$
$$= 83 \text{ kip feet}$$

From ACI Section 13.6.4, the column strip positive moment at midspan is

$$M_{cm} = 0.60M_m$$
$$= 107 \text{ kip feet}$$

From ACI Section 13.6.6, the middle strip positive moment at midspan is

$$M_{mm} = 0.40M_m$$
$$= 71 \text{ kip feet}$$

7.3.4 Deflection criteria

The ACI Code provides two methods for control of deflections in two-way slab systems. Deflections may be calculated[10,11] and compared with the maximum allowable values given in ACI Table 9.5(b) Alternatively, the deflection requirements may be deemed satisfied if the minimum thickness requirements[10,12,13] of ACI Table 9.5(c) or ACI Equations (9-11) or (9-12) are adopted.

The allowable values of deflection for two-way slabs are identical with those for one-way slabs. For members not supporting deflection sensitive elements, the allowable, immediate deflections due to live load are

- $\ell/180$ ... flat roofs
- $\ell/360$ ... floors

where ℓ = span length

For roof or floor construction supporting non-sensitive elements, the allowable short-term plus long-term deflection due to loading applied after the construction of the elements is

- $\ell/240$

For roof or floor construction supporting deflection sensitive elements, the corresponding allowable deflection is

- $\ell/480$

The empirical minimum thickness requirements given in ACI Table 9.5(c) have evolved over the years based on previous experience. For slabs with Grade 60 reinforcement and without interior beams and drop panels, the relevant values are shown in Figure 7–16. Members conforming to these span/depth ratios are considered to have sufficient stiffness to prevent allowable deflections being exceeded. In accordance with ACI Section 9.5.3.2 the minimum thickness shall in no case be less than 5 inches.

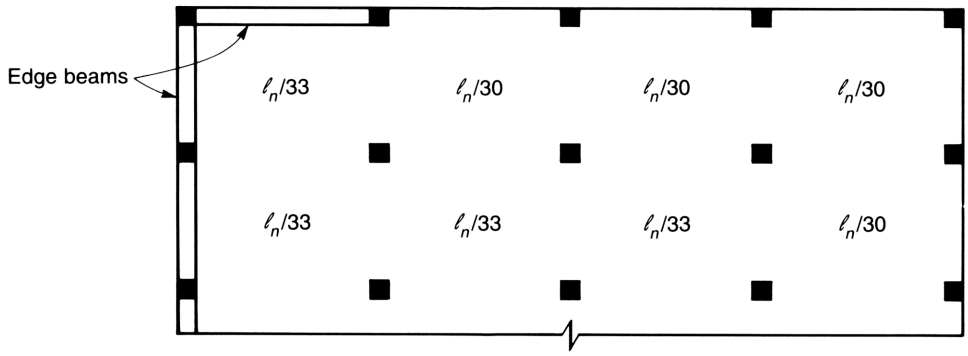

Figure 7–16 Minimum thickness of flat plates with Grade 60 reinforcement

The values given in ACI Table 9.5(c) are also applicable to systems with interior beams when the average value in a panel of the ratio of equivalent beam stiffness to slab stiffness is given by

$$\alpha_m \quad \leq 0.2$$

Dimensions of the equivalent beam and slab are shown in Figure 7–6. For α_m greater than 0.2 but not greater than 2.0, the minimum thickness is specified by ACI Equation (9–11) as

$$h = \ell_n(0.8 + f_y/200{,}000)/[36 + 5\beta(\alpha_m - 0.2)]$$

where β = ratio of long side to short side of column

For α_m greater than 2.0, the minimum thickness is specified by ACI Equation (9–12) as

$$h = \ell_n(0.8 + f_y/200{,}000)/(36 + 9\beta)$$

Example 7.6 (Design of a flat plate for deflection, shear, and flexure[14])

A reinforced concrete flat plate floor plan is shown in Figure 7–17. The plate is reinforced with Grade 60 bars and has a concrete strength of 4000 pounds per square inch. The plate carries a superimposed dead load, including its self weight, of 135 pounds per square foot and a superimposed live load of 30 pounds per square foot. Normal weight concrete is used throughout and the columns are of equal height above and below the plate.

Determine the minimum permissible plate thickness required by the code when actual deflections are not calculated. A slab thickness of 8½ inches is adopted and an effective depth for positive and negative moments of 7 inches is provided. For an interior bay, compute the shear in the slab at the critical section for column D–2 and compare this with the allowable shear stress. Compute the positive and negative moments of the column strip and of the middle strip for the north–south direction of grid D. Using only No.5 bars, compute the required flexural reinforcing in the column strip on grid D.

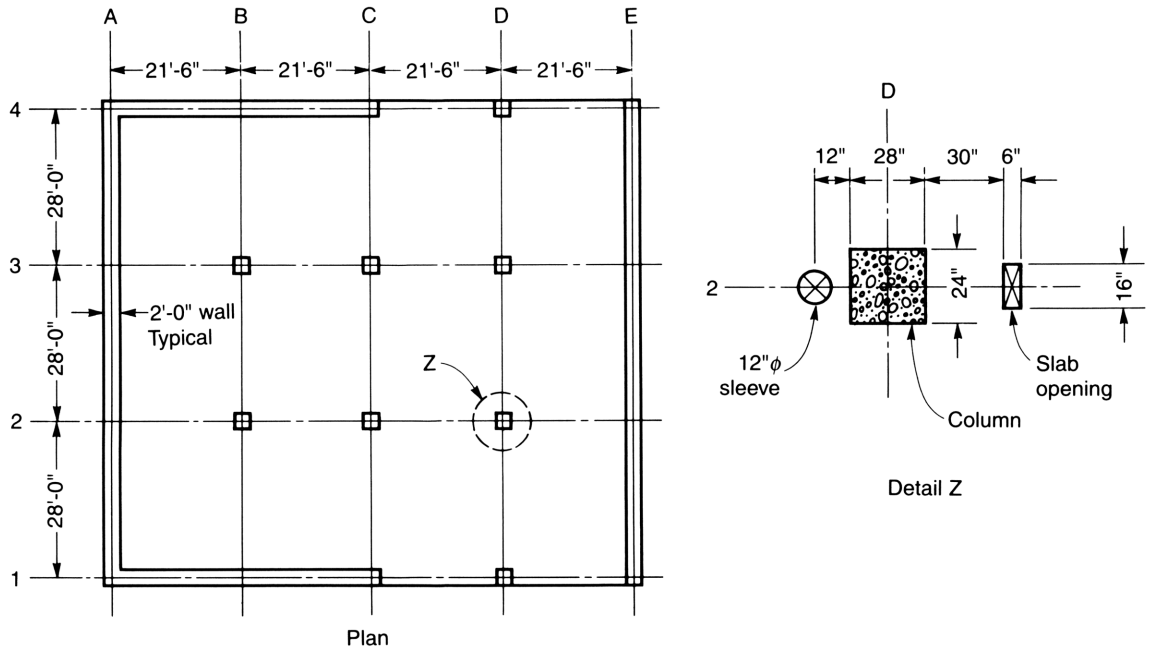

Figure 7–17 Details for Example 7–6

Solution

The factored load is obtained from ACI Section 9.2 Equation (9–1) as

$$\begin{aligned} w_u &= 1.4\ w_D + 1.7 w_L \\ &= 1.4 \times 135 + 1.7 \times 30 \\ &= 240 \text{ pounds per square foot} \end{aligned}$$

For a flat plate floor, the minimum slab thickness is given by ACI Table 9.5(c) as

$$\begin{aligned} h &= \ell_n/30 \ \ldots \text{ in an exterior panel} \\ &= (\ell_1 - c_1)/30 \\ &= (28 \times 12 - 24)/30 \\ &= 10.4 \text{ inches} \end{aligned}$$

The minimum permissible plate thickness required by the code when actual deflections are not calculated is 10.4 inches.

The critical section for punching shear in a flat plate, as defined by ACI Section 11.12.1.2, is located on the perimeter of a rectangle with sides of length

$$\begin{aligned} b_1 &= c_1 + d \\ &= 24 + 7 \\ &= 31 \text{ inches} \\ b_2 &= c_2 + d \\ &= 28 + 7 \\ &= 35 \text{ inches} \end{aligned}$$

The length of the critical perimeter is

$$\begin{aligned} b_o &= 2(b_1 + b_2) \\ &= 2(31 + 35) \\ &= 132 \text{ inches} \end{aligned}$$

Since the openings in the slab are located within a column strip, the critical perimeter must be reduced as defined in ACI Section 11.12.5. The ineffective length produced by the rectangular opening is given by

$$\begin{aligned} x_1 &= 16 \times 17.5/44 \\ &= 6.36 \text{ inches} \end{aligned}$$

The half angle subtended by the twelve inch diameter sleeve at the center of the column is given by

$$\begin{aligned} \theta &= \sin^{-1}[6/(12 + 14)] \\ &= 13.34^\circ \end{aligned}$$

The ineffective length produced by the sleeve is

$$\begin{aligned} x_2 &= 2 \times 17.5\tan\theta \\ &= 8.30 \text{ inches} \end{aligned}$$

The effective perimeter is, then

$$\begin{aligned} b_e &= b_o - x_1 - x_2 \\ &= 132 - 6.36 - 8.30 \\ &= 117.34 \end{aligned}$$

The ratio of the long side to the short side of the column is

$$\begin{aligned} \beta &= c_2/c_1 \\ &= 28/24 \\ &= 1.17 \\ &< 2 \end{aligned}$$

Hence the shear capacity of the flat plate for two-way action is given by ACI Equation (11-37) as

$$\phi V_c = 4\phi b_e d(f'_c)^{0.5}$$

where ϕ = strength reduction factor
= 0.85 from ACI Section 9.3

and $\phi V_c = 4 \times 0.85 \times 117.34 \times 7(4000)^{0.5}/1000$
$= 176.63$ kips

The factored applied shear at the critical perimeter is

$$\begin{aligned} V_u &= w_u(\ell_1 \times \ell_2 - b_1 \times b_2) \\ &= 0.24(28 \times 21.5 - 31 \times 35/144) \\ &= 142.67 \text{ kips} \\ &< \phi V_c \end{aligned}$$

Hence, the shear capacity of the slab is satisfactory.

The clear span in the north–south direction is

$$\begin{aligned} \ell_n &= \ell_1 - c_1 \\ &= 28 - 2 \\ &= 26 \text{ feet} \\ &> 0.65\ell_1 \end{aligned}$$

Then in accordance with ACI Section 13.6.2.5 the value to be adopted for the clear span is

$$\ell_n = 26 \text{ feet}$$

For an interior bay, the widths of a column strip and of a middle strip in the north–south direction are defined by ACI Section 13.2 as

$$\begin{aligned} w_c &= \ell_2/2 \\ &= 21.5/2 \\ &= 10.75 \text{ feet} \end{aligned}$$

and

$$\begin{aligned} w_m &= \ell_2 - w_c \\ &= 21.5 - 10.75 \\ &= 10.75 \text{ feet} \end{aligned}$$

The total factored static moment is given by

$$\begin{aligned} M_o &= w_u \ell_2 \ell_n^2/8 \\ &= 0.24 \times 21.5 \times 26^2/8 \\ &= 436 \text{ kip feet} \end{aligned}$$

Strip	Section	Line 1	Span 12	Line 2	Span 23
Full Width	Distribution Coeff Moment	0.26 113	0.52 227	0.70 305	0.35 153
Column Strip	Distribution Coeff Moment	1.00 113	0.60 136	0.75 229	0.60 92
Middle Strip	Distribution Coeff Moment	0.00 0.00	0.40 91	0.25 76	0.40 61

Table 7–6 Distribution of moments for Example 7–6

This moment is distributed in accordance with ACI Section 13.6.3.3 to give positive and negative moment across the full width between columns and then distributed to the column and middle strips in accordance with ACI Section 13.6.4 and ACI Section 13.6.6 as indicated in Table 7–6.

The minimum reinforcement for temperature and shrinkage stresses is defined for Grade 60 reinforcement in ACI Section 7.12 as

$$A_{s(min)} = 0.0018bh$$
$$= 0.0018 \times 12 \times 8.5$$
$$= 0.184 \text{ square inches per foot}$$

Maximum bar spacing is defined in ACI Section 13.4.2 as

$$s_{max} = 2h$$
$$= 2 \times 8.5$$
$$= 17 \text{ inches}$$

The maximum reinforcement in accordance with ACI Section 10.3.3 is

$$A_{s(max)} = 0.6375bd\beta_1 f'_c \times 87{,}000/f_y(87{,}000 + f_y)$$
$$= 1.76 \text{ square inches per foot}$$

The required reinforcement area is obtained by using a calculator program and is given by

$$A_s = 0.85bdf'_c[1 - (1 - M_u/0.383f'_c bd^2)^{0.5}]/f_y$$

The reinforcement area and the required spacing of No.5 bars are indicated in Table 7.7.

Section	Line 1	Span 12	Line 2	Span 23
Moment Kip/ft	10.55	12.65	21.29	8.52
A_s Required	0.35	0.42	0.73	0.28
Spacing No.5 Bars ins	10	8	5	12
A_s Provided	0.37	0.46	0.74	0.31

Table 7–7 Reinforcement details for Example 7–6

Additional reinforcement is required over the exterior column to resist that fraction of the unbalanced moment transferred by flexure to the column. These additional bars are placed within an effective slab width given by ACI Section 13.5.3.2 as

$$b_c = c_2 + 3h$$
$$= 28 + 3 \times 8.5$$
$$= 53.5 \text{ inches}$$

The required fraction of unbalanced moment at column D1 is given by ACI Equation (13–1) as

$$\gamma_f = 1/[1 + 0.67 \times (b_1/b_2)^{0.5}]$$
$$= 1/\{1 + 0.67 \times [(24 + 7/2)/35]^{0.5}\}$$
$$= 0.63$$

The moment to be transferred is, then

$$\gamma_f M_u = 0.63 \times 113$$
$$= 71 \text{ kip feet}$$

Using a calculator program, the additional reinforcement required is

$$A_s = 2.39 \text{ square inches}$$

This requires a total of eight additional No.5 bars, at 7.5 inches centers, over the exterior column. The reinforcement layout and development lengths are determined from ACI Figure 13.3.8 and are shown in Figure 7–18.

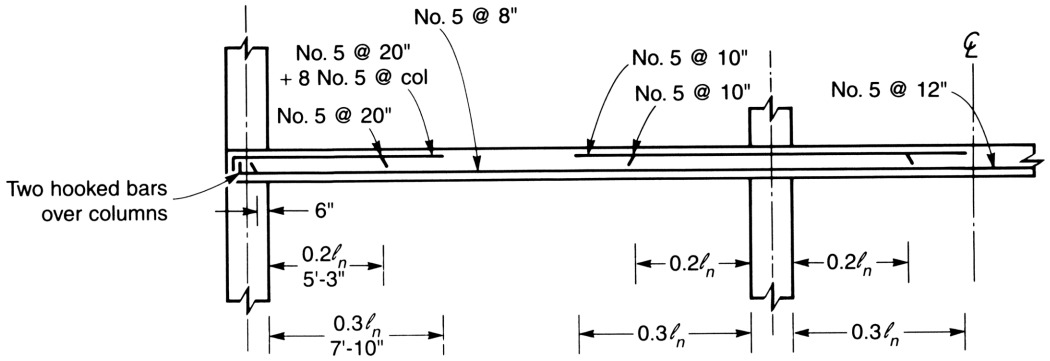

Figure 7–18 Reinforcement details for Example 7–6

7.3.5 Reinforcement detailing

ACI Figure 13.3.8 provides detailing rules for flat plates and slabs and these are illustrated in Figure 7–19.

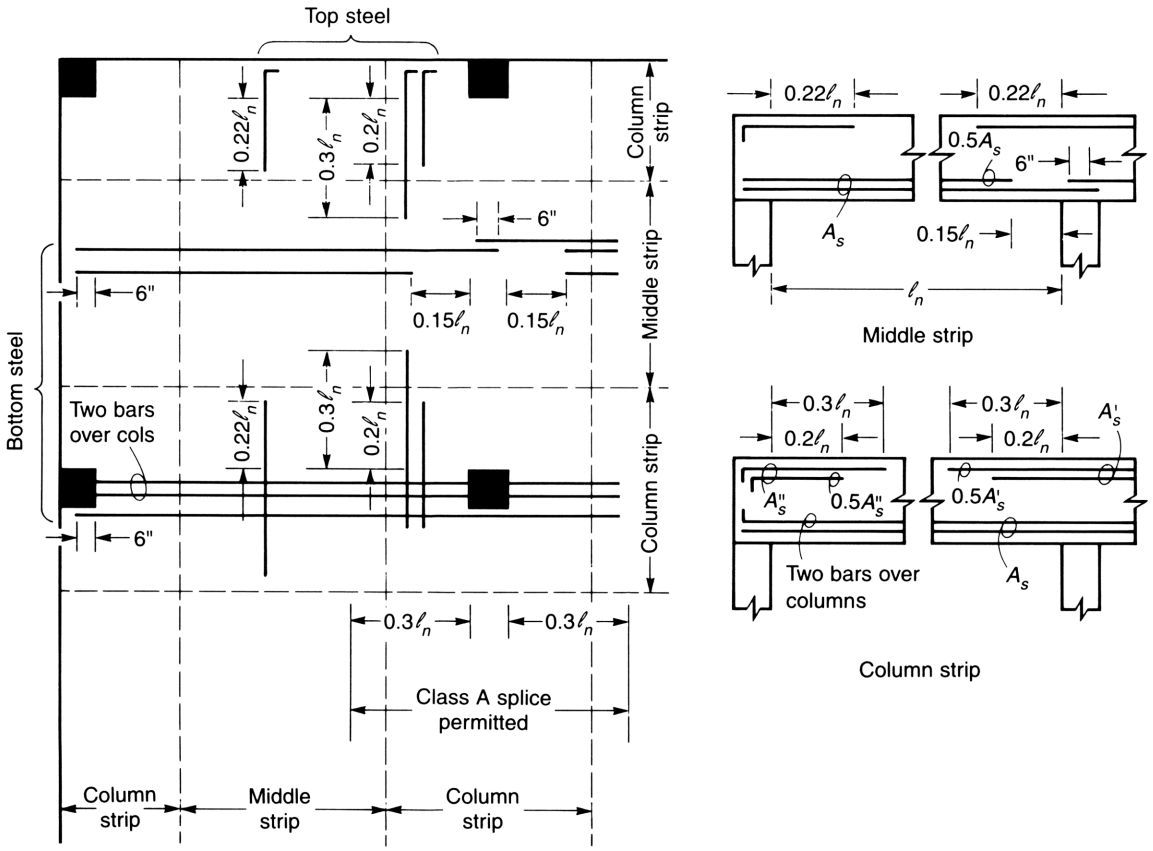

Figure 7–19 Flat plate reinforcement detailing

7.4 SUPPLEMENTARY PROBLEMS

Problem 7–1 (Design of a one-way slab)
A reinforced concrete slab spans between 12 inch wide beams at 20 feet centers over 3 bays. The discontinuous ends of the slab are unrestrained. The superimposed dead load, due to finishes and services is 40 pounds per square foot and the superimposed live load is 50 pounds per square foot. The concrete strength is 4000 pounds per square inch and all reinforcement consists of Grade 60 No.4 bars. The slab does not support any deflection sensitive partitions or ceilings.

Determine the minimum permissible thickness of the slab and the effective depth. Determine the required spacing of No.4 bars in the end span, the central span, and at the interior supports.

Problem 7.2 (Design of a flat plate for flexure)
A reinforced concrete flat plate floor without beams has 15 inches square columns at 15 feet centers in both directions and supports a factored load of 200 pounds per square foot. For an interior bay, compute the positive and negative moments of the column strip and of the middle strip for an end span, interior span, and interior support.

Problem 7.3 (Design of a flat plate for shear)
A reinforced concrete flat plate floor without beams has 15 inches diameter columns at 15 feet centers in both directions and supports a factored load of 200 pounds per square foot. For an interior column, compute the minimum required effective depth to resist punching shear if the concrete strength is 4000 pounds per square inch.

Problem 7.4 (Design for shear due to unbalanced moment)
The 15 inches square corner column of a flat plate floor supports a factored shear force of 50 kips and an unbalanced moment of 40 kip feet about both axes. The depth of the slab is 7 inches and the concrete strength is 4000 pounds per square inch. Determine if the shear capacity of the slab is adequate.

Problem 7.5 (Design for punching shear and moment)
A reinforced concrete flat plate floor with an effective depth of 7 inches, has 15 inches square columns at 20 feet centers in both the north-south and the east-west direction. The concrete strength is 4000 pounds per square inch. The total factored live load plus dead load (including self weight of slab and partitions) is 250 pounds per square foot. For an interior column, compute the punching shear force and shear capacity of the slab. For an interior bay, compute the positive and negative factored moments in the column strip and middle strip in the east-west direction.

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8

Design of footings

8.1 GENERAL

8.1.1 Types of footings

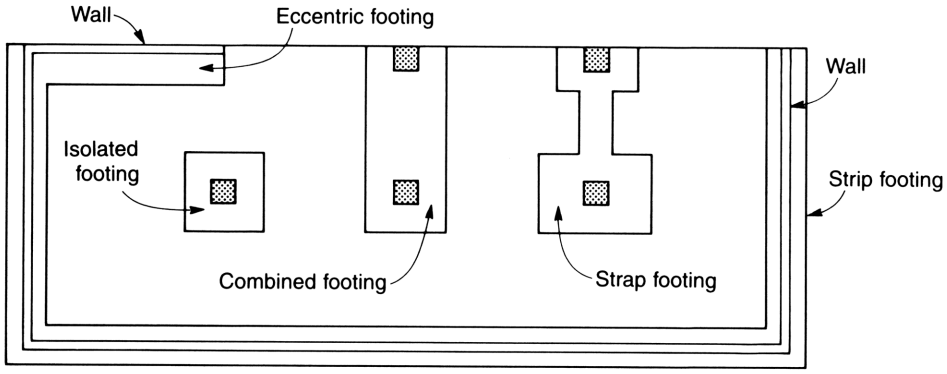

Figure 8–1 Types of footings

Reinforced concrete footings are provided in order to transfer loading from the structure to the ground. The different types of footings are illustrated in Figure 8–1 and consist of:

- Strip footings which are continuous footings supporting walls. The strip footing may be designed as a slab which cantilevers from the wall face.
- Isolated footings which support a single, central column. Isolated footings may be used when adequate space is available.

- Combined footings which support two or more columns. Combined footings may be used when one column is adjacent to the property line and space is not available for an isolated footing.
- Strap footings which support two columns. Strap footings may be used when one column is adjacent to the property line and space is not available for an isolated footing.
- Eccentric footings which support walls which are adjacent to the property line when space is not available for a strip footing.

8.1.2 Reinforcement detailing

The maximum spacing of the principal reinforcement in footings is limited by ACI Sections 7.6.5 and 10.5.4 to

$$s = 3h \leq 18 \text{ inches}$$

where h = footing thickness

The maximum spacing of distribution reinforcement, which is required to resist shrinkage and temperature stresses, is limited by ACI Section 7.12.2.2 to

$$s = 5h \leq 18 \text{ inches}$$

The minimum spacing of reinforcement must be adequate to allow full consolidation of the concrete around the bars. The minimum clear spacing between parallel bars in a layer is specified by ACI Sections 7.6.1 and 3.3.2 as

$$s_{(\text{min clear})} = d_b \geq 1 \text{ inch} \geq 1.33 \times \text{maximum aggregate size}$$

where d_b = diameter of bar

The minimum reinforcement ratio specified for distribution steel in ACI Section 7.12.2.1 is

$$A_s/A_g = 0.0020 \text{ ... for Grade 40 and 50 reinforcement}$$
$$A_s/A_g = 0.0018 \text{ ... for Grade 60 reinforcement}$$
$$A_s/A_g = 0.0018 \times 60{,}000/f_y \text{ ... for } f_y > 60{,}000 \text{ pounds per square inch} \geq 0.0014$$

where A_s = area of distribution reinforcement
A_g = gross area of footing cross section

The minimum cover to reinforcement when the concrete is cast against earth, as is the situation in the soffit of footings, is specified in ACI Section 7.7.1 as three inches. For concrete exposed to earth or weather, the minimum cover for No. 5 bars and smaller is specified as 1½ inches, and for larger bars as two inches.

The minimum reinforcement required in the vertical wall of an eccentric footing is specified by ACI Section 14.3. For bars larger than number 5, the reinforcement ratios, based on the gross concrete area, for vertical and horizontal reinforcement are given by

ρ_{vert} = 0.15 percent
ρ_{hor} = 0.25 percent

For bars not larger than number 5, the corresponding ratios are

ρ_{vert} = 0.12 percent
ρ_{hor} = 0.20 percent

For walls exceeding ten inches in thickness, two layers of reinforcement are required with one layer consisting of not less than one-half and not more than one-thirds of the total reinforcement and the other layer making up the balance of the reinforcement.

8.2 DESIGN REQUIREMENTS

8.2.1 Strip footing

A strip footing transfers the loads from a wall to the supporting soil. The size of the footing is determined by the allowable soil bearing pressure which is computed from the unfactored service load on the footing. The footing is designed for one-way beam action in flexure and shear using factored loads, and the width of the footing must be adequate to develop the flexural reinforcement. Sections 15.4 and 11.12 of the ACI Code specify the critical sections in the footing for flexure and shear. For a concrete wall the location of the critical section for flexure is defined in ACI Section 15.4.2 as being at the face of the wall. The critical section for shear is specified in ACI Sections 15.5.2 and 11.1.3.1 as being a distance from the wall face equal to the effective depth of the footing. For a masonry wall the location of the critical section for flexure is defined in ACI Section 15.4.2 as being halfway between the center and the face of the wall. The critical section for shear is specified in ACI Sections 15.5.2 and 11.1.3.1 as being a distance from the wall face equal to the effective depth of the footing.

Example 8–1 (Strip footing supporting a concrete wall)
A six inch wide concrete wall is supported by a twelve inch deep strip footing as shown in Figure 8–2. The wall applies to the base a service dead load of 10 kips per foot and a service live load of 5.4 kips per foot. The strength of the concrete in the base is 3000 pounds per square inch and the reinforcement is Grade 60. Determine the required footing width for an allowable soil bearing pressure of 4000 pounds per square inch and the required spacing of No. 4 reinforcing bars. Check the adequacy of the base for shear.

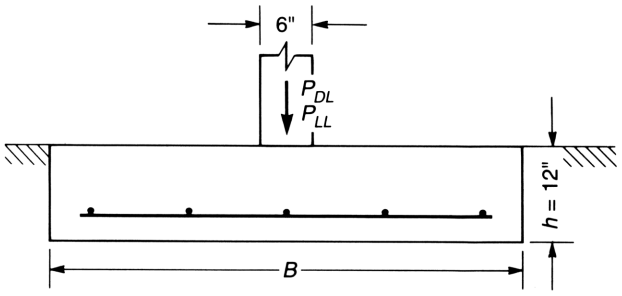

Figure 8–2 Details for Example 8–1

Solution

The weight of the footing is

$$w = 150 \times 12/12$$
$$= 150 \text{ pounds per square foot}$$

Considering a one foot length of the wall and footing, the required width of footing is

$$B = 1.0(P_{DL} + P_{LL} + wB)/q_a$$
$$= (10{,}000 + 5850 + 150B)/4000$$
$$= 4 \text{ feet}$$

The net factored pressure acting on the footing is

$$q_u = (1.4P_{DL} + 1.7P_{LL})/(1.0 \times B)$$
$$= (1.4 \times 10 + 1.7 \times 5.4)/4$$
$$= 5.80 \text{ kips per square foot}$$

The critical section for flexure is located at the face of the wall and the factored moment is

$$M_u = q_u b(B/2 - c_1/2)^2/2$$

where c_1 = width of wall

$$= 5.80 \times 1.0 \times (4/2 - 0.5/2)^2/2$$
$$= 8.88 \text{ kip feet}$$

The required clear cover to a member which is cast against earth is given in ACI Section 7.7.1 as

$$c_c = 3 \text{ inches}$$

and

$$d = h - c_c - d_b/2$$
$$= 12 - 3 - 0.5/2$$
$$= 8.75 \text{ inches}$$

The required reinforcement ratio is derived from ACI Section 10.2 and is

$$\rho = 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$
$$= 0.22 \text{ percent}$$

The maximum reinforcement ratio is given by ACI Section 10.3 as

$$\rho_{max} = 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y)$$
$$= 1.60 \text{ percent}$$
$$> \rho \ldots \text{ satisfactory}$$

The required reinforcement area is

$$A_s = \rho bd$$
$$= 0.0022 \times 12 \times 8.75$$
$$= 0.23 \text{ square inches per foot}$$

The minimum reinforcement area is given by ACI Section 7.12.2.1 as

$$A_{s(min)} = 0.18 \text{ percent of the gross area}$$
$$= 0.0018 \times b \times h$$
$$= 0.0018 \times 12 \times 12$$

$= 0.26$ square inches ... governs
$> A_s$

Providing No.4 bars at nine inches on center gives

$A'_s = 0.26$ square inches ... satisfactory

For calculating development length, ACI Equation (12–1) is applicable with

$2.5 = (c + K_{tr})/d_b$
$1 = \alpha = \beta = \lambda$
$\gamma = 0.8$ ... for No. 4 bars

and the development length of the No.4 bars in 3000 pounds per square inch concrete is

$$\ell_d = 26.3d_b$$
$$= 26.3 \times 0.5$$
$$= 13.2 \text{ inches}$$
$$> 12 \text{ inches ... satisfactory}$$

Allowing for an end cover of three inches, the available anchorage length for the bars is

$$\ell_{da} = B/2 - c_1/2 - c_e$$

where c_1 = width of wall
c_e = end cover

$$= 48/2 - 6/2 - 3$$
$$= 18 \text{ inches}$$
$$> 13.2 \text{ inches ... satisfactory}$$

The longitudinal distribution reinforcement area is given by ACI Section 7.12.2.1 as

$A_{s(min)}$ = 0.18 percent of the gross area
$= 0.0018 \times B \times h$
$= 0.0018 \times 48 \times 12$
$= 1.04$ square inches

Providing five No.4 bars gives

$A'_s = 1.00$ square inches ... satisfactory

The critical section for flexural shear is located a distance from the wall face equal to the effective depth of the footing. The distance of the critical section from the edge of the footing is

$$x = B/2 - c_1/2 - d$$
$$= 4/2 - 0.5/2 - 8.75/12$$
$$= 1.02 \text{ feet}$$

The factored shear force at this section is

$$V_u = q_u \times b \times x$$
$$= 5.99 \times 1.0 \times 1.02$$
$$= 6.11 \text{ kips}$$

The shear capacity of the footing for flexural shear is given by ACI Equation (11-3) as

$$\begin{aligned}\phi V_c &= 2\phi db(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 8.75 \times 12(3000)^{0.5}/1000 \\ &= 9.78 \text{ kips} \\ &> V_u\end{aligned}$$

In accordance with ACI Section 11.5.5.1 the flexural shear capacity is adequate and stirrups are not required.

8.2.2 Isolated column footing

An isolated column footing transfers the loads from a single column to the supporting soil. The size of the footing is determined by the allowable soil bearing pressure. The footing is designed for flexure, punching or two-way shear, and flexural or one-way shear. The depth of the footing is generally governed by punching shear. The ACI Code, in Sections 15.4 and 15.5, specifies the critical sections in the footing for flexure and shear. For a reinforced concrete column, the critical section for flexure is defined in ACI Section 15.4.2 as being located at the face of the column. The location of the critical sections for flexural shear and punching shear are specified in ACI Sections 11.1.3.1, 11.12.1.2, and 15.5.2 and are illustrated in Figure 8-3.

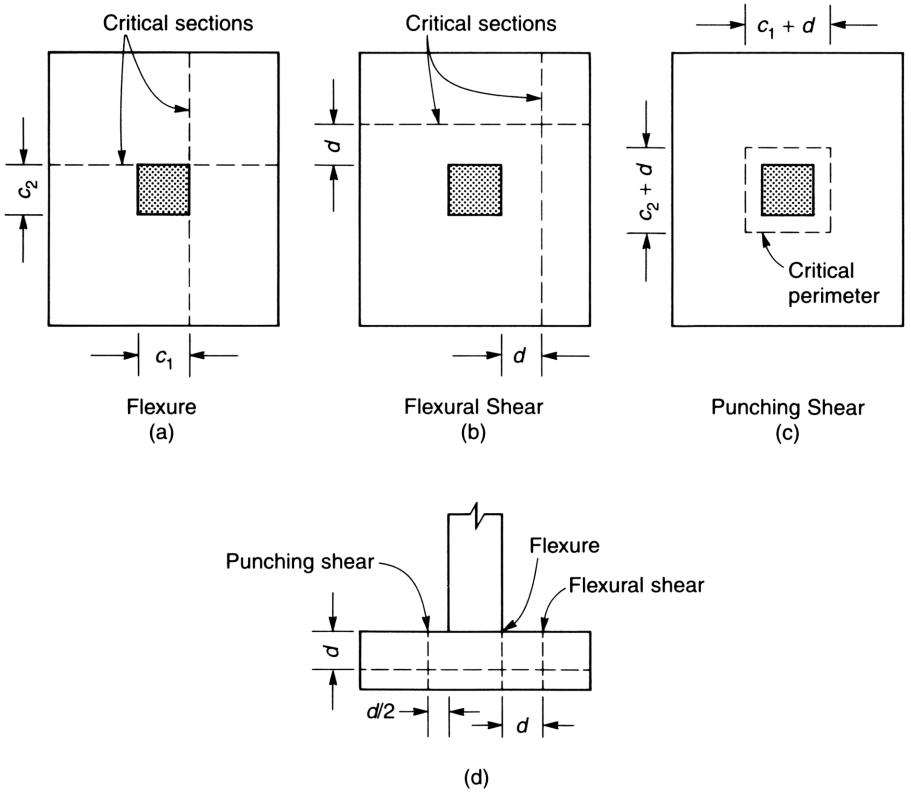

Figure 8-3 Critical sections: footing with reinforced concrete column

The length of the critical perimeter for punching shear is given by

$$b_o = 2(c_1 + c_2) + 4d$$

where c_1 = short side of column
c_2 = long side of column
d = effective depth of footing reinforcement.

The critical section for a footing supporting a column with a steel base plate is specified as half way between the face of the column and the edge of the base plate. The maximum moment is computed at this section, the flexural shear at a distance d from this section and the punching shear at a distance of d/2 from this section as shown in Figure 8–4.

For a masonry column the location of the critical section for flexure is defined in ACI Section 15.4.2 as being halfway between the center and the face of the wall. The critical sections for shear are specified in ACI Sections 15.5.2 and 11.1.3.1 as being measured from the column face as shown in Figure 8–5.

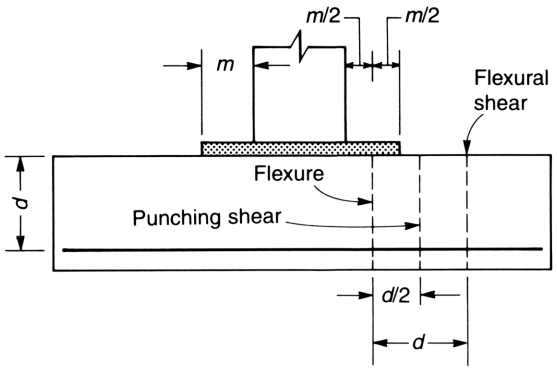

Figure 8–4 Critical sections: footing with steel base plate

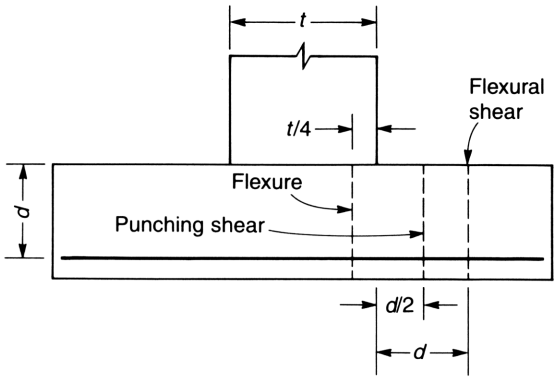

Figure 8–5 Critical sections: footing with masonry column

Reinforcement is designed for the maximum moment at the reaction plane and is distributed uniformly across the base in the case of a square footing. For a rectangular footing, reinforcement parallel to the shorter side should be concentrated in a central band width equal to the length of the shorter side as illustrated in Figure 8–6. The area of reinforcement required in the central band is given by ACI Equation (15–1) as

$$A_b = 2A_s/(\beta + 1)$$

where A_s = total required reinforcement area
β = ratio of the long side to the short side of the footing
$= \ell_2/\ell_1$

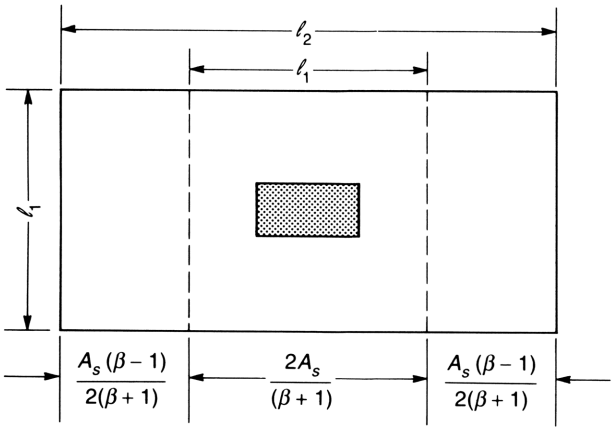

Figure 8–6 Rectangular footing: reinforcement areas required

The capacity of a footing for flexural shear is given by ACI Equation (11–3) as

$$\phi V_c = 2\phi bd(f'_c)^{0.5}$$

where ϕ = strength reduction factor
= 0.85 from ACI Section 9.3
b = width of footing
d = effective depth
f'_c = concrete strength

The capacity of a footing for punching shear is given by ACI Equation (11–35) as

$$\phi V_c = (2 + 4/\beta_c)\phi b_o d(f'_c)^{0.5}$$
$$\leq 4\phi b_o d(f'_c)^{0.5} \ldots \text{ ACI Equation (11–37)}$$

where β_c = ratio of long side to short side of column
$= c_2/c_1$
b_o = length of critical perimeter for punching shear
$= 2(c_1 + c_2) + 4d$

Load transfer between a reinforced concrete column and the footing may be provided by the bearing capacity of the column and the footing. The bearing capacity of the column concrete at the interface is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_1$$

where ϕ = strength reduction factor
= 0.7 from ACI Section 9.3
A_1 = area of column
f'_c = strength of column concrete

The bearing capacity of the footing concrete at the interface is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_1 (A_2/A_1)^{0.5}$$
$$\leq 0.85\phi f'_c A_1 \times 2$$

where f'_c = strength of footing concrete
ϕ = strength reduction factor
A_2 = area of the base of the pyramid, with side slopes of 1:2, formed within the footing by the column base.

In accordance with ACI Section 15.8.1.2, when the bearing strength of the concrete is exceeded, reinforcement must be provided at the interface to transfer the excess load and the capacity of this reinforcement is

$$\phi P_s = \phi A_s f_y$$

where ϕ = strength reduction factor
= 0.7 from ACI Section 9.3
f_y = reinforcement yield strength
A_s = reinforcement area

In addition, in accordance with ACI Section 15.8.2.1, a minimum area of reinforcement must be provided across the interface given by

$$A_{s(min)} = 0.005A_1$$

Example 8–2 (Concrete footing with steel column)

Figure 8–7 indicates the loads acting on a steel column with a reinforced concrete footing. Check the adequacy of the footing and determine the required base plate thickness and reinforcement area, using Grade 60 bars, for a concrete strength of 2000 pounds per square inch and a base plate of Grade A36 steel.

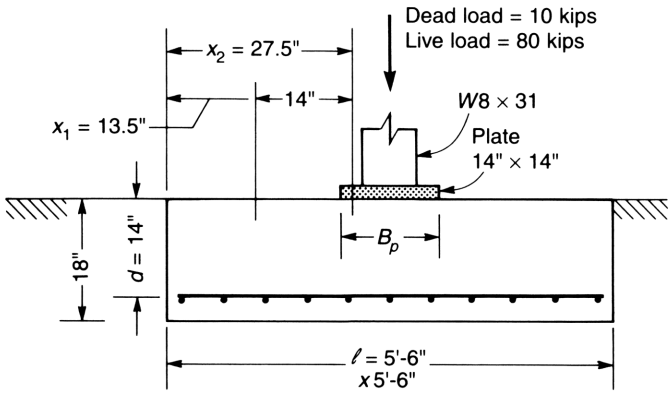

Figure 8–7 Details for Example 8–2

Solution

The base plate thickness may be computed by the design method presented in the AISC Manual[1] Section 3–106. The bearing pressure on the base plate due to the applied service loads is

$$f_p = (P_D + P_L)/A_p$$
$$= (10 + 80)/(14 \times 14)$$
$$= 0.46 \text{ kips per square inch}$$

The relevant design parameters are

$$m = (B_p - 0.95d)/2$$
$$= (14 - 0.95 \times 8)/2$$
$$= 3.2$$

$$n = (B_p - 0.80b_f)/2$$
$$= (14 - 0.80 \times 8)/2$$
$$= 3.8 \text{ ... governs}$$

Hence, the required base plate thickness is given by

$$t_p = 2n(f_p/F_y)^{0.5}$$
$$= 2 \times 3.8 \times (0.46/36)^{0.5}$$
$$= 0.86 \text{ inches}$$
$$= \tfrac{7}{8}\text{-inch}$$

The factored applied load is given by ACI Equation (9–1) as

$$P_u = 1.4P_D + 1.7P_L$$
$$= 1.4 \times 10 + 1.7 \times 80$$
$$= 150 \text{ kips}$$

The area of the base of the pyramid, with side slopes of 1:2, formed within the footing by the base plate area is

$$A_2 = (B_p + 4d)^2$$
$$\leq \ell^2$$

where d = effective depth = 14 inches

ℓ = size of footing = 66 inches

Then $A_2 = (14 + 4d)^2$
$= 4356$ ... maximum

$$(A_2/A_p)^{0.5} = (4356/14^2)^{0.5}$$
$$> 2$$

Hence, the bearing capacity of the footing concrete is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_p \times 2$$
$$= 0.85 \times 0.7 \times 2000 \times 196 \times 2/1000$$
$$= 466 \text{ kips}$$
$$> P_u \text{ ... satisfactory}$$

The net factored pressure acting on the underside of the footing is

$$q_u = P_u/A_f$$
$$= 150/5.5^2$$
$$= 4.96 \text{ kips per square foot}$$

The critical section for flexural shear is located a distance from the edge of the footing given by

$$x_1 = \ell/2 - (B_p + b_f)/4 - d$$
$$= 5.5/2 - (14 + 8)/48 - 14/12$$
$$= 1.125 \text{ feet}$$

The factored applied shear at the critical section is

$$\begin{aligned} V_u &= q_u \ell x_1 \\ &= 4.96 \times 5.5 \times 1.125 \\ &= 30.69 \text{ kips} \end{aligned}$$

The flexural shear capacity of the footing is given by Equation (11–3) as

$$\begin{aligned} \phi V_c &= 2\phi \ell d(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 5.5 \times 12 \times 14 \times (2000)^{0.5}/1000 \\ &= 70.25 \text{ kips} \\ &> V_u \ldots \text{ satisfactory} \end{aligned}$$

The length of one side of the critical perimeter for punching shear is

$$\begin{aligned} b_o/4 &= (B_p + b_f)/2 + d \\ &= (14 + 8)/24 + 14/12 \\ &= 2.08 \text{ feet} \end{aligned}$$

The factored applied shear at the critical perimeter is

$$\begin{aligned} V_u &= P_u - q_u(b_o/4)^2 \\ &= 150 - 4.96 \times 2.08^2 \\ &= 128.47 \text{ kips} \end{aligned}$$

The ratio of the base plate sides is

$$\begin{aligned} \beta_c &= 1 \\ &< 2 \end{aligned}$$

Then, the punching shear capacity of the footing is given by ACI Equation (11–35) as

$$\begin{aligned} \phi V_c &= 4\phi b_o d(f'_c)^{0.5} \\ &= 4 \times 0.85 \times 4 \times 2.08 \times 12 \times 14 \times (2000)^{0.5}/1000 \\ &= 212.53 \text{ kips} \\ &> V_u \ldots \text{ satisfactory} \end{aligned}$$

The critical section for flexure is located a distance from the edge of the footing given by

$$\begin{aligned} x_2 &= \ell/2 - (B_p + b_f)/4 \\ &= 5.5/2 - (14 + 8)/48 \\ &= 2.29 \text{ feet} \end{aligned}$$

The factored applied moment at this section is

$$\begin{aligned} M_u &= q_u \ell x_2^2/2 \\ &= 4.96 \times 5.5 \times 2.29^2/2 \\ &= 71.53 \text{ kip feet} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.2 and is

$$\begin{aligned} \rho &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.126 \text{ percent} \end{aligned}$$

The maximum reinforcement ratio is given by ACI Section 10.3 as

$$\begin{aligned}\rho_{max} &= 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y)\\ &= 1.07 \text{ percent}\\ &> \rho \ldots \text{satisfactory}\end{aligned}$$

The required reinforcement area is

$$\begin{aligned}A_s &= \rho \ell d\\ &= 0.00126 \times 66 \times 14\\ &= 1.16 \text{ square inches per foot}\end{aligned}$$

The minimum reinforcement area in a footing slab for Grade 60 bars is given by ACI Section 7.12.2.1 as

$$\begin{aligned}A_{s(min)} &= 0.18 \text{ percent of the gross area}\\ &= 0.0018 \times \ell \times h\\ &= 0.0018 \times 66 \times 18\\ &= 2.14 \text{ square inches} \ldots \text{governs}\\ &> A_s\end{aligned}$$

Provide eleven No.4 bars to give an area of

$$A'_s = 2.20 \text{ square inches} \ldots \text{satisfactory}$$

Allowing for an end cover of three inches, the available anchorage length for the bars is

$$\begin{aligned}\ell_{da} &= x_2 - c_e\\ &= 2.29 \times 12 - 3\\ &= 24.5 \text{ inches}\end{aligned}$$

For calculating development length, ACI Equation (12–1) is applicable with

$$\begin{aligned}2.5 &= (c + K_{tr})/d_b\\ 1 &= \alpha \quad = \beta \quad = \lambda\\ \gamma &= 0.8 \ldots \text{for No. 4 bars}\end{aligned}$$

and the development length of the No.4 bars in 2000 pounds per square inch concrete is

$$\begin{aligned}\ell_d &= 32.2 d_b\\ &= 32.2 \times 0.5\\ &= 16.1 \text{ inches}\\ &> 12 \text{ inches} \ldots \text{satisfactory}\\ &< \ell_{da} \ldots \text{satisfactory}\end{aligned}$$

Hence, the footing is adequate.

Example 8–3 (Concrete column and footing[2])

The factored load acting on a concrete column and footing are shown in Figure 8–8. Check the adequacy of the footing for shear and flexure using Grade 60 bars and a concrete strength of 4000 pounds per square inch. Ignoring vertical reinforcement, check the adequacy of the column size shown to transfer the 800 kips load to the footing.

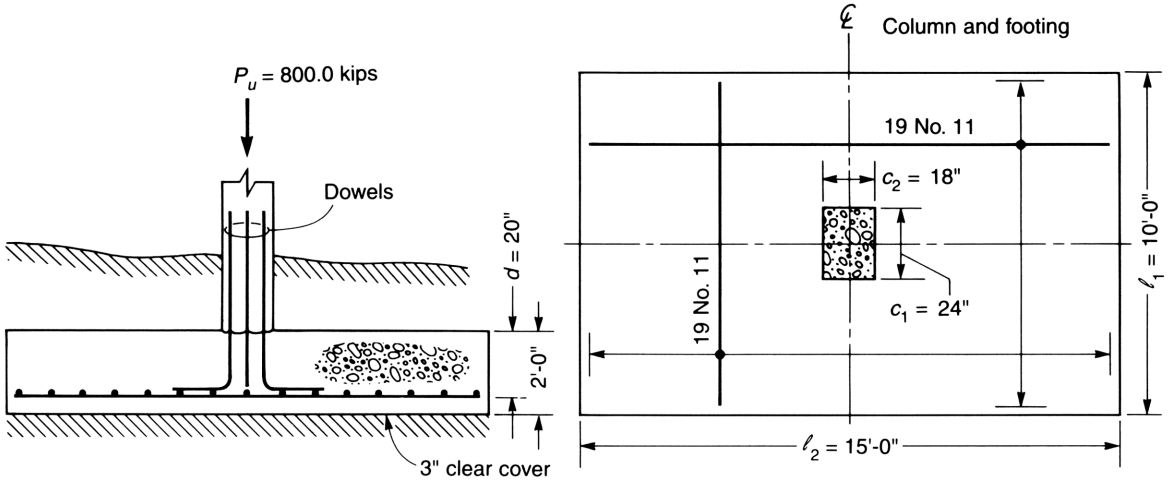

Figure 8–8 Details for Example 8–3

Solution

The net factored loading on the footing is

$$\begin{aligned} q_u &= P_u/A_f \\ &= 800/(15 \times 10) \\ &= 5.33 \text{ kips per square foot} \end{aligned}$$

The critical section for flexural shear in a footing, as defined by ACI Sections 15.5.2 and 11.1.3.1 is located a distance from the long side of the column equal to the effective depth. The distance of the critical section from the short edge of the footing is, then

$$\begin{aligned} x &= \ell_2/2 - c_2/2 - d \\ &= 15/2 - 18/24 - 20/12 \\ &= 5.083 \text{ feet} \end{aligned}$$

The shear capacity of the footing for beam action is given by ACI Equation (11–3) as

$$\begin{aligned} \phi V_c &= 2\phi bd(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 10 \times 12 \times 20(4000)^{0.5}/1000 \\ &= 258.04 \text{ kips} \end{aligned}$$

The factored applied shear at the critical section is

$$\begin{aligned} V_u &= q_u bx \\ &= 5.33 \times 10 \times 5.083 \\ &= 270.92 \text{ kips} \\ &> \phi V_c \end{aligned}$$

Hence beam action shear is unsatisfactory according to the simplified Equation and it is necessary to check with the more detailed ACI Equation (11–5). Then, the shear capacity is given by

$\phi V_c = \phi[1.9(f'_c)^{0.5} + 2500\rho V_u d/M_u]bd$

where ρ = reinforcement ratio
$= A_s/\ell_1 d$
$= 29.64/(10 \times 12 \times 20)$
$= 0.0124$

and M_u = factored moment occurring with V_u
$= q_u b x^2/2$
$= 5.33 \times 10 \times 5.083^2/2$
$= 688.55$ kip feet

Hence $V_u d/M_u = 270.92 \times 20/(12 \times 688.55)$
$= 0.66$
< 1 ... satisfactory

and $\phi V_c = 0.85(1.9(4000)^{0.5} + 2500 \times 0.0124 \times 0.66)10 \times 12 \times 20/1000$
$= 286.6$ kip feet
$> V_u$ ... satisfactory
$< 3.5 \times 258.04/2$

Hence the flexural shear capacity is satisfactory.

The critical section for punching shear in a footing, as defined by ACI Sections 15.5.2 and 11.12.1.2, is located on the perimeter of a rectangle with sides of length

$b_1 = c_1 + d$
$= 24 + 20$
$= 44$ inches

and $b_2 = c_2 + d$
$= 18 + 20$
$= 38$ inches

The length of the critical perimeter is

$b_o = 2(b_1 + b_2)$
$= 2(44 + 38)$
$= 164$ inches

The ratio of the long side to the short side of the column is

$\beta_c = c_1/c_2$
$= 24/18$
$= 1.33$
< 2

Hence the punching shear capacity of the footing is given by ACI Equation (11–37) as

$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$
$= 4 \times 0.85 \times 164 \times 20(4000)^{0.5}/1000$
$= 705.31$ kips

The factored applied shear at the critical perimeter is

$$\begin{aligned} V_u &= q_u(\ell_1\ell_2 - b_1b_2) \\ &= 5.33(10 \times 15 - 44 \times 38/144) \\ &= 737.61 \text{ kips} \\ &> \phi V_c \end{aligned}$$

Hence, the punching shear capacity is inadequate.

The critical section for flexure in a footing, as defined by ACI Section 15.4.2, is located along the long side of the column. The factored applied moment at this section, for the longitudinal direction, is

$$\begin{aligned} M_u &= q_u\ell_1(\ell_2/2 - c_2/2)^2/2 \\ &= 5.33 \times 10(15/2 - 18/24)^2/2 \\ &= 1214 \text{ kip feet} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.2 and is

$$\begin{aligned} \rho_r &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.0059 \\ &< \rho \text{ ...satisfactory} \end{aligned}$$

The maximum allowable reinforcement ratio is given by ACI Section 10.3 as

$$\rho_{max} = 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y)$$

where $\beta_1 = 0.85$ as defined in ACI Section 10.2.7.3

and $\rho_{max} = 0.0214$

$> \rho$ ...satisfactory

The minimum reinforcement area in a footing slab for Grade 60 bars is given by ACI Section 7.12.2.1 as

$$\begin{aligned} A_{s(min)} &= 0.18 \text{ percent of the gross area} \\ &= 0.0018 \times \ell_1 \times h \\ &= 0.0018 \times 120 \times 24 \\ &= 5.18 \text{ square inches} \\ &< A_s \text{ ... satisfactory} \end{aligned}$$

ACI Section 12.2.2 is applicable with clear cover exceeding d_b and clear spacing exceeding $2d_b$ and the development length of the No.11 bars in 4000 pounds per square inch concrete is

$$\begin{aligned} \ell'_d &= 47.4d_b \\ &= 47.4 \times 1.41 \\ &= 66.8 \text{ inches} \end{aligned}$$

Allowing for the excess reinforcement provided, the required development length in accordance with ACI Section 12.2.5 is given by

$$\begin{aligned} \ell_d &= \ell'_d\rho_r/\rho \\ &= 66.8 \times 0.0059/0.0124 \\ &= 31.8 \text{ inches} \end{aligned}$$

Allowing for an end cover of three inches, the available anchorage length for the bars is

$$\begin{aligned} \ell_{da} &= (\ell_2 - c_2)/2 - c_e \\ &= (180 - 18)/2 - 3 \\ &= 78 \text{ inches} \\ &> \ell_d \ldots \text{ satisfactory} \end{aligned}$$

Hence, the flexural reinforcement is adequate in the longitudinal direction.

The factored applied moment at the critical section for the transverse direction is

$$\begin{aligned} M_u &= q_u \ell_2 (\ell_1/2 - c_1/2)^2/2 \\ &= 5.33 \times 15(10/2 - 2/2)^2/2 \\ &= 640 \text{ kip feet} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.2 and is

$$\begin{aligned} \rho_r &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.00201 \end{aligned}$$

The required reinforcement area is

$$\begin{aligned} A_r &= \rho_r \times \ell_2 \times d \\ &= 0.00201 \times 180 \times 20 \\ &= 7.24 \text{ square inches} \\ &< A_s \ldots \text{ satisfactory} \end{aligned}$$

The ratio of the long side to the short side of the footing is

$$\begin{aligned} \beta &= \ell_2/\ell_1 \\ &= 15/10 \\ &= 1.5 \end{aligned}$$

From ACI Equation (15–1) the area of reinforcement required in a central band width equal to the length of the short side of the footing is

$$\begin{aligned} A_b &= A_r \times 2/(\beta + 1) \\ &= 7.24 \times 2/(1.5 + 1) \\ &= 5.79 \text{ square inches} \end{aligned}$$

The number of No.11 bars provided in the central band width of ten feet is

$$\begin{aligned} n &= 19 \times 10/14.5 \\ &= 13 \end{aligned}$$

and the area of reinforcement provided over this band width is

$$\begin{aligned} A'_b &= 20.28 \text{ square inches} \\ &> 5.79 \ldots \text{ satisfactory} \end{aligned}$$

The available anchorage length exceeds the required development length and the flexural reinforcement is adequate in the transverse direction.

Neglecting vertical reinforcement, the column load must be transferred to the footing by bearing on the concrete. The bearing capacity of the column concrete at the interface is given by ACI Section 10.17.1 as

$$\phi P_n = 0.85\phi f'_c A_1$$

where ϕ = strength reduction factor
= 0.7 from ACI Section 9.3

and A_1 = loaded area
= 24 × 18
= 432 square inches

Hence $\phi P_n = 0.85 \times 0.7 \times 4 \times 432$
= 1028 kips
> 800 kips ... satisfactory

The area of the base of the pyramid, with side slopes of 1:2, formed within the footing by the loaded area is

$$A_2 = (c_1 + 4d)(c_2 + 4d)$$
$$= (24 + 4 \times 20)(18 + 4 \times 20)$$
$$= 10{,}192 \text{ square inches}$$

Hence $(A_2/A_1)^{0.5} = (10{,}192/432)^{0.5}$
= 4.9
> 2

Then the bearing capacity of the footing concrete is given by ACI Section 10.17.1 as

$$\phi P_n = 2 \times 0.85\phi f'_c A_1$$
$$= 2 \times 1028$$
$$= 2056 \text{ kips}$$
> 800 kips ... satisfactory

Hence, the concrete alone is adequate to transfer the column load to the footing. ═══

8.2.3 Isolated column footing with eccentric load

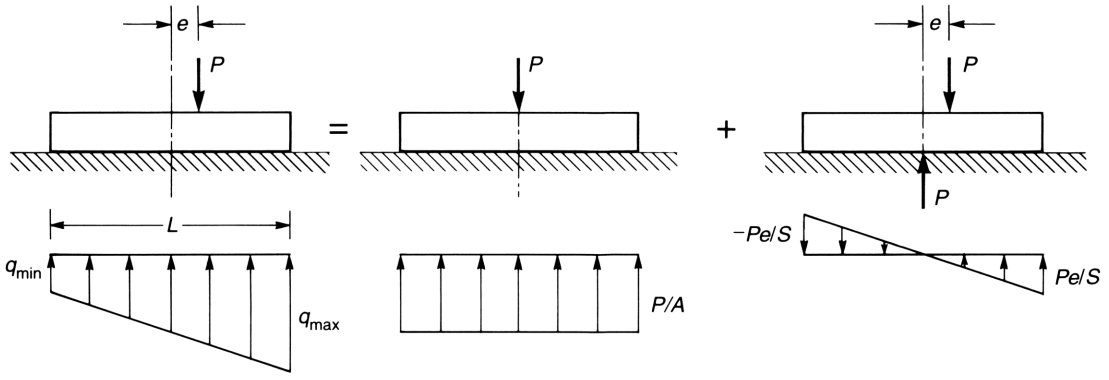

Figure 8–9 Footing with eccentric load

An applied load with an eccentricity less than L/6 produces the pressure distribution shown in Figure 8-9. The maximum and minimum bearing pressure under the footing is given by

$$q = P/A \pm Pe/S$$
$$= P(1 \pm 6e/L)/BL$$

where e = eccentricity of the applied load P
B = width of footing
L = length of footing

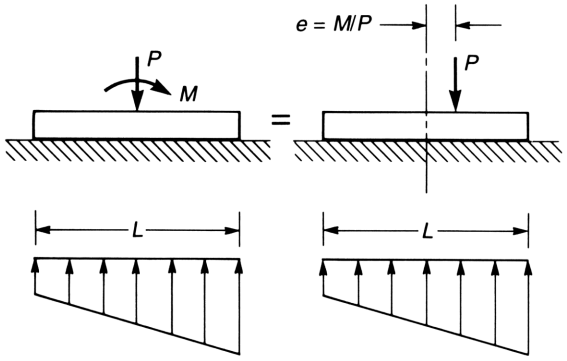

Figure 8-10 Footing with applied moment

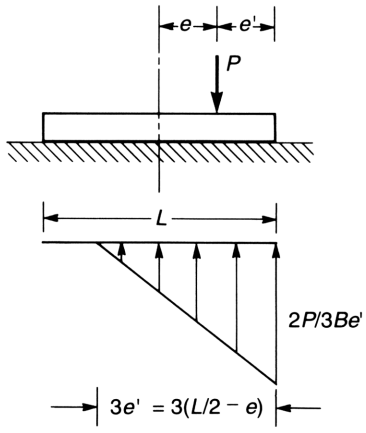

Figure 8-11 Footing with eccentricity >L/6

An axial load plus bending moment applied to a footing, as shown in Figure 8-10, produces an equivalent eccentricity of

$$e = M/P$$

The bearing pressure under the footing is similarly obtained as

$$q = P(1 \pm 6e/L)/BL$$

When the eccentricity exceeds L/6, as shown in Figure 8-11, no tension is possible between the soil and the footing and the bearing pressure under the footing is given by

$$q_{max} = 2P/3Be'$$

where $e' = L/2 - e$

When the magnitude of the eccentricity, as shown in Figure 8-12, is given by

$$e = L/6$$

the bearing pressure under the footing is

$$q_{max} = 2P/BL$$
$$q_{min} = 0$$

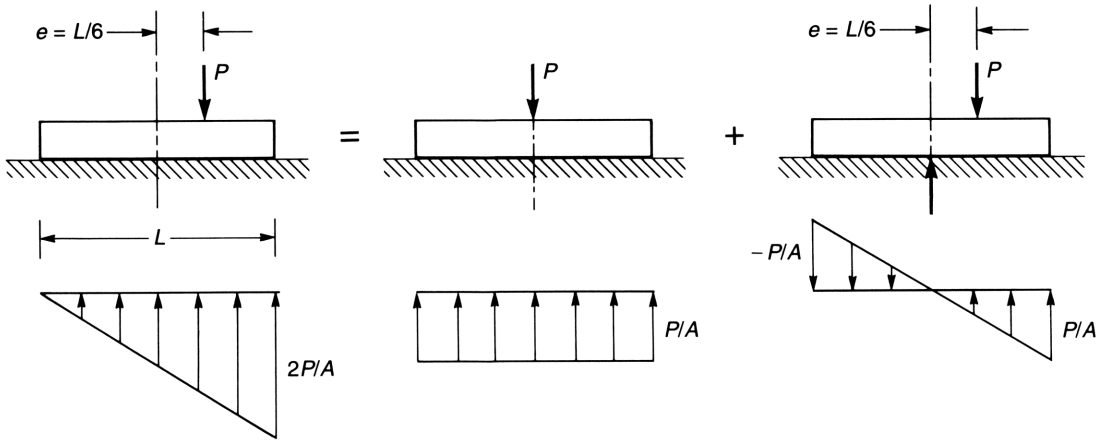

Figure 8–12 Footing with eccentricity = L/6

Example 8–4 (Footing with applied moment)

A twelve inch square column is supported by a three foot wide rectangular base as shown in Figure 8–13. The column applies to the base an axial service load of 80 kips and a service bending moment of 29 kip feet parallel to the longitudinal axis of the base. The strength of the concrete in the base is 3000 pounds per square inch and the reinforcement is Grade 60.

Neglecting the self weight of the base, determine the required base length for an allowable soil bearing pressure of 5000 pounds per square foot. Check the adequacy of the base for shear and for flexure.

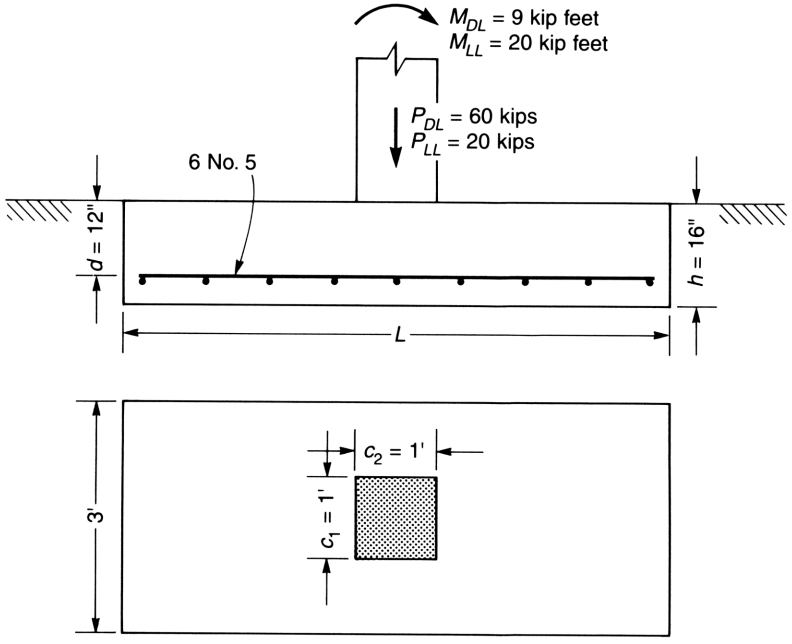

Figure 8–13 Details for Example 8–4

Solution

Assuming the equivalent eccentricity does not exceed L/6, the maximum bearing pressure is given by

$$q_{max} = P/A + M/S$$
$$= 80/3L + 6 \times 29/3L^2$$
$$= 5 \text{ kips per square foot}$$

Hence $L = 7$ feet

The factored loads are given by ACI Equation (9–1) as

$$P_u = 1.4 \times 60 + 1.7 \times 20$$
$$= 118.0 \text{ kips}$$
$$M_u = 1.4 \times 9 + 1.7 \times 20$$
$$= 46.6 \text{ kip feet}$$

The maximum factored earth pressure is

$$q_{max} = P_u/A + M_u/S$$
$$= 118/21 + 46.6/24.5$$
$$= 5.62 + 1.90$$
$$= 7.52 \text{ kips per square foot}$$

The minimum factored earth pressure is

$$q_{min} = 5.62 - 1.90$$
$$= 3.72 \text{ kips per square foot}$$

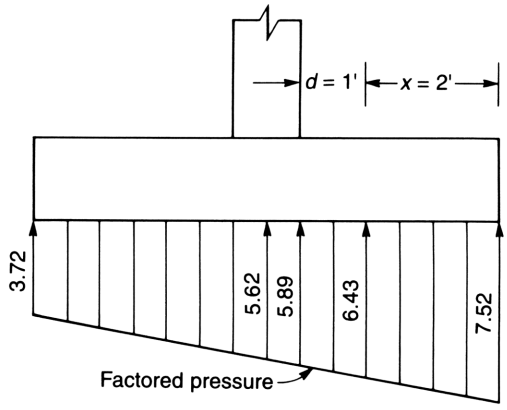

Figure 8–14 Factored pressure distribution

The factored pressure distribution is shown in Figure 8–14.

The critical section for flexural shear is located a distance from the column face equal to the effective depth of the base. The distance of the critical section from the edge of the base is

$$x = L/2 - c_2/2 - d$$
$$= 7/2 - 1/2 - 12/12$$
$$= 2.0 \text{ feet}$$

The factored shear force at this section is

$$V_u = 3 \times 2.0\,(7.52 + 6.43)/2$$
$$= 41.85 \text{ kips}$$

The shear capacity of the footing for flexural shear is given by ACI Equation (11–3) as

$$\phi V_c = 2\phi db(f'_c)^{0.5}$$
$$= 2 \times 0.85 \times 36 \times 12(3000)^{0.5}/1000$$
$$= 40.22 \text{ kips}$$
$$< V_u$$

Hence, the more detailed ACI Equation (11–5) must be used to check the shear and this gives the shear capacity as

$$\phi V_c = \phi[1.9(f'_c)^{0.5} + 2500\rho V_u d/M_u]bd$$

where ρ = reinforcement ratio
$= A_s/bd$
$= 1.96/(36 \times 12)$
$= 0.0045$

M_u = factored moment occurring at the critical section with V_u
$= bx^2(7.52 \times 2 + 6.43)/6$
$= 3 \times 2^2(7.52 \times 2 + 6.43)/6$
$= 42.94$ kip feet

Hence $V_u d/M_u = 41.85 \times 1.0/42.94$
$= 0.975$
< 1 ... satisfactory

Then $\phi V_c = 0.85[1.9(3000)^{0.5} + 2500 \times 0.975 \times 0.0045] \times 36 \times 12/1000$
$= 42.24$ kips
$> V_u$ ... satisfactory
$< 3.5\phi db(f'_c)^{0.5}/1000$ ... satisfactory

Hence, the flexural shear capacity is adequate.

The critical perimeter for punching shear is located a distance from the column face equal to half the effective depth of the base. The length of the critical perimeter is

$$b_o = 4(c_1 + d)$$
$$= 4\ (12 + 12)$$
$$= 96 \text{ inches}$$

The factored shear force at the critical perimeter is

$$V_u = P_u - q(c_1 + d)^2$$
$$= 118 - 5.62 \times 2^2$$
$$= 95.52 \text{ kips}$$

The ratio of the column sides is

$$\beta_c = 1$$
$$< 2$$

Then, the punching shear capacity of the footing is given by ACI Equation (11–37) as

$$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$$
$$= 4 \times 0.85 \times 96 \times 12(3000)^{0.5}/1000$$
$$= 214.53 \text{ kips}$$
$$> V_u \text{ ... satisfactory}$$

Hence, the punching shear capacity is adequate.

The critical section for flexure is located at the face of the column and is given by

$$M_u = 3^3(7.52 \times 2 + 5.89)/6$$
$$= 94.19 \text{ kip feet}$$

The design moment strength is determined in accordance with ACI Section 10.2 and is given by

$$\begin{aligned} \phi M_n &= 0.9A_s f_y d(1 - 0.59\rho f_y/f'_c)/12 \\ &= 100.17 \text{ kip feet} \\ &> M_u \text{ ... satisfactory} \end{aligned}$$

and

$$\begin{aligned} \rho_{max} &= 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y) \\ &= 0.016 \\ &> \rho \text{ ... satisfactory} \end{aligned}$$

In accordance with ACI Section 7.12.2.1 the minimum area of reinforcement required is

$$\begin{aligned} A'_s &= 0.0018 \times bh \\ &= 0.0018 \times 36 \times 16 \\ &= 1.04 \text{ square inches} \\ &< A_s \text{ ... satisfactory} \end{aligned}$$

Hence, the flexural capacity is adequate

8.2.4 Combined footing

The rectangular footing shown in Figure 8–15 has one column adjacent to the property line of the building and this limits the allowable length of the footing. The centroid of the footing is designed to coincide with the centroid of the service loads on the two columns thus providing a uniform soil bearing pressure for service loads. The factored bearing pressure under factored loads will not, however, be uniform unless the ratio of the factored load to service load for both columns is identical. In the longitudinal direction, the footing is designed as a beam spanning over two supports. In the transverse direction, that portion of the footing under each column acts in a manner similar to that of a footing under an isolated column. The load from each column may be considered to be distributed in the longitudinal direction over a width of

$$b_b = c_2 + 1.5d$$

where c_2 = column width in the longitudinal direction

This width of the footing is designed for the bending moment caused by the column factored load acting on this portion of the footing only.

Alternatively, the total area of reinforcement required in the transverse direction may be determined by assuming that the footing cantilevers about the face of both columns. This reinforcement is then concentrated under each column in a band width equal to the length of the shorter side. The area of reinforcement required in the band under column 1 is given by ACI Equation (15–1) as

$$A_{1b} = 2A_s P_{1u}/(\beta + 1)(P_{1u} + P_{2u})$$

where A_s = total required reinforcement area
P_{1u} = factored load on column 1
P_{2u} = factored load on column 2
β = ratio of the long side to the short side of the footing

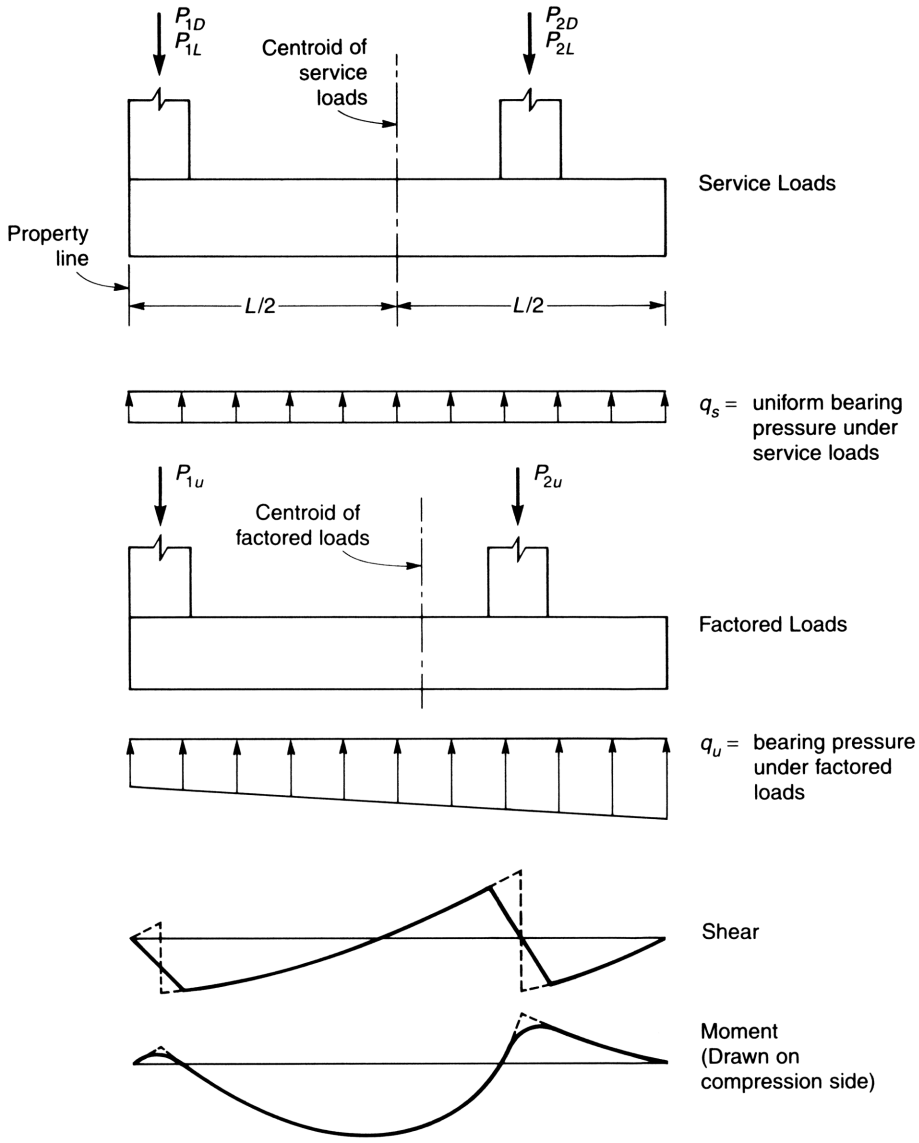

Figure 8–15 Combined footing

Example 8–5 (Combined footing)

Figure 8–16 indicates the service loads supported by two 12 inch × 12 inch columns and their location with respect to the property line. The depth of the footing is 24 inches and the concrete strength is 3000 pounds per square inch. Determine the dimensions of a combined footing which will provide an allowable soil bearing pressure of 2000 pounds per square foot. If the ratio of factored load to service load for both columns is 1.5, determine if the Grade 60 reinforcement indicated and the depth of the footing are adequate.

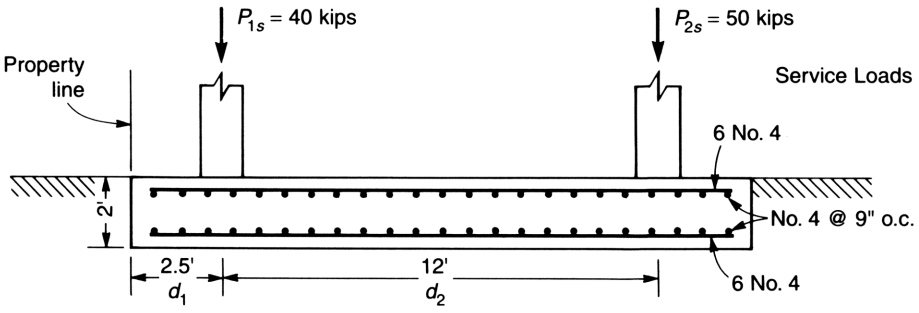

Figure 8–16 Combined footing for Example 8–5

Solution

Allowing for the weight of the footing, the equivalent soil pressure produced by the column service loads is

$$\begin{aligned} q &= q_s - 150h \\ &= 2000 - 150 \times 2 \\ &= 1700 \text{ pounds per square foot} \end{aligned}$$

The centroid of the service loads is located a distance from the property line given by

$$\begin{aligned} x_o &= (2.5\ P_{1s} + 14.5P_{2s})/(P_{1s} + P_{2s}) \\ &= 2.5 \times 40 + 14.5 \times 50)/(40 + 50) \\ &= 9.17 \text{ feet.} \end{aligned}$$

Then, to produce a uniform pressure under the footing, the length of the footing is

$$\begin{aligned} L &= 2x_o \\ &= 2 \times 9.17 \\ &= 18.34 \text{ feet} \end{aligned}$$

The required width of footing to produce an allowable soil pressure of 2000 pounds per square foot is

$$\begin{aligned} B &= (P_{1s} + P_{2s})/qL \\ &= (40{,}000 + 50{,}000)/(1700 \times 18.34) \\ &= 2.89 \text{ feet} \end{aligned}$$

Since the ratio of factored load to service load for both columns is 1.5, the net factored pressure acting on the underside of the footing for factored loads is

$$\begin{aligned} q_u &= (P_{1u} + P_{2u})/L \\ &= 1.5(40 + 50)/18.34 \\ &= 7.36 \text{ kips per linear foot} \end{aligned}$$

The shear force and bending moment diagrams for the footing, for factored loading, are shown in Figure 8–17. Allowing for 3 inches cover to the longitudinal reinforcement, the effective depth is given by

$$
\begin{aligned}
d &= h - 3 - d_b/2 \\
&= 24 - 3 - 0.5/2 \\
&= 20.75 \text{ inches}
\end{aligned}
$$

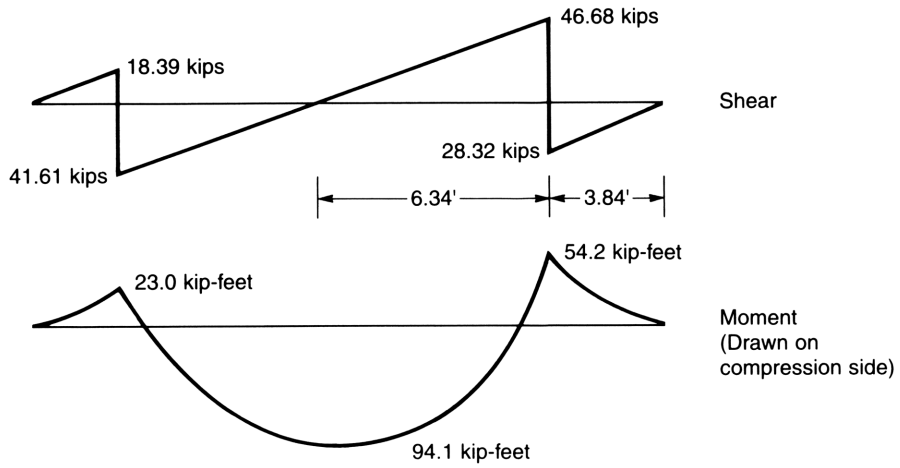

Figure 8–17 Shear force and bending moment diagrams for Example 8–5

The critical section for flexural shear is located a distance equal to the effective depth from column 2. The factored applied shear at this critical section is given by

$$
\begin{aligned}
V_u &= V_{2u} - q_u(d + c_2/2)/12 \\
&= 46.68 - 7.36(20.75 + 6)/12 \\
&= 30.27 \text{ kips}
\end{aligned}
$$

The flexural shear capacity of the footing is given by Equation (11–3) as

$$
\begin{aligned}
\phi V_c &= 2\phi B d(f'_c)^{0.5} \\
&= 2 \times 0.85 \times 2.89 \times 12 \times 20.75(3000)^{0.5}/1000 \\
&= 66.31 \text{ kips} \\
&> V_u \ldots \text{ satisfactory}
\end{aligned}
$$

The length of the critical perimeter for punching shear is given by

$$
\begin{aligned}
b_o &= 4(c_2 + d) \\
&= 4(12 + 20.75) \\
&= 131 \text{ inches}
\end{aligned}
$$

The column side ratio is

$$
\begin{aligned}
\beta_c &= 1 \\
&< 2
\end{aligned}
$$

Then the punching shear capacity of the footing is given by ACI Equation (11–37) as

$$
\begin{aligned}
\phi V_c &= 4\phi b_o d(f'_c)^{0.5} \\
&= 4 \times 0.85 \times 131 \times 20.75 \times (3000)^{0.5}/1000 \\
&= 506 \text{ kips} \\
&> P_{2u} \ldots \text{ satisfactory}
\end{aligned}
$$

The minimum reinforcement ratio in a footing slab for Grade 60 bars is given by ACI Section 7.12.2.1 as

$$\rho_{min} = 0.18 \text{ percent of the gross area}$$

The minimum area of reinforcement required is

$$\begin{aligned} A_{s(min)} &= \rho_{min} hB \\ &= 0.18 \times 24 \times 2.89 \times 12/100 \\ &= 1.5 \text{ square inches.} \end{aligned}$$

The sum of the reinforcement areas in the top and bottom of the footing, which consists of a total of twelve No.4 bars, is

$$\begin{aligned} 2A_s &= 2.4 \text{ square inches} \\ &> 1.5 \ldots \text{ satisfactory} \end{aligned}$$

The maximum reinforcement ratio is given by ACI Section 10.3 as

$$\begin{aligned} \rho_{max} &= 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y) \\ &= 1.6 \text{ percent} \end{aligned}$$

The reinforcement ratio provided in the top of the footing is

$$\begin{aligned} \rho &= A_s/Bd \\ &= 1.2 \times 100/(2.89 \times 12 \times 20.75) \\ &= 0.17 \text{ percent} \\ &< \rho_{max} \ldots \text{ satisfactory.} \end{aligned}$$

Neglecting compression reinforcement, the design moment strength of the reinforcement in the top of the footing is given by ACI Section 10.2 as

$$\begin{aligned} \phi M_n &= 0.9A_s f_y d(1 - 0.59\rho f_y/f'_c)/12 \\ &= 109.8 \text{ kip feet} \\ &> 94.1 \ldots \text{ satisfactory} \end{aligned}$$

The reinforcement provided in the bottom of the footing is clearly adequate.

The critical section for flexure in the transverse direction is at the face of the columns and the factored moment at this section is

$$\begin{aligned} M_u &= q_u L(B - c_1)^2/8 \\ &= 7.36 \times 18.34 \ (2.89 - 1)^2/8 \\ &= 60.27 \text{ kip feet} \end{aligned}$$

The required reinforcement ratio in the bottom of the footing is derived from ACI Section 10.2 and is

$$\begin{aligned} \rho &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.0142 \text{ percent} \end{aligned}$$

Hence the required reinforcement area is

$$A_s = \rho bd$$
$$= 0.0142 \times 18.34 \times 12 \times 20.75/100$$
$$= 0.65 \text{ square inches}$$

The reinforcement area provided in the bottom of the footing in the transverse direction is

$$A_s' = 0.2 \times 18.34 \times 12/9$$
$$= 4.89 \text{ square inches}$$
$$> A_s \ldots \text{satisfactory}$$

The ratio of the long side to the short side of the footing is

$$\beta = 18.34/2.89$$
$$= 6.34$$

The area of reinforcement required under column 2 in a band of width B is given by ACI Equation (15-1) as

$$A_b = 2A_sP_{2u}/(\beta + 1)(P_{1u} + P_{2u})$$
$$= 2 \times 0.65 \times 50/(7.34 \times 90)$$
$$= 0.098 \text{ square inches}$$

The area of reinforcement provided in a band of width B is

$$A_b' = 0.2 \times 2.89 \times 12/9$$
$$= 0.77 \text{ square inches}$$
$$> A_b \ldots \text{satisfactory}$$

Example 8–6 (Two-column footing[3])

Figure 8-18 indicates the service loads supported by two 24 inch × 24 inch columns. The depth of the footing is 36 inches, the width is 16 feet, and the concrete strength is 4000 pounds per square inch. Determine if the depth of the footing is adequate and design the flexural reinforcement using No. 6 Grade 60 reinforcement.

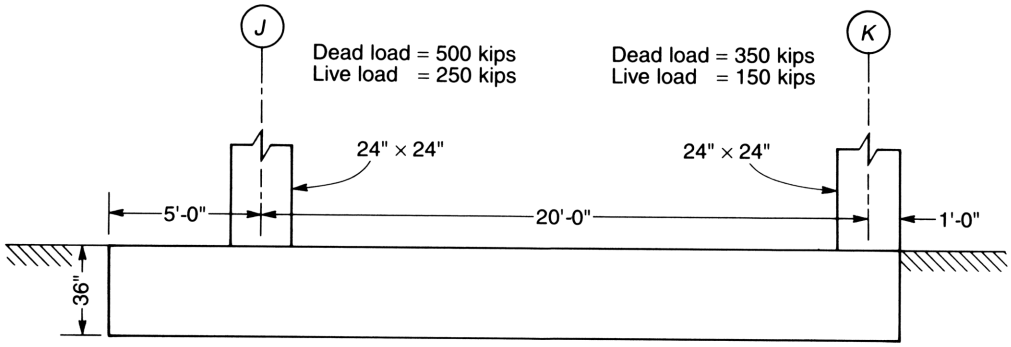

Figure 8–18 Details for Example 8–6

Solution

The factored load on column J is obtained from ACI Equation (9–1) as

$$W_J = 1.4 \times 500 + 1.7 \times 250$$
$$= 1125 \text{ kips}$$

The factored load on column K is

$$W_K = 1.4 \times 350 + 1.7 \times 150$$
$$= 745 \text{ kips}$$

The centroid of the factored loads is located a distance from column K given by

$$x_o = 20W_J/(W_J + W_K)$$
$$= 20 \times 1125/(1125 + 745)$$
$$= 12 \text{ feet}$$

Hence, the centroid of the factored loads coincides with the centroid of the footing giving a uniform factored soil pressure under the footing of

$$q = (W_J + W_K)/l$$
$$= (1125 + 745)/26$$
$$= 71.92 \text{ kips per linear foot}$$

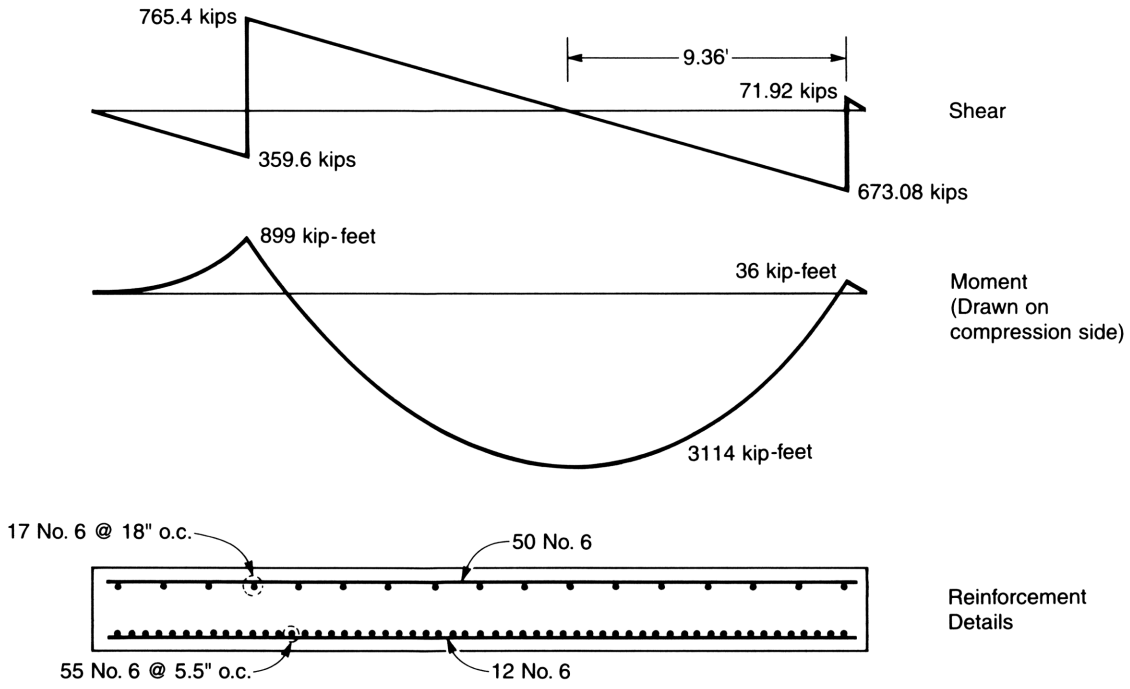

Figure 8–19 Shear and bending moment diagrams for Example 8–6

The shear force and bending moment diagrams for the footing are shown in Figure 8–19. The point of zero shear is located a distance from the center of column K given by

$$\begin{aligned} x &= 673.08/q \\ &= 673.08/71.92 \\ &= 9.36 \text{ feet} \end{aligned}$$

The maximum negative moment occurs at this point and is given by

$$\begin{aligned} M_u &= xW_K - q(x + 1)^2/2 \\ &= 9.36 \times 745 - 71.92(9.36 + 1)^2/2 \\ &= 3114 \text{ kip feet} \end{aligned}$$

The footing depth may be checked by determining the capacity of the footing in two-way shear and flexural shear. Assuming three inch clear cover with No. 6 bars in both directions gives an average effective depth in each direction of

$$\begin{aligned} d &= h - c_c - d_b \\ &= 36 - 3 - 0.75 \\ &= 32.25 \end{aligned}$$

The critical section for punching shear, as defined by ACI Sections 15.5.2 and 11.12.1.2, is located a distance equal to half the effective depth from the face of the column. For column K the critical perimeter forms three sides of a rectangle with a length of

$$\begin{aligned} b_o &= (c_1 + d) + 2(c_2 + d/2) \\ &= (24 + 32.25) + 2(24 + 32.25/2) \\ &= 136.5 \text{ inches} \end{aligned}$$

The ratio of the long side of the column to the short side is

$$\begin{aligned} \beta_c &= 24/24 \\ &= 1 \\ &< 2 \end{aligned}$$

Hence the punching shear capacity of the footing is given by ACI Equation (11–38) as

$$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$$

where $\phi = 0.85$ from ACI Section 9.3.2

and

$$\begin{aligned} \phi V_c &= 4 \times 0.85 \times 136.5 \times 32.25(4000)^{0.5} \\ &= 947 \text{ kips} \\ &> W_K \end{aligned}$$

At column J the critical perimeter is

$$\begin{aligned} b_o &= 4(c_1 + d) \\ &= 4(24 + 32.25) \\ &= 225 \text{ inches} \end{aligned}$$

The punching shear capacity at column J is

$$\begin{aligned}\phi V_c &= 947 \times 225/136.5\\ &= 1561 \text{ kips}\\ &> W_J\end{aligned}$$

Hence the footing depth is adequate for punching shear.
The critical section for flexural shear in a footing, as defined by ACI Sections 15.5.2 and 11.1.3.1, is located a distance equal to the effective depth from the face of the column. The factored applied shear at this critical section is

$$\begin{aligned}V_u &= 765.4 - q(d + c_2/2)\\ &= 765.4 - 71.92(32.25 + 12)/12\\ &= 500 \text{ kips}\end{aligned}$$

The shear capacity of the footing for beam action is given by ACI Equation (11–3) as

$$\begin{aligned}\phi V_c &= 2\phi bd(f'_c)^{0.5}\\ &= 2\times 0.85 \times 16 \times 12 \times 32.25(4000)^{0.5}/1000\\ &= 666 \text{ kips}\\ &> V_u\end{aligned}$$

Hence the footing depth is adequate for flexural shear.

Neglecting compression reinforcement, the required reinforcement ratio in the top of the footing in the longitudinal direction is derived from ACI Section 10.2 and is

$$\rho = 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$

where $M_u = 3114$ kip feet
and $\rho = 0.0036$

In accordance with ACI Section 10.3 the maximum allowable reinforcement ratio is given by

$$\rho_{max} = 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y)$$

where $\beta_1 = 0.85$ as defined in ACI Section 10.2.7.3
and $\rho_{max} = 0.0214$
$> \rho$ ... satisfactory

In accordance with ACI Section 7.12.2.1 the minimum allowable reinforcement ratio, expressed as a percentage of the gross concrete area, is given by

$$\begin{aligned}\rho_{min} &= 0.18 \text{ percent}\\ &< \rho \text{ ... satisfactory}\end{aligned}$$

Thus the required reinforcement area is given by

$$\begin{aligned}A_s &= \rho bd\\ &= 0.0036 \times 192 \times 32.25\\ &= 22.16 \text{ square inches}\end{aligned}$$

This may be provided by fifty No.6 bars at a spacing of 3.8 inches giving an area of

$$A_s = 22.0 \text{ square inches}$$

The critical section for positive moment, as defined by ACI Section 15.4.2, is located along the face of column J. The factored applied moment at this section is given by

$$M_u = q \times 4^2/2$$
$$= 71.92 \times 4^2/2$$
$$= 575 \text{ kip feet}$$

Neglecting compression reinforcement, the required reinforcement ratio in the bottom of the footing is derived from ACI Section 10.2 and is

$$\rho = 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$

where $M_u = 575$ kip feet

and $\rho = 0.00064$
$< \rho_{max}$ ... satisfactory
$< \rho_{min}$

Hence, in accordance with ACI Section 10.5.3 the required reinforcement area is given by

$$A_s = 1.33\rho bd$$
$$= 1.33 \times 0.00064 \times 192 \times 32.25$$
$$= 5.31 \text{ square inches}$$

This may be provided by twelve No.6 bars giving an area of

$$A_s = 5.28 \text{ square inches}$$

The bar spacing is

$$s = 186/11$$
$$= 16 \text{ inches}$$
$$< 3h$$
$$< 18 \text{ inches}$$

Hence the requirements of ACI Section 7.6.5 are satisfied.

The factored applied moment at the critical section for the transverse direction is

$$M_u = (W_J + W_K)/b \times 7^2/2$$
$$= 1{,}870 \times 7^2/(16 \times 2)$$
$$= 2863 \text{ kip feet}$$

The required reinforcement ratio in the bottom of the footing in the transverse direction is derived from ACI Section 10.2 and is

$$\rho = 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$
$$= 0.0020$$
$$< \rho_{max} \text{ ... satisfactory}$$
$$> \rho_{min} \text{ ... satisfactory}$$

and $A_s = 20.08$ square inches

The ratio of the long side to the short side of the footing is

$$\beta = \ell/b$$
$$= 26/16$$
$$= 1.63$$

From ACI Equation (15–1) the area of reinforcement required in a band width equal to the length of the short side of the footing is given by

$$\begin{aligned} A_b &= 2A_s/(\beta + 1) \\ &= 2 \times 20.08/(1.63 + 1) \\ &= 15.27 \text{ square inches} \end{aligned}$$

This is equivalent to an area per linear foot of band width of

$$\begin{aligned} A_t &= A_b/b \\ &= 15.27/16 \\ &= 0.954 \text{ square inches per linear foot} \end{aligned}$$

Because of the proximity of the two band widths from column J and column K, it is convenient to provide this reinforcement over the whole length of the footing. Hence, use No.6 bars at a spacing of 5.5 inches to give an area of

$$A_t = 0.96 \text{ square inches per linear foot, a total of fifty-five bars.}$$

Provide No.6 bars at eighteen inch centers in order to locate the top longitudinal bars. Complete reinforcement details are shown in Figure 8–19.

8.2.5 Strap Footing

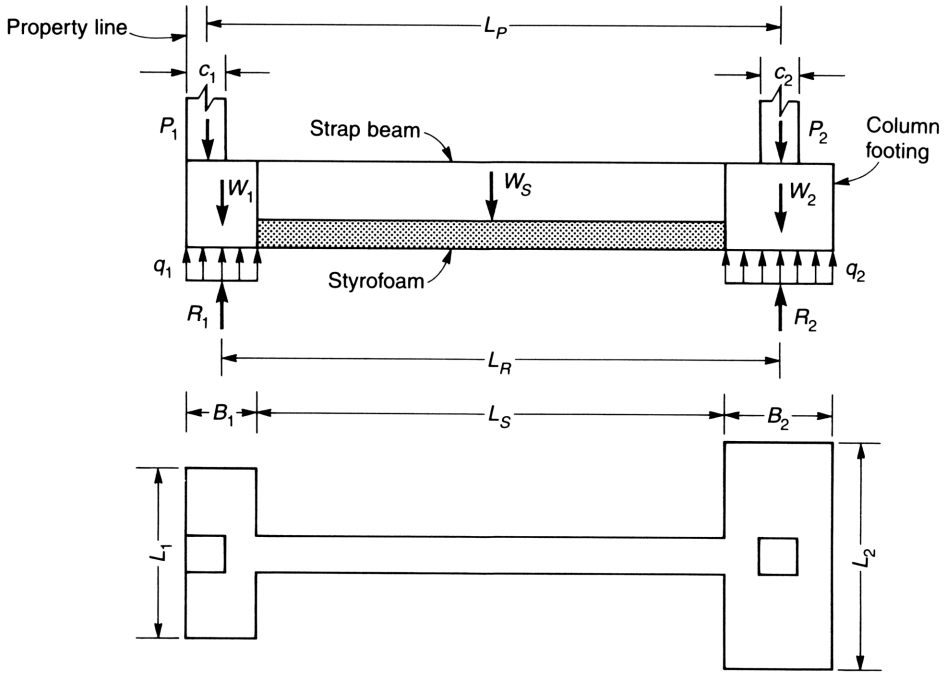

Figure 8–20 Strap footing

The strap footing shown in Figure 8–20 consists of two columns on individual pad footings which are connected by a strap beam. The soffit of the strap is not subject to soil pressure as it is poured on a layer of styrofoam. It is assumed that the strap and footings act as a rigid body with a uniform bearing pressure under the footings. Then, the soil reactions, R_1 and R_2, act at the center of the footings and the pressure under the footings is given by

$$q_1 = q_2 = q$$

The base area of each footing is

$$A_1 = B_1 L_1$$
$$A_2 = B_2 L_2$$

The soil reactions are given by

$$R_1 = qA_1$$
$$R_2 = qA_2$$

By locating the footing under column 2 symmetrical with respect to the center of the column, the lines of action of R_2 and P_2 coincide then, from the Figure, the distance between R_1 and R_2 is

$$L_R = L_P + c_1/2 - B_1/2$$

The strap length is

$$L_S = L_R - (B_1 + B_2)/2$$

Two equations of statics are available for solving the two unknowns R_1 and R_2. However, the values of R_1 and R_2 also influence the required dimensions for the members and an iterative technique is required.

Appropriate dimensions are initially selected for B_1 and B_2, the depth of the footings and strap, and the width of the strap. Hence, the weight of the strap, W_S, may be determined. An initial estimate is made of the soil reaction R_1 and corresponding values of A_1 and W_1 are determined and an estimate is made of W_2. Resolving vertically gives

$$R_2 = P_1 + P_2 - R_1 + W_1 + W_2 + W_S$$

Hence, A_2 may be determined and the initial estimate of W_2 revised. Taking moments about the center of footing 2 gives

$$R_1 = [P_1 L_P + W_1 L_R + W_S (L_S + B_2)/2]/L_R$$

The initial estimate of R_1 may now be revised and the process repeated until convergence is reached.

Example 8–7 (Strap footing)

Design the strap beam of the strap footing shown in Figure 8–21. The allowable soil bearing pressure is 3000 pounds per square foot, concrete strength is 3000 pounds per square inch, and all reinforcement is Grade 60.

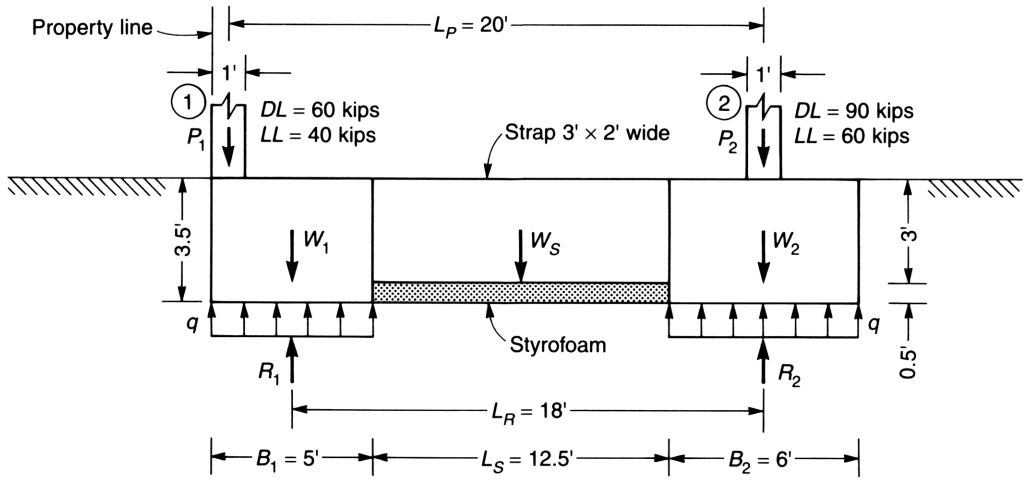

Figure 8–21 Details for Example 8–7

Solution

As an initial estimate, assume that the value of the soil reaction under column 1 is given by

$$\begin{aligned} R_1 &= 1.42P_1 \\ &= 1.42(60 + 40) \\ &= 142 \text{ kips} \end{aligned}$$

The area of the footing required for column 1 is

$$\begin{aligned} A_1 &= R_1/q \\ &= 142/3 \\ &= 47.33 \text{ square feet} \end{aligned}$$

The weight of this footing is

$$\begin{aligned} W_1 &= 0.15A_1h_1 \\ &= 0.15 \times 47.33 \times 3.5 \\ &= 24.81 \text{ kips} \end{aligned}$$

The weight of the strap beam is

$$\begin{aligned} W_S &= 0.15L_SB_Sh_S \\ &= 0.15 \times 12.5 \times 2 \times 3 \\ &= 11.25 \text{ kips} \end{aligned}$$

As an initial estimate, assume that the weight of the footing required for column 2 is

$$W_2 = 30.6 \text{ kips}$$

Then, resolving vertically, the value of the soil reaction under column 2 is

$$\begin{aligned} R_2 &= P_1 + P_2 - R_1 + W_1 + W_2 + W_S \\ &= 100 + 150 - 142 + 24.81 + 30.6 + 11.25 \\ &= 174.7 \text{ kips} \end{aligned}$$

The area of the footing required for column 2 is

$$\begin{aligned} A_2 &= R_2/q \\ &= 174.7/3 \\ &= 58.23 \end{aligned}$$

A revised estimate of the weight of this footing is

$$\begin{aligned} W_2 &= 0.15A_2h_2 \\ &= 0.15 \times 58.23 \times 3.5 \\ &= 30.57 \text{ kips.} \\ &\approx 30.6 \end{aligned}$$

Hence, the initial estimate of W_2 is sufficiently accurate.
Taking moments about the center of footing 2 gives

$$\begin{aligned} R_1 &= [P_1L_P + W_1L_R + W_S(L_S + B_2)/2]/L_R \\ &= (100 \times 20 + 24.81 \times 18 + 11.25 \times 18.5/2)/18 \\ &= 141.7 \text{ kips} \\ &\approx 142 \end{aligned}$$

Hence, the initial estimate of R_1 is sufficiently accurate.
The factored loading on column 1 is given by ACI Equation (9–1) as

$$\begin{aligned} P_{1u} &= 1.4P_D + 1.7P_L \\ &= 1.4 \times 60 + 1.7 \times 40 \\ &= 152 \text{ kips} \end{aligned}$$

The ratio of factored load to service load for column 1 is

$$\begin{aligned} P_{1u}/P_{1s} &= 152/100 \\ &= 1.52 \end{aligned}$$

and the same ratio is applicable for column 2 since the ratio of dead load to live load is the same for both columns. The factored load on column 2 is

$$\begin{aligned} P_{2u} &= 1.52(90 + 60) \\ &= 228 \text{ kips} \end{aligned}$$

Hence, neglecting the effect of the footings' factored dead loads, a uniform factored pressure acts on the underside of both footings under factored loading. The factored weights of the footings are

$$\begin{aligned} W_{1u} &= 1.4W_1 \\ &= 34.73 \text{ kips} \\ W_{2u} &= 1.4W_2 \\ &= 42.80 \text{ kips} \\ W_{Su} &= 1.4W_S \\ &= 15.75 \text{ kips} \end{aligned}$$

The total factored load is

$$\begin{aligned} P_u &= P_{1u} + P_{2u} + W_{1u} + W_{2u} + W_{Su} \\ &= 473.28 \text{ kips} \end{aligned}$$

Hence the uniform factored soil pressure at both footings is

$$\begin{aligned} q_u &= P_u/(A_1 + A_2) \\ &= 473.28/(47.33 + 58.23) \\ &= 4.48 \text{ kips per square foot} \end{aligned}$$

Hence the net factored pressures acting on the underside of both footings are

$$\begin{aligned} q_{1u} &= (q_u A_1 - W_{1u})/B_1 \\ &= (4.48 \times 47.33 - 34.73)/5 \\ &= 35.46 \text{ kips per linear foot} \\ q_{2u} &= (q_u A_2 - W_{2u})/B_2 \\ &= (4.48 \times 58.23 - 42.80)/6 \\ &= 36.35 \text{ kips per linear foot} \end{aligned}$$

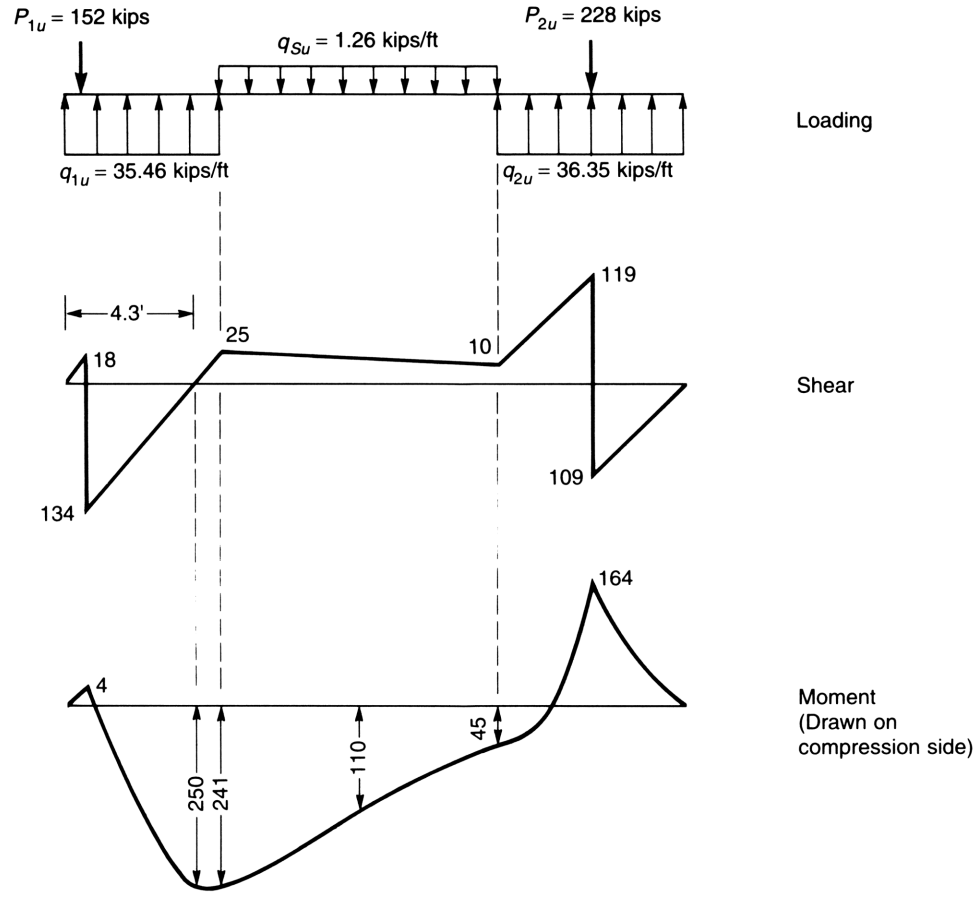

Figure 8–22 Shear and bending moment diagrams for Example 8–7

The factored self weight of the strap beam is

$$q_{Su} = W_{Su}/L_S$$
$$= 15.75/12.5$$
$$= 1.26 \text{ kips per linear foot}$$

The loading, shear and moment diagrams are shown in Figure 8–22
The maximum negative moment in the strap occurs at the interface with footing 1 and has the value

$$M_u = 241 \text{ kip feet}$$

The required flexural reinforcement ratio in the top of the strap is derived from ACI Section 10.2 and is

$$\rho = 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y$$

where $d = 33.625$ inches ... using No. 6 bars with 2 inches cover
and $\rho = 0.202$ percent

The minimum reinforcement ratio in the strap beam for Grade 60 bars and 3000 pounds per square inch concrete is given by ACI Sections 10.5.1 and 10.5.3 as the lesser of

$$\rho_{min} = 200/f_y$$
$$= 0.333 \text{ percent}$$

or $$\rho_{min} = 1.333\rho$$
$$= 0.269 \text{ percent ... governs}$$

Hence the reinforcement area required in the top of the strap beam is given by

$$A_s = bd\rho_{min}$$
$$= 24 \times 33.625 \times 0.00269$$
$$= 2.17 \text{ square inches}$$

Providing five No.6 bars gives a reinforcement area

$$A'_s = 2.20 \text{ square inches}$$
$$> A_s \text{ ... satisfactory}$$

The maximum shear in the strap occurs at the interface with footing 1 and has the value

$$V_u = 25 \text{ kips}$$

The design shear strength of the concrete section is given by ACI Equation (11–3) as

$$\phi V_c = 2\phi (f'_c)^{0.5} b_w d$$
$$= 2 \times 0.85 \times (3000)^{0.5} \times 24 \times 33.625/1000$$
$$= 75 \text{ kips}$$
$$> 2V_u \text{ ... satisfactory}$$

Hence, in accordance with ACI Section 11.5.5 no shear reinforcement is required. ═══

8.2.6 Eccentric Footing

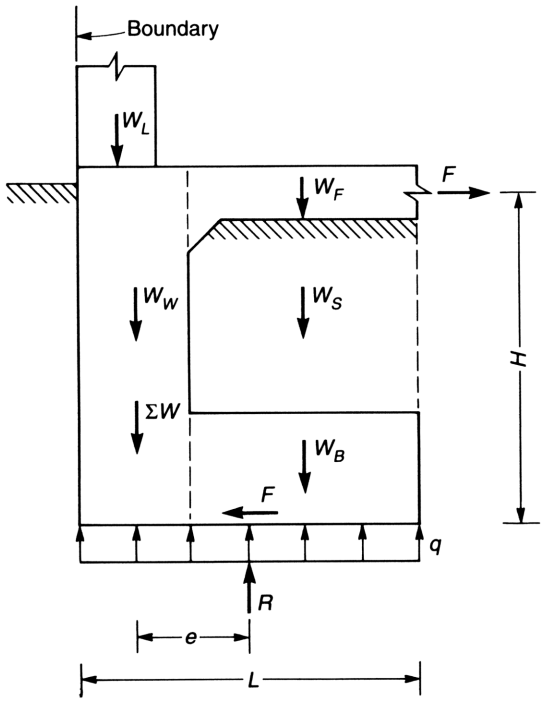

Figure 8–23 Eccentric footing

An eccentric footing, as shown in Figure 8–23, may be utilized when the wall of a building must be located on the property line. It is assumed that the footing acts as a rigid body with a uniform soil pressure under the base and that the lateral soil pressures on either side of the footing are balanced. The total gravity load acting on the footing is

$$\Sigma W = W_L + W_F + W_W + W_S + W_B$$

The soil reaction under the base is

$$R = \Sigma W$$

The required length of the base is

$$L = R/q$$

where q = allowable soil pressure.

The counterclockwise couple produced by R and ΣW is

$$M_R = Re$$

where e is the eccentricity between R and ΣW.

The frictional force produced at the underside of the base is

$$F = \mu R$$

where μ is the coefficient of friction.

The frictional force is balanced by an equal and opposite tensile force in the top slab and this produces the clockwise couple

$$M_F = FH$$

Equating the two couples gives

$$\begin{aligned} M_F &= M_R \\ FH &= Re \\ H &= Re/F \\ &= e/\mu \end{aligned}$$

By assuming an initial value for H, W_F, W_S and W_B, values of L and e may be determined and a revised value of H computed. The process is repeated until convergence is reached.

Example 8–8(Design of eccentric footing)

Design the eccentric footing shown in Figure 8–24 to support a superimposed dead plus live load of 6.2 kips per linear foot. The coefficient of friction is 0.3 and the unit soil weight 120 pounds per cubic foot. The allowable soil bearing pressure is 2500 pounds per square foot. Concrete strength is 3000 pounds per square inch and all reinforcement is Grade 60.

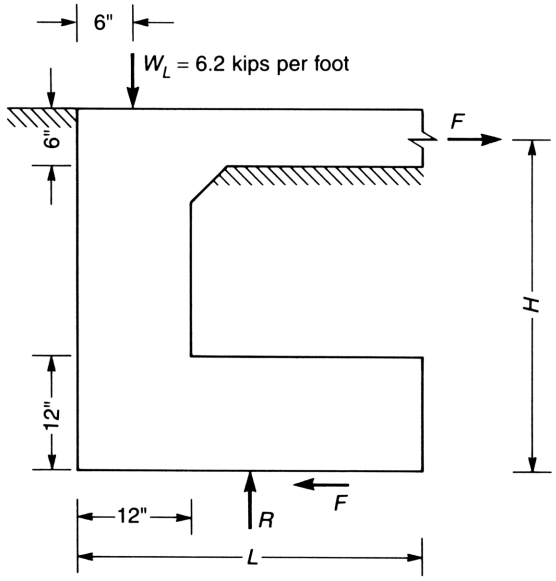

Figure 8–24 Details for Example 8–8

Solution

As initial estimates, assume that

$$\begin{aligned} H &= 2.75 \text{ feet} \\ W_F &= 0.15 \text{ kips} \\ W_B &= 0.30 \text{ kips} \\ W_S &= 0.36 \text{ kips} \end{aligned}$$

Then $W_W = 0.15 \times 3 \times 1$

$= 0.45$ kips

$$\begin{aligned}\Sigma W &= R = W_L + W_F + W_B + W_S + W_W \\ &= 6.2 + 0.15 + 0.30 + 0.36 + 0.45 \\ &= 7.46 \text{ kips}\end{aligned}$$

The base length required is given by

$$\begin{aligned}L &= R/q \\ &= 7.46/2.5 \\ &= 2.984 \text{ feet}\end{aligned}$$

The eccentricity between the lines of action of R and ΣW is

$$\begin{aligned}e &= L[1 - (W_F + W_B + W_S)/\Sigma W]/2 - 0.5 \\ &= 2.984(1 - 0.81/7.46)/2 - 0.5 \\ &= 0.83\end{aligned}$$

The required value for H is given by

$$\begin{aligned}H &= e/\mu \\ &= 0.83/0.3 \\ &= 2.77 \text{ feet} \\ &\approx 2.75\end{aligned}$$

Hence the initial estimate of H is sufficiently accurate.
Revised values for the footing and soil weights are

$$\begin{aligned}W_F &= 0.15 \times 1.984 \times 0.5 \\ &= 0.149 \\ &\approx 0.150 \\ W_B &= 0.15 \times 1.984 \times 1.0 \\ &= 0.298 \\ &\approx 0.300 \\ W_S &= 0.12 \times 1.984 \times 1.52 \\ &= 0.362 \\ &\approx 0.360 \\ W_W &= 0.15 \times 3.02 \times 1 \\ &= 0.453 \\ &\approx 0.450\end{aligned}$$

The net pressure acting on the base of the footing is

$$\begin{aligned}q' &= q - (W_F + W_S + W_B)/(L - 1) \\ &= 2.5 - 0.408 \\ &= 2.09 \text{ kips per linear foot.}\end{aligned}$$

The bending moment at the face of the front wall is

$$\begin{aligned}M &= q'(L - 1)^2/2 \\ &= 4.11 \text{ kip feet}\end{aligned}$$

Minimum reinforcement is adequate and is given by ACI Section 7.12.2.1 as

$$\begin{aligned} A_s &= 0.0018bh \\ &= 0.0018 \times 12 \times 12 \\ &= 0.26 \text{ square inches per foot} \\ &= \text{No.4 bars at 18 inch spacing top and bottom.} \end{aligned}$$

Using six No.3 bars transversely provides a reinforcement ratio of

$$\begin{aligned} \rho &= 0.66 \times 100/(12 \times 12 \times 1.984) \\ &= 0.23 \text{ percent} \\ &> 0.18 \text{ percent} \ldots \text{ satisfactory} \end{aligned}$$

Minimum reinforcement in the front wall is governed by ACI Section 14.3. Vertical reinforcement consisting of No.4 bars at 18 inch spacing front and back provides a reinforcement ratio of

$$\begin{aligned} \rho_{vert} &= 0.4 \times 100/(12 \times 18) \\ &= 0.19 \text{ percent} \\ &> 0.12 \text{ percent} \ldots \text{ satisfactory} \end{aligned}$$

Horizontal reinforcement consisting of eight No.3 bars provides a reinforcement ratio of

$$\begin{aligned} \rho_{hor} &= 0.88 \times 100/(12 \times 36) \\ &= 0.204 \text{ percent} \\ &> 0.20 \text{ percent} \ldots \text{ satisfactory} \end{aligned}$$

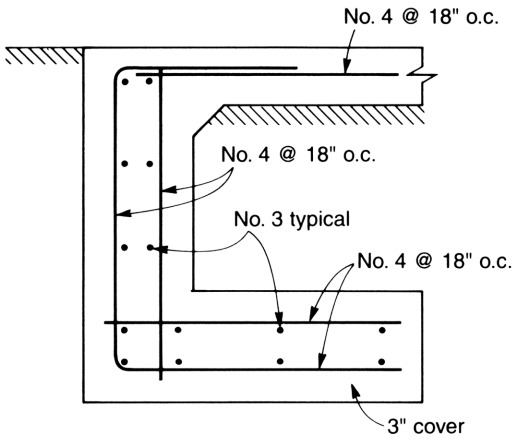

Figure 8–25 Reinforcement details for Example 8–8

In accordance with ACI Section A.3.2, the area of tensile reinforcement required in the floor slab is given by

$$\begin{aligned} A_s &= F/f_s \\ &= 0.3 \times 7.46/24 \\ &= 0.093 \text{ square inches per foot} \end{aligned}$$

Using No.4 bars at 18 inch spacing provides an area

$$A_s' = 0.13 \text{ square inches} > 0.093 \ldots \text{ satisfactory}$$

The reinforcement layout is shown in Figure 8–25.

Example 8–9 (Eccentric footing)

The tilt-up concrete wall panel shown in Figure 8–26 is located six inches from the property line and supports an axial load of five kips per linear foot. The wall panel is supported on the eccentric footing indicated and the ground floor slab is tied to the wall panel with Grade 60 reinforcement to limit the soil bearing pressure to 3000 pounds per square foot. Determine the required depth H of the footing below the level of the ground floor slab to produce a maximum soil bearing pressure of 3000 pounds per square foot. Calculate the area of reinforcement required to tie the wall panel to the ground floor slab.

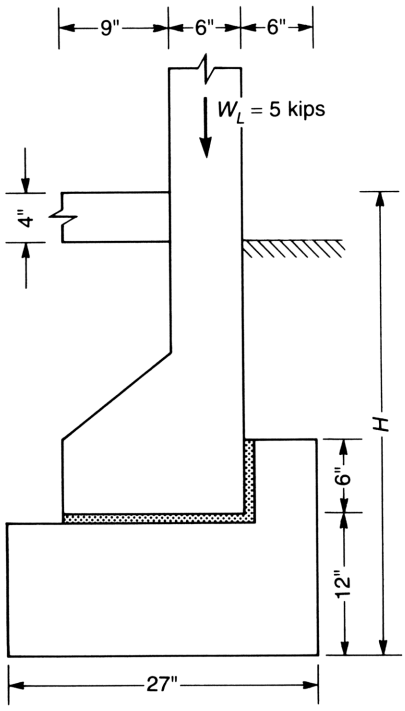

Figure 8–26 Details for Example 8–9

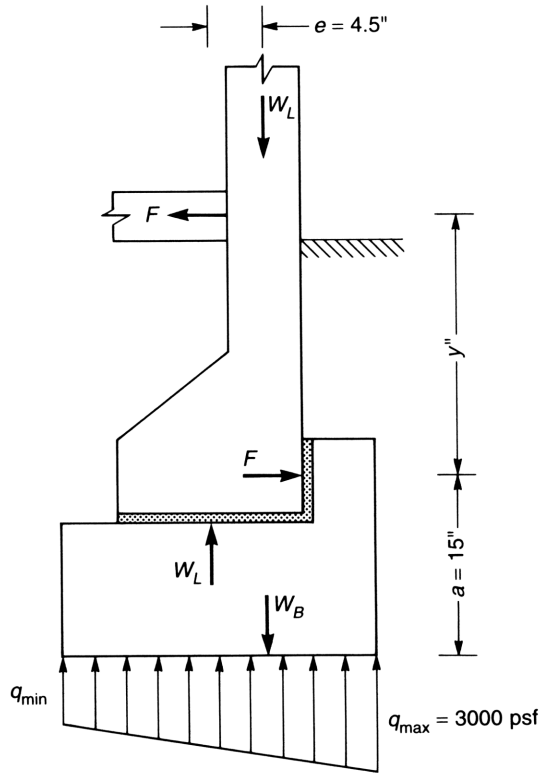

Figure 8–27 Forces acting on footing

Solution

The forces acting on the footing are shown in Figure 8–27 and assume a uniform distribution of pressure under the wall and between the side of the wall and the footing. The axial load in the wall panel is

$$W_L = 5 \text{ kips per foot}$$

The weight of the base is

$$W_B = 0.15(2.25 \times 1 + 0.5 \times 0.5)$$
$$= 0.375 \text{ kips per foot}$$

The total vertical load is

$$R = W_L + W_B$$
$$= 5.375 \text{ kips per foot}$$

The eccentricity between the axial load in the wall panel and the center of pressure under the wall is

$$e = 15/2 - 6/2$$
$$= 4.5 \text{ inches}$$

The clockwise couple produced by the axial load in the wall is

$$M_W = eW_L$$

This is balanced by the counterclockwise couple

$$M_F = yF$$

Equating the two couples gives

$$F = eW_L/y$$
$$= 4.5 \times 5/y$$
$$= 22.5/y \text{ kips}$$

The maximum pressure under the base is given by

$$q_{max} = R/A + Fa/S$$
$$= 5.375/2.25 + 22.5 \times 1.25/0.844y$$
$$= 3.0 \text{ kips per square foot}$$

Hence $y = 54.5$ inches

Then, the required depth of the footing below the level of the ground floor slab is

$$H = y + 15 + 4/2$$
$$= 71.5 \text{ inches}$$

In accordance with ACI Section A.3.2, the area of tensile reinforcement required in the floor slab is

$$A_S = F/f_s$$
$$= 22.5/(54.5 \times 24)$$
$$= 0.017 \text{ square inches per foot}$$

8.3 SUPPLEMENTARY PROBLEMS

Problem 8–1 (Strip footing)
The 14 inch deep strip footing shown in Figure 8–28 supports an eight inch masonry wall which applies the indicated service loads to the footing. The allowable soil bearing pressure is 5000 pounds per square foot. Using a concrete strength of 3000 pounds per square inch and Grade 60 reinforcement, design the footing.

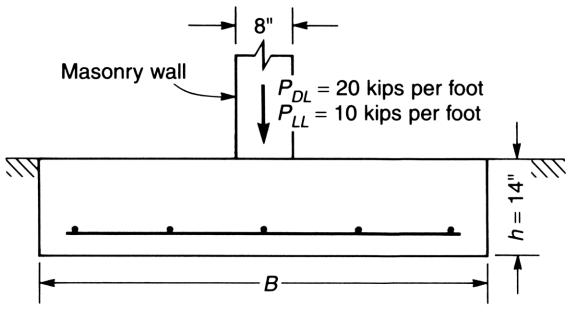

Figure 8–28 Details for Problem 8–1

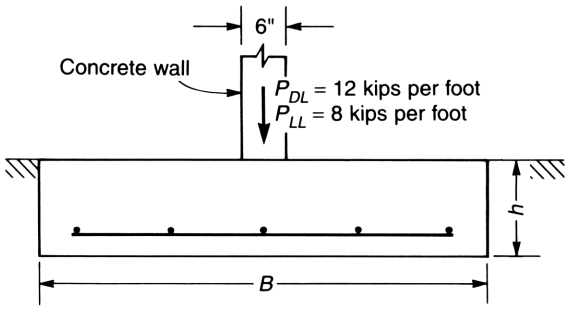

Figure 8–29 Details for Problem 8–2

Problem 8–2 (Strip footing)
The strip footing shown in Figure 8–29 supports a six inch concrete wall which applies the indicated service loads to the footing. The allowable soil bearing pressure is 4000 pounds per square foot. Using a concrete strength of 4000 pounds per square inch and Grade 60 reinforcement, determine the minimum width and depth of the footing and the required reinforcement.

Problem 8–3 (Square isolated column footing)
The six foot square footing shown in Figure 8–30 supports a concrete column which applies the indicated factored axial load to the footing. Using a concrete strength of 3000 pounds per square inch and Grade 60 reinforcement, determine the minimum depth of the footing and the required reinforcement.

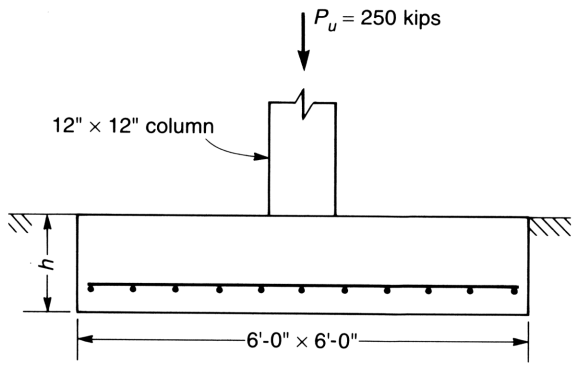

Figure 8–30 Details for Problem 8–3

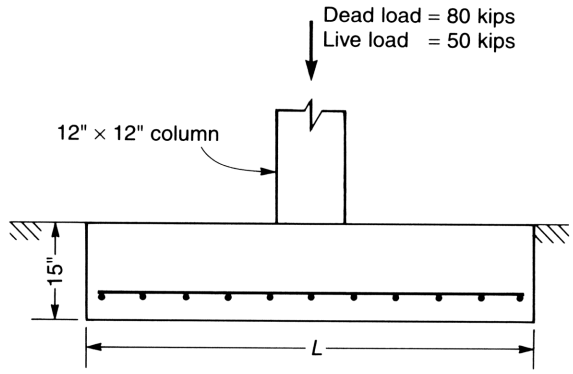

Figure 8–31 Details for Problem 8–4

Problem 8–4 (Rectangular isolated column footing)
The four foot wide and fifteen inch deep footing shown in Figure 8–31 supports a concrete column which applies the indicated service loads to the footing. The allowable soil bearing pressure is 4500 pounds per square foot. Using a concrete strength of 3000 pounds per square inch and Grade 60 reinforcement, determine the minimum required length of the footing, the required reinforcement, and check the adequacy of the footing for shear.

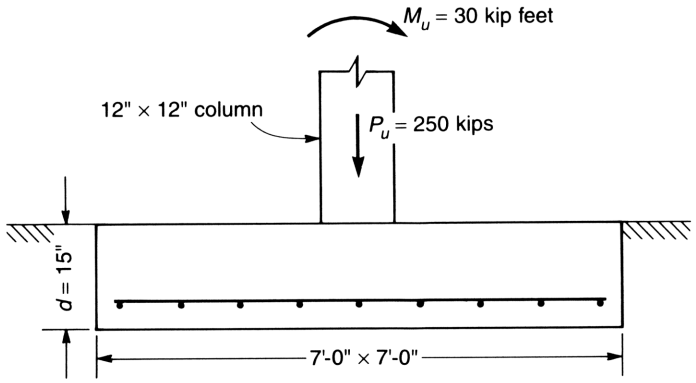

Figure 8–32 Details for Problem 8–5

Problem 8–5 (Isolated column footing with eccentric load)
The seven foot square footing shown in Figure 8–32 supports a concrete column which applies the indicated factored loads to the footing. Using a concrete strength of 3000 pounds per square inch and Grade 60 reinforcement, determine if the effective depth of 15 inches is adequate and calculate the reinforcement required.

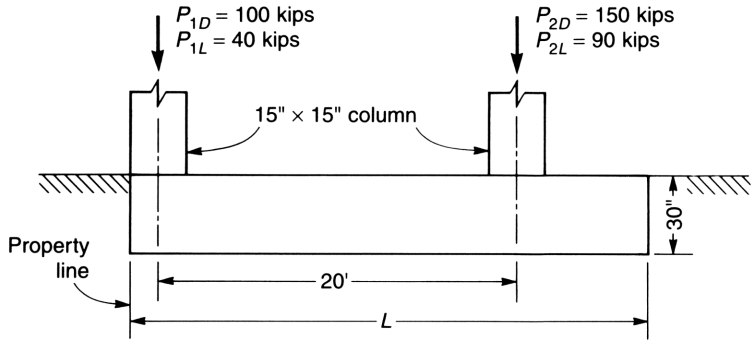

Figure 8–33 Details for Problem 8–6

Problem 8–6 (Combined footing)
The plan dimensions of the two-column footing shown in Figure 8–33 are to be determined to provide a uniform soil bearing pressure of 4000 pounds per square foot under the indicated service loads. Using a concrete strength of 4000 pounds per square inch and Grade 60 reinforcement, determine the required longitudinal reinforcement and check the adequacy of the footing for shear.

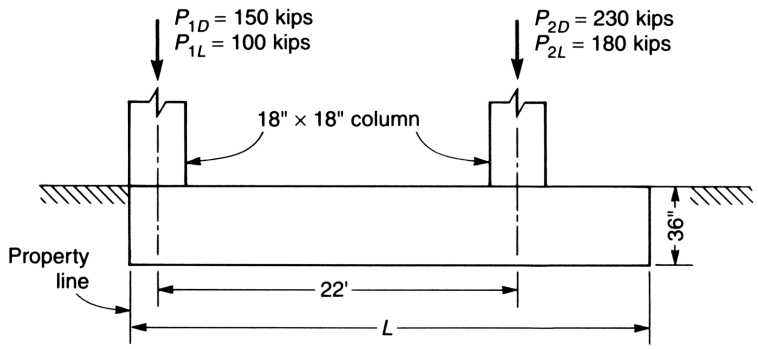

Figure 8–34 Details for Problem 8–7

Problem 8–7 (Combined footing)

The plan dimensions of the two-column footing shown in Figure 8–34 are to be determined to provide a uniform soil bearing pressure of 4000 pounds per square foot under the indicated service loads. Using a concrete strength of 4000 pounds per square inch and Grade 60 reinforcement, determine the required longitudinal reinforcement and check the adequacy of the footing for shear.

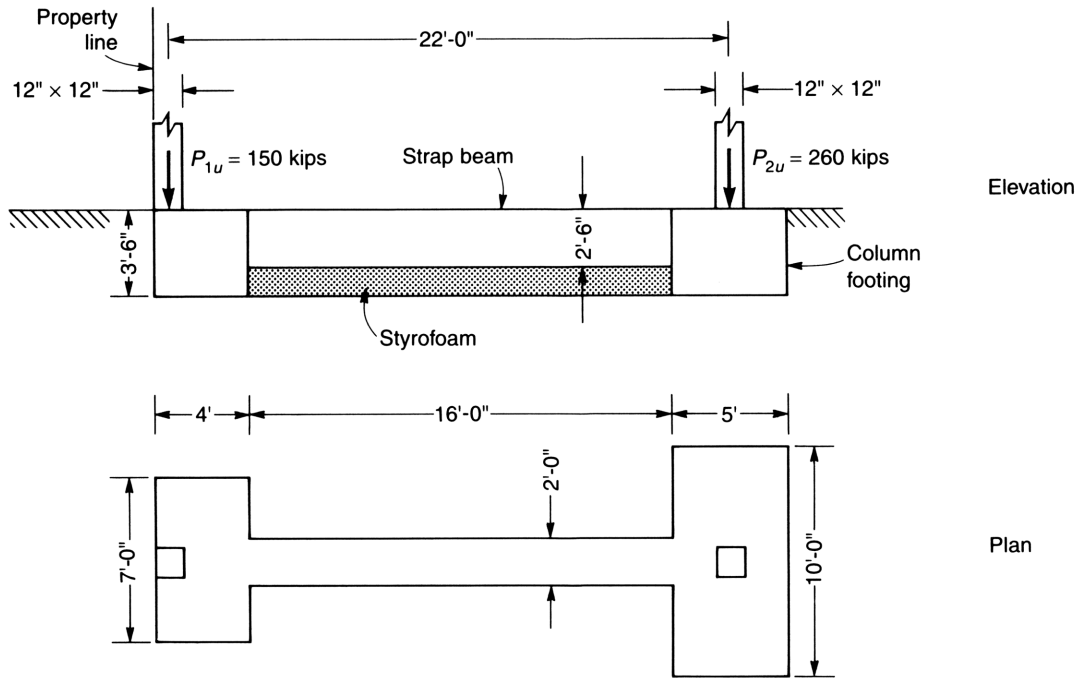

Figure 8–35 Details for Problem 8–8

Problem 8–8 (Strap footing)
Design the strap beam of the strap footing for the factored loads shown in Figure 8–35. The concrete strength is 4000 pounds per square inch, and all reinforcement is Grade 60.

References

1. American Institute of Steel Construction. *Manual of steel construction, ninth edition. Allowable Stress Design.* Chicago, IL, 1989.

2. Board of Registration for Professional Engineers and Land Surveyors. *Structural Engineer Examination–1991.* Sacramento, CA, 1991.

3. Board of Registration for Professional Engineers and Land Surveyors. *Structural Engineer Examination–1988.* Sacramento, CA, 1988.

9

Retaining walls

9.1 GENERAL

9.1.1 Types of retaining wall

Retaining walls are used to contain earth embankments when allowing the embankment to slope at its natural angle of repose will occupy too much area. The different types of walls are illustrated in Figure 9–1 and consist of:

- A mass gravity wall which may be constructed from masonry or mass concrete and relies for stability on its own size and weight. The resultant of the self weight and applied loads must lie within the middle third of the section if tensile stresses are to be avoided.
- A cantilever retaining wall consists of a vertical wall cantilevering from a horizontal footing. The footing may project either forwards from the wall or backwards under the fill. In the latter case, the weight of earth supported on the footing assists in providing stability.
- A counterfort retaining wall may be constructed to a greater height than is suitable for a cantilever retaining wall. The wall is designed as a continuous slab spanning between the vertical counterforts.
- A cantilever sheet pile wall consists of interlocking units which are driven into the earth a sufficient depth to provide stability and anchorage for the vertical cantilever.
- An anchored sheet pile wall may be constructed to a greater height than is suitable for a cantilever retaining wall. The provision of a tie reduces the bending moment in the sheet pile wall.

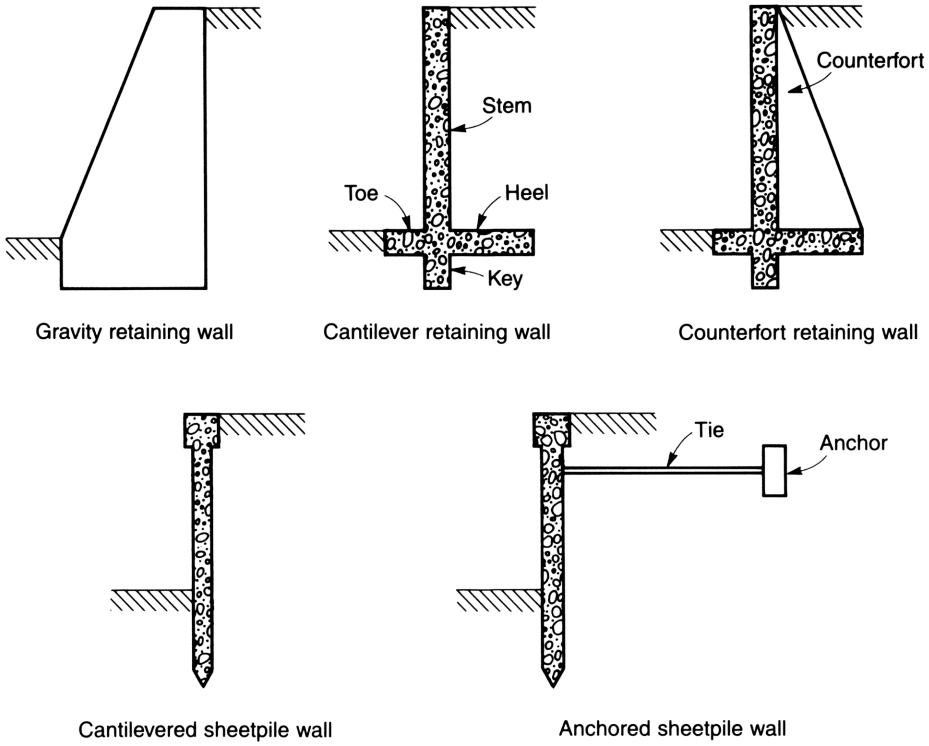

Figure 9–1 Types of retaining walls

9.1.2 General design considerations

Failure of a retaining wall may occur by sliding forward of the wall on the soil, by rotation of the wall about the toe, and by a slip circle shear failure in the earth behind the wall. Factors of safety are adopted[1,2] in order to prevent these types of failure. Retaining walls are usually designed to resist sliding by at least 1.5 times the lateral force. Similarly, the resistance to rotation must be at least 1.5 times the overturning moment.

Failure of a sheetpile wall may occur because of insufficient penetration of the piling to develop the necessary resistance to sliding. In the case of an anchored sheetpile wall, failure may occur because of insufficient distance between the anchorage and the piling to develop the necessary passive earth resistance in front of the anchor.

Active earth pressure behind the wall and passive pressure in front of the wall may conveniently be obtained using Rankine's theory. Alternatively[1,2], walls retaining drained earth may be designed for a pressure equivalent to that exerted by a fluid weighing not less than 30 pounds per cubic foot and having a depth equal to that of the retained earth. A live load surcharge behind the wall may be represented by an additional height of fill which produces the same uniform pressure behind the wall and a minimum height of two feet is recommended[2]. Adequate drainage is necessary behind a retaining wall to prevent a build up of hydrostatic pressure.

9–2 CANTILEVER AND GRAVITY WALLS

9.2.1 Design of Cantilever retaining walls

The forces acting on a reinforced concrete cantilever retaining wall are shown in Figure 9–2. The weight of the stem wall, base and shear key in pounds per foot length are

$$W_W = 150L_W(H - h)$$
$$W_B = 150hL_B$$
$$W_K = 150L_W(H_K - h)$$

The live load surcharge behind the stem and the weight of the fill supported by the heel are

$$W_L = wL_H$$
$$W_S = \gamma_S L_H(H - h)$$

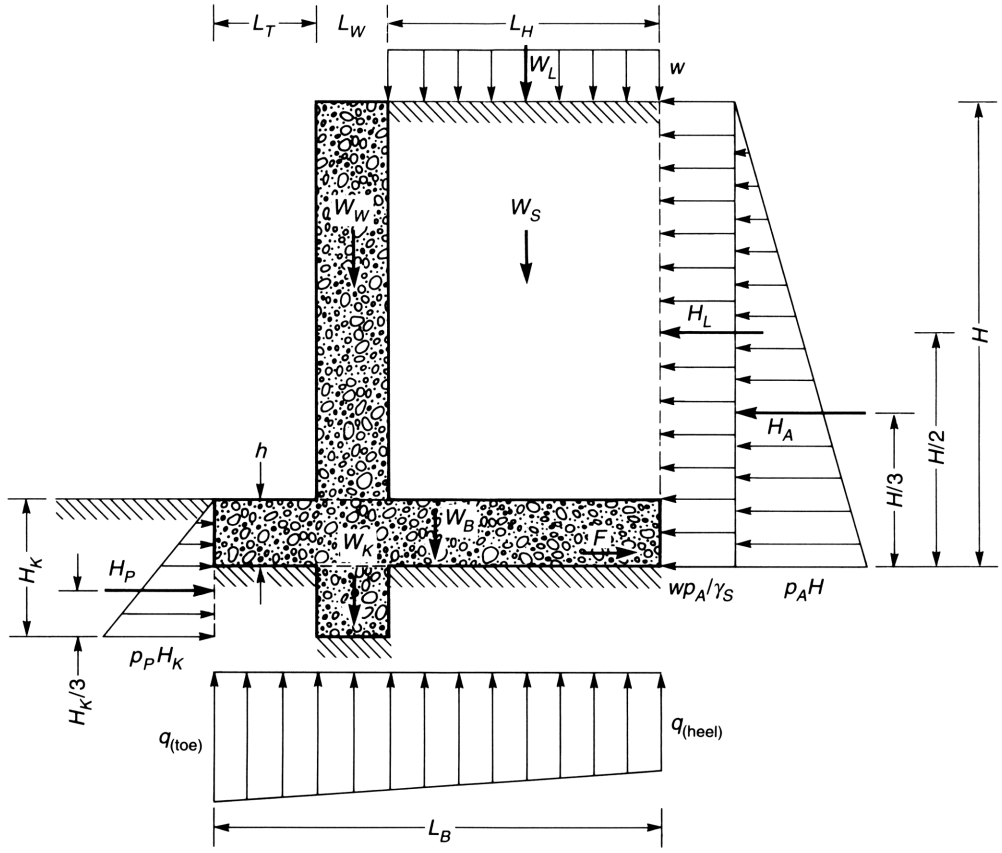

Figure 9–2 Cantilever retaining wall

The total active earth pressure behind the wall, acting at a height of H/3 above the base, is

$$H_A = K_A \gamma_S H^2/2$$
$$= p_A H^2/2$$

where p_A = lateral pressure equivalent to a fluid of density p_A pounds per cubic foot
$= K_A \gamma_S$
γ_S = density of fill
K_A = Rankine coefficient of active earth pressure
$= (1 - \sin\phi)/(1 + \sin\phi)$
ϕ = angle of internal friction

Hence, the equivalent fluid pressure for a fill material with a density of 110 pounds per cubic foot and an angle of internal friction of 35 degrees is

$$p_A = 110\ (1 - \sin 35°)/(1 + \sin 35°)$$
$$= 30 \text{ pounds per square foot per foot}$$

The live load surcharge of w pounds per square foot produces a lateral pressure of

$$p_L = wK_A$$
$$= \bar{H}p_A$$

where $\bar{H}$ = equivalent additional height of fill
$= wK_A/p_A$
$= w/\gamma_S$

This produces a uniform pressure over the height of the fill of $p_L = \bar{H}p_A$ and the total surcharge pressure, acting at a height of H/2 above the base, is

$$H_L = p_L H$$

The frictional force produced at the underside of the base is

$$F = \mu \Sigma W$$

where μ = coefficient of friction.

When the frictional force is insufficient to provide an adequate factor of safety against sliding, a shear key is provided to mobilize the passive pressure of the undisturbed soil in front of the wall. The passive earth pressure in front of the wall, acting at a height of $H_K/3$ above the bottom of the key, is

$$H_P = K_P \gamma_S H_K^2/2$$
$$= p_P H_K^2/2$$

where p_P = lateral pressure equivalent to a fluid of density p_P pounds per cubic foot
$= K_P \gamma_S$
K_P = Rankine coefficient of passive earth pressure
$= (1 + \sin\phi)/(1 - \sin\phi)$

Hence, the equivalent fluid pressure for a material with a density of 110 pounds per cubic foot and an angle of internal friction of 35 degrees is

$$p_P = 110\ (1 + \sin 35°)/(1 - \sin 35°)$$
$$= 400 \text{ pounds per square foot per foot}$$

To provide an adequate factor of safety against sliding a depth of shear key must be provided to ensure that

$$F + H_P \geq 1.5(H_A + H_L)$$

Similarly, the required factor of safety against overturning about the toe is 1.5.

By determining the eccentricity of the resultant of all forces about the center of the base, the bearing pressure under the base may be obtained.

To design the stem wall and the base, the required strength is determined from ACI Equation (9–4) as

$$U = 1.4D + 1.7L + 1.7H$$

where D = force due to dead load
L = force due to live load
H = force due to earth pressure

9.2.2 Reinforcement detailing

The minimum reinforcement required in the stem wall is specified by ACI Section 14.3. For Grade 60 bars larger than No.5, the reinforcement ratios, based on the gross concrete area, for vertical and horizontal reinforcement are given by

$$\rho_{vert} = 0.15 \text{ percent}$$
$$\rho_{hor} = 0.25 \text{ percent}$$

For bars not larger than No.5, the corresponding ratios are

$$\rho_{vert} = 0.12 \text{ percent}$$
$$\rho_{hor} = 0.20 \text{ percent}$$

For walls exceeding ten inches thickness, two layers of reinforcement are required as detailed in Figure 9–3.

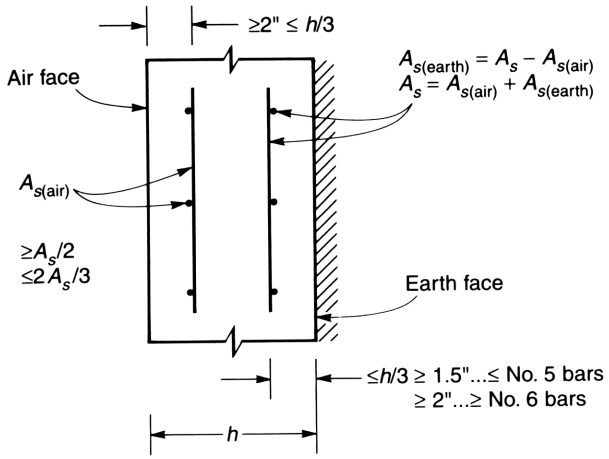

Figure 9–3 Wall reinforcement

Example 9–1 (Cantilever retaining wall)

The fill behind the retaining wall in Figure 9–4 has a unit weight of 110 pounds per cubic foot with an equivalent fluid pressure of 30 pounds per square foot per foot. Passive earth pressure may be assumed equivalent to a fluid pressure of 300 pounds per square foot per foot and the coefficient of friction at the underside of the base is 0.4. All concrete has a compressive strength of 3000 pounds per square inch and reinforcement consists of Grade 60 deformed bars.

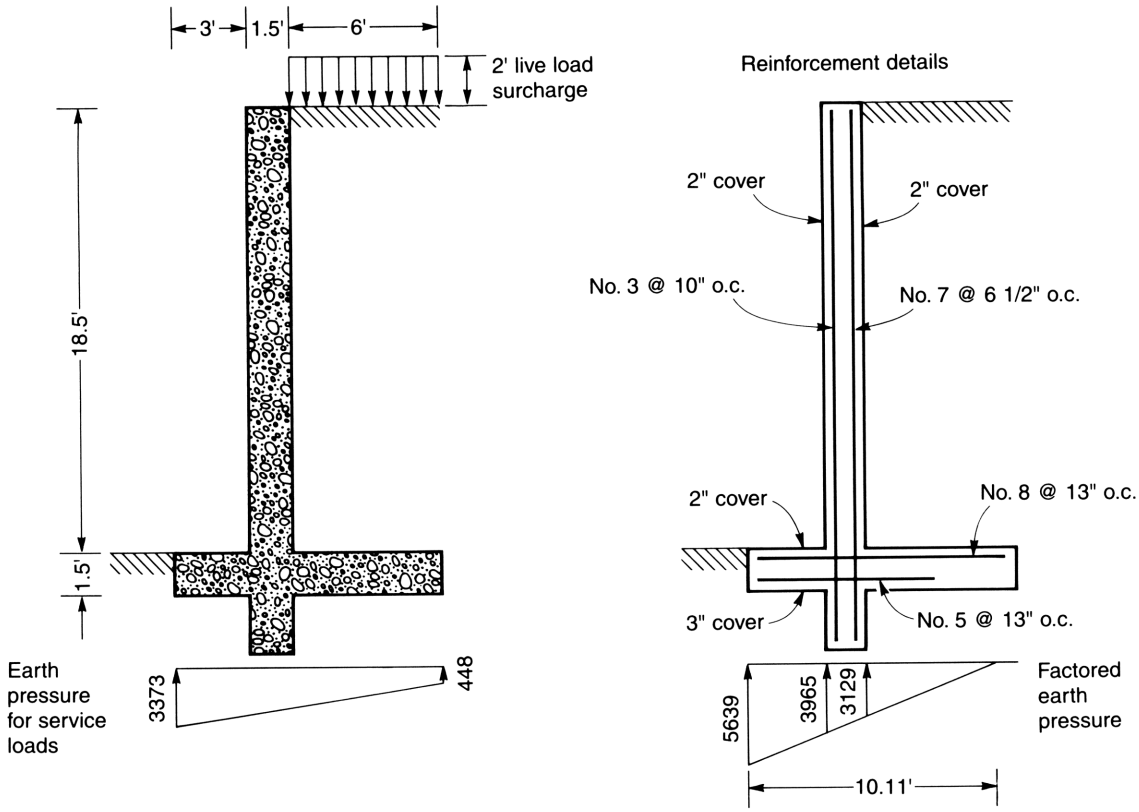

Figure 9–4 Details for Example 9–1

Neglecting the effects of the shear key, determine the factor of safety against overturning and the bearing pressure distribution under the base. Using factored loading, determine if the reinforcement indicated in the footing and stem is adequate. Combining the effects of friction and passive pressure, calculate the minimum depth of the shear key to provide a factor of safety of 1.5 against sliding.

Solution

The horizontal service loads acting on the wall are given by

$$\begin{aligned} H_A &= \text{lateral pressure from backfill} \\ &= p_A H^2/2 \\ &= 30 \times 20^2/2 \\ &= 6000 \text{ pounds} \end{aligned}$$

H_L = lateral pressure from surcharge
= Hp_AH
= 2 × 30 × 20
= 1200 pounds

The overturning moment about the toe is

M_O = $H_A \times H/3 + H_L \times H/2$
= 6000 × 20/3 + 1200 × 20/2
= 52,000 pounds feet

The vertical service loads acting on the wall are given by

W_W = weight of stem wall
= 150 × 1.5 × 18.5
= 4163 pounds
W_B = weight of base
= 150 × 1.5 × 10.5
= 2363 pounds
W_S = weight of backfill
= 110 × 6 × 18.5
= 12,210 pounds
W_L = weight of surcharge
= 110 × 6 × 2
= 1320 pounds.

The total vertical load is

ΣW = 20,056

The distance of the resultant vertical load from the toe is

$\bar{x}$ = (4163 × 3.75 + 2363 × 5.25 + 12,210 × 7.5 + 1320 × 7.5)/20,056
= 6.5 feet

The factor of safety against overturning is

$\bar{x}\Sigma W/M_O$ = 6.5 × 20,056/52,000
= 2.5

The eccentricity of the applied loads about the toe is given by

e' = $(\bar{x}\Sigma W - M_O)/\Sigma W$
= (6.5 × 20,056 - 52,000)/20,056
= 3.91 feet

The eccentricity about the mid point of the base is

e = 10.5/2 - 3.91
= 1.34 feet
< $L_B/6$ ... no tension under the base

The pressure under the toe is

$$\begin{aligned} q_{(toe)} &= \Sigma W(1 + 6e/L_B)/BL_B \\ &= 20{,}056\ (1 + 6 \times 1.34/10.5)/(1 \times 10.5) \\ &= 3373 \text{ pounds per square foot} \end{aligned}$$

The pressure under the heel is

$$\begin{aligned} q_{(heel)} &= \Sigma W\ (1 - 6e/L_B)/BL_B \\ &= 448 \text{ pounds per square foot} \end{aligned}$$

In accordance with ACI Section 9.2.4 the factored total overturning moment about the toe is

$$\begin{aligned} \gamma M_O &= 1.7 \times 52{,}000 \\ &= 88{,}400 \text{ pounds feet} \end{aligned}$$

The factored total vertical load is

$$\begin{aligned} \Sigma(\gamma W) &= 1.4(W_W + W_B + W_S) + 1.7W_L \\ &= 5828 + 3308 + 17{,}094 + 2244 \\ &= 28{,}474 \text{ pounds} \end{aligned}$$

The factored restoring moment is

$$\begin{aligned} \gamma M_R &= 5828 \times 3.75 + 3308 \times 5.25 + 17{,}094 \times 7.5 + 2244 \times 7.5 \\ &= 184{,}257 \text{ pounds feet} \end{aligned}$$

The eccentricity of the factored loads about the toe is given by

$$\begin{aligned} e' &= (\gamma M_R - \gamma M_O)/\Sigma(\gamma W) \\ &= 3.37 \text{ feet} \\ &< L_B/3 \ldots \text{ outside middle third.} \end{aligned}$$

The pressure under the toe is given by

$$\begin{aligned} q_{(toe)} &= 2\Sigma(\gamma W)/3e' \\ &= 2 \times 28{,}474/(3 \times 3.37) \\ &= 5639 \text{ pounds per square foot.} \end{aligned}$$

The pressure distribution under the base is shown in Figure 9–4 and the maximum factored bending moment in the heel is

$$\begin{aligned} M_u &= 3(\gamma W_S + \gamma W_L + 6\gamma W_B/10.5) - 3129 \times 5.61^2/6 \\ &= 3\ (17{,}094 + 2244 + 1890) - 16{,}413 \\ &= 47{,}271 \text{ pounds feet.} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.2 and is

$$\begin{aligned} \rho &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.38 \text{ percent} \end{aligned}$$

The maximum reinforcement ratio is given by ACI Section 10.3 as

$$\begin{aligned} \rho_{max} &= 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y) \\ &= 1.60 \text{ percent} \\ &> \rho \ldots \text{ satisfactory} \end{aligned}$$

The minimum reinforcement ratio is given by ACI Section 7.12 as

$$\begin{aligned} \rho_{min} &= 0.18 \text{ percent of the gross area} \\ &< \rho \ldots \text{ satisfactory} \end{aligned}$$

Thus, the required reinforcement area is

$$\begin{aligned} A_s &= \rho bd \\ &= 0.0038 \times 12 \times 15.5 \\ &= 0.71 \text{ square inches per foot} \end{aligned}$$

Shear is not critical and No.9 bars at a spacing of 13 inches provides a reinforcement area of

$$\begin{aligned} A_s' &= 0.72 \text{ square inches per foot} \\ &> A_s \ldots \text{ satisfactory} \end{aligned}$$

The maximum factored bending moment in the toe is

$$\begin{aligned} M_u &= 3^2(3965 + 2 \times 5639)/6 - 1.5 \times 3\gamma W_B/10.5 \\ &= 22{,}865 - 1418 \\ &= 21{,}447 \text{ pounds feet} \end{aligned}$$

The required reinforcement ratio is

$$\rho = 0.188 \text{ percent} \ldots \text{ satisfactory}$$

The required reinforcement area is

$$\begin{aligned} A_s &= 0.00188 \times 12 \times 14.69 \\ &= 0.33 \text{ square inches per foot} \end{aligned}$$

Shear is not critical and No.5 bars at a spacing of 11 inches provides a reinforcement area of

$$\begin{aligned} A_s' &= 0.34 \text{ square inches per foot} \\ &> A_s \ldots \text{ satisfactory} \end{aligned}$$

The maximum factored bending moment in the stem is

$$\begin{aligned} M_u &= 1.7(30 \times 18.5^3/6 + 60 \times 18.5^2/2) \\ &= 71{,}274 \text{ pounds feet} \end{aligned}$$

The required reinforcement ratio is

$$\rho = 0.58 \text{ percent} \ldots \text{ satisfactory}$$

The required reinforcement area is

$$\begin{aligned} A_s &= 0.0058 \times 12 \times 15.5 \\ &= 1.08 \text{ square inches per foot} \end{aligned}$$

Shear is not critical and No.7 bars at a spacing of 6.5 inches provides a reinforcement area of

$$\begin{aligned} A_s' &= 1.12 \text{ square inches per foot} \\ &> A_s \ldots \text{ satisfactory} \end{aligned}$$

Based on the gross concrete area, the reinforcement ratio required for the vertical reinforcement in the outer face of the wall is given by ACI Section 14.3 as

$$\rho = 0.0012/2 \ldots \text{bars not larger than No.5}$$

The required vertical reinforcement area is

$$\begin{aligned} A_s &= 0.0012 \times 12 \times 18/2 \\ &= 0.13 \text{ square inches per foot} \end{aligned}$$

Providing No.3 bars at a spacing of 10 inches gives

$$\begin{aligned} A_s' &= 0.13 \text{ square inches per foot} \\ &= A_s \ldots \text{satisfactory} \end{aligned}$$

The applied horizontal force on the retaining wall is

$$\begin{aligned} H &= H_A + H_L \\ &= 6000 + 1200 \\ &= 7200 \text{ pounds} \end{aligned}$$

The frictional resistance available is

$$\begin{aligned} F &= \mu \Sigma W \\ &= 0.4 \times 20{,}056 \\ &= 8022 \text{ pounds} \end{aligned}$$

To provide a factor of safety of 1.5 against sliding requires a passive resistance of

$$\begin{aligned} H_P &= 1.5H - F \\ &= 1.5 \times 7200 - 8022 \\ &= 2777 \text{ pounds} \end{aligned}$$

To provide this passive force, the total required depth of the shear key plus footing is

$$\begin{aligned} H_K &= (2H_P/p_P)^{0.5} \\ &= (2 \times 2777/300)^{0.5} \\ &= 4.3 \text{ feet} \end{aligned}$$

The depth of the shear key is

$$\begin{aligned} D &= H_K - 1.5 \\ &= 2.8 \text{ feet} \end{aligned}$$

Example 9–2 (Cantilever retaining wall with masonry stem)

Determine the bearing pressure and factors of safety for overturning and sliding of the retaining wall shown in Figure 9–5 if the backfill weighs 120 pounds per cubic foot, equivalent fluid pressure in the backfill behind the wall is 30 pounds per square foot per foot, and the coefficient of friction at the underside of the base is 0.40. The nominal eight inch solid grouted concrete masonry stem wall, has a specified strength of 1500 pounds per square inch and is constructed without special inspection. The base concrete has a compressive strength of 3000 pounds per square inch. Using Grade 60 deformed bars, determine if the reinforcement indicated in the stem and base is adequate and compute the bearing pressure distribution under the base.

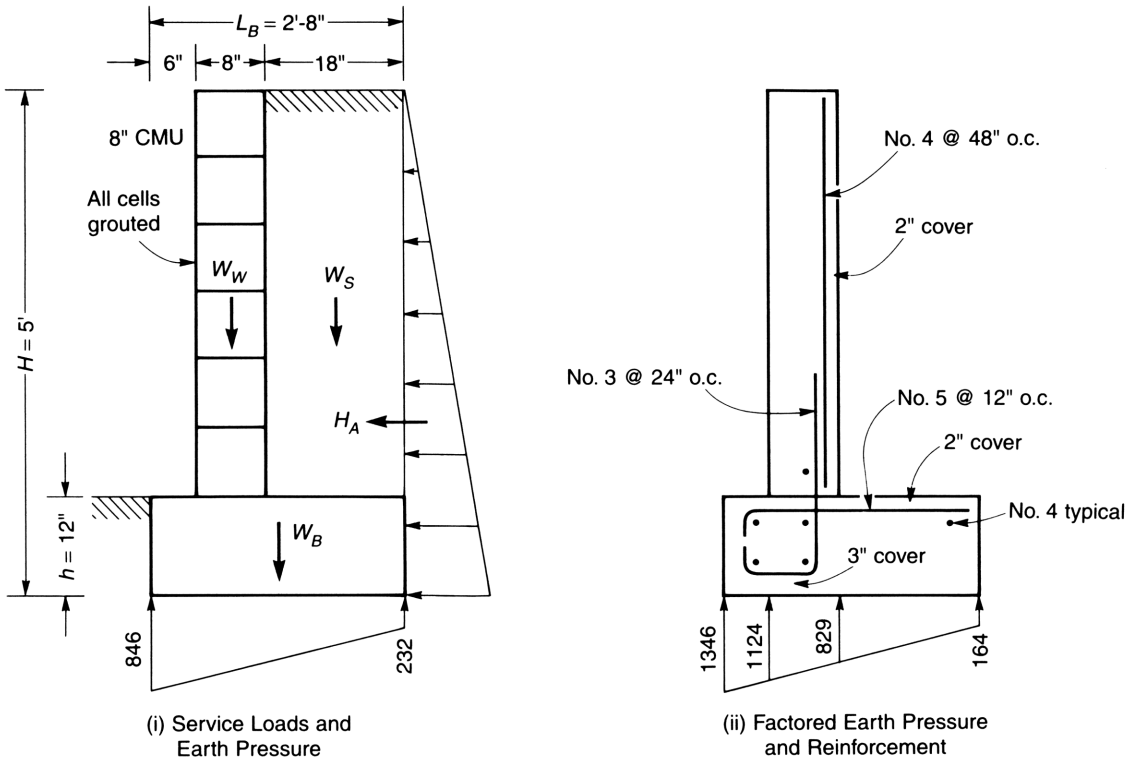

Figure 9–5 Details for Example 9–2

Solution

The service loads acting on the retaining wall are given by

H_A = lateral pressure from backfill
$= p_A H^2/2$
$= 30 \times 5^2/2$
$= 375$ pounds

W_W = weight of stem wall
$= 80(H-h)$
$= 80 \times 4$
$= 320$ pounds

W_B = weight of base
$= 150 h L_B$
$= 150 \times 1.0 \times 2.67$
$= 400$ pounds

W_S = weight of backfill
$= \gamma_S L_H (H-h)$
$= 120 \times 1.5 \times 4$
$= 720$ pounds

The restoring and overturning moments about the bottom corner of the toe are shown below.

Part	W	x	Wx	H	y	Hy
Backfill	720	1.92	1380	375	1.67	625
Wall	320	0.83	267			
Base	400	1.33	533			
Total	1440		2180	375		625

Table 9–1 Details for Example 9–2

The factor of safety against overturning is given by

$$\Sigma(Wx)/\Sigma(Hy) = 2180/625 = 3.5$$

The frictional force produced at the underside of the base is given by

$$F = \mu\Sigma W = 0.40 \times 1440 = 576 \text{ pounds}$$

The factor of safety against sliding is given by

$$F/\Sigma H = 576/375 = 1.5$$

The eccentricity of the applied loads about the toe is given by

$$e' = [\Sigma(Wx) - \Sigma(Hy)]/\Sigma W = (2180 - 625)/1440 = 1.08 \text{ feet}$$

The eccentricity about the base centroid is

$$e = L_B/2 - e' = 2.67/2 - 1.08 = 0.25 \text{ feet} < L_B/6 \ldots \text{ no tension under the base}$$

The pressure under the toe is

$$q_{(toe)} = \Sigma W(1 + 6e/L_B)/BL_B = 1440\ (1 + 6 \times 0.25/2.67)/(1 \times 2.67) = 846 \text{ pounds per square foot}$$

The pressure under the heel is

$$q_{(heel)} = \Sigma W(1 - 6e/L_B)/BL_B = 232 \text{ pounds per square foot}$$

The critical section for moment and shear in the stem is at the base of the stem and, due to service loads, these values are

$$\begin{aligned} V &= p_A(H - h)^2/2 \\ &= 30(5 - 1)^2/2 \\ &= 240 \text{ pounds per foot} \\ M &= V(H - h)/3 \\ &= 240(5 - 1)/3 \\ &= 320 \text{ pounds feet per foot} \end{aligned}$$

Using the working stress design method, in accordance with UBC Section 2107.2, the allowable stresses, without special inspection provided, are

$$\begin{aligned} F_b &= 0.5 \times 0.33f'_m &&= 250 \text{ pounds per square inch} \\ F_v &= 0.5(f'_m)^{0.5} &&= 19.36 \text{ pounds per square inch} \\ F_s &= 24{,}000 \text{ pounds per square inch} \\ E_m &= 750f'_m &&= 1125 \text{ kips per square inch} \\ E_s &= 29{,}000 \text{ kips per square inch} \end{aligned}$$

The relevant section properties of the stem are

$$\begin{aligned} b &= 12 \text{ inches} \\ d &= 7.63 - 2.0 - 0.5/2 &&= 5.38 \text{ inches} \end{aligned}$$

Providing No.4 bars at 48 inch spacing, the masonry stresses may be obtained from UBC Section 2107.2, and are

$$\begin{aligned} n &= E_s/E_c &&= 25.8 \\ p &= A_s/bd &&= 0.049/(12 \times 5.38) = 0.076 \text{ percent} \\ k &= (n^2p^2 + 2np)^{0.5} - np &&= 0.179 \\ j &= 1 - k/3 &&= 0.940 \\ f_b &= 24M/jkbd^2 &&= 131 \text{ pounds per square inch} \\ &< F_b \ldots \text{ satisfactory} \\ f_s &= 12M/jdA_s &&= 15{,}492 \text{ pounds per square inch} \\ &< F_s \ldots \text{ satisfactory} \\ f_v &= V/bjd &&= 4 \text{ pounds per square inch} \\ &< F_v \ldots \text{ satisfactory} \end{aligned}$$

Hence, the reinforcement indicated in Figure 9–5 is adequate

To determine the reinforcement in the base, factored loads must be used. In accordance with ACI Section 9.2.4, the factored total vertical load, restoring moment and overturning moment about the toe are

$$\begin{aligned} \Sigma(\gamma W) &= 1.4 \times 1440 &&= 2016 \text{ pounds} \\ \Sigma(\gamma Wx) &= 1.4 \times 2180 &&= 3052 \text{ pounds feet} \\ \Sigma(\gamma Hy) &= 1.7 \times 625 &&= 1063 \end{aligned}$$

The eccentricity of the factored loads about the toe is given by

$$\begin{aligned} e' &= [\Sigma(\gamma Wx) - \Sigma(\gamma Hy)]/\Sigma(\gamma W) \\ &= (3052 - 1063)/2016 \\ &= 0.99 \text{ feet} \end{aligned}$$

The eccentricity about the mid point of the base is

$$\begin{aligned} e &= L_B/2 - e' \\ &= 2.67/2 - 0.99 \\ &= 0.35 \text{ feet} \\ &< L_B/6 \ldots \text{ no tension under the base} \end{aligned}$$

The pressure under the toe is

$$\begin{aligned} q_{(toe)} &= \Sigma(\gamma W)(1 + 6e/L_B)/BL_B \\ &= 2016\ (1 + 6 \times 0.35/2.67)/(1 \times 2.67) \\ &= 1346 \text{ pounds per square foot} \end{aligned}$$

The pressure under the heel is

$$\begin{aligned} q_{(heel)} &= \Sigma(\gamma W)(1 - 6e/L_B)/BL_B \\ &= 164 \text{ pounds per square foot} \end{aligned}$$

The critical section for shear and moment in the heel, in accordance with ACI Section R11.1.3, is at the rear face of the wall and, due to factored loads, these values are

$$\begin{aligned} V_u &= \gamma W_S + 18\gamma W_B/32 - 1.5(829 + 164)/2 \\ &= 1.4 \times 720 + 18 \times 1.4 \times 400/32 - 745 \\ &= 578 \text{ pounds per foot} \\ M_u &= 0.75\ (\gamma W_S + 18\gamma W_B/32) - 1.5^2\ (829 + 2 \times 164)/6 \\ &= 558 \text{ pounds feet per foot} \end{aligned}$$

The shear capacity of the heel is given by ACI Equation (11–3) as

$$\begin{aligned} \phi V_c &= 2\phi b_w d(f'_c)^{0.5} \\ &= 2 \times 0.85 \times 12 \times 9.69 \times (3000)^{0.5} \\ &= 10{,}827 \text{ pounds} \\ &> V_u \ldots \text{ satisfactory} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.2, and is

$$\begin{aligned} \rho &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.011 \text{ percent} \end{aligned}$$

The minimum reinforcement ratio in a footing slab for Grade 60 bars is given by ACI Section 7.12 as

$$\begin{aligned} \rho_{min} &= 0.18 \text{ percent of the gross area} \ldots \text{ governs} \\ &> \rho \end{aligned}$$

Hence the required reinforcement area is

$$\begin{aligned} A_S &= bh\rho_{min} \\ &= 12 \times 12 \times 0.18/100 \\ &= 0.26 \text{ square inches per foot} \end{aligned}$$

Provide No.5 bars at twelve inch spacing to give

$$A_s' = 0.31 \text{ square inches per foot}$$

The critical section for shear and moment in the toe is at the front face of the stem and, due to factored loads, these values are

$$\begin{aligned} V_u &= 0.5(1346 + 1124)/2 - 6\gamma W_B/32 \\ &= 513 \text{ pounds per foot} \\ &< \phi V_c \ldots \text{ satisfactory} \\ M_u &= 0.5^2 (1124 + 2 \times 1346)/6 - 0.25 \times 6\gamma W_B/32 \\ &= 133 \text{ pounds feet per foot} \end{aligned}$$

The required reinforcement ratio is

$$\begin{aligned} \rho &= 0.85f_c'[1 - (1 - 2M_u/0.765bd^2f_c')^{0.5}]/f_y \\ &= 0.0034 \text{ percent} \\ \rho_{min} &= 0.18 \text{ percent of the gross area } \ldots \text{ governs} \end{aligned}$$

The required minimum reinforcement is supplied by the reinforcement in the top of the base. To provide starter bars for the stem reinforcement, use No.3 bars at 24 inches spacing to give

$$\begin{aligned} A_s' &= 0.055 \text{ square inches per foot} \\ &> 0.049 \ldots \text{ satisfactory} \end{aligned}$$

9.2.3 Gravity retaining wall

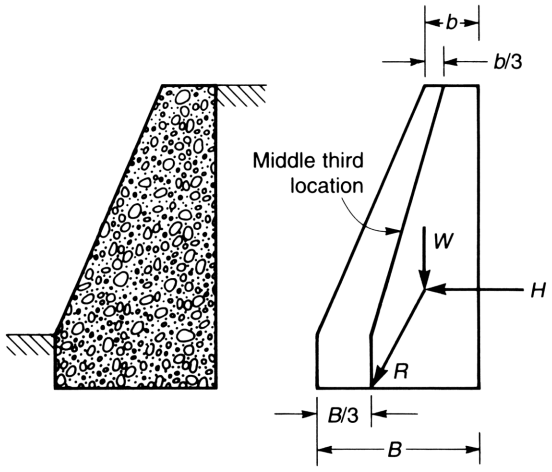

Figure 9–6 Mass gravity retaining wall

In a gravity retaining wall, as shown in Figure 9–6, stability and acceptable earth pressure are provided by the size and mass of the structure. The resultant of the self weight and applied loads must be adjusted, over the full height of the structure to ensure that the allowable tensile stress of the material is not exceeded at any section. When specifications require zero tension at any section, the locus of the resultant thrust must lie within the middle third at any section, as shown in Figure 9–6.

Example 9–3 (Mass concrete gravity dam)

The mass concrete gravity dam shown in Figure 9–7 is 18 feet high. Determine the minimum base width if no tensile stresses are allowed in the concrete and the water level is at the top of the dam wall.

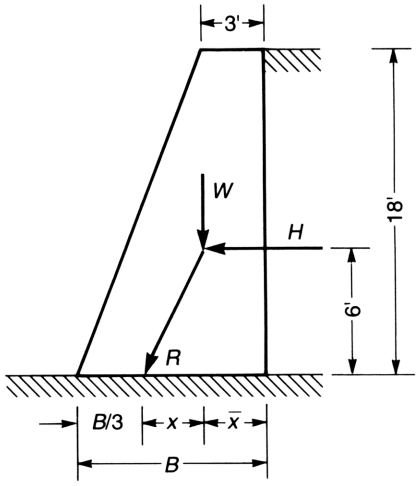

Figure 9–7 Details for Example 9–3

Solution

The total weight of the wall is

$$W = 144 \times 18\,(3 + B)/2$$
$$= 1296\,(3 + B) \text{ pounds per foot}$$

The total water pressure acting on the wall at a height of 6 feet above the base is

$$H = 62.4 \times 18^2/2$$
$$= 10{,}108 \text{ pounds per foot}$$

The centroid of the wall mass is located a distance from the heel given by

$$\bar{x} = (9 + 3B + B^2)/3(3 + B) \text{ feet}$$

The line of action of the resultant thrust cuts the base at a distance from the centroid given by

$$x = 6H/W$$
$$= 6 \times 10{,}108/1296(3 + B)$$
$$= 46.80/(3 + B)$$

To ensure no tensile stress in the base

$$x + \bar{x} \leq 2B/3$$

Hence the minimum base width is given by

$$0 = 46.80/(3 + B) + (9 + 3B + B^2)/3(3 + B) - 2B/3$$

and $$B = 10.81 \text{ feet.}$$

The minimum required base width is 10.81 feet.

9.2.4 Counterfort retaining wall

Counterforts are added to a high cantilever retaining wall in order to reduce the bending moment in the stem. At the top of the wall, the stem spans horizontally between counterforts and at the bottom of the wall, the stem cantilevers from the base. In effect, the stem between counterforts may be considered as a panel fixed at its base and at its two vertical edges and unsupported at the top edge, as shown in Figure 9–8.

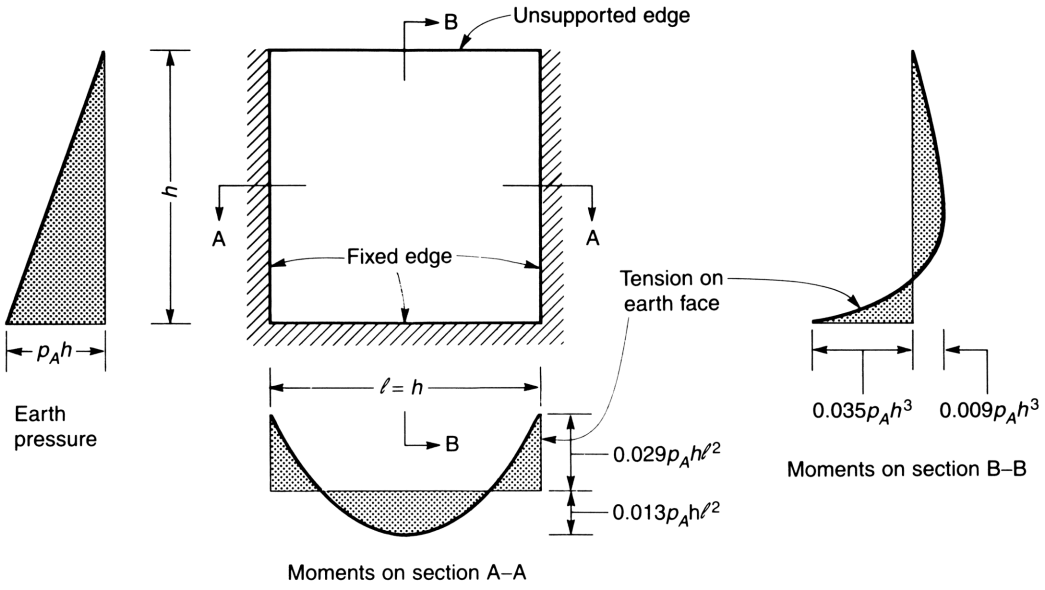

Figure 9–8 Bending moments on a square panel

The spacing of the counterforts is usually between 0.5 to 1.0 times the stem height, and bending moment coefficients are available for a wide range of aspect ratios[3,4]. The appropriate values for a square panel are given in Figure 9–8. For aspect ratios between 0.5 and 1.0, the stem may conservatively be designed as a slab spanning longitudinally across counterforts using a value for the earth pressure at a depth of 0.6 times the height of the wall. The approximate moment coefficients given in ACI Section 8.3.3 may be used to obtain the relevant moments. At the base of the stem the cantilever moment may be similarly determined using a value for the earth pressure at a depth of 0.6 times the height of the wall. The base slab is designed for the net

pressure, as a slab spanning longitudinally across counterforts. A counterfort acts as a T-beam with the stem and resists the bending moment produced by the earth pressure at the base of the stem. The couple, produced by the tension in the reinforcement at the rear of the counterfort and the compression in the stem concrete, resists this moment. The lever arm of the couple acts at right angles to the reinforcement. In addition, horizontal ties are required in the counterfort to resist the thrust produced by the earth pressure acting on the stem.

Example 9–4 (Counterfort retaining wall[5])

Details of a counterfort retaining wall are shown in Figure 9–9. The fill behind the wall has an equivalent fluid pressure of 50 pounds per square foot per foot. All concrete has a compressive strength of 3000 pounds per square inch and reinforcement consists of Grade 60 deformed bars. The distance from nearest face of concrete to center of reinforcing bars is 3.5 inches for bars in an outer layer and 4.25 inches for bars in an inner layer. Compute the reinforcement required for bars A, B, C, and D.

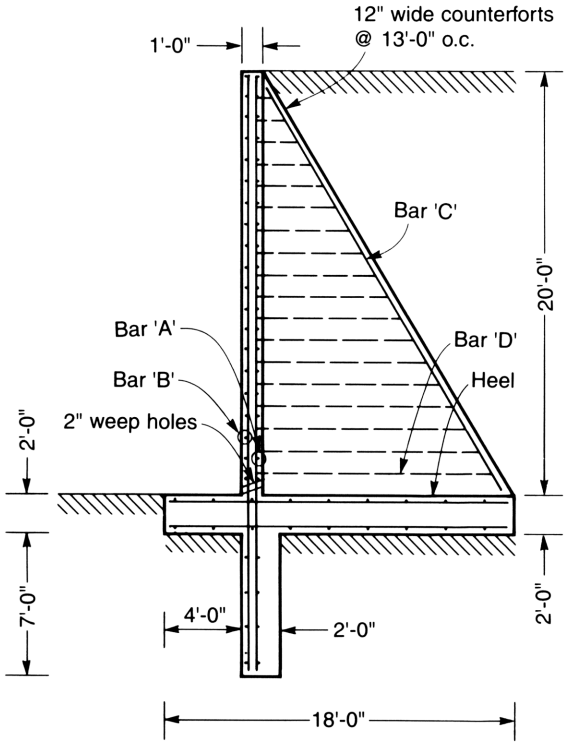

Figure 9–9 Details for example 9–4

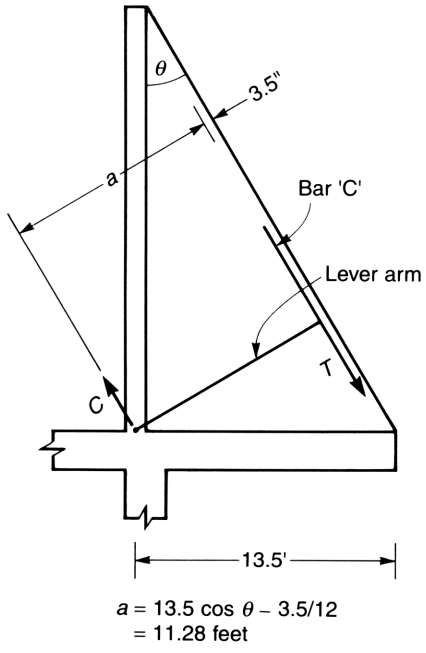

Figure 9–10 Lever arm of resisting couple

Solution

The lateral earth pressure at a depth of 0.6 times the stem height is

$$\begin{aligned} q &= 0.6p_A h \\ &= 0.6 \times 50 \times 20/1000 \\ &= 0.6 \text{ kips per square foot} \end{aligned}$$

and this is the pressure for which the stem reinforcement should be designed.

The maximum factored bending moment acting on a 12 inches horizontal strip of the stem is obtained from ACI Sections 8.3.3 and 9.2.4. At the counterfort supports, the moment is

$$\begin{aligned} M_u &= 1.7 \times q\ell_n^2/11 \\ &= 1.7 \times 0.6 \times 12^2/11 \\ &= 13.35 \text{ kip feet} \end{aligned}$$

The effective depth of the stem reinforcement A at the counterfort is

$$\begin{aligned} d &= 12 - 3.5 \\ &= 8.5 \text{ inches} \end{aligned}$$

The required reinforcement area is derived from ACI Section 10.2, and is

$$\begin{aligned} A_s &= 0.85bdf'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.36 \text{ square inches} \\ A_{s(min)} &= 0.0018 \times 12 \times 12 \\ &= 0.26 \\ &< 0.36 \end{aligned}$$

Provide No.5 bars at 10 inches spacing to give an area of

$$A_{(prov)} = 0.37 \text{ square inches ... satisfactory}$$

Between counterforts the span moment is

$$\begin{aligned} M_u &= 1.7 \times q\ell_n^2/16 \\ &= 1.7 \times 0.6 \times 12^2/16 \\ &= 9.18 \text{ kip feet} \end{aligned}$$

The effective depth of the stem reinforcement B at the counterfort is

$$\begin{aligned} d &= 12 - 3.5 \\ &= 8.5 \text{ inches} \end{aligned}$$

The required reinforcement area is derived from ACI Section 10.2, and is

$$\begin{aligned} A_s &= 0.85bdf'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.25 \text{ square inches} \\ A_{s(min)} &= 0.0018 \times 12 \times 12 \\ &= 0.26 \text{ ... governs} \end{aligned}$$

Provide No.5 bars at 14 inches spacing to give an area of

$$A_{(prov)} = 0.27 \text{ square inches} \ldots \text{satisfactory}$$

The factored bending moment produced by the earth pressure at the base of the stem over one bay is

$$M = 1.7p_A sh^3/6$$
$$= 1.7 \times 50 \times 13 \times 20^3/(6 \times 1000)$$
$$= 1473 \text{ kip feet}$$
$$= \phi f_y A_s a$$

where a = lever arm of the resisting couple for bars C shown in Figure 9–10

and $1473 = 0.9 \times 60 \times A_s \times 10.43$

$A_s = 2.62$ square inches

Provide two No.7 bars and two No.8 bars to give an area of

$$A_{(prov)} = 2.78 \text{ square inches} \ldots \text{satisfactory}$$

The earth pressure acting 6 inches above the base of the stem is

$$q = p_A(h - 0.5)$$
$$= 50 \times (20 - 0.5)/1000$$
$$= 0.975 \text{ kips per square foot}$$

The factored thrust acting on a 12 inches wide horizontal strip at the base of the stem over one bay, which is resisted by bars D, is given by

$$V = 1.7q(s - b)$$
$$= 1.7 \times 0.975(13 - 1)$$
$$= 19.89 \text{ kips}$$
$$= \phi f_y A_s$$

and $A_s = 19.89/(0.9 \times 60)$

$= 0.37$ square inches

Provide No.4 stirrups (two arms) at 12 inches centers to give an area of

$$A_{(prov)} = 2 \times 0.20$$
$$= 0.40$$
$$> 0.37 \ldots \text{satisfactory}$$

9.3 SHEETPILE WALLS

9.3.1 Cantilevered sheetpile wall

The forces acting on a cantilevered sheetpile wall are shown in Figure 9–11. An initial estimate[6] is required of the penetration of the sheetpiling and of the location of the point R about which the sheetpiling rotates. Point O occurs where the passive pressure in front of the sheetpiling equals the active pressure behind the sheetpiling. The forces acting on the wall are given by

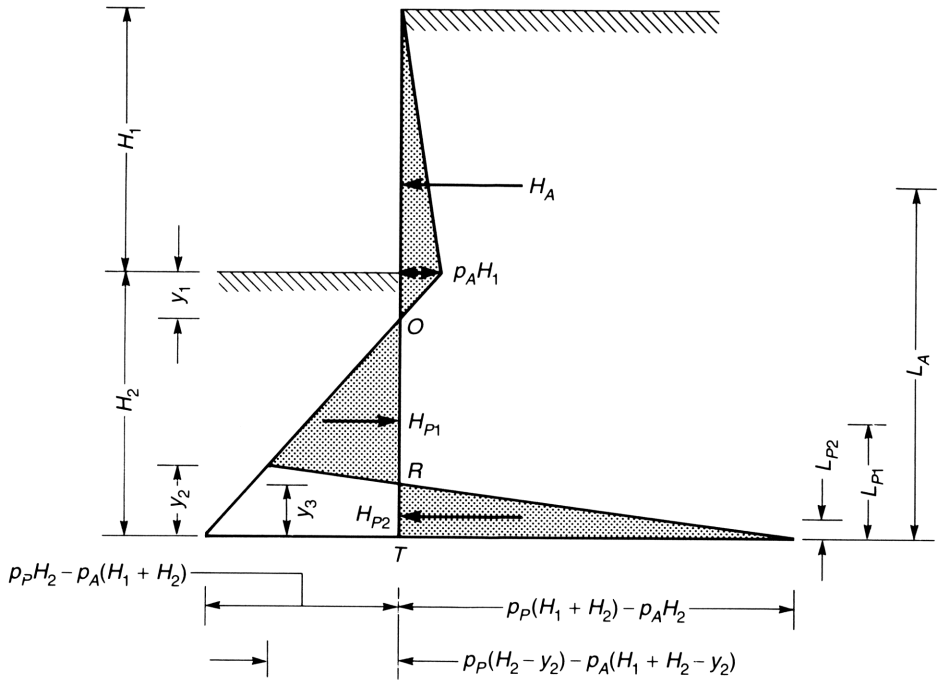

Figure 9–11 Cantilevered sheetpile wall

$$
\begin{aligned}
H_A &= p_A H_1(H_1 + y_1)/2 \\
H_{P1} &= [p_P(H_2 - y_2) - p_A(H_1 + H_2 - y_2)](H_2 - y_1 - y_3)/2 \\
H_{P2} &= [p_P(H_1 + H_2) - p_A H_2]y_3/2
\end{aligned}
$$

where

$$
\begin{aligned}
y_1 &= p_A H_1/(p_P - p_A) \\
y_3 &= y_2[p_P(H_1 + H_2) - p_A H_2]/(p_P - p_A)(H_1 + 2H_2 - y_2)
\end{aligned}
$$

Resolving horizontally gives

$$0 = H_A - H_{P1} + H_{P2}$$

Taking moments about the toe of the pile at T gives

$$0 = H_A L_A - H_{P1} L_{P1} + H_{P2} L_{P2}$$

Adjustments are made to the initial estimates until the above equations are satisfied. The shear and bending moment in the sheetpiling may then be obtained.

9.3.2 Anchored sheetpile retaining wall

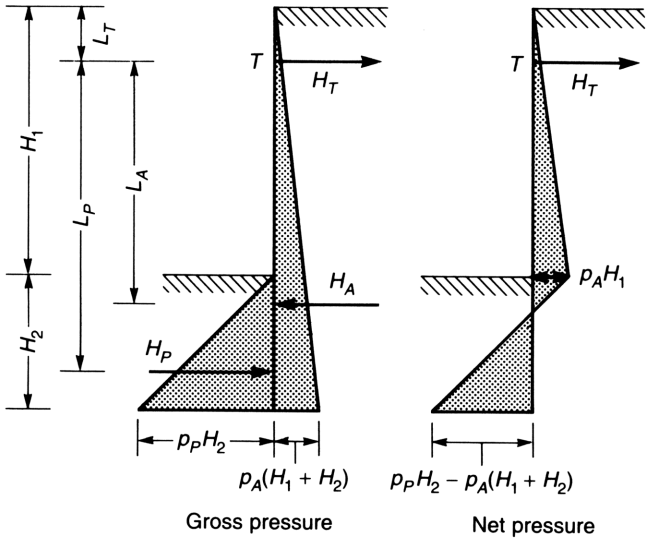

Figure 9–12 Anchored sheetpile retaining wall

The forces acting on an anchored sheetpile wall are shown in Figure 9–12. An initial estimate[6,7] is required of the penetration of the sheetpiling and the forces acting on the wall are given by

H_A = total active pressure behind wall
$= p_A(H_1 + H_2)^2/2$
H_P = total passive pressure in front of wall
$= p_PH_2^2/2$
H_T = force in tie
$= H_A - H_P$

Taking moments of all forces about the tie point T gives

$$H_AL_A - H_PL_P = 0$$

Adjustments are made to the initial estimate of the penetration until the above equation is satisfied.

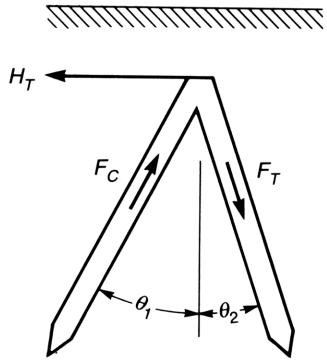

Figure 9–13 Anchor piles

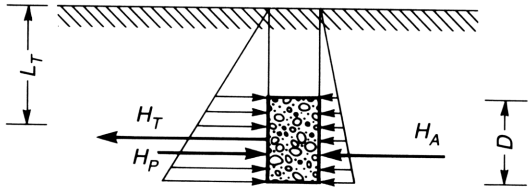

Figure 9–14 Dead-man

The tie may be secured by means of a soil anchor, anchor piles or a dead man. Anchor piles are shown in Figure 9–13 and the forces in the piles are given by

$$\begin{aligned} F_C &= \text{force in compression pile} \\ &= H_T/(\sin\theta_1 + \cos\theta_1 \tan\theta_2) \\ F_T &= \text{force in tension pile} \\ &= H_T/(\sin\theta_2 + \cos\theta_2 \tan\theta_1) \end{aligned}$$

A dead man anchor is shown in Figure 9–14 and may consist of a continuous beam or an isolated anchorage. The resistance of a continuous beam consists of the passive pressure in front of the beam less the active pressure behind the beam. The dead man must be located a sufficient distance from the wall that the passive wedge in front of the dead man does not intersect the active wedge behind the wall. Resolving horizontally for a continuous anchor with a tie force per unit length of H_T gives

$$\begin{aligned} 0 &= H_P - H_A + H_T \\ 0 &= DL_T(p_P - p_A) - H_T \end{aligned}$$

Hence, the size of the anchor may be determined.

Example 9–5 (Anchored sheetpile retaining wall)

Determine the required penetration of the sheetpile retaining wall shown in Figure 9–15 if active earth pressure may be assumed equivalent to a fluid pressure of 30 pounds per square foot per foot and passive pressure may be assumed equivalent to a fluid pressure of 400 pounds per square foot per foot. Calculate the force in the tie and the location and magnitude of the maximum shear and the maximum moment in the sheetpiling.

Solution

The total active pressure on the back of the wall is

$$\begin{aligned} H_A &= 30(14 + H_2)^2/2 \\ &= 2940 + 420H_2 + 15H_2^2 \end{aligned}$$

The total passive pressure on the front of the wall is

$$\begin{aligned} H_P &= 400H_2^2/2 \\ &= 200\ H_2^2 \end{aligned}$$

The distance between the line of action of the active pressure and the tie is

$$\begin{aligned} L_A &= 2(14 + H_2)/3 \\ &= 9.33 + 0.67H_2 \end{aligned}$$

The distance between the line of action of the passive pressure and the tie is

$$L_P = 14 + 0.67H_2$$

Taking moments about the tie point T gives

$$\begin{aligned} 0 &= H_A L_A - H_P L_P \\ 0 &= (2940 + 420H_2 + 15H_2^2)(9.33 + 0.67H_2) - 200H_2^2(14 + 0.67H_2) \end{aligned}$$

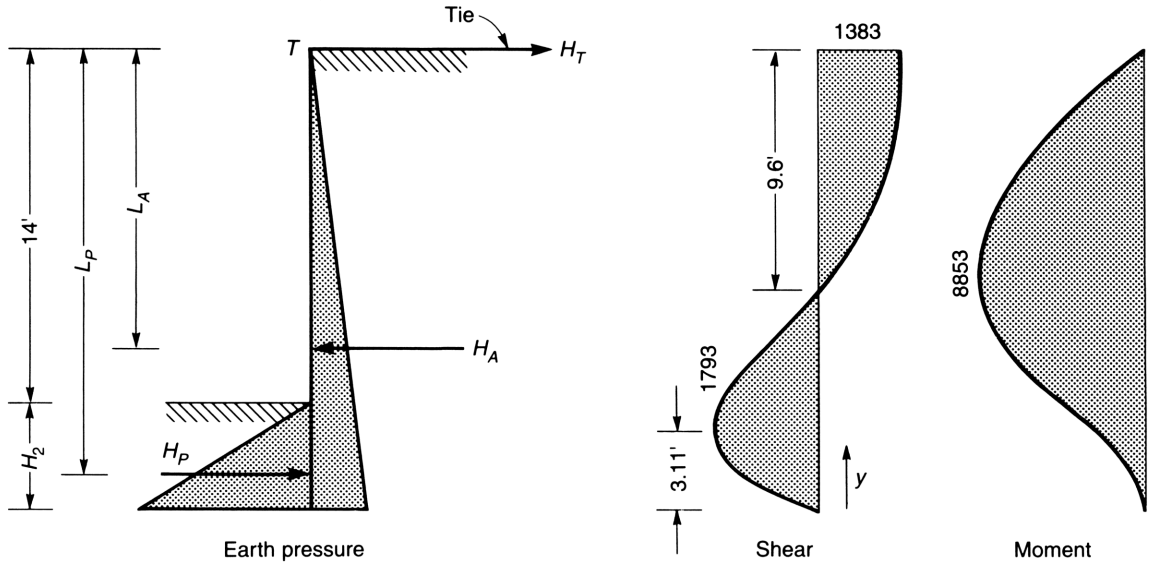

Figure 9–15 Details for Example 9–5

Solving for H_2 gives the required penetration as

$$H_2 = 4.25 \text{ feet}$$

The force in the tie is

$$\begin{aligned} H_T &= H_A - H_P \\ &= 4996 - 3613 \\ &= 1383 \text{ pounds per foot} \end{aligned}$$

and

$$\begin{aligned} p_P H_2 &= \text{passive pressure at toe of wall} \\ &= 400 \times 4.25 \\ &= 1700 \text{ pounds per foot} \end{aligned}$$

$$\begin{aligned} p_A(14 + H_2) &= \text{active pressure at toe of wall} \\ &= 30(14 + 4.25) \\ &= 548 \text{ pounds per square foot} \end{aligned}$$

The shear force at a distance y from the toe is

$$\begin{aligned} V &= y(1700 - 200y) - y(548 - 15y) \\ &= 1152y - 185y^2 \end{aligned}$$

The maximum shear occurs when

$$\begin{aligned} dV/dy &= 0 \\ &= 1152 - 370y \end{aligned}$$

Hence

$$y = 3.11 \text{ feet}$$

and

$$V_{max} = 1793 \text{ pounds per foot}$$

The shear force at a distance x from the top of the wall is

$$V = H_T - p_A x^2/2$$
$$= 1383 - 15x^2$$

The maximum moment occurs when

$$V = 0$$

Hence $$x = (1383/15)^{0.5}$$
$$= 9.6 \text{ feet}$$

The maximum moment is

$$M_{max} = H_T x - p_A x^3/6$$
$$= 1383 \times 9.6 - 30 \times 9.6^3/6$$
$$= 8853 \text{ pounds feet per foot}$$

Example 9–6 (Anchored bulkhead)

The ties for the anchored bulkhead shown in Figure 9–16 are located four feet from the top of the sheetpiling and are spaced at fifteen feet centers. The end of the tie is secured to two anchor piles raked as indicated. The active earth pressure may be assumed equivalent to a fluid pressure of 30 pounds per square foot per foot and passive pressure may be assumed equivalent to a fluid pressure of 400 pounds per square foot per foot.

For the depth of penetration indicated, calculate the value of the total active pressure on the back of the wall, the value of the total passive pressure on the front of the wall, and the moment of all forces about the tie point. Determine the force in the tie, the maximum shear and the maximum moment in the wall, and the forces in the compression and tension anchor piles.

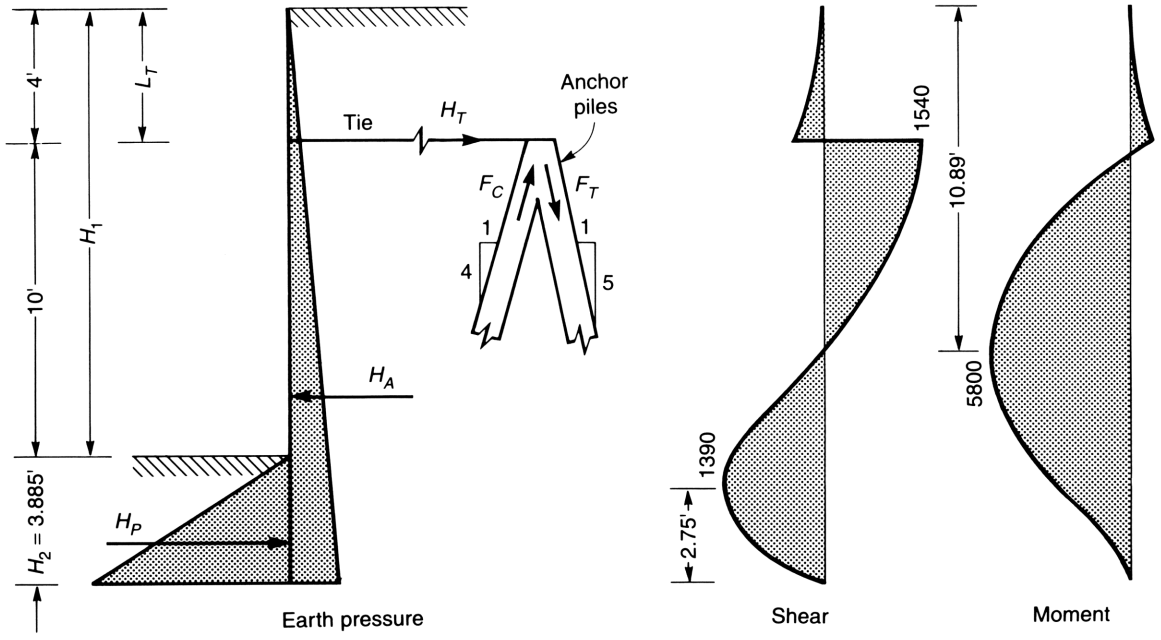

Figure 9–16 Details for Example 9–6

Solution

The total active pressure on the back of the wall is

$$\begin{aligned} H_A &= p_A(H_1 + H_2)^2/2 \\ &= 30\ (14 + 3.885)^2/2 \\ &= 4798 \text{ pounds per foot} \end{aligned}$$

The total passive pressure on the front of the wall is

$$\begin{aligned} H_P &= p_P H_2^2/2 \\ &= 400 \times 3.885^2/2 \\ &= 3018 \text{ pounds per foot} \end{aligned}$$

The bending moment at the tie point is

$$\begin{aligned} M_T &= H_A[2(H_1 + H_2)/3 - L_T] - H_P(H_1 - L_T + 2H_2/3) \\ &= 4798\ (2 \times 17.885)/3 - 4) - 3018(10 + 2 \times 3.885/3) \\ &= 7 \text{ pounds feet per foot} \\ &\approx 0 \end{aligned}$$

Hence the penetration of the sheetpiling is adequate.

The force in the tie is

$$\begin{aligned} H_T &= H_A - H_P \\ &= 4798 - 3018 \\ &= 1780 \text{ pounds} \end{aligned}$$

The maximum shear occurs at the tie point which is four feet from the top of the sheetpiling and is given by

$$\begin{aligned} V_{max} &= H_T - p_A L_T^2/2 \\ &= 1780 - 240 \\ &= 1540 \text{ pounds per foot} \end{aligned}$$

The shear force at a distance x from the top of the sheetpiling, when x exceeds L_T, is

$$\begin{aligned} V &= H_T - p_A x^2/2 \\ &= 1780 - 15x^2 \end{aligned}$$

The maximum moment occurs when

$$V = 0$$

Hence

$$\begin{aligned} x &= (1780/15)^{0.5} \\ &= 10.89 \text{ feet} \end{aligned}$$

The maximum moment is

$$\begin{aligned} M_{max} &= H_T(x - L_T) - p_A x^3/6 \\ &= 1780\ (10.89 - 4) - 30 \times 10.89^3/6 \\ &= 5800 \text{ pounds feet per foot} \end{aligned}$$

The force in the compression pile is

$$\begin{aligned} F_C &= 15H_T/(\sin\theta_1 + \cos\theta_1 \tan\theta_2) \\ &= 15 \times 1780/(\sin 14° + \cos 14° \tan 11.3°) \\ &= 15 \times 1780/(0.243 + 0.970 \times 0.200) \\ &= 61{,}094 \text{ pounds} \end{aligned}$$

The force in the tension pile is

$$\begin{aligned} F_T &= 15H_T/(\sin\theta_2 + \cos\theta_2 \tan\theta_1) \\ &= 15 \times 1780/(\sin 11.3° + \cos 11.3° \tan 14°) \\ &= 15 \times 1780/(0.196 + 0.981 \times 0.250) \\ &= 60{,}524 \text{ pounds} \end{aligned}$$

9.4 SUPPLEMENTARY PROBLEMS

Problem 9–1 (Cantilever retaining wall)
The fill behind the retaining wall in Figure 9–17 has a unit weight of 110 pounds per cubic foot with an equivalent fluid pressure of 30 pounds per square foot per foot. Passive earth pressure may be assumed equivalent to a fluid pressure of 300 pounds per square foot per foot and the coefficient of friction at the underside of the base is 0.4. All concrete has a compressive strength of 3000 pounds per square inch and reinforcement consists of Grade 60 deformed bars. Neglecting the effects of the shear key, determine the factor of safety against overturning and the bearing pressure distribution under the base. Using factored loading, determine the reinforcement required in the footing and stem. Combining the effects of friction and passive pressure, calculate the minimum depth of the shear key to provide a factor of safety of 1.5 against sliding.

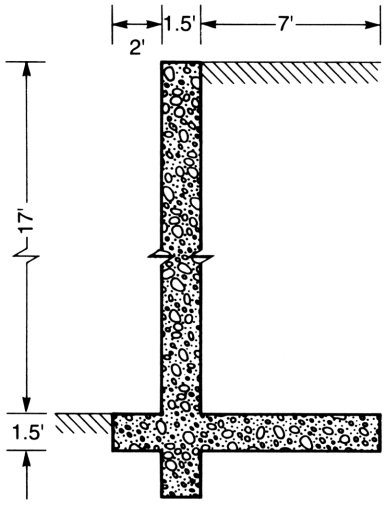

Figure 9–17 Details for Problem 9–1

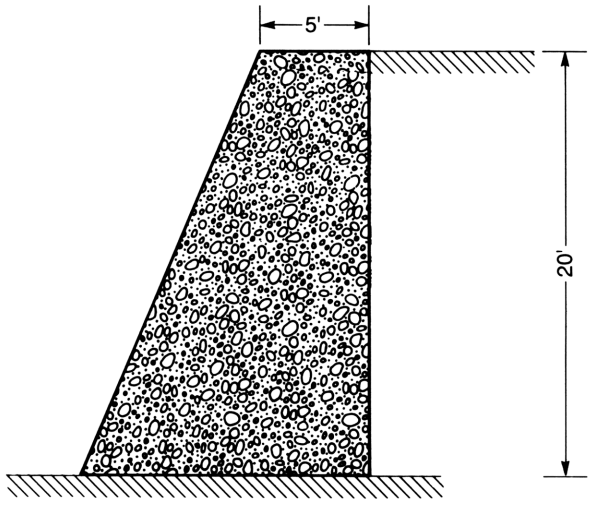

Figure 9–18 Details for Problem 9–2

Problem 9.2 (Mass concrete gravity dam)
The mass concrete gravity dam shown in Figure 9–18 is 20 feet high. Determine the minimum base width if no tensile stresses are allowed in the concrete and the water level is at the top of the dam wall.

Problem 9.3 (Counterfort retaining wall)
Details of a counterfort retaining wall are shown in Figure 9–19. The fill behind the wall has an equivalent fluid pressure of 40 pounds per square foot per foot. All concrete has a compressive strength of 3000 pounds per square inch and reinforcement consists of Grade 60 deformed bars. Determine the horizontal reinforcement required at the base of the stem in the front and earth faces, the reinforcement required in the rear of the counterforts, and the horizontal tie reinforcement required at the base of the counterforts.

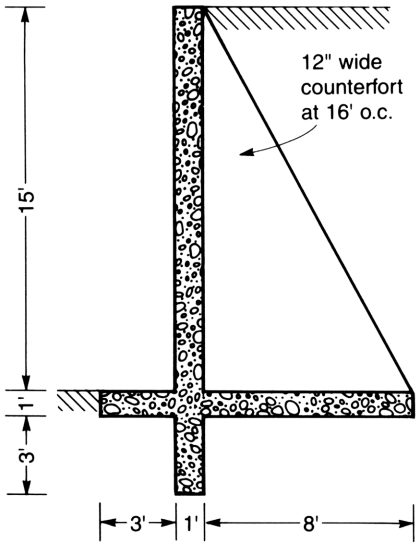

Figure 9–19 Details for Problem 9–3

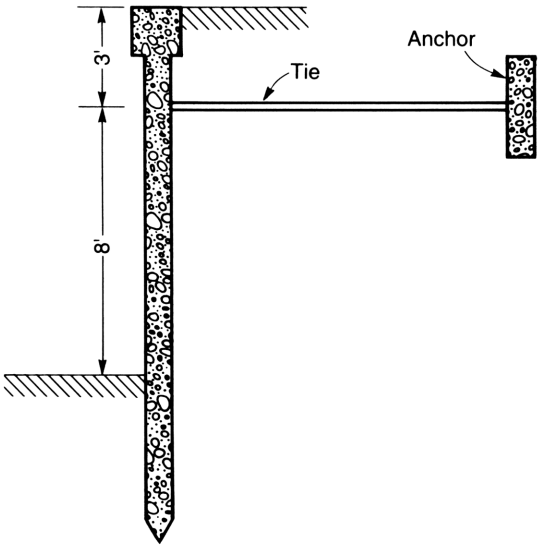

Figure 9–20 Details for Problem 9–4

Problem 9.4 (Anchored sheetpile retaining wall)
Determine the required penetration of the sheetpile retaining wall shown in Figure 9–20 if active earth pressure may be assumed equivalent to a fluid pressure of 35 pounds per square foot per foot and passive pressure may be assumed equivalent to a fluid pressure of 450 pounds per square foot per foot. Calculate the force in the tie and the location and magnitude of the maximum shear and the maximum moment in the sheetpiling.

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10

Piled foundations

10.1 GENERAL

10.1.1 Using piled foundations

A piled foundation constitutes a deep foundation and consists of a pile cap which transfers the structure loads to the supporting piles. A piled foundation is utilized when it provides the most economical solution to supporting the structure loads. This generally occurs when the upper soil layers are unsuitable for load bearing and piles are used to transmit the loads to a stable soil layer some distance below the surface.

Reinforced concrete piles may be either precast piles which are driven to a suitable foundation depth, or cast in situ piles which are constructed in a pre–drilled hole. The compressive force in a pile is transferred to the soil either by end bearing on a suitable soil layer, by the frictional resistance between the surface of the pile and the surrounding soil, or by a combination of both features. The tensile force in a pile may be transferred to the soil by skin friction, or by underreaming the tip of the pile into a suitable rock layer. Vertical forces applied to the pile cap produce axial forces in both vertical and inclined piles. Lateral forces applied to the pile cap produce axial forces in inclined piles and axial forces and bending moments in vertical piles.

The pile cap is assumed infinitely rigid. Reinforcement requirements may be determined using the deep beam analogy method or by the truss analogy method[1,2]. In addition, a pile cap must be designed for flexural shear and punching shear around the heads of piles and the base of columns.

10.2 PILE GROUP WITH VERTICAL PILES

10.2.1 Vertical loading only

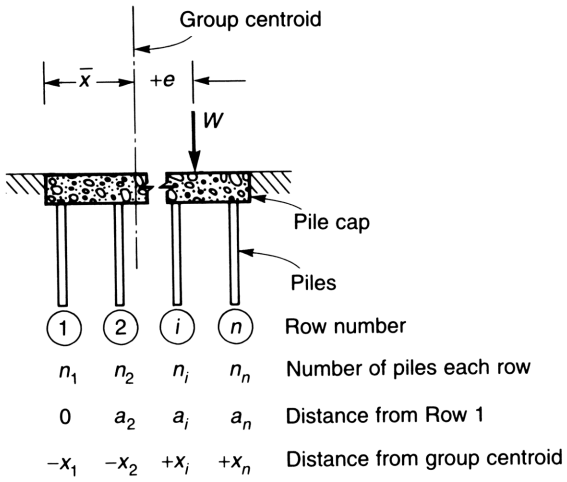

Figure 10–1 Pile group with vertical piles and vertical load

Using the notation in Figure 10–1, which shows a group of vertical piles with a rigid pile cap and an imposed vertical eccentric load, the location of the pile group centroid is

$$\bar{x} = \Sigma(an)/\Sigma n$$

The axial force on each pile in row i, neglecting the self weight of the pile and the pile cap, is given by

$$P_i = W/\Sigma n + Wex_i/\Sigma(nx^2)$$

For a symmetrical group, the location of the group centroid is

$$\bar{x} = a_n/2$$

When the applied load coincides with the pile group centroid, the axial force on each pile is given by

$$P = W/\Sigma n$$

When the pile cap is subjected to a bending moment M the eccentricity is given by

$$e = M/W$$

When each row in the group has an identical number of piles n_R and the number of piles in each line is n_L the load applicable to each line is

$$W' = W/n_R$$

The axial force on a pile in row i is

$$P_i = W'/n_L + W'ex_i/\Sigma x^2$$

Example 10–1 (Pile group with vertical piles and vertical load)
Figure 10–2 shows the loading acting on one line of a symmetrical group of piles supporting a structure. The vertical load indicated includes the self weight of the rigid pile cap. Determine the maximum axial force produced in a pile.

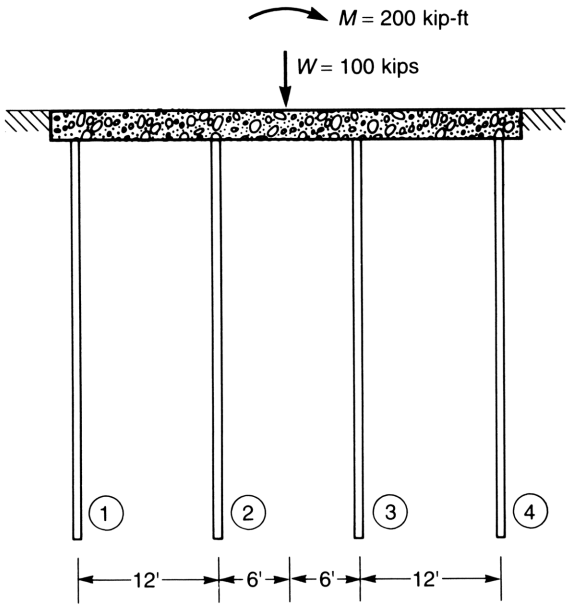

Figure 10–2 Details for Example 10–1

Solution
The equivalent eccentricity of the moment acting on the pile group is

$$\begin{aligned} e &= M/W \\ &= 200/100 \\ &= 2 \text{ feet} \end{aligned}$$

The line of piles is symmetrical with an inertia of

$$\begin{aligned} \Sigma x^2 &= 2(6^2 + 18^2) \\ &= 720 \end{aligned}$$

The maximum axial force occurs in pile 4 and, neglecting its self weight, is given by

$$\begin{aligned} P_4 &= W/n_L + Wex_4/\Sigma x^2 \\ &= 100/4 + 100 \times 2 \times 18/720 \\ &= 30 \text{ kips} \end{aligned}$$

10.2.2 Vertical and lateral loading

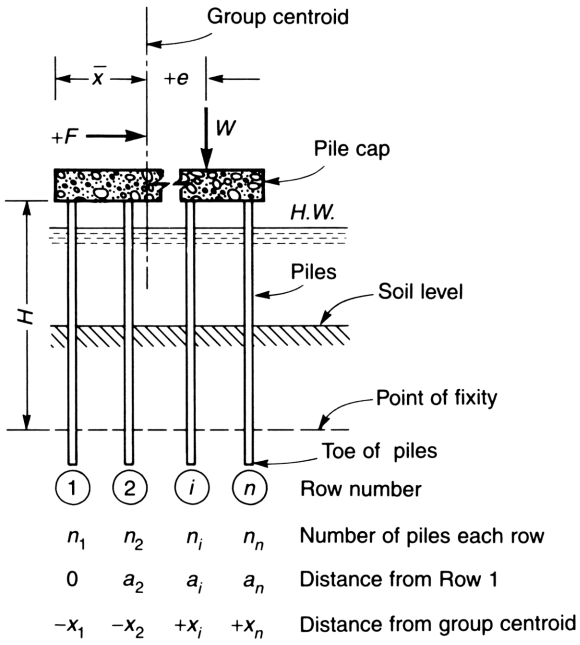

Figure 10–3 Pile group with vertical piles and lateral load

Figure 10–3 illustrates a group of vertical piles subjected to a horizontal load, as in the case of a pier or wharf. The piles are considered fixed ended at the pile cap and at a depth of penetration H below the bottom of the rigid pile cap, with a point of contraflexure occurring at a depth of H/2. The total moment acting on the pile group is

$$M_T = We + FH$$

The axial force on each pile in row i, neglecting the self weight of the pile and the pile cap, is given by

$$P_i = W/\Sigma n + M_T x_i/\Sigma(nx^2)$$

The shearing force on each pile is

$$V = F/\Sigma n$$

The maximum bending moment, at the point of fixity and at the pile cap, in each pile is

$$M_{max} = VH/2$$

Example 10–2 (Pile group with vertical piles and vertical and lateral load)

Figure 10–4 shows the loading acting on one line of a group of symmetrical piles supporting a dock. The vertical load indicated includes the self weight of the rigid pile cap and the piles may be considered fixed at a depth of 20 feet below the bottom of the cap. Determine the shear force and maximum bending moment in each pile and the maximum axial force produced in a pile.

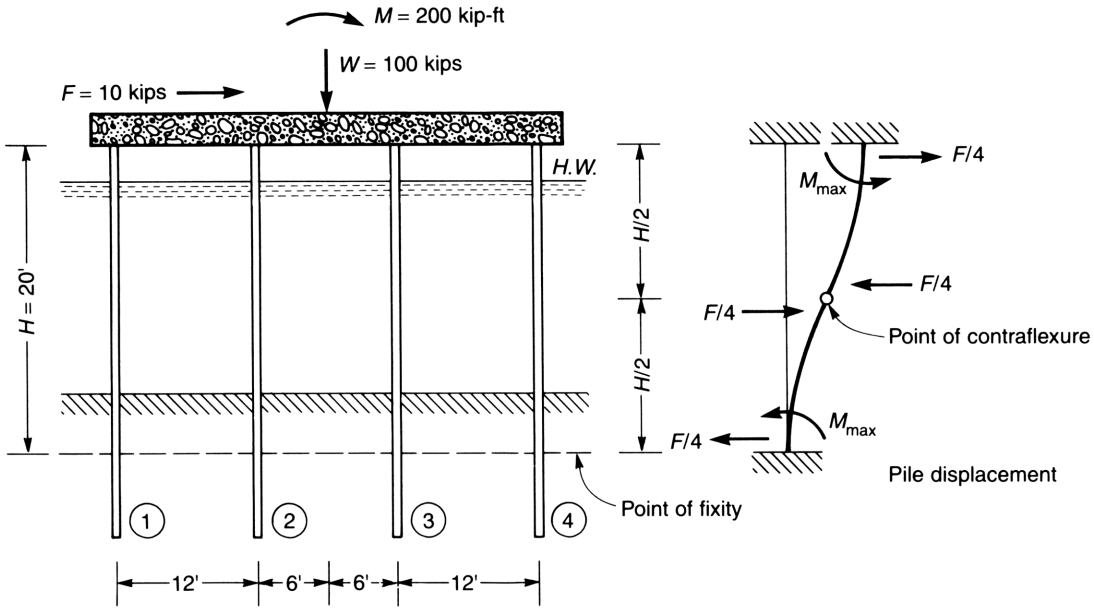

Figure 10–4 Details for Example 10–2

Solution

The total moment acting on the pile group is

$$\begin{aligned} M_T &= M + FH \\ &= 200 + 10 \times 20 \\ &= 400 \text{ kip feet} \end{aligned}$$

The pile group is symmetrical with an inertia of

$$\begin{aligned} \Sigma x^2 &= 2(6^2 + 18^2) \\ &= 720 \end{aligned}$$

The maximum axial force occurs in pile 4 and, neglecting the self weight, is given by

$$\begin{aligned} P_4 &= W/n_L + M_T x_4/\Sigma x^2 \\ &= 100/4 + 400 \times 18/720 \\ &= 35 \text{ kips} \end{aligned}$$

The shear force in each pile is

$$\begin{aligned} V &= F/\Sigma n \\ &= 10/4 \\ &= 2.5 \text{ kips} \end{aligned}$$

The maximum bending moment in a pile is determined by assuming a point of contraflexure at a depth of H/2, as shown in Figure 10–4. The maximum moment is

$$\begin{aligned} M_{max} &= VH/2 \\ &= 2.5 \times 20/2 \\ &= 25 \text{ kip feet} \end{aligned}$$

10.3 PILE GROUP WITH INCLINED PILES

10.3.1 No rotation of pile cap

The elastic center method[3,4] may be used to determine the forces in inclined piles. All piles are assumed to be hinged at each end and the rigid pile cap is assumed to rotate about the elastic center thus producing forces in the piles which balance the moment of the external load about the elastic center. For simple pile arrangements, the location of the elastic center may be determined by inspection, as shown in Figure 10–5.

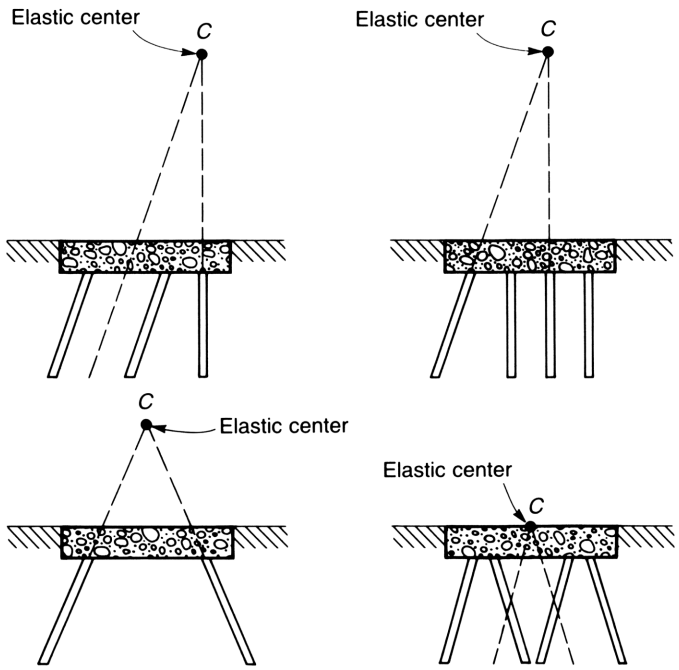

Figure 10–5 Location of elastic center

Forces in the piles are produced by both translation of the pile cap and rotation about the elastic center. When the line of action of the external load passes through the elastic center, no rotation of the pile cap occurs and the forces in the piles, due to the translation of the pile cap, may be obtained by resolution.

Example 10–3 (Forces in raked piles without rotation of the pile cap)
Determine the forces, due to the indicated loads, in the raked piles shown in Figure 10–6. The piles have a batter of 1:4 and the line of action of the external loads passes through the elastic center

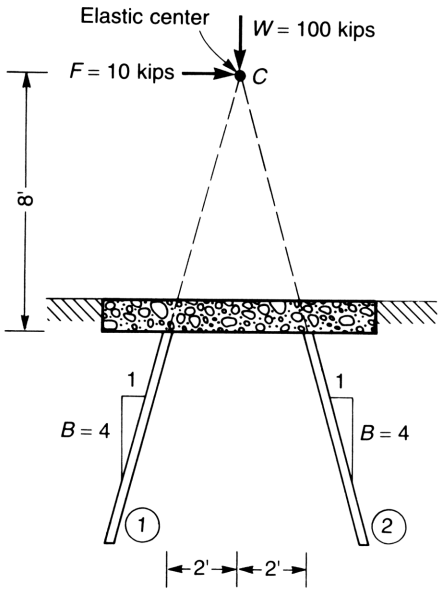

Figure 10–6 Details for Example 10–3

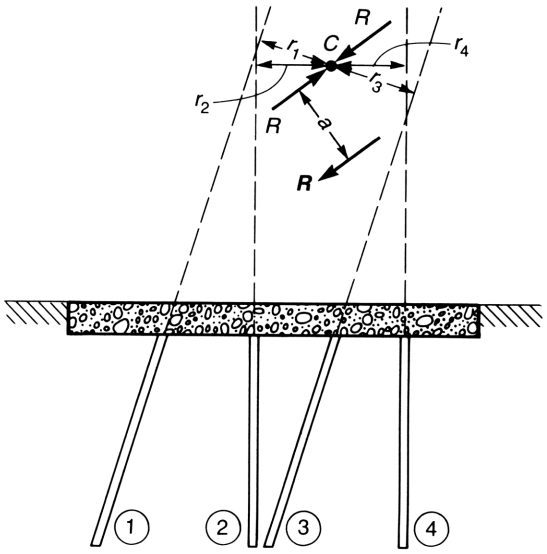

Figure 10–7 Pile cap rotation

Solution

Resolving forces vertically, the vertical component of the axial force in pile 1 is given by

$$\begin{aligned} V_1 &= W/2 - FB/2 \\ &= 100/2 - 10 \times 4/2 \\ &= 30 \text{ kips} \end{aligned}$$

The axial force in pile 1 is

$$\begin{aligned} P_1 &= V_1(1 + B^2)^{0.5}/B \\ &= 30(17)^{0.5}/4 \\ &= 30.92 \text{ kips} \end{aligned}$$

The vertical component of the axial force in pile 2 is

$$\begin{aligned} V_2 &= W/2 + FB/2 \\ &= 70.00 \text{ kips} \end{aligned}$$

The axial force in pile 2 is

$$\begin{aligned} P_2 &= V_2(1 + B^2)^{0.5}/B \\ &= 72.15 \text{ kips} \end{aligned}$$

10.3.2 Rotation of pile cap

When the line of action of the external loads does not pass through the elastic center, additional axial forces are produced in the piles due to the rotation of the pile cap about the elastic center. As shown in Figure 10–7, the external load R may be replaced by a force R through the elastic center plus a couple of magnitude

$$M = Ra$$

where a is the perpendicular distance from the line of action of R to the elastic center at C. The force R through the elastic center produces a translation of the pile cap and the axial forces in the piles are determined by resolution as previously described.

The couple M causes a rotation θ of the pile cap which produces an axial deformation in each pile of magnitude

$$\delta = r_i\theta$$

where r_i is the distance from the elastic center perpendicular to each pile
The axial force produced in each pile is

$$P_i = EA\delta/L$$
$$= EAr_i\theta/L$$

This axial force produces a moment about the elastic center of

$$M_P = P_i r_i$$
$$= EAr_i^2\theta/L$$

For equilibrium, the sum of the moments of the pile forces about the elastic center must equal the external couple about the elastic center. Hence,

$$M = \theta\Sigma(EAr^2/L)$$
$$\theta = M/\Sigma(EAr^2/L)$$
$$P_i = (MEAr_i/L)/\Sigma(EAr^2/L)$$
$$= Mr_i/\Sigma r^2 \text{ ... for identical piles}$$

Example 10–4 (Forces in raked piles with rotation of the pile cap)
Determine the forces, due to the indicated loads, in the raked piles shown in Figure 10–8. The piles have a batter of 1:4 and have identical lengths and cross sectional areas.

Solution
The axial forces due to the translation are obtained from Example 10–3 as

$$P_1 = P_2$$
$$= 30.92/2$$
$$= 15.46 \text{ kips}$$
$$P_3 = P_4$$
$$= 72.15/2$$
$$= 36.08 \text{ kips.}$$

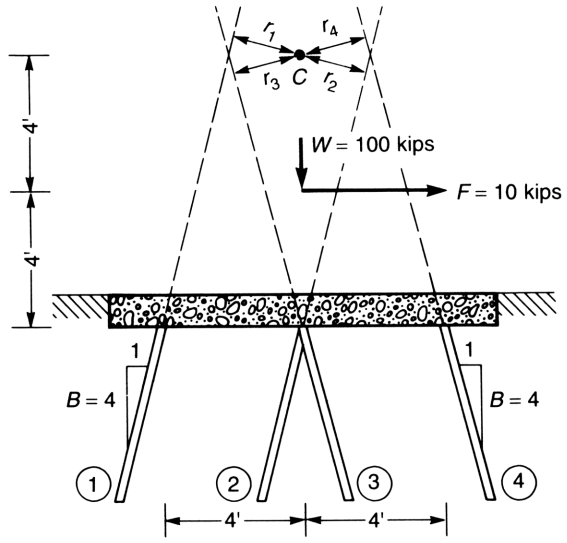

Figure 10–8 Details for Example 10–4

The couple about the elastic center due to the applied load is

$$\begin{aligned} M &= 4F \\ &= 4 \times 10 \\ &= 40 \text{ kip feet} \end{aligned}$$

The distance from the elastic center perpendicular to each pile is

$$\begin{aligned} r_1 = r_2 = r_3 = r_4 &= 2\cos 14.04^\circ \\ &= 1.94 \text{ feet} \\ \Sigma r^2 &= 4(1.94)^2 \\ &= 15.05 \end{aligned}$$

The axial forces in the piles due to rotation of the pile cap are

$$\begin{aligned} P_1 &= P_3 \\ &= + Mr/\Sigma r^2 \\ &= 40 \times 1.94/15.05 \\ &= 5.15 \text{ kips ... compression} \\ P_2 &= P_4 \\ &= 5.15 \text{ kips ... tension} \end{aligned}$$

The final pile forces are given by the sum of the forces due to translation and rotation and are

$$\begin{aligned} P_1 &= 15.46 + 5.15 \\ &= 20.61 \text{ kips} \\ P_2 &= 15.46 - 5.15 \\ &= 10.31 \text{ kips} \\ P_3 &= 36.08 + 5.15 \\ &= 41.23 \text{ kips} \\ P_4 &= 36.08 - 5.15 \\ &= 30.93 \text{ kips} \end{aligned}$$

Example 10–5 (Retaining wall on piles)

The fill behind the retaining wall in Figure 10–9 has a unit weight of 110 pounds per cubic foot with an equivalent fluid pressure of 30 pounds per square foot per foot. The live load surcharge behind the wall is equivalent to an additional two feet of fill. The 12 inch square piles are spaced at 5 feet on center longitudinally.

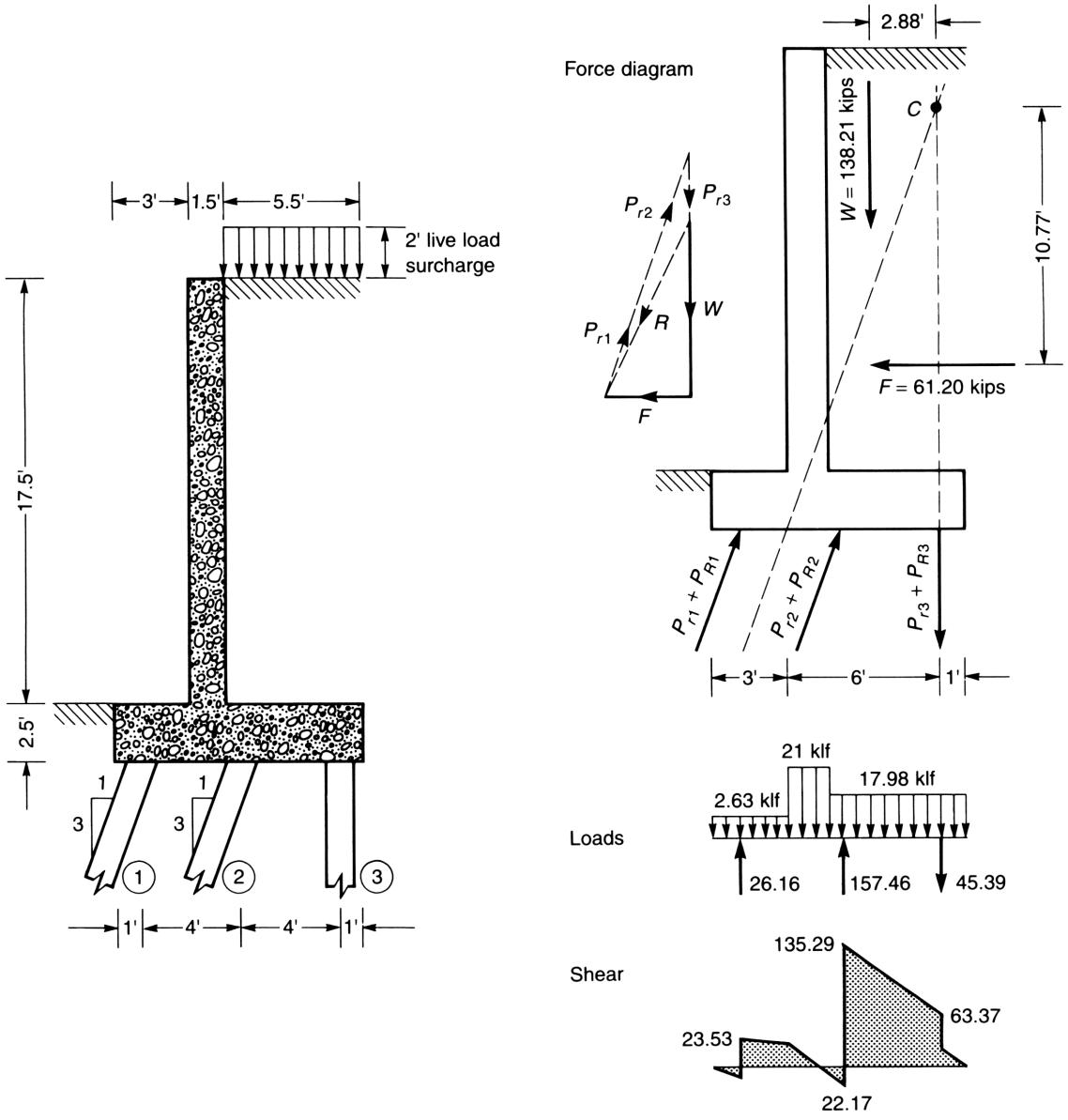

Figure 10–9 Details for Example 10–5

Determine the location of the elastic center and the moment of the factored loads about the elastic center. Determine the forces in the piles and the maximum shear in the base of the retaining wall.

Solution

The height of the elastic center above the base is

$$L_y = 3 \times 6$$
$$= 18 \text{ feet}$$

The horizontal distance of the elastic center from the toe is

$$L_x = 9 \text{ feet}$$

The factored vertical loads for a 5 foot length of wall are given by

W_W = factored weight of stem wall
$= 5 \times 1.4 \times 150 \times 1.5 \times 17.5/1000$
$= 27.56$ kips

W_B = factored weight of base
$= 5 \times 1.4 \times 150 \times 2.5 \times 10/1000$
$= 26.25$ kips

W_S = factored weight of backfill
$= 5 \times 1.4 \times 110 \times 5.5 \times 17.5/1000$
$= 74.11$ kips

W_L = factored weight of surcharge
$= 5 \times 1.7 \times 110 \times 5.5 \times 2/1000$
$= 10.29$ kips

W = the total factored vertical load
$= 27.56 + 26.25 + 74.11 + 10.29$
$= 138.21$ kips

The factored horizontal loads for a 5 foot length of wall are given by

H_A = factored lateral pressure from backfill
$= 5 \times 1.7 \times 30 \times 20^2/2000$
$= 51.00$ kips

H_L = factored lateral pressure from surcharge
$= 5 \times 1.7 \times 2 \times 30 \times 20/1000$
$= 10.20$ kips

F = the total factored horizontal load
$= 51.00 + 10.20$
$= 61.20$ kips

The clockwise moment of the factored loads about the elastic center is

$$M = 11.33\ H_A + 8H_L - 5.25\ W_W - 4W_B - 1.75(W_S + W_L)$$
$$= 578.0 + 81.6 - 144.7 - 105.0 - 147.7$$
$$= 262.2 \text{ kip feet}$$

Resolving forces, the axial force in pile 1 due to translation of the pile cap is given by

$$P_{T1} = F(1 + 3^2)^{0.5}/2$$
$$= 61.2(10)^{0.5}/2$$
$$= 96.77 \text{ kips} \ldots \text{ compression}$$

The angle of inclination of pile 1 and pile 2 to the vertical is

$$\begin{aligned}\theta &= \tan^{-1}0.33\\ &= 18.43°\end{aligned}$$

The distance from the elastic center perpendicular to each pile is

$$\begin{aligned}r_1 &= r_2\\ &= 2\cos\theta\\ &= 2\cos 18.43°\\ &= 1.90 \text{ feet}\\ r_3 &= 0\\ \Sigma r^2 &= 2(1.90)^2\\ &= 7.2\end{aligned}$$

The axial force in pile 1 due to rotation of the pile cap is

$$\begin{aligned}P_{R1} &= Mr_1/\Sigma r^2\\ &= 262.2 \times 1.90/7.2\\ &= 69.20 \text{ kips ... tension}\end{aligned}$$

The total axial force in pile 1 is

$$\begin{aligned}P_1 &= P_{T1} + P_{R1}\\ &= 96.77 - 69.20\\ &= 27.57 \text{ kips ... compression.}\end{aligned}$$

The axial force in pile 2 due to translation is

$$P_{T2} = 96.77 \text{ kips ... compression}$$

The axial force in pile 2 due to rotation is

$$P_{R2} = 69.20 \text{ kips ... compression}$$

The total axial force in pile 2 is

$$\begin{aligned}P_2 &= P_{T2} + P_{R2}\\ &= 96.77 + 69.20\\ &= 165.97 \text{ kips ... compression.}\end{aligned}$$

The axial force in pile 3 due to translation is

$$\begin{aligned}P_{T3} &= W - 3F\\ &= 138.21 - 3 \times 61.20\\ &= -45.39 \text{ kips ... tension}\end{aligned}$$

The axial force in pile 3 due to rotation is

$$\begin{aligned}P_{R3} &= Mr_3/\Sigma r^2\\ &= 0\end{aligned}$$

The total axial force in pile 3 is

$$\begin{aligned}P_3 &= P_{T3} + P_{R3}\\ &= -45.39 \text{ kips ... tension}\end{aligned}$$

The factored distributed load on the heel of the retaining wall for a 5 foot length of wall is

$$w_u = (W_B L_H / L_B + W_S + W_L)/L_H$$

where L_H = length of heel
= 5.5 feet
L_B = length of base
= 10 feet

and $w_u = (26.25 \times 5.5/10 + 74.11 + 10.29)/5.5$
= 17.98 kips per linear foot

The loading diagram and corresponding shear force diagram are shown in Figure 10–9. The maximum shear force occurs at pile 2 and has a value of

$$V_2 = 5w_u + P_3$$
$$= 5 \times 17.98 + 45.39$$
$$= 135.29 \text{ kips}$$

10.4 PILE CAP DESIGN

10.4.1 Beam analogy method

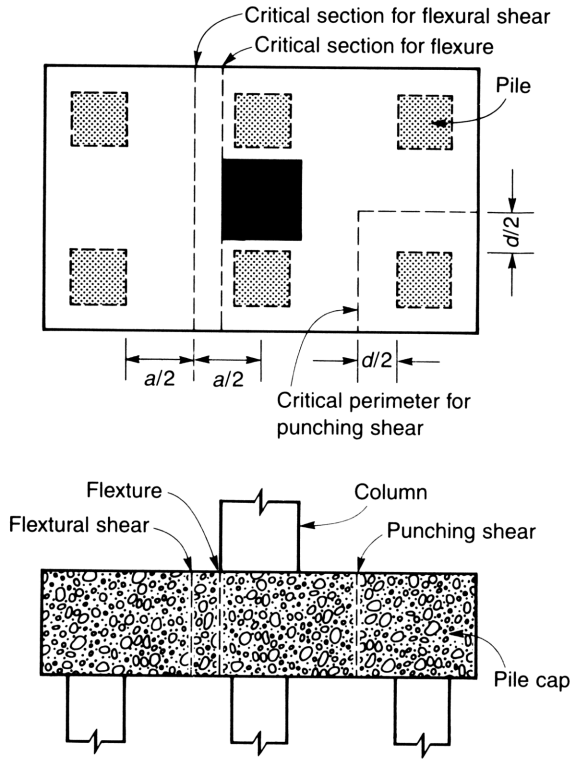

Figure 10–10 Pile cap beam analogy

The pile cap depth is governed by shear requirements or by the necessity to provide adequate anchorage length for the reinforcement projecting from the piles or for the column starter bars.

The critical perimeter for punching shear is located a distance of d/2 from the face of the column or from the face of a pile. Punching shear at the column will generally not control since some piles are normally within the critical perimeter. As shown in Figure 10–10, punching shear for a corner pile is most severe. The capacity of the pile cap for punching shear is the minimum value given by ACI Equations (11–36), (11–37), and (11–38) as

$$\phi V_c = (2 + 4/\beta_c)\phi b_o d(f'_c)^{0.5}$$
$$\phi V_c = (\alpha_s d/b_o + 2)\phi b_o d(f'_c)^{0.5}$$
$$\phi V_c = 4\phi b_o d(f'_c)^{0.5}$$

where β_c = ratio of long side to short side of a pile
ϕ = strength reduction factor
= 0.85 from ACI Section 9.3
b_o = length of critical perimeter for punching shear
α_s = 40 ... interior piles
= 30 ... edge piles
= 20 ... corner piles
d = effective depth

In accordance with ACI Section 11.8.1, the pile cap may be classified as a deep flexural member when the ratio of the clear distance between piles to the effective depth is

$$\ell_n/d < 5$$

In this case, ACI Section 11.8.5 specifies that the critical section for flexural shear is located a distance from the face of a pile given by

$$x = a/2$$
$$\leq d$$

where a = shear span
= distance from face of nearest pile to center of column.

The capacity of the pile cap for flexural shear, across the full width of the cap, is given by ACI Equation (11–28) as

$$\phi V_c = 2\phi db(f'_c)^{0.5}$$

Alternatively, advantage may be taken of the enhancement in shear strength due to the short shear span by utilizing ACI Equation (11–29) to give a capacity of

$$\phi V_c = \phi(3.5 - 2.5M_u/V_u d)(1.9(f'_c)^{0.5} + 2500\rho V_u d/M_u)bd$$
$$\leq 6\phi db(f'_c)^{0.5}$$

where ρ = reinforcement ratio
V_u = factored shear at critical section
M_u = factored moment at the same section

and $2.5 \geq (3.5 - 2.5M_u/V_u d)$

The location of the critical section for flexure is, in accordance with ACI Section 15.4.2, at the face of the column. The area of tension reinforcement required is obtained using the principles of ACI Section 10.3.

Example 10.6 (Pile cap design by beam analogy)

An 18 inch square column carrying a factored axial load of 900 kips is supported, as shown in Figure 10–11, on a pile cap and a group of six 12 inch square piles. The concrete strength is 3000 pounds per square inch and reinforcement consists of Grade 60 bars.

Determine the minimum allowable effective depth of the pile cap and the required number of No.7 reinforcing bars required in the longitudinal direction.

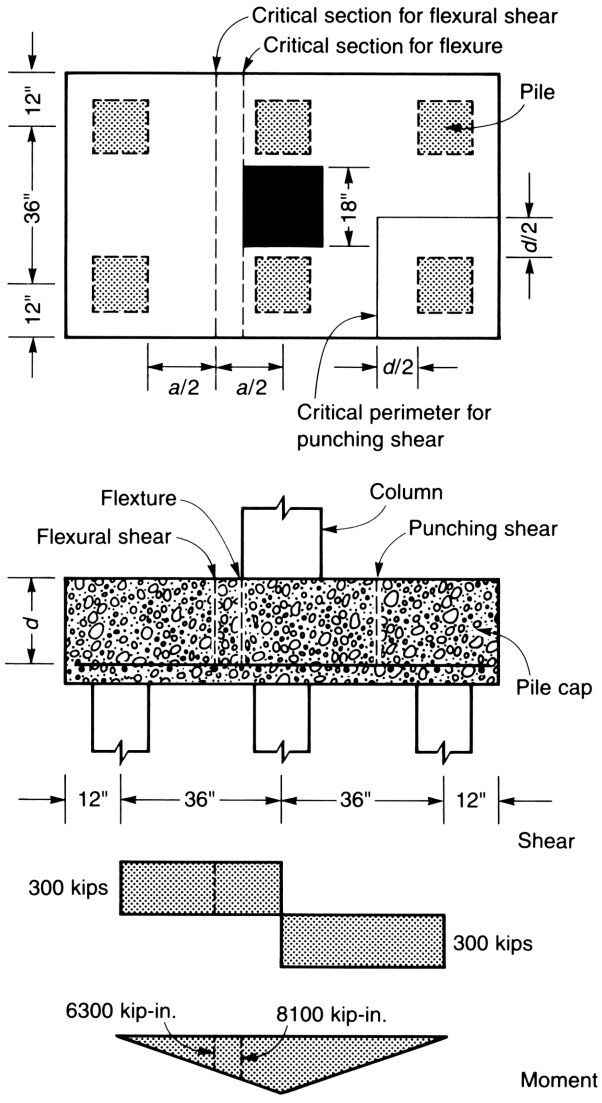

Figure 10–11 Details for Example 10–6

Solution

Assuming that the pile cap may be classified as a deep flexural member, as specified in ACI Section 11.8.1, the critical section for flexural shear is given by ACI Section 11.8.5 as a distance from the face of the outer piles of

$$\begin{aligned} x &= a/2 \\ &= (36 - 6)/2 \\ &= 15 \text{ inches} \end{aligned}$$

The factored shear at this location is

$$\begin{aligned} V_u &= W_u/3 \\ &= 900/3 \\ &= 300 \text{ kips} \end{aligned}$$

The factored moment at the same location is

$$\begin{aligned} M_u &= W_u(x + 6)/3 \\ &= 900(15 + 6)/3 \\ &= 6300 \text{ kip inches} \end{aligned}$$

The reinforcement ratio may be assumed to be

$$\begin{aligned} \rho &= A_s/bd \\ &= 0.0020 \end{aligned}$$

The flexural shear capacity of the section is obtained from ACI Equation (11–29) as

$$\begin{aligned} \phi V_c &= \phi(3.5 - 2.5M_u/V_u d)(1.9(f'_c)^{0.5} + 2500\rho V_u d/M_u)bd \\ &= 0.85(3.5–2.5\times6300/300d)(1.9(3000)^{0.5}+2500\times0.002\times300d/6300)60d \\ &= 51d(3.5 - 52.5/d)(104.06 + 0.238d) \\ &= V_u \\ &= 300{,}000 \text{ pounds} \end{aligned}$$

Solving for the effective depth gives

$$d = 30 \text{ inches.}$$

$$V_u < 6\phi db(f'_c) \text{ ... satisfactory}$$

and

$$2.5 > (3.5 - 2.5M_u/V_u d) \text{ ... satisfactory}$$

Punching shear for the column is not applicable since the central piles are located within the critical perimeter of the column. The critical perimeter for a corner pile is

$$\begin{aligned} b_o &= 2(18 + d/2) \\ &= 2(18 + 30/2) \\ &= 66 \text{ inches} \end{aligned}$$

The capacity of the pile cap for punching shear at a corner pile is governed by ACI Equation (11–37) which gives

$$\begin{aligned} \phi V_c &= 4\phi b_o d(f'_c)^{0.5} \\ &= 4 \times 0.85 \times 66 \times 30(3000)^{0.5} \\ &= 368{,}730 \text{ pounds} \\ &> W_u/6 \text{ ... satisfactory} \end{aligned}$$

The ratio of the clear distance between piles to the effective depth is

$$\begin{aligned} \ell_n/d &= 24/30 \\ &= 0.8 \\ &< 5 \end{aligned}$$

Hence, in accordance with ACI Section 11.8.1 the pile cap may be classified as a deep flexural member and the minimum effective depth of 30 inches is satisfactory.

The critical section for flexure is located at the face of the column and the factored moment at this section is

$$\begin{aligned} M_u &= W_u(36 - 9)/(3 \times 12) \\ &= 675 \text{ kip feet} \end{aligned}$$

The required reinforcement ratio is derived from ACI Section 10.3 and is

$$\begin{aligned} \rho &= 0.85f'_c[1 - (1 - 2M_u/0.765bd^2f'_c)^{0.5}]/f_y \\ &= 0.0029 \end{aligned}$$

This is greater than the initial assumed value which is, therefore, satisfactory.
The maximum reinforcement ratio is given by ACI Section 10.3 as

$$\begin{aligned} \rho_{max} &= 0.75 \times 0.85 \times 87\beta_1 f'_c/f_y(87 + f_y) \\ &= 0.0160 \\ &> \rho \ldots \text{ satisfactory} \end{aligned}$$

The minimum reinforcement ratio is given by ACI Section 7.12 as

$$\begin{aligned} \rho_{min} &= 0.18 \text{ percent of the gross area} \\ &< \rho \ldots \text{ satisfactory} \end{aligned}$$

Thus, the required reinforcement area is

$$\begin{aligned} A_s &= \rho bd \\ &= 0.0029 \times 60 \times 30 \\ &= 5.22 \text{ square inches} \end{aligned}$$

Provide nine No.7 bars to give an area of

$$\begin{aligned} A'_s &= 5.40 \text{ square inches} \\ &> A_s \ldots \text{ satisfactory} \end{aligned}$$

10.4.2 Truss analogy method

When the piles are symmetrically arranged about the column and do not exceed five in number, the truss analogy method may be used as shown in Figure 10–12. The pile cap is assumed to act as a three dimensional triangulated space truss. The upper node of the truss is located at the center of the column with the lower nodes at the centers of the piles. The column load is transmitted to the piles by direct thrust on inclined planes through the concrete with the reinforcement between the piles forming horizontal ties. The reinforcement must be placed

within a band width[5] not exceeding 3.0 times the width of the piles. Since the reinforcement is fully stressed at the center of the piles, a full anchorage length must be provided beyond this point.

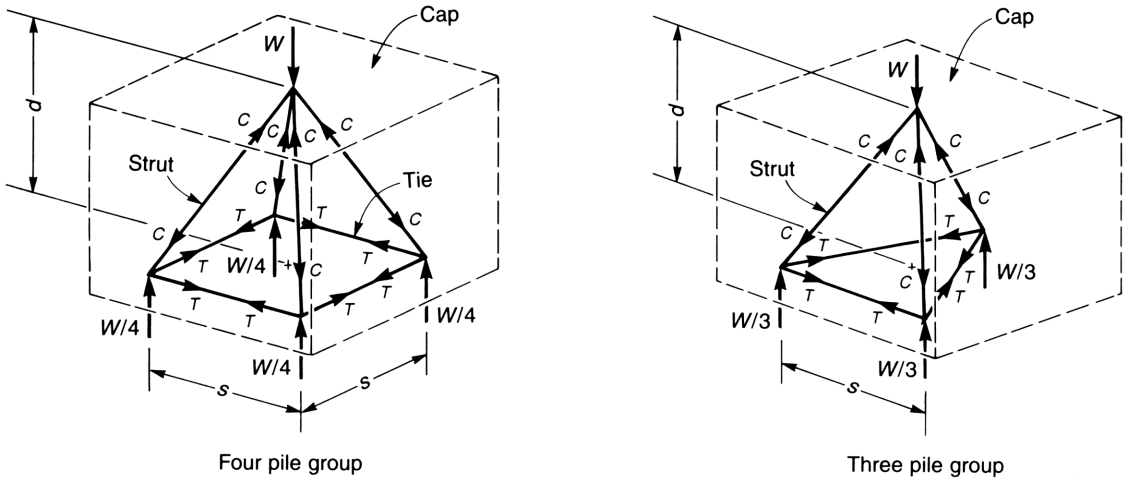

Figure 10–12 Pile cap truss analogy

By resolving forces at the lower nodes, the tensile force in the reinforcement of a four pile group is given by

$$T = Ws/8d$$

In a three pile group the tensile force is

$$T = Ws/9d$$

In a two pile group the tensile force is

$$T = Ws/4d$$

In a five pile group the tensile force is

$$T = Ws/10d$$

The area of reinforcement required is

$$A_s = T_u/\phi f_y$$

where ϕ = strength reduction factor for tension
= 0.90 from ACI Section 9.3
T_u = factored tie force
f_y = reinforcement yield strength
W = axial load in column
s = distance between pile centers.

Example 10–7 (Pile cap design by truss analogy)
Determine the area of Grade 60 reinforcement required in the pile cap shown in Figure 10–13 and check the shear capacity. The concrete strength is 3000 pounds per square inch.

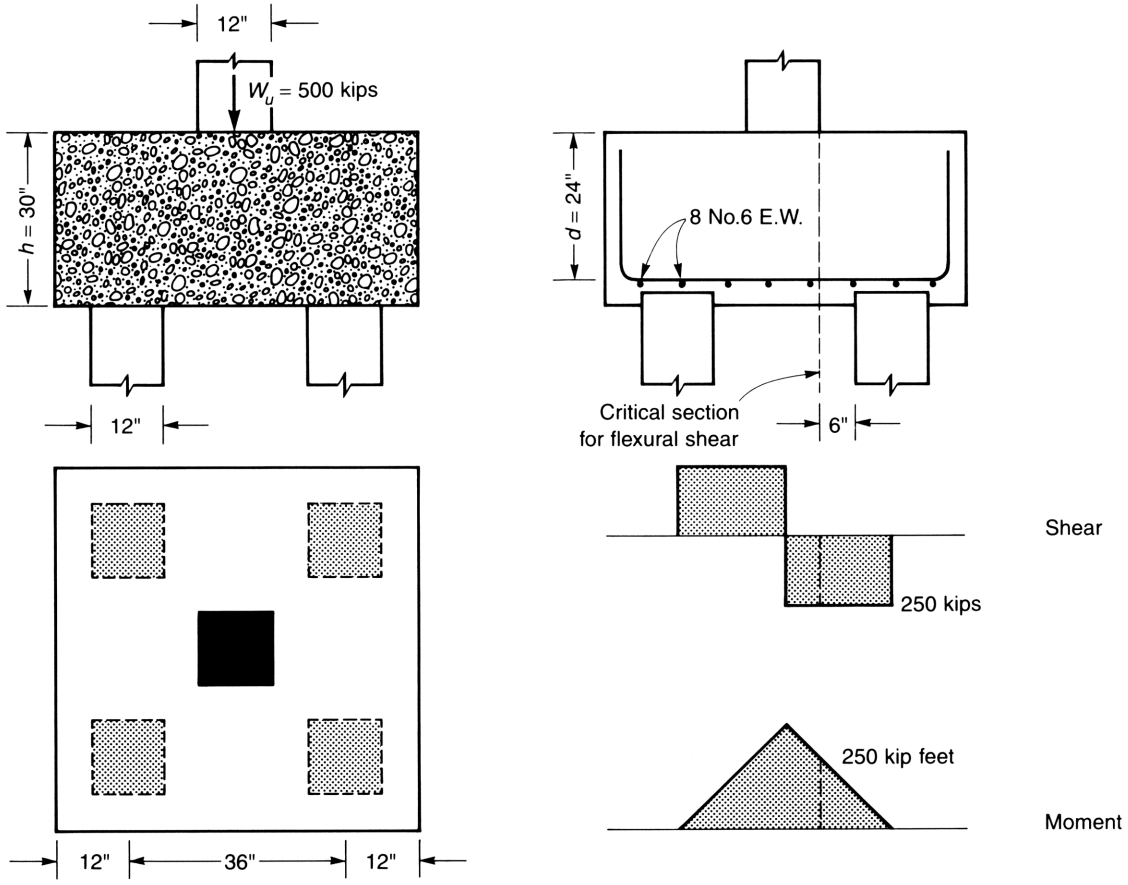

Figure 10–13 Details for Example 10–7

Solution
The force in each of the reinforcement ties is

$$\begin{aligned} T_u &= W_u s/8d \\ &= 500 \times 3/(8 \times 2) \\ &= 93.75 \text{ kips} \end{aligned}$$

The reinforcement area required for each tie is

$$\begin{aligned} A_s &= T_u/\phi f_y \\ &= 93.75/(0.9 \times 60) \\ &= 1.74 \text{ square inches} \end{aligned}$$

Since there are two ties in each direction, the total reinforcement area in each direction is

$$\begin{aligned} A_s &= 2 \times 1.74 \\ &= 3.48 \text{ square inches} \end{aligned}$$

Providing eight No.6 bars, as shown in Figure 10–13, gives an area of

$$\begin{aligned} A_s' &= 3.52 \text{ square inches} \\ &> A_s \ldots \text{ satisfactory} \end{aligned}$$

This gives a reinforcement ratio, based on the gross cross sectional area of

$$\begin{aligned} \rho' &= A_s'/bh \\ &= 3.52/(60 \times 30) \\ &= 0.0020 \end{aligned}$$

The minimum reinforcement area in a pile cap for Grade 60 bars, based on the gross cross sectional area, is given by ACI Section 7.12 as

$$\begin{aligned} \rho'_{min} &= 0.0018 \\ &< 0.0020 \ldots \text{ satisfactory} \end{aligned}$$

The ratio of the clear distance between piles to the effective depth is

$$\begin{aligned} \ell_n/d &= 2/2 \\ &= 1 \\ &< 5 \end{aligned}$$

Hence, in accordance with ACI Section 11.8.1 the pile cap may be classified as a deep flexural member and the critical section for flexural shear, is given by ACI Section 11.8.5, as a distance from the face of the pile

$$\begin{aligned} x &= a/2 \\ &= (1.5 - 0.5)/2 \\ &= 0.5 \text{ feet} \end{aligned}$$

The factored shear at this location, as shown in Figure 10–13, is

$$\begin{aligned} V_u &= W_u/2 \\ &= 500/2 \\ &= 250 \text{ kips} \end{aligned}$$

The factored moment at the same section, as shown in Figure 10–13, is

$$\begin{aligned} M_u &= W_u\,(x + 0.5)/2 \\ &= 500 \times 1.0/2 \\ &= 250 \text{ kip feet} \end{aligned}$$

The reinforcement ratio provided is

$$\begin{aligned} \rho &= A_s'/bd \\ &= 3.52/(60 \times 24) \\ &= 0.0024 \end{aligned}$$

The flexural shear capacity is obtained from ACI Equation (11–29) as

$$\begin{aligned}\phi V_c &= \phi(3.5 - 2.5M_u/V_u d)(1.9(f'_c)^{0.5} + 2500\rho V_u d/M_u)bd \\ &= 0.85(3.5-2.5\times250/250d)(1.9(3000)^{0.5}+2500\times0.0024\times250d/250)60d \\ &= 0.85 \times 2.25(104 + 12)1440 \\ &= 0.85 \times 261 \times 1440 \\ &= 319{,}464 \text{ pounds} \\ &> V_u \ldots \text{ satisfactory} \\ &< 6\phi bd(f'_c)^{0.5} \ldots \text{ satisfactory}\end{aligned}$$

Hence, the flexural shear capacity is adequate.

Punching shear for the column is not applicable since the piles are located within the critical perimeter of the column. The length of the critical perimeter for a corner pile is

$$\begin{aligned}b_o &= 2(18 + d/2) \\ &= 2(18 + 24/2) \\ &= 60 \text{ inches}\end{aligned}$$

The capacity of the pile cap for punching shear at a corner pile is governed by ACI Equation (11-37) which gives

$$\begin{aligned}\phi V_c &= 4\phi b_o d(f'_c)^{0.5} \\ &= 4 \times 0.85 \times 60 \times 24(3000)^{0.5} \\ &= 268{,}165 \text{ pounds} \\ &> W_u/4 \ldots \text{ satisfactory}\end{aligned}$$

Hence, the punching shear capacity of the pile cap is adequate.

10.5 SUPPLEMENTARY PROBLEMS

Problem 10–1 (Pile group with vertical piles and vertical load)
Figure 10–14 shows the loading acting on one line of a symmetrical group of piles supporting a structure. The vertical load indicated includes the self weight of the rigid pile cap. Determine the maximum axial force produced in a pile.

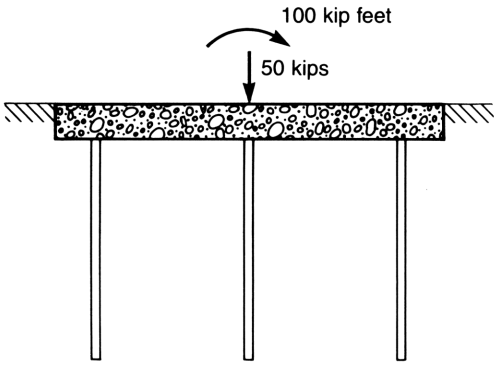

Figure 10–14 Details for Problem 10–1

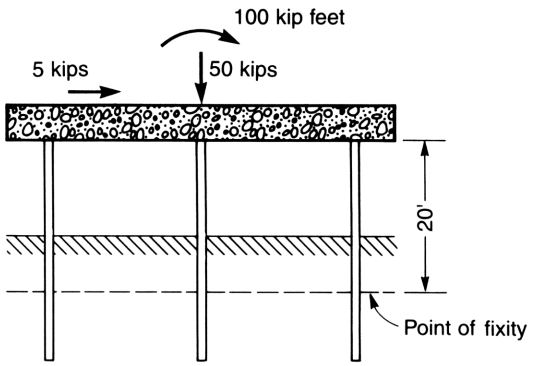

Figure 10–15 Details for Problem 10–2

Problem 10–2 (Pile group with vertical piles and vertical and lateral load)
Figure 10–15 shows the loading acting on one line of a group of symmetrical piles supporting a dock. The vertical load indicated includes the self weight of the rigid pile cap and the piles may be considered fixed at a depth of 20 feet below the bottom of the cap. Determine the shear force and maximum bending moment in each pile and the maximum axial force produced in a pile.

Problem 10–3 (Forces in raked piles without rotation of the pile cap)
Determine the forces, due to the indicated loads, in the raked piles shown in Figure 10–16. The piles have a batter of 1:3 and the line of action of the external loads passes through the elastic center.

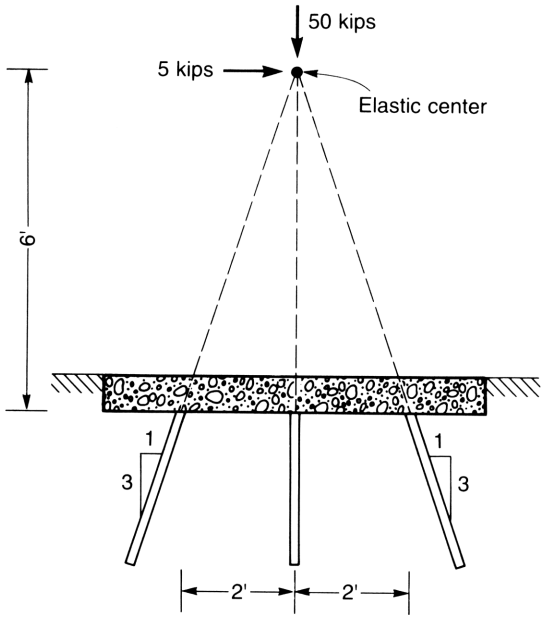

Figure 10–16 Details for Problem 10–3

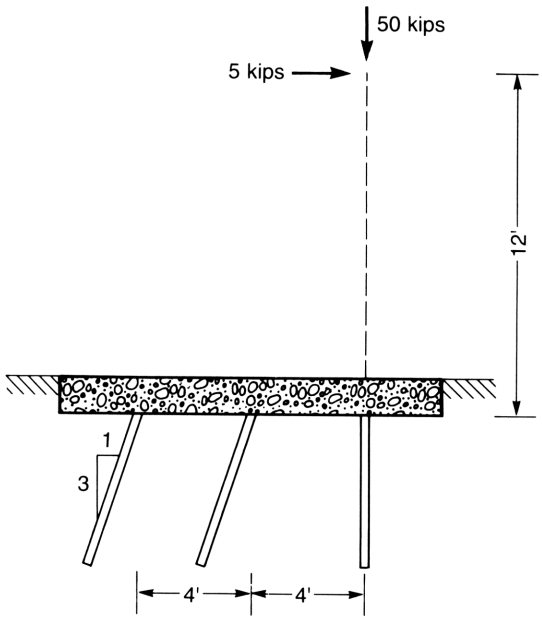

Figure 10–17 Details for Problem 10–4

Problem 10–4 (Forces in raked piles with rotation of the pile cap)
Determine the forces, due to the indicated loads, in the raked piles shown in Figure 10–17. The piles have a batter of 1:3 and have identical lengths and cross sectional areas.

Problem 10–5 (Pile cap design by truss analogy)
Determine the area of Grade 60 reinforcement required in the pile cap shown in Figure 10–18. The concrete strength is 3000 pounds per square inch, the effective depth of the pile cap is 24 inches, and the factored load on the column is 200 kips.

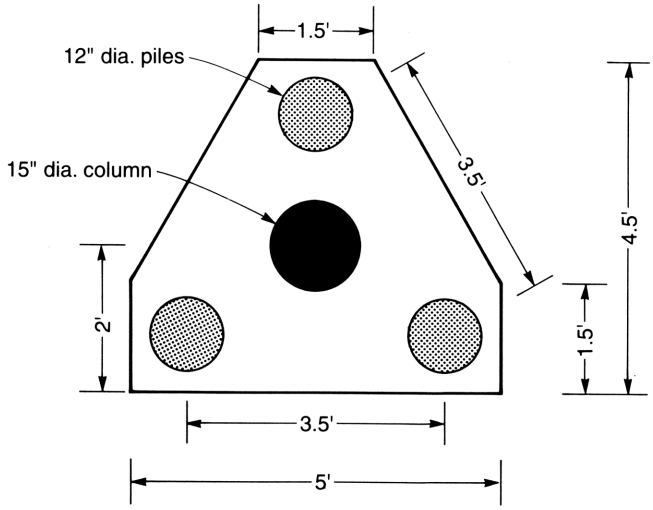

Figure 10–18 Details for Problem 10–5

References

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2. Adebar, P., Kuchma, D., Collins, M.P. Strut-and-tie models for the design of pile caps. *Proceedings American Concrete Institute,* Vol 87, No.1, 1990. pp 81–92.

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5. British Standards Institution. *BS 8110: Structural use of concrete.* London, 1985.

APPENDIX

Structural design programs for the HP-28S calculator

CONCRETE DESIGN PROGRAMS

A.1: TRD: Design reinforced concrete beam with tension reinforcement only. ... 364

A.2: TRA: Analysis of reinforced concrete beam with tension reinforcement only. ... 366

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A.5: TBD: Design of reinforced concrete T-beam. (Flange thickness less than depth of the equivalent rectangular stress block) ... 372

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A.1: TRD: DESIGN REINFORCED CONCRETE BEAM WITH TENSION REINFORCEMENT ONLY

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME ' CONC CRDIR	Create parent directory, CONC.
USER CONC ' TRD MEMORY CRDIR	Create subdirectory, TRD.
USER TRD	Open current directory, TRD.
≪ ' 12 * M/(B * D^2) ' EVAL ENTER ' K STO	Define the parameter, K. Store K.
≪ ' (1-√(1-2 * K/(.9 * .85 * FC))) * .85 * FC/FY ' EVAL ENTER ' R STO	Define the reinforcement ratio, R. (ie. ρ of reference 1). Store R.
≪ IF FC 4 < THEN .85 ELSE IF FC 8 > THEN .65 ELSE ' .85-(FC-4)/20 ' EVAL END END ENTER ' B1 STO	Define the factor, B1 (i.e. β_1 of reference 2). Store B1.
≪ ' .75 * .85 * 87 * B1 * FC/(FY * (87+FY)) ' EVAL ENTER ' RMAX STO	Define the maximum reinforcement ratio, RMAX (i.e. ρ_{max} of reference 3). Store RMAX.
≪ ' .2/FY ' EVAL ENTER ' RMIN STO	Define the minimum reinforcement ratio, RMIN (i.e. ρ_{min} of reference 4). Store RMIN.
≪ ' R * B * D ' EVAL ENTER ' AS STO	Define reinforcement area, AS. Store AS.
≪ ' .85 * 2 * B * D * √(1000 * FC)/1000 ' EVAL ENTER ' PHVC STO	Define concrete design shear strength, PHVC (i.e. ϕV_c of reference 5). Store PHVC.

≪ ' 12 * 50 * B/(1000 * FY) ' EVAL ENTER	Define minimum shear reinforcement AVS (ie. A_v/s of reference 6).
' AVS STO	Store AVS.
≫ {STO M FY FC B D} MENU ENTER	Set up an input menu for data entry
' DAT STO	Store the data menu

EXAMPLE

MEMORY HOME USER CONC TRD	Recall program TRD
DAT	Prepare data entry.
53.5 M	Required factored moment, kip ft.
60 FY	Reinforcement yield strength, ksi
3 FC	Concrete compressive strength, ksi.
12 B	Beam width, in.
14.4 D	Effective depth of reinforcement, in.
USER NEXT NEXT	Display output functions.
K	= 0.258 ksi
R	Reinforcement ratio, ρ = 0.0051
NEXT NEXT	
B1	Compression zone factor β_1 = 0.85
RMAX	Maximum reinforcement ratio, ρ_{max} = 0.016
RMIN	Minimum reinforcement ratio ρ_{min} = 0.0033
AS	Reinforcement area A_s = 0.878 in^2
PHVC	Concrete design shear strength, ϕV_c = 16.09 kip
AVS	Minimum required shear reinforcement A_v/s = 0.12 in^2/ft.

A.2: TRA: ANALYSIS OF REINFORCED CONCRETE BEAM WITH TENSION REINFORCEMENT ONLY

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' TRA MEMORY CRDIR	Create subdirectory, TRA.
USER TRA	Open current directory, TRA.
≪ ' AS/(B * D) ' EVAL ENTER ' R STO	Define the reinforcement ratio, R. Store R.
≪'.9 * AS * FY * D * (1-.59 * R * FY/ FC)/12 ' EVAL ENTER ' M STO	Define the design moment strength, M. (i.e ϕM_n of reference 1). Store M.
≪ IF FC 4 < THEN .85 ELSE IF FC 8 > THEN .65 ELSE ' .85-(FC-4)/20 ' EVAL END END ENTER ' B1 STO	Define the factor, B1 (i.e. β_1 of reference 2). Store B1.
≪ ' .75 * .85 * 87 * B1 * FC/(FY * (87+ FY)) ' EVAL ENTER ' RMAX STO	Define the maximum reinforcement ratio, RMAX (i.e. ρ_{max} of reference 3). Store RMAX.
≪ ' .2/FY ' EVAL ENTER ' RMIN STO	Define the minimum reinforcement ratio, RMIN (i.e. ρ_{min} of reference 4). Store RMIN
≪ ' .85 * 2 * B * D * √(1000 * FC)/1000 ' EVAL ENTER ' PHVC STO	Define concrete design shear strength, PHVC (i.e. ϕV_c of reference 5). Store PHVC
≪ ' 12 * 50 * B/(1000 * FY) ' EVAL ENTER ' AVS STO	Define minimum shear reinforcement, AVS (i.e. A_v/s of reference 6). Store AVS
≪ {STO AS B D FY FC} MENU ENTER ' DAT STO	Set up an input menu for data entry. Store the data menu.

EXAMPLE

MEMORY HOME USER CONC TRA	Recall program TRA.
DAT	Prepare for data entry.
.878 AS	Reinforcement area, in^2.
12 B	Beam width, in.
14.4 D	Effective depth of reinforcement, in.
60 FY	Reinforcement yield strength, ksi.
3 FC	Concrete compressive strength, ksi.
USER NEXT NEXT	Display output functions.
R	Reinforcement ratio, $\rho = 0.0051$.
NEXT NEXT	
M	Design moment strength $\phi M_n = 53.5$ kip ft.
B1	Compression zone factor, $\beta_1 = 0.85$
RMAX	Maximum reinforcement ratio, $\rho_{max} = 0.016$
RMIN	Minimum reinforcement ratio, $\rho_{min} = 0.0033$
PHVC	Concrete design shear strength, $\phi V_c = 16.09$ kip.
AVS	Minimum required shear reinforcement, $A_v/s = 0.12$ in^2/ft.
	The actual reinforcement ratio lies between the maximum and minimum allowable values and is satisfactory.

A.3: CRD: DESIGN REINFORCED CONCRETE BEAM WITH COMPRESSION REINFORCEMENT

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' CRD MEMORY CRDIR	Create subdirectory, CRD.
USER CRD	Open current directory, CRD.
≪ IF FC 4 < THEN .85 ELSE IF FC 8 > THEN .65 ELSE ' .85-(FC-4)/20 ' EVAL END END ENTER ' B1 STO	Define compression zone factor, B1 (i.e. β_1 of reference 2). Store B1.
≪ ' .75 * .85 * 87 * B1 * FC/(FY * (87+FY)) ' EVAL ENTER ' RMAX STO	Define the maximum reinforcement ratio for a singly reinforced section, RMAX Store RMAX.
≪ ' RMAX * B * D ' EVAL ENTER ' AMAX STO	Define the maximum reinforcement area for a singly reinforced section, AMAX. Store AMAX.
≪ ' .9 * AMAX * FY * D * (1-.59 * RMAX * FY/ FC)/12 ' EVAL ENTER ' MMAX STO	Define the maximum design moment strength for a singly reinforced section. Store MMAX.
≪ ' M-MMAX ' EVAL ENTER ' MR STO	Define the residual moment MR (i.e. factored applied moment - MMAX). Store MR.
≪ ' 12 * MR/(.9 * FY * (D-D1)) ' EVAL ENTER ' ASTA STO	Define the additional area of tension reinforcement required, ASTA. Store ASTA.
≪ ' AMAX + ASTA ' EVAL ENTER ' AST STO	Define the total required area of tension reinforcement, AST. Store AST

```
≪ ' FY * AMAX/(0.85 * FC * B) ' EVAL
ENTER
```
Define the stress block depth A by equating tensile force at balanced strain conditions to the total compressive force.

```
' A STO
```
Store A.

```
≪ ' A/B1 ' EVAL ENTER
' C STO
```
Define the neutral axis depth, C.
Store C.

```
≪ IF ' 87 * (1-D1/C) ' EVAL FY >
THEN FY ELSE ' 87 * (1-D1/C) ' EVAL
END ENTER
```
Define the stress in the compression reinforcement, FSC.

```
' FSC STO
```
Store FSC.

```
≪ ' ASTA * FY/FSC ' EVAL ENTER
```
Define the area of compression reinforcement, ASC.

```
' ASC STO
```
Store ASC.

```
≪ ' .85 * 2 * B * D * √(1000 * FC)/1000 '
EVAL ENTER
' PHVC STO
```
Define concrete design shear strength, PHVC (i.e. ϕV_c of reference 5).
Store PHVC.

```
≪ ' 12 * 50 * B/(1000 * FY) ' EVAL
ENTER
' AVS STO
```
Define minimum shear reinforcement AVS (i.e. A_v/s of reference 6).
Store AVS.

```
≪ {STO M FY FC B D D1} MENU
ENTER
' DAT STO
```
Set up an input menu for data entry.

Store the data menu.

EXAMPLE

```
MEMORY HOME USER CONC CRD
NEXT DAT
```
Recall program CRD
Prepare for data entry.

```
190  M
60   FY
5    FC
12   B
12   D
2.4  D1
USER NEXT NEXT
B1
```
Required factored moment, kip ft.
Reinforcement yield strength, ksi.
Concrete compressive strength, ksi.
Beam width, in.
Effective depth of reinforcement, in.
Depth to compression reinforcement, in.
Display output functions.
Compression zone factor, $\beta_1 = 0.80$.

RMAX	Maximum reinforcement ratio, ρ_{max} = 0.025
AMAX	Maximum tension reinforcement area for singly reinforced beam = 3.62 in^2
MMAX	Maximum design moment strength for a singly reinforced section = 160.76 kip ft.
MR	Residual moment = 29.24 kip ft.
NEXT NEXT	
ASTA	Additional tension reinforcement area = 0.68 in^2
AST	Tension reinforcement area = 4.30 in^2
A	Stress block depth = 4.26 in
C	Neutral axis depth = 5.33 in
FSC	Stress in compression reinforcement = 47.80 ksi.
ASC	Compression reinforcement area = 0.85 in^2
PHVC	Concrete design shear strength, ϕV_c = 17.31 kip.
AVS	Minimum required shear reinforcement, A_v/s = 0.12 in^2/ft.

A.4: CRA: ANALYSIS OF REINFORCED CONCRETE BEAM WITH COMPRESSION REINFORCEMENT

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' CRA MEMORY CRDIR	Create subdirectory, CRA.
USER CRA	Open current directory, CRA.
≪ IF FC 4 < THEN .85 ELSE IF FC 8 > THEN .65 ELSE ' .85-(FC-4)/20 ' EVAL END END ENTER	Define compression zone factor, B1 (i.e. β_1 of reference 2)
' B1 STO	Store B1

```
« IF ' 87 * (1-D1/C) ' EVAL FY >
THEN FY ELSE ' 87 * (1-D1/C) ' EVAL
END ENTER
' FSC STO
```

Define the stress in compression reinforcement, FSC (ie. f'_{sb} of reference 1).
Store FSC.

```
« ' 75 * .85 * 87 * B1 * FC/(FY * (87+
FY))+ASC * FSC/(B * D * FY) ' EVAL
ENTER
' RMAX STO
```

Define the maximum reinforcement ratio, RMAX (i.e. ρ_{max} of reference 3).

Store RMAX

```
« ' AST/(B * D) ' EVAL ENTER
' R STO
```

Define the reinforcement ratio, R.
Store R.

```
« ' .85 * FC * B * C * B1 ' EVAL
ENTER
' CC STO
```

Define the compressive force in the concrete compression zone, CC.
Store CC.

```
« ' ASC * FSC ' EVAL ENTER

' CS STO
```

Define the compressive force in the compression reinforcement, CS.
Store CS.

```
« ' AST * FY ' EVAL ENTER

' T STO
```

Define the tensile force, T, in the tension reinforcement.
Store T.

```
« ' .9 * (CC * (D-B1 * C/2)+CS *
(D-D1))/12 ' EVAL ENTER
' M STO
```

Define the design moment strength, M(i.e. ϕM_n of reference 1).
Store M.

```
« ' .85 * 2 * B * D * √(1000 * FC)/1000 '
EVAL ENTER
' PHVC STO
```

Define concrete design shear strength, PHVC (i.e. ϕV_c of reference 5)
Store PHVC

```
« ' 12 * 50 * B/(1000 * FY) ' EVAL
ENTER
' AVS STO
```

Define minimum shear reinforcement, AVS (i.e. A_v/s of reference 6).
Store AVS.

```
« {STO AST ASC FY FC B D D1 C}
MENU ENTER
' DAT STO
```

Set up an input menu for data entry.

Store the data menu.

EXAMPLE

MEMORY HOME USER CONC CRA	Recall program CRA
NEXT DAT	Prepare for data entry.
9.36 AST	Tension reinforcement area, in^2
6.24 ASC	Compression reinforcement area in^2
60 FY	Reinforcement yield strength, ksi.
4 FC	Concrete compressive strength, ksi.
24 B	Beam width, in.
27 D	Effective depth of reinforcement, in.
NEXT	
3 D1	Depth to compression reinforcement.
4.98 C	Initial estimate of depth to neutral axis, in.
USER NEXT NEXT NEXT	Display output functions.
B1	Compression zone factor, $\beta_1 = 0.85$.
NEXT NEXT NEXT	
FSC	Compression reinforcement stress =34.59 ksi
RMAX	Maximum reinforcement ratio,ρ_{max} = 0.0269
R	Reinforcement ratio, $\rho = 0.014 < \rho_{max}$...ok
CC	Compressive force in the concrete compressive zone = 345.4 kip.
CS	Compressive force in the compression reinforcement = 215.8 kip.
T	Tensile force in the tension reinforcement = 561.6 kip $\approx$ CC + CS Hence, the initial estimate for C is correct.
NEXT NEXT NEXT	
M	Design moment strength,ϕM_n=1033 kip ft
PHVC	Concrete design shear strength, ϕV_c = 69.67 kip.
AVS	Minimum required shear reinforcement, $A_v/s = 0.24$ in^2/ft.

A.5: TBD: DESIGN OF REINFORCED CONCRETE T-BEAM

(Note: Assumes flange thickness is less than depth of the equivalent rectangular stress block)

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' TBD MEMORY CRDIR	Create subdirectory, TBD.
USER TBD	Open current directory, TBD.
≪ ' .85 * FC * (B-BW) * HF ' EVAL ENTER ' CF STO	Define the compressive force in the flange, CF (i.e. C_f of reference 1). Store CF
≪ ' CF/FY ' EVAL ENTER	Define the reinforcement area, ASF, required to balance the flange force (i.e. A_{sf} of reference 1).
' ASF STO	Store ASF.
≪ ' .9 * ASF * FY * (D-.5 * HF)/12 ' EVAL ENTER ' MF STO	Define the design moment strength, MF, due to ASF. Store MF.
≪ ' M-MF ' EVAL ENTER	Define the residual moment MW (i.e. the factored applied moment - MF)
' MW STO	Store MW.
≪ ' 12 × MW/(BW * D^2) ' EVAL ENTER ' KW STO	Define the parameter, KW. Store KW.
≪ ' (1-√(1-2 * KW/(.9 * .85 * FC))) * .85 * FC/FY ' EVAL ENTER ' RW STO	Define the web reinforcement ratio, RW. Store RW.
≪ ' RW * BW * D ' EVAL ENTER	Define the reinforcement area, ASW required to resist the residual moment.
' ASW STO	Store ASW.

```
≪ ' ASF + ASW ' EVAL ENTER
```
Define the total required area of tension reinforcement, AS.

```
' AS STO
```
Store AS.

```
≪ IF FC 4 < THEN .85 ELSE IF FC 8
> THEN .65 ELSE ' .85-(FC-4)/20 '
EVAL END END ENTER
' B1 STO
```
Define the factor, B1 (i.e. β_1 of reference 2).

Store B1.

```
≪ ' .85 * 87 * B1 * FC/(FY * (87+FY)) '
EVAL ENTER
' RB STO
```
Define the web balanced reinforcement ratio, RB (i.e. $\overline{\rho}_b$ of reference 3).
Store RB.

```
≪ ' ASF/(BW * D) ' EVAL ENTER
```
Define the reinforcement ratio, RF, due to ASF (i.e. ρ_f of reference 1).

```
' RF STO
```
Store RF.

```
≪ ' .75 * BW * (RB+RF)/B ' EVAL
ENTER
' RMAX STO
```
Define the maximum reinforcement ratio, RMAX (i.e. ρ_{max} of reference 3).
Store RMAX

```
≪ ' AS/(B * D) ' EVAL ENTER
```
Define the reinforcement ratio, R(i.e. ρ of reference 1).

```
' R STO
```
Store R.

```
≪ ' ASW * FY/(.85 * FC * BW) ' EVAL
ENTER
```
Define the depth of the equivalent rectangular stress block, A (i.e. a of reference 2).

```
' A STO
```
Store A.

```
≪ ' .85 * 2 * BW * D * √(1000 * FC)/
1000 ' EVAL ENTER
' PHVC STO
```
Define concrete design shear strength, PHVC (i.e. ϕV_c of reference 5).
Store PHVC.

```
≪ ' 12 * 50 * BW/(1000 * FY) ' EVAL
ENTER
' AVS STO
```
Define the minimum shear reinforcement, AVS (i.e. A_v/s of reference 6).
Store AVS.

```
≪ {STO M FY FC B BW D HF} MENU
ENTER
' DAT STO
```
Set up an input menu for data entry.

Store the data menu.

EXAMPLE

MEMORY HOME USER CONC TBD	Recall program TBD.
NEXT DAT	Prepare for data entry.
400 M	Required factored moment, kip ft.
60 FY	Reinforcement yield strength, ksi.
4 FC	Concrete compressive strength, ksi.
30 B	Flange width, in.
10 BW	Web width, in.
19 D	Effective depth of reinforcement, in.
NEXT	
2.5 HF	Flange depth, in.
USER NEXT NEXT NEXT	Display output functions.
CF	Flange compressive force C_f = 170 kip.
ASF	Reinforcement area, A_{sf} = 2.83 in^2
MF	Design moment due to A_{sf} = 226 kip ft.
MW	Residual moment = 174 kip ft.
KW	KW = 0.577
RW	Web reinforcement ratio = 0.012
NEXT NEXT NEXT	
ASW	Reinforcement area, ASW = 2.27 in^2
AS	Total reinforcement area = 5.1 in^2
B1	Compression zone factor, β_1 = 0.85
RB	Reinforcement ratio, $\bar{\rho}_b$ = 0.0285
RF	Reinforcement ratio, ρ_f = 0.0149
RMAX	Reinforcement ratio, ρ_{max} = 0.0109
NEXT NEXT NEXT	
R	Actual reinforcement ratio, ρ = 0.0090 $< \rho_{max}$ .....ok
A	Depth of stress block, a = 4.01" > h_f ...ok
PHVC	Concrete design shear strength ϕV_c = 20.43 kip.
AVS	Minimum required shear reinforcement A_v /s = 0.0996 in^2 / ft.

A.6: TBA: ANALYSIS OF REINFORCED CONCRETE T-BEAM

(Note: Assumes flange thickness is less than depth of the equivalent rectangular stress block)

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' TBA MEMORY CRDIR	Create subdirectory, TBA.
USER TBA	Open current directory, TBA
≪ ' .85 * FC * (B-BW) * HF ' EVAL ENTER ' CF STO	Define the compressive force in the flange, CF(i.e. C_f of reference 1). Store CF.
≪ ' CF/FY ' EVAL ENTER	Define the reinforcement area, ASF, required to balance the flange force (i.e. A_{st} of reference 1).
' ASF STO	Store ASF.
≪ ' .9 * ASF * FY * (D-.5 * HF)/12 ' EVAL ENTER ' MF STO	Define the design moment strength, MF, due to ASF. Store MF.
≪ ' AS - ASF ' EVAL ENTER	Define the reinforcement area, ASW, required to resist the residual moment.
' ASW STO	Store ASW.
≪ ' ASW/(BW * D) ' EVAL ENTER	Define the web reinforcement ratio, RW, required to resist the residual moment.
' RW STO	Store RW.
≪ ' .9 * ASW * FY * D * (1-.59 * RW * FY/FC)/12 ' EVAL ENTER	Define the residual moment, MW.
' MW STO	Store MW.
≪ ' MF + MW ' EVAL ENTER	Define the design moment strength, M. (i.e.ϕM_n of reference 1).
' M STO	Store M

```
« IF FC 4 < THEN .85 ELSE IF FC 8
> THEN .65 ELSE ' .85-(FC-4)/20 '
EVAL END END ENTER
' B1 STO
```

Define the factor, B1 (i.e. β_1 of reference 2).

Store B1.

```
« ' .85 * 87 * B1 * FC/(FY * (87+FY)) '
EVAL ENTER
' RB STO
```

Define the web balanced reinforcement ratio, RB(i.e. $\overline{\rho}_b$ of reference 3).
Store RB.

```
« ' ASF/(BW * D) ' EVAL ENTER

' RF STO
```

Define the reinforcement ratio, RF, due to ASF (i.e. ρ_f of reference 1).
Store RF.

```
« ' .75 * BW * (RB+RF)/B ' EVAL
ENTER
' RMAX STO
```

Define the maximum reinforcement ratio, RMAX (i.e. ρ_{max} of reference 3).
Store RMAX.

```
« ' AS/(B * D) ' EVAL ENTER

' R STO
```

Define the reinforcement ratio, R(i.e. ρ of reference 1).
Store R.

```
« ' RW  * FY * D/(.85 * FC) ' EVAL
ENTER

' A STO
```

Define the depth of the equivalent rectangular stress block, A (i.e. a of reference 2).
Store A

```
« ' .85 * 2 * BW * D * √(1000 * FC) '
EVAL ENTER
' PHVC STO
```

Define concrete design shear strength, PHVC (i.e. A_v/s of reference 6).
Store PHVC

```
« ' 12 * 50 * BW/(1000 * FC) ' EVAL
ENTER
' AVS STO
```

Define the minimum shear reinforcement, AVS (i.e. A_v/s of reference 6).
Store AVS.

```
« {STO AS FY FC B BW D HF} MENU
ENTER
' DAT STO
```

Set up an input menu for the data entry.

Store the data menu

EXAMPLE

MEMORY HOME USER CONC TBA	Recall program TBA.
NEXT DAT	Prepare for data entry.
5 AS	Reinforcement area, in^2.
60 FY	Reinforcement yield strength, ksi.
4 FC	Concrete compressive strength, ksi.
30 B	Flange width, in.
10 BW	Web width, in.
19 D	Effective depth of reinforcement, in.
NEXT	
2.5 HF	Flange depth, in.
USER NEXT NEXT NEXT	Display output functions.
CF	Flange compressive force, C_f = 170 kip.
ASF	Reinforcement area, A_{sf} = 2.83 in^2.
MF	Design moment due to A_{sf} = 226 kip ft.
ASW	Reinforcement area, ASW = 2.17 in^2.
RW	Web reinforcement ratio = 0.011.
MW	Residual moment = 167 kip ft.
NEXT NEXT NEXT	
M	Design moment strength, ϕM_n = 393 kip ft.
B1	Compression zone factor, β_1 = 0.85.
RB	Reinforcement ratio, ρ_b = 0.0285.
RF	Reinforcement ratio, ρ_f = 0.0149.
RMAX	Reinforcement ratio, ρ_{max} = 0.0109
R	Actual reinforcement ratio, ρ = 0.00877 < ρ_{max} .....ok.
NEXT NEXT NEXT	
A	Depth of stress block, a = 3.82 in > h_f ....ok.
PHVC	Concrete design shear strength, ϕV_c = 20.43 kip.
AVS	Minimum required shear reinforcement, A_v/s = 0.0996 in^2/ft

A.7: DEF: DEFLECTION OF A SIMPLY SUPPORTED REINFORCED CONCRETE BEAM WITH UNIFORMLY DISTRIBUTED LOADING

PROGRAM LISTING:

KEYSTROKES	COMMENTARY
MEMORY HOME USER CONC	Open parent directory, CONC.
' DEF MEMORY CRDIR	Create subdirectory, DEF.
USER DEF	Open current directory, DEF.
≪ ' 33 * (WC^3 * FC)^0.5/1000' EVAL ENTER ' EC STO	Define the modulus of elasticity of the concrete EC (i.e. E_c of reference 7). Store EC.
≪ ' ES/EC ' EVAL ENTER ' N STO	Define the modular ratio N (i.e. n of reference 8). Store N.
≪ ' AS/(B * D) ' EVAL ENTER ' P STO	Define the reinforcement ratio P (i.e. ρ of reference 9) Store P.
≪ ' √(2 * N * P+P^2 * N^2)-N * P ' EVAL ENTER ' K STO	Define the neutral axis depth ratio K. Store K.
≪ ' B * (K * D)^3/3+N * AS * (D-K * D)^2 ' EVAL ENTER ' IR STO	Define the moment of inertia of the cracked section IR (i.e. I_{cr} of reference 10). Store IR.
≪ ' B * H^3/12 ' EVAL ENTER ' IG STO	Define the moment of inertia of the gross section IG (i.e. I_g of reference 10). Store IG.
≪ ' 7.5 * FC^0.5 ' EVAL ENTER ' FR STO	Define the modulus of rupture FR (i.e. f_r of reference 10) Store FR

≪ ' 2 * FR * IG/(H * 12000) ' EVAL ENTER	Define the cracking moment MR (i.e. M_{cr} of reference 10)
' MR STO	Store MR.
≪ ' (MR/MA)^3 * IG +(1-(MR/MA)^3) * IR ' EVAL ENTER	Define the effective moment of inertia IE (i.e. I_e of reference 10)
' IE STO	Store IE.
≪ ' 180 * MA * L^2/(EC * IE) ' EVAL ENTER	Define the deflection DEF
' DEF STO	Store DEF.
≪ {STO L B D H AS ES FC WC MA} MENU ENTER	Set up an input menu for data entry.
' DAT STO	Store the data menu.

EXAMPLE

MEMORY HOME USER CONC DEF	Recall program DEF
NEXT DAT	Prepare for data entry.
15 L	Span length, ft.
12 B	Beam width, in.
16 D	Effective depth of reinforcement, in.
18 H	Beam overall depth, in.
3.81 AS	Area of reinforcement, in^2.
29,000 ES	Modulus of elasticity of reinforcement, kip/in^2.
3000 FC	Concrete compressive strength, psi.
150 WC	Unit weight of concrete, pcf.
93.09 MA	Maximum bending moment, kip ft.
USER NEXT NEXT NEXT	Display output functions.
EC	Modulus of elasticity of concrete = 3320 kip/in^2.
N	Modular ratio = 8.73
NEXT NEXT NEXT	
P	Reinforcement ratio = 0.0198
K	Neutral axis depth ratio = 0.44
IR	Moment of inertia of cracked section = 4067 in^4.

IG	Moment of inertia of gross concrete section = 5832 in^4.
FR	Concrete modulus of rupture = 411 psi
MR	Cracking moment = 22.18 kip ft.
NEXT NEXT NEXT	
IE	Effective moment of inertia = 4091 in^4.
DEF	Short term deflection = 0.28 in.

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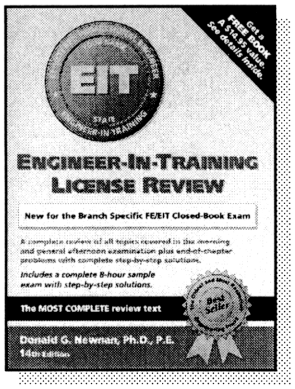

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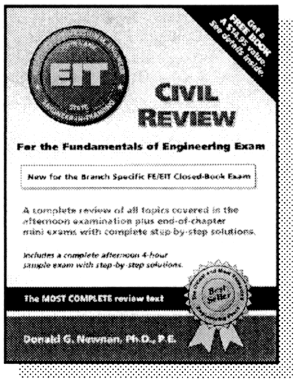

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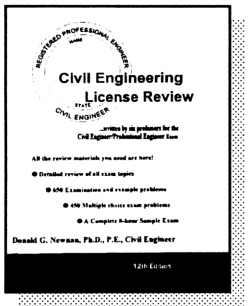

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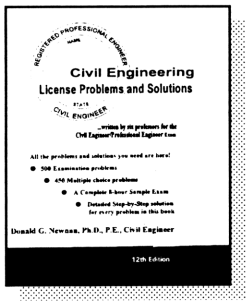

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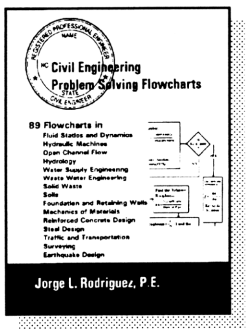

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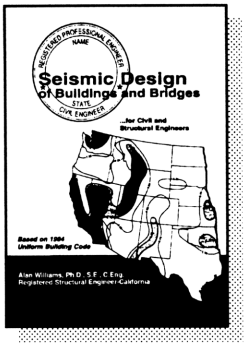

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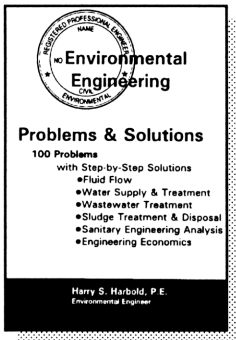

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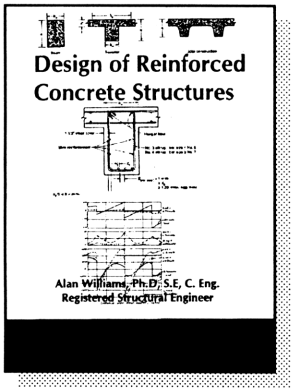

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