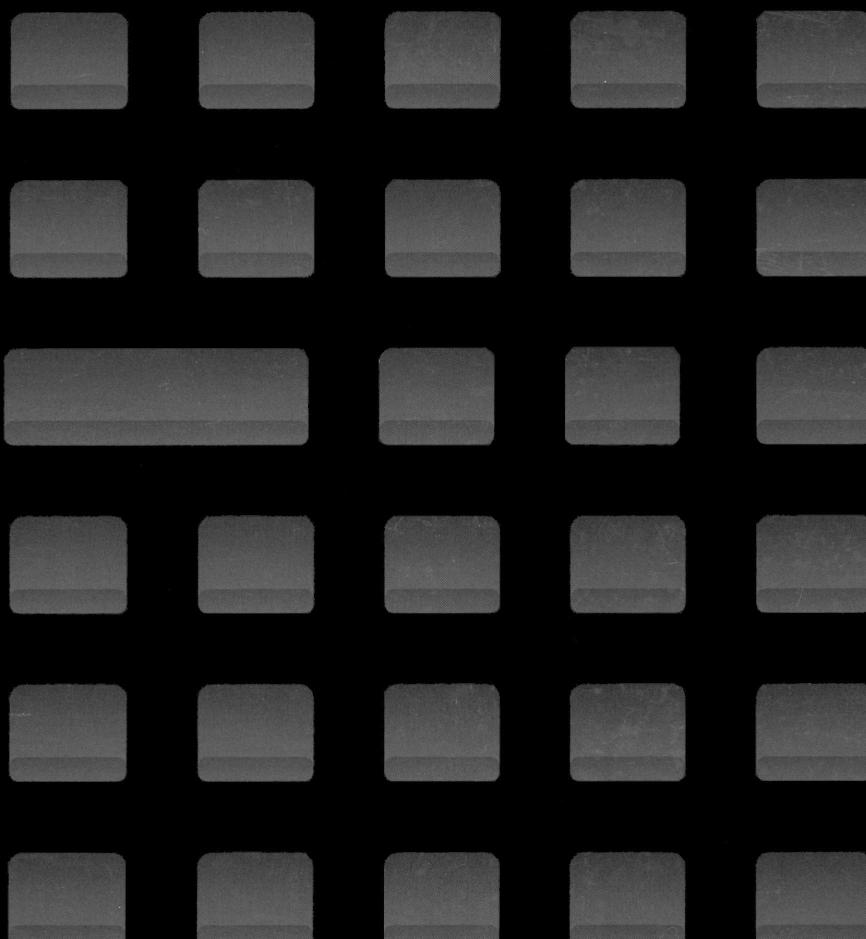
An Easy Course in

Financial Calculations

Plain talk on the how's and why's of
"doing the numbers" in borrowing or investing



by Chris Coffin

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In Memory of Janet Cryer

Contents

0. Why a Book on "Finance?"

Key points to be covered here:

- "Finance" here means borrowing or lending money at interest.
- The interest calculations are what introduce the complexities.

You'll hear the word "finance" used in all sorts of ways. In its most general sense, to ***finance*** something simply means to pay for it, or to provide for its funding.

But that's too broad for the purposes of this book. You don't need a book (or anything more complicated than grade-school arithmetic) simply to figure out if you can pay for something. You just add up what's in your wallet and your bank account(s). If you have enough money, you can finance it yourself. If you don't, you need someone else to help you, by giving or lending you the difference.

Gifts are nice, of course, although they're rare. But you can usually find someone willing to lend you money—if you promise to pay it back, with interest. That is, you can usually ***rent*** money.

That's the sense of the word being used here: *Financing is the "renting" of money—i.e. borrowing or lending it at interest.* And it's the calculation of that interest that produces all the involved math, which is why you're here: The purpose of this book is to help you understand how interest works and how to set up and work a financial calculation—anything involving borrowing, lending and interest.

Of course, you don't have to do the actual math longhand these days. That's what a calculator is for (and the terminology and methods you see here will definitely help you use and understand your financial calculator, no matter what make or model you have). But until someone invents a mind-reading device, no technology will give you the right answers until you ask it the right questions.

This book is a short course in asking the right questions.

The first questions to ask are about interest and how it works....

1. How Interest Works

Key points to be covered here:

- Interest is rent paid on borrowed money.
- The rent is to compensate a lender for his/her opportunity cost.
- Past-due interest must therefore compound.
- Compound interest calculations are not simple, because the compounding math uses exponents (powers).
- Interest rent is due and payable at different times for different forms of rental (i.e. different types of loans).
- The period of compounding is critical: for a given annualized rate, faster compounding produces more interest.

Why Interest Isn't Simple

Interest is the rent you pay to use someone else's money for awhile.

Rent is a very simple concept. So why should renting money be any more complicated than, say, renting a car? "For this model car, it's such-and-such rent per day." "For this much loan, it's such-and-such interest per day." What's the big deal?

To answer that, consider first why you pay rent at all—on anything. Why should the harmless use of someone else's property (assuming you return it in the same condition) cost you anything?

If your neighbor isn't using his car today, it's no big deal if you use it instead, and he isn't going to ask you for any rent on it. But what if he needs it that day, too—for his own livelihood? Your having his car costs him the time and expense necessary to go borrow, buy or rent another one. Either that or he must give up a day's earnings.

Rent is the price you pay to compensate the lender for doing without his own property while you use it.

The fancy term for this is ***opportunity cost***. While a lender is without his property, he loses the opportunity to use it for his own benefit. You, the borrower, reimburse him that cost by paying rent.

The amount you pay depends on the benefit the lender is foregoing—and for how long. A neighbor who isn't using his car today probably doesn't expect to get a lot of benefit from it sitting in his garage—no skin off his nose if you borrow it rent-free. But if he's an on-call doctor or salesman—or a car rental agent—not having his car for the day could cost him a lot. And two days could cost him twice as much, etc., so you really must expect to compensate him for that cost.

So, what about the rental of money, rather than a car?

Like a car, money is just another form of property, with similar limits on casual lending. Though your neighbor might be willing to lend you, say, $50 until next payday, no strings attached. But he probably won't lend you $5,000, interest free, until this time next year, because he'd miss that much money over that much time. That is, he'd feel the opportunity cost of doing without his property, and rightly would expect you to compensate him by paying him rent on it—interest.

Of course, businesses aren't like neighbors. A bank won't lend you $50 interest-free, even for day—just as a car rental agency won't lend you even a Yugo for a day for free—because their very reason for existence is to earn income from their property (even small portions) —all the time (even short intervals). For commercial lenders, property is never considered "idle" in a garage or wallet. Every day you borrow it is a day they could have rented it instead to someone else; the rent you pay reflects what another borrower would have paid.

But there is one big difference in the way money is rented as compared to a car: the way the rent (interest) is charged.

To see the difference, first consider the rare occasions where the two situations are actually alike. Take one of those usurious little household deals a kid might cut with a brother or sister: "I'll lend you $5, but you'll owe me 5 cents for every day until you pay it all back."

In its terms (though not its rates) that deal sounds just like a car rental agency: "We'll lend you this car, but you'll owe us $50 for every day until you return it."

The key thing to notice is that the amount of interest charged doesn't depend on when you pay it—only on how much time elapses before you fully repay the principal. You can wait and settle up all at the end, paying all principal and interest together, or you can fork over each day's interest as it happens. Either way, it's a straight "so much per day" multiplication—easy to remember and easy to compute via simple arithmetic. That's why it's called ***simple interest***.

You certainly won't find many commercial money loans that work like that, though. Any interest owed to a money lender represents more potential loans (more "cars on his lot" that could be rented out). So he wants it paid right as it's charged—every period. If you don't pay it then, he'll treat this as if you had but then he immediately "re-lent" it back to you—a fresh little loan in itself. Thus, any interest that's not paid right when it's due begins to earn interest "on itself"—***compound interest***.

Again: With simple interest, the timing of the interest payments does not affect the amount of interest charged. With compound interest, that timing is all-important.

Compare in the charts here how these two forms of interest would treat a hypothetical loan. The loan is $100, at an interest rate of 1% per month for 12 months, with the option either to pay the interest as you go or defer it—pay it all at the end, along with the principal.

As you look at the charts, notice a few things:

- What does 1% per month mean in each form of interest? with simple interest, the 1% is applied only to the original principal amount. In compound interest, the 1% applies to all money currently owed—principal plus any unpaid interest.
- As long as compound interest is paid as it's charged, it never has a chance to start compounding (i.e. "earning interest on itself"). So in the pay-as-you-go scenario, there's no effective difference between simple and compound interest. This is not true in the deferred interest payment scenarios, as you can see by the difference in the final lump-sum payments: $112.00 vs. $112.68.
- Notice the upward acceleration of the balance in the compound interest case. Interest earning interest on itself is a classic example of exponential growth (what Einstein called the most powerful force in the universe). In such growth, the math involves exponents—"powers" of numbers, where you can no longer rely on simple, in-your-head, per-diem multiplication such a car rental.
- As you can see, with exponential growth, you also get more digits in your numbers—down to fractions of cents. This introduces further questions: How many digits to carry? How and when to round off to pennies for the purposes of real-world payment? Life gets complicated with compound interest.

Simple Interest

	"Pay the interest as you go"				"Pay the interest all at the end"			
Mo.	Start. balance	Interest charged	Pmt. made	End. balance	Start. balance	Interest charged	Pmt. made	End. balance
1	100.00	1.00	1.00	100.00	100.00	1.00	0.00	101.00
2	100.00	1.00	1.00	100.00	101.00	1.00	0.00	102.00
3	100.00	1.00	1.00	100.00	102.00	1.00	0.00	103.00
4	100.00	1.00	1.00	100.00	103.00	1.00	0.00	104.00
5	100.00	1.00	1.00	100.00	104.00	1.00	0.00	105.00
6	100.00	1.00	1.00	100.00	105.00	1.00	0.00	106.00
7	100.00	1.00	1.00	100.00	106.00	1.00	0.00	107.00
8	100.00	1.00	1.00	100.00	107.00	1.00	0.00	108.00
9	100.00	1.00	1.00	100.00	108.00	1.00	0.00	109.00
10	100.00	1.00	1.00	100.00	109.00	1.00	0.00	110.00
11	100.00	1.00	1.00	100.00	110.00	1.00	0.00	111.00
12	100.00	1.00	101.00	0.00	111.00	1.00	112.00	0.00

Compound Interest

	"Pay the interest as you go"				"Pay the interest all at the end"			
Mo.	Start. balance	Interest charged	Pmt. made	End. balance	Start. balance	Interest charged	Pmt. made	End. balance
1	100.00	1.00	1.00	100.00	100.00	1.000	0.00	101.00
2	100.00	1.00	1.00	100.00	101.00	1.010	0.00	102.01
3	100.00	1.00	1.00	100.00	102.01	1.020	0.00	103.03
4	100.00	1.00	1.00	100.00	103.03	1.030	0.00	104.06
5	100.00	1.00	1.00	100.00	104.06	1.041	0.00	105.10
6	100.00	1.00	1.00	100.00	105.10	1.051	0.00	106.15
7	100.00	1.00	1.00	100.00	106.15	1.062	0.00	107.21
8	100.00	1.00	1.00	100.00	107.21	1.072	0.00	108.29
9	100.00	1.00	1.00	100.00	108.28	1.083	0.00	109.37
10	100.00	1.00	1.00	100.00	109.36	1.094	0.00	110.46
11	100.00	1.00	1.00	100.00	110.46	1.105	0.00	111.57
12	100.00	1.00	101.00	0.00	111.56	1.116	112.68	0.00

Counting the Opportunity Cost

With compound interest, the math is harder, admittedly—but not so much as to outweigh its benefits. Compound interest is the convention nowadays because it reflects one real-world fact that simple interest does not: As soon as one "rental" period has past, any unpaid interest is also lost opportunity, because money is nearly always rentable. Compound interest accounts for this; simple interest does not.

This is relevant for everyone, not just for banks and other commercial lenders. You become a lender whenever you put money into an interest-bearing account—a "24/7" opportunity these days. Any de-lay in getting your money in there really costs you. You may not wor-ry all that much about the pennies, but it's true.

Opportunity cost may be present in purchase transactions, too. When you pay cash for groceries, for example, it's an immediate trade—no opportunity cost to the grocer, because there's no time elapsing with unpaid debt. With the currency you tender on the spot, the grocer could, in theory, lend out of the till, at interest, the very next moment. But suppose instead that your local grocer lets you "run a tab" until the end of each month—a nice, friendly courtesy. Clearly the grocer is standing the small opportunity cost of his/her uncollected funds for a little while in the hopes of keeping you as a regular customer.

Then what if the end of the month comes, and you don't pay? The grocer might agree to carry the debt still, but at interest, so that he/she is no longer incurring opportunity cost on the funds. Instead, you're compensating him/her via the (compound) interest you'll pay

How much interest? At least as much as could be readily earned somewhere else—usually by lending to the local bank in an interest-bearing account. That's the minimum opportunity he/she is losing by carrying your tab.

This is usually a good measure of minimum opportunity cost for your own money, too: a local interest-bearing account. It represents the rate at which your money could immediately and reliably earn if "rented out"—the "baseline against which you can measure any other proposed use of it.

Granted, when you need groceries, you don't stop to weigh the alternative of going hungry for the sake of stashing your cash to earn interest in a bank. You need to live today as well as plan for tomorrow. But too many people overlook the second value of money:

(i) You can trade it directly for goods and services, as needed;

(ii) Before you need to trade it away, you can rent it out —put it to work earning more of itself.

To overlook that second value is to cost yourself opportunity. Yes, it's negligible for small amounts of money and time, but not when the numbers get big. Even a few dollars earn a lot of interest over a long time; and even one day's earnings on a huge sum is significant.

Interest as Debt

Before you can actually calculate compound interest, of course, you have to know its rules and assumptions. Interest is rent on borrowed money—on debt—so the terms of rental must define the debt:

- When is a debt incurred?
- When is that debt due?
- When is that debt payable?

Consider once again your cash purchase of goods at the grocery store. You incur the debt (you're "charged") when you receive the goods. And if you're on a cash basis with the grocer, that debt is both due immediately and payable immediately.

If you have an account with this grocer and you're running a monthly tab, the due date is the end of the month. In most cases, that's also when it's payable—when your grocer actually demands payment. He may occasionally agree to carry the tab farther ("past due") but at interest, to compensate for his/her extra opportunity cost. In that case, the payable date (when you'll actually have to pay) is postponed beyond the due date.

That's a good way to arrive at these general definitions:

> *A debt is **incurred** when the value offered in exchange has been either delivered or promised in a binding contract.*
>
> *The **due** date is the last day the debt will be carried by the creditor at no interest charge.*
>
> *The **payable** date is the date when payment is finally expected (demanded).*

Those are all pretty clear for a simple, one-time purchase of goods. But they apply, too, for less straightforward cases.

For example, what if the transaction in question is a rental? You're borrowing something of value (the "principal") that you must return later. This is, by definition, a debt from the very outset; the principal portion of a rental debt is incurred immediately. And when is it due? Also immediately—but it's not payable then. (What would be the point of any rental you had to return the instant you signed it out?) Rather, it's due in the sense of the above definition: The lender won't carry the debt any longer without charging you interest (rent) on it to cover his opportunity cost.

What about the rent? It, too, is debt incurred at some point(s) during a rental transaction—charged at a certain rate per certain time period. When is this debt incurred, and when is it due?

It depends on what you're renting. Generally, rent debt is incurred at the beginning of the specified rental period. When you rent a car by the day, for example, you incur a rent debt for every 24 hour period—charged at the beginning of that period. You owe the first day's rent the moment you sign out the car, the second day's rent 24 hours later, etc. But it's not *due* then. You might drive for a week or more in your rental car (or stay a week in a hotel) while your tab runs—no problem, no extra carrying charges. The rental debt isn't due until the end of the entire transaction, when you finally repay the principal (i.e. return the car or vacate the hotel room).

In other cases, though, such as housing or equipment leases, rent debt is due and payable at the beginning of the period—immediately as it is incurred (or quite soon—say, the 10th of the month). You're not allowed to delay paying the rent.

Then there's the rental of money. Like other rentals, the rent debt (interest) on money is incurred at the beginning of the specified rental period. But unlike other rentals, it's typically due at the end of the period—and even then, it may not be payable (demanded) until long after that. It all depends on the type of "rental agreement." Compare:

- A bond demands payment of the rent right as it comes due (usually at the end of every 6- or 12-month period).
- A mortgage also demands payment of the rent right when due (usually monthly)—along with a little bit of the principal, too.
- A bank savings account may incur daily rental debt (to you—as depositor you're the lender), but may be payable quarterly.
- A CD may incur rent debt monthly, but that interest isn't payable until the principal is also demanded, which may be many months or even years later.

In the first two types of loans, no interest compounding ever happens. It's "pay-as-you-go" (recall page 13); each rental payment is demanded (payable) right when due. But in the other two cases, there are time lapses between the interest due dates and payable dates.

In the savings account, the first day's interest is incurred the moment you open the account with a deposit. And that interest comes due at the end of that first day—but it's not payable for another 89 days, *during which time it will accrue more interest "on itself."* The second day's interest is incurred on the morning of Day 2 (and is due at the end of that day), but it's not payable for another 88 days, and will thus compound meanwhile. And so on.

The monthly interest on the CD works similarly: Each month's interest goes unpaid and compounding for all the additional months remaining until the CD matures. Then all debt, both principal and interest, become payable at once.

In short, both these rental agreements feature past-due debt as part of the deal—*pre-calculated and agreed to*—by both borrower and lender (no surprises, no bickering). As each period's interest becomes past due (i.e. as the period expires), that unpaid interest is then regarded as a new loan in its own right; it, too, begins to incur periodic rental charges, thus earning interest "on itself"—compounding.

Quoting and Calculating Compound Interest
How Much—and How Often?

So how do they pre-calculate the interest in such "past-due rental" arrangements? How do you allow for the compounding effect of the interest? It all depends—not only on the interest rate but also on the frequency at which it is compounded

Take an example: Suppose a bank is offering a 1-year CD at an annual interest rate of 6%, with interest compounded quarterly but not paid until the CD matures after one year. The bank is also offering a savings account paying 6% annual interest, but this interest is compounded monthly. Which is the better deal?

First of all, what does it mean to say that interest is "6% annually, compounded quarterly" or "6% annually, compounded monthly"? It certainly doesn't mean that every dollar of debt earns 6 cents each quarter or month.

These quoted rates are a sort of shorthand, a nominal convention in the financial industries, that don't tell the true rates in so many words. The phrase "6% annually, compounded quarterly" actually means, "Divide the 6% annual rate by the number of quarters in the year (4), and use the result as the quarterly rate." So the actual rate used to accrue interest will be 6%/4 per month, or 1.5% per quarter.

Likewise, "6% annually, compounded monthly" actually means, "Divide the 6% annual rate by the number of months in the year (12), and use the result as the monthly rate." So the actual rate used to accrue interest will be 6%/12 per month, or 0.5% per month.

So the first thing when trying to determine which rate is a better deal is to recognize that it's really a comparison of 1.5% per quarter with 0.5% per month. Then just do a hypothetical experiment, watching as each of those rates accrues interest on some money....

For convenience, assume that each account begins with $100 and earns interest for one year:

	"6% annual interest, compounded quarterly"				"6% annual interest, compounded monthly"			
Qtr.	Start. balance	Interest charged	Pmt. made	End. balance	Start. balance	Interest charged	Pmt. made	End. balance
					100.00	0.500	0.00	100.50
					100.50	0.503	0.00	101.00
1	100.00	1.500	0.00	101.50	101.00	0.505	0.00	101.51
					101.51	0.508	0.00	102.02
					102.02	0.510	0.00	102.53
2	101.50	1.523	0.00	103.02	102.53	0.513	0.00	103.04
					103.04	0.515	0.00	103.55
					103.55	0.518	0.00	104.07
3	103.02	1.545	0.00	104.57	104.07	0.520	0.00	104.59
					104.59	0.523	0.00	105.11
					105.11	0.526	0.00	105.64
4	104.57	1.569	106.14	0.00	105.64	0.528	106.17	0.00

You can see that a monthly compounding period accrues more interest in a year than does the quarterly compounding—for the same annual rate. But this only makes sense: The sooner interest can begin to earn more interest on itself (i.e. the shorter the compounding period), the faster that extra interest will accrue.

In other words, *for the same **nominal rate*** (the 6% "shorthand" number in this example), *you get a higher **effective rate** with more frequent compounding.*

So don't be fooled by comparing nominal rates. Nominal Rate A may be higher than Nominal Rate B, but if B compounds a lot more often, it can actually be the higher effective rate. And it's only the effective rate—what actually happens to $100 over a year—that matters.

Key points that were covered here:

- "Finance" here means borrowing or lending money at interest.
- The interest calculations are what introduce the complexities.
- Interest is rent paid on borrowed money.
- The rent is to compensate a lender for his/her opportunity cost.
- Past-due interest must therefore compound.
- Compound interest calculations are not simple, because the compounding math uses exponents (powers).
- Interest rent is due and payable at different times for different forms of rental (i.e. different types of loans).
- The period of compounding is critical: for a given annual rate, faster compounding produces more interest.

Quiz

1. In nearly every rental agreement, the rent charge is incurred at the start of each rental period. Why? The lender hasn't been without his property for that period yet, nor have you had the benefit yet. (The answer is on page 22.)

2. Is there opportunity cost in accepting a check? (Page 23.)

3. A savings account rate is a good guide as a *minimum* rate of reimbursement of opportunity cost. Why minimum? (Page 24.)

4. For the same nominal annual rate, monthly compounding yields more interest than quarterly—and daily more than monthly, etc. So where does this stop? Can't you always get more interest by compounding faster—every minute, or second—or nanosecond? (Page 24.)

Quiz Answers

1. The lender's use of the asset is not generally the same as yours. Look at a car agency, which rents its property by the day. To you the value of a car is in driving it around all day. To the agency, the car's value is in its rental—a customer decision/transaction that happens in seconds—an *opportunity* that may not come again until the next rental period (probably why that period is so designated in the first place). Your keeping the car even a few minutes into a new day could force the agency to turn away another customer who needed a car. This would indeed cost them an entire day's rent, so they charge it to you.

   Moreover, in every rental transaction involving a physical asset, there are handling and preparation costs to the lender that can't be conveniently incurred on any arbitrary schedule. If the cleaners and mechanics all work the night shift at the car agency, and you return a vehicle in mid-morning, even if another prospective renter walks in right behind you, the car can't be cleaned and inspected then; it's still effectively out of service until the next day.

   This not-so-fluid trait in a rental market is even more pronounced with real estate, because the renter, too, needs time to prepare for the transaction: contract, alterations, moving, etc. So those rental contracts are in terms of months, or at least weeks.

   And how about money lending? Money is much more fluid stuff: no cleaning or repairing—and minimal transport. And with global markets and electronic commerce, it's now easy to re-lend the idle asset in some fashion no matter when the previous borrower returns it. But even money isn't instantaneously fluid. The larger the amount, the more time is needed for the lender to prepare the asset, qualify the borrower, and come to terms and a contract. So mortgages and other large loans are usually much more rigid in terms and repayment schedule than a personal savings account. The lender needs time to prepare for the next such transaction.

2. Suppose you pay for your groceries by check. It's a piece of paper attesting that the grocer can redeem it at your bank at any time for cash, a short-term promissory note of debt—which bears no interest, since it's redeemable immediately. Theoretically the grocer could run down to your bank that very minute, get cash for your check, then lend that money out, at interest.

   That would cost time and effort, though, and the grocer doesn't do this (a fact we all count on occasionally). The interest lost on the uncollected debt over a few days' "float time" is far less than the expense of hustling to the bank with each individual check. The float time is truly an opportunity cost to the grocer, but like most vendors who accept checks, the grocer simply builds this small cost into the prices of his/her goods.

   You might argue that the grocer could endorse your check and lend it out on the spot, the same as cash. OK: technically, a properly endorsed check is "money," too—negotiable just like cash. And in theory, this could go on from party to party, indefinitely—but it doesn't. Faith falters eventually: will the bank still honor this note? Doubt forces the final bearer to present it for cash, and for the time required to do that, he bears the opportunity cost of uncollected funds. A check is a short-term promissory note, and like any such interest-free note, the ultimate recipient incurs opportunity cost in the delay from acceptance to redemption.

   Trivial? If you're dealing with a few small checks, maybe. But at a major financial institution, it's no laughing matter. To a bank that routinely clears millions of dollars in checks, even a single day's delay adds up to thousands of dollars in opportunity cost—interest that could have been earned by lending out those funds. And banks really do lend instantly, flashing short-term, inter-bank loans around the globe at the click of a button. When widespread and undue delays happen accidentally, there are fees, fines or firings. When such delays are rigged intentionally—in someone's favor—it's a felony, called "check kiting."

3. When trying to gauge opportunity cost—i.e. what else you might do with your money—generally you'll want to choose that with the highest return for similar *risk* and *liquidity*.

It wouldn't be right, after all, to cite a lottery jackpot as the opportunity lost for every dollar of unpaid debt. Yes, technically, it's potential, but a lottery ticket is astronomically more risky than a passbook savings account; it's just not realistic. "Could have" is not "would have."

Similarly, you shouldn't cite the rate earned on a 1-year, $2500 CD as the opportunity you passed up by lending your neighbor $100 for a couple of weeks. The CD has no more risk than your neighbor's good word, but it's still not comparable to your neighborly loan, because it's not as liquid. You simply wouldn't have earned that CD rate for so little money and for so short a time.

So when analyzing your opportunity cost as a lender, first you have to look at what the loan is, so that you can then focus on the market alternatives that carry similar risk and liquidity. For small amounts of ready cash, it's hard to beat a savings or money market account as a measure of comparable investment. But for a more complex venture involving a larger amount and/or a longer time horizon, maybe the opportunity cost might be on a par with a bond, a note or a stock.

4. Yes, there's a limit to how much compound interest you can get simply by accelerating the period to shorter and shorter intervals. This limit is called (surprise, surprise) ***continuous compounding***. To see how it works, look at the math behind compound interest. (No need to memorize this or anything, but it's instructive to see.)

If, at the start of a new compounding period, a debt's balance, B, is accruing interest at a percentage rate of $I\%$ per year, then for a single period, the formula for the interest accrued, A, is $A = Br/P$, where P is the number of compounding periods in the year and r

is the decimal version of $I\%$: $r = I\%/100$. The ending balance, E, after that period is given by $E = B+A = B+Br/P = B(1+r/P)$.

Start with \$100 in an account earning 6% annually, compounding monthly. Then $B = 100$, $I = 6$ (hence $r = .06$), and $P = 12$. So for month #1, the interest accrued, A, would be $100(.06)/12 = \$0.50$. The ending balance, E_1, is $B(1+r/P) = 100(1+.06/12) = \$100.50$.

All fine and good. What about the second month? Its beginning balance is the ending balance of month 1, E_1. So for month 2, $E_2 = E_1(1+r/P) = B(1+r/P)(1+r/P) = B(1+r/P)^2$. And for month 3, $E_3 = E_2(1+r/P) = B(1+r/P)^2(1+r/P) = B(1+r/P)^3$, etc. So the general formula for one year, which is P periods, is $E_P = B(1+r/P)^P$.

Test this formula on the table on page 20, which compares monthly and quarterly compounding: Under quarterly compounding, $E_4 = 100(1+.06/4)^4 = 100(1+.015)^4 = 106.14$. Under monthly compounding, $E_{12} = 100(1+.06/12)^{12} = 100(1+.005)^{12} = 106.17$. So the formula works—and it's a lot handier than tables.

Now (finally) to the question at hand. You already know that for the same nominal interest rate (e.g. 6% in the example on page 20), you earn more interest over a given amount of time (a year) if you shorten the compounding period, so there are more periods per year. (Note how this worked in quarterly vs. monthly.) That is, you let n get larger and larger in the formula: $E_P = B(1+r/P)^P$.

So *what's the limit*—if you let P grow to "infinity" (that would be continuous compounding) for the formula $(1+r/P)^P$? Turns out it's $\boldsymbol{e}^r$, where $\boldsymbol{e}$ is the natural logarithm base ($\boldsymbol{e} = 2.718281828\ldots$).

So the most interest you can earn in a year on, say, \$100 at 6% annual interest is compounding it constantly: $100\boldsymbol{e}^{.06} = \$106.18$. For 2 years it's $100\boldsymbol{e}^{2(.06)}$; for 0.3 years, it's $100e^{.3(.06)}$, and so on.

In general, the formula for the ending balance—after continuous compounding interest on a beginning balance—is $E = B\boldsymbol{e}^{rt}$, where t is the time (expressed in years), and r is the annual interest rate (expressed as a decimal).

2. THE TIME VALUE OF MONEY

Key points to be covered here:

- "Time is money." Interest, that all-purpose rental compensation for lost opportunity, is directly related to time. A dollar today is worth more than $1 tomorrow, because it could earn for you meanwhile. So in any financial transaction, you can't directly compare cash flows that don't occur at the same time.
- However, if a transaction's interest rate is specified, or if you can identify a suitable opportunity cost rate, you can then adjust the amounts of cash flows in exchange for moving them forward or backward in time, to allow for direct comparison. In any valid financial transaction, the cash flows sum to zero if they are all so adjusted to a single point in time.
- For all these reasons, it's convenient to represent a financial transaction as a set of vertical lines (the cash flows) placed appropriately along a horizontal line (the time line).

When is a Dollar Really a Dollar?

As you've read, opportunity cost is the potential benefit you forego by not having the use of some asset. If the asset in question is money, the potential is, at minimum, what it can earn via rental—interest.

But interest is, of course, a *rate*; it's expressed in terms of "x% of the outstanding debt *per period*." The opportunity of money—its ability to earn more of itself—is directly dependent on the time involved. The more time elapses, the more money your money earns (and the interest rate lets you calculate precisely how much more).

In this sense, "time is money" becomes more than a cliche: $1 today is worth more than $1 tomorrow, because today's dollar could earn some interest overnight. This is the ***time value of money***.

That may seem rather obvious, but look at its more profound implications—which are too often overlooked:

- Because value changes with time, you cannot determine how much money you have unless you know *when* you have it.
- Therefore you cannot directly compare any two amounts of money unless they're both held at the same time. To do otherwise would be to "compare apples with oranges."

Of course, the difference between today's value and tomorrow's can be negligible (and so overlooked), unless you're dealing with a large sum—or many such "tomorrows." For those transactions, you simply cannot ignore the time value of money.

So how to account for it? If a dollar today isn't a dollar tomorrow, what *is* it?

Squaring the TVM Deal

Turning Apples into Oranges (or Vice Versa)

If your brother (the same enterprising dude who lent you gum money at usurious rates when you were kids) needs a 2-year loan of \$1000, and you've got the money available—sitting in a fund earning 6% annually, compounded monthly (i.e. 0.5% per month)—what amount should he pay you after the 2 years? And what if he offers to pay you exactly \$1,100 instead? How much should you lend him in that case?

That \$1000 in your account today will grow over 2 years—and you can calculate precisely how much. It's a fairly easy general formula that lets you equate sums of money that are separated in time.

The amount you have today, the Present Value (***PV***), grows into the amount you'll have later, the Future Value (***FV***), as follows:

$$\boldsymbol{FV} = \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}}$$

Here $\boldsymbol{i}$ is the *periodic* (not annual) interest rate, expressed as the decimal. That is, it's the fraction of the ***PV*** earned *per period*—as rental of that ***PV***—over $\boldsymbol{n}$ such periods. (The derivation of this formula, and its resulting form when using an annual rate, $\boldsymbol{r}$, is on page 25.)

Thus, if you know ***PV***, you can easily use $\boldsymbol{i}$ and $\boldsymbol{n}$ to calculate ***FV***. The formula lets you account for the time value of money (TVM), adjusting a known present value *forward in time* to exactly balance the value being offered in the future. By transforming today's "orange" into the future "apple," you can truly compare it to the repayment "apple" being offered then. In a fair trade, those two simultaneous values must match—just as in any other monetary trade: value given for value received.

The alchemy:

$$\begin{aligned} \boldsymbol{FV} &= \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= 1000(1+.005)^{24} \\ &= 1{,}127.16 \end{aligned}$$

So \$1,127.16 is what your brother should, by all rights, commit to repay you in two years' time, if you lend him a full \$1,000 today.

But what if he'll promise only \$1,100 by way of repayment? That's not enough to cover your opportunity cost of doing without \$1,000 for 2 years. Of course, it's not that he's not offering you any interest; he's repaying more than the \$1,000 principal, just not enough. He's covering your opportunity cost for a *portion* of the \$1,000. Can you figure out exactly what portion?

No problem: Rearranging the formula, you can compute an unknown ***PV*** from a known ***FV*** (i.e. convert the apple backwards in time to become an orange):

$$\begin{aligned} \boldsymbol{PV} &= \boldsymbol{FV} \div (1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= 1{,}100 \div (1+.005)^{24} \\ &= 975.90 \end{aligned}$$

So if he'll promise only \$1,100 in 2 years, you can offer to lend him just \$975.90 now. Your opportunity cost for precisely that much loan will be completely covered.

Reminder: These calculations hinge upon the rate you could otherwise earn with your money—that 6% annually, compounded monthly. If you have a different opportunity (with risk and liquidity equivalent to the loan to your brother), that offers a higher rate of interest, then the above calculations will give entirely different results, since ***i*** would be higher. Remember: the rate of a liquid account is the default *minimum* to use as the opportunity cost; it could well be higher.

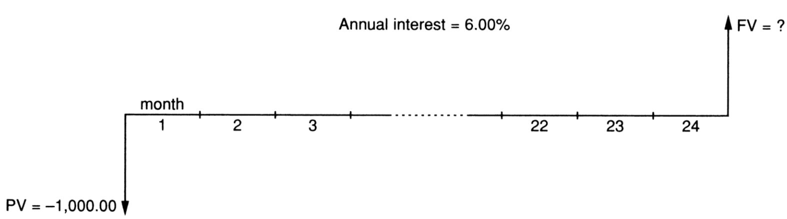
(Come to think of it, maybe you should use a higher rate—say, that of a 2-year CD, which, like your brother's loan, is less liquid than a withdraw-anytime fund account. Or, if there's any real chance your brother could default, you might want to use a higher rate to match other 2-year investments with comparable risk—say a bond or note.)

Drawing the Picture
Cash Flow Diagrams

Now suppose your brother really needs to borrow the entire $1,000 (nothing less), and he still can't commit to repay more than $1,100 at the 24-month mark. But maybe he'd have that much in 18 months—or maybe if you were willing to wait until, say, 30 months, he'd have time to gather enough extra to make the wait worth your while.

In other words: *What if* he does repay the $1,100 in 18 months? Does this meet your entire opportunity cost?" Or *what if* you let the loan term extend to 30 months—how much extra will he need then to cover your entire opportunity cost?"

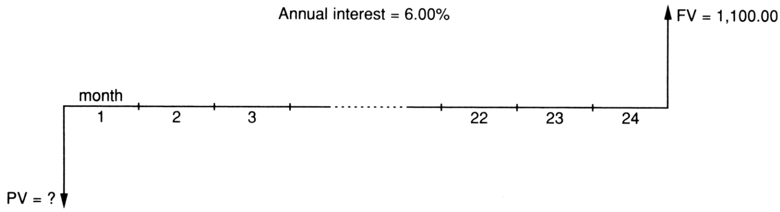
Playing "what-if" like this is a typical pattern of thinking in financial scenarios. You usually explore a lot of hypothetical options before committing real dollars. As you've seen, the flexibility offered by the PV-FV formula is ideal for such experimentation, because it lets you transform values to move them forward or backward in time. And to help you better envision the situation and play "what-if," it's convenient to use a ***cash flow diagram***. For example, here is the cash flow diagram for the original loan to your brother (page 28) along with a summary of the values you'd need for the PV-FV formula:



	n	*i%*	*PV*	*FV*
	24	0.50	−1,000.00	$\boxed{1,127.16}$
P (or ***P/YR***) = 12 Periods/Year				
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	2	6.00		
	Yrs	***I%YR***		

The unknown here (the boxed item) was the Future Value (***FV***)—the repayment amount.

And here's the alternative—his proposed $1,100 payback (page 29):



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*FV*
24	0.50	$\boxed{-975.90}$	1,100.00
2	6.00		
Yrs	*I%YR*		

The unknown here was the Present Value (***PV***)—the loan amount.

Notice a couple of things about cash flow diagrams in general:

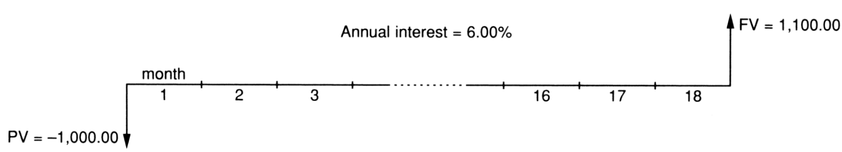
- Cash flows are represented as vertical lines. (And use the lengths of the lines to show the relative magnitudes of the cash flows. This helps distinguish them and further clarifies the situation.)
- The timing of each cash flow is shown by its position along the horizontal time line. (Use tick marks show the time periods.)
- The direction of each cash flow line indicates whether you are receiving money or paying money. Thus if your brother were to draw these loan scenario diagrams, his cash flow lines would be exactly opposite (inverted) from yours. Pick one perspective, either borrower or lender, and stick with it throughout the analysis.
- In any cash flow diagram that represents a financial transaction, there must always be at least one cash flow in each direction (up and down). This only makes sense: There's no deal—no trade or transaction—with value flowing in only one direction.
- The PV-FV formula lets you "slide" any given cash flow forward or back along the time line, keeping it entirely *equivalent in value* to its original time by growing it or shrinking its dollar amount, thus accounting for the time value of money.

That last trait is the most subtly powerful: Sure, a cash flow diagram lets you see the whole transaction at a glance—very convenient. But it also lets you play "what-if" very easily, with absolute confidence that the altered transaction is entirely equivalent to—just as "fair and square" as—the original.

How so? You just use the PV/FV formula to slide cash flows back or forward in time until they *all happening at the same time*—the same horizontal point on the diagram (and it doesn't matter what point you choose). This, as you're learned, is the *only* situation where you can directly compare dollar values. Then you simply add up all those simultaneous cash flows. If the transaction is fair and valid, the positive (upward) flows will exactly balance the negative (downward) flows; the net sum will be *zero*.

Try this slide-and-sum-to-zero method on the questions at hand:

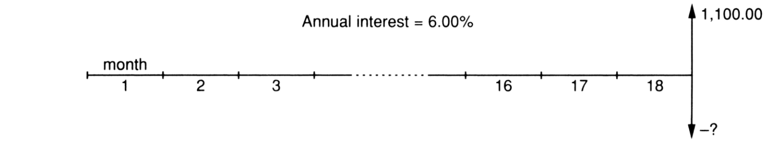
What if your brother pays you $1,100 after just 18 months? Does this meet your entire opportunity cost? Here's the situation from your point of view:



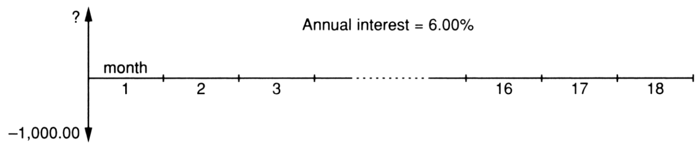
The question: If money is worth 6% annually, compounded monthly, *is this a fair transaction?* To find out, you must get all cash flows to happen simultaneously by sliding them along the time line, using the PV-FV formula to maintain equivalence to their original values.

There are many ways to do this, of course. You could move each cash flow to, say, the end of month 24, or month 11, or *whenever*. Mathematically it doesn't matter, so long as the net sum of their values at any given time point is zero.

Probably the easiest two ways are to keep one cash flow right as it is and move the other to that time point. In other words, either this:



…or this:



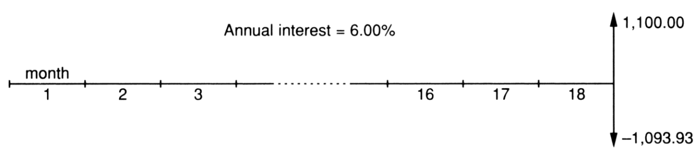
The PV-FV formula gives this as the equivalent to your loan after 18 months:

$$\begin{aligned} \boldsymbol{FV} &= \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= -1000(1+.005)^{18} \\ &= -1{,}093.93 \end{aligned}$$

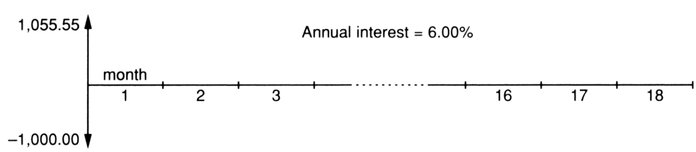
And the equivalent to your brother's repayment, as moved back to the start of the loan, is:

$$\begin{aligned} \boldsymbol{PV} &= \boldsymbol{FV}\div(1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= 1{,}100\div(1+.005)^{18} \\ &= 1{,}005.55 \end{aligned}$$

Now the pictures are complete. You have either this:



…or this:



Neither one of these scenarios sums to zero, as they would for an exactly fair transaction. (And you wouldn't expect just one to sum to zero; either they both do, or neither does—right?) The net sum at the start of the loan is \$5.55; the net sum after 18 months is \$6.07.

What do these numbers mean? First of all, since the diagrams are from your point of view, and the net sums are positive, you're getting the better of the deal here. You're receiving slightly more than 6% interest on your money (and your brother is paying slightly more than 6%; his diagrams would show slightly *negative* net sums).

How much more interest? You could use a trial-and-error approach with various higher rates until the sums both came to exactly zero (and they would indeed do so at once). Or you could do some fancy math, rearranging the PV-FV formula to find the $\boldsymbol{i}$ that produces an $\boldsymbol{FV}$ of exactly –$1,100 or a $\boldsymbol{PV}$ of exactly $1,000, respectively.

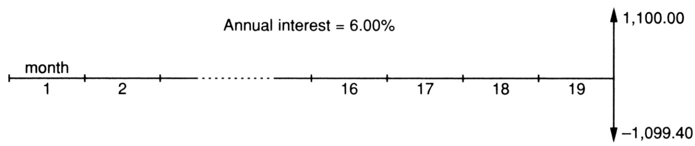
But a simpler, more meaningful way to look at this is to focus on the dollar value rather than an interest rate: Your brother is effectively handing you a gift of $5.55—free and clear—right now. That is, if your cost of money is truly 6%, the loan to him is $5.55 less than it should be today. You're not quite giving him the same value that you are receiving in trade.

Question: If, instead of $5.55 today, he were to give you the same net gift value 18 months down the road—adjusting the value forward to account for your time value of money—how much would it be? Yup: $6.07. Do the math:

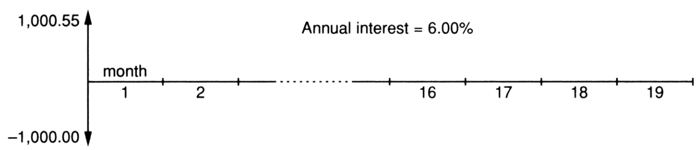
$$\begin{aligned} \boldsymbol{FV} &= \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= 5.55(1+.005)^{18} \\ &= 6.07 \end{aligned}$$

This is no coincidence, of course: While it's usually most convenient to compute the net value of a transaction from your viewpoint as of the beginning (a sum called the "Net Present Value")—since that's when you're making the decisions—it's still entirely valid to similarly compute and state the net value to you at *any* time point in the transaction. Every value on a cash flow diagram, no matter if arrived at through sliding-and-summing, is an equivalent value that can be further "slid" around (and even further summed) as you wish. So long as the underlying assumption—the opportunity cost of money—holds, all such adjustments, additions and simplifications are valid.

Well, since you want to be fair to your brother, can you find a repayment schedule for his $1,100 that's at least a little closer to correct for a 6% rate? How about having him wait, say, one more month before the payment—pay off at 19 months? Here are the pictures that the PV-FV formula would help you complete:



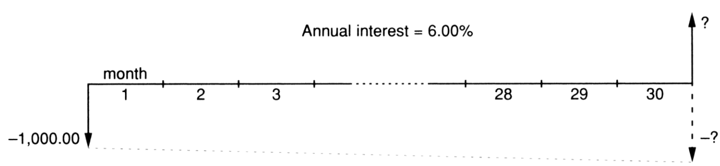
or



So the up-front net value to you is $0.55; the 19-month equivalent value is $0.60. These are pretty close. Maybe this ought to be the early payoff scenario your brother considers.

Now, how about the other "What-If" question posed? What if you let the loan term extend to 30 months? What repayment will he need then to cover your entire opportunity cost for that longer loan?

Here's the question in picture form:



To balance this transaction, clearly his payment must be the mirror opposite of the 30-month FV of your –$1,000 loan.

Do the math: $$\begin{aligned} \boldsymbol{FV} &= \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}} \\ &= -1{,}000(1+.005)^{30} \\ &= -1{,}161.40 \end{aligned}$$

So he'll need somehow to scrounge up another \$61.40 in the time between the end of Month 18 (where ostensibly he could have paid his flat \$1,100) and the end of Month 30.

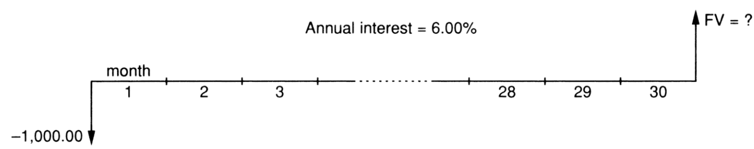
This is a good point for a word about using a calculator. Although this book does not give machine-specific keystrokes, almost certainly you're reading it in the hopes of learning better how to use a financial calculator, with specialized PV-FV sorts of calculations already built in (either a stand-alone handheld unit or a software tool installed on a computer). It's pretty tedious to crunch numbers with the PV-FV formula otherwise.

But be aware: Most of the popular financial calculators and software packages are smart enough to do some of the thinking and transacting for you. Specifically, in making it as easy as possible for you to complete a given cash-flow diagram in as few steps as possible, in calculations such as the one above, they automatically change the sign of FV in relation to PV—to reflect the reality that a valid transaction has to have value flowing in both directions.

What does this mean for you? Take the last example above: The PV-FV formula all by itself will give you an FV of the same sign (±) as the PV. You're just translating a value forward in time, letting it grow to reflect your opportunity cost; that's all the formula is supposed to do. But many financial calculators *change the sign* of the FV result, in an attempt to save you the thought process elucidated above: "OK, the FV of my –\$1,000 is –\$1,161.40, so that means the necessary balancing cash flow at that point is a positive \$1,161.40." The calculator changes the sign to complete the transaction picture more quickly, showing you the necessary *balancing* cash flow value —the opposite of the slide-and-sum net value: $\boldsymbol{FV} = -\boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}}$ (Compare this "sign-flipped" version with the original formula—top of the page.)

Because such a calculator design is so common nowadays, this book will assume it. That is, when you see the values for a transaction listed in the form below, they will indeed reflect the cash flow diagram sign conventions for the valid *finished* transaction (from one viewpoint, of course), flipping the sign of the PV or FV calculated, to reflect the final balancing cash flow.

For example, here's how the previous example would be diagrammed and listed for calculation:



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

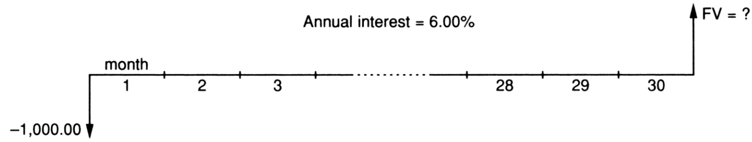
n	***i%***	***PV***	***FV***
30	0.50	–1,000.00	$\boxed{1{,}161.40}$
2.5	6.00		
Yrs	***I%YR***		

Notice now that PV and FV are of *opposite* sign—just as the valid transaction denoted on the cash flow diagram requires—even though the PV-FV formula has PV and FV with the same sign (see page 36).

To find out whether your machine obeys the cash flow diagram sign conventions for a finished transaction, try this: Key in the values shown above, using a negative value (–$1,000.00) for ***PV*** and solving for the unknown (***FV***). Your machine should have keys or buttons to store and calculate with these quantities (including a [***PMT***] key, discussed later, which you should leave at zero for now).

If you get a positive $1,161.40 for ***FV***, your machine is indeed obeying the cash flow diagram sign conventions—"flipping the sign," as assumed in this book, to complete the diagram. If you get a negative ***FV*** (–$1,161.40), this means your machine is only doing the math on top of page 36; you'll have to interpret that result yourself, mentally flipping the sign (i.e. thinking through the paragraph after the math on page 36) to correctly complete a valid cash flow diagram.

One other point about the notation used in this book for the purposes of your calculator—take this same example once more:



P (or **P/YR**) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***FV***
30	0.50	–1,000.00	$\boxed{1,161.40}$
2.5	6.00		
Yrs	***I%YR***		

Notice the redundancy of the numbers given for ***n*** and ***i*** in the calculator data box. The upper two values are the *periodic* values—what the formula really uses (although, in a nod to convention, the interest is given in percentage form and is so denoted as ***i%***; the decimal equivalent is what the formula uses). The lower two values are the *annualized* values—more often the terms in which you might encounter or think about a financial transaction.

This redundancy is offered in this book simply because some calculator models demand inputs in periodic form, some in annual form, and some allow either—offering different keys for the two forms. Of course, the two forms are directly related by the Periods/Year value, shown at the left of the data box (and that value is used by some calculator models to help translate from periodic to annualized for you). So when encountering these calculator data boxes in the solutions to the problems throughout this book, just use the values that apply to your particular calculator model, and ignore the others.*

*Important note: You'll see calculator values for ***n*** and ***i%*** given here in varying precisions—1 to 3 decimal places—but the calculator will probably be using more places internally. For example, an annual interest rate of 8.000%, compounded monthly, would appear here as 0.667% per month, but—as noted to the left of each data set—the actual rate used by the machine would be 8%/12 or 0.666666666...% (to as many decimal places as your machine would carry it—more than three, surely). For a more in-depth look at rounding issues, see Chapter 5.

Key points that were covered here:

- "Time is money." Interest, that all-purpose rental compensa tion for lost opportunity, is directly related to time. A dollar today is worth more than $1 tomorrow, because it could earn for you meanwhile. So in any financial transaction, you can't directly compare cash flows that don't occur at the same time.
- However, if a transaction's interest rate is specified, or if you can identify a suitable opportunity cost rate, you can then adjust the amounts of cash flows in exchange for moving them forward or backward in time, to allow for direct comparison. In any valid financial transaction, the cash flows sum to zero if they are all so adjusted to a single point in time.
- For all these reasons, it's convenient to represent a financial transaction as a set of vertical lines (the cash flows) placed appropriately along a horizontal line (the time line).
- Most calculators produce results that reflect the sign conventions of cash flow diagrams. That is, they compute directly the unknown value necessary to balance the picture and make the transaction valid. This results in a change of sign (±) from the

Quiz

1. Can you think of situations where an interest rate is negative—where a Future Value is less than its corresponding Present Value? Does this change how you use the PV-FV formulas for "sliding" cash flows forward or back in time? (The answer is on page 40.)

2. In that $1,000, 24-month loan to your brother, what if he offered to pay back $550 after 12 months? How much would he still owe you at 24 months? (Page 41.)

3. Can you solve question **2** in more than one way? (Page 43.)

Quiz Solutions

1. Inflation is one situation where the rate of growth of monetary value can indeed be negative. That is, the buying power of a dollar can actually shrink, even as the face value is accruing interest.

Consider a loaf of bread that costs \$1.00 today. If the inflation rate, $\boldsymbol{r}$, is 5%, that means the same sort of loaf will cost \$1.05 this time next year. What will happen to the buying power—expressed in terms of today's value—of that dollar? That is, *how many loaves of bread will that same dollar buy* next year? It won't buy a whole loaf anymore. It will buy precisely 100/105ths of a loaf: $1/(1+\boldsymbol{r})$. And so its buying power has changed by $1/(1+\boldsymbol{r}) - 1$, or $-\boldsymbol{r}/(1+\boldsymbol{r})$. It has gone *down* by 5/105ths.

Does this change how you use the PV-FV formula? Not at all:

$$\boldsymbol{FV} = \boldsymbol{PV}(1+\boldsymbol{i})^{\boldsymbol{n}}$$

But in this case, $\boldsymbol{i} = -5/105$ (expressed as a decimal). Used in the formula, this rate would tell you what will happen to the buying power—expressed in today's dollars—of money left under a mattress for $\boldsymbol{n}$ years, devaluing at an annual inflation rate of 5%.

And what if you don't leave it under the mattress? If you put it to work earning interest, does the buying power still shrink? It depends on how much interest it is earning. If it's earning at exactly the rate of inflation, the buying power stays constant; your dollar grows just as fast as the price on the loaf of bread. Any faster, and you're actually earning interest—at what's called a "real" interest rate: the amount of growth over and above what's needed to keep pace with inflation.

To compute the real interest rate, $\boldsymbol{i}$, from a gross earnings rate, $\boldsymbol{g}$, and an inflation rate, $\boldsymbol{r}$ (all rates in decimal form), using the same bread loaf example, note that the buying power of a dollar next year will be $(1+\boldsymbol{g})/(1+\boldsymbol{r})$. The change in that buying power—the real interest rate, $\boldsymbol{i}$—is therefore $(1+\boldsymbol{g})/(1+\boldsymbol{r}) - 1$, or $(\boldsymbol{g} - \boldsymbol{r})/(1+\boldsymbol{r})$.

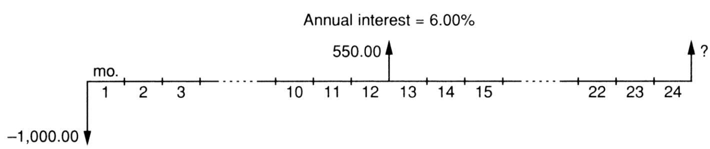
Does this general formula for a real interest rate make sense?

Sure: When g is equal to r, the real interest rate, i, is zero—no gain or loss of buying power when the earning rate of money exactly matches inflation.

If g is less than r, the buying power shrinks; the real interest rate, i, is negative, because inflation has the upper hand.

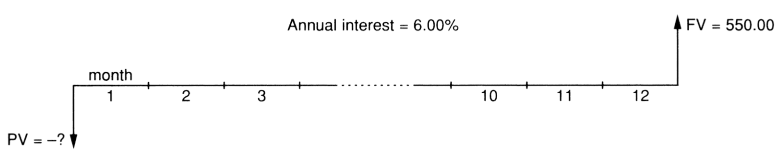
Obviously, the best circumstance is where g is greater than r, so that even if the real interest rate, i—which is the true growth rate of your money's buying power (expressed in today's dollars)—is being somewhat slowed by inflation, it's not stopped altogether.

2. Here's the picture of the situation. You need to find the cash flow at the 24-month point that will balance the whole thing:



Probably the easiest way to do this is in two parts—two separate PV-FV calculations.

First, "slide" the $550 payment back to the beginning of the time line, so that you'll be able to combine it with the loan amount. Here's that calculation:

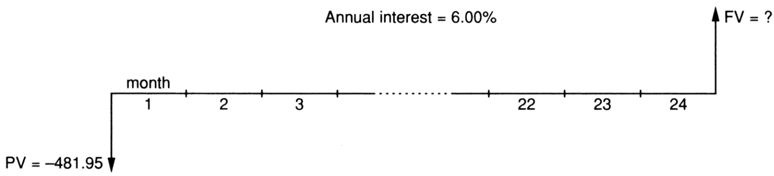


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*FV*
12	0.50	$\boxed{-518.05}$	550.00
1	6.00		
Yrs	*I%YR*		

So $550 in 12 months is the same as $518.05 now. It's as if you just lent your brother $1,000 and he immediately handed back $518.05. That makes it effectively a $481.95 loan, rather than a thousand-dollar loan, for 24 months. And you know what to do with that, right?



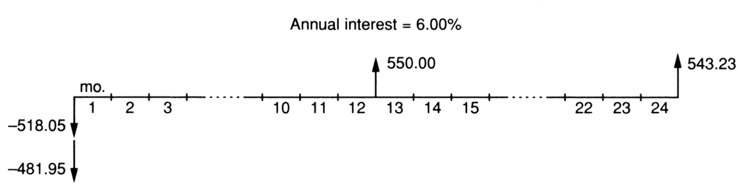
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	***i%***	***PV***	***FV***
24	0.50	–481.95	$\boxed{543.23}$
2	6.00		
Yrs	***I%YR***		

Did you notice how the sign conventions used by the calculator worked in this case? In moving back the $550 to the beginning, it flipped the result to negative indicate that the $550 corresponded to a *loan* of exactly $518.05. In other words, "that much" of the loan is taken care of by "that much" repayment.

So it's just as if you made two separate simultaneous loans, but since the governing interest rate is the same for both, you can diagram together and either combine or treat them separately:

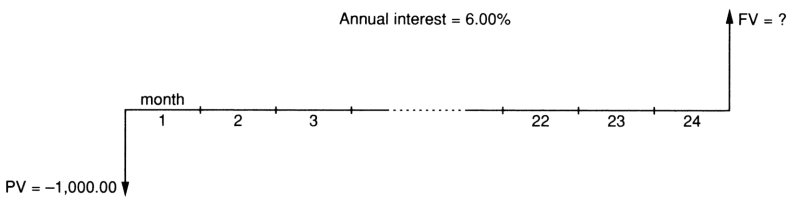


One loan was for $518.05 for just 12 months, with a repayment amount of $550.00.

The other loan was for $481.95 for 24 months with a repayment amount of $543.23.

3. Of course you could have done it another way. You could have done it dozens of other ways, choosing any one of 25 different time points at which to sum the cash flows. For example, you could have sent both the original loan amount and the \$550 forward to the end of the 24th month—one "slide" at a time.

First, "slide" the loan amount forward 24 months:

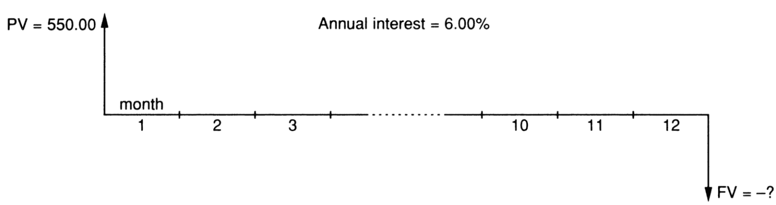


P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***FV***
24	0.50	–1,000.00	$\boxed{1{,}127.16}$
2	6.00		
Yrs	***I%YR***		

Next, "slide" the \$550 repayment amount forward 12 months:



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***FV***
12	0.50	550.00	$\boxed{-583.92}$
1	6.00		
Yrs	***I%YR***		

The net sum of the above two results at the end of Month 24 indicates the necessary balancing cash flow: \$1,127.16 – \$583.92 = \$543.24. (Rounding makes this result differ by a penny from the other method. If you carry more than two decimal places, you'll get more exact correspondence.)

3. Even Cash Flow Situations

Key points to be covered here:

- Just as the PV-FV formula (the "sign-flipped" version) calculates the balancing cash flow for a simple PV-FV transaction, so the PMT formula does for a transaction that includes a uniform series of cash flows—one per period.
- The assumptions behind the math of the PMT formula demand that the ***PMT*** cash flow occurs either at the end of each period or at the beginning. Therefore, at one end of the timeline or the other, a ***PMT*** cash flow will coincide with the ***PV*** or ***FV***, but you must never net these two together when using the PMT formula; it needs to use them separately in its math.
- If you adhere to that rule—and properly draw the cash flow diagram, with its sign conventions—the PMT formula is a flexible, powerful tool that will let you solve for any of five financial values, given the other four and the annuity mode.

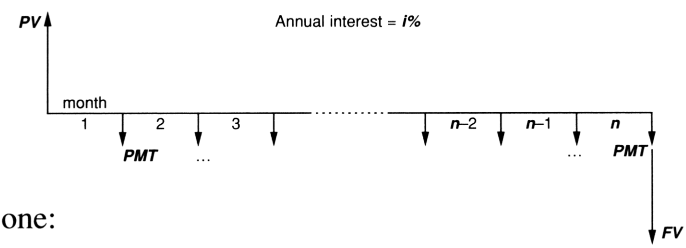
Adding PMT's to the Picture

The PV-FV formula accounts for the time value of money as you slide a cash flow around on the time line. And the "sign-flipped" version tells you the balancing cash flow required to complete a valid transaction. But this works only for individual cash flows; for every cash flow in the scenario, you have to use the PV-FV formula to slide that cash flow to the desired point in time for summation.

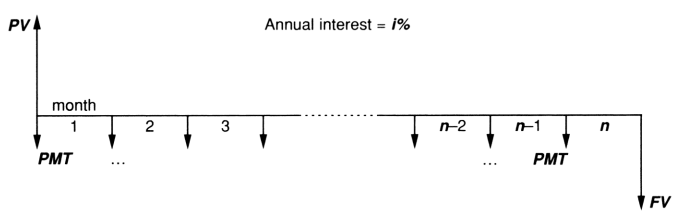
Well, what if the scenario in question is a 72-month car lease, or a 25-year projection of monthly retirement income, or a 30-year mortgage? Care to try to use the PV-FV formula to find the remaining balance (FV) after 360 monthly mortgage payments? (You'd have to do 360 separate calculations.)

Fortunately, there is an easier way to deal with such scenarios. It's a formula that computes the balancing cash flow required to complete a valid transaction—similar to the "sign-flipped" PV-FV formula—but not only for ***PV*** and ***FV***. It can also account for other cash flows in between, *so long as those cash flows consist of one identical flow (**PMT**) per period—always in the same amount and direction.*

That is, it's a *PMT-formula* that helps you complete this picture:



or this one:



Here is the PMT formula:

$$\boldsymbol{PV} + (1+\boldsymbol{iA})(\boldsymbol{PMT})\left[\frac{1-(1+\mathrm{i})^{-n}}{\boldsymbol{i}}\right] + \frac{\boldsymbol{FV}}{(1+\boldsymbol{i})^{n}} = 0$$

This formula is so useful and fundamental that it's built into every financial calculator; you don't need to memorize it (nor know how it was derived—the math is a bit more complicated than the simpler PV-FV formula). But you do need to know the names of all the variables and what they mean. Those are the values you must decide on before using them in the formula.

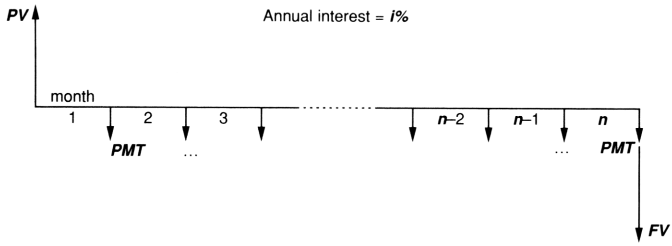
You already know four of the variables:

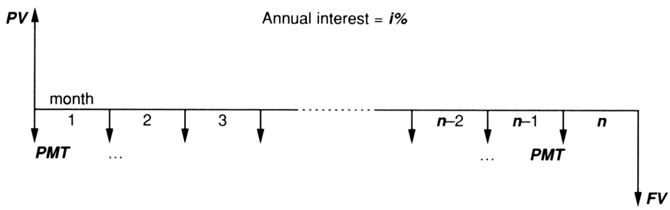
> ***n*** is the total number of time periods contained on the time line. The period in question may be a year, a month, a day, etc., but it is the compounding period for the interest rate used.
>
> ***i*** is the interest rate *per period* (expressed as a decimal).
>
> ***PV*** is the cash flow (positive or negative, according to the viewpoint you adopt throughout the analysis) that occurs at the beginning of the time line.
>
> ***FV*** is the cash flow (positive or negative, according to your viewpoint) that occurs at the end of the time line;

The new item is ***PMT***. This number denotes a *series* of cash flows, where an identical cash flow occurs once each period—same amount, same direction (positive or negative), same time in each period (at the beginning or end).

The first thing to keep in mind about ***PMT*** is that *zero is an entirely acceptable value for it*. That is, the above PMT formula is valid even when there are no cash flows at all happening within the transaction —just a ***PV*** and an ***FV***. You can see that when ***PMT*** is zero, the PMT formula above reduces immediately to the convenient, "sign-flipped" version of the PV-FV formula: $\boldsymbol{FV} = -\boldsymbol{PV}(1+\boldsymbol{i})^{n}$

The other thing to keep in mind about ***PMT*** is that, when it is *non-zero*, its timing during the period (either beginning or end) is critical. Here again are the two allowed scenarios:





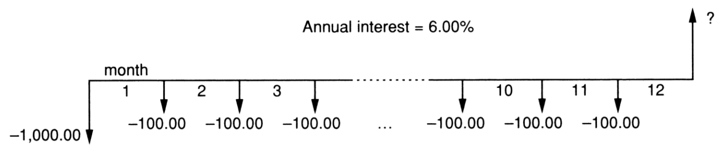
The PMT formula works *only* in the above two circumstances (which cover nearly all installment loans and leases). The ***PMT*** cash flows must occur one each period, at the beginning or at the end of each period—same timing, every period. They can't vary back and forth, nor can they happen at mid-period or any other time.

That means that at either the beginning of the time line or the end, there must be a ***PMT*** cash flow occurring *in addition to* the ***PV*** or ***FV***, respectively. Notice above how the cash flow arrows show this, rather than netting the two simultaneous flows into one. Don't confuse yourself (or your calculator) by lumping those two amounts together! You must distinguish them in order for the PMT formula's math to work correctly. (Remember: From a financial point of view, you can treat simultaneous cash flows either together or separately on the same cash flow diagram.)

To clarify this point, look at an example:

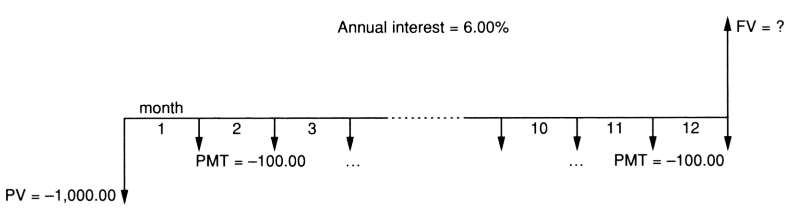
Suppose on December 31, you deposit $1,000 to start a savings account earning interest of 6% annually, compounded monthly. At the end of each month, you plan to invest another $100. How much will you be able to withdraw on the following December 31?

Hmm… Are you really going to deposit $100 on December 31, only to withdraw it moments later, along with the rest of the account? Wouldn't make much sense, would it? In fact, the actual diagram of the situation is probably this:



There are just 11 monthly investments of $100 here—but you've got 12 periods—and that's a problem: The PMT formula demands the same payment in every period—and at the same point in the period.

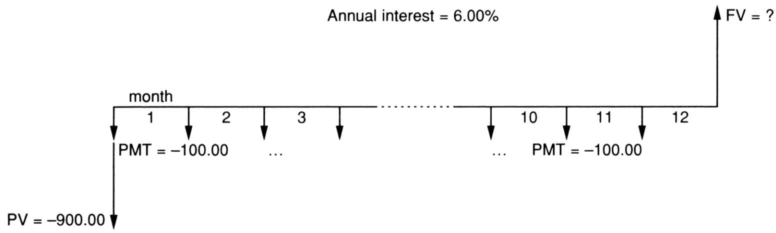
So for the purposes of using the convenient PMT formula, you can just *pretend* to make a momentary investment on December 31. It won't make any difference mathematically; it nets right back to the actual picture above, simply increasing the proceeds of the account closing by the same $100 you just invested. *But it satisfies the math used by the PMT formula:*



	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	12	0.50	–1,000.00	–100.00	$\boxed{2{,}295.23}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	6.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

Again: This ***FV*** result reflects the $100 you stuck in the account just before closing it. So if you're trying to figure the net account proceeds of the real-life transaction (as the first diagram opposite shows) it's $100 *less* than your ***FV*** result: $2,195.23. You made the December 31 investment just to avoid having to find a slow, tedious alternative to the PMT formula.

Could you have done it such a way that the ***FV*** result does exactly reflect the net account's proceeds? Yep—try this:



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	***i%***	***PV***	***PMT***	***FV***
12	0.50	–900.00	–100.00	$\boxed{2{,}195.23}$
1	6.00		(***BEGINning***	
Yrs	***I%YR***		of the period)	

See the logic behind this version? Again, you didn't change anything mathematically, but you satisfied the requirement of the PMT formula by allowing one PMT cash flow at the same point in every period.

Of course, there are many straightforward loans and other scenarios where you don't have to stop and mess about like this, trying to adjust the actual picture in a mathematically neutral way just to satisfy the PMT formula. This example was just to emphasize the rules so you never forget:

PMT is the uniform cash-flow that occurs once each period, and its value must be broken out separately and correctly on every diagram in order for the PMT formula to work correctly.

Annuity Mode
The Sixth Variable

By now, of course, one question is surely occurring to you:

How do you indicate in the PMT formula whether the ***PMT*** occurs at the beginning or at the end of each period? Answer: That's the purpose of the *annuity mode* variable, ***A***. Here's the formula again:

$$PV + (1+iA)(PMT)\left[\frac{1-(1+i)^{-n}}{i}\right] + \frac{FV}{(1+i)^n} = 0$$

If a ***PMT*** occurs at the beginning of each period (called "annuity in advance"), then you must use an ***A*** value of 1; if a ***PMT*** comes at the end of each period ("annuity in arrears"), you use an ***A*** value of 0.

On most calculators, of course, you don't explicitly key in a 0 or 1. Usually there's a little indicator, annunciator or other toggled "mode" that you must adjust, which internally translates into a 0 or 1 in the formula. But make no mistake: the annuity mode *is* an input value that you must give for every use of the ***PMT*** formula with your calculator (unless ***PMT*** = 0). That's why the annuity mode is noted as part of the solution in the calculator data boxes—right beneath the ***PMT*** value itself:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	12	0.50	–900.00	–100.00	$\boxed{2,195.23}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	6.00		***(BEGINNING***	
	Yrs	***I%YR***		of the period)	

All in all, then, you have five actual numerical values in the general PMT formula—plus the annuity mode. Your calculator will allow you to solve for any one of those five, given the other four values and the proper annuity mode. (And note that some machines let you key in the Periods/Year, so that you can deal directly with annualized values for ***n*** and/or ***i***.)

This method—to give all values but one, then solve for the unknown —is the key to problem-solving with the PMT formula. So long as you adhere to the assumptions of the formula and draw a correct cash flow diagram, your calculator can use the PMT formula to quickly help you complete the diagram for a valid, balanced transaction.

Time for practice now—lots of it—with the following quiz. As you work through each problem and solution, notice how seemingly distinct terminologies of various financial transactions melt away when you diagram them for solution. No matter the jargon: if you can draw a picture of it that satisfies the PMT formula, you can solve it.

Key points that were covered here:

- Just as the PV-FV formula (the "sign-flipped" version) calculates the balancing cash flow for a simple PV-FV transaction, so the PMT formula does for a transaction that includes a uniform series of cash flows—one per period.
- The assumptions behind the math of the PMT formula demand that the ***PMT*** cash flow occurs either at the end of each period or at the beginning. Therefore, at one end of the timeline or the other, a ***PMT*** cash flow will coincide with the ***PV*** or ***FV***, but you must never net these two together when using the PMT formula; it needs to use them separately in its math.
- If you adhere to that rule—and properly draw the cash flow diagram, with its sign conventions—the PMT formula is a flexible, powerful tool that will let you solve for any of five financial values, given the other four and the annuity mode.

Quiz

1. Find the monthly payment amount on a 30-year mortgage of \$100,000 at 8%. What if the rate were 9%? What if it were 7%? (The solution is on page 56.)

2. What's the monthly payment on a 20-year mortgage of \$100,000 at 8%? What if it's a 15-year mortgage? What's the term if the payment is exactly \$800/mo.? (Page 57.)

3. A 130,000 mortgage that would normally amortize (pay off) fully in 30 years of monthly payments (\$953.89/mo.) is scheduled instead to come due in a lump-sum balloon payment after 5 years. What is that balloon amount? What if the payment were exactly \$1,000/mo.? What payment amount is necessary for the balloon to be exactly \$115,000? (Page 58.)

4. An adjustable-rate mortgage (ARM) of \$125,000 carries a 6.75% annual rate for the first year, then adjusts to 7.5% in years 2 and 3, then to 8.25% thereafter, for the rest of a 30-year term (27 more years). Find the PMT amounts in each of the three rate periods. (Page 59.)

5. To qualify for a mortgage through a certain lender, a prospective borrower may use no more than 40% of monthly gross income toward principal, interest, taxes and insurance, and other installment payments (car, credit card, etc.). A couple has a combined gross income of \$6,000/month, a \$350/month car lease payment, and \$150/month in credit card payments. They're shopping for a home in an area where taxes and insurance typically total about \$200/month. If current 30-year fixed mortgage rates are at 7.75% with a 10% down-payment, how much home can they buy? What if they can put down 20% and get a 7.375% rate? (Page 61.)

6. Mortgage interest is often tax-deductible, and many calculators have special keys or operations to allow you to compute the principal and interest amounts paid over a given number of periods (e.g. the 12 monthly payments in a tax year). But these are also straightforward to calculate via the PMT formula. Find the principal and interest paid over the first 12 payments of a 30-year, \$100,000 mortgage at 8%. (Page 62.)

7. You take out a second mortgage of \$40,000 for home renovation, with a monthly payment calculated to fully pay off the loan in 30 years, but the balance due in 5 years (i.e. "a 30-year amortization, 5-year balloon"). The mortgage rate is 7.75%, with a loan origination fee of 2%. What rate are you really paying? (Page 63.)

8. If you're financing \$150,000 at 7.5% for 30 years, how much shorter would the term be if you paid \$50/month more than the required payment? How about \$100/month more? (Page 64.)

9. If you're financing \$150,000 at 7.5% for 30 years, how much shorter would the term be if you paid half the required monthly payment in the middle of the month and the other half at the end of the month, rather than the whole payment at the end of the month—24 half-payments per year? (Page 65.)

10. If you're financing \$150,000 at 7.5% for 30 years, how much will you shorten the term if you pay half the required monthly payment every 2 weeks—26 half-payments per year? (Page 65.)

11. How long will you need to make monthly payments on a 30-year, \$100,000 mortgage at 8% before you've paid down the loan balance to \$75,000? How about \$50,000? \$25,000? (Page 66.)

12. How long does it take to pay off a \$10,000 loan at 10% with monthly payments of \$75? (Page 68.)

13. A construction loan for $150,000 is a 7.5% mortgage written so that there are no payments for 9 months, then a normal 30-year stream of payments. What's the payment amount? (Page 69.)

14. You're 5 years into a 30-year mortgage on $110,000 at 8.5%. Rates have now dropped to 7.5%, and you can refinance with a 1% fee plus $300 for the appraisal. How long must you keep the house after the refinancing to make it worthwhile? (Page 70.)

15. You need to lease a $250,000 piece of equipment. If the lease rate is 18% annually, find the monthly payment on a 10-year term with a 10% residual buyout. What if it's a $1 buyout? (Page 72.)

16. 10 years ago, you bought a certain ballplayer's rookie card for $1. He's now a major star—future Hall of Fame material—and that rookie card is now appraised at $50. What annual appreciation rate does this represent? (Page 72.)

17. You just won the state lottery. Choose which you would rather receive: A lump-sum cash payment today of $1 million, taxable at a rate of 50%; or 20 annual tax-free payments of $50,000. Do expectations as to inflation rates affect your answer? (Page 73.)

18. What monthly amount should doting grandparents begin to contribute to a newborn's taxable college fund account if they wish to have the future equivalent of $150,000 available on that child's 18th birthday? Assume an inflation rate of 3%, an account growth rate of 10%, and a tax rate of 20%. What monthly amount would be necessary if the account were non-taxable? (Page 74.)

19. You're the holder of a note due in 3 years. The face amount due is $75,000. You need cash now, and you're willing to sell the note at a discount. What should the buyer pay if he wants to yield 15%? What if the note were due instead as three annual installments of $25,000? (Page 75.)

20. You've found the new \$24,000 car you want to be driving (for personal use) for the next 3 years, and now you're trying to decide whether to buy it or lease it. Either way, you figure that the resale value after 3 years will be about \$16,000. Your choices:

To buy, you'd pay \$1,200 down and finance the balance over 72 months at 6.5% annual interest rate, with a \$300 loan fee.

To lease it, you'd pay \$2,400.00 down, plus an acquisition fee of \$400, then 36 monthly payments of \$395 (due at the beginning of each month). Assuming no disposition fees for excessive mileage or wear-and-tear after 36 months, you could then buy it out for \$12,000.00. (Page 75.)

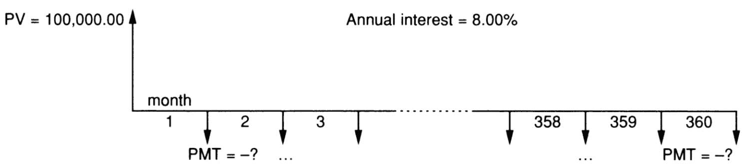
21. A married couple is planning for retirement in 25 years. The husband, age 35, has put his savings of \$16,000 into a taxable mutual fund that will earn 10% per year (compounded monthly), after taxes, long-term. His government pension income will be about \$4,000/month if he begins to draw it at age 65. The wife, age 30, has put her savings of \$12,500 into the same taxable mutual fund. Her pension will be about \$3,600 if she, too, begins to draw it at age 65. Beginning immediately, each spouse will open a non-taxable retirement account earning 10%/year (compounded monthly). Each will contribute the yearly maximum, \$2,000, in monthly installments. Will they have enough when they retire so they can draw \$10,000/month until the husband is 95? (Page 77.)

22. On their 18th birthday, twins Amy and Brad each become eligible to open a tax-deferred retirement account, with a maximum yearly contribution of \$2,000. Amy opens hers immediately and contributes the maximum amount yearly through her 30th birthday, then doesn't invest anything more in the account. Brad waits until his 30th birthday to open his account, then makes the maximum contribution every year through his 64th birthday. If both accounts earn a steady 12% annually over all the years, how do the two balances compare on the twins' 65th birthday? (Page 80.)

Quiz Solutions

1. Here's the situation for the 8% case, followed by calculator inputs and results for all three rates. In all three calculations, the unknown (the boxed value) being calculated is the ***PMT*** amount.

Note: With most calculators, after you solve for the ***PMT*** using the first rate (8%), you need only vary the rate (***i%***) to get the other answers. You probably don't need to clear and re-enter the other known values, which are the same in all three calculations.



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.667	100,000.00	$\boxed{-733.76}$	0.00
30	8.000		***(END***	
Yrs	***I%YR***		of the period)	

On a $100,000, 30-year mortgage at 8%, the PMT is $733.76/mo.

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.750	100,000.00	$\boxed{-804.62}$	0.00
30	9.000		***(END***	
Yrs	***I%YR***		of the period)	

On a $100,000, 30-year mortgage at 9%, the PMT is $804.62/mo.

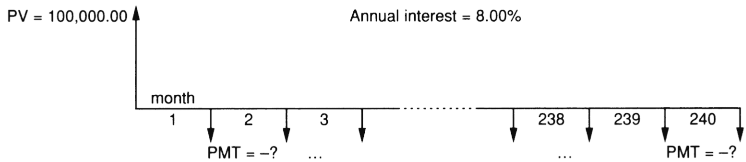
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.583	100,000.00	$\boxed{-665.30}$	0.00
30	7.000		***(END***	
Yrs	***I%YR***		of the period)	

On a $100,000, 30-year mortgage at 7%, the PMT is $665.30/mo.

2.

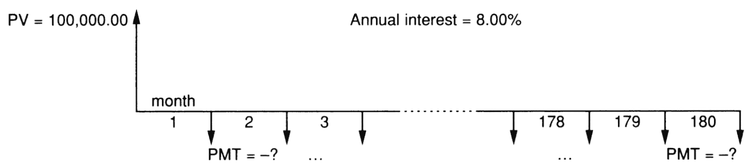


P (or ***P/YR***) = 12 Periods/Year

*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*

n	***i%***	***PV***	***PMT***	***FV***
240	0.667	100,000.00	–836.44	0.00
20	8.000		***(END***	
Yrs	***I%YR***		of the period)	

On a $100,000, 20-year mortgage at 8%, the PMT is $836.44/mo.

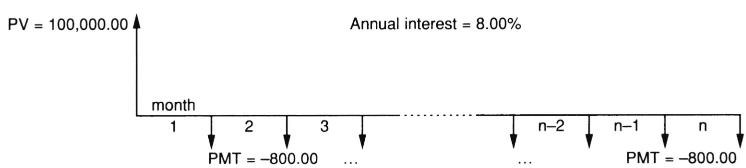


P (or ***P/YR***) = 12 Periods/Year

*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*

n	***i%***	***PV***	***PMT***	***FV***
180	0.667	100,000.00	–955.65	0.00
15	8.000		***(END***	
Yrs	***I%YR***		of the period)	

On a $100,000, 15-year mortgage at 8%, the PMT is $955.65/mo.



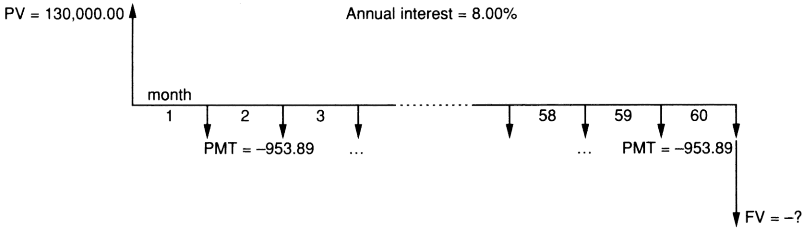
P (or ***P/YR***) = 12 Periods/Year

*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*

n	***i%***	***PV***	***PMT***	***FV***
269.7	0.667	100,000.00	–800.00	0.00
22.47	8.000		***(END***	
Yrs	***I%YR***		of the period)	

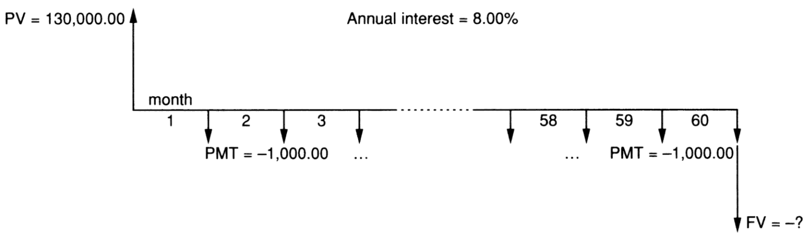
A $100,000 mortgage at 8%, with an $800 monthly PMT, will take 270 months to pay off. (The last payment is less than $800 —see Chapter 5 for more about partial periods/payments.)

3. Find the balloon amount:



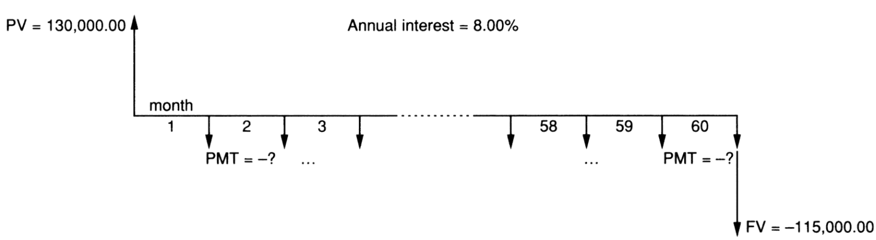
	n	i%	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	60	0.667	130,000.00	–953.89	$\boxed{-123{,}591.10}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	5	8.000		***(END***	
	Yrs	***I%YR***		**of the period)**	

Re-calculate with the prescribed $1,000 PMT:



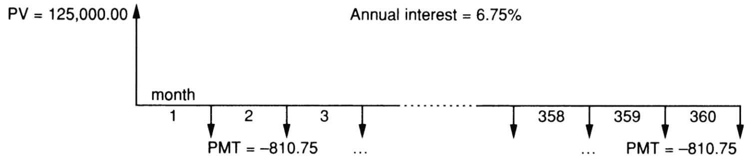
	n	i%	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	60	0.667	130,000.00	–1,000.00	$\boxed{-120{,}203.09}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	5	8.000		***(END***	
	Yrs	***I%YR***		**of the period)**	

Set the prescribed $115,000 balloon and find the necessary PMT:



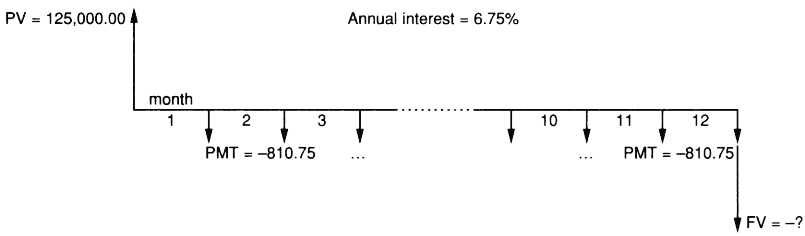
	n	i%	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	60	0.667	130,000.00	$\boxed{-1{,}070.81}$	–115,000.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	5	8.000		***(END***	
	Yrs	***I%YR***		**of the period)**	

4. First, calculate the monthly payment amount during year 1:



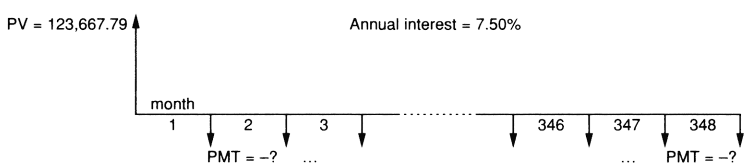
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.563	125,000.00	$\boxed{-810.75}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	6.750		***(END*** of the period)	
	Yrs	***I%YR***			

Then find the remaining balance at the end of year 1:



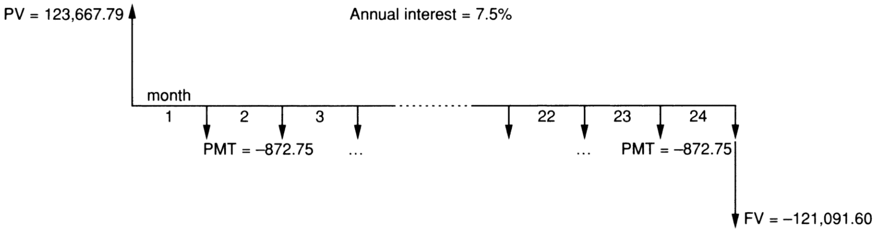
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	12	0.563	125,000.00	–810.75	$\boxed{-123{,}667.79}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	6.750		***(END*** of the period)	
	Yrs	***I%YR***			

That remaining balance becomes the starting balance—i.e. the "amount financed"—for a new *29-year* loan (at 7.5%):



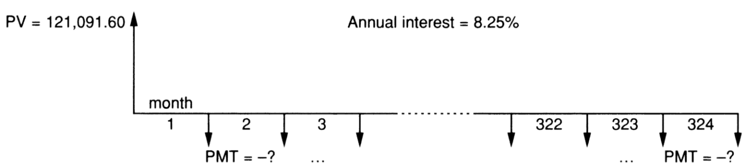
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	348	0.625	123,667.79	$\boxed{-872.75}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	29	7.500		***(END*** of the period)	
	Yrs	***I%YR***			

The \$872.75 monthly payment is in effect for 2 years—i.e. until the end of year 3. Find the remaining balance at that time:



	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	24	0.625	123,667.79	–872.75	$\boxed{-121{,}091.60}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	2	7.500		**(*END***	
	Yrs	***I%YR***		**of the period)**	

That remaining balance becomes the starting balance—i.e. the "amount financed"—for a new *27-year* loan(at 8.25%):



	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	324	0.688	121,091.60	$\boxed{-933.95}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	27	8.250		**(*END***	
	Yrs	***I%YR***		**of the period)**	

Note: All dollar amount results in the above solution were rounded to whole cents before being used in the next calculation (though no interest rates were rounded). If you simply re-use your calculator's 10- or 12-digit results in subsequent calculations, you'll get slightly different answers for the remaining balances (though these are not different enough to affect the ***PMT*** results to within a penny):

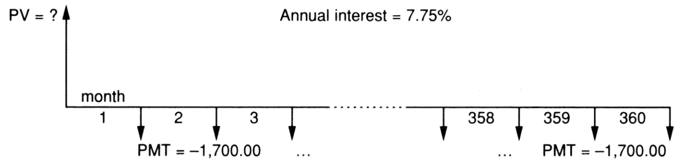
FV after year 1: –\$123,667.81

FV after year 3: –\$121,091.67

Strictly speaking, these numbers are more accurate mathematically, but they're not correct in the practical sense. For example, nobody writes a mortgage payment check for \$810.7476207..., so for the *best* answer, the remaining balance after year 1 should be computed using a ***PMT*** of exactly –\$810.75. For a more in-depth look at rounding issues, see Chapter 5.

5. You're looking for the maximum allowable amount for principal and interest (P&I), the two parts to a mortgage payment (PMT). If principal, interest, taxes and insurance (PITI), plus other debt payments, can be no more than 40% of \$6,000, or \$2,400, then PITI can be no more than \$2,400–500, or \$1,900. And with taxes and insurance at \$200/mo., that leaves \$1,700 available for P&I.

So for a 10% down-payment, here's the financing picture:



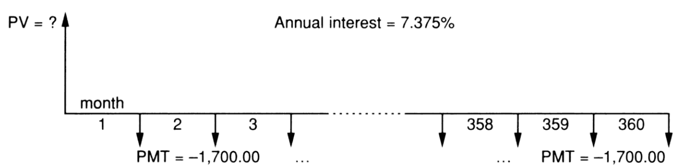
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.646	$\boxed{237{,}293.54}$	–1,700.00	0.00
30	7.750		(***END***	
Yrs	***I%YR***		of the period)	

If the amount financed, \$237,293.54, is 90% of the home value, the couple can qualify for financing for a house priced no higher than \$263,659.49 (That's \$237,293.54 ÷ 0.90.) They will need \$26,365.95 for a down-payment, plus additional for closing costs.

For a 20% down-payment, here's the financing picture:



P (or ***P/YR***) = 12 Periods/Year

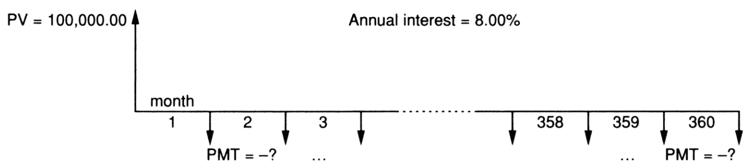
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.615	$\boxed{246{,}135.97}$	–1,700.00	0.00
30	7.375		(***END***	
Yrs	***I%YR***		of the period)	

If the amount financed, \$246,135.97, is 80% of the home value, the couple can qualify for financing for a house priced no higher than \$307,669.97 (That's \$246,135.97 ÷ 0.80.) They will need \$61,533.99 for a down-payment, plus additional for closing costs.

6. Mortgage payments (PMT's) pay both principal and interest, so the sum of the year's PMT's is the sum of the amounts paid to principal and interest.

First, find the PMT amount:



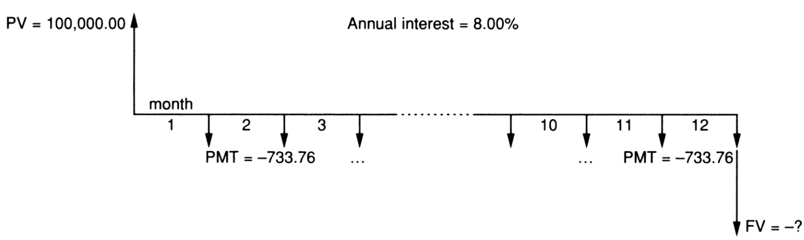
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.667	100,000.00	$\boxed{-733.76}$	0.00
30	8.000		(***END***	
Yrs	***I%YR***		of the period)	

OK, so you paid \$733.76/month—a 12-month total of \$8,805.12 in principal and interest. How much of that was principal? It was the *difference* between the year's starting loan balance (\$100,000) and its ending balance.

Find that ending balance:



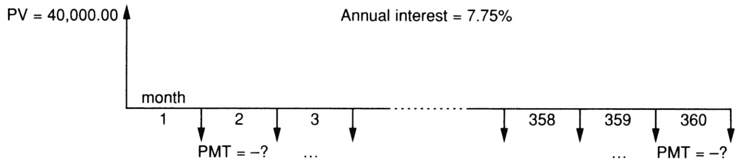
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
12	0.667	100,000.00	–733.76	$\boxed{-99{,}164.69}$
1	8.000		(***END***	
Yrs	***I%YR***		of the period)	

You paid \$(100,000.00 – 99,164.69) = \$835.31 toward principal in the first year. So the rest of the \$8,805.12 was all paid toward interest: \$8,805.12 – \$835.31 = \$7,969.81.

7. First, find the PMT amount:

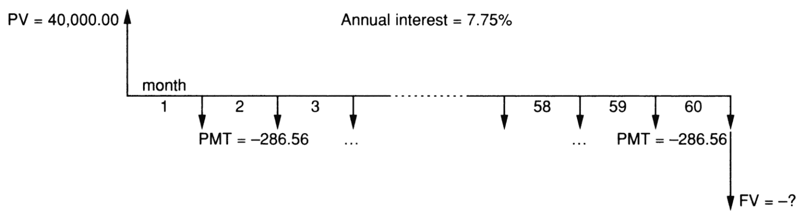


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
360	0.646	40,000.00	$\boxed{-286.56}$	0.00
30	7.750		(*END*	
Yrs	*I%YR*		of the period)	

Now find the 5-year balloon amount:

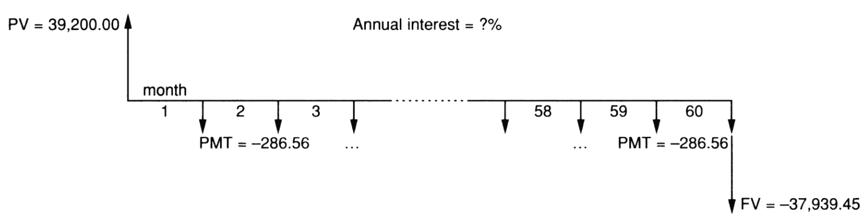


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
60	0.646	40,000.00	–286.56	$\boxed{-37{,}939.45}$
5	7.750		(*END*	
Yrs	*I%YR*		of the period)	

And now that you know all the cash flow amounts, you can draw the full picture and calculate what rate you really paid:



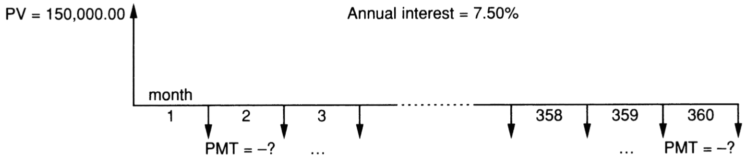
P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
60	$\boxed{0.688}$	39,200.00	–286.56	–37,939.45
5	$\boxed{8.250}$		(*END*	
Yrs	*I%YR*		of the period)	

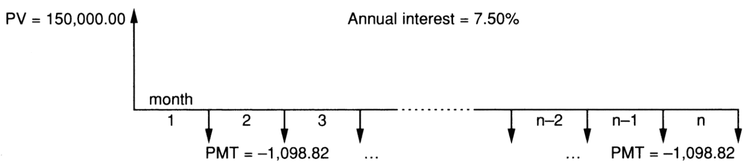
8.25%—quite a boost from 7.75%. Over a relatively short time (5 years), an up-front loan fee can really affect the cost of money.

8. First, find the ***PMT*** amount:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.625	150,000.00	$\boxed{-1{,}048.82}$	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	30	7.500		***(END***	
	Yrs	***I%YR***		of the period)	

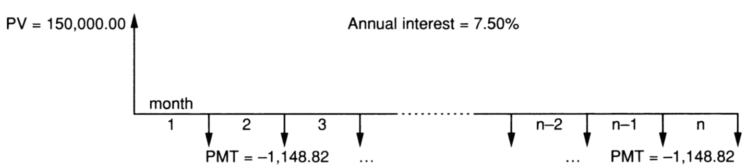
Now increase that ***PMT*** amount by $50 and solve for ***n***:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	$\boxed{307.9}$	0.625	150,000.00	–1,098.82	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	$\boxed{25.66}$	7.500		***(END***	
	Yrs	***I%YR***		of the period)	

So paying an extra $50/month will shorten the loan term by about 308 months (about 25 years and 8 months).

Try $100/month extra:

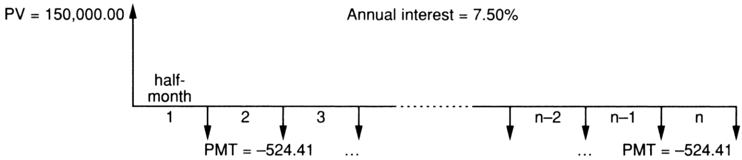


	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	$\boxed{271.7}$	0.625	150,000.00	–1,148.82	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	$\boxed{22.64}$	7.500		***(END***	
	Yrs	***I%YR***		of the period)	

Paying an extra $100/month will shorten the loan term by about 272 months (about 22 years and 8 months).

9. First, you have to find the normal monthly ***PMT*** amount. You just did this in the previous problem (opposite page): –\$1,048.82

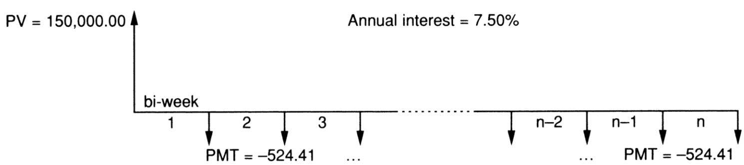
Half of this would be –\$524.41. So here's the picture:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 24 Periods/Year	$\boxed{718.9}$	0.3125	150,000.00	–524.41	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	$\boxed{29.95}$	7.500		(***END***	
	Yrs	***I%YR***		**of the period)**	

This doesn't shorten the term much—less than a month, actually.

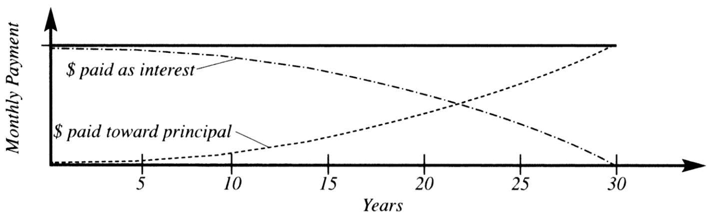
10. This is the same as the previous problem, except there are two more periods per year:



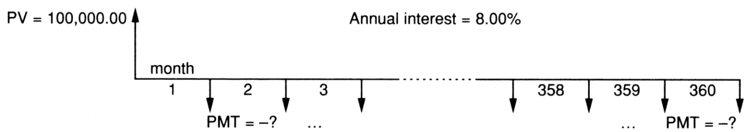
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 26 Periods/Year	$\boxed{605.3}$	0.288	150,000.00	–524.41	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	$\boxed{23.28}$	7.500		(***END***	
	Yrs	***I%YR***		**of the period)**	

This shortens the term considerably—nearly 7 years—quite a contrast from the case above. That only makes sense though: Above, you weren't actually paying any more on the loan over any given month or year. You were just paying half of it sooner each month. By contrast, with 26 half-payments per year, you are paying some \$1,048.82 *extra* per year against the principal. That makes a big difference in reducing the loan balance quickly.

11. As you know, a monthly mortgage payment combines payments to principal and interest. The interest portion of the PMT amount covers all the interest charged that month. (In mortgages, interest on the loan is never allowed to start compounding; it's paid in full every month.) Then, on top of that, a little bit of principal is paid. As time goes on and the balance shrinks, there is less interest charged each month, so more of the PMT amount can go to-ward principal. Thus you get an accelerating rate of payoff of the principal as you near the end of the loan term:

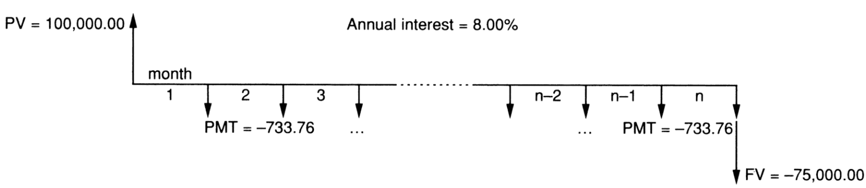


First, find the ***PMT*** amount:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.667	100,000.00	$\boxed{-733.76}$	0.00
Interest rate is used in decimal form: ***i*** *=* ***i%****/100 =* ***I%****/(100****P****)*	30	8.000		***(END***	
	Yrs	***I%YR***		of the period)	

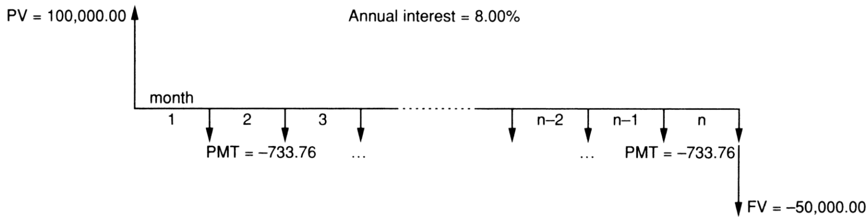
Now specify the remaining balance and find out how long it will take to reach that point—where the loan is 25% paid:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	$\boxed{187.9}$	0.667	100,000.00	–733.76	–75,000.00
Interest rate is used in decimal form: ***i*** *=* ***i%****/100 =* ***I%****/(100****P****)*	$\boxed{15.65}$	8.000		***(END***	
	Yrs	***I%YR***		of the period)	

It takes over half of the loan's term to pay off a quarter of it.

Calculate likewise for the 50%-paid and 75%-paid points:

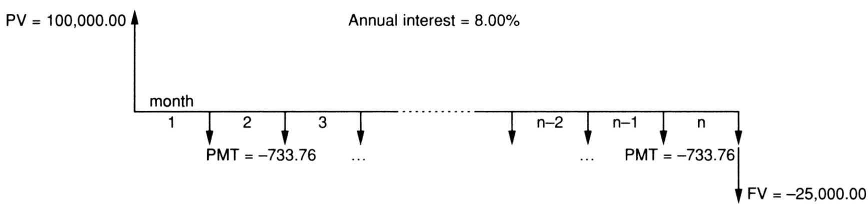


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
$\boxed{268.9}$	0.667	100,000.00	–733.76	–50,000.00
$\boxed{22.40}$	8.000		(*END*	
Yrs	*I%YR*		of the period)	

It takes nearly 3/4 of the loan's term to pay off half of it.



P (or *P/YR*) = 12 Periods/Year

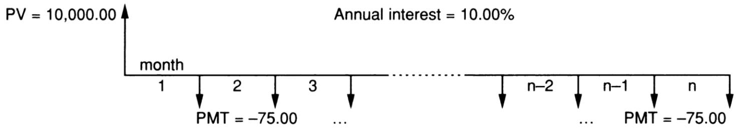
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
$\boxed{321.2}$	0.667	100,000.00	–733.76	–25,000.00
$\boxed{26.77}$	8.000		(*END*	
Yrs	*I%YR*		of the period)	

You'll pay off a full quarter of the loan in the last 3 years and 3 months. This is where the amount paid to principal really begins to accelerate.

Note that the amount of the loan doesn't affect the payoff curve at all; you'd find the same quarter-, half- and three-quarter-paid points for a loan of $1000 or $1 million, so long as it's monthly payments for 30 years at 8%. But varying the interest rate or term does affect the "way-points" and the shape of the curve: A longer or shorter term will"squeeze" or "stretch" the curve proportionally. A higher interest rate will flatten the curve along most of its length, delaying the way-points.

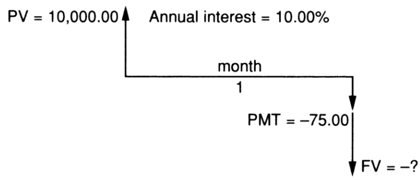
12. The problem seems simple enough:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	??	0.667	10,000.00	–75.00	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	??	10.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

You won't be able to complete this calculation, though—you'll probably get an error message on your calculator. Why? Because *you'll never pay off this loan*—$75/month doesn't even cover the interest. The remaining balance every month will be *growing*, not shrinking toward zero. This is called ***negative amortization***.

To demonstrate this, find the remaining balance after 1 month:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	1	0.833	10,000.00	–75.00	–10,008.33
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)		10.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

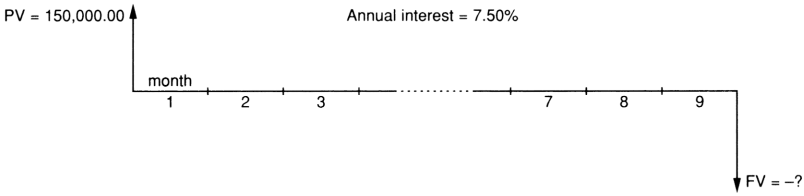
The amount the balance grew in the one month ($8.33) represents the interest earned in excess of the PMT. To make *interest-only* PMT's (i.e. not touch the principal at all), the ***PMT*** amount would therefore need to be $8.33 more than $75/month, or $83.33. And in order for the loan's principal ever to be paid off, you'll have to pay at least a penny more to start chipping away at it: $83.34. And with just that minimal starting amount, it'll take awhile:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	1,137	0.833	10,000.00	–83.34	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	94.7	10.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

In fact, it will take almost 95 years.

13. The key here is to realize that the amount you'll actually need to finance for 360 PMT's will be more than the original $150,000—increased by the amount of unpaid interest that has accrued over those 9 preliminary months.

So, first you need to calculate that accrued, unpaid interest:

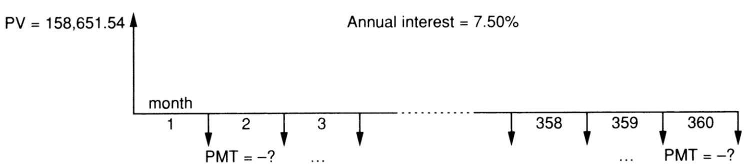


P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
9	0.625	150,000.00	0.00	$\boxed{-158{,}651.54}$
0.75	7.500			
Yrs	***I%YR***			

So $158,651.54 is the amount you'll need to pay off in 360 equal installments, and now you can readily calculate that ***PMT*** value:



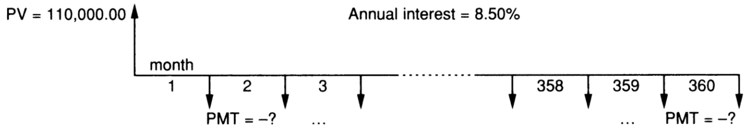
P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
360	0.625	158,651.54	$\boxed{-1{,}109.31}$	0.00
30	7.500		***(END***	
Yrs	***I%YR***		of the period)	

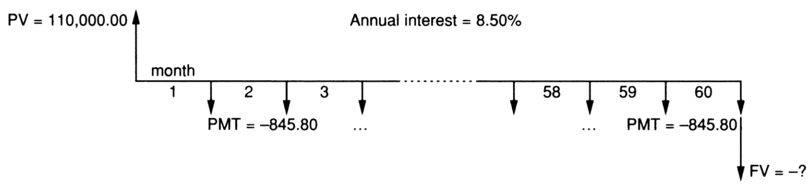
14. The question is, how long will you need to make your new PMT's in order for your savings—the difference between the new PMT's and the old ones—to balance the cost of refinancing?

The first thing to do, then, is to figure out what your refinancing will cost, and that means calculating the amount you'll need to refinance—the remaining balance on the mortgage. And to do that, first you have to find the original PMT amount:



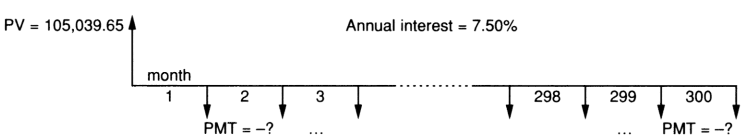
	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	360	0.708	110,000.00	$\boxed{-845.80}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	8.500		***(END***	
	Yrs	***I%YR***		of the period)	

Now figure the balance after 5 years:



	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	60	0.708	110,000.00	–845.80	$\boxed{-105{,}039.65}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	5	8.500		***(END***	
	Yrs	***I%YR***		of the period)	

So the refinance cost will be 1% of $105,039.65, or $1,050.40, plus $300, for a total of $1,350.40. Now, what's your new PMT?



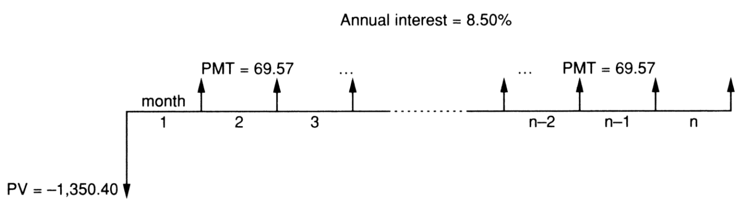
	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	300	0.625	105,039.65	$\boxed{-776.23}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	25	7.500		***(END***	
	Yrs	***I%YR***		of the period)	

So you'll be saving $69.57/month on your mortgage PMT after refinancing, but it will cost you $1,350.40 to do the deal. OK, to figure out how many months you'll need to pay on the new mortgage to recoup your expenses, do you simply divide $1,350.40 by $69.57, concluding that you'll be at break-even after 20 months?

No—not if you want to be accurate. Yes, straight division is often used as a quick "ball-park" method, but it's not correct. Don't forget the *time value* of money. You'll pay the refinancing costs all at once, but you'll recover those expenses over time. Remember: You can't compare values until you adjust them to the same point on the time line.

So, what rate should you use to do the "sliding?" What is money worth to you? Well, it's worth more than a savings account rate, isn't it? You pay 8.5% to borrow, so there's your opportunity cost rate right there: If you weren't putting the $1,350.40 into the refinancing, you could earn 8.5% with it, by paying down your current mortgage balance that much.

So here's the calculation you really need to find your break-even point for refinancing:

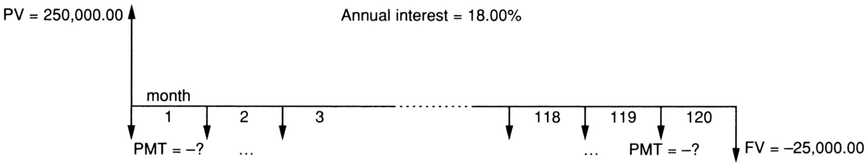


	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	$\boxed{20.96}$	0.708	–1,350.40	69.57	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	$\boxed{1.75}$	8.500		**(*END***	
	Yrs	***I%YR***		**of the period)**	

As you can see, if you account for the *opportunity cost* of your $1,350.40 investment in the refinancing, you'll be at break-even after *21* months.

15. A lease is very similar to a mortgage; you're renting equipment rather than money, but the idea is the same. Just don't forget that, unlike money, when you rent other more tangible capital assets, the rental payments are due at the beginning of each period.

So here's the situation with the 10% buyout:

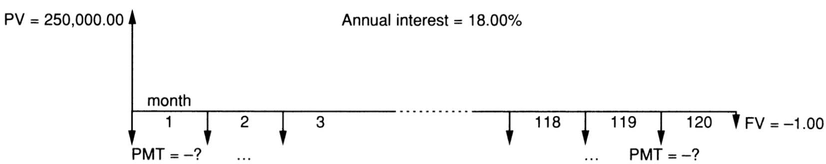


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
120	1.500	250,000.00	$\boxed{-4,363.71}$	–25,000.00
10	18.00		(*BEGINning*	
Yrs	*I%YR*		of the period)	

And here's the situation with the $1 buyout:

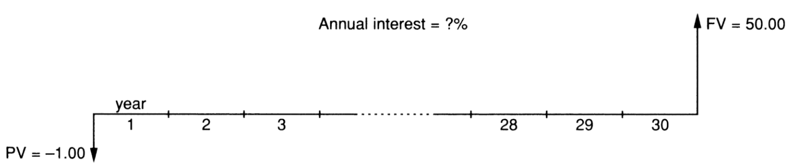


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
120	1.500	250,000.00	$\boxed{-4,438.06}$	–1.00
10	18.00		(*BEGINning*	
Yrs	*I%YR*		of the period)	

16. This is just a simple PV-FV calculation, where the interest rate is the unknown:



P (or *P/YR*) = 1 Period/Year

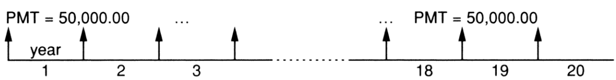
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
30	$\boxed{47.88}$	–1.00	0.00	50.00
30	$\boxed{47.88}$			
Yrs	*I%YR*			

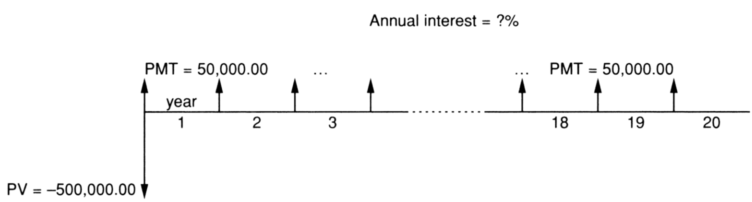
17. You'll notice that nothing was said about what opportunity cost—what discount rate—you ought to use to compare them. Why not make that your unknown? Find the rate of interest that would equate the two scenarios and then ask yourself if it's realistic:

The first scenario is simple: Of the million dollar prize, if you opt to take a lump-sum now, taxes will eat half; you'll get $500,000.

Alternatively, you can get this 20-year cash flow stream, tax-free:



So, what interest rate would equate the two? At what rate would you have to earn on your $500,000 lump-sum in order to receive the other payment stream?



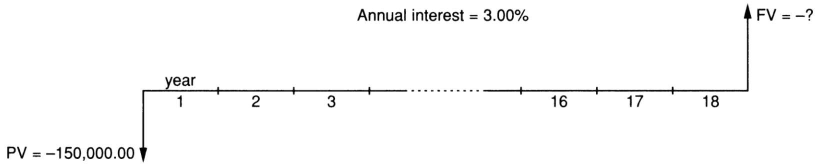
P (or ***P/YR***) = 1 Period/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
20	$\boxed{8.92}$	–500,000.00	50,000.00	0.00
20	$\boxed{8.92}$		(***BEGIN***ning	
Yrs	***I%YR***		of the period)	

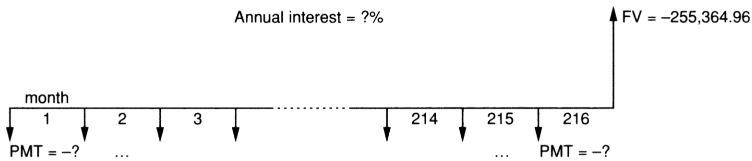
Is that a rate you might expect to earn—reliably enough to get this annual income? Maybe. But don't you need to try to predict inflation in order to make your decision? No. For one thing, the interest rates available in the markets already reflect inflation expectations; they're "built in." But really, the only question you need ask is whether you can make the your lump-sum earn better than $50,000/year in *face-value* income—unadjusted for inflation—because that's what you're comparing it to; the lottery folks aren't offering to inflation-adjust your 20 annual installments.

18. First, you'd better figure out what $150,000 in today's dollars' buying power will have to be in 18 years, at 3% inflation:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 1 Period/Year	18	3.000	–150,000.00	0.00	$\boxed{255,364.96}$
*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*	18	3.000			
	Yrs	***I%YR***			

Now you just need to complete this picture:



Now, what interest rate will this college savings account earn? If the interest is taxable, then a decent approximation of the after-tax interest (see Chapter 5 for more on this topic) is to reduce the interest rate by the tax rate. That is, if you earn 10 cents on the dollar before a 20% tax, then that tax will take 2 of the 10 cents. So your after-tax interest rate is 8 cents on the dollar, or 8%.

Now you're ready to calculate:

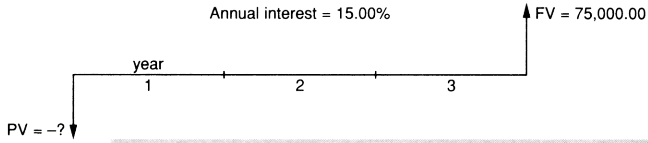
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	216	0.67	0.00	$\boxed{-528.39}$	255,364.96
*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*	18	8.000		(***BEGIN****ning*	
	Yrs	***I%YR***		of the period)	

The grandparents should invest $528.39/month, starting today.

But of course, if the interest isn't taxable, the account would grow at the full 10%, and they wouldn't need to invest as much:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	216	0.833	0.00	$\boxed{-421.70}$	255,364.96
*Interest rate is used in decimal form: **i** = **i%**/100 = **I%**/(100**P**)*	18	10.00		(***BEGIN****ning*	
	Yrs	***I%YR***		of the period)	

19. Just draw the picture and find the unknown. In this case, it's ***PV***:

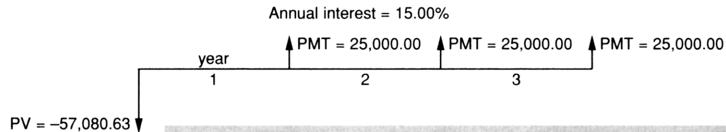


P (or ***P/YR***) = 1 Period/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
3	15.00	$\boxed{-49{,}313.72}$	0.00	75,000.00
3	15.00			
Yrs	*I%YR*			

That's what the buyer should pay if the face amount is due all at once, after the three years. If it's due instead in three annual installments, then here's the picture.



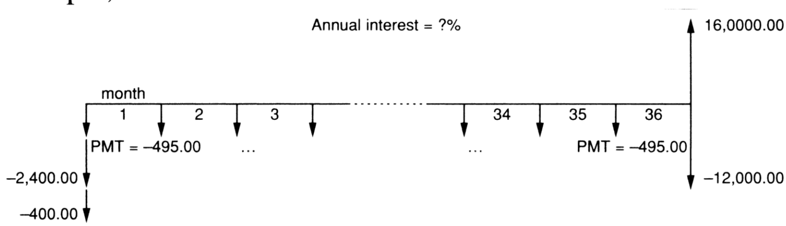
P (or ***P/YR***) = 1 Period/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
3	15.00	$\boxed{-57{,}080.63}$	25,000.00	0.00
3	15.00		**(*END***	
Yrs	*I%YR*		**of the period)**	

You don't have to discount the note as deeply in the installments scenario, because the buyer (i.e. the new lender) will be getting a good part of his money back sooner.

20. This "lease-vs.-buy" decision boils down to comparing the costs, but as you know, you can't compare them until you "slide-and-sum" them to the same point in time—preferably the present—using some reasonable cost of money as your discount rate. For example, the Lease situation would look like this:



You need to "slide-and-sum" all the future cash flows back to the beginning, then add the up-front cash flows you already know.

What's a good discount rate? You could use a money market or savings rate, of course, but note that your "cost of money" in one scenario is defined: 6.5%. Why not use it as the discount rate?

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 1 Period/Year	36	0.542	$\boxed{9{,}664.60}$	–395.00	4,000.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	3	6.500		**(*BEGIN*ning**	
	Yrs	***I%YR***		**of the period)**	

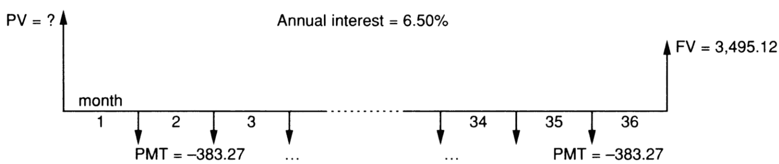
Now, recall that your calculator flips the sign of the answer when using the PMT formula—to give you the recommended balancing cash flow. It's recommending a positive $9,664.60, so the "slide-and-sum" total of all those future cash flows must be –$9,664.60. Adding the up-front leasing costs (–$2,800), the net present value of your 3-year leasing scenario is –$12,464.60.

To calculate the Buy scenario, first you have to figure out the loan payment and the remaining balance after 3 years:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	72	0.542	22,800.00	$\boxed{-383.27}$	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	6	6.500		**(*END***	
	Yrs	***I%YR***		**of the period)**	

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	36	0.542	22,800.00	–383.27	$\boxed{-12{,}504.88}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	3	6.500		**(*END***	
	Yrs	***I%YR***		**of the period)**	

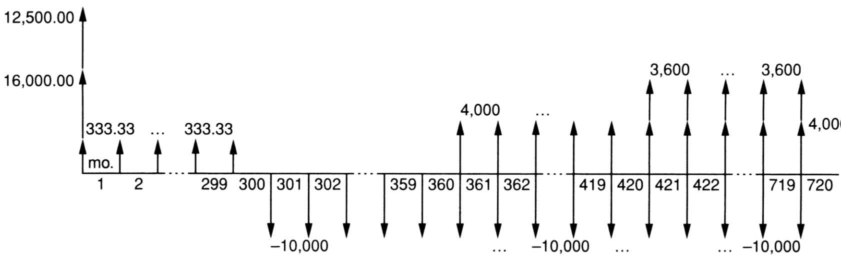
Now you can find the "slide-and-sum" total of the buy scenario's future cash flows:



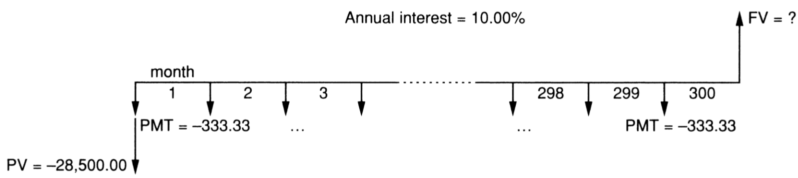
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	36	0.542	$\boxed{9{,}627.72}$	–383.27	3,495.12
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	3	6.500		**(*END***	
	Yrs	***I%YR***		**of the period)**	

After "unflipping" the sign of this result and adding the up-front costs (–$1,500), the net present value of the 3-year purchase scenario looks to be –$11,127.72. Buying is less costly than leasing.

21. Even though you have to calculate such a complicated scenario in parts, it often helps to see the whole thing mapped out first:



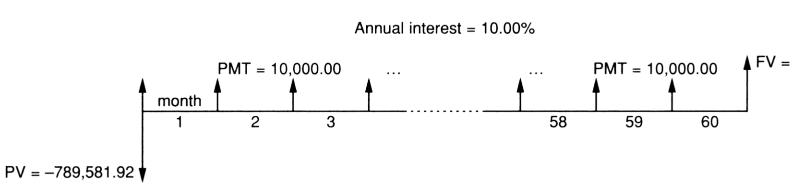
To begin, find out how much the couple will have in their retirement accounts and mutual funds the day they retire. Since all these accounts grow at the same rate, you can simply treat them all together, in one account:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	300	0.833	–28,500.00	–333.33	$\boxed{789{,}581.92}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	25	10.00		***(BEGINning***	
	Yrs	***I%YR***		**of the period)**	

That's their nest-egg on the day of their retirement. For the first 5 years after that, until his pension begins, they'll be drawing on this alone.

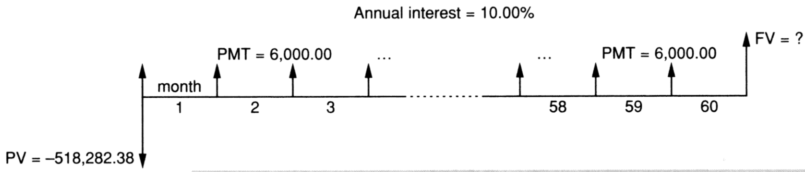
So, find out how much they'll have left when his pension begins:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	60	0.833	–789,581.92	10,000.00	$\boxed{518{,}282.38}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	5	10.00		***(BEGINning***	
	Yrs	***I%YR***		**of the period)**	

When his $4,000/month pension begins, they'll be drawing that much less—just $6,000/month—from their account, for 5 years, until her pension begins.

So, find out how much they'll have left when her pension begins:

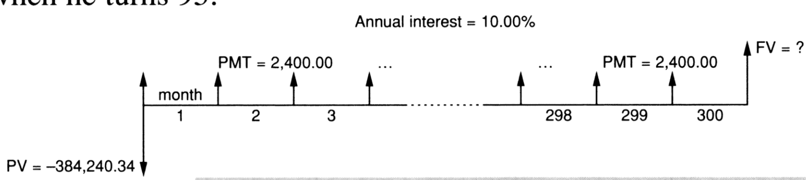


n	*i%*	*PV*	*PMT*	*FV*
60	0.833	–518,282.38	6,000.00	$\boxed{384,240.34}$
5	10.00		(***BEGIN**ning*	
Yrs	***I%YR***		of the period)	

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

When her $3,600/month pension begins, they'll be drawing that much less—just $2,400/month—from their account, for the next 25 years (his 95th birthday). Find out how much they'll have left when he turns 95:



n	*i%*	*PV*	*PMT*	*FV*
300	0.833	–384,240.34	2,400.00	$\boxed{1,421,827.8}$
25	10.00		(***BEGIN**ning*	
Yrs	***I%YR***		of the period)	

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

What's this? They'll have *plenty* left when he's 95—more, in fact than when they retired! What gives? It's this: At $2,400/month, they're not even drawing all the interest the account is still earning, so it's growing—*negative amortization* (which you encountered in problem **12**).

There may be regulations requiring them to draw down their retirement accounts (rather than let them grow)—and besides, this was money they planned to spend on themselves, not leave as an inheritance. So what level income should they draw—for 35 years of retirement—that empties their retirement account when he's 95? (Apparently, they'll live on just pensions thereafter.)

If you just try to guess at the correct income level (e.g. $11,000/month), you'll have to do the above solution all over again, working forward, finding each FV, making it the new PV, etc. Want to do that *over and over* until your guess gets a final FV of zero?

No? Well, why not "slide-and-sum"—to the beginning of their retirement years—all the pension income and add their "contributions" to the nest-egg? That way you'll have one net lump sum to amortize in a stream of level PMT's—easy to calculate.

"Slide-and-sum" her pension income (and mind the "sign flips"):

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	300	0.833	$\boxed{-399{,}471.45}$	3,600.00	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	25	10.00		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	120	0.833	$\boxed{-147{,}567.54}$	0.00	399,471.45
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	10	10.00			
	Yrs	***I%YR***			

"Slide-and-sum" his pension income (and mind the "sign flips"):

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.833	$\boxed{-459{,}601.64}$	4,000.00	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	30	10.00		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	60	0.833	$\boxed{-392{,}297.52}$	0.00	459,601.64
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	5	10.00			
	Yrs	***I%YR***			

Now net these two results together with the nest-egg to see what's really available to draw on:

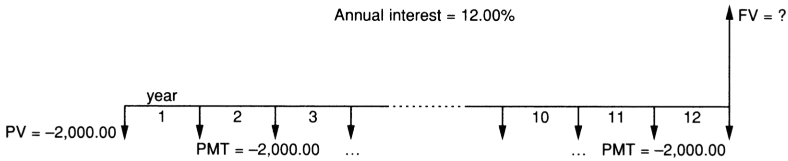
$789,581.92 + $147,567.54 + $279,340.63 = $1,216,490.09

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	420	0.833	–1,216,490.09	$\boxed{10{,}371.40}$	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	35	10.00		(***BEGINning*** of the period)	
	Yrs	***I%YR***			

Between their own nest egg and their pension incomes, they can draw $10,371.40/month for 35 years.

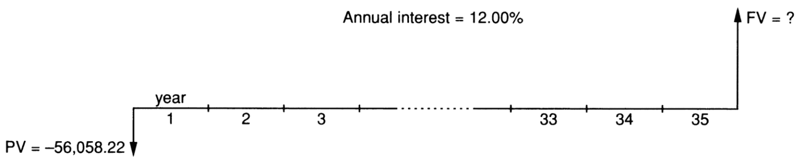
22. The question you're asking is: How do 13 annual early investments stack up against 35 such annual investments made later?

Here's Amy's picture during her investing years:



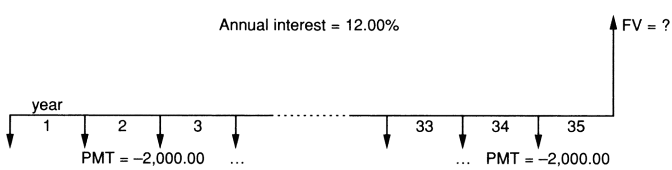
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 1 Period/Year	12	12.00	–2,000.00	–2,000.00	$\boxed{56{,}058.22}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	12	12.00		**(*END***	
	Yrs	***I%YR***		**of the period)**	

Then, after her 30th birthday, she just lets the account grow—no more investing—until her 65th birthday:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 1 Period/Year	35	12.00	–56,058.22	0.00	$\boxed{2{,}959{,}853}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	35	12.00			
	Yrs	***I%YR***			

Pretty nice nest egg. So just imagine what a pile her brother must have, seeing as how he invested so much more. Go ahead and run his numbers now:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 1 Period/Year	35	12.00	0.00	–2,000.00	$\boxed{966{,}926.23}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	35	12.00		**(*BEGINning***	
	Yrs	***I%YR***		**of the period)**	

You may be tempted to re-calculate that last result, but it's true; these numbers don't lie. This is an absolutely astounding illustration of the power of time and compounding:

Amy invests just $26,000 over 12 years; Brad invests nearly three times that, $70,000—over nearly three times the years. Yet because Amy started sooner, *she* is the multimillionaire by age 65—with over three times *more* savings than her brother!

Of course, it's not as if Brad is exactly prodigal; he saved soon enough and often enough to have a comfortable retirement. But look what the extra time of *earlier* investing does—and then in all the many years after age 30 (when one's life and family commitments often increase the demands on one's income), there's no need to save further.

If only every 18-year-old could see this and take it to heart! It's literally just a dollar an hour out of a paycheck, from age 18 to age 30, then not another dime, and he/she can be a millionaire by age 55 (and almost three times over by age 65)!

It's the <u>time</u> value of money.

4. Uneven Cash Flow Situations

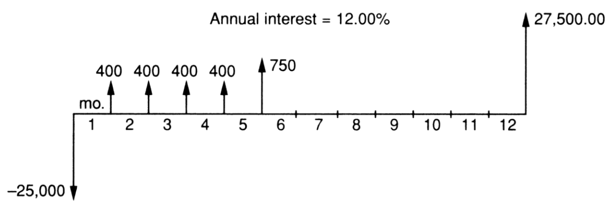
Key points to be covered here:

- The PMT formula is great for a wide variety of common financial transactions, but it applies only to a uniform, periodic cash flow. To balance non-uniform transactions, you must resort to the tedious slide-and-sum-to-zero, treating each cash flow individually. But your calculator can apply the more convenient PMT math to areas of the time line wherever flows are uniform. So you describe uneven cash flow situations to your calculator in terms of *groups* of identical cash flows.
- Once you draw the picture for your calculator that way, you can find the net sum (no sign flip) of the cash flows at any point in time. Finding the interest rate is trickier: Uneven cash flow streams can lead to multiple possibilities for the IRR. In those cases, testing the NPV with a variety of rates—or evaluating the "end-to-end" rate with methods such as MIRR or FMRR —may be more informative.

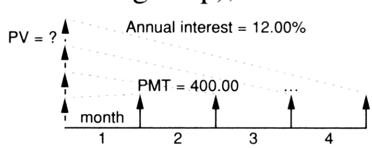
When Things Aren't So Smooth

As you've seen, to balance any financial transaction, you must use the fundamental Time Value of Money (PV-FV) relation to "slide" all its cash flows to the same point on the time line, then sum them. The result must be zero. If it isn't, the missing cash flow must be the opposite of the sum. (And most calculators actually give you this opposite, using a "sign-flipped" version of the PV-FV formula.)

Such sliding and summing is tedious for transactions involving many cash flows. Fortunately (as you've also seen) there's a mathematical shortcut available for situations with uniform periodic cash flows throughout the time line: the PMT formula. This cuts through the tedium in many installment loans and investments. But what about other sorts of transactions, where cash doesn't flow uniformly? After all, you can't apply the PMT formula to a situation like this, can you?



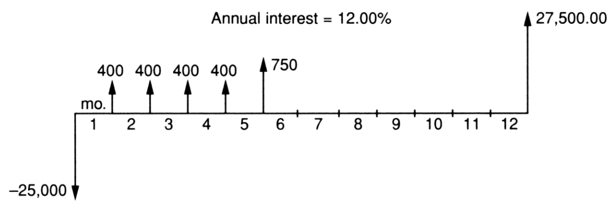
No, you can't apply the PMT formula to the whole time line at once, but for any section of it where cash flows are uniform and periodic, the PMT math could at least supply a shortcut there—to sum all those to the beginning or end of that section. For example, you could "slide-and-sum" the first 4 months to the beginning of the time line (ignoring your calculator's sign flip), like this:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	4	1.00	$\boxed{-1{,}560.79}$	400.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	--	12.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

Depending on how many such "smooth" sections you have in a transaction, such a section-by-section approach can save you a lot of time compared to a "slide-and-sum" process with each cash flow—especially since your calculator can do it for you. Nearly every financial calculator now can do *discounted cash flow analysis*. That is, it can analyze a transaction that doesn't necessarily have uniform payments throughout.* In doing so, it saves time and memory by using the section-by-section approach described above, which allows use of the PMT math wherever possible.

So when you want to describe an uneven cash flow situation to your calculator, you do it in terms of cash flow *groups:*

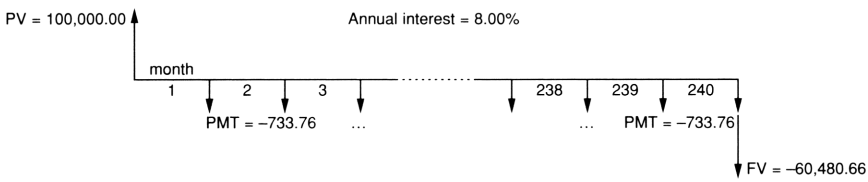


0. The initial cash flow ("Group 0") is –25,000;
1. The amount of the next cash flow group ("Group 1") is 400, and there are 4 such cash flows in a row (one per period);
2. The amount of the next cash flow group ("Group 2") is 750, and there is 1 such cash-flow.
3. The amount of the next cash flow group ("Group 3") is 0, and there are 6 such cash flows in a row (one per period);
4. The amount of the next cash flow group ("Group 4") is 27,500, and there is 1 such cash-flow.

*Don't confuse non-uniform *cash flows* with non-uniform *periods*: The cash flow *amounts* can vary, but not the time intervals between them. To analyze any cash flow scenario—either with the PMT formula or discounted cash flow analysis, you must have uniform time periods between cash flows. All TVM analysis is based upon the PV-FV formula, which assumes a steady interest rate, ***i***, *per period*. If that formula is to give correct results (time is money), the period length must not vary.

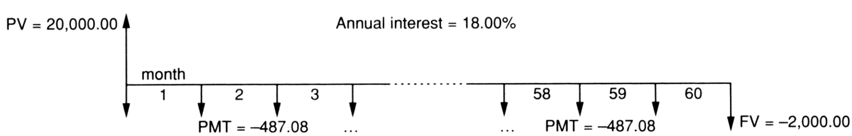
Of course, the exact keystrokes for entering this scenario in a particular calculator will vary slightly from model to model, but nearly every calculator needs it entered in terms of groups like this.

The point is, it's just a different way to describe a cash flow diagram —rather than indicating all the periods at once (as with the PMT formula) or each cash flow individually (as with the PV-FV formula). For example, you could just as well describe this mortgage (with annuity in arrears) as follows:



0. The initial cash flow is \$100,000.00 (***PV***);
1. The amount of the next cash flow group is –\$733.76 (***PMT***); there are 239 (***n***–1) such cash flows in a row (1 per period);
2. The amount of the next cash flow group is –\$61,214.42 (***FV+PMT***), and there is 1 such cash-flow.

Similarly, you can describe this lease (annuity in advance) as follows:



0. The initial cash flow is \$19,512.92 (***PV+PMT***);
1. The amount of the next cash flow group is –\$487.08 (***PMT***); there are 59 (***n***–1) such cash flows in a row (1 per period);
2. The amount of the next cash flow group is –\$2,000.00 (***FV***), and there is 1 such cash-flow.

So the *description* of an uneven (non-uniform) cash flow scenario is pretty straightforward: just think in terms of groups of cash flows. But then what about the calculations themselves? What sorts of unknowns can you solve for with discounted cash flow analysis—as compared to the PMT formula, where you could solve for any one of five variables, given the other four (and the annuity mode)?

Obviously, you can't solve for ***PMT***. In an uneven scenario, there is no such thing—no uniform periodic cash flow throughout the time line. That's why you're not using the PMT formula in the first place.

You won't be solving for the number of periods, either, because that's never going to be the unknown here.* Specifying all the groups and their sizes, you effectively determine the length of the overall time line: The total number of periods is the sum of all group sizes.

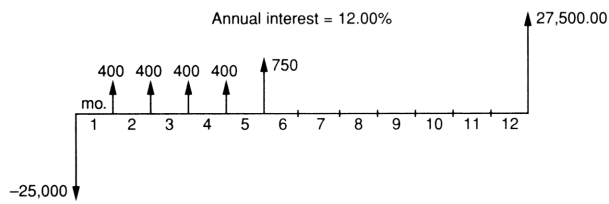
So what other items on a diagram does that leave you? What are the values you most typically need to calculate? There are three....

- The Net Present Value (NPV) is the result of a "slide-and-sum" analysis that sends all known cash flows back to the beginning of the time line. As you've seen, this is handy to know when you're looking for an initial cash flow to balance the transaction.
- Alternatively, the Net Future Value (NFV) is the result of a "slide-and-sum" analysis that sends all known cash flows forward to the end of the time line. This is handy to know when you're looking for a final cash flow to balance the transaction.

*OK, granted: With some careful calculating work, you could allow either the amount of a given cash flow group (i.e. the equivalent of a PMT for that group) or the size of that group to be the unknown, then sum everything else to the beginning or end of that section of the timeline and use the PMT formula to find the unknown. This is perfectly feasible—and quite instructive for the purposes of studying and better understanding the slide-and-sum-to-zero principle (see the upcoming quiz for an example)—but it's time consuming and seldom of interest. It's far more likely that you'll want to solve for a present value, future value or an interest rate.

- Of course, any sort of “slide-and-sum” analysis presupposes a known interest rate (also called the *discount rate*, because sliding a cash flow back in time discounts its face value). But that rate itself may be the unknown item: Maybe you know all the cash flows, but you need to calculate the rate that would make them sum to zero when you “slide” them to any one point in time. In even-flow scenarios, using the PMT formula, this rate is generally called the interest rate; when using discounted cash flow analysis for uneven-flow scenarios, it’s also known as the Internal Rate of Return. (Whatever you call it, it still represents the same thing: the *opportunity cost*—the time value—of money.)

Take a look at each of these quantities and how you would calculate them—using this same cash flow situation:



Here is the “calculator-ready” description of the situation.

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: $\boxed{1,679.24}$

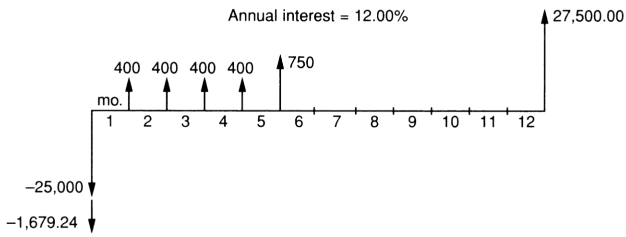
NFV: $\boxed{1,892.21}$

irr%	***Group #***	***Amount***	***Flows in group***
1.00	**0**	–25,000.00	**1**
12.00	**1**	400.00	4
IRR%	**2**	750.00	1
	3	0.00	6
	4	27,500.00	1

As in the calculator data sets for PMT problems, an unknown value is denoted in a box. Here, for example, the ***IRR%*** is known. The computed unknown is the net of the cash flows after they’re adjusted to, and summed at, one point in time. The two most convenient such sums are at the two ends of the time line: ***NPV*** and ***NFV***.

Don't confuse ***NPV*** with the ***PV*** of the PMT formula (nor ***NFV*** with ***FV*** of the PMT formula). The "Net" in Net Present Value is a reminder that its result isn't interpreted. It's just the "slide-and-sum" net total of all cash flows listed in the description. *Unlike the PMT formula, it does not flip the sign of the result to directly indicate the necessary balancing cash flow at that point—you have to do it yourself.*

Thus in the example on the previous page, the cash flow necessary to balance the transaction (make it fair by bringing the net sum to zero) would be an additional *negative* flow (an investment) of $1,679.24 at the beginning of the time line:



P (or ***P/YR***) = 12 Periods/Year

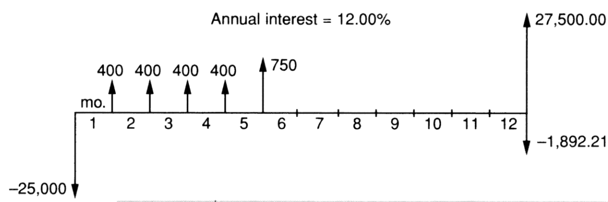
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	***Group #***	***Amount***	***Flows in group***
1.00	**0**	–26,679.24	**1**
12.00	**1**	400.00	4
IRR%	**2**	750.00	1
	3	0.00	6
	4	27,500.00	1

NPV: 0.00

NFV: 0.00

Or, it could be an additional *negative* flow (i.e. an investment) in the amount of $1,892.21 at the end of the time line:



P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

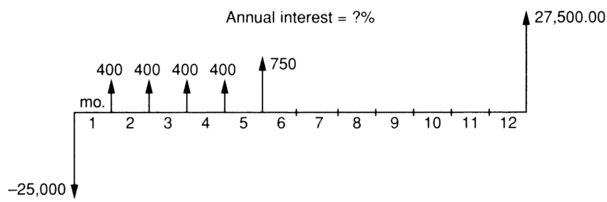
irr%	***Group #***	***Amount***	***Flows in group***
1.00	**0**	–25,000.00	**1**
12.00	**1**	400.00	4
IRR%	**2**	750.00	1
	3	0.00	6
	4	25,607.79	1

NPV: 0.00

NFV: 0.00

And what if the internal rate of return is the unknown? In the previous example, you fixed it at 12% per year (1% per month), which was clearly too low for the initial scenario, as indicated by the resulting positive ***NPV*** and ***NFV*** you calculated. If the original cash flows are fixed (i.e. –\$25,000 at the start and \$27,500 at the end), what interest rate would balance the transaction—allow NPV (or NFV) to sum exactly to zero now—without any other balancing flow?

Here's the picture, followed by the calculator data/results:



P (or ***P/YR***) = 12 Periods/Year

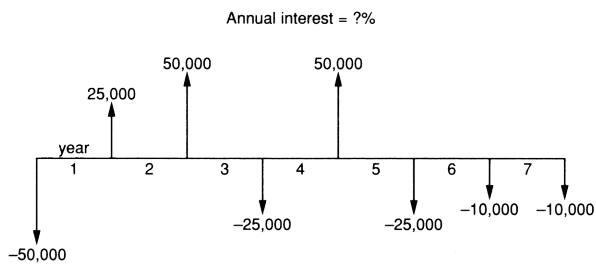
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV:	0.00
NFV:	0.00

irr%	***Group #***	***Amount***	***Flows in group***
1.586	**0**	–25,000.00	**1**
19.031	**1**	400.00	4
IRR%	**2**	750.00	1
	3	0.00	6
	4	27,500.00	1

When IRR Doesn't Tell You Much

It's all well and good when you get one clear answer for IRR, as in the previous example, but it doesn't always happen. Why? Because in an uneven cash flow transaction, there are so many possible combinations of cash flows that sometimes you'll get a situation where there is more than one interest rate that can set the "slide-and-sum" net to zero.* It's just a mathematical fact. Look at this example:



P (or ***P/YR***) = 1 Period/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: 0.00

NFV: 0.00

irr%	***Group #***	***Amount***	***Flows in group***
??	**0**	–50,000.00	**1**
??	**1**	25,000.00	1
IRR%	**2**	50,000.00	1
	3	–25,000.00	1
	4	50,000.00	1
	5	–25,000.00	1
	6	–10,000.00	2

If you try to calculate the IRR of this on your calculator, depending on the make and model, you may get:

- An error message; or
- One of the valid solutions—with no hint that there is another; or
- A message informing you that there is more than one possible answer (and maybe a prompt letting you choose among them by entering a guess to guide the machine to the nearest solution).

Try your calculator and see what you get....

*You can often anticipate which sorts of cash flow scenarios will produce multiple IRR solutions: They tend to be situations where the negative and positive flows are fairly close in total value and interspersed along the time line—as opposed to most negative flows (investments) up front and most positive flows (returns) later on.

For the record: The two possible IRR% values for this example are (to 3 decimal places) 17.767% and –9.489%. *Either* of those rates produces an NPV (or NFV, or any other "slide-and-sum" net) of zero.

So now what? Which rate do you believe? As an investment, are you making or losing money here?

On the *face* of it, it looks as if you're coming out slightly ahead, since the total of the positive flows, $125,000 is slightly more than the total of the negative flows, $120,000. It seems that you'll walk away at the end of the seven-year investment with $5,000 more than when you started. But you know you can't compare values unless they're adjusted to the same point in time.

So probably the best question to ask is: "What is the NPV of the situation for a given discount rate?" Try a whole range of rates with this problem. Here's a table of the results you should get:

Rate (%)	NPV	Rate (%)	NPV
–10	$ –641.48	5	$ 4,543.06
–9	566.21	6	4,317.30
–8	1,586.81	7	4,058.16
–7	2,440.05	8	3,769.60
–6	3,143.58	9	3,455.15
–5	3,713.21	10	3,118.04
–4	4,163.09	11	2,761.13
–3	4,505.90	12	2,387.05
–2	4,753.02	13	1,998.12
–1	4,914.66	14	1,596.47
0	5,000.00*	15	1,184.01
1	5,017.27	16	762.44
2	4,973.87	17	333.32
3	4,876.47	18	–101.96
4	4,731.07	19	–542.15

*Notice: A 0% discount rate—where there is no time value of money—is the *only* time when the NPV (or any adjusted, simultaneous sum) will be the same as the unadjusted (non-simultaneous) sum of the cash flows.

As you see, the ***NPV*** stays positive for the entire range of discount rates between the two solutions (where ***NPV*** goes to zero). So if your opportunity cost of money is anywhere from –9.489% and 17.767%, this deal is favorable to you.

What's a realistic opportunity cost in this case? Certainly it's not negative. Remember that when choosing a discount rate, you ask what you could reasonably expect to get for an investment of similar risk and liquidity. For the 7-year situation in question, it doesn't look all that liquid—maybe it's a housing project with multiple funding sources or something like that—and the risk seems substantial too: clearly it's not an installment loan or note. Under those circumstances the 17.767% seems to be in the ballpark, and you can probably regard that as the logical rate of return here.

But something should be bothering you about all this: If someone were to tell you that you would invest $50,000 and later walk away with $55,000—a gain of just $5,000 over seven years—but the annualized rate of return would be 17.767%, you'd probably want to buy him a new calculator (or call the fraud squad) rather than to invest.

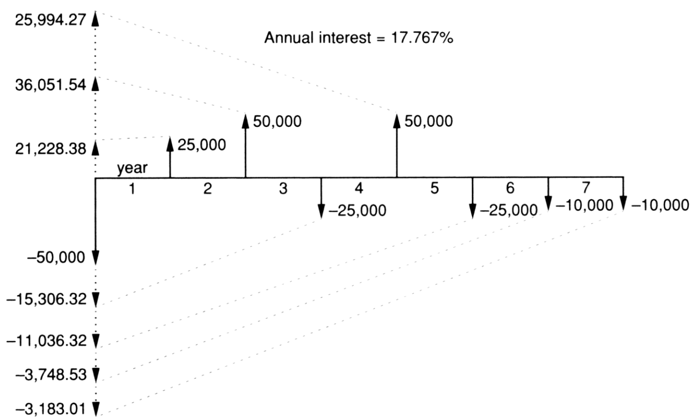
But he'd be right.

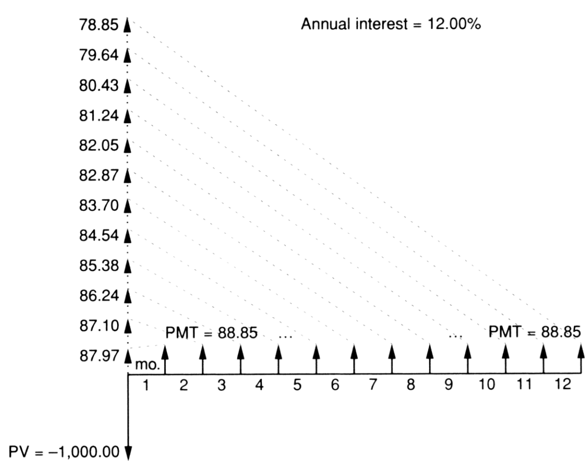
Why? Because he is NOT claiming that you get a 17.767% return on your *entire* $50,000 for the *entire* 7 years. Remember: it's called the *Internal* Rate of Return—the rate of return only so long as the money is within the investment. Once it's returned to you, you might put it under the mattress, lose it at the track, or reinvest it somewhere else, at an entirely different rate.

*The interest rate governing a cash flow situation—the **i** in a PMT scenario or the **IRR** in an uneven cash flow scenario—makes no assumption about what you do with your money after you have received it back in a positive cash flow.*

Telling you that you'll walk away with $55,000 was assuming you'd put your positive cash-flows under your mattress—no further growth at all—a worst-case scenario (unless you play the ponies). Clearly you could do a whole lot better than that with your interim proceeds, but IRR is not concerned with that. It's just the rate at which your money would grow *while in the investment*.

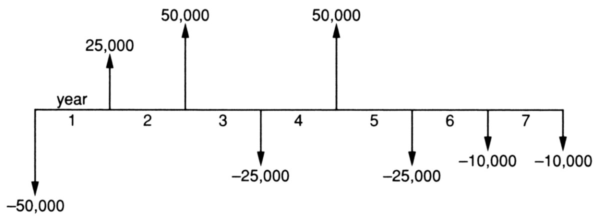
And you can prove it "manually:" You can trace each positive cash flow to match up with negative cash flows, using the computed IRR rate to adjust them to the same point in time. This is just the old slide-and-sum game all over again. It works just as well in this oddball situation here as in a loan or other more conventional investment. Compare the two situations below: different structure, same method.



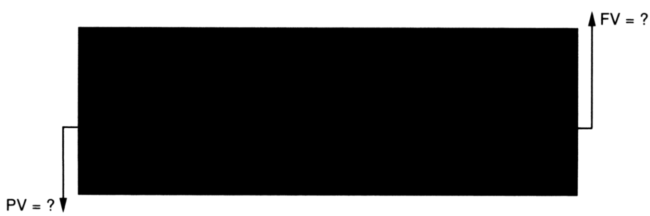


But what if you really do want to change the question from "What's my money earning while in the investment?" to "What's my money effectively doing—as measured from one end of the investment to the other?" In other words: "How much do I need to invest at the beginning and how much will I walk away with at the end?"

Asking that question is really asking to replace the picture of the true scenario,



with a sort of "black-box" substitute—as if you simply put an amount of money into a black box at the start of the time line and withdraw another amount at the end:



In this latter picture, you can easily calculate the effective return (it's just the ***i*** in a simple PV-FV computation), but that rate will be entirely different than the *internal* rate of return of the underlying investment, because in computing the ***PV*** and ***FV*** in the latter picture, you need to decide what your cash will be doing when it's not in the underlying investment. Besides the underlying investment, a "black-box" scenario involves one or more other investments—to handle your cash while it's not tied up in the actual venture....

Modified Internal Rate of Return (MIRR)

For starters, consider where you're going to get the money for all the negative flows required throughout the investment in question—a total of $120,000 in this case. You'll be committed to supplying the various investment amounts when scheduled, so it's no question that you have the money—else you wouldn't be considering the investment in the first place.

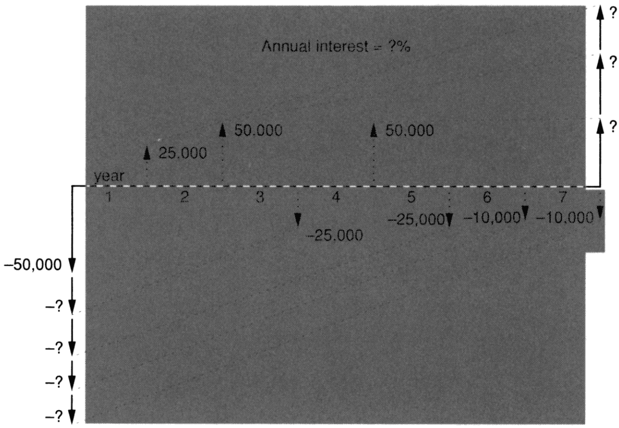
So start by assuming that you have $120,000 in your pocket. You need only $50,000 of it right away. What are you going to do with the other $70,000? It'll be a full 3 years until any more is needed, and even then you'll need only another $25,000.

Clearly you don't want to let all that money sit and do nothing for years at a time. You'll want to put it to work—in something secure enough and liquid enough to assure you of its availability when needed. This is likely to be some kind of interest-bearing account, at a conservative rate that reflects its safe, secure status—say, 6%.

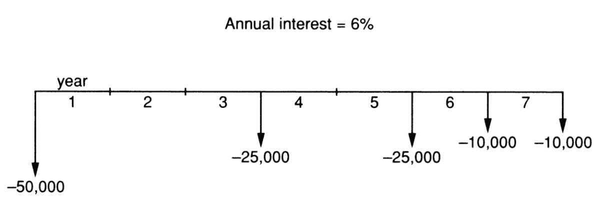
Next question: What are you going to do with the interim returns that come back to you within the investment's time frame? At the very least you would put them right back in that "safe-rate" account for the rest of the time line, no?

Or, you might decide to find another investment with a higher "risk-rate" yield than that of the safe-rate account. After all, these funds are not contractually committed to anyone now. They're your returns on this investment; you can do what you like with them—and since you've already risked them in the underlying investment, why not let them ride in some other similar risk venture for the rest of the time line? Allowing for a little more liquidity than in the original (since you want to withdraw right at the end of the time line), that might be a rate of, say, 15%.

So here's the picture: At the start of the time line, you'll stash all your needed capital in the safe-rate account, ready to draw on for the scheduled negative cash flow; and you'll plow your positive cash-flows back into a risk-rate account until the end of the time line. This is what you want to get to in order to evaluate the money-in/money-out nature of your investment, using the "black box" model:



Now for the specifics: What are those numbers at the ends of the time line? Obviously, they're not simply the sums of the positive and negative flows (–\$120,000 or \$125,000), because you're sliding (discounting) them forward or back along the time line. Look at your safe-rate account first. Since that account will accrue some interest for you, you don't need to start with a full \$120,000 in that account; it's less—and to find it, you can do an NPV calculation using the safe rate:



P (or ***P/YR***) = 1 Period/Year

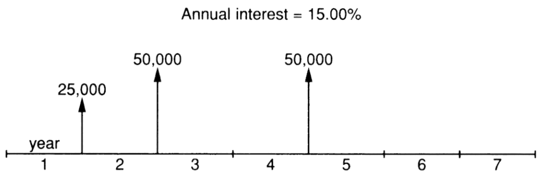
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: –103,372.11

NFV:

irr%	*Group #*	*Amount*	*Flows in group*
6.00	**0**	–50,000.00	1
6.00	**1**	0.00	2
IRR%	**2**	–25,000.00	1
	3	0.00	1
	4	–25,000.00	1
	5	–10,000.00	2

Likewise, your reinvested returns will have grown to be more than \$125,000 by the end of the time line. To compute that, you can do a slide-and-sum forward to the end of the time line *using the risk rate*:



P (or **P/YR**) = 1 Period/Year

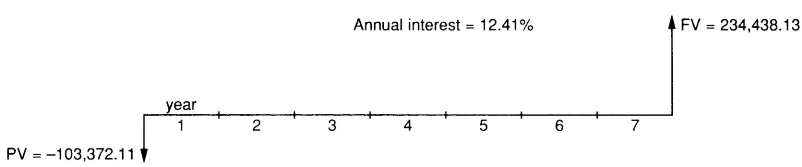
IRR used in periodic, decimal form: ***irr** = **irr%**/100 = **IRR%**/(100**P**)*

NPV: [ ]

NFV: 234,438.13

irr%	***Group #***	***Amount***	***Flows in group***
15.00	**0**	0.00	**1**
15.00	**1**	25,000.00	1
IRR%	**2**	50,000.00	1
	3	0.00	1
	4	50,000.00	1
	5	0.00	3

So here's the completed black-box picture, and now it's a simple PV-FV sort of calculation to find an effective, "end-to-end" rate of return, called a ***Modified Internal Rate of Return***—"modified" so that your funds were put to work earning interest whenever not needed in the original venture:



P (or **P/YR**) = 1 Period/Year

Interest rate is used in decimal form: ***i** = **i%**/100 = **I%**/(100**P**)*

n	***i%***	***PV***	***PMT***	***FV***
7	12.410	–103,372.11	0.00	234,438.13
7	12.410			
Yrs	***I%YR***			

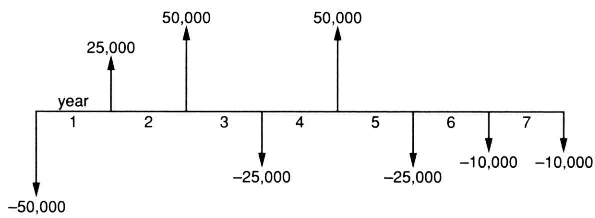
Financial Management Rate of Return (FMRR)

MIRR is a simple, easily calculated way to get a handle on the "end-to-end" prospects for an investment, but it's not quite accurate. An astute money manager would take a look at MIRR and notice that it's not quite modeling how he/she would really manage the funding for your investment to improve the overall yield —at no greater risk.

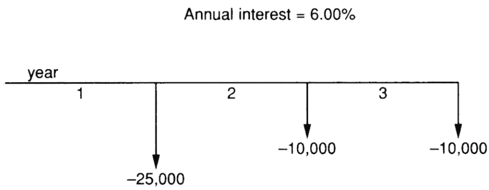
Specifically, notice that you can tie up fewer funds up front (funds that earn the relatively low safe rate) if you *fund later investments out of earlier returns*, wherever possible. It's just common sense, really —what most you'd probably be tempted to do anyway: take money "out of the till" as you go, rather than tap a separate, sequestered account for such interim outlays.

You can calculate the beginning and ending amounts implied by this more accurate method and then find the rate the reflects that growth, which is called Financial Management Rate of Return (FMRR). But it's a bit more complex to do so (hence the decent approximate model of MIRR), because you work backward, one negative cash flow at a time, calculating what portion(s) of the prior positive cash flow(s) is(are) needed to fund it.

For example, this is what you'd have to do in the case of your housing project to compute the FMRR (using the same 6% safe rate and 15% risk rate as in the MIRR). Here's the overall picture again:



How much of the positive $50,000 at the end of Year 4 will you need to set aside in the safe-rate account to fund the negative cash flows at the ends of Years 5, 6 and 7?



P (or **P/YR**) = 1 Period/Year

IRR used in periodic, decimal form:
irr = *irr%*/100 = ***IRR%***/(100***P***)

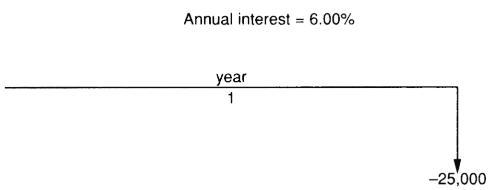
irr%	*Group #*	*Amount*	*Flows in group*
6.00	**0**	0.00	**1**
6.00	**1**	–25,000.00	1
IRR%	**2**	–10,000.00	2

NPV: –40,881.06

NFV:

Only $9,118.94 of the cash flow at the end of Year 4 is available for the risk-rate account; the other $40,881.06 must go into the safe fund.

How much of the $50,000 income at the end of Year 2 must you set aside to fund the investment, –$25,000, at the end of Year 3?



P (or **P/YR**) = 1 Period/Year

IRR used in periodic, decimal form:
irr = *irr%*/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
6.00	**0**	0.00	**1**
6.00	**1**	–25,000.00	1
IRR%			

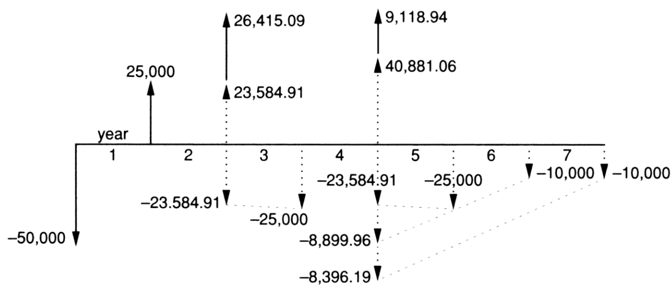
NPV: –23,584.91

NFV:

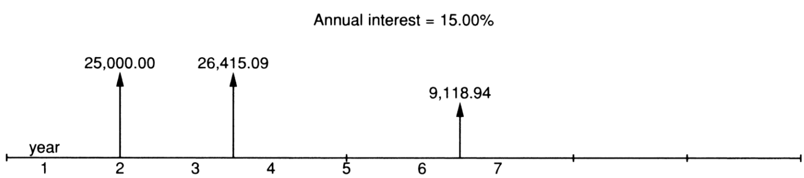
Only $26,415.09 of the cash flow at the end of Year 2 is available for the risk-rate account; the other $23,584.91 must go into the safe fund.

Then of course, all of the $25,000 income at the end of Year 1 is available for the risk-rate account—there's no other future investment to be funded. And as for the initial $50,000 investment, there's no getting around that—it's not funded by any prior income.

So now look at how you've simplified the picture: You've *summed to zero* certain cash flow pairs—the investment and the prior income that funded it:



As it happens, there's only one unfunded investment left—the $50,000 up front. That's your net cash needed for the deal—no need to do any NPV calculation to figure that out. And what's the net cash out at the end? Just send the excess income forward at the risk rate:



P (or **P/YR**) = 1 Period/Year

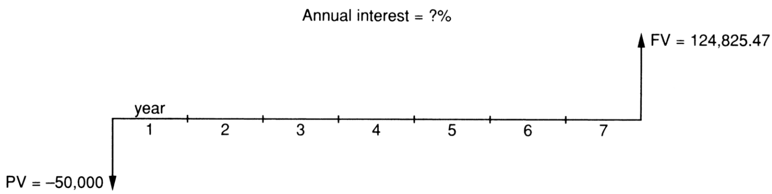
IRR used in periodic, decimal form:
***irr** = **irr%**/100 = **IRR%**/(100**P**)*

NPV: [ ]

NFV: 124,825.47

irr%	*Group #*	*Amount*	*Flows in group*
15.00	**0**	0.00	**1**
15.00	**1**	25,000.00	1
IRR%	**2**	26,415.09	1
	3	0.00	1
	4	9,118.94	1
	5	0.00	3

And now get the FMRR—the rate that balances this picture:



P (or *P/YR*) = 1 Period/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
7	$\boxed{13.962}$	–50,000.00	0.00	124,825.47
7	$\boxed{13.962}$			
Yrs	*I%YR*			

Some things to note about MIRR and FMRR as they relate to one another and to IRR:

- Again: IRR is the rate of growth of money while it's invested in a given cash flow scenario. By contrast, MIRR and FMRR measure the blended rate of growth produced not only by the investment scenario itself but by the accounts (safe-rate and risk-rate) where money is "parked" while not in use in the main scenario. Thus MIRR and FMRR make more assumptions (the two "parking" rates) and are not as readily calculated, but they do provide a more realistic "end-to-end" assessment of the scenario, regarding the cash you must put into it up front vs. the cash you can expect out of it at the end.

- Since at least part of your money will be "parked" in the more conservative safe-rate account for at least part of the time line, it follows that the overall effective rate given by MIRR or FMRR is less than the IRR. This doesn't mean you're making less money. A higher rate can earn less money than a lower rate when it's acting for a shorter time; the IRR acts on each dollar only until it's repaid in a positive cash flow somewhere along the line.

- A comparison between IRR and either MIRR or FMRR is meaningless anyway. They're all rates, but they don't all measure the same thing. IRR is not an *end-to-end* measure of your money's effective earning rate. If you want to fairly compare the MIRR or FMRR model to some sort of "raw scenario," find the *end-to-end* return on your money if you simply pocket the income as it flows to you (and invest it out-of-pocket as needed)—i.e. safe and risk rates of *0%*. That's as neutral an assumption as you can make—and look what it gets you: In the housing example, that's $50,000 out-of-pocket up front and $55,000 into pocket 7 years later. (Go ahead—do the calculation: ***i*** = 1.371% per year.)

- Comparing your results for MIRR and FMRR in the example used, notice that FMRR is the higher rate. Why? Because you're committing less cash up front ($50,000 vs. $103,372.11) *relative* to the total you can expect to have at the end ($124,825.47 vs. $234,438.13). And less cash in is less cash risked, too—another reason why FMRR is the model that more accurately reflects real investment practices: To get into the housing deal, the FMRR method requires you to commit less than half the cash required under the assumptions of MIRR.

- "But," you protest, "I make more money with the MIRR model!" Only because you invest more. To compare the two models fairly, ask which one makes you more money when you invest the same amount in either. If you invest $103,372.11 in the MIRR model, sure, you'll have $234,438.13 at the end. With the FMRR model, though, you can take the extra $53, 372.11 that's not committed to the housing project and just "park" it in the risk-rate account for the duration (which is, after all, the assumption under either model for cash not needed in the housing deal). The future value of that "parked" cash after 7 years at 15% is $141,970.87, which, when added to the $124,825.47 from the FMRR's housing deal itself, gives a total end amount of $266,796.34 for the FMRR —better than the MIRR by over $32,000. *Less cash committed leaves more cash to do something else at least as fruitful.*

Key points that were covered here:

- The PMT formula is great for a wide variety of common financial transactions, but it applies only to a uniform, periodic cash flow. To balance non-uniform transactions, you must resort to the tedious slide-and-sum-to-zero, treating each cash flow individually. But your calculator can apply the more convenient PMT math to areas of the time line wherever flows are uniform. So you describe uneven cash flow situations to your calculator in terms of *groups* of identical cash flows.
- Once you draw the picture for your calculator that way, you can find the net sum (no sign flip) of the cash flows at any point in time. Finding the interest rate is trickier: Uneven cash flow streams can lead to multiple possibilities for the IRR. In those cases, testing the NPV with a variety of rates—or evaluating the "end-to-end" rate with methods such as MIRR or FMRR —may be more informative.

Quiz

1. At your bank, you have arranged a home equity line of credit in the amount of $5,000. The account charges an interest rate of 12.99%, compounded daily (effective immediately for each withdrawal). During the holiday shopping season, you drew on the account as follows:

November 27:	$ 500
November 29:	$ 750
December 4:	$1,000
December 11:	$ 650
December 14:	$ 350

How much credit was still available as of December 20? If you then withdraw another $400 on December 21, what will the payoff balance be on January 10? (The solution is on page 106.)

2. Use discounted cash-flow analysis to help solve problem **21** on page 55. (The solution is on page 107.)

3. Use discounted cash-flow analysis to help compare the twins' retirement accounts, problem **22**, page 55. (Solution on page 108.)

4. You're a creditor with some long-overdue receivables on your books from a certain customer who has hit hard times. The customer, who owes you \$6,000 (past-due 9 months) and \$12,000 (past-due 6 months), wants to work out a payment schedule for what he owes you, but in order to stay in business, he'll need to place fresh \$8,000 orders from you twice more—3 months from now and 6 months from now. What he proposes is that you let him pay a flat amount at the end of every month for the next 12 months so that he is out of your debt entirely by then. If you value your working capital at 12% annually, what monthly payment should you demand? (The solution is on page 108.)

5. What blended rate is the lender earning over the full 30 years for the ARM in problem **4**, page 52? (The solution is on page 109.)

6. A home has two mortgages. The first is \$100,000 on a 30-year term with an \$877.57 payment and no balloon. The second began 15 years after the first and is \$50,000 on a 10-year term with a \$658.22 payment and a \$10,000 balloon. Now, after the 20th year, the owner wants to consolidate the loans and get financing for remodeling. You offer to "wrap" his current mortgages (cover all their obligations) and lend him \$50,000 in new money, for 120 monthly payments of \$2,200 and then a \$25,000 balloon. What's your yield? What's the borrower's overall interest rate? He paid 1 point on the first mortgage, 2 points on the second. (The solution is on page 110.)

7. You're trying to decide how well a small portfolio of analyst-recommended stocks would have performed over 10 years, as compared to just buying and holding a broadly indexed mutual fund.

From a starting balance of $9,400 on 1/2/91, after paying all applicable taxes (and reinvesting all your other net gains), the mutual fund balance as of 1/2/01, was $26,376.10.

The net cash-flow history (i.e. after brokerage/transaction fees) of the portfolio is as follows:

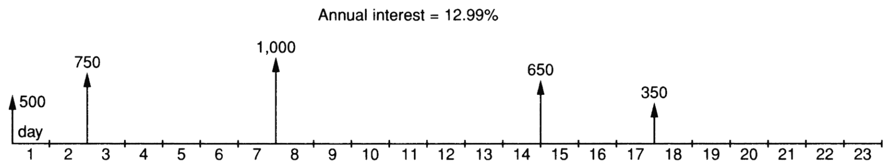
Date	(Invest)/Return	Reason
1/2/91	$ (2,500)	Bought shares.
5/6/91	(5,200)	Bought shares.
9/27/91	(3,200)	Bought shares.
4/23/92	6,000	Sold shares.
6/15/92	(400)	Paid taxes on gains.
7/28/92	(5,600)	Bought shares.
8/5/94	7,200	Sold shares.
9/15/94	(900)	Paid taxes on gains.
12/2/94	(6,500)	Bought shares.
2/17/99	16,500	Sold shares.
4/15/99	(2,850)	Paid taxes on gains.
10/13/99	(13,500)	Bought shares.
1/2/01	$ 28,250	Market value.

Which investment strategy had the better performance? (The solution is on page 111.)

8. Refer again to the couple planning for retirement (problem **21** on page 55). Suppose their taxable mutual fund were to earn 12% after taxes, while their non-taxable retirement funds still earned 10%. What yearly contribution to their retirement funds would then be sufficient to meet their retirement goal—i.e. enough to draw $10,000/month through his 95th birthday? (The solution is on page 113.)

Quiz Solutions

1. Here's the picture:



So, what's the Net Future Value of the account on December 20?

P (or ***P/YR***) = 365 Periods/Year

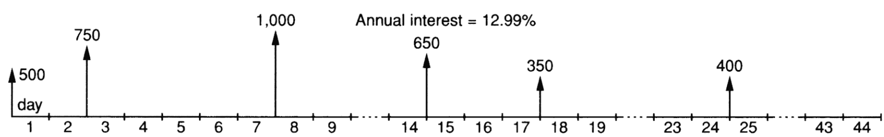
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV:

NFV: 3,268.28

irr%	***Group #***	***Amount***	***Flows in group***
0.3559	**0**	500.00	1
12.99	**1**	0.00	1
IRR%	**2**	750.00	1
	3	0.00	4
	4	1,000.00	1
	5	0.00	6
	6	650.00	1
	7	0.00	2
	8	350.00	1
	9	0.00	6

You owe $3,268.28 as of December 20; you still have $1,731.72 of the $5,000 credit line available. Then you draw $400 more the next day. What are the total damages on January 10? ***NFV*** again:



P (or ***P/YR***) = 365 Periods/Year

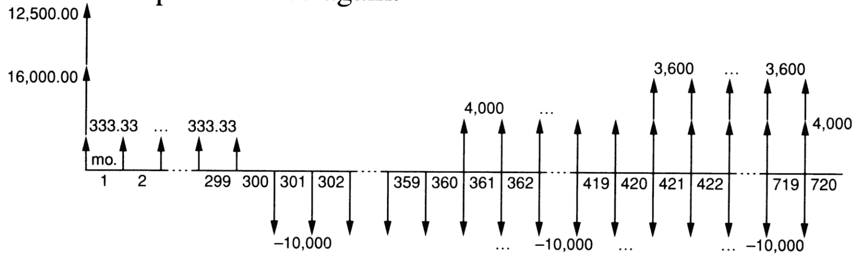
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV:

NFV: 3,695.65

irr%	***Group #***	***Amount***	***Flows in group***
0.3559	**0**	500.00	1
12.99	**1**	0.00	1
IRR%	**2**	750.00	1
	3	0.00	4
	4	1,000.00	1
	5	0.00	6
	6	650.00	1
	7	0.00	2
	8	350.00	1
	9	0.00	6
	10	400.00	1
	11	0.00	20

2. Here's the picture once again:



Just find the Net Present Value of the whole picture, using the growth rate of their account(s) as the discount rate. If the NFV (or NPV) is positive, the couple will have enough money for their retirement goal; if it's negative, they won't:

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	***Group #***	***Amount***	***Flows in group***
0.8333	**0**	28,833.33	1
10.00	**1**	333.33	299
IRR%	**2**	–10,000.00	60
	3	–6,000.00	60
	4	–2,400.00	300
	5	0.00	1

NPV: [ ]

NFV: [1,421,827.89]

They'll have far more than enough. How much could they draw from their accounts during retirement to exactly meet their goal? As you read on page 79, the idea is to "slide-and-sum" all their retirement resources together at the start of their retirement years.

You could use ***NFV*** to find the balance in their savings accounts, but it's simpler to do a PMT-formula calculation (as on page 79): $789,581.92. But ***NPV*** comes in handy, though, to find the discounted value of their two future pension income streams:

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

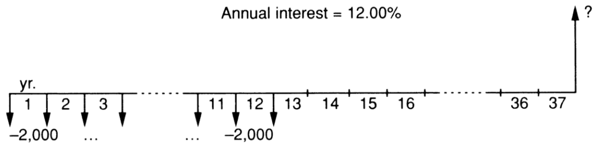
irr%	***Group #***	***Amount***	***Flows in group***
0.8333	**0**	0.00	1
10.00	**1**	0.00	59
IRR%	**2**	4,000.00	60
	3	7,600.00	300

NPV: [426,908.17]

NFV: [ ]

The sum of these two results ($1,216,490.09) is the couple's net resources available upon their retirement. To calculate the level monthly income they could draw from that over 35 years (again, see page 79), is just a simple ***PMT*** calculation: $10,371.40

3. Here's Amy's scenario—where NFV comes in handy:



P (or ***P/YR***) = 1 Period/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
12.00	**0**	2,000.00	**1**
12.00	**1**	2,000.00	12
IRR%	**2**	0.00	35

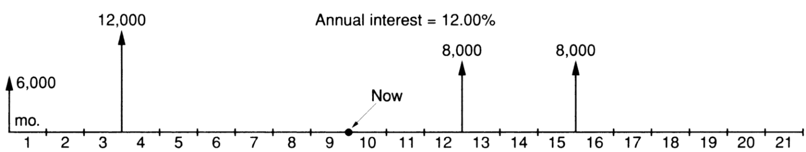
NPV:

NFV: 2,959,852.61

(Gads, what an effortless pile!)

As for Brad's scenario, it's simpler just to use a PMT calculation, as shown at the bottom of page 80.

4. Here's the picture:



If you "slide-and-sum" everything to the present time (which is not the beginning of the time line here; the present is marked as NOW above), then that sum will be what your customer needs to pay off in 12 level PMT's—an easy calculation.

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
1.00	**0**	6,000.00	**1**
12.00	**1**	0.00	2
IRR%	**2**	12,000.00	1
	3	0.00	6

NPV:

NFV: 19,300.35

This is what the customer already owes you for the past invoices.

P (or *P/YR*) = 12 Periods/Year

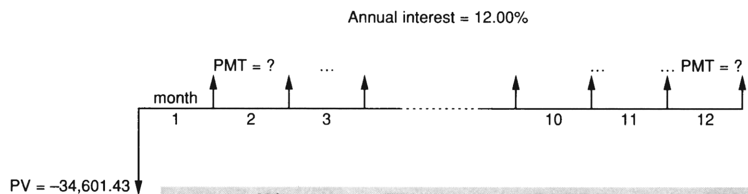
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
1.00	**0**	0.00	1
12.00	**1**	0.00	2
IRR%	**2**	8,000.00	1
	3	0.00	2
	4	8,000.00	1

NPV: $\boxed{15{,}301.08}$

NFV:

That's the present value of what he'll owe you for the two future orders. Now sum the two results ($19,300.35 + $15,301.08 = $34,601.43) and amortize over 12 months in a PMT calculation:

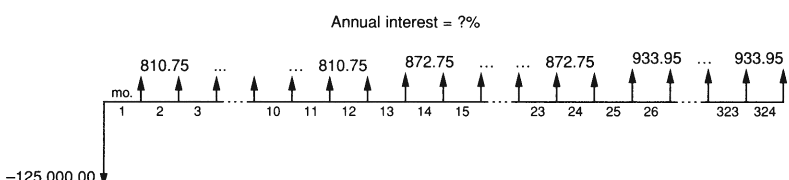


P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	***i%***	***PV***	***PMT***	***FV***
12	1.000	–34,601.43	$\boxed{3{,}074.30}$	0.00
1	12.00		***(END***	
Yrs	***I%YR***		**of the period)**	

5. You would first need to calculate the PMT's for all three rates, as shown on pages 59-60. But then the blended rate is a straight-forward IRR calculation:



P (or *P/YR*) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

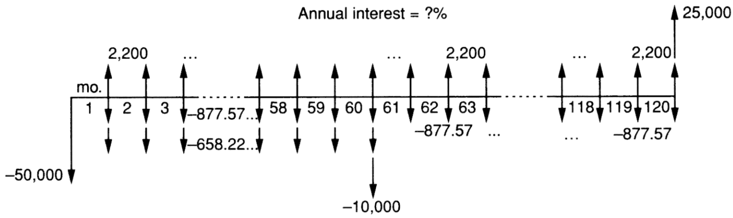
irr%	*Group #*	*Amount*	*Flows in group*
$\boxed{0.664}$	**0**	–125,000.00	1
$\boxed{7.968}$	**1**	810.75	12
IRR%	**2**	872.75	24
	3	933.95	324

NPV: 0.00

NFV: 0.00

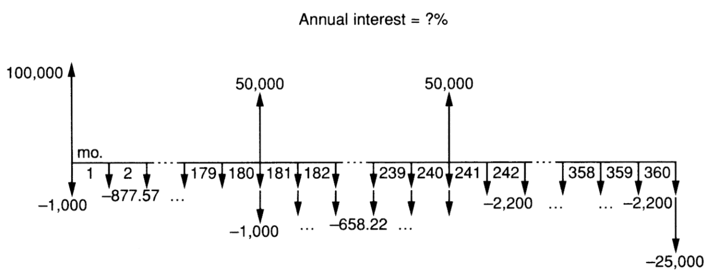
The blended rate is toward the higher end of the rate range, since the highest rate (8.25%) is in effect for 27 of the 30 years.

6. Here's the picture from your point of view, and once you "draw" this picture for your calculator, it's a straightforward IRR calculation:



	irr%	*Group #*	*Amount*	*Flows in group*
P (or **P/YR**) = 12 Periods/Year	1.414	**0**	–50,000.00	**1**
IRR used in periodic, decimal form:	16.97	**1**	664.21	59
irr = ***irr%***/100 = ***IRR%***/(100***P***)	***IRR%***	**2**	–9,335.79	1
		3	1,322.43	59
NPV: 0.00		**4**	26,322.43	1
NFV: 0.00				

The homeowner's total picture is quite different. It has a longer term, and it includes loan fees:



	irr%	*Group #*	*Amount*	*Flows in group*
P (or **P/YR**) = 12 Periods/Year	0.876	**0**	99,000.00	**1**
IRR used in periodic, decimal form:	10.51	**1**	–877.57	179
irr = ***irr%***/100 = ***IRR%***/(100***P***)	***IRR%***	**2**	48,122.43	1
		3	–1,535.79	59
NPV: 0.00		**4**	48,464.21	1
NFV: 0.00		**5**	–2,200.00	119
		6	–27,200.00	1

7. Here's the actively managed portfolio situation. You can confirm these date calculations as you wish; many calculators have time and date calculations tools.

P (or *P/YR*) = 365 Periods/Year

IRR used in periodic, decimal form:
irr = *irr%*/100 = *IRR%*/(100*P*)

irr%	Group #	Amount	Flows in group
0.029	0	–2,500.00	1
10.52	1	0.00	123
IRR%	2	–5,200	1
	3	0.00	143
	4	–3,200	1
	5	0.00	208
	6	6,000.00	1
	7	0.00	52
	8	–400.00	1
	9	0.00	42
	10	–5,600.00	1
	11	0.00	737
	12	7,200.00	1
	13	0.00	40
	14	–900.00	1
	15	0.00	77
	16	–6,500.00	1
	17	0.00	1537*
	18	16,500.00	1
	19	0.00	56
	20	–2,850.00	1
	21	0.00	180
	22	–13,500.00	1
	23	0.00	446
	24	28,250.00	1

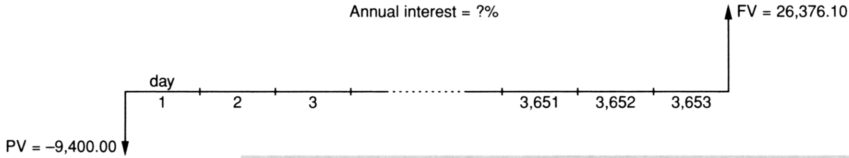
NPV: 0.00

NFV: 0.00

All this work—just to figure out how you've done (not to mention the research time and your added risk).

*Note: Depending on which model financial calculator you're using, it may or may not allow you to specify more than 999 cash flows in any one group. If not, just split this group of 1537 cash flows into two consecutive groups (e.g. 999 and 538).

By contrast, the mutual fund's performance is an easy calculation—just a quick PMT-formula exercise, actually:



P (or ***P/YR***) = 365 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
3653	$\boxed{0.028}$	–9,400.00	0.00	26,376.10
10	$\boxed{10.31}$			
Yrs	***I%YR***			

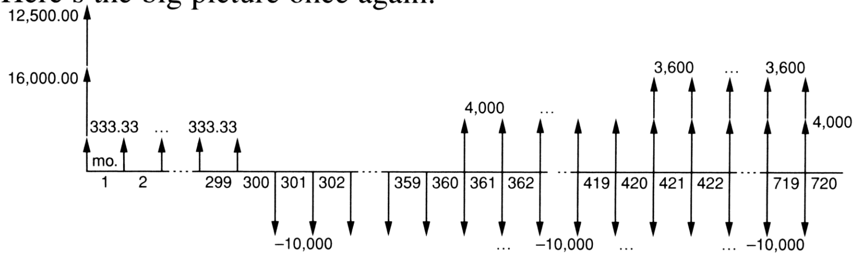
OK, you did slightly better rate-wise with the active portfolio. *But did you make more money?* Remember: A lesser rate can make more money if it acts over a longer time.

Your entire $9,400 passive mutual fund investment was working at its 10.31% rate for the entire 10 years, netting you $16,976.10.

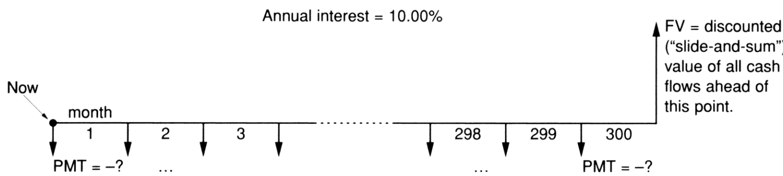
By contrast, your active portfolio was in and out of the market constantly, varying tremendously the amount of money at work at any given time. To figure out what kind of lump sum you'd have needed at the start to fund your portfolio—and how much you could have walked away with (note that your proceeds are not all in the market even at the end of the 10 years)—you'd have to do an FMRR calculation, using, say your broker's money market sweep account as both your safe rate and risk rate (since that's the place you park both seed cash and proceeds). More work.

OK: Assuming an after-tax money market rate of 4%, you'd have needed to deposit $10,776.19 in your brokerage account to fund your 10-year venture. And afterwards, you'd have walked away with $28,805.45—a net gain of $18,029.26. So, yes: You netted $1053.16 more in actively managing a portfolio than in a mutual fund—about $105.32 per trade. So if you value your time at, say, $25/ hour, you couldn't have invested more than about 4 hours of research per trade (42 hours over 10 years, including all this time to calculate your success) in trying to beat the pros—who worked at it full-time (say, 25,000 hours each over 10 years). Hmmm....

8. Here's the big picture once again:



Probably the most straightforward approach is to "slide-and-sum" everything except their retirement investments to the beginning of the retirement years, then treat that net as the future value those investment would need to grow into:

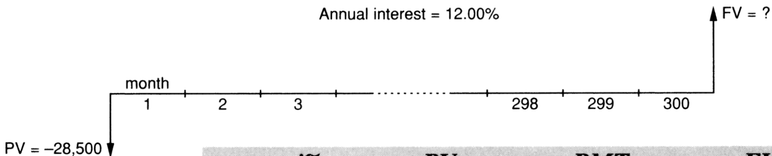


But in your "sliding-and-summing," what should you use for a discount rate? Some of the accounts value their money at 10%, some at 12%. How do you discount the future pension incomes? Do you assume that they augment (i.e. "slow the drain" on) the 10% accounts or the 12% accounts? That is, when the first pension kicks in, does this reduce the amount they draw out of the 12% accounts or the 10% accounts—or some of each? Given the choice, which accounts should they drain first? Does it matter?

The answer is a bit counterintuitive. You might think that the longer you could let the faster-earning account grow, the better; i.e., draining the slower-earning account isn't as big a loss.

You might think that, but it turns out that the opposite is true: The pension income is worth more as an augmentation of the 10% account. It's just a simple mathematical comparison of present values: A future cash flow is worth more as a present value today when it is discounted at a lesser rate. If you can't do as much with it in the meantime, your opportunity cost—the discount—is less. You can see this most clearly as the discount rate goes to zero: If money can't earn anything on itself, then its present value is the same as its future value—the unadjusted sum of the cash flows. But if it can earn on itself, then there's a definite difference between its value in your hand now and the value of receiving it later—so you must be willing to discount the later flows for the right to receive them now.

So here's the full strategy for solving this problem: two separate calculations. First, the 12% account will grow untouched into a nest-egg lump-sum at retirement:

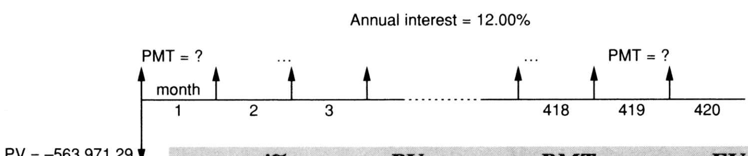


n	*i%*	*PV*	*PMT*	*FV*
300	1.00	–28,500	0.00	$\boxed{563,971.29}$
25	12.00			
Yrs	***I%YR***			

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

And then that lump-sum is steadily depleted in monthly draws over 35 years (with no help from the pension incomes):

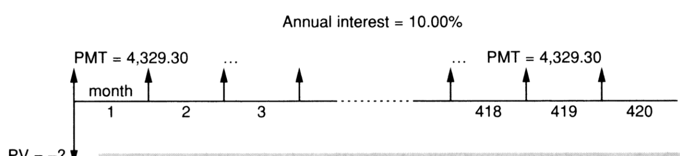


n	*i%*	*PV*	*PMT*	*FV*
420	1.000	–563,971.29	$\boxed{5,670.70}$	0.00
35	12.00		(***BEGIN***ning	
Yrs	***I%YR***		of the period)	

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

So the rest of the required $10,000 monthly retirement income ($4,329.30) must come from the 10% account:

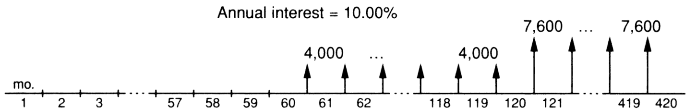


n	*i%*	*PV*	*PMT*	*FV*
420	0.833	$\boxed{-507,795.45}$	4,329.30	0.00
35	10.00		(***BEGIN***ning	
Yrs	***I%YR***		of the period)	

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

Now the 10% account will be augmented by the pension incomes, so you need to subtract the portion attributable to them:



P (or **P/YR**) = 12 Periods/Year

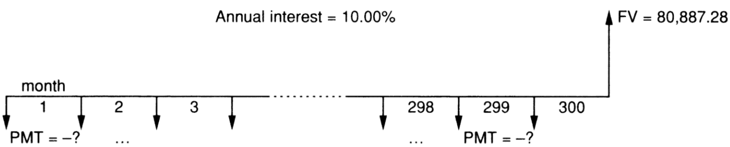
IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	***Group #***	***Amount***	***Flows in group***
0.8333	**0**	0.00	**1**
10.00	**1**	0.00	59
IRR%	**2**	4,000.00	60
	3	7,600.00	300

NPV: $\boxed{426{,}908.17}$

NFV: $\boxed{}$

So of the $507,795.45 the 10% account will need to "have on hand" as of the date of the couple's retirement, fully $426,908.17 is the present value of the pension incomes. The retirement investments need only provide the difference: $80,887.28.



P (or **P/YR**) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	***i%***	***PV***	***PMT***	***FV***
300	0.833	0.00	$\boxed{-60.46}$	80,887.28
25	10.00		**(*BEGINning***	
Yrs	***I%YR***		of the period)	

Given their expected pensions, just $60.46 per month is all they need save (between them!) in their 10% retirement accounts, if they just let their lump-sum savings of $28,500 grow at 12%.*

*Now, suppose you don't buy the argument made above—that the pension incomes are worth more if used to augment the 10% account than the 12% account. So redo the analysis. Subtract the present value of the pension incomes from the 12% account before amortizing it over 35 years. You'll find that it'll supply just $8,896.30 of the $10,000 monthly income; the 10% retirement account must supply the other $1,103.70. Discounted to the start of retirement, this would require a lump-sum balance there of $129,456.00, and all of that would have to come from the retirement investments, which would thus have to be $96.76/month—over 50% more than the other scenario. No doubt about it: Using the pension income to augment the 10% account is better.

5. Real-World Complications

Key points to be covered here:

- Since they're usually used to plan and evaluate future events, financial calculations rely on a certain amount of estimation and assumption. So it's seldom worth the effort to strive for perfect accuracy, which will be swamped anyway by the errors in your assumptions as to timing, inflation, etc.
- Still, there are real-world complications that you can model fairly well on a cash flow diagram: an interest rate that compounds on an interval different from the payment period; partial periods at the beginning or end of a cash flow stream; non-periodic cash flows; rounding results to two decimal places (dollars and cents); inflation and taxes. None of the models or solutions given here are perfect, but they'll get you closer.

How Exact Do You Need to Be?

As you've no doubt come to suspect now and then, the real world of finance isn't determined by a calculator or computer. It's a world of human beings—run by human beings, for human beings—complete with exceptions, approximations and hoary traditions. Logic goes only so far, and change comes slowly.*

It shouldn't come as too much of a shock, then, to learn that in many cases you may need to do some preparatory work or side calculations to translate a real-life transaction into terms your calculator can deal with—then translate its results back into meaningful terms for you.

This isn't hard, really—usually just common sense. For example, in the previous chapter you examined alternative ways to evaluate a transaction when the IRR was ambiguous: You inspected the NPV to see if one IRR solution or the other was reasonable. Also, you saw how to use MIRR or FMRR to find a cash-in/cash-out, "end-to-end" sort of value on the transaction.

In particular, you saw how FMRR modeled reality more closely than MIRR, thus giving you a more accurate analysis. This chapter is all about that kind of thinking: how to better model real-life situations on the calculator.

How close a model do you need to construct? Well, how accurate is "good enough?" It depends on the type of transaction and the values you're dealing with. If you're wondering about a mortgage payment, a round-off difference of a fraction of a penny won't mean much, even over 30 years. But if you fail to choose the correct annuity mode before calculating, or if you fail to note that the interest compounds daily rather than monthly, that can make a big difference.

*It took stock markets until the year 2000 to price stock shares in decimal fractions, rather than 8ths or 16ths—denominations not used in currency for centuries.

Rounding Issues

Start with the little stuff. No matter what currency you use in finance, there's usually a fairly blunt limit to its precision: Tenths, hundredths or maybe thousandths of the basic unit is about the extent of the denomination. By contrast, your calculator or computer can carry 10, 12, 15 or more decimal places. That means it will often come up with answers far more precise than you really need.

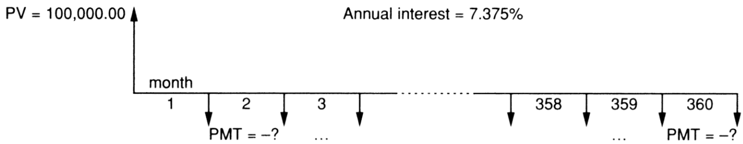
The first defense, of course, is setting your machine to show you only the relevant decimal places—usually just 2 places if you're working in dollars and cents. This doesn't mean the machine is rounding—it still uses all 10 or 12 digits internally. It's just editing the displayed version for your eyes only.

For example, take the fraction 3/8. Go ahead—do the division now on your calculator.... The actual result is 0.375, but if your machine is set to show you just two decimal places, you'll see 0.38. But in all likelihood, *the unrounded value in the machine is still 0.375*. You can prove this easily, too: Without re-entering that result, just multiply it now by 8.... If the machine is rounding only the displayed version—not the internal value itself—you'll get 3.00, since (0.375)(8) = 3.00. If the machine is really rounding the internal value (unlikely), it will give 3.04, because (0.38)(8) = 3.04.

The point is this: When you use the result of one calculation as a known value in the next calculation, it can make a difference whether you simply leave that value in the machine or re-enter it by hand. The difference arises because, if you read from the display, which may be rounding for your eyes (as you asked it to), you may not be keying in the same value as the machine is holding internally. This isn't usually a huge deal, but it can be disconcerting to see small differences between your calculated results and, say, a bank statement. At least be aware of it, so that you can be confident that you haven't done anything fundamentally wrong in your calculations.

Look at a couple of examples where the rounding of intermediate results makes a difference:

Find the monthly payment on your plain 30-year, $100,000 mortgage at 7.375%. (This should be old hat by now.)



	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	360	0.615	100,000.00	–690.68	0.00
	30	7.375		***(END***	
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	***Yrs***	***I%YR***		of the period)	

No big deal, right? OK, prove it: Clear your calculator and re-solve the problem, but this time enter the ***PMT*** as a known value and find the remaining balance (***FV***) after 30 years. If the ***PMT*** amount, –$690.68, is correct, ***FV*** should calculate out to be zero, no? Try it:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	360	0.615	100,000.00	–690.68	6.38
	30	7.375		***(END***	
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	***Yrs***	***I%YR***		of the period)	

What gives? Why is there a slight credit balance—due back to you as a refund (which you know from its positive sign)? It's because the ***PMT*** value that would really pay the loan off to exactly zero in 360 months isn't –690.6800000…, which is what you entered as –690.68. Rather, it's –690.6751476..., nearly a half-penny less. So if you're actually overpaying by nearly half a cent per month (and there's no alternative, since you can't write a check in denominations finer than cents), over 30 years a small excess accrues.

How does a mortgage lender handle such inaccuracies? It's usually taken care of in the final payment, or it may be credited toward the lien release fee. However, since most mortgages don't run to term before being paid off via refinancing, a balloon, or sale of the property, it's still important for the lender to keep accurate (i.e. "to the whole penny") track of the remaining balance at any given period along the way.* Since the inaccuracy creeps in a little bit each period, the remaining balance at any point will show some discrepancy from the mathematical ideal.

Take the previous example once more and compare the balance after the first 3 payments, as calculated in one shot…

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	3	0.615	100,000.00	$\boxed{-690.68}$	–99,770.30
*Interest rate is used in decimal form: **i** = **i**%/100 = **I**%/(100**P**)*	.25	7.375		(***END***	
	Yrs	***I%YR***		of the period)	

…or month by month.…

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	1	0.615	100,000.00	$\boxed{-690.68}$	–99,923.90
*Interest rate is used in decimal form: **i** = **i**%/100 = **I**%/(100**P**)*	.083	7.375		(***END***	
	Yrs	***I%YR***		of the period)	

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	1	0.615	99,923.90	$\boxed{-690.68}$	–99,847.34
*Interest rate is used in decimal form: **i** = **i**%/100 = **I**%/(100**P**)*	.083	7.375		(***END***	
	Yrs	***I%YR***		of the period)	

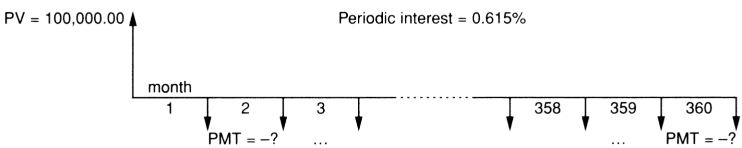
	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	1	0.615	99,847.34	$\boxed{-690.68}$	–99,770.31
*Interest rate is used in decimal form: **i** = **i**%/100 = **I**%/(100**P**)*	.083	7.375		(***END***	
	Yrs	***I%YR***		of the period)	

Already, there's a penny of difference (and 357 more months to go).

*It's also important for the lender to report accurately to you, the borrower, the exact (i.e. to the whole penny) amount of interest and principal you paid each year.

Another, less trivial rounding matter, is the question of precision in the interest rate. Unlike dollar values, the interest rate is not at all limited to two decimal places, and your failure to include too few of the relevant digits can affect the outcome by more than mere pennies.

Look again at the same mortgage as on the previous page. Suppose you look at the calculation data box on the previous page and forget that the periodic interest rate shown is a three-place rounded "display" version (rounded merely for space considerations). And you blithely key it into your calculator as you see it: 0.615. What calculated PMT amount will result? Try it....



n	***i%***	***PV***	***PMT***	***FV***
360	0.615	100,000.00	$\boxed{-691.02}$	0.00
30			***(END***	
Yrs	***I%YR***		of the period)	

If you were to mistakenly make that payment—an overpayment of 34 cents a month—then over the term of the loan, you would have paid about $453 too much:

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
360	0.615	100,000.00	–691.02	$\boxed{453.21}$
30	7.375		***(END***	
Yrs	***I%YR***		of the period)	

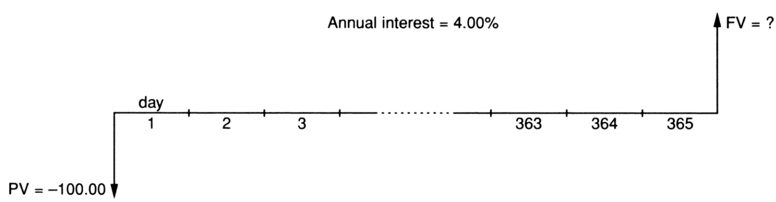
Finding Equivalent Interest Rates

As you know, the PMT formula and the general discounted cash flow calculation method both work under the assumption that interest is compounding with the same frequency as cash is flowing. You have a PMT or cash flow (even if it's \$0) exactly once each period; and interest compounds exactly once over that period. The mathematics behind the formulas simply don't work unless this is true.

So, what do you do when you want to make payments or investments on a different schedule than the applicable interest? You have to convert that interest rate to an equivalent rate that compounds with the same frequency as your desired cash flows.

Take an example: You want to save toward a house down-payment. Your savings account pays 4% annually, compounded daily, but your paycheck comes just once a month. How much should you save from each paycheck to accumulate \$12,500 in 5 years?

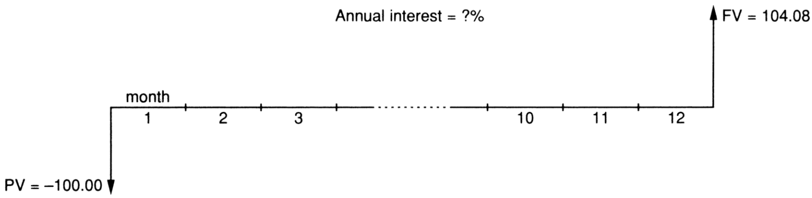
The first question to ask is, "How much does 4% annually, compounded daily, actually accrue in interest over a year?" You know it's slightly more than 4%, the nominal rate, right? (Recall the discussion of interest basics in Chapter 1.) How much more? Some calculators have specialized menus or side-calculation tools to figure this out, but it's easy to use the PMT formula to do it. Just choose a convenient starting balance of \$100 and see how it grows:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 365 Periods/Year	365	0.011	−100.00	0.00	$\boxed{104.08}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	4.000			
	Yrs	***I%YR***			

Notice that you can read the effective annual interest rate right from the FV result: It's 4.08% per year (and you could view more decimal places if you wanted: 104.0808493…)

The point is, for a monthly-compounding rate to be equivalent, it must accrue exactly the same amount of interest over a year. And now that you know what that amount is, you can ask, "What monthly compounding rate will earn that same 4.08...% per year?" Here's the calculation (and notice that you need only change the value of ***n*** and then calculate ***i%***—no need to touch ***PV***, ***PMT*** or ***FV***):

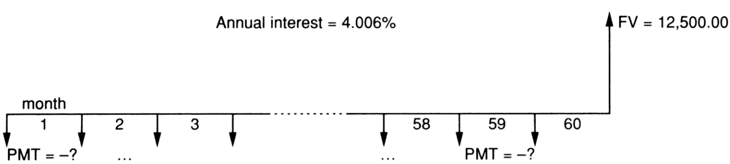


P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** *=* ***i%****/100 =* ***I%****/(100****P****)*

n	***i%***	***PV***	***PMT***	***FV***
12	$\boxed{0.334}$	–100.00	0.00	104.08
1	$\boxed{4.006}$			
Yrs	***I%YR***			

And now that you've got your equivalent interest rate figured out (no need to touch ***i%*** again) the savings calculation is straightforward:



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** *=* ***i%****/100 =* ***I%****/(100****P****)*

n	***i%***	***PV***	***PMT***	***FV***
60	0.334	0.00	$\boxed{-187.88}$	12,500.00
5	4.006		**(*BEGINning***	
Yrs	***I%YR***		**of the period)**	

To gather your down-payment cash, you'll need to save $187.88/mo. for five years.

Partial Periods

Interest conversion calculations are the best way to reconcile the cash flow periods with the compounding period—and the two must match for the formulas to work. But sometimes you have situations where the cash flows don't seem to be periodic at all. What then?

First of all, look closely to be sure that there's no *finer periodicity* you can use. For example, if you have income of $100 on Monday, $200 on Friday, $50 on Sunday and $125 on Wednesday, etc., this isn't a periodic pattern of income. But if you define a period as 1 day, then you have a periodic cash flow scenario that's readily analyzed via discounted cash flow analysis:

	Group#	**Amount**	**Flows in Group**
Monday	**0**	$ 100	**1**
(Tues.-Thurs.)	**1**	0	3
Friday	**2**	200	1
(Saturday)	**3**	0	1
Sunday	**4**	50	1
(Mon.-Tues.)	**5**	0	2
Wednesday	**6**	125	1

Moral of the story: If you can find a smaller period for which your non-zero cash flows all occur at intervals that are whole multiples of that smaller period, then your worries are over: You can use $0 cash flows for the "in-between" periods, as illustrated above.

Now, if you stop and think about it for a moment, you'll realize that you could actually apply this technique to *any* cash flow scenario. No matter how irregular the timing of the non-zero cash flows, you can always find a smaller unit of time (hours? minutes? seconds?) that is an exact divisor of each of the various intervals between those cash flows—much like finding a greatest common factor among numbers.

But there comes a point when this is more trouble than help.

For example, suppose your 30-year, $100,000 mortgage (8%) closed on January 20, but the PMT's are scheduled for automatic withdrawal from your bank account on the last day of each calendar month, and so your first scheduled PMT is for February 28.

Now, do you *really* want to change an entire 30-year scenario from a monthly periodicity (interest and PMT's) into a daily one—all for the sake of 10 extra days at the start? Probably not.* You'd have this cash flow situation to analyze:

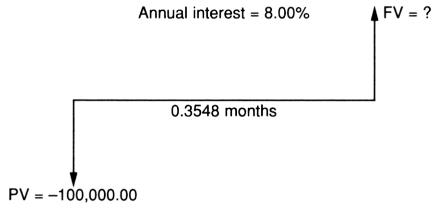
	Group#	Amount	Flows in Group
Jan. 20: Loan closes	**0**	$ 100,000	**1**
(Jan 21 - Feb 27)	**1**	0	38
Feb 28: PMT 1	**2**	(PMT)	1
(Mar. 1 - Mar. 30)	**3**	0	30
Mar. 31: PMT 2	**4**	(PMT)	1
(Apr. 1 - Apr. 29)	**5**	0	30
Apr. 30: PMT 3	**6**	(PMT)	1
(May 1 - May. 30)	**7**	0	30
May 31: PMT 4	**8**	(PMT)	1
:	:	:	:

Suddenly a simple PMT-formula mortgage morphs into a monster scenario with 721 cash flow groups—all because of a lousy few extra days up front. There *must* be a way around this.

There is....

*Most certainly not: A daily accounting would actually be too detailed (too accurate) to be correct, because mortgage conventions just don't work that way. Since the smallest unit of time they deal with is a month, they treat all calendar months as being of equal length. Doing things daily, you couldn't. If you want to preserve monthly (rather than daily) mortgage payments, either you have to change the payment amount each month to reflect the length of that month (so February's payment would be a bit less than March's)—in which case the PMT formula will be useless —or you have to make each PMT the same 30 days apart. For example: January 30, March 1 (or February 29), March 31 (or March 30), April 30 (or April 29), etc. Effectively you would be shortening the term of the loan by about 5 months (360 PMT's x 30 days is some 150 days less than 30 calendar years)*What a scheduling hassle*—brought to you by the peculiarities of the common calendar. Monthly conventions are indeed approximations, but their simplifications are certainly worth it.

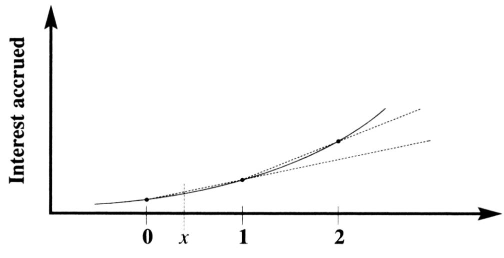
Go back to basics: Just use the mortgage rate to "slide" (adjust) the starting loan amount forward to the end of January. You have a fraction of a month (11/31, or 0.3548, according to the actual calendar*) to "slide" that cash flow forward:



OK, that makes sense. But how do you adjust a cash flow by a fraction of a period? The interest that would accrue on a given amount of money over a part of a period hasn't really been defined.

As defined, the interest rate is 8%, compounded monthly. In other words, for every $100 owed, the interest per month is $8/12, or $0.67. Likewise—scaling up—the interest on the $100,000 is $8,000/12 per month, or $666.67. So it seems reasonable that the interest for half a month ought to be half of that, $333.33; or the interest on a quarter of a month should be a quarter of it, $166.67; etc. Any fraction, x, of a month should accrue exactly that fraction, x, of the entire month's interest—just plain "straight-line" prorating: $\boldsymbol{FV} = \boldsymbol{PV}(1+x\boldsymbol{i})$

But this isn't what you'll get if you use the PMT formula (which was designed for a whole number of periods, $\boldsymbol{n}$) with a fraction, x, instead of $\boldsymbol{n}$: $\boldsymbol{FV} = \boldsymbol{PV}(1+\boldsymbol{i})^x$ Compare the two formulas on this graph:



*Whereas, if you use a "smoothed calendar," with all months of the same length—about 30.44 days—this fraction is about 10.44/30.44.

Over any *fraction* of period, the straight-line method (the dotted line slopes) actually produces slightly *more* interest than the compounding path of the PMT formula (solid line)—while (of course) over any duration beyond one period, the compounding produces more.

Which method is "correct?" It depends on the conventions and definitions of the financial transaction in question. For the sake of illustration, try each method on your mortgage and compare the results. First, use the PMT formula, which uses the compounding method for partial-period interest, to do the "sliding" shown opposite:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	0.355	0.667	–100,000.00	0.00	$\boxed{100{,}236.05}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$		8.000			
	Yrs	***I%YR***			

Now for the straight-line method. Few calculators offer a specialized built-in tool for computing straight-line partial-period interest, so you will probably have to do its arithmetic "manually:" $FV = PV(1+xi)$

$$FV = 100{,}000[1 + (11/31)(.08/12)] = \$100{,}236.56$$

That's about 51 cents different than the compounded result above—not a deal-breaker or anything, but it's definitely different (i.e. not a discrepancy introduced by rounding or something like that).

So now—choosing one of the above results as the "amount financed" —you can finally answer the question: What's your mortgage PMT?

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.667	100,236.05	$\boxed{-735.50}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	8.000		***(END***	
	Yrs	***I%YR***		of the period)	

or

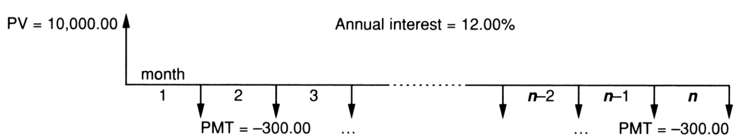
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.667	100,236.56	$\boxed{-735.50}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	8.000		***(END***	
	Yrs	***I%YR***		of the period)	

No difference—to within a penny—on your monthly PMT, though the final remaining balances will differ slightly (recall the discussion on rounding and precision on pages 118-121).

So what's the big deal about partial periods, if the difference between straight-line and compounding is so negligible? Why not just use the handy PMT formula for any circumstance and be done with it? It's just an easy and close approximation isn't it?

Well, notice that all of the above discussion was about "sliding" a single cash flow along the time line for a fraction of a period; you were just concerned with ***PV*** and ***FV***. That is, for the "sliding" calculation itself, the ***PMT*** value was *zero*. When ***PMT*** is not zero, using the PMT formula with an ***n*** value that is not a whole number is <u>very ambiguous</u>. Sure, you'll get answers, *but what do they mean?*

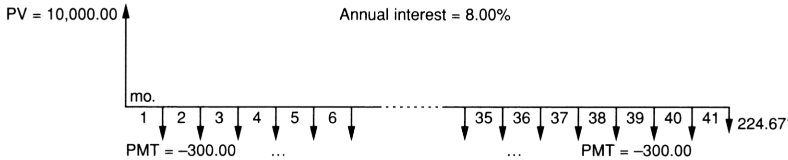
This question usually arises when you're calculating the ***n*** value as the unknown. Take a simple example: Suppose you borrow \$10,000, at an interest rate of 12%, and you want to make monthly PMT's of \$300 to repay it. How long will it take you to discharge the debt?



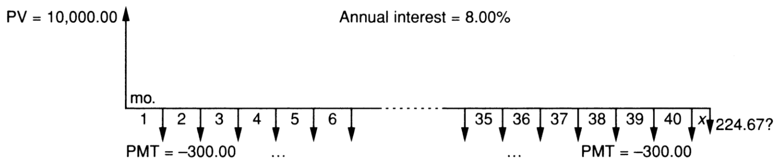
	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	$\boxed{40.7}$	1.000	10,000	–300.00	0.00
	$\boxed{3.39}$	12.000		***(END***	
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	***Yrs***	***I%YR***		of the period)	

Easy, right? But what *exactly* does that ***n*** value (40.7489…) mean? It tells you that the loan will take more than 40 but less than 41 periods and PMT's to pay off. What does that mean? Does it mean that at some point, you'll be making precisely that fraction, *x*, of a PMT (i.e. .7489 of a normal PMT, or \$224.67)? Or is there some cash flow happening after that fraction of a month? Or both? And when might these occur—at the start of the time line or at the end? Or one at each end (and if so, which at which)? Surely one of these scenarios is mathematically borne out as the true picture given by this ***n*** value.

Try the far end first. Maybe there's a partial PMT after a full period:



Or maybe a partial PMT after a partial period, x (where $x = 0.7489$).



To test these, first calculate the remaining balance after 40 periods:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	40	1.000	10,000.00	–300.00	$\boxed{-222.76}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	3.33	12.00		(***END***	
	Yrs	***I%YR***		**of the period)**	

Now try the first picture above: If you "slide" this \$222.76 forward a full period, what will you owe as a final loan pay-off amount?

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	1	1.000	222.76	0.00	$\boxed{-224.95}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)		12.00			
	Yrs	***I%YR***			

That's not a full \$300 payment—but you already knew that: If you needed a full 41st PMT after a full 41st period, your calculator would have given you an even 41 for the ***n*** value in the first place. (So a full PMT after a partial period, x, won't work, either—it's way too much.) But \$224.95 is not the telltale partial PMT value (\$224.67) you're seeking, either.

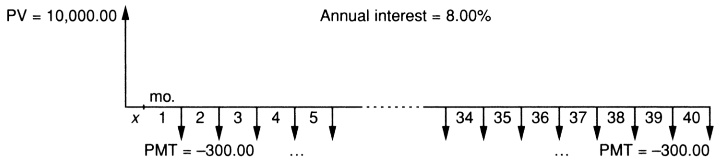
Try the second picture above: Try "sliding" the 40-month balance (\$222.76) forward by just .7489 periods:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	.7489	1.000	222.76	0.00	$\boxed{-224.39}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)		12.00			
	Yrs	***I%YR***			

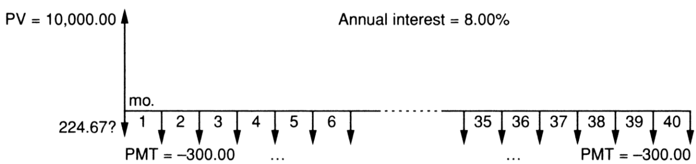
Nope—that's not the telltale partial PMT value (\$224.67), either.

So it doesn't look like there are proportional ***PMT*** and ***n*** fractions (.7489) both happening at the end of the time line—nor is either one happening by itself there. Maybe one or both happen up front.

You know that a partial period alone won't do it—that would produce a larger loan to pay off in exactly 40 periods, which you already know $300/month won't cover (try it if you wish):



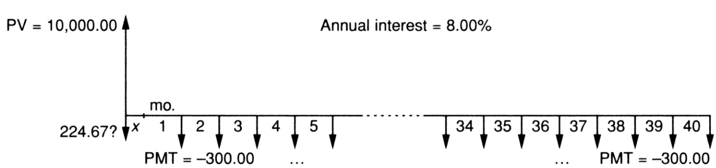
But maybe a partial PMT by itself would balance the picture:



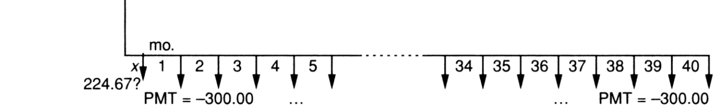
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	40	1.000	9,775.33	–300.00	111.78
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	3.33	12.00		(***END***	
	Yrs	***I%YR***		of the period)	

No, then you've overpaid by nearly $112.

Maybe the partial payment and period both occur up front—this:



Or this:



If one of those two scenarios is correct, then it should give an adjusted starting balance for a 40-month loan that is exactly paid off with $300 PMT's:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	40	1.000	$\boxed{9,850.41}$	–300.00	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	3.33	12.00		(***END***	
	Yrs	***I%YR***		of the period)	

So the telltale adjusted starting balance is $9,850.41. See if either of the latter two scenarios can produce it:

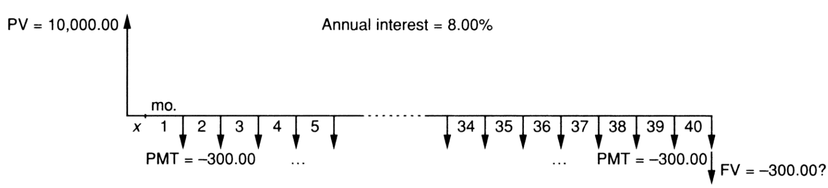
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	.7489	1.000	9,775.33	0.00	$\boxed{-9,848.44}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)		12.00			
	Yrs	***I%YR***			

No, that one doesn't match.

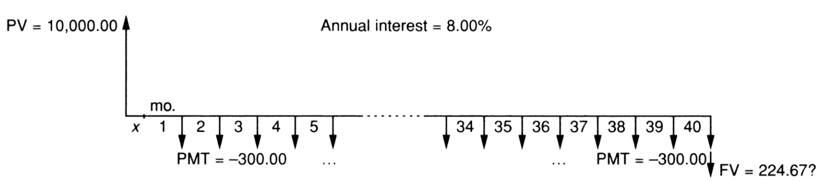
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	.7489	1.000	10,000.00	0.00	$\boxed{-10,074.80}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)		12.00			
	Yrs	***I%YR***			

And $10,074.80, less the magic partial PMT amount of 224.67, gives $9,850.13. That's close to $9,850.41, but it's not a balance.

There are other possibilities, too. Maybe the partial period happens up front and a PMT, either full or partial, happens at the end—this:



Or this:



Go ahead and test them if you wish.... They won't balance, either.

Whenever the ***PMT*** value is non-zero, it's very difficult to interpret for practical purposes a non-zero fractional portion of the ***n*** value. Yes, such a value is useful to indicate approximately when a loan will be paid off. You know above, for example, that the loan will take 40 normal payments plus a partial amount (and this is typically the use made of such calculations). But any practical conclusion more specific than this is pretty elusive.*

When ***PMT*** *is non-zero, the fractional portion of the* **n** *value used in the PMT formula exactly indicates neither the duration nor location in time of a fractional period, nor the exact amount of a fractional PMT.* (When ***PMT*** is zero, of course, the location in time of the fractional period is immaterial; the PV-FV math produces the same result either way.)

Most financial calculators come equipped only with the plain PMT formula—which, as you've seen, makes absolutely no assumptions about where and how the fractional portion of the ***n*** value applies. But some models are programmed to detect a fractional portion do try to do a bit of "reasoning." Typically these models assume that the fractional period—*but no fractional* ***PMT*** —occurs at the beginning of the time line. (This assumption deals with the most common situation, as you saw on pages 125-127: A mortgage requires its PMT's to fall at a certain time of month, but the loan has closed at another time of the month; there's a partial month to elapse before the whole number of periods can begin.)

*Actually, the discrepancies of the last three pages are hard to interpret only in the practical sense of payments and schedules; they're easy to explain mathematically: Any ***n*** value produced by the PMT formula uses the compounding path—the curved (solid) line shown in the graph on page 126. By contrast, all the above attempts to interpret the fractional portion of ***n*** are based upon *prorating* the period or ***PMT***—the straight-line model (dotted lines in the graph). As the graph shows, the two models are different and will of course give different values for any fractional value of ***n***.

Taxes and Inflation

You've already seen the basic effect that inflation has upon an interest or growth rate (pages 40-41): To find the real periodic interest rate, $\boldsymbol{i}$, from a gross earnings rate, $\boldsymbol{g}$, and an inflation rate, $\boldsymbol{r}$ (with all rates in decimal form), the buying power of a dollar after one period is $(1+\boldsymbol{g})/(1+\boldsymbol{r})$. So the real interest rate (the change in buying power) is $(1+\boldsymbol{g})/(1+\boldsymbol{r}) - 1$, or $(\boldsymbol{g} - \boldsymbol{r})/(1+\boldsymbol{r})$.

It's a fairly safe, accurate assumption to model inflation like this—as a rate that acts smoothly over time. Although certain quarters may see slightly higher inflation rates, it's not as if prices stay steady all year, then jump all at once on a given date or dates.

Taxes don't necessarily work that way, though, which you must consider when trying to do any sort of after-tax calculation under a reasonably accurate model of reality. The most straightforward periodic taxes are probably income taxes on a wage or salary and property taxes paid monthly along with your mortgage payment. But other than those calculation-friendly circumstances, tax payments happen on schedules different than the other cash flows you're dealing with in a transaction—and they vary according to the type of tax, also.

Sales Taxes: Generally, when you pay a sales tax, you do so right when you make a purchase, so it shows up on your cash flow diagram simply as a small additional negative cash flow that happens at the same time as the purchase.

If you're in a position to *collect* sales taxes, however, the timing is different: You collect the tax on one date (a positive cash flow), but you're required to pay it to the government on some other date (a negative cash flow). There's nothing complex about this in principle, but as tax collections accrue it can get to be a bookkeeping headache to get all the little cash flows at their proper spots on the time line.

Income Taxes: These are the main hassles—lots to decide. First you must identify which cash flows—or *parts* of cash flows—represent taxable income (or tax-deductible expenses—including other taxes, such as sales or property taxes). Then you must decide when you'd actually pay the taxes (or get the benefit of the deductions). Such questions may require your accountant or tax attorney to help you answer; the examples here are strictly hypothetical.

Suppose that on 5/23/00, you invest $10,000 for one year in a taxable account at 6%, compounded daily. If your marginal income tax bracket is 33%, what are the effective before-tax and after-tax rates?

Do the before-tax case first. As you'll recall, an effective rate is the amount by which $100 would actually grow over a year's time at the indicated nominal rate and compounding. That's about 6.18% in this case, as you can read from the ***FV*** result in this calculation:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 365 Periods/Year	365	0.016	–100.00	0.00	106.18
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	6.000			
	Yrs	***I%YR***			

For the after-tax case, assume that all interest (but only the interest) is taxable at your normal marginal rate, 33%. The interest is credited to your account each quarter. When do you pay the taxes on it?

For "thumbnail" calculations, a convenient shortcut is to assume that the tax is withheld out of every period's interest income. This simply reduces the nominal interest rate by the tax percentage. So if the government is going to take 33% of the interest here, the 6% before-tax interest rate becomes 3.96% after taxes: (6)(1 – 0.33) = 3.96

Thus:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 365 Periods/Year	365	0.011	–100.00	0.00	104.04
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	3.960			
	Yrs	***I%YR***			

That's an effective after-tax rate of about 4.04%.

Of course, you probably don't deposit taxes every day on the interest earned that day (and you probably don't take the taxes directly from the account in question). Want a more accurate model? OK....

Suppose you owe estimated tax deposits on 6/15/00 (for income in the second calendar quarter '00), 9/15/00 (for Q3 '00), 12/15/00 (for Q4 '00), 4/16/01 (for Q1 '01), and 6/15/01 (for Q2 '01). This means you have to calculate the interest earned:

- from 5/24/00 through 6/30/00 (38 days), deposited 6/15/00;
- from 7/1/00 through 9/30/00 (92 days), deposited 9/15/00;
- from 10/1/00 through 12/31/00 (92 days), deposited 12/15/00;
- from 1/1/01 through 3/31/01 (90 days), deposited 4/16/01;
- from 4/1/01 through 5/23/01 (53 days), deposited 6/15/01.

(Note that you withdraw the principal $10,000 along with the final interest, on 5/23/01 but don't deposit the Q2 '01 taxes until 6/15/01.)

To find the interest earned from 5/24/00 through 6/30/00:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 365 Periods/Year	38	0.016	–10,000.00	0.00	$\boxed{10{,}062.66}$
		6.000			
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	***Yrs***	***I%YR***			

That's $62.66 in interest you'll earn in Q2'00. You must deposit as taxes an amount that is 33% of this, or $20.68, on 6/15/00.

To find the interest earned from 7/1/00 through 9/30/00:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 365 Periods/Year	92	0.016	–10,062.66	0.00	$\boxed{10{,}215.98}$
		6.000			
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	***Yrs***	***I%YR***			

That's $153.32 in interest you'll earn in Q3'00. You must deposit as taxes an amount that is 33% of this, or $50.60, on 9/15/00.

To find the interest earned from 10/1/00 through 12/31/00:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 365 Periods/Year	92	0.016	–10,215.98	0.00	$\boxed{10{,}371.64}$
		6.000			
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	***Yrs***	***I%YR***			

That's $155.66 in interest you'll earn in Q4'00. You must deposit as taxes an amount that is 33% of this, or $51.37, on 12/15/00.

To find the interest earned from 1/1/01 through 3/31/01:

	n	i%	PV	PMT	FV
P (or P/YR) = 365 Periods/Year	90	0.016	–10,371.64	0.00	$\boxed{10,526.21}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$		6.000			
	Yrs	I%YR			

That's $154.57 in interest you'll earn in Q1'01. You must deposit as taxes an amount that is 33% of this, or $51.01, on 4/16/01.

To find the interest earned from 4/01/01 through 5/23/01:

	n	i%	PV	PMT	FV
P (or P/YR) = 365 Periods/Year	53	0.016	–10,526.21	0.00	$\boxed{10,618.31}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$		6.000			
	Yrs	I%YR			

That's $92.10 in interest you'll earn in Q2'01. You must deposit as taxes an amount that is 33% of this, or $30.39, on 6/15/01.

Here, then, is the cash flow situation and its resulting ***IRR%***:

P (or P/YR) = 365 Periods/Year

IRR used in periodic, decimal form: $irr = irr\%/100 = IRR\%/(100P)$

irr% / IRR%	Group #	Amount	Flows in group
irr%: $\boxed{011}$	**0**	–10,000.00	**1**
IRR%: $\boxed{4.028}$	**1**	0.00	22
	2	–20.68	1
	3	0.00	91
	4	–50.60	1
	5	0.00	90
	6	–51.37	1
	7	0.00	121
	8	–51.01	1
	9	0.00	36
	10	10,618.31	1
	11	0.00	23
	12	–30.39	1

NPV: 0.00

NFV: 0.00

But now keep in mind that this ***IRR%*** is the nominal rate; you must let the daily ***irr%*** rate compound for a year to get the annual effective rate. (Be sure to keep a lot of decimal places from the irr% calculation, as this daily rate is particularly sensitive to rounding effects.)

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 365 Periods/Year	365	0.011	–100.00	0.00	$\boxed{104.11}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	4.028			
	Yrs	***I%YR***			

Looks like about 4.11%.

All that work—and not very different from the "thumbnail" nominal estimate (4.04%). And since the "thumbnail" method errs on the conservative side (as it assumes you pay the taxes—from the account itself—every day), you can see why it's used a lot.

For example, when financial planners compare the yield of a tax-free municipal bond to the taxable equivalent yield, they usually use the "thumbnail" calculation, $A = B(1–T)$, where A is the after-tax yield, B is the before-tax yield, and T is the marginal income tax bracket (as a decimal).

The same goes for a first-pass reckoning of your after-tax interest rate cost of a mortgage (and if you've adjusted the withholding allowances of your salary income, the thumbnail is all the more accurate): If your mortgage interest is entirely tax-deductible, then $A = B(1–T)$, where A is the after-tax interest rate, B is the before-tax rate, and T is your marginal income tax bracket (as a decimal).

Besides interest, of course, you may have other sources of taxable income (and tax-deductible expenses) as well....

Suppose that you've moved to a new town for a 5-year contract job, and you're shopping for housing. It looks like you can either rent or buy the house of your choice.

The house would rent on a 60-month lease for $1,250/month, adjusted yearly for inflation (3%), with first and last payments due up front.

The house is for sale for $250,000. With 10% down, you can finance it over 30 years at 7.5%, with a 1-point mortgage fee and about $750 in other closing costs. Property taxes and insurance would be about $200/month and $50/month, respectively during the first year, with the house value, taxes and insurance increasing at the inflation rate. Realty fees and other costs upon resale would be 6% of the sales price, plus $300.

You have $30,000 in ready cash in an investment account earning a reliable average of 10% after all taxes. Your marginal tax bracket is 33% for ordinary income; 28% for capital gains. Should you buy the house or just rent it? ("To buy or not to buy—that is the question.")

This isn't a question of which investment yields you the better rate of return on your money. It's which leaves you with more money. A lesser rate can net you more if you simply invest more.

In this case, of course, you're talking about housing expense, so you can pretty well expect to be comparing two *negative* values—two costs—not two sources of income. But the method and reasoning is exactly the same: Compare the Net Present Values of the two scenarios, using your cash account's rate as the discount rate.

Start with the Rent option: The monthly rental will be $1,250.00 in year 1; $1287.50 in year 2; $1,326.13 in year 3; 1,365.91 in year 4; $1,406.89 in year 5 (all of which you can confirm via simple multiplication by 1.03, or by using a PV-FV computation).

So the up-front move-in cash flow will be \$1,250+\$1,406.89, for a total of \$2,656.89.

Here, then, is the situation as a whole—and the necessary NPV calculation for the Rent scenario:

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: -63,165.71

NFV:

irr%	***Group #***	***Amount***	***Flows in group***
0.8333	**0**	–2,656.89	1
10.000	**1**	–1,250.00	11
IRR%	**2**	–1,287.50	12
	3	–1,326.13	12
	4	–1,365.91	12
	5	–1,406.89	11

This is the amount you'd have to start with in your investment account in order to be able to fund your housing rental expenses.

Now how about the Buy option? Here's where the tax considerations start coming into play. First, though you'll need to do some calculations to establish some of the other cash flows in the picture:

Total cash to purchase would be the 10% down (\$25,000), plus the 1-point fee on the amount financed (\$2,250), plus \$750. That's a total of \$28,000 for the initial cash flow.

Next, calculate the house's resale price:

P (or ***P/YR***) = 1 Period/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	***i%***	***PV***	***PMT***	***FV***
5	3	–250,000.00	0.00	289.818.52
5	3			
Yrs	***I%YR***			

So your gross gain would be \$39,818.52. The other expenses of the sale would be \$300 plus 6% of \$289,818.52, or \$17,689.11.

Calculate the mortgage payment:

P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
360	0.625	225,000.00	$\boxed{-1{,}573.23}$	0.00
30	7.500		**(*END***	
Yrs	***I%YR***		**of the period)**	

And the remaining mortgage balance after one year:

P (or *P/YR*) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
12	0.625	225,000.00	−1,573.23	$\boxed{-222{,}925.91}$
1	7.5		**(*END***	
Yrs	***I%YR***		**of the period)**	

(Do likewise for the balances after each of the other years.)

The tax effects: For purposes of illustration (these are hypothetical only—consult a tax advisor if you're doing this for real), suppose that only your mortgage fee, all your mortgage interest and all your property taxes are tax-deductible; and that your gross gain upon re-sale is fully taxable as capital gains. So the tax effects of these items are the products of their amounts and their applicable tax brackets:

The tax on your gain at resale = (39,818.52)·(0.28) = $11,149.19

Your tax savings on the mortgage fee = ($2,250)·(0.33) = $742.50

Tax effects of the interest & prop. taxes = ($2,250)·(0.33) = $742.50

Yr.	Total PMT's	Ending Balance	Mortgage Interest Paid*	Mortgage Interest Tax Sav.	Property Taxes Paid**	Property Taxes Tax Sav.	Insur.**
1	18,878.76	222,925.91	16,804.67	5,545.54	2,400.00	792.00	600.00
2	18,878.76	220,690.79	16,643.64	5,492.40	2,568.00	847.44	618.00
3	18,878.76	218,282.16	16,470.13	5,435.14	2,747.76	906.76	636.54
4	18,878.76	215,686.55	16,283.15	5,373.44	2,940.10	970.23	655.64
5	18,878.76	212,889.42	16,081.63	5,306.94	3,145.91	1038.15	675.31

*Interest paid in any given year is the sum of the payments, less the amount paid to principal (the difference between the starting and ending mortgage balances).

**To find inflated prices in subsequent years, simply multiply the previous year by 1+r, where r is the annual rate of increase, in decimal form.

Now the final questions before you draw your cash flow diagram are where to apply these tax effects.

Look at it this way: If you draw a salary, you can adjust your withholding allowances to closely match your projected deductions for any given tax year, thus distributing the effect with your every paycheck. So it's a fairly good model of reality simply to distribute the effects of all your deductions on, say, a monthly basis (with the mortgage fee distributed over the first year).

As for the tax on your gain at resale, assume you pay it immediately. Yes you might hold the entire gain for awhile before depositing the tax, but it wouldn't be very long—and anyway this assumption errs on the conservative side.

So here's the final situation—taking into account all tax effects and proceeding year by year through the 5-year term of ownership:

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: –60,406.86

NFV:

irr%	***Group #***	***Amount***	***Flows in group***
0.8333	**0**	–28,000.00	**1**
10.000	**1**	–1,233.23	12
IRR%	**2**	–1,310.41	12
	3	–1,326.77	12
	4	–1,344.24	12
	5	–1,362.91	11

From the looks of this, you'll be better off buying the house than renting it. Either way will cost you, of course, but this is the lesser cost.

Quiz

1. A 30-year mortgage for $100,000 at 8% has monthly payments of "$736.55," says one lender; "$734.04," says another; "$733.79," says another. Your calculator says $733.76. Another calculator says $728.91. What gives? (The solution is on page 144.)

2. Refer back to problem **6** on page 53 (and its solution on page 62). Compare that result to what you get if you go a month at a time, calculating remaining balance, interest and principal paid as of the end of each month for 12 months. (Page 145.)

3. Your bank offers to consolidate an outstanding credit card balance ($7,600) into a 3-year installment loan with monthly payments of $260. The credit card charges .0375%/day. Should you accept the bank's offer? What if the interest portion of the loan payments is tax deductible? Your income tax bracket is 37%. (Page 146.)

4. A 30-year Canadian mortgage for $200,000 requires monthly payments, but the quoted nominal rate of 8% is allowed to compound only twice—rather than 12 times—per year. Calculate the monthly payment. (Page 147.)

5. Look again at problem **10** on page 53. This mortgage structure is often used when borrowers wish to accelerate their payoff schedule. But the solution on page 65 assumed that the quoted nominal mortgage rate was allowed to compound 26 times a year (instead of the normal 12 times)—to match the payment schedule. This raises the effective rate. How much? How can you re-do the calculation so that the effective rate does not change? (Page 147.)

6. If you've just made your monthly payment of \$880 on your 6% mortgage, and your remaining balance is now \$124,847.03, how long until it's paid off? How much is the final payment? When do you pay it? (Page 148.)

7. Your 60-month car loan (\$12,500 @ 5.9%) closes on January 6, but your first monthly payment is on the last day of February. What's your monthly payment? (Page 149.)

8. A mortgage rate that compounds monthly is used to figure PMT's as if the actual calendar months were all the same length. But they're not. If, on June 30, 2002, you borrow \$100,000 on a 30-year mortgage at 8%, what annual nominal mortgage rate are you really paying over the first 12 months, if you actually count the days in each month? (Use the results of problem **2** on page 145, to find the remaining balance after the 12th PMT.) What if your mortgage starts instead on June 30, 2003? (Page 150.)

9. You're 35 years old and plan to retire when you turn 60. From then on, through your 95th birthday, you want to be assured of a monthly pre-tax income of \$6,000 per month from your retirement account. Assuming an after-tax yield of 10%, what lump-sum amount would you need to start with today? If inflation is a steady 3%, what is the equivalent in today's dollars of your beginning retirement income? How about your income on your 95th birthday? How could you recast the whole problem to equalize these values, so that you would maintain the same standard of living throughout your retirement years? (Page 152.)

Quiz Solutions

1. All these calculations are dutifully dividing the quoted annual interest rate by 12 to get the monthly rate for the calculation:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.667	100,000.00	$\boxed{-733.76}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	8.000		***(END*** of the period)	
	Yrs	***I%YR***			

The difference in most cases is in how many decimal places they're keeping for that monthly interest rate. To be exact, 8%/12 = 0.66666666....%, but the first lender apparently rounds this to 0.67% for his calculation; the second lender uses 0.667%; the third, 0.6667%. Your calculator carries at least 10 or 12 decimal places, so it's probably using 0.6666666667% or something even more exact, giving you the $733.76, as shown above.

Which is "correct?" It's whatever the lender states he's using, so long as he discloses the payment and also the effective annual rate—the amount of interest that actually accrues on $100 over 12 compounding periods. (For example, in the case of the lender using 0.67%, that effective annual rate would be 8.34%; using 0.6666666667%, it would be 8.30%.)

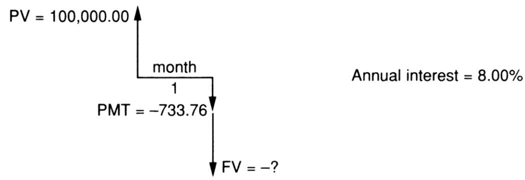
As for the other calculator result ($728.91), that was basic "pilot error"—apparently done using Annuity in Advance (i.e. with the PMT made at the *beginning* of each month)—incorrect for a standard mortgage:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	0.667	100,000.00	$\boxed{-728.91}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	8.000		***(BEGINning*** of the period)	
	Yrs	***I%YR***			

If that payment amount were to be accepted as a correct *end*-of month payment, it would essentially reduce the mortgage rate to about 7.93%, as you can verify:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	360	$\boxed{0.661}$	100,000.00	–728.91	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	30	$\boxed{7.930}$		***(END*** of the period)	
	Yrs	***I%YR***			

2. As you saw on page 62, the ***PMT*** amount is 733.76. Then the calculation for each month's ending balance (***FV***) uses the previous month's ending balance as the ***PV***:



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
1	0.667	100,000.00	–733.76	$\boxed{-99{,}932.91}$
	8.000		**(*END***	
Yrs	***I%YR***		**of the period)**	

P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

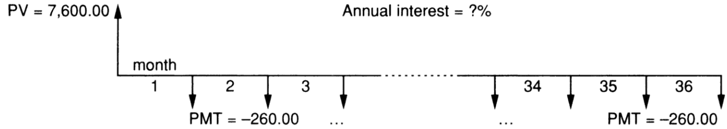
n	*i%*	*PV*	*PMT*	*FV*
1	0.667	99,932.91	–733.76	$\boxed{-99{,}865.37}$
	8.000		**(*END***	
Yrs	***I%YR***		**of the period)**	

Continuing likewise for 12 months, you should get these results:

Month	Starting Bal.	Ending Bal.	Principal Paid	Interest Paid
1	$ 100,000.00	$99,932.91	$67.09	$666.67
2	99,932.91	99,865.37	$67.54	$666.22
3	99,865.37	99,797.38	$67.99	$665.77
4	99,797.38	99,728.94	$68.44	$665.32
5	99,728.94	99,660.04	$68.90	$664.86
6	99,660.04	99,590.68	$69.36	$664.40
7	99,590.68	99,520.86	$69.82	$663.94
8	99,520.86	99,450.57	$70.29	$663.47
9	99,450.57	99,379.81	$70.76	$663.00
10	99,379.81	99,308.58	$71.23	$662.53
11	99,308.58	99,236.88	$71.70	$662.06
12	99,236.88	**99,164.70**	$72.18	$661.58
	— TOTALS —		**$835.30**	**$7,969.82**

These are slightly different—and more correct—results than the quick calculations on page 62. Each month's numbers must be rounded to dollars and cents (2 decimal places), which the procedure on page 62 did not do; it did all 12 months at once, in a calculation that probably carried 10 or 12 digits. This is why many financial calculators feature separate *amortization* tools that do this tedious, period-by-period arithmetic and rounding for you.

3. The bank's interest rate is easily calculated:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	36	$\boxed{1.172}$	7,600.00	–260.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	3	$\boxed{14.07}$		***(END***	
	Yrs	***I%YR***		of the period)	

But that 1.172(309376...) is the monthly interest rate. Find its daily equivalent to compare it to the credit card's daily rate:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	12	1.172	–100.00	0.00	$\boxed{115.01}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	14.07			
	Yrs	***I%YR***			

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 365 Periods/Year	365	$\boxed{0.038}$	–100.00	0.00	115.01
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	$\boxed{14.07}$			
	Yrs	***I%YR***			

The credit card's rate (0.0375) is slightly better in this case.

And what if the loan interest is tax-deductible? The interest over 36 months is 36(260) – 7,600 = \$1,760. If the tax bracket is 37%, that means that your real cost for interest is 37% less, or \$651.20. Your effective monthly PMT is therefore reduced by an average of \$651.20/36 or \$18.09/month, to \$241.91.

Repeating the above comparison calculation with this reduced effective PMT would give an equivalent daily rate of 0.025..., so in this case the loan is a much better deal.* The tax impacts can make all the difference.

*Note here that for the purposes of quick comparison, you have averaged the interest's tax savings evenly over the 3 years. In reality, you'd get more benefit in the first year than in the second, and even less in the third, as your portion paid to principal would accelerate. So your average figure is a conservative estimate of the tax benefits: the actuality of more benefits to you sooner would further reduce the rate you're paying.

4. Before you can do a straightforward ***PMT*** calculation, you need to find the monthly equivalent of the semi-annual Canadian rate:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 2 Periods/Year	2	4.000	–100.00	0.00	$\boxed{108.16}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	8.000			
	Yrs	***I%YR***			

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	12	$\boxed{0.656}$	–100.00	0.00	108.16
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	$\boxed{7.870}$			
	Yrs	***I%YR***			

Now the ***PMT*** calculation:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	360	0.656	200,000.00	$\boxed{-1{,}449.42}$	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	30	7.870		(***END***	
	Yrs	***I%YR***		of the period)	

5. First, compare the effective rates:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	12	0.625	–100.00	0.00	$\boxed{107.76}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	7.500			
	Yrs	***I%YR***			

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 26 Periods/Year	26	0.288	–100.00	0.00	$\boxed{107.78}$
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	7.500			
	Yrs	***I%YR***			

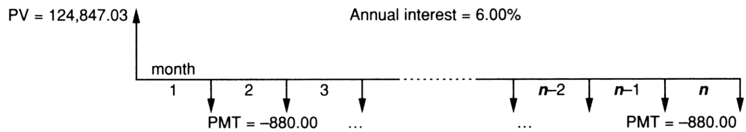
A couple of basis points (hundredths of %) more in the effective yearly rate—not a deal breaker, but it's definitely different. To make the scheme truly rate-neutral, you'd need to use the 26-period rate that produces the exact *same* effective yearly rate:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 26 Periods/Year	26	$\boxed{0.288}$	–100.00	0.00	107.76
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	1	$\boxed{7.487}$			
	Yrs	***I%YR***			

If you use this rate, the mortgage payoff is some 4 weeks earlier:

	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 26 Periods/Year	$\boxed{603.6}$	0.288	150,000.00	–524.41	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	$\boxed{23.21}$	7.487		(***END***	
	Yrs	***I%YR***		of the period)	

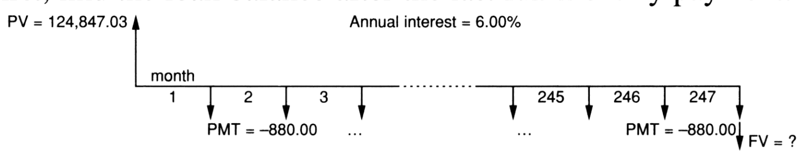
6. Here's the situation:



	n	$i\%$	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	$\boxed{247.8}$	0.500	124,847.03	–880.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	$\boxed{20.6}$	6.000		(***END***	
	Yrs	***I%YR***		of the period)	

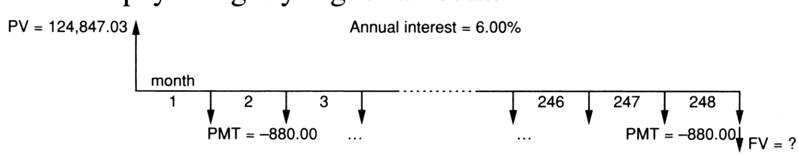
So you've got another 247 full monthly payments ahead of you… and then something else. What else? Look at your choices.

First, find the loan balance after the last full monthly payment:



	n	$i\%$	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	247	0.500	124,847.03	–880.00	$\boxed{-657.13}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	20.6	6.000		(***END***	
	Yrs	***I%YR***		of the period)	

At that point, you could pay off the loan entirely with an extra payment $657.13 (paid on top of your 247th regular installment of $880). Or, if you want to wait until the end of the 248th month, you could pay a slightly higher amount:



	n	$i\%$	PV	PMT	FV
P (or ***P/YR***) = 12 Periods/Year	248	0.500	124,847.03	–880.00	$\boxed{219.58}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	20.7	6.000		(***END***	
	Yrs	***I%YR***		of the period)	

The positive balance after a proposed 248th regular payment tells you that $880 is $219.58 too much; you're due a refund of that much. So what you really owe is $880.00–$219.58, or $660.42.*

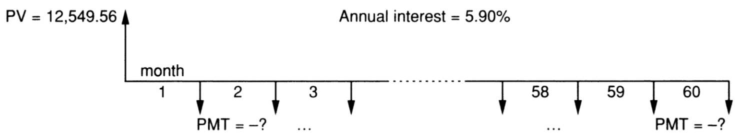
*Note that you could have arrived at this same number also by letting the previous month's balance ($657.13) simply accrue interest (0.5%) for a month.

7. This is one of those cases of a mortgage with some odd days up front. The question is, should the partial period's interest be accrued on a straight-line (prorated) basis, or via the compounding curve (what the PMT formula uses)? Try both and compare the results.

Prorating (straight-line) first, you have 25/31 of a month extra to let 5.9% (compounded monthly) accrue on the $12,500 loan.

$$(.059 \div 12)(25 \div 31)(12{,}500) = \$49.56$$

So figure your loan based on a starting balance of $12,549.56:

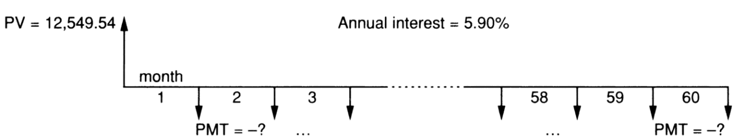


	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	60	0.492	12,549.56	$\boxed{-242.04}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	5	5.900		***(END***	
	Yrs	***I%YR***		**of the period)**	

Now try the compounding curve for partial period interest:*

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	0.806	0.492	12,500.00	0.00	$\boxed{-12{,}549.54}$
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$		5.900			
	Yrs	***I%YR***			

This 2 cents of difference actually changes the monthly ***PMT*** by a penny (and the final payoff adjustment, too, no doubt)!



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	60	0.492	12,549.54	$\boxed{-242.03}$	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	5	5.900		***(END***	
	Yrs	***I%YR***		**of the period)**	

*Reminder: As you read earlier in this chapter, most calculators will use the normal PMT formula (i.e. compounding) for partial period interest. A few may offer you the option of straight-line (i.e. prorating), also.

8. Here's your actual-days PMT schedule. (***PMT*** = \$733.76.)

PMT	Date	Days Since Previous PMT
1	7/31/02	31
2	8/31/02	31
3	9/30/02	30
4	10/31/02	31
5	11/30/02	30
6	12/31/02	31
7	1/31/03	31
8	2/28/03	28
9	3/31/03	31
10	4/30/03	30
11	5/31/03	31
12	6/30/03	30

And, as you found in problem 2 (page 145), the loan balance after the 12th PMT is \$99,164.70. So, find the IRR of this situation:

P (or ***P/YR***) = 365 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: 0.00

NFV: 0.00

irr%	***Group #***	***Amount***	***Flows in group***
0.022	**0**	100,00.00	1
7.973	**1**	0.00	30
IRR%	**2**	–733.76	1
	3	0.00	30
	4	–733.76	1
	5	0.00	29
	6	–733.76	1
	7	0.00	30
	8	–733.76	1
	9	0.00	29
	10	–733.76	1
	11	0.00	30
	12	–733.76	1
	13	0.00	30
	14	–733.76	1
	15	0.00	27
	16	–733.76	1
	17	0.00	30
	18	–733.76	1
	19	0.00	29
	20	–733.76	1
	21	0.00	30
	22	–733.76	1
	23	0.00	29
	24	–99,898.46	1

Now, keep in mind that this 0.0218437...% is a daily-compounded rate. To find the equivalent monthly-compounded rate, you'll need to do the usual conversion procedure:

P (or **P/YR**) = 365 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	***i%***	***PV***	***PMT***	***FV***
365	0.0218	–100.00	0.00	$\boxed{108.30}$
1	7.973			
Yrs	***I%YR***			

P (or **P/YR**) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

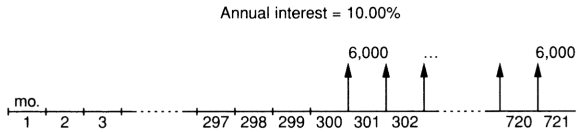
n	***i%***	***PV***	***PMT***	***FV***
12	$\boxed{0.667}$	–100.00	0.00	108.30
1	$\boxed{7.999}$			
Yrs	***I%YR***			

So the plain-vanilla standard mortgage rate is actually slightly *lower* by the actual calendar than by the fictitious 12-uniform-month calendar assumed in a PMT calculation. No big deal—everyone accepts the simplifications offered by the standard—but it *is* an approximation, a slight overstatement of reality.

Would things be any different if the starting date of the mortgage were a year later? Yes, 2004 is a leap year. So the February PMT would have 28, rather than 27, days of $0.00 cash flows before it. So adjust that value in the cash flow list (Group #**15**) and find the IRR: 0.0217847...%

Converting this in the manner shown above would then give an equivalent monthly-compounded rate of 7.977%. So a leap year is an extra good deal for the borrower.

9. The solution to the first question is simple: Just find the Net Present Value of your required future income.



P (or **P/YR**) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
0.8333	**0**	0.00	**1**
10.000	**1**	0.00	299
IRR%	**2**	6,000.00	421

NPV: 58,384.63

NFV:

So all you need to sock away now is $58,384.63 and you're all set for retirement.... Or are you? What standard of living will $6,000/month buy you in 25 years, assuming a 3% inflation rate? Or, to put it differently, how much will you need to spend then to buy what $6,000 buys you now? It's easy enough to calculate:

P (or **P/YR**) = 12 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	*i%*	*PV*	*PMT*	*FV*
300	0.25	–6,000.00	0.00	12,690.12
25	3.00			
Yrs	***I%YR***			

You won't be living *half* so well on $6,000 at age 60 as you do now, at age 35. And what about when you're 95?

P (or **P/YR**) = 1 Periods/Year

Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)

n	*i%*	*PV*	*PMT*	*FV*
720	0.25	–6,000.00	0.00	36,216.44
60	3.00			
Yrs	***I%YR***			

Your $6,000 income then will buy you less than a *sixth* of what it does today. Not so good.

What to do about this? You can change your assumptions about how your account grows. Instead of computing how its *face value* grows (10%/year, compounded monthly), you can look instead at how its *buying power* grows.

Recall the discussion on pages 40-41 about how the inflation rate, $\boldsymbol{r}$, affects the true interest rate, $\boldsymbol{i}$—the rate at which your buying power ("today's dollars") is really accruing in an account earning a face-value growth rate, $\boldsymbol{g}$: The formula is $\boldsymbol{i} = (\boldsymbol{g} - \boldsymbol{r})/(1+\boldsymbol{r})$, where all rates are in periodic decimal form.

In this case, $\boldsymbol{i} = (0.00833–0.0025)/(1+0.0025)= 0.005818786...$

On a nominal annualized basis, that's less than 7%—quite a different picture from the 10% you though you were earning. So redo your NPV calculation with this true interest rate, and you'll see how much you really need to salt away now in order to maintain the same standard of living throughout retirement that \$6,000 buys you now:

P (or **P/YR**) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

irr%	*Group #*	*Amount*	*Flows in group*
0.5819	**0**	0.00	**1**
6.98	**1**	0.00	299
IRR%	**2**	6,000.00	421

NPV: 166,118.50

NFV:

You'll need nearly three times as much as you first thought (and that's assuming just a moderate 3% inflation rate)! Only if you put away \$166,118.50 now, can you then withdraw the necessary \$12,690.12 (recall your calculation on the previous page—what you'll need to buy the same groceries as now) in the first month of your retirement. And you can increase that draw by 3% per year (0.25%/month) through your 95th birthday, when (as you also calculated) your final monthly draw will be \$36,216.44.

That's an important point about calculating with real interest rates: The value of every future cash flow in the scenario is in terms of today's dollars' buying power. To translate it back to the face value you'd actually pay or receive at that time, you need to inflate it. Every month's income in your retirement scenario has the same \$6,000 value—in terms of today's buying power—but that's not the number of dollars—face value—you'll draw then.

A. Appendix: Special Problems

Key points to be covered here:

Certain financial transactions involve calculations beyond the scope of the PMT formula or discounted cash flow analysis. The most common among these are *bonds*, *increasing (or decreasing) annuities* and *leases with skipped and/or advance payments*.

For these, your calculator generally requires either a separate set of built-in features (such as a menu of tools dedicated to bonds) or a customized automated solution (e.g. a program that you key in or download). Of course, a book cannot provide such, but this appendix does outline the calculations involved, so that you can tackle them "manually" if you wish—or at least understand them more completely.

Bonds

A bond is a *negotiable, interest-bearing note*, a binding loan contract obligating the note's issuer (the borrower) to pay the note's bearer (the lender) a certain amount of interest periodically, and then repay the principal at a certain future date. Since the note is negotiable, the original bearer may sell it on the open market, in which case the buyer becomes the new bearer, and all the issuer's obligations under the terms of the note are now directed toward that new bearer.

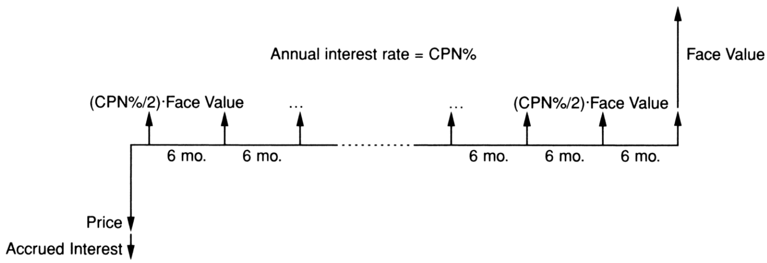
There are many kinds of bonds, but most adhere to some common conventions:

- The issue date is the date on which the borrower sells the bond to the first bearer, at a mutually agreed price,which need not match the face value, or *maturity value*— the "principal" amount due to be repaid by the issuer at the end of the bond's term.
- The bond accrues interest on the principal, at a yearly rate called the *coupon rate*, which may compound either annually or semi-annually. The bond is thus classified as either an *annual-coupon* bond or *semiannual-coupon* bond.
- The bond issuer pays to the current bearer all accrued interest—referred to as a *coupon payment*—either annually or semiannual ly (matching the compounding period). Thus a coupon payment usually marks a 6-month or yearly anniversary of the issue date.
- By convention, coupon interest accrues and payments are made using either an *actual calendar basis* (365 days) or a 30/360 calendar basis (30 days/ month; 360 days/year). Each convention has rules for treating the asymmetries of the actual calendar.*

*For example, what date is 6 months after August 30th? Or March 31? Counting the actual days in a half year is different than a smoother—but more approximate—reckoning in uniform 30-day months. (When you think about it, there's a basic "smoothing" assumption in monthly mortgage payments, too: February is a short month, but you still owe the same payment as for a longer month, such as March.)

- Interest compounds semiannually or annually, but because a bond is negotiable and may be sold most any day of the year, the interest that has accrued since the last coupon payment up to the date of the sale is rightfully due to the seller and must be computed and paid at the time of the sale—in addition to the selling price.

Look at a picture of a bond investment—your perspective as a buyer of a typical semiannual-coupon bond.



Notice that it doesn't really matter whether you're the original buyer (i.e. buying it from the issuer) or a subsequent buyer; the diagram is still be valid. If you're the original buyer, the accrued interest is simply zero, and the settlement date is, by definition, a coupon date.

The calculations most commonly needed by bond investors are these:

- The *yield-to-maturity*. This is what the bearer's will earn as the "internal rate of return" on his/her money if he/she holds the bond (receiving coupon payments as scheduled) until the maturity date, when he/she will receive the principal ("face value") of the note.*
- The *price* to pay for a bond on a given date to achieve a desired yield-to-maturity (the "*PV*" necessary to achieve a certain "*i*").
- The *accrued interest* due to the seller on a given date—in addition to the price.

*The terms of many bonds also include "call" dates, usually coupon dates prior to maturity, on which the bond bearer may "call" it due (demand payment of the face value). A *yield-to-call* calculation is just a variation on yield-to-maturity (merely a different date), and most automated bond programs will do this, too.

Obviously, you could, given enough time—and a complete listing of the rules for the calendar basis of a given bond—work out these calculations using the PMT formula or discounted cash flow analysis. But it would be tedious and counterproductive. Far better to let software or hardware manufacturers build specialized tools into your technology of choice (calculator, PDA, computer, etc.)—especially if you do enough bond buying and selling to need these calculations anyway. In short: Don't try these manually.

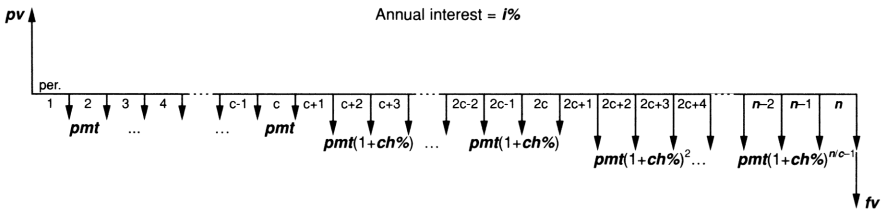
One other note about bond convention and terminology: The actual money involved in a bond may be thousands or millions of dollars, but usually you'll hear it referred to, priced and calculated on the *basis* or *par value* of 100 (and many calculator/computer formulas will use this convention, too). That is, the face value—the amount of money to be repaid as principal at maturity—is assigned a value of 100 and then all other quantities are weighted appropriately, as *percentages* of the face value—a very convenient shorthand to give everyone a better feel for the relative amounts involved.

So the price of, say, a $400,000 bond (that's its face value) might be quoted as 97.5, and its accrued interest for a given settlement (sales) date might be, say, 2.75. This means you would pay an actual price of .975(400,000) or $390,000, plus .0275(400,000), or $11,000, in accrued interest.

Increasing/Decreasing Annuities

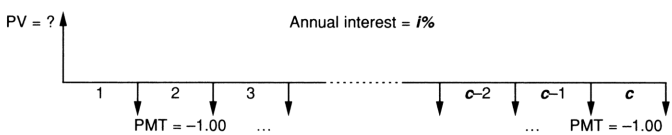
Often when trying to plan for retirement or otherwise account for inflation, you may want to calculate values of an increasing (or decreasing) annuity—i.e. a stream of increasing or decreasing cash flows—over ***n*** periods, with a discount rate, ***i%***, as usual.

It's as if you want to calculate ***PV***, ***FV***, or ***PMT***, except that in this special case (call them ***pv***, ***fv*** and ***pmt***) the ***pmt*** increases or decreases periodically (after every ***c*** periods) by a given percentage, ***ch%***:



You can't use the normal PMT formula directly on the above situation, of course. But you can use it indirectly. To calculate the present value, ***pv***, of an increasing(or decreasing) annuity (paid in arrears —at the end of each period) given a starting periodic cash flow value, ***pmt***, which increases (or decreases) every ***c*** periods by ***ch%***, (positive for increasing, negative for decreasing), follow these steps:

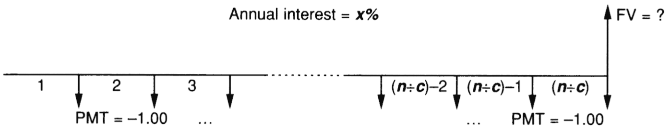
A. Find the ***PV*** of ***c*** periods of $1 ***PMT***'s, using ***i%***:



n	***i%***	***PV***	***PMT***	***FV***
c	***i%***	$\boxed{A}$	–1.00 (***END*** of the period)	0.00

B. Compute $\left[\dfrac{(1+ch\%\div 100)}{(1+i\%\div 100)^c} - 1\right] \cdot 100$. Call this result $x\%$.

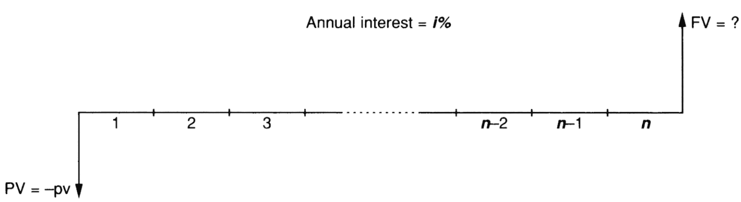
Now find the ***FV*** of $n \div c$ periods of \$1 ***PMT***'s, using $x\%$:



n	*i%*	*PV*	*PMT*	*FV*
n÷c	*x%*	0.00	–1.00 (*END* of the period)	$\boxed{B}$

C. Multiplying results ***A*** and ***B*** together would give you the present value of an increasing stream of payments that started with \$1. But you want to start the stream with ***pmt*** dollars ("***pmt*** times as much"), so you also multiply by ***pmt***,to get ***pv***: (***A***)(***B***)(***pmt***) = ***pv***

D. And of course, you could then find ***fv*** by a simple TVM calculation, using ***i%*** over ***n*** periods, as usual:



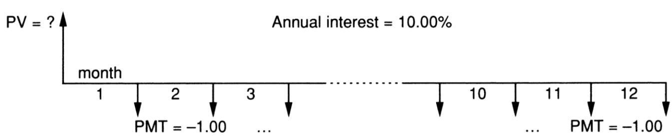
n	*i%*	*PV*	*PMT*	*FV*
n	*i%*	*pv*	0.00	$\boxed{fv}$

Try an example: What balance must you have in your retirement account (earning 10% annually, compounded monthly) at age 60 to be able to withdraw \$10,000 every month thereafter, increased annually for 3% inflation, until you're 95? What will that account balance be when you're 80? If you have \$1 million in the account at age 60, what's your inflation-adjusted monthly income until you're 95?

Here are the known values—grind away:

pmt = 10,000	(the starting periodic annuity amount)
n = 35(12) = 420	(total periods in analysis)
i% = 10÷12 = 0.8333...	(prevailing "value of money" interest rate)
c = 12	(# of level payments before each increase)
c% = 3	(percentage of payment increase)

A. Find the ***PV*** of ***c*** periods of \$1 ***PMT***'s, using ***i%***:



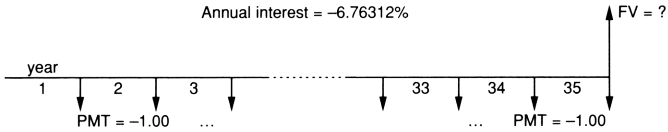
	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	12	0.833	$\boxed{11.37451}$	–1.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	1	10.00		***(END***	
	Yrs	***I%YR***		**of the period)**	

B. Compute $\left[\dfrac{(1+\boldsymbol{ch\%}\div 100)}{(1+\boldsymbol{i\%}\div 100)^{\boldsymbol{c}}} - 1\right] \cdot 100$:

$$\left[\frac{(1+3\div 100)}{(1+.83333\div 100)^{12}} - 1\right] \cdot 100 = -6.76312$$

Call this result ***x%***.

Now find the ***FV*** of ***n÷c*** periods of \$1 ***PMT***'s, using ***x%***:



	n	*i%*	*PV*	*PMT*	*FV*
	35	–6.76312	0.00	–1.00	$\boxed{13.51133}$
P (or ***P/YR***) = 1 Period/Year	35	–6.76312		***(END***	
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	***Yrs***	***I%YR***		of the period)	

C. Now multiply results ***A*** and ***B*** together with ***pmt***,to get ***pv***:

(***A***)(***B***)(***pmt***) = (11.37451)(13.51133)(\$10,000) = \$1,536,847.42

This is the balance needed in your retirement account at age 60 in order for you to draw the specified inflation-adjusted income.

Notice that if you already know the ***pv*** available to you, you can use the same relation to find an unknown ***pmt*** amount:

(***A***)(***B***)(***pmt***) = ***pv*** means that ***pmt*** = ***pv***÷[(***A***)(***B***)]

So if your lump-sum retirement account balance is \$1 million on the day you retire at age 60, then you can draw

\$1 million÷[(11.37451)(13.51133)]

or \$6,506.83/month, adjusted by 3% annually for inflation, until you're 95.

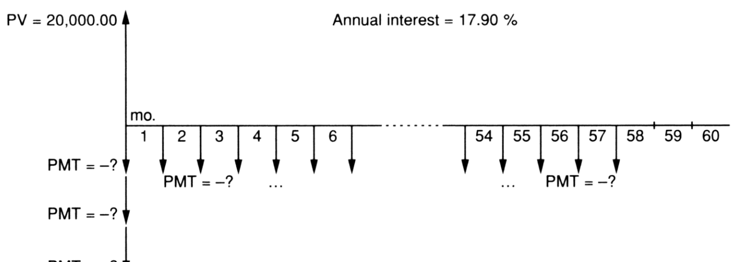
Leases with Skipped and/or Advance Payments

A lease is a long-term rental of some physical asset where the payments are often computed by "amortizing" the capital value (a hypothetical market or sales price) of the asset, using a "lease rate" of interest. This is exactly analogous to amortizing a real estate mortgage loan, using a mortgage interest rate—and you can use the same sort of straightforward ***PMT*** calculation to do it.

However, due to the nature of the equipment being leased, some lease contract terms get a little more intricate. They may specify that a few of the "***n***" lease payments must be made in advance. Some contracts may also allow certain payments to be skipped periodically (such as with heavy equipment that can be used only seasonally). That's a new wrinkle on the standard calculation of a plain old ***PMT***, but you can still do it via the PMT formula.

Take an example of advance payments first; it's easier. Suppose a 5-year lease for $20,000 of computer equipment calls for 2 payments in advance, then monthly installments for the other 58 payments. The lease rate is 17.9%. What's the payment amount?

Here's a picture of the situation:



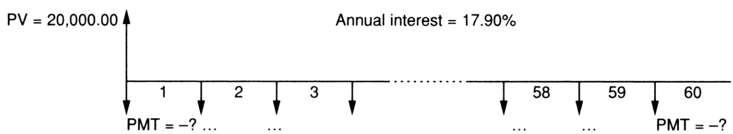
Note that a lease (which is a rental, remember) generally is structured with Annuity in Advance—payments due at the beginning of each period. So the first regular PMT occurs at the start of the time line, along with the two advance PMT's.

Clearly, if you could guess the value of the two advance PMT's, you could net their sum in with ***PV*** and solve for ***PMT***:

	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	58	1.492	20,000–2(PMT)	–??	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	4.83	17.90		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

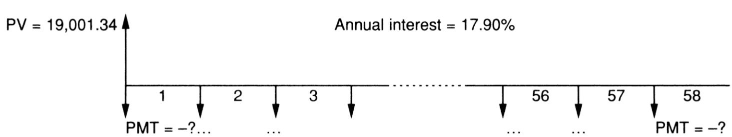
Unfortunately, of course, that means you'd have to assume the very PMT value you need to calculate.

Actually, that *is* one way to do this: trial and error. You make a *guess* for the ***PMT*** value, use that guess in the ***PV*** (see above), then calculate what ***PMT*** results. Start with the normal 60-month lease PMT:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	60	1.492	20,000	–499.33	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	5	17.90		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

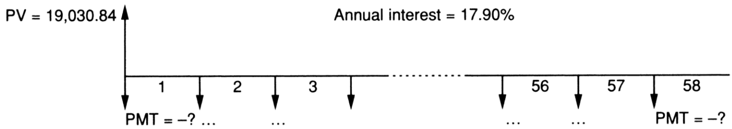
Now take this PMT value, –$499.33, as your "guess" and use it in the ***PV*** value: 20,000 – 2(499.33) = 19,001.34. Then solve for ***PMT***....



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	58	1.492	19,001.34	–484.58	0.00
Interest rate is used in decimal form: ***i*** = ***i%***/100 = ***I%***/(100***P***)	4.83	17.90		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

If your guess value for the PMT had been exactly correct, the calculated ***PMT*** value would have matched it. It didn't, but it's close.

So now use this new result, –484.58, as your next “guess” for the ***PMT*** used to adjust the ***PV*** value: 20,000 – 2(484.58) = 19,030.84



P (or ***P/YR***) = 12 Periods/Year

Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$

n	*i%*	*PV*	*PMT*	*FV*
58	1.492	19,030.84	$\boxed{-485.33}$	0.00
4.83	17.90		(***BEGIN***ning of the period)	
Yrs	***I%YR***			

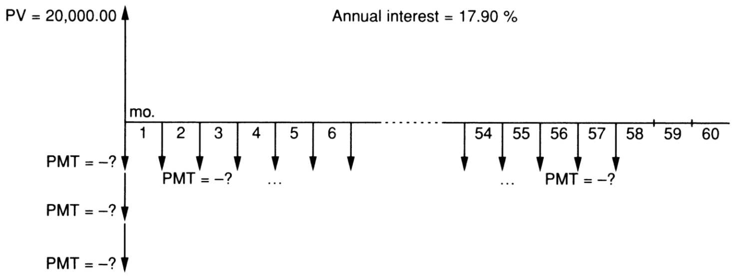
Now you’re getting very close—your guess was less than $1 off. Just keep on iterating in this way, using each calculated ***PMT*** value as the next guess. Here’s a summary of all the results you’ll get:

your ***PMT*** guess	***PV*** used = 20,000 – 2(***PMT*** guess)	calculated ***PMT*** result
–499.33	19,001.34	–484.58
–484.58	19,030.84	–485.33
–485.33	19,029.34	–485.29
–485.29	19,029.42	–485.29

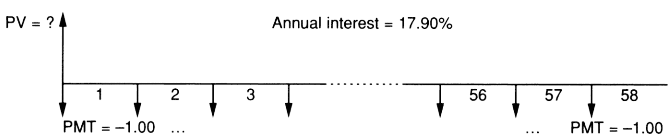
At last, your guess is confirmed by a calculated result: The required monthly payment on a 5-year lease of a $20,000 asset, at an annual lease rate of 17.9%, with 2 payments in advance, is $485.29.

Now, that’s the long way to do it—trial and error. It works, but obviously it’s a bit tedious. Can you calculate it more directly? Yep.

The whole premise of any valid financial transaction, including this lease, is that the sum of all cash flows, when adjusted to any common point on the time line, must equal zero:



Suppose you decide to "slide-and-sum" all the cash flows to the beginning of the time line. *Question:* What if the ***PMT*** amount were \$1? How much of the ***PV*** would these PMT's (including the two in advance)—after proper adjustments—balance at the beginning of the time line? You can calculate this:

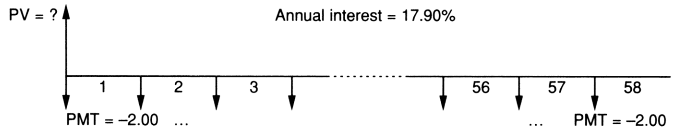


	n	***i%***	***PV***	***PMT***	***FV***
P (or ***P/YR***) = 12 Periods/Year	58	1.492	$\boxed{39.2123}$	–1.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	4.83	17.90		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

Now just add the two \$1 PMT's in advance (no need to adjust them—they're already at the beginning of the time line), and you get an adjusted value of about \$41.2123 for the whole set of \$1 PMT's.

Clearly \$1 PMT's aren't enough; they only balance \$41.21 of the \$20,000 PV.

All right, what if the PMT's were \$2? Again, you can calculate this:



	n	*i%*	*PV*	*PMT*	*FV*
P (or ***P/YR***) = 12 Periods/Year	58	1.492	$\boxed{78.4246}$	–2.00	0.00
Interest rate is used in decimal form: $i = i\%/100 = I\%/(100P)$	4.83	17.90		**(*BEGIN*ning**	
	Yrs	***I%YR***		**of the period)**	

Now just add the two \$2 PMT's in advance, and you get an adjusted value of \$82.4246 for the whole set of \$2 PMT's.

Notice: The discounted value of the \$2 PMT stream is <u>exactly</u> <u>twice</u> as much as that of the \$1 PMT stream. *That is, the discounted value of any set of PMT's is proportional (i.e. linear with respect) to the PMT value itself.**

*This is clear also, from looking at the general PMT formula itself (back on page 46): It's just the mathematical way of saying, respectively, "The sum of the ***PV***, plus all the ***PMT***'s and the ***FV***—after they've each been adjusted ("slid-and-summed") using the prevailing interest rate back to the beginning of the time line—is zero." And the middle term of the formula—the one representing the ***PMT***'s contributions to that discounted sum—is *directly proportional* (linear with respect) to the ***PMT*** value itself: If you double the ***PMT*** value, you double the value of this entire term; if you triple the ***PMT*** value, that triples the whole term. *Whatever* you do to the ***PMT*** value—multiply it by 12, or 4.5, or 125.78, or any other factor—it has exactly that effect on the whole term.

This is an extremely handy relationship. It means that all you need to do to balance any set of identical cash flows (PMT's) against the rest of the scenario at the beginning of the time line is to compute the discounted value of a set of $1 PMT's, then divide that result into the sum you're trying to balance.

You're trying to balance a sum of $20,000, and you know that $1 PMT's will balance about $41.2123 of it. You also know that $"*x*" PMT's will balance "*x*" times as much—the proportionality at work. So just divide $20,000 by 41.2123, and you'll get the correct "*x*:"

$$\$20{,}000 \div 41.2123 = \$485.29$$

<u>To</u> <u>summarize</u> the technique for computing an unknown PMT stream where the timing of the PMT's don't allow you to use the PMT formula directly (such as in this lease with advance PMT's):

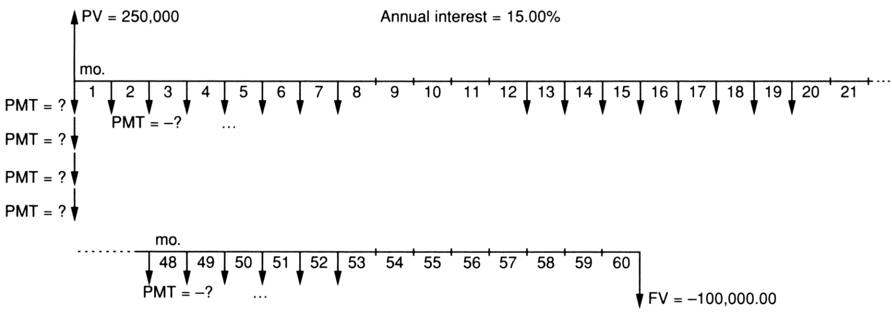
A. You "slide-and-sum"—to the beginning of the time line—all cash flows except the unknown PMT set.

B. You "slide-and-sum"—to the beginning of the time line—a set of $1 PMT's.

C. You divide result **A** by result **B**. Voilá.

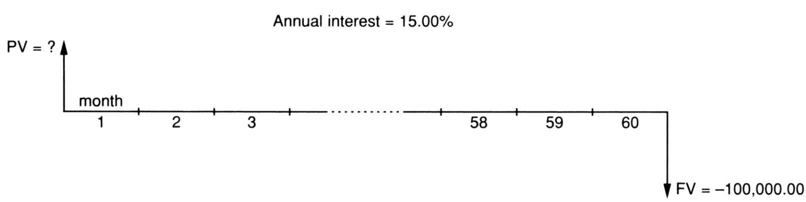
Practice this technique now with the other complex leasing situation: skipped payments. Suppose you're leasing a $250,000 piece of farm equipment for 5 years, at a lease rate of 15%, with, say three PMT's in advance,and after 5 years, the buyout price would be $100,000. But because of the seasonal nature of the equipment's usefulness, the leasing company structures it so that you can skip the payments during the final four months of every year.

Here's the picture:



This is one situation, where you really don't want to have to use the iterative, trial-and-error approach; the proportionality technique really comes in handy. Grind away:

A. "Slide-and-sum"—to the beginning of the time line—all cash flows except the unknown PMT set.



	n	*i%*	*PV*	*PMT*	*FV*
P (or **P/YR**) = 12 Periods/Year	60	1.250	$\boxed{47,456.76}$	0.00	–100,000.00
Interest rate is used in decimal form: i = i%/100 = I%/(100P)	5	15.00		(***BEGIN***ning of the period)	
	Yrs	***I%YR***			

The residual accounts for $47,456.76 of the $250,000; the ***PMT***'s must balance the rest: $250,000 – $47,456.76, or $202,543.24

B. Now "slide-and-sum" a set of $1 PMT's: In this case, you can't conveniently use the PMT formula, but you can use discounted cash flow analysis:

P (or ***P/YR***) = 12 Periods/Year

IRR used in periodic, decimal form:
irr = ***irr%***/100 = ***IRR%***/(100***P***)

NPV: –30.53821

NFV:

irr%	***Group #***	***Amount***	***Flows in group***
1.250	**0**	–4.00	1
15.00	**1**	–1.00	7
IRR%	**2**	0.00	4
	3	–1.00	8
	4	0.00	4
	5	–1.00	8
	6	0.00	4
	7	–1.00	8
	8	0.00	4
	9	–1.00	5

C. Now just divide result **A** by result **B**:

$202,543.24 ÷ (–30.53821) = –$6,632.45

Voilá. There's the required lease payment.

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