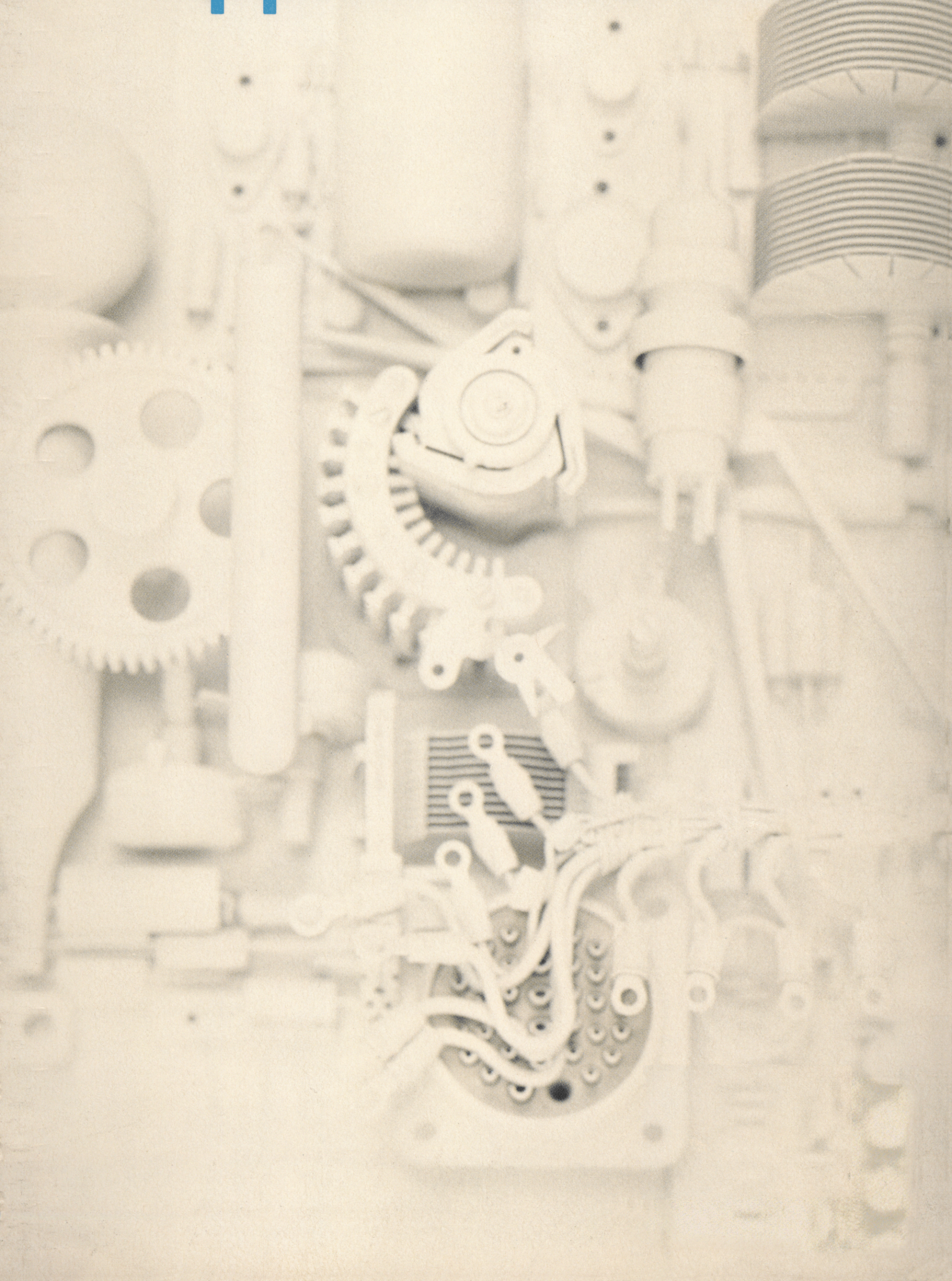
HEWLETT-PACKARD

HP-21 Applications Book



Introduction

The **HP-21 Applications Book** gives you a sample of the problem-solving capabilities of the HP-21 Scientific Pocket Calculator. Hewlett-Packard has found that there are a number of solutions that calculator owners repeatedly seek within their specific application areas or fields of interest. In this book, we have presented key sequence routines that solve over 40 of these common problems. The routines are organized into sections covering the areas of Statistics, Mathematics, Finance, Navigation, Surveying and Conversions. No matter what your technical or professional field, whether you're a student or a scientist, statistician or surveyor, business-person or engineer, we're confident that you'll find these pages filled with useful information.

Each keystroke routine is furnished with a full explanation, including a description of the problem, any pertinent equations, instructions for keying in the problem (Standard Key Sequence Format), and an example or two, with solutions. We suggest that you first read the introductory material explaining the Standard Key Sequence Format. Then locate the routine you want with the Table of Contents or Index and go to work!

Because of the easy "cookbook" style of the solutions presented here, no knowledge of the operation of the calculator is required. All you need are an HP-21 Scientific Pocket Calculator and this *Applications Book* to begin getting answers to common problems.

Eventually, of course, Hewlett-Packard hopes you'll become an expert calculator user by working through the excellent *HP-21 Owner's Handbook* that was packaged with your HP-21. But even experts can profit from the keystroke solutions here.

With the *HP-21 Applications Book,* Hewlett-Packard provides you with answers to some of your more common calculating problems–immediately. But also we hope you'll use this book as a springboard. You can alter or combine these keystroke routines, or develop your own routines, for your more specific applications. In this way you will realize the maximum return from the investment in your HP-21 Scientific Pocket Calculator.

We hope you find the **HP-21 Applications Book** a useful tool, and always welcome your comments, requests, and suggestions–these are our most important source of ideas for future publications like this one.

Table of Contents

Standard Key Sequence Format

Shown below is the key sequence routine for computing the roots of the Quadratic Equation:

$$Ax^2 + Bx + C = 0$$

Using A, B, and C, which you supply as data, the routine produces an intermediate result D. If $D < 0$, one root is the complex conjugate of the other; the real and imaginary parts of one complex root develop on lines 9 and 10. Except for the opposite sign of the imaginary part, the other complex root is identical. If $D \geqslant 0$, the roots are real and develop on lines 6 and 8 (if $-B/2A \geqslant 0$) or on lines 7 and 8 (if $-B/2A < 0$).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	B	↑	2	÷	CHS		$-B/2$	
2	A	STO	÷				$-B/2A$	
3		↑	↑	×			$(B/2A)^2$	
4	C	RCL	÷	STO	−		D	If display is negative go to 9
5			$\sqrt{x}$	x⇄y			$(-B/2A)$	If display is negative go to 7
6		+					x_1	Go to 8
7		x⇄y	−				x_1	
8		RCL	x⇄y	÷			x_2	
9		CHS		$\sqrt{x}$			i v	Imaginary part
10		x⇄y					u	Real part

To execute the sequence, start with line 1 and read from left to right, making the appropriate keystrokes as you proceed. Interpret the respective columns as follows:

Data: Information to be supplied by you, the user. In the sample case, lines 1, 2 and 4 prompt the reader to enter coefficients A, B and C. To enter negative data, it is merely necessary to press CHS after pressing the data value.

Operations: The keys to be pressed after you enter any requested data item for the line. [↑] is the symbol used to denote the [ENTER↑] key of the HP-21. All other key designations are identical to the HP-21 keys. Ignore any blank positions in the operations column. The blue prefix key is represented as a solid key with no lettering (e.g., the first stroke of line 5). The next key to be pressed is denoted by the corresponding functional name (e.g., $\sqrt{x}$, second stroke, line 5) which, on the key is printed in blue.

Display: Intermediate or final results which you should, in most cases, jot down. In the sample case, D is developed so that the reader can decide which line (5 or 9) to execute next.

Remarks: Conditional and unconditional jumps to specified lines or other information for the reader. In the sample case, the reader is prompted to continue with line 9 (ignoring lines 5 through 8) if D is negative. If the condition fails, execution continues on the next line. In the sample case, the reader proceeds to line 5, if D is zero or positive.

Thus, lines are read in sequential order except where the remarks column directs otherwise (as in line 4 of the sample case). To assist the reader in distinguishing lines to be repeated, a sequence of lines making up an iterative process is outlined with a bold border. The following sequence for computing chi-square statistic for goodness of fit illustrates this convention.

Formula:

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

where O_i = observed frequency

E_i = expected frequency

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	↑					
2	O_i	↑						Perform lines 2-4 for i = 1, 2, ..., n
3	E_i	STO	−	↑	×	RCL		
4		÷	+					
5							χ^2	

In a few cases, an iterative process is embedded in a series of lines which are themselves iterated.

Chapter 1. Statistics

Mean, Standard Deviation, and Sums (Ungrouped data)

This procedure will calculate the mean, standard deviation, sum, and sum of squares for one variable.

Formulas:

Mean:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Standard deviation:

$$s = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1}}$$

Sum of values:

$$Sx = \sum_{i=1}^{n} x_i$$

Sum of squared values:

$$SS = \sum_{i=1}^{n} x_i^2$$

Example:

Find $\bar{x}$, s, Sx, and SS for the values $\{2, 3.4, 3.41, 7, 11, 23\}$. (n = 6)

Answer:

SS = 726.19

Sx = 49.81

$\bar{x}$ = 8.30

s = 7.91

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO					
2	x_i		M+	↑	×	+		Perform line 2 for i = 1, 2, ..., n
3							SS	
4		RCL					Sx	Stop if $\bar{x}$ and s not needed
5		↑	↑					
6	n	STO	÷				$\bar{x}$	
7		R↓	↑	×	RCL	÷		
8		−	RCL	1	−	÷		
9			$\sqrt{x}$				s	

Mean, Standard Deviation and Sums (grouped data)

This procedure will compute mean, standard deviation, sum, and sum of squares for one variable grouped data.

Formulas:

Given a set of data points

$$x_1, x_2, \ldots, x_n$$

with respective frequencies

$$f_1, f_2, \ldots, f_n$$

Let $k = \sum_{i=1}^{n} f_i$.

Then

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

$$s = \sqrt{\frac{\sum_{i=1}^{n} f_i x_i^2 - \frac{\left(\sum_{i=1}^{n} f_i x_i\right)^2}{k}}{k-1}}$$

$$Sx = \sum_{i=1}^{n} f_i x_i$$

$$SS = \sum_{i=1}^{n} f_i x_i^2$$

Example:

Compute mean, standard deviation, and sums for

f_i	3	3	1	2	1
x_i	1	2	3	4	5

(k =10)

Answer:

SS = 81.00

Sx = 25.00

$\overline{x}$ = 2.50

s = 1.43

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO					
2	x_i	↑	↑					Perform lines 2-3 for i = 1, ..., n
3	f_i	x		M+	x	+		
4							SS	
5		RCL					Sx	Stop if $\bar{x}$ and s not needed
6		↑	↑					
7	k	STO	÷				$\bar{x}$	
8		R↓	↑	x	RCL	÷		
9		–	RCL	1	–	÷		
10			$\sqrt{x}$				s	

Linear Regression and Correlation Coefficient

The following procedure will perform a least-squares fit of the line $y = ax + b$ to a set of data points $\{(x_i, y_i), i = 1, ..., n\}$. It will compute, in addition to the constants a (slope) and b (y-intercept), the coefficient of determination and the correlation coefficient.

Formulas:

$$y = ax + b$$

$$a = \frac{\Sigma x_i y_i - \dfrac{\Sigma x_i \, \Sigma y_i}{n}}{\Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n}} \qquad \text{(slope)}$$

$$b = \bar{y} - a\bar{x} \qquad \text{(y-intercept)}$$

where

$$\bar{x} = \frac{\Sigma x_i}{n}$$

$$\bar{y} = \frac{\Sigma y_i}{n}$$

Coefficient of determination (measure of goodness of fit—the closer to 1 the better the fit).

$$r^2 = \frac{\left[\Sigma x_i y_i - \dfrac{\Sigma x_i \Sigma y_i}{n}\right]^2}{\left[\Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n}\right]\left[\Sigma y_i^2 - \dfrac{(\Sigma y_i)^2}{n}\right]}$$

Correlation coefficient

$$r = \sqrt{r^2} \text{ (using the sign of a)}$$

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Answers:

SSx = 19956.00000	Sxy = 22200.00
Sx = 354.0000000	y = –1.03x + 121.04 = ax + b
SSy = 35451.00000	r^2 = 0.92
Sy = 481.0000000	r = –0.96

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO	DSP	.	9		Record sums to full 10-digit
								accuracy
2	x_i		M+	↑	x	+		Perform line 2 for i = 1, ..., n
3							SSx	
4		RCL					Sx	
5		CLX	STO					
6	y_i		M+	↑	x	+		Perform line 6 for i = 1, ..., n
7							SSy	
8		RCL					Sy	
9		CLX	↑					
10	x_i	↑						Perform lines 10-11 for i = 1, ..., n
11	y_i	x	+					
12		DSP	.	2			Sxy	
13		RCL						
14	Sx	STO	x					
15	n	÷	–					
16	SSx	RCL	↑	x				
17	n	÷	–	÷	STO		a	
18	Sx	x						
19	Sy	x⇄y	–					
20	n	÷					b	Stop if r^2 is not needed
21		R↓	RCL	x				
22	SSy	↑						
23	Sy	↑	x					
24	n	÷	–	÷			r^2	
25			$\sqrt{x}$				r	If a ⩾ 0, stop
26		CHS					r	If a < 0, r < 0

Exponential Curve Fit

This procedure will fit an exponential curve of the form $y = ae^{bx}$ $(a > 0)$ to a given set of data points $\{(x_i, y_i), i = 1, ..., n\}$.

By writing the equation as $\ln y = bx + \ln a$, the problem can be solved as a linear regression problem on the transformed variables $(x_i, \ln y_i)$. The procedure computes a and b as well as the coefficient of determination, r^2.

Note: y_i must be positive, $i = 1, ..., n$.

Formulas:

Coefficients a, b

$$b = \frac{\Sigma x_i \ln y_i - \frac{1}{n}(\Sigma x_i)(\Sigma \ln y_i)}{\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2} \qquad \text{(continuous growth rate)}$$

$$a = \exp\left[\frac{\Sigma \ln y_i}{n} - b\,\frac{\Sigma x_i}{n}\right] \qquad \text{(y-intercept)}$$

Coefficient of determination

$$r^2 = \frac{\left[\Sigma x_i \ln y_i - \frac{1}{n}\Sigma x_i \Sigma \ln y_i\right]^2}{\left[\Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}\right]\left[\Sigma(\ln y_i)^2 - \frac{(\Sigma \ln y_i)^2}{n}\right]}$$

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Answers:

SSx = 19956.00000

Sx = 354.0000000

SSy = 123.4548955

Sy = 29.32528695

Sxy = 1449.92

$y = 149.07\ e^{-0.02x} = a\ e^{bx}$

$r^2 = 0.89$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO	DSP	.	9		Record sums to full 10-digit accuracy
2	x_i		M+	↑	x	+		Perform line 2 for i = 1, ..., n
3							SSx	
4		RCL					Sx	
5		CLX	STO					
6	y_i		LN		M+	↑		Perform lines 6-7 for i = 1, ..., n
7		x	+					
8							SSy	
9		RCL					Sy	
10		CLX	↑					
11	x_i	↑						Perform lines 11-12 for i = 1, ..., n
12	y_i		LN	x	+			
13		DSP	.	2			Sxy	
14		RCL						
15	Sx	STO	x					
16	n	÷	–					
17	SSx	RCL	↑	x				
18	n	÷	–	÷	STO		b	
19	Sx	x						
20	Sy	x⇄y	–					
21	n	÷	e^x				a	Stop if r^2 is not needed
22		R↓	RCL	x				
23	SSy	↑						
24	Sy	↑	x					
25	n	÷	–	÷			r^2	

Power Curve Fit

This procedure will fit a power curve of the form $y = ax^b$ $(a > 0)$ to a given set of data points $\{(x_i, y_i), i = 1, ..., n\}$.

By writing the equation as $\ln y = \ln a + b \ln x$, the problem can be solved as a linear regression problem on the transformed variables $(\ln x_i, \ln y_i)$. The procedure computes the constants a and b as well as the coefficient of determination, r^2.

Note: x_i and y_i must both be positive, $i = 1, ..., n$.

Formulas:

Regression coefficients

$$b = \frac{\Sigma (\ln x_i)(\ln y_i) - \dfrac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}}{\Sigma (\ln x_i)^2 - \dfrac{(\Sigma \ln x_i)^2}{n}}$$

$$a = \exp\left[\frac{\Sigma \ln y_i}{n} - b\,\frac{\Sigma \ln x_i}{n}\right]$$

Coefficient of determination

$$r^2 = \frac{\left[\Sigma (\ln x_i)(\ln y_i) - \dfrac{(\Sigma \ln x_i)(\Sigma \ln y_i)}{n}\right]^2}{\left[\Sigma (\ln x_i)^2 - \dfrac{(\Sigma \ln x_i)^2}{n}\right]\left[\Sigma (\ln y_i)^2 - \dfrac{(\Sigma \ln y_i)^2}{n}\right]}$$

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Answers:

SSx = 105.1698117

Sx = 27.00621375

SSy = 123.4548955

Sy = 29.32528695

Sxy = 112.45

$y = 987.66\ x^{-0.70} = a\ x^b$

$r^2 = 0.80$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO	DSP	.	9		Record sums to full 10-digit accuracy
2	x_i		LN		M+	↑		Perform lines 2-3 for i = 1, ..., n
3		x	+					
4							SSx	
5		RCL					Sx	
6		CLX	STO					
7	y_i		LN		M+	↑		Perform lines 7-8 for i = 1, ..., n
8		x	+					
9							SSy	
10		RCL					Sy	
11		CLX	↑					
12	x_i		LN					Perform lines 12-13 for i = 1, ..., n
13	y_i		LN	x	+			
14		DSP	.	2			Sxy	
15		RCL						
16	Sx	STO	x					
17	n	÷	–					
18	SSx	RCL	↑	x				
19	n	÷	–	÷	STO		b	
20	Sx	x						
21	Sy	x⇄y	–					
22	n	÷	e^x				a	Stop if r^2 is not needed
23		R↓	RCL	x				
24	SSy	↑						
25	Sy	↑	x					
26	n	÷	–	÷			r^2	

Logarithmic Curve Fit

This procedure will fit a logarithmic curve of the form $y = a + b \ln x$ to a set of data points $\{(x_i, y_i), i = 1, ..., n\}$. This problem is solved as a linear regression problem on the transformed variables $(\ln x_i, y_i)$. The procedure computes the coefficient of determination, r^2, as well as the regression constants a and b.

Note: x_i must be positive, $i = 1, ..., n$.

Formulas:

Regression coefficients

$$b = \frac{\Sigma\, y_i \ln x_i - \frac{1}{n} \Sigma \ln x_i\, \Sigma\, y_i}{\Sigma\, (\ln x_i)^2 - \frac{1}{n} (\Sigma \ln x_i)^2}$$

$$a = \frac{1}{n} (\Sigma\, y_i - b\, \Sigma \ln x_i)$$

Coefficient of determination

$$r^2 = \frac{\left[\Sigma\, y_i \ln x_i - \frac{1}{n} \Sigma \ln x_i\, \Sigma\, y_i\right]^2}{\left[\Sigma\, (\ln x_i)^2 - \frac{1}{n} (\Sigma \ln x_i)^2\right] \left[\Sigma y_i^2 - \frac{1}{n} (\Sigma\, y_i)^2\right]}$$

Example:

x_i	26	30	44	50	62	68	74
y_i	92	85	78	81	54	51	40

Answers:

SSx = 105.1698117

Sx = 27.00621375

SSy = 35451.00000

Sy = 481.0000000

Sxy = 1811.11

y = 244.48 + (–45.56) ln x = a + b ln x

r^2 = 0.85

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO	DSP	.	9		Record sums to full 10-digit accuracy
2	x_i		LN		M+	↑		Perform lines 2-3 for i = 1, ..., n
3		x	+					
4							SSx	
5		RCL					Sx	
6		CLX	STO					
7	y_i		M+	↑	x	+		Perform line 7 for i = 1, ..., n
8							SSy	
9		RCL					Sy	
10		CLX	↑					
11	x_i		LN					Perform lines 11-12 for i = 1, ..., n
12	y_i	x	+					
13		DSP	.	2			Sxy	
14		RCL						
15	Sx	STO	x					
16	n	÷	–					
17	SSx	RCL	↑	x				
18	n	÷	–	÷	STO		b	
19	Sx	x						
20	Sy	x⇄y	–					
21	n	÷					a	Stop if r^2 is not needed
22		R↓	RCL	x				
23	SSy	↑						
24	Sy	↑	x					
25	n	÷	–	÷			r^2	

Random Number Generator

Random numbers are used in a variety of disciplines. They provide the element of chance in computer games involving, for instance, card dealing or dice rolling. They can provide typical, but varying, values of parameters in simulation or modelling programs. For the statistician, they allow irregular and unbiased sampling from a population.

This procedure calculates uniformly distributed pseudo random numbers u_i in the range $0 < u_i < 1$ using the multiplicative linear congruential method:

$$u_{i+1} = \text{fractional part of } (997\ u_i)$$

where i = 0, 1, 2,...

$$u_0 = 0.5284163.$$

The period has length 500000 (i.e., 500000 different numbers can be generated before repeating). The least significant digits (the right-hand digits) of u_i are not as random as the most significant digits (the left-hand digits). Thus random digits, if needed, should be taken from the most significant end of the numbers.

If a different sequence of numbers is desired, a different starting value u_0 can be chosen such that $0 < u_0 < 1$. Note that if $10^7 \times u_0$ is not divisible by 2 or 5, then the period of the generator has length 500000.

Example:

Let $u_0 = 0.5284163$. Find the first five random numbers in this sequence $\{u_i, i = 1, ..., 5\}$.

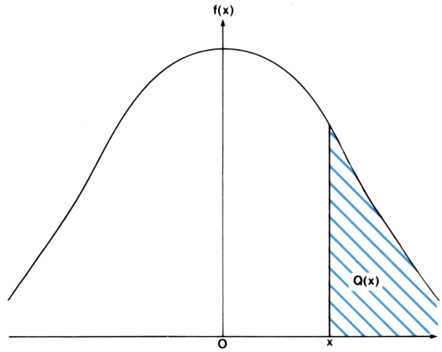
Answers:

0.83, 0.56, 0.27, 0.04, 0.20.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	997	STO						
2	u_0							Perform lines 3-4 for j = 1, 2, ...
3		RCL	x				t_j	Let i_j = integer part of t_j*
4	i_j	–					u_j	Go to line 3
								*E.g., if t_1 = 526.83, then i_1 = 526

Normal Distribution

Given a standard normal variable x, this program computes two functions for a normal distribution: the density function f(x) and the upper tail area Q(x).



Note:

This procedure works only for $x \geqslant 0$. For $x < 0$, f and Q can be found using the relations $f(-x) = f(x)$ and $Q(-x) = 1 - Q(x)$.

Formulas:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} .$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt .$$

For $x \geqslant 0$, polynomial approximation is used to compute Q(x):

$$Q(x) = f(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) + \epsilon(x)$$

where $|\epsilon(x)| < 7.5 \times 10^{-8}$

$$t = \frac{1}{1 + rx}, \quad r = 0.2316419$$

$b_1 = .31938153$, $b_2 = -.356563782$

$b_3 = 1.781477937$, $b_4 = -1.821255978$

$b_5 = 1.330274429$

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions,* National Bureau of Standards, 1968.

Examples:

1. x = 1.18
2. x = 2.28

Answers:

1. f(x) = 0.20

   Q(x) = 0.12
2. f(x) = 0.03

   Q(x) = 0.01

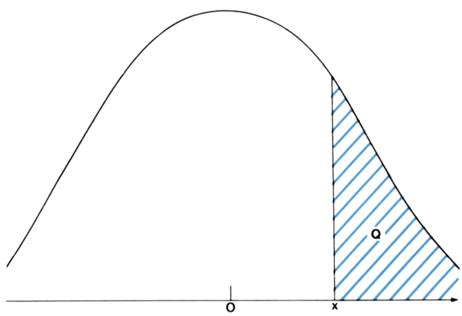
LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	STO	↑	×	2	÷		
2		CHS	e^x		π	2		
3		×		$\sqrt{x}$	÷		f(x)	Stop if Q(x) is not needed
4		RCL	x⇄y	STO	R↓			
5	r	×	1	+	1/x	↑		
6		↑	↑				t	
7	b_5	×						
8	b_4	+	×					
9	b_3	+	×					
10	b_2	+	×					
11	b_1	+	×	RCL	×		Q(x)	

Inverse Normal Integral

This procedure determines the value of x such that

$$Q = \int_x^\infty \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}\,dt$$

where Q is given and $0 < Q \leqslant 0.5$.



Formulas:

The following rational approximation is used:

$$x = t - \frac{c_0 + c_1\,t + c_2\,t^2}{1 + d_1\,t + d_2\,t^2 + d_3\,t^3} + \epsilon(Q)$$

where $|\epsilon(Q)| < 4.5 \times 10^{-4}$

$$t = \sqrt{\ln \frac{1}{Q^2}}$$

$c_0 = 2.515517 \qquad d_1 = 1.432788$

$c_1 = 0.802853 \qquad d_2 = 0.189269$

$c_2 = 0.010328 \qquad d_3 = 0.001308$

Reference:

Abramowitz and Stegun, *Handbook of Mathematical Functions,* National Bureau of Standards, 1968.

Examples:

1. Q = 0.12
2. Q = 0.05

Answers:

1. x = 1.18
2. x = 1.65

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	Q	↑	×	$^1/_x$		LN		
2			$\sqrt{x}$	↑	↑	↑	t	
3	d_3	×						
4	d_2	+	×					
5	d_1	+	×	1	+	STO		
6		CLX						
7	c_2	×						
8	c_1	+	×					
9	c_0	+	RCL	÷	−		x	

Chi-Square Evaluation

This procedure calculates the value of the χ^2 statistic for the goodness of fit test. This statistic measures the closeness of the agreement between observed and expected frequencies.

Formulas:

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

where O_i = observed frequency

E_i = expected frequency.

Notes:

1. In order to apply this test to a set of given data, it may be necessary to combine some classes to make sure that each expected frequency is not too small (say, not less than 5).

2. If the expected frequencies E_i are all equal to some value E, then E should be computed beforehand as

$$E = \frac{\Sigma O_i}{n}$$

and then input at each step as the expected frequency E_i.

Example:

O_i	8	50	47	56	5	14
E_i	9.6	46.75	51.85	54.4	8.25	9.15

Answer:

$\chi^2 = 4.84$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	↑					
2	O_i	↑						Perform lines 2-4 for i = 1, ..., n
3	E_i	STO	−	↑	×	RCL		
4		÷	+					
5							χ^2	

Paired t Statistic

Given a set of paired observations $\{(x_i, y_i), i = 1, ..., n\}$ drawn from two normal populations with unknown means μ_1, μ_2, this procedure will compute the paired t-statistic. This test statistic, which has n−1 degrees of freedom, can be used to test the null hypothesis Ho: $\mu_1 = \mu_2$.

Formulas:

$$D_i = x_i - y_i$$

$$\overline{D} = \frac{1}{n} \sum_{i=1}^{n} D_i$$

$$s_D = \sqrt{\frac{\Sigma D_i^2 - \frac{1}{n}(\Sigma D_i)^2}{n - 1}}$$

$$s_{\overline{D}} = \frac{s_D}{\sqrt{n}}$$

$$t = \frac{\overline{D}}{s_{\overline{D}}},$$

Example:

x_i	14	17.5	17	17.5	15.4
y_i	17	20.7	21.6	20.9	17.2

Answers:

$\bar{D} = -3.20$

$S_{\bar{D}} = 0.45$

$t = -7.16$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO					
2	x_i	↑						Perform lines 2-4 for i = 1, ..., n
3	y_i	–		M+	↑	x		
4		+						
5		RCL	↑	↑				
6	n	÷	STO				$\bar{D}$	
7		R↓	↑	x				
8	n	↑	R↓	÷	–	R↓		
9		R↓	1	–	x	÷		
10			$\sqrt{x}$				$s_{\bar{D}}$	
11		RCL	x⇄y	÷			t	

Chapter 2. Mathematics

Quadratic Equation

This program performs a general solution for the quadratic equation. Either real or imaginary roots may be found.

Formulas:

A general quadratic equation is of the form:

$$Ax^2 + Bx + C = 0.$$

The equation has two roots, x_1 and x_2.

Let
$$D = \left(\frac{B}{2A}\right)^2 - \frac{C}{A}$$

If $D \geqslant 0$, then

$$x_1 = \begin{cases} -\dfrac{B}{2A} + \sqrt{D} & \text{if } \dfrac{-B}{2A} \geqslant 0 \\ -\dfrac{B}{2A} - \sqrt{D} & \text{if } -\dfrac{B}{2A} < 0 \end{cases}$$

and $x_2 = \dfrac{C}{AX_1}$

If $D < 0$, then $x_1, x_2 = -\dfrac{B}{2A} \pm i\sqrt{-D}$

$$= u \pm iv$$

The coefficient A cannot be zero.

Examples:

Find the solutions to the following equations:

1. $x^2 - 3x - 4 = 0$
2. $2x^2 + 3x + 4 = 0$

Answers:

1. $D = 6.25$ $x_1 = 4, x_2 = -1$

2. $D = -1.44$ $x_1, x_2 = -.75 \pm 1.20i$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	B	↑	2	÷	CHS		−B/2	
2	A	STO	÷				−B/2A	
3		↑	↑	x			$(B/2A)^2$	
4	C	RCL	÷	STO	−		D	If display is negative go to 9
5			$\sqrt{x}$	x⇄y			(−B/2A)	If display is negative go to 7
6		+					x_1	Go to 8
7		x⇄y	−				x_1	
8		RCL	x⇄y	÷			x_2	
9		CHS		$\sqrt{x}$			i v	Imaginary part
10		x⇄y					u	Real part

Polynomial Evaluation

In many instances, it is necessary to evaluate a polynomial of the form

$$f(x) = c_0x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n$$

given values for x and the constants c_0 through c_n. By restructuring the problem in the form

$$f(x_0) = (\ldots(((c_0x_0 + c_1)x_0 + c_2)x_0 + c_3)x_0 + \ldots)x_0 + c_n$$

maximum benefit can be gained from the HP-21's operational stack.

Example:

If $f(x) = x^5 + 5x^4 - 3x^2 - 7x + 11$, find f(2.5). (Note that c_2=0)

Answer:

Restructure as $f(x) = ((((x + 5)x + 0)x - 3)x - 7)x + 11$

$f(2.5) = 267.72$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x_0	↑	↑	↑				
2	c_0							
3		×						Perform 3-4 for i = 1, 2, ..., n
4	c_i	+						

Complex Arithmetic

These procedures evaluate the basic complex number operations.

Complex Addition

Formula:

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) = u + iv$$

Example:

$(3 + 4i) + (7.4 - 5.6i) = 10.40 - 1.60i$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a_1	↑						
2	a_2	+					u	
3	b_1	↑						
4	b_2	+					v	

Complex Subtraction

Formula:

$$(a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2) = u + iv$$

Example:

$(3 + 4i) - (7.4 - 5.6i) = -4.40 + 9.60i$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a_1	↑						
2	a_2	−					u	
3	b_1	↑						
4	b_2	−					v	

Complex Multiplication

Formula:

$$\prod_{k=1}^{2}(a_k + ib_k) = \prod_{k=1}^{2} r_k \; e^{\,i\sum_{k=1}^{2}\theta_k} = u + iv$$

$$\text{where } a_k + ib_k = r_k\, e^{i\theta_k}$$

Example:

$(3.1 + 4.6i) \times (5 - 12i) = 70.70 - 14.20i$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b_1	↑						
2	a_1	■	→P	x⇄y				
3	b_2	↑						
4	a_2	■	→P	R↓	+	R↓		
5		x	x⇄y	R↓	■	→R	u	
6		x⇄y					v	

Complex Division

Formula:

$$\frac{(a_1 + ib_1)}{(a_2 + ib_2)} = \frac{r_1}{r_2}\, e^{i(\theta_1 - \theta_2)} = u + iv$$

where
$$a_1 + ib_1 = r_1\, e^{i\theta_1}$$
$$a_2 + ib_2 = r_2\, e^{i\theta_2} \neq 0$$

Example:

$$\frac{(3 + 4i)}{7 - 2i} = .25 + .64i$$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b_2	↑						
2	a_2	■	→P					
3	b_1	↑						
4	a_1	■	→P	x⇄y	R↓	x⇄y		
5		÷	R↓	–	R↓	R↓		
6		R↓	■	→R			u	
7		x⇄y					v	

Complex Functions

Complex Reciprocal

Formula:

$$\frac{1}{z} = \frac{1}{a + ib} = \frac{1}{r}\, e^{-i\theta}, z \neq 0$$

$$= u + iv$$

Example:

$$\frac{1}{2 + 3i} = .15 - .23i$$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	CHS	↑					
2	a		→P	1/x		→R	u	
3		x⇄y					v	

Complex Square

Formula:

$$(a + ib)^2 = r^2\, e^{i2\theta}$$

Example:

$$(7 - 2i)^2 = 45.00 - 28.00i$$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	↑						
2	a		→P	x⇄y	2	x		
3		x⇄y	↑	x		→R	u	
4		x⇄y					v	

Complex Square Root

Formula:

$$\sqrt{a + ib} = \pm\left(\sqrt{r}\, e^{i\theta/2}\right) = \pm\,(u + iv)$$

Example:

$$\sqrt{7 + 6i} = \pm\,(2.85 + 1.05i)$$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	↑						
2	a		→P		√x	x⇄y		
3		2	÷	x⇄y		→R	u	
4		x⇄y					v	

Vector Operations

Vector Addition

Suppose vector V_k (in two-dimensional space) has magnitude m_k and direction θ_k. Find the sum of two vectors:

$$V = \sum_{k=1}^{2} V_k = x\vec{i} + y\vec{j}$$

Example:

m_k	θ_k
2	30°
6.2	–45°

Answer:

$6.12\vec{i} - 3.38\vec{j}$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	θ_1	↑						
2	m_1		→R					
3	θ_2	↑						
4	m_2		→R	x⇄y	R↓	+	x	
5		R↓	+				y	

Vector Angles

Suppose $\vec{x} = (x_1, x_2, x_3)$

$\vec{y} = (y_1, y_2, y_3)$

then the angle between these two vectors is

$$\theta = \cos^{-1} \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2} \sqrt{y_1^2 + y_2^2 + y_3^2}}$$

Example:

Find the angle between

$\vec{x} = (5, -6.2, -7)$

$\vec{y} = (3.15, 2.22, -0.3)$

Answer:

θ = 84.28 degrees = 1.47 radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		CLX	STO					
2	x_i	↑	↑	×		M+		
3		CLX						
4	y_i	×	+					Perform 2-4 for i = 1, 2, ...
5	0	↑						
6	y_i	↑	×	+				Perform 6 for i = 1, 2, ...
7			$\sqrt{x}$	RCL		$\sqrt{x}$		
8		×	÷		$\cos^{-1}$		θ	

Vector Cross Product

Formula:

If $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ are two vectors, then the cross product $\vec{z}$ is also a vector.

$$\begin{aligned}\vec{z} &= \vec{x} \times \vec{y} \\ &= (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \\ &= (z_1, z_2, z_3)\end{aligned}$$

Example:

If $\vec{x} = (2.34, 5.17, 7.43)$

$\vec{y} = (.072, .231, .409)$

Find $\vec{x} \times \vec{y}$

Answer:

$\vec{x} \times \vec{y} = (.40, -.42, .17)$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x_2	↑						
2	y_3	×						
3	x_3	↑						
4	y_2	×	−				z_1	
5	x_3	↑						
6	y_1	×						
7	x_1	↑						
8	y_3	×	−				z_2	
9	x_1	↑						
10	y_2	×						
11	x_2	↑						
12	y_1	×	−				z_3	

Vector Dot Product

Formulas:

Given two vectors $\vec{x}$, $\vec{y}$ in an n-dimensional vector space

$$\vec{x} = (x_1, x_2, ..., x_n)$$

$$\vec{y} = (y_1, y_2, ..., y_n)$$

the dot product is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + ... + x_n y_n$$

Example:

If $\vec{x} = (2.34, 5.17, 7.43, 9.11, 11.41)$

$\vec{y} = (.072, .231, .409, .703, .891)$

then $\vec{x} \cdot \vec{y} = 20.97$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x_1	↑						
2	y_1	×						
3	x_i	↑						Perform 3-4 for i = 2, 3, ..., n
4	y_i	×	+					

Triangle Solutions

Triangles are commonly encountered in engineering, surveying, aviation and navigation. The following keystrokes can be helpful in solving for the unknowns. When working with angles in degrees, minutes and seconds, the Angle Conversions in Chapter 6 may be helpful.

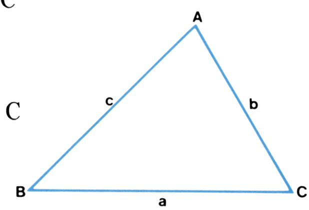
The basic formulas used to solve a triangle are:

1. law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

2. law of cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$



Note: Triangle solution routines work in any angular mode. When the calculator is in DEG mode, all angles are in decimal degrees.

Given a, b, C; find A, B, c

Given two sides and their included angle, this procedure solves the triangle for the remaining parameters.

Formulas:

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

$$A = \tan^{-1}\left(\frac{a \sin C}{b - a \cos C}\right)$$

$$B = \cos^{-1} [-\cos (A + C)]$$

Example:

Given $b = 224$
$C = 28.67°$
$a = 132$

Find c, A, B

Answer:

c = 125.36
A = 30.34°
B = 120.99°

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	STO						
2	C	↑						
3	a		→R	RCL	x⇄y	–		
4			→P				c	
5		x⇄y					A	
6	C	+	COS	CHS		COS^{-1}	B	

Given a, b, c; find A, B, C

Given three sides, this procedure solves the triangle for the remaining parameters.

Formulas:

$$A = 2 \cos^{-1}\left(\sqrt{\frac{S(S-a)}{bc}}\right)$$

$$\text{where } S = (a + b + c)/2$$

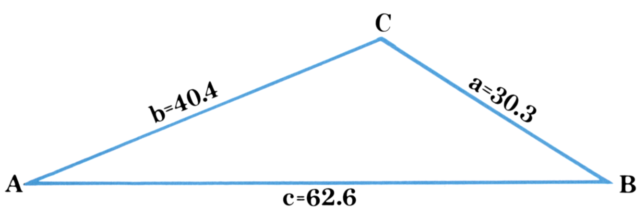
$$B = \tan^{-1}\left(\frac{b \sin A}{c - b \cos A}\right)$$

$$C = \cos^{-1}\ [-\cos (A + B)]$$

Example:

Given $a = 30.3$
$b = 40.4$
$c = 62.6$

Find A, B, C.



Answer:

A = 23.66° = 0.41 radians
B = 32.35° = 0.56 radians
C = 123.99° = 2.16 radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	STO						
2	b	↑						
3	c	+	+	2	÷		S	
4		↑	↑	RCL	−	×		
5	b	÷						
6	c	÷	■	$\sqrt{x}$	■	$\cos^{-1}$		
7		2	×	STO			A	
8	b	■	→R					
9	c	x⇄y	−	■	→P	x⇄y	B	
10		RCL	+	COS	CHS	■		
11		$\cos^{-1}$					C	

Given a, A, C; find B, b, c

Given two angles and a non-included side, this procedure solves the triangle for the remaining parameters.

Formulas:

$$b = \frac{a \sin (A + C)}{\sin A}$$

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

$$B = \tan^{-1}\left(\frac{b \sin C}{a - b \cos C}\right)$$

Example: (For this example, change to RAD mode).

Given $a = 17.5$
$C = 1.09$ radians
$A = 0.72$ radians

Find B, b, c.

Answer:

$b = 25.78$
$c = 23.53$
$B = 1.33$ radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	STO						
2	C	↑						
3	A	+	SIN					
4	A	SIN	÷	RCL	x		b	
5	C	x⇄y	■	→R	RCL	x⇄y		
6		–	■	→P			c	
7		x⇄y					B	

Given a, B, C; find A, b, c

Given two angles and their included side, this procedure solves the triangle for the remaining parameters.

Formulas:

$$c = \frac{a \sin C}{\sin (B + C)}$$

$$b = \sqrt{a^2 + c^2 - 2\,ac \cos B}$$

$$A = \cos^{-1} [-\cos (B + C)]$$

Example:

Given $a = 25.2$
$B = 35.33°$
$C = 68.50°$

Find A, b, c.

Answer:

$c = 24.15$
$b = 15.01$
$A = 76.17°$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑						
2	B	STO						
3	C	SIN	RCL					
4	C	+	SIN	÷				
5	a	x					c	
6		RCL	x⇄y	■	→R			
7	a	x⇄y	–	■	→P		b	
8		x⇄y	RCL	+	COS	CHS		
9		■	COS^{-1}				A	

Given B, b, c; find a, A, C

Given two sides and a non-included angle, this procedure solves the triangle for the remaining parameters.

Formulas:

$$a = \frac{c \sin (B + M_1)}{\sin M_1}$$

where

$$M_1 = \begin{cases} \sin^{-1}\left(\dfrac{c \sin B}{b}\right) & \text{or} \\ \sin^{-1}\left(-\dfrac{c \sin B}{b}\right) \end{cases}$$

$$A = \tan^{-1}\left(\frac{a \sin B}{c - a \sin B}\right)$$

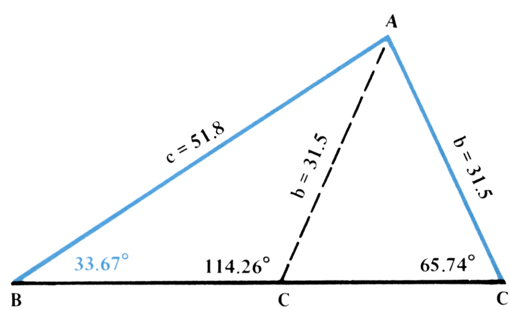
$$C = \cos^{-1}\,[-\cos (A + B)]$$

Note: If B is acute and $b < c$, two solutions exist.

Example:

Given $b = 31.5$
$c = 51.8$
$B = 33.67°$

Find a, A, C.



Answer:

a = 56.05
A = 80.59°
C = 65.74°

Alternate answer:

a = 30.17
A = 32.07°
C = 114.26°

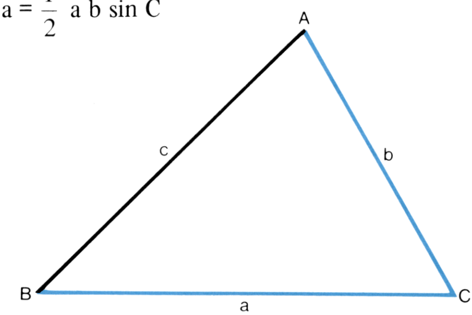
LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	STO						
2	c	↑						
3	B	SIN	x	RCL	÷			
4		SIN^{-1}	STO				M_1	
5		SIN						
6	B	RCL	+	SIN	x⇄y	÷		
7	c	x					a	
8	B	x⇄y		→R				
9	c	x⇄y	–		→P	x⇄y	A	
10	B	+	COS	CHS		COS^{-1}	C	If b ≥ c, stop
11		RCL	CHS	STO				Go to 5 for alternate solution

Given a, b, C; find area

Given two sides and a non-included angle, this procedure solves the triangle for the area.

Formula:

$$\text{area} = \frac{1}{2}\, a\, b \sin C$$



Example:

If $a = 5.3174$,

$b = 7.0898$

$C = \frac{\pi}{4}$

Answer:

area = 13.33

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑						
2	b	↑						
3	C	SIN	×	×	2	÷	area	

Given a, b, c; find area

Given three sides, this procedure solves for the area of the triangle.

Formula:

$$\text{area} = \sqrt{S(S-a)(S-b)(S-c)}$$

where

$$S = \frac{1}{2}(a + b + c).$$

Example:

a = 5.31
b = 7.08
c = 8.86

Answer:

area = 18.80

(S = 10.63).

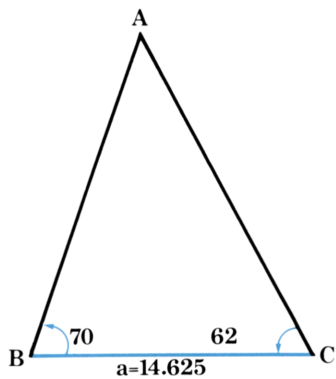
LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	STO						
2	b	↑						
3	c	↑	CHS	R↓	+	+		
		2	÷	↑	↑	RCL		
		CHS	x⇄y	STO	+	↑		
		R↓	R↓	+	×	RCL		
4	b	−	×	RCL	×			
		$\sqrt{x}$					area	

Given a, B, C; find area

Given two angles and the included side, this procedure solves for the area of the triangle.

Formula:

$$\text{area} = \frac{a^2}{2} \frac{\sin B \sin C}{\sin (B + C)}$$



Example:

If $B = 70°$

$C = 62°$

$a = 14.625$

Answer:

area = 119.40

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑	x	2	÷			
2	B	STO	SIN					
3	C		M+	SIN	x	x		
4		RCL	SIN	÷			area	

Given vertices; find area

Given the coordinates of the vertices, this procedure solves for the area of the triangle.

Formula:

Given the three vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) of a triangle

$$\text{area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \right]$$

Example:

Compute the area of the triangle with vertices (0,0), (4,0), (4,3).

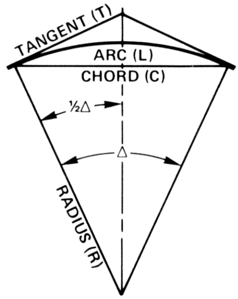
Answer:

6.00

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x_1	↑						
2	y_2	↑						
3	y_3	STO	−	×				
4	x_3	↑						
5	y_1		M−					
6	y_2	−	×	+	RCL			
7	x_2	×	+	2	÷		area	

Curve Solutions

Curve solutions are used for design and layout of small curves on machine parts to large highway curves. It is often necessary to solve for the remaining parameters of the curve given any two of them. When working with angles in degrees, the Angle Conversions in Chapter 6 may be helpful.



Notation:

T = Tangent distance

C = Chord length

L = Arc length

R = Radius

Δ = Central angle

Given Δ and R, find the remaining parameters plus the sector and segment area.

Formulas:

$T = R \tan(\Delta/2)$

$C = 2 R \sin(\Delta/2)$

$L = \Delta \pi R/180$

Sector area = LR/2

Segment area = Sector area – $1/2\ R^2 \sin(\Delta)$

Example:

R = 223.181

Δ = 45.5064°

Answers:

1/2 Δ = 22.7532°

T = 93.602

C = 172.636

L = 177.258

Sector area (∇) = 19,780

Segment area (⌓) = 2,015

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	R	↑	↑	↑				
2	Δ	STO	2	÷	TAN	x	T	Δ in decimal degrees
3		CLX	RCL	2	÷	SIN		
		x	2	x			C	
4		CLX	RCL	x		π		
		x	1	8	0	÷	L	
5		x	2	÷			Sector Area	
6		x⇄y	↑	x	RCL	SIN		
7		x	2	÷	–		Segment Area	

Given R and C, find the remaining parameters plus the sector and segment area.

Formulas:

$R = C/(2 \sin(\Delta/2))$

$\Delta = 2 \sin^{-1} (1/2\ C/R)$

$T = R \tan(\Delta/2)$

$L = \Delta\ \pi\ R/180$

Sector area = $LR/2$

Segment area = Sector area − $1/2\ R^2 \sin \Delta$

Example:

$R = 223.181$

$C = 172.636$

Answers:

$\Delta = 45.5064°$ (Convert to degrees, minutes and seconds, see Chapter 6 Conversions)

$\Delta/2 = 22.7531°$

$T = 93.602$

$L = 177.258$

Sector area (∇) = 19,780

Segment area (⌓) = 2,015

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	R	↑	↑	↑				
2	C	x⇄y	÷	2	÷			
		SIN⁻¹	2	x	STO		Δ	Δ in decimal degrees
3		2	÷	TAN	x		T	
4		CLX	RCL	x		π		
		x	1	8	0	÷	L	
5		x	2	÷			Sector Area	
6		x⇄y	↑	x	RCL	SIN		
7		x	2	÷	−		Segment Area	

Given Δ and C, find the remaining parameters plus the sector and segment area.

Formulas:

$$R = C/(2 \sin(\Delta/2))$$

$$T = R \tan(\Delta/2)$$

$$L = \Delta \pi R/180$$

$$\text{Sector area} = LR/2$$

$$\text{Segment area} = \text{Sector area} - 1/2\, R^2 \sin \Delta$$

Example:

$C = 172.636$

$\Delta = 45.5064°$

Answers:

$\Delta/2 = 22.7532°$ (To convert to degrees, minutes, and seconds, see Chapter 6 Conversions)

$R = 223.181$

$T = 93.602$

$L = 177.258$

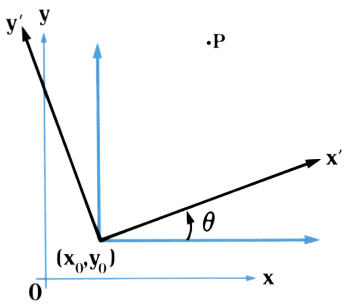
Sector area (∇) = 19,780

Segment area (⌓) = 2,015

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	C	↑						
2	Δ	STO	2	÷	SIN	2		Δ in decimal degrees
		x	÷	↑	↑	↑	R	
3		RCL	2	÷	TAN	x	T	
4		CLX	RCL	x		π		
		x	1	8	0	÷	L	
5		x	2	÷			Sector Area	
6		x⇄y	↑	x	RCL	SIN		
7		x	2	÷	–		Segment Area	

Coordinate Translation and Rotation

Suppose point P has coordinates (x, y) with respect to the rectangular coordinate system (x, y axes). Let (x_0, y_0) be the center of a new coordinate system rotated through an angle θ. Find the new coordinates (x', y') of P with respect to the new system having x', y' axes.



Formulas:

$$x' = (x - x_0)\cos\theta + (y - y_0)\sin\theta$$

$$y' = -(x - x_0)\sin\theta + (y - y_0)\cos\theta$$

Example:

Translate the point (1, 3) to a new coordinate system with center at (−1, 4) at 30° to the old system.

Answer:

$x' = 1.23$

$y' = -1.87$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	y	↑						(x, y) = old coordinates
2	y_0	−						(x_0, y_0) = new origin
3	x	↑						
4	x_0	−		→P	x⇄y			
5	θ	−	x⇄y		→R		x'	
6		x⇄y					y'	

Trigonometric Functions

The HP-21 supplies the basic trigonometric functions of SIN, COS, TAN, SIN^{-1}, COS^{-1}, and TAN^{-1}. Other trigonometric functions may be found through simple manipulations of these basic functions.

Formula:

Let p = principal value

q = secondary value = $\cos^{-1}(-1) - \sin^{-1}(x)$

We set the calculator to DEG or RAD mode, as desired..

Secondary value of arc sin x

Example:

$x = -0.77$, find secondary value of arc sin x.

Answer:

$q = 230.35^\circ = 4.02$ radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x		SIN^{-1}				p	
2		1	CHS		COS^{-1}	x⇄y		
3		−					q	

Secondary value of arc cos x

Formula:

$$q = 2\cos^{-1}(-1) - \cos^{-1}(x)$$

Example:

$x = 0.76$, find secondary value of arc cos x.

Answer:

$q = 319.46^\circ = 5.58$ radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x		COS^{-1}					
2		1	CHS		COS^{-1}	↑	p	
3		+	x⇄y	−			q	

Secondary value of arc tan x

Formula:

$$q = \tan^{-1}(x) + \cos^{-1}(-1)$$

Example:

x = 2, find secondary value of arc tan x.

Answer:

q = 243.43° = 4.25 radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x		TAN^{-1}				p	
2		1	CHS		COS^{-1}	+	q	

Cotangent

Formula:

$$\cot x = \frac{1}{\tan x}$$

Example:

x = 37

Answer:

cot x = 1.33 (in DEG mode) or

−1.19 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	TAN	$1/x$				cot x	

Cosecant

Formula:

$$\csc x = \frac{1}{\sin x}$$

Example:

x = 30

Answer:

csc x = 2.00 (in DEG mode) or
−1.01 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	SIN	1/x				csc x	

Secant

Formula:

$$\sec x = \frac{1}{\cos x}$$

Example:

x = 45

Answer:

sec x = 1.41 (in DEG mode) or
1.90 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	COS	1/x				sec x	

Versine

Formula:

$$\text{vers } x = 1 - \cos x$$

Example:

x = 38

Answer:

vers x = 0.21 (in DEG mode) or
0.04 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	COS	1	x⇄y	–		vers x	

Coversine

Formula:

$$\text{covers } x = 1 - \sin x$$

Example:

x = 38

Answer:

covers x = 0.38 (in DEG mode) or
0.70 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	SIN	1	x⇄y	–		covers x	

Haversine

Formula:

$$\text{hav } x = \frac{1 - \cos x}{2}$$

Example:

x = 42.3

Answer:

hav x = 0.13 (in DEG mode) or
0.56 (in RAD mode).

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	COS	1	x⇄y	−	2		
2		÷					hav x	

Arc cotangent

Formula:

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

Example:

x = 0.35

Answer:

$\cot^{-1} x = 70.71°$ or 1.23 radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	1/x		TAN^{-1}			$\cot^{-1}$ x	

Arc cosecant

Formula:

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

Example:

x = 3.45

Answer:

$\csc^{-1} x = 16.85°$ or 0.29 radians

LINE	DATA	OPERATIONS	DISPLAY	REMARKS
1	x	[1/x] [ ] [SIN⁻¹] [ ] [ ]	$\csc^{-1} x$	

Arc secant

Formula:

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

Example:

x = 1.1547

Answer:

$\sec^{-1} x = 30°$ or 0.52 radians

LINE	DATA	OPERATIONS	DISPLAY	REMARKS
1	x	[1/x] [ ] [COS⁻¹] [ ] [ ]	$\sec^{-1} x$	

Hyperbolic Functions

These procedures evaluate three hyperbolic functions and their inverses.

Hyperbolic Sine

Formula:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Example:

sinh 3.2 = 12.25

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	e^x	↑	$1/x$	−	2		
2		÷					sinh x	

Hyperbolic Cosine

Formula:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Example:

cosh 3.2 = 12.29

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	e^x	↑	$1/x$	+	2		
2		÷					cosh x	

Hyperbolic Tangent

Formula:

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Example:

tanh 3.2 = 1.00

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	e^x	↑	$1/x$	STO	−		
2		RCL	↑	$1/x$	+	÷	tanh x	

Inverse Hyperbolic Sine

Formula:

$$\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1})$$

Example:

$\sinh^{-1} 51.777 = 4.64$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	↑	↑	×	1	+		
2			√x	+		LN	$\sinh^{-1} x$	

Inverse Hyperbolic Cosine

Formula:

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) \qquad (x \geqslant 1)$$

Example:

$\cosh^{-1} 12.29 = 3.20$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	↑	↑	×	1	−		
2			√x	+		LN	$\cosh^{-1} x$	

Inverse Hyperbolic Tangent

Formula:

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}$$

$$(-1 < x < 1)$$

Example:

$\tanh^{-1} 0.777 = 1.04$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		1	↑					
2	x	STO	+	1	RCL	−		
3		÷		LN	2	÷	$\tanh^{-1} x$	

Integer Base Conversions

Note:

Base conversion algorithms are given for positive values only. To convert a negative number, change sign, convert, and change sign of result.

Decimal Integer to Integer in Any Base

$$I_{10} \rightarrow J_b$$

In the following key sequence, f + 1 is the number of digits in J_b. d_i ($i = 1, ..., f + 1$) represents the i^{th} digit in J_b, counting from left to right, i.e.

$$J_b = (d_1\ d_2\ \text{---}\ d_{f+1})_b$$

For large numbers, $J_b = (d_1\ d_2\ \text{---}\ d_{f+1})_b \cdot b^f$, see example 3.

Example 1:

Convert 1206 to hexadecimal (base 16).

(The hexadecimal digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A(10), B(11), C(12), D(13), E(14), F(15).)

Answer:

$1206_{10} = 4B6_{16}$ (f = 2)

Example 2:

Convert 513 to octal (base 8).

Answer:

$513_{10} = 1001_8$

Example 3:

Convert 6.023×10^{23} to octal.

Answer:

$6.023 \times 10^{23} = 1.7743_8 \times 8^{26}$

Note:

If we consider 6.023×10^{23} to be a scientific measurement good only to four significant digits, it is meaningless for the octal representation to contain more than 5 significant digits. Therefore, we stop before the loop is completed.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	8	↑	↑	6	.	0		
2		2	3	EEX	2	3		
3		STO		LN	x⇄y			
4		LN	÷				26.33	f = 26 (Note: this gives the ex-
								ponent in base 8.)
5		CLX						
6	26	x⇄y	↑	↑	RCL	R↓		
7		R↓	x⇄y		y^x	÷	1.99	d_1 = 1
8	1	–	x				7.94	d_2 = 7
9	7	–	x				7.54	d_3 = 7
10	7	–	x				4.34	d_4 = 4
11	4	–	x				2.69	d_5 = 3 (rounded), stop.

In General the Key Sequence is:

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	↑	↑					b = base
2	I	STO		LN	x⇄y			I = Integer number
3		LN	÷				D	Let f be the largest integer ≤ D
4		CLX						
5	f	x⇄y	↑	↑	RCL	R↓		
6		R↑	x⇄y		y^x	÷	E_1	d_i = integer part of E_i (i = 1, 2,..., f
7	d_1	–	x				E_2	
8	d_i	–	x				E_{i+1}	Perform 8 for i = 2, 3, ..., f
9		DSP	•	0			d_{f+1}	

Integer in Base b to Decimal

$$(d_1\ d_2\ \text{---}\ d_{n-1}\ d_n)_b \rightarrow I_{10}$$

Examples:

1. $730020461_8 = 123740465_{10}$
2. $7DOF_{16} = 32015_{10}$

   (A = 10, B = 11, C = 12, D = 13, E = 14, F = 15 in the hexadecimal system.)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	b	↑	↑	↑				
2	d_1	x						
3	d_i	+	x					Perform 3 for i = 2, ..., n − 1.
4	d_n	+						

Iterative Techniques

Note:

We will deal here with equations of the form

$$x = f(x)$$

for cases where it is difficult to separate all x's to one side of the equal sign. The iterative approach is illustrated through the solution of selected equations.

Example 1: Find x such that $x = e^{-x}$

Method:

Choose $x_a = 5$ as an approximation for the solution. Then after 44 iterations, the answer is x = 0.567143290.

Note:

The algorithm will converge to 0.567143290 in about 50 iterations for any value of x_a.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	5	CHS	e^x	DSP	.	9	0.006737947	
2		CHS	e^x					Perform 2 forty-three times
3							0.567143290	

Example 2: $4 = x - \frac{1}{x}$

Method:

Rewrite the equation as

$$x = \frac{1}{x} + 4.$$

Choose an approximate solution for x, say $x_a = 4$.

Answer:

4.236067978

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	4	$^1/_x$	4	+	DSP	.		
2		9						
3		$^1/_x$	4	+				Perform 3 seven times
4							4.236067978	

Example 3: $x^x = 1000$

Method:

Rewrite the equation in the form

$$x = \ln 1000/\ln x\ .$$

Pick an approximation x_a for x, say $x_a = 4$. If we use the following algorithm, convergence is from both sides, and takes a long time.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	1000		LN	↑	↑	↑		
2	4	DSP	.	9				
3			LN	÷			4.982892143	
4			LN	÷			4.301189432	
5			LN	÷			4.734933900	
6			LN	÷			4.442378437	
7			LN	÷			4.632377942	
8			LN	÷			4.505830645	
9							4.588735608	
10							4.533824354	
11							4.569933525	etc. (final result is 4.555535705)

Progressions

Evaluation of the basic types of progressions (arithmetic, geometric and harmonic) can be done very easily using RPN and the four-register operational stack.

Formulas:

Arithmetic Progression

$$a, a + d, a + 2d, \ldots, a + (n - 1)d$$

Geometric Progression

$$a, ar, ar^2, \ldots, ar^{n-1}$$

Harmonic Progression

$$\frac{a}{b}, \frac{a}{b + c}, \frac{a}{b + 2c}, \ldots, \frac{a}{b + (n - 1)c}$$

n = number of terms

a = first term in arithmetic and geometric progressions

l = last term

d = difference between two successive terms in an arithmetic progression

r = ratio between two successive terms in a geometric progression

S = sum of a progression

Step through an arithmetic progression

Formula:

$$a, a + d, a + 2d, \ldots, a + (n - 1)d$$

Example:

Display the progression with a = 0, d = 17.

Answer:

0.00, 17.00, 34.00, 51.00, 68.00, 85.00, 102.00, 119.00, ...

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	d	↑	↑	↑				
2	a							
3		+						Perform 3 as many times as desired

Step through a geometric progression

Formula:

$$a, ar, ar^2, \ldots, ar^{n-1}$$

Example:

Step through the powers of 8.

Answers:

8.00, 64.00, 512.00, 4096.00, 32768.00, ...

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	r	↑	↑	↑				
2	a							
3		×						Perform 3 as many times as desired

Step through a harmonic progression

Formula:

$$\frac{a}{b}, \frac{a}{b+c}, \frac{a}{b+2c}, \ldots, \frac{a}{b+(n-1)c}$$

Note: A harmonic progression can be obtained by multiplying the constant a by the reciprocals of the terms of the arithmetic progression b, b + c, b + 2c, ..., b + (n − 1) c. In the following algorithm, x_i (i = 1, 2, ...) represents the i^{th} term of the progression.

Example:

Step through the harmonic progression where a = 1, b = 2, and c = 3.

Answers:

0.50, 0.20, 0.13, 0.09, ...

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	STO						
2	c	↑	↑	↑				
3	b	↑	$1/x$	RCL	x		x_1	
4		CLX	+	+	↑	$1/x$		Perform 4-5 for i = 2, 3, ...
5		RCL	x				x_i	

n^{th} term of an arithmetic progression

Formula:

Given the number of terms, the last term of an arithmetic progression is given by

$$n^{th} \text{ term} = a + (n-1)d$$

Example:

Find the 25^{th} term of the arithmetic progression with a = 2, d = 3.14.

Answer:

77.36

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	n	↑	1	−				
2	d	x						
3	a	+					nth term	

n^{th} term of a geometric progression

Formula:

Given the number of terms, the last term of a geometric progression is given by

$$n^{th} \text{ term} = ar^{n-1}$$

Example:

Find the 14^{th} term of the geometric progression with a = 2, r = 3.14.

Answer:

5769197.69

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	r	↑						
2	n	↑	1	−				If r > 0, go to 4
3		x⇄y	CHS	x⇄y		y^x		If n is even, go to 5.
								Otherwise, go to 6
4			y^x					Go to 6
5		CHS						
6	a	x					nth term	

Arithmetic progression sum (given the last term)

Formula:

Given the last term, the sum of an arithmetic progression to n terms is

$$S = \frac{n}{2}(a + l)$$

Example:

If a = 3.5, l = 25, and n = 11, find the sum.

Answer:

S = 156.75

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑						
2	ℓ	+						
3	n	×	2	÷			S	

Arithmetic progression sum (given the difference)

Formula:

Given the first term and the difference between two successive terms, the sum of an arithmetic progression to n terms is:

$$S = na + \frac{n(n-1)d}{2}$$

Example:

If a = 3.5, n = 11, and d = 2.15, find the sum of 11 terms.

Answer:

S = 156.75

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	n	↑	↑	1	−			
2	d	×						
3	a	↑	2	×	+	×		
4		2	÷				S	

Sum of a geometric progression (r < 1)

Formula:

The sum of a geometric progression to n terms with r < 1 is

$$S = \frac{a(1 - r^n)}{1 - r}$$

Example:

If a = 1, r = –2.1, and n = 6, find the sum of 6 terms.

Answer:

S = –27.34

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑	↑					
2	n	↑						
3	r	STO						If r > 0, go to 5
4		CHS	x⇄y		y^x			If n is odd, go to 6;
								Otherwise, go to 7
5		x⇄y		y^x				Go to 7
6		CHS						
7		x	–	1	RCL			
8		–	÷				S	

Sum of a geometric progression (r > 1)

Formula:

The sum of geometric progression to n terms with r > 1 is

$$S = \frac{a\,(r^n - 1)}{r - 1}$$

Example:

If a = 1, r = 2.1, n = 6, find the sum.

Answer:

S = 77.06

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑	↑					
2	r	STO						
3	n		y^x	x	x⇄y	–		
4		RCL	1	–	÷		S	

Sum of an infinite geometric progression (–1 < r < 1)

Formula:

$$S = \frac{a}{1 - r}$$

Example:

If a = 2 and r = .5, find the sum.

Answer:

S = 4.00

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	a	↑	1	↑				
2	r	–	÷				S	

Factoring Integers—Determining Primes

Prime Numbers under 100

2	13	31	53	73
3	17	37	59	79
5	19	41	61	83
7	23	43	67	89
11	29	47	71	97

With the list memorized or in sight, it is easy to factor any integer x less than 10000 (and many other integers even greater). In the following program, omit the numbers 2 and 5 from the list of primes if the integer ends in 1, 3, 7 or 9.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	DSP		5				If x is even, let P = 2;
								Otherwise P = 3.
2		↑	↑	↑		$\sqrt{x}$	Max	
3								If P ⩾ Max, stop
4		R↓						
5	P	÷					Q	Read note

Note: If Q is not an integer, let P = next prime number, go to 3. If Q is a prime, then both P and Q are factors, stop. Otherwise P is a factor, let P = current prime, go to 2.

Example:

Factor 4807.

Answer:

4807 = 11 x 19 x 23

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	4807	DSP	•	5	↑	↑		
2		↑	[shaded]	√x			69.33253	←MAX, P = 3
3		R↓	3	÷			1602.33333	P = 7
4		R↓	7	÷			686.71429	P = 11
5		R↓	1	1	÷		437.00000	←Q is an integer,
								so 11 is a factor
6		↑	↑	↑	[shaded]	√x	20.90454	P = 11
7		R↓	1	1	÷		39.72727	P = 13
8		R↓	1	3	÷		33.61538	P = 17
9		R↓	1	7	÷		25.70588	P = 19
10		R↓	1	9	÷		23.00000	Q = 23 is a prime, 19 and
								23 are factors, stop

Example:

Factor 2909.

Answer:

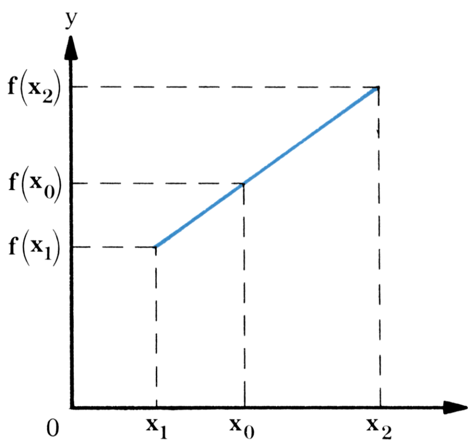
2909 is a prime.

Interpolation

Linear Interpolation

If f(x) is a function of x and $x_1 < x_0 < x_2$, $f(x_0)$ can be approximated by

$$f(x_0) = \frac{(x_2 - x_0)\, f(x_1) + (x_0 - x_1)\, f(x_2)}{x_2 - x_1}$$



Example:

Suppose a table shows

x	f(x)
1.2	0.30119
1.3	0.27253

Interpolate f(x) to 5 decimal places for x = 1.27.

Answer:

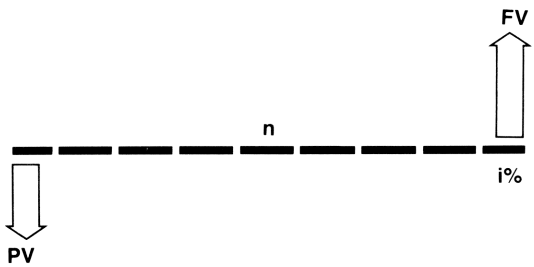
0.28113

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x_2	↑						
2	x_0	STO	−					
3	$f(x_1)$	×	RCL					
4	x_1	STO	−					
5	$f(x_2)$	×	+					
6	x_2	RCL	−	÷			$f(x_0)$	

Chapter 3. Finance

Compound Interest

This procedure applies to an amount of principal that has been placed into an account and compounded periodically with no further deposits. With any three of the variables given, a fourth may easily be calculated.



Note:

The above diagram is representative of diagrams which will be used in this section. The horizontal line represents the time period(s) involved, while the arrows represent the cash flows.

Notation:

n = number of time periods

i = periodic interest rate expressed as a decimal

PV = present value or principal

FV = future value or amount

I = interest amount

Future Value

Formula:

$$FV = PV\,(1 + i)^n$$

Example:

Find the future amount of $1000 invested at 6% compounded annually for 5 years.

Answer:

$1338.23 (Note: i = 0.06)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	↑	1	+				
2	n		y^x					
3	PV	x					FV	

Present Value

Formula:

$$PV = \frac{FV}{(1 + i)^n}$$

Example:

What sum invested today, at 6% compounded annually, will amount to $1500 in 5 years?

Answer:

$1120.89 (Note: i = 0.06)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	FV	↑						
2	i	↑	1	+				
3	n		y^x	÷			PV	

Number of Time Periods

Formula:

$$n = \frac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + i)}$$

Example:

If you deposit $250 in a savings account at 6% annual interest, how long will it take for your money to double?

Answer:

11.90 years (Note: i = .06)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	FV	↑						
2	PV	÷		LN	1	↑		
3	i	+		LN	÷		n	

Rate of Return

Formula:

$$i = \left(\frac{FV}{PV}\right)^{1/n} - 1$$

Example:

Find the annual rate of return if $2000 doubles in 10 years, compounded monthly.

Answer:

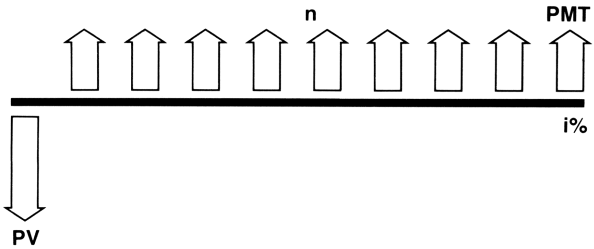
6.95% (.0695) annually

(Note: n = 120 months; FV = 4000; answer must be multiplied by 12 to find an annual rate of return.)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		DSP	.	4				
2	FV	↑						
3	PV	÷						
4	n	$^1/x$		y^x	1	−	i	

Loans (direct reduction)

Given any three of the variables listed below, these procedures calculate the fourth for a direct reduction loan (the type of loan commonly used for mortgages).



Notation:

n = number of payment periods

i = periodic interest rate expressed as a decimal

PMT = payment

PV = present value or principal

Payment Amount

Formula:

$$PMT = \frac{PV \cdot i}{1 - (1 + i)^{-n}}$$

Example:

What monthly payment is required to pay off a \$5000 loan at 9.5% interest in 36 months?

Answer:

\$160.16 (Note: i = 0.095/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO						
2	PV	x	RCL	1	+			
3	n	CHS		y^x	1	x⇄y		
4		−	÷				PMT	

Present Value:

Formula:

$$PV = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Example:

You are willing to pay \$125 per month for 36 months. If the current interest rate is 9.5%, how much can you borrow?

Answer:

\$3902.23 (Note: i = 0.095/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO	1	+				
2	n	CHS		y^x	1	$x \rightleftarrows y$		
3		−	RCL	÷				
4	PMT	x					PV	

Number of Time Periods

Formula:

$$n = -\frac{\ln(1 - i\,PV/PMT)}{\ln(1+i)}$$

Example:

How many payments does it require to pay off a loan of \$4000 at 9.5% annual interest, with payments of \$175 per month?

Answer:

25.31 months (Note: i = .095/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO						
2	PV	x						
3	PMT	÷	1	$x \rightleftarrows y$	−			
4		LN	RCL	1	+			
5		LN	÷	CHS			n	

Accumulated interest

Formula:

The interest paid from payment j to payment k is

$$I_{j-k} = PMT\left[k-j - \frac{(1+i)^{k-n}}{i}\left(1-(1+i)^{j-k}\right)\right]$$

Compute the monthly payment, PMT, by the formula given above under "Payment Amount."

Example:

Consider a house costing \$30,000 with a 30 year, 8% mortgage. The monthly payment is \$220.13. Find the interest paid on the first 36 monthly payments ($i = .08/12$, $j = 0$, $k = 36$, $n = 360$).

Answer:

I_{0-36} = \$7108.72

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO	1	+				
2	j	↑						
3	k	−		y^x	1	x⇄y		
4		−	RCL	÷	RCL	1		
5		+						
6	k	STO						
7	n	−		y^x	x	RCL		
8	j	−	x⇄y	−				
9	PMT	x					I_{j-k}	

Remaining balance

Formula:

The remaining balance at payment k (k = 1, 2, 3, ..., n) is

$$PV_k = \frac{PMT}{i}\left[1 - (1+i)^{k-n}\right]$$

Example:

Using the previous example, find the remaining balance at payment 36.

Answer:

$29184.13

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO	1	+				
2	k	↑						
3	n	–		y^x	1	x⇄y		
4		–						
5	PMT	x	RCL	÷			PV_k	

Amortization Schedule

An amortization schedule consists of the interest paid, the payment to principal, and the remaining balance for each payment k = 1, 2, ...

I_k = interest paid in k^{th} payment

PMT = payment

PP_k = payment to principal of k^{th} payment

PV_k = remaining balance after k^{th} payment

PV_0 = amount of loan

i = periodic interest rate expressed as a decimal

These quantities are calculated with the following formulas:

1. $I_k = iPV_{k-1}$
2. $PP_k = PMT - I_k$
3. $PV_k = PV_{k-1} - PP_k$

Example:

Generate an amortization schedule for the first two months of a $40,000 loan at 9% with monthly payments of $321.85.

Answers:

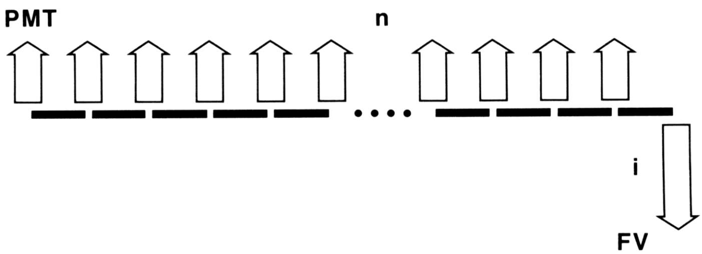
I_1 = \$300.00	I_2 = \$299.84
PP_1 = \$21.85	PP_2 = \$22.01
PV_1 = \$39978.15	PV_2 = \$39956.14

(Note: i = .09/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	PV_0	↑	↑					
2	i	STO	x				I_1	
3	PMT	x⇄y	–				PP_1	
4		–					PV_1	
5		↑	↑	RCL	x		I_k	Perform 5-7 for k = 2, ..., n
6	PMT	x⇄y	–				PP_k	
7		–					PV_k	

Periodic Savings

These procedures calculate the payment amount or future value of a schedule of periodic payments into a savings account, given the interest rate and two of the other three variables.



Notation:

n = number of payments

i = periodic interest rate expressed as a decimal

PMT = payment (at the beginning of the period)

FV = future value

Note:

Payments are assumed to occur at the beginning of the time period (annuity due or "payments in advance").

Payment Amount

Formula:

$$PMT = \frac{FV \cdot i}{(1+i)^{n+1} - (1+i)}$$

Example:

In 3 years you will need \$5000. How much should you deposit each month, if you will receive 6% annual interest, compounded monthly?

Answer:

\$126.48 (Note: n = 36, i = .06/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO						
2	FV	x	1	RCL	+	STO		
3	n	↑	1	+		y^x		
4		RCL	–	÷			PMT	

Future Value

Formula:

$$FV = \frac{PMT}{i} \left[(1 + i)^{n+1} - (1 + i) \right]$$

Example:

You are depositing \$1000 per year in a savings account earning 7.5% interest compounded annually. How much will you have in 10 years?

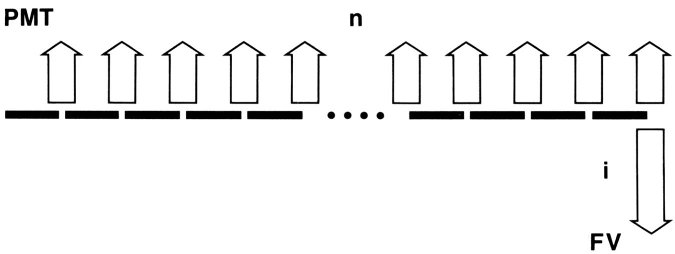
Answer:

\$15,208.12 (Note: $i = .075$)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO	1	+	↑	↑		
2	n	↑	1	+		y^x		
		x⇄y	−					
4	PMT	x	RCL	÷			FV	

Sinking Fund

These procedures calculate the payment amount or future value of a schedule of periodic payments into a fund, given the interest rate and two of the other three variables. Payments are assumed to occur at the end of the time period (ordinary annuity).



Notation:

n = number of time periods

i = periodic interest rate, expressed as a decimal

PMT = payment

FV = future value

Payment Amount

Formula:

$$PMT = \frac{FV \cdot i}{(1 + i)^n - 1}$$

Example:

Calculate the sinking fund annual payment amount necessary to accumulate $25,000 in 15 years at 5¾%.

Answer:

$1094.69 (Note: i = .0575)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO						
2	FV	x	1	RCL	+			
3	n		y^x	1	−	÷	PMT	

Future Value

Formula:

$$FV = PMT \left[\frac{(1 + i)^n - 1}{i} \right]$$

Example:

If you had secretly deposited \$50 each anniversary from your wedding date for 25 years in an 8% certificate account, how much would you have to buy a fur coat for your wife?

Answer:

\$3655.30 (Note: $i = .08$)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	STO	1	+				
2	n		y^x	1	−	RCL		
3		÷						
4	PMT	x					FV	

Percent

Percent change

Formula:

$$\% \text{ change} = \frac{V_N - V_O}{V_O} \times 100$$

where V_O = Original Value (base number)

V_N = New Value

Example:

If sales last year were \$8.6 million, and this year were \$9.3 million, what is the percent change?

Answer:

8.14%

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	V_N	↑						
2	V_O	STO	−	RCL	÷	EEX		
3		2	x				% change	

Markup percent

Formula:

To make a gross profit of G%, add A% to the cost price. To find A for a given G:

$$A = \frac{100G}{100 - G}$$

Example:

To make a profit of 30%, what is the percentage of markup?

Answer:

42.86%

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	G	STO	EEX	2	x	EEX		
2		2	RCL	−	÷		A	

Percent profit

Formula:

If A% is added to the cost price, the profit will be G% of the selling price.

$$G = \frac{100A}{A + 100}$$

Example:

If we add 30% to our cost price, what percent of the selling price will be the profit?

Answer:

23.08%

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	A	STO	EEX	2	x	EEX		
2		2	RCL	+	÷		G	

Nominal Rate Converted to Effective Annual Rate

This procedure calculates the effective or compounded annual interest rate when the number of periods per year and the nominal annual interest rate are known.

Formula:

$$\text{Effective} = (1 + i)^n - 1$$

Example:

What is the effective annual rate of interest if the nominal (annual) rate of 6% is compounded monthly?

Answer:

6.17% (.0617) (Note: n = 12, i = .06/12)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		DSP	.	4				
2	i	↑	1	+				
3	n		y^x	1	−		Effective	

This procedure converts a nominal annual interest rate to the continuous effective rate.

Formula:

$$\text{Effective} = e^i - 1$$

Example:

A bank offers a savings plan with a 5.75% annual nominal interest rate. What is the annual effective rate if compounding is continuous?

Answer:

5.92% (.0592) (Note: i = .0575)

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		DSP	•	4				
2	i	e^x	1	−			Effective	

Depreciation

Machines, buildings, delivery trucks, showcases, tools and other tangible assets all decline in value with the passing of time. To provide for the replacement of obsolete or worn-out equipment, you usually set aside a fixed amount of money each year that is equal to the loss in value of that article during the year. The three common methods of depreciation are described below.

Straight Line Depreciation

The straight-line method is simply a matter of dividing the total depreciable amount by the number of useful years, then subtracting that amount each year from the item's value. The advantage of the straight-line method is its simplicity—it's easy to figure and it's consistent.

Formulas:

$$D = \frac{PV}{n}$$

$$RDV_k = PV - kD$$

where PV = original value of asset (less salvage value)

n = depreciable life of asset

D = depreciation per year

RDV_k = remaining depreciable value at time period k

Example:

A duplex costing \$40,000 (exclusive of land) is depreciated over 25 years, using the straight line method. What is the depreciation amount and remaining depreciable value after 5 years?

Answers:

D = \$1600

RDV_5 = \$32,000

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	PV	↑	↑					
2	n	÷					D	
3	k	x	-				RDV_k	

Declining Balance Depreciation

The declining balance method is one form of accelerated depreciation; as such it provides for more depreciation in the earlier years of ownership and less depreciation in the later years than the straight-line method. With declining balance (sometimes called the fixed-rate method), a constant percentage is applied each year to the remaining balance (book value) to find the depreciable amount. The salvage value is not subtracted initially, but the asset may not be depreciated below this salvage value.

Formulas:

$$D_k = PV \cdot \frac{R}{n}\left(1 - \frac{R}{n}\right)^{k-1}$$

$$RDV_k = PV\left(1 - \frac{R}{n}\right)^{k}$$

where PV = original value of asset

n = depreciable life of asset

R = depreciation rate (given by user)

D_k = depreciation at time period k

RDV_k = remaining depreciable value at time period k

Example:

A fleet car has a value of \$2500 and a life expectancy of 6 years. Use the double declining balance method (R = 2) to find the amount of depreciation and remaining depreciable value after 4 years.

Answers:

RDV_4 = \$493.83
D_4 = \$246.91

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	k	↑	1	↑				
2	R	↑						
3	n	÷	–	STO	x⇄y			
4		y^x						
5	PV	×					RDV_k	
6		RCL	÷	1	RCL	–		
7		×					D_k	

Sum of the Years Digits Depreciation (SOYD)

Like the declining balance method, the sum-of-the-years-digits method is an accelerated form of depreciation, allowing more depreciation in the early years of an asset's life than allowed under the straight line method. The SOYD method is based on the sum of the digits from one to the number of years of the asset's life.

Formula:

$$D_k = \frac{2(n - k + 1)}{n(n + 1)} PV$$

$$RDV_k = S + (n - k)\, D_k/2$$

where PV = original value of asset

n = depreciable life of asset

S = salvage value

D_k = depreciation at time period k

RDV_k = remaining depreciable value at time period k

Example:

Apartments valued at \$88,000 are depreciated over 25 years using SOYD depreciation. What is the depreciation amount and remaining depreciable value after 10 years?

Answers:

D_{10} = \$4332.31

RDV_{10} = \$32492.31

Note:

The salvage value is zero.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	n	↑	↑					
2	k	−	STO	1	+	2		
3		×	x⇄y	↑	↑	×		
4		+	÷					
5	PV	×					D_k	
6		RCL	×	2	÷			
7	S	+					RDV_k	

Discounted Cash Flow Analysis

The primary purpose of this procedure is to compute the net present value (NPV) of a series of cash flows. The NPV is found by discounting the cash flows at the desired rate of return, and then subtracting the initial investment.

In general, an initial investment is made in some enterprise that is expected to bring in periodic cash flows. After discounting, a negative value for "NPV" indicates that the enterprise would not be profitable, while a positive value for "NPV" means that the enterprise will show a profit to the extent that a rate of return "i" on the initial investment has been exceeded.

Notation:

PV_o = original investment

PV_k = net cash flow of the k^{th} period

i = discount rate per period (as a decimal)

NPV_k = net present value at period k

Formula:

$$NPV_k = -PV_o + \sum_{j=1}^{k} \frac{PV_j}{(1+i)^j}$$

Example:

A small shopping complex, which costs $137,000, is estimated to have annual cash flows as follows:

Year	Cash Flow ($)
1	10,000
2	13,000
3	19,000
4	152,000 (property sold in 4^{th} year)

The desired minimum yield is 10%. Will this rate be achieved by the above cash flows?

Answers:

$NPV_1 = -127909.09$

$NPV_2 = -117165.29$

$NPV_3 = -102890.31$

$NPV_4 = 927.74$

Because the final NPV is positive, the investment more than achieves the desired yield.

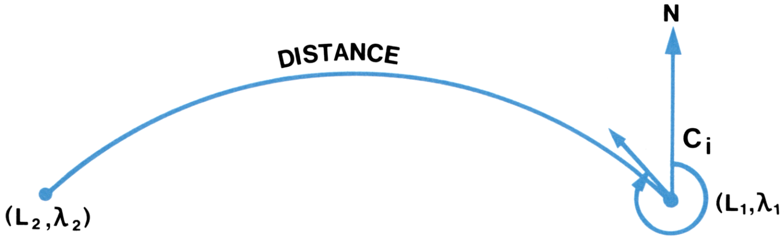
LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	i	↑	1	+	STO			
2	PV_1	x⇄y	÷					
3	PV_o	–					NPV_1	
4	PV_j	RCL						Perform 4-5 for j = 2, 3, ..., k
5	j		y^x	÷	+		NPV_j	

Chapter 4. Navigation

The following procedures use angles, latitudes and longitudes expressed in decimal degrees. Angle Conversions (Ch. 6) may be used to convert degrees, minutes, and seconds to degrees.

Great Circle Navigation

This procedure calculates the great circle distance and initial course for the great circle between two points on a spherical earth. Since the great circle is the shortest path between two points, it is usually the minimum-time path as well.



Formulas:

L_1 = latitude of initial point

λ_1 = longitude of initial point

L_2 = latitude of final point

λ_2 = longitude of final point

DIST = great circle distance (nautical miles)

C_i = initial course

$$\text{DIST} = 60 \cos^{-1} [\sin(L_1)\sin(L_2) + \cos(L_1)\cos(L_2)\cos(\lambda_1 - \lambda_2)]$$

$$C_i = \cos^{-1}\left(\frac{\sin(L_2) - \cos(\text{DIST}/60)\sin(L_1)}{\sin(\text{DIST}/60)\cos(L_1)}\right)$$

If $\sin(\lambda_1 - \lambda_2) < 0$, then $C_i = 360 - C_i$

Notes:

No endpoint of a leg should be at either the North or South pole.

If it is desired to go more than half way around the earth, subtract DIST from 21600 to get the distance and add 180° to C_i to get the initial course (subtracting 360° if $C_i > 360$).

Points located at diametrically opposite sides of the earth should not be used since there are an infinite number of great circle courses through such points.

C_i cannot always be calculated along lines of longitude ($\lambda_1 = \lambda_2$).

Northern latitudes and western longitudes are input and output as positive values; southern latitudes and eastern longitudes are input and output as negative values.

Example:

A ship is proceeding from Manila to Los Angeles. The captain wishes to sail a great circle course from L12.7533°N, λ124.3350°E (input as negative), off the entrance to San Bernardino Strait, to L33.8133°N, λ120.1183°W, five miles south of Santa Rosa Island.

Find the initial great circle course and great circle distance.

Answers:

DIST = 6185.88 nautical miles

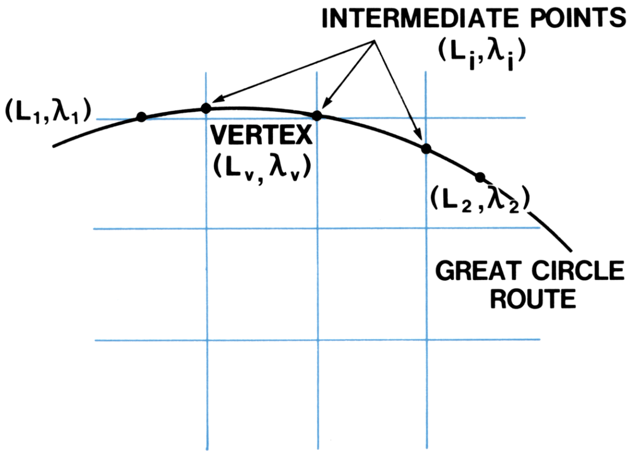
$C_i = 50.32°$

Make sure that the calculator is in DEG mode.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	L_1	↑	COS	x⇄y	SIN			L_1 in decimal deg.
2	L_2	STO	SIN	x	x⇄y	RCL		L_2 in decimal deg.
3		COS	x					
4	λ_1	↑						λ_1 in decimal deg.
5	λ_2	−	COS	x	+			λ_2 in decimal deg.
6		COS^{-1}	STO	6	0	x	Distance (n.m.)	
7		RCL	SIN	RCL	COS			
8	L_1	STO	SIN	x				
9	L_2	SIN	x⇄y	−	x⇄y	RCL		
10		COS	x	÷		COS^{-1}		
11		STO						
12	λ_1	↑						
13	λ_2	−	SIN					If negative, go to line 15.
14		RCL					Course (deg.)	Stop
15		3	6	0	RCL	−	Course (deg.)	Stop

Great Circle Computation

This procedure computes the latitude corresponding to a specified longitude on a great circle passing through two given points.



Notes:

The program does not compute along lines of longitude ($\lambda_1 = \lambda_2$).

Examples:

A ship is proceeding from Manila to Los Angeles. The captain wishes to use great-circle sailing from L12.7533°N, λ124.3350°E (input as negative), off the entrance to San Bernardino Strait, to L33.8133°N, λ120.1183°W, five miles south of Santa Rosa Island. What is the latitude on the dateline (λ = 180°)?

Answer:

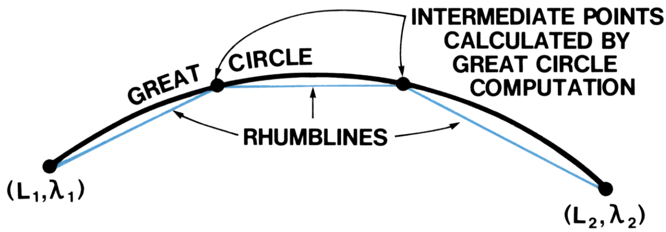
39.69°N

Make sure that the calculator is in DEG mode.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	L_1	TAN						L_1 in decimal deg.
2	L_2	TAN						L_2 in decimal deg.
3	λ_i	STO						λ_i in decimal deg.
4	λ_1	−	SIN	x	x⇄y	RCL		λ_1 in decimal deg.
5	λ_2	STO	−	SIN	x	−		λ_2 in decimal deg.
6		RCL						
7	λ_1	−	SIN	÷		TAN^{-1}	L_i	λ_1, L_i in decimal deg.

Rhumb Line Navigation

This procedure calculates the rhumb line distance and course for the rhumb line between two points on a spherical earth. On long passages, rhumb lines should be used to sail between successive intermediate points along the great circle which passes through the source and destination.



Formulas:

L_1 = latitude of initial point

λ_1 = longitude of initial point

L_2 = latitude of final point

λ_2 = longitude of final point

$$C = \text{course} = \left| \tan^{-1} \left(\frac{\pi \sin^{-1} \quad \sin \quad (\lambda_1 - \lambda_2)/2}{90 \ln \dfrac{\tan(45 + L_2/2)}{\tan(45 + L_1/2)}} \right) \right|$$

If $\sin^{-1}\ [\sin(\lambda_1 - \lambda_2)] < 0$, then $C = 360 - C$

$$\text{Distance} = \begin{cases} 60\,(\lambda_2 - \lambda_1)\cos(L_1), \text{ if } \cos(C) = 0 \\ 60\,\dfrac{L_2 - L_1}{\cos(C)}, \text{ if } \cos(C) \neq 0 \end{cases}$$

Notes:

Northern latitudes and western longitudes are input and output as positive values; southern latitudes and eastern longitudes are input and output as negative values.

No course should pass through the North or South pole.

This procedure gives incorrect results when computing distances due east or due west across the dateline. To obtain correct results, compute up to the dateline and then proceed on the other side.

Errors in distance calculations may be encountered as cos (C) approaches zero (i.e., $C \rightarrow 90°$ or $C \rightarrow 270°$).

Example:

Find the distances and headings for a flight from Anchorage, Alaska to Juneau, Alaska to Seattle, Washington.

Anchorage	L61.22°N	λ149.90°W
Juneau	L58.30°N	λ134.42°W
Seattle	L47.60°N	λ122.33°W

Answers:

Anchorage - Juneau	C = 110.55°	DIST = 499.17 nautical miles
Juneau - Seattle	C = 145.93°	DIST = 775.03 nautical miles

Make sure that the calculator is in DEG mode.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	L_1	↑	2	÷	4	5		L_1 in decimal deg.
2		+	TAN					
3	L_2	↑	2	÷	4	5		L_2 in decimal deg.
4		+	TAN	x⇄y	÷			
5		LN	9	0	x			
6	λ_1	↑						λ_1 in decimal deg.
7	λ_2	−	STO	2	÷	SIN		λ_2 in decimal deg.
8			SIN⁻¹		π	x		
9		x⇄y		→P	x⇄y	↑		
10		x		√x	RCL	SIN		
11			SIN⁻¹					If negative, go to line 13
12		x⇄y					Course (deg.)	Go to line 14
13		x⇄y	3	6	0	+	Course (deg.)	
14		COS					cos C	If zero, go to line 18
15	L_1	↑						
16	L_2	x⇄y	−	x⇄y	÷	6		
17		0	x				Distance (n.m.)	Stop
18	L_1	COS						
19	λ_1	↑						
20	λ_2	x⇄y	−	x	6	0		
21		x					Distance (n.m.)	Stop

Sight Reduction Table

This procedure calculates the computed altitude and azimuth of a celestial body given the observer's latitude and the declination and local hour angle of the body. If the observer is not actually located where he thinks he is, the celestial body will appear to be at some other altitude and azimuth. A knowledge of the difference between observed and computed altitudes provides a first-order correction to the observer's position estimate.

Formulas:

DEC = declination of celestial body

LHA = local hour angle of body

L = observer's latitude

Zn = azimuth of body

Hc = computed altitude of body

$$Hc = \sin^{-1}\ [\sin(DEC)\sin(L) + \cos(DEC)\cos(L)\cos(LHA)]$$

$$Z = \cos^{-1}\left(\frac{\sin(DEC) - \sin(L)\sin(Hc)}{\cos(Hc)\cos(L)}\right)$$

$$Zn = \begin{cases} Z, \text{ if } \sin(LHA) < 0 \\ 360 - Z, \text{ if } \sin(LHA) > 0 \end{cases}$$

Notes:

Northern latitudes, northern declinations, and western hour angles are input as positive values; southern latitudes, southern declinations and eastern hour angles are input as negative values.

This procedure may also be used for star identification by entering the observed azimuth in place of local hour angle and observed altitude in place of declination. The outputs are then declination and local hour angle instead of altitude and azimuth, respectively. The star may be identified by comparing this computed declination to the list of stars in The Nautical Almanac.

Example:

Compute the altitude and azimuth of the Sun if its LHA is 333.0317°W and its declination is 12.4683°S (input as negative). The assumed latitude is 34.1850°S (input as negative).

Answers:

Hc = 57.27°

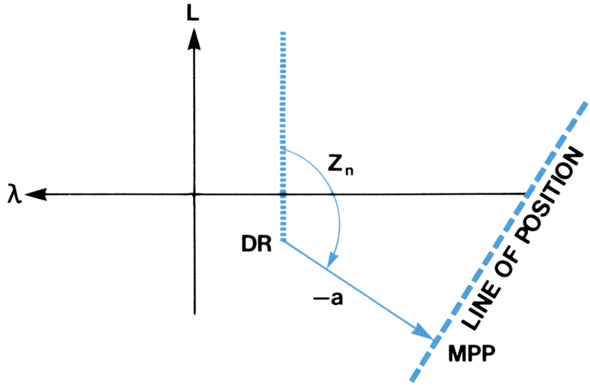
Zn = 54.97°

Make sure that the calculator is in DEG mode.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	DEC	↑	COS	x⇄y	SIN			
2	L	STO	SIN	x	x⇄y	RCL		
3		COS	x					
4	LHA	COS	x	+		SIN^{-1}	Hc (deg.)	
5		↑	COS	x⇄y	SIN			
6	DEC	SIN	x⇄y					
7	L	STO	SIN	x	–	x⇄y		
8		RCL	COS	x	÷			
9		COS^{-1}					Z	
10	LHA	SIN						If negative, go to line 12.
11		3	6	0	x⇄y	–	Zn (deg.)	Stop
12		x⇄y					Zn (deg.)	Stop

Most Probable Position

This procedure computes the most probable position (MPP) from a single observation of a celestial object by dropping a perpendicular from the dead reckoning position (DR) to the line of position (LOP) of the object.



Formulas:

L_{ap} = latitude of observer's assumed position

λ_{ap} = longitude of observer's assumed position

L_{mpp} = latitude of most probable position

λ_{mpp} = longitude of most probable position

Hc = computed altitude of object

Ho = corrected sextant height

a = altitude intercept: (−) = toward, (+) = away

a = Hc − Ho

Zn = azimuth of object

$$\lambda_{mpp} = \lambda_{ap} + \frac{a \sin (Zn)}{\cos (L_1)}$$

$$L_{mpp} = L_{ap} - a \cos (Zn)$$

Notes:

Northern latitudes and western longitudes are input and output as positive values; southern latitudes and eastern longitudes are input and output as negative values.

Example:

A navigator determines his DR to be L40.20°S (input as negative), λ159.95°E (input as negative). He observes Procyon to be 11.1883° above the horizon. The computed altitude is 10.95° at azimuth 73.4°. What is his MPP?

Answers:

L_{mpp} = 40.13°S (output as negative)

λ_{mpp} = 160.25°E (output as negative)

Make sure that the calculator is in DEG mode.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	Zn	↑						Zn in decimal deg.
2	Hc	↑						Hc in decimal deg.
3	Ho	−		→R				Ho in decimal deg.
4	L_{ap}	STO	x⇄y	−			L_{mpp} (deg.)	L_{ap} in decimal deg.
5		x⇄y	RCL	COS	÷			
6	λ_{ap}	+					λ_{mpp} (deg.)	λ_{ap} in decimal deg.

Chapter 5. Surveying

Surveyors will also find useful routines in the Mathematics, Navigation and Conversions chapters, e.g., Triangle Solutions, Curve Solutions, Great Circle Navigation, and Angle Conversions.

Field Angle Traverse

This procedure can be used to calculate coordinates of points in a traverse from field angles or deflections and horizontal distances.

Formulas:

N, E = coordinates of point

PRE N, PRE E = coordinates of previous point

AR, AL = angle right, angle left

DR, DL = deflection right, deflection left

AZ = azimuth

REF AZ = reference azimuth

HD = horizontal distance

LAT = latitude

DEP = departure

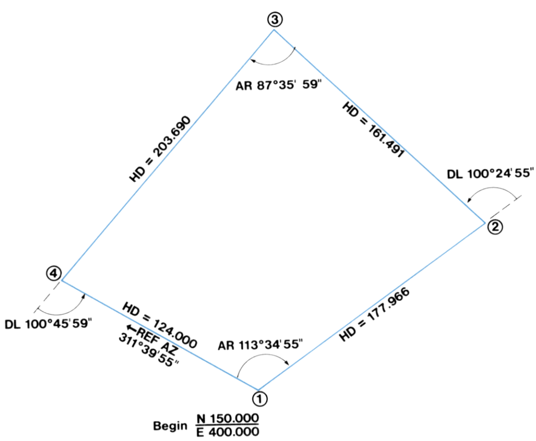
LAT = HD cos (AZ)

DEP = HD sin (AZ)

$N_i = N_{i-1} + \text{LAT}$

$E_i = E_{i-1} + \text{DEP}$

Example:



The field angles and distances for a traverse are shown on the figure. If the coordinates of the starting point are N 150, E 400, calculate the coordinates of the other points.

Answers:

First convert the angles in degrees, minutes and seconds to decimal degrees using the angle conversions (ch. 6) procedure.

Point	N	E
1	150	400
2	224.52	561.61
3	356.53	468.59
4	232.34	307.14
1	149.91	399.77

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	REF AZ	↑	1	8	0	+		REF AZ in decimal degrees
		STO						
2	AR	↑	1	8	0	+		On line 2, key in AR, AL, DR, or
or 2	AL	CHS	↑	1	8	0		DL in decimal degrees
		+						
or 2	DR	↑						
or 2	DL	CHS						
3		RCL	+	STO				
4	HD	RCL	x⇄y		→R		LAT	
5	PRE N	+					N	
6		x⇄y					DEP	
7	PRE E	+					E	Repeat lines 2-7.

Area of Traverse from Coordinates

The area of a closed traverse can be calculated from the coordinates of the points using this procedure.

Formulas:

$$N_1, E_1 = \text{starting coordinates}$$

$$\text{AREA} = \frac{1}{2}\{E_1(N_2 - N_1) + [E_2(N_3 - N_1) + E_3(N_4 - N_2) + \ldots + E_{n-1}(N_n - N_{n-2})] + E_n(N_1 - N_{n-1})\}$$

Example:

N	E
100	100
100	500
500	500
500	100

Answer:

AREA = 160000 sq. feet

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1			CLR	STO				
2	E_1	↑						
3	N_1	↑						
4	NEXT E	↑	R↓	x	RCL	x⇄y		
5		–	STO	CLX				
6	NEXT N	↑	R↓	x	RCL	+		
7		STO	R↓	x⇄y				Return to line 4 until starting
								coordinates have been reentered.
8		RCL	2	÷			AREA	

Slope Distance Reduction

This procedure can be used to reduce a sloped distance and zenith angle or vertical angle to the horizontal distance and elevation change.

Formulas:

ZA = zenith angle

VA = vertical angle

SD = slope distance

HD = horizontal distance

ΔEL = elevation change

$HD = SD \sin(ZA)$ $\Delta EL = SD \cos(ZA)$

$HD = SD \cos(VA)$ $\Delta EL = SD \sin(VA)$

Example:

With an EDM (electronic distance meter) and theodolite you measure a slope distance of 857.49 feet with a zenith angle of 87°49′33″. What is the horizontal distance and elevation change?

Answer:

First convert the angle in degrees, minutes and seconds to decimal degrees using the Angle Conversions (Ch. 6) procedure.

HD = 856.87 feet

ΔEL = 32.53 feet

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	ZA	↑						ZA in decimal degrees
2	SD		→R				ΔEL	
3		x⇄y					HD	
or								
1	VA	↑						VA in decimal degrees
2	SD		→R				HD	
3		x⇄y					ΔEL	

Horizontal Distance to Latitude and Departure

This procedure calculates the latitude and departure given the azimuth and horizontal distance for a course. If the coordinates of the starting point are known, the coordinates of the final point can be calculated.

Formulas:

LAT = latitude (difference of northings)

DEP = departure (difference of eastings)

HD = horizontal distance

AZ = azimuth

N_1, E_1 = coordinates of starting point

N_2, E_2 = coordinates of final point

$$LAT = HD \cos (AZ)$$

$$DEP = HD \sin (AZ)$$

$$N_2 = N_1 + LAT$$

$$E_2 = E_1 + DEP$$

Example:

For a course from N 100, E 500, the distance is 583 feet along an azimuth of 43.47°. Find the latitude, departure and coordinates of the final point.

Answers:

LAT = 423.10

DEP = 401.09

N_2 = 523.10

E_2 = 901.09

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	AZ	↑						AZ in decimal degrees
2	HD	[blue key]	→R				LAT	
3		x⇄y					DEP	
4	E_1	+					E_2	
5		x⇄y						
6	N_1	+					N_2	

Coordinates to Distance and Azimuth

This procedure inverses between two points to give distance and azimuth.

Formulas:

N_1, E_1 = coordinates of point 1

N_2, E_2 = coordinates of point 2

AZ = azimuth

HD = horizontal distance

$$AZ = \tan^{-1}\left(\frac{E_2 - E_1}{N_2 - N_1}\right)$$

$$HD = \sqrt{(N_2 - N_1)^2 + (E_2 - E_1)^2}$$

Example:

Inverse from point N 1000, E 1000 to point N 1500, E 2000.

Answer:

HD = 1118.03 feet

AZ = 63.43 degrees

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	E_2	↑						
2	E_1	−						
3	N_2	↑						
4	N_1	−	[shaded key]	→P			HD	
5		x⇄y					AZ	If negative, press CHS, 3, 8, 0, +

Bearing to Azimuth—Azimuth to Bearing

This procedure can be used to convert azimuths to quadrant bearings or quadrant bearings to azimuths.

Example:

Convert S 80°47′53″W to an azimuth and back to a quadrant bearing.

Answer:

First use Angle Conversions (Chapter 6) to convert bearing in degrees, minutes and seconds to decimal degrees.

AZ = 260.80 degrees

BRG = 80.80 degrees

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	NE BRG						AZ	No conversion required
or 1	SE BRG	CHS	↑	1	8	0		
		+					AZ	
or 1	SW BRG	↑	1	8	0	+	AZ	
or 1	NW BRG	CHS	↑	3	6	0		
		+					AZ	
1	AZ							
or 2	0-90						NE BRG	No conversion required
or 2	90-180	CHS	↑	1	8	0		
		+					SE BRG	
or 2	180-270	↑	1	8	0	–	SW BRG	
or 2	270-360	CHS	↑	3	6	0		
		+					NW BRG	

Chapter 6. Conversions

Angle Conversions

Degrees, minutes, and seconds to decimal degrees

Example:

$46°17'32.6'' = 46.29°$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	Seconds	↑	6	0	STO	÷		
2	Minutes	+	RCL	÷				
3	Degrees	+					Degrees	

Decimal degrees to degrees, minutes and seconds

Note:

x = decimal degrees.

Example:

$23.32916667° = 23°19'45''$

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	X	↑						Let D = integer part of X
2	D						Degrees	
3		–	6	0	STO	x	A	Let M = integer part of A
4	M						Minutes	
5		–	RCL	x			B	Let S = nearest integer to B
5	S						Seconds	

Radians to degrees

Examples:

1. 1 radian = 57.29577951°
2. $\frac{3}{4}\pi$ radians = 135°

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		1	8	0	π	÷		
2		↑	↑	↑				
3		CLX						
4	Radians	x					Degrees	Stop. For new case, go to 3

Degrees to radians

Examples

1. 1° = .0174532925 radians
2. 266° = 4.64 radians

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		π	1	8	0	÷		
2		↑	↑	↑				
3		CLX						
4	Degrees	x					Radians	Stop. For new case, go to 3

Grads to degrees

Example:

300 grads = 270°

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	Grads	↑	.	9	x		Degrees	

Degrees to grads

Example:

360° = 400 grads

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	Degrees	↑	.	9	÷		Grads	

Feet and Inches Conversions

Feet and Inches to Decimal Feet

Example:

43′ 5-3/8″ = 43.45 feet

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	inches	↑	1	2	÷			
2	feet	+						

Decimal Feet to Feet and Inches

Note:

x = decimal feet

Example:

7.373 feet = 7′ 4-1/2″

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1	x	↑						Let F = integer part of x, e.g.,
								integer part of 7.375 is 7.
2	F						feet	
3		−	1	2	×			

Formulas with Two Constants.

Celsius to Fahrenheit (a°C → b°F)

Formula:

$$b = \frac{9}{5} a + 32$$

Examples:

a	−30	0	28	100	539
b	−22.	32.	82.4	212.	1002.2

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		3	2	STO	9	↑		
2		5	÷	↑	↑	↑		
3		CLX						
4	a	x	RCL	+			b	Stop. For new case, go to 3

Fahrenheit to Celsius (b°F → a°C)

Formula:

$$a = \frac{5}{9} (b - 32)$$

Examples:

b	−460	−40	0	32	212
a	−273.33	−40.	−17.78	0.	100.

LINE	DATA	OPERATIONS					DISPLAY	REMARKS
1		3	2	STO	5	↑		
2		9	÷	↑	↑	↑		
3		CLX						
4	b	RCL	−	x			a	Stop. For new case, go to 3

Useful Conversion Factors

The following factors are provided to 10 digits of accuracy where possible. Exact values are marked with an asterisk. For more complete information on conversion factors, refer to *Metric Practice Guide E380-74* by the American Society for Testing and Materials (ASTM).

Length

1 inch	= 25.4 millimeters*
1 foot	= 0.304 8 meter*
1 mile (statute)†	= 1.609 344 kilometers*
1 mile (nautical)†	= 1.852 kilometers*
1 mile (nautical)†	= 1.150 779 448 miles (statute)†

Area

1 square inch	= 6.451 6 square centimeters*
1 square foot	= 0.092 903 04 square meter*
1 acre	= 43 560 square feet
1 square mile†	= 640 acres

Volume

1 cubic inch	= 16.387 064 cubic centimeters*
1 cubic foot	= 0.028 316 847 cubic meter
1 ounce (fluid)†	= 29.573 529 56 cubic centimeters
1 ounce (fluid)†	= 0.029 573 530 liter
1 gallon (fluid)†	= 3.785 411 784 liters*

Mass

1 ounce (mass)	= 28.349 523 12 grams
1 pound (mass)	= 0.453 592 37 kilogram*
1 ton (short)	= 0.907 184 74 metric ton*

Energy

1 British thermal unit	= 1 055.055 853 joules
1 kilocalorie (mean)	= 4 190.02 joules
1 watt-hour	= 3 600 joules*

Force

1 ounce (force)	= 0.278 013 85 newton
1 pound (force)	= 4.448 221 615 newtons

Power

1 horsepower (electric)	= 746 watts*

Pressure

1 atmosphere	= 760 mm Hg at sea level
1 atmosphere	= 14.7 pounds per square inch
1 atmosphere	= 101 325 pascals

Temperature

Fahrenheit	= 1.8 Celsius + 32
Celsius	= 5/9(Fahrenheit − 32)
kelvin	= Celsius + 273.15
kelvin	= 5/9 (Fahrenheit + 459.67)
kelvin	= 5/9 Rankine

† U.S. values chosen. * Exact values.

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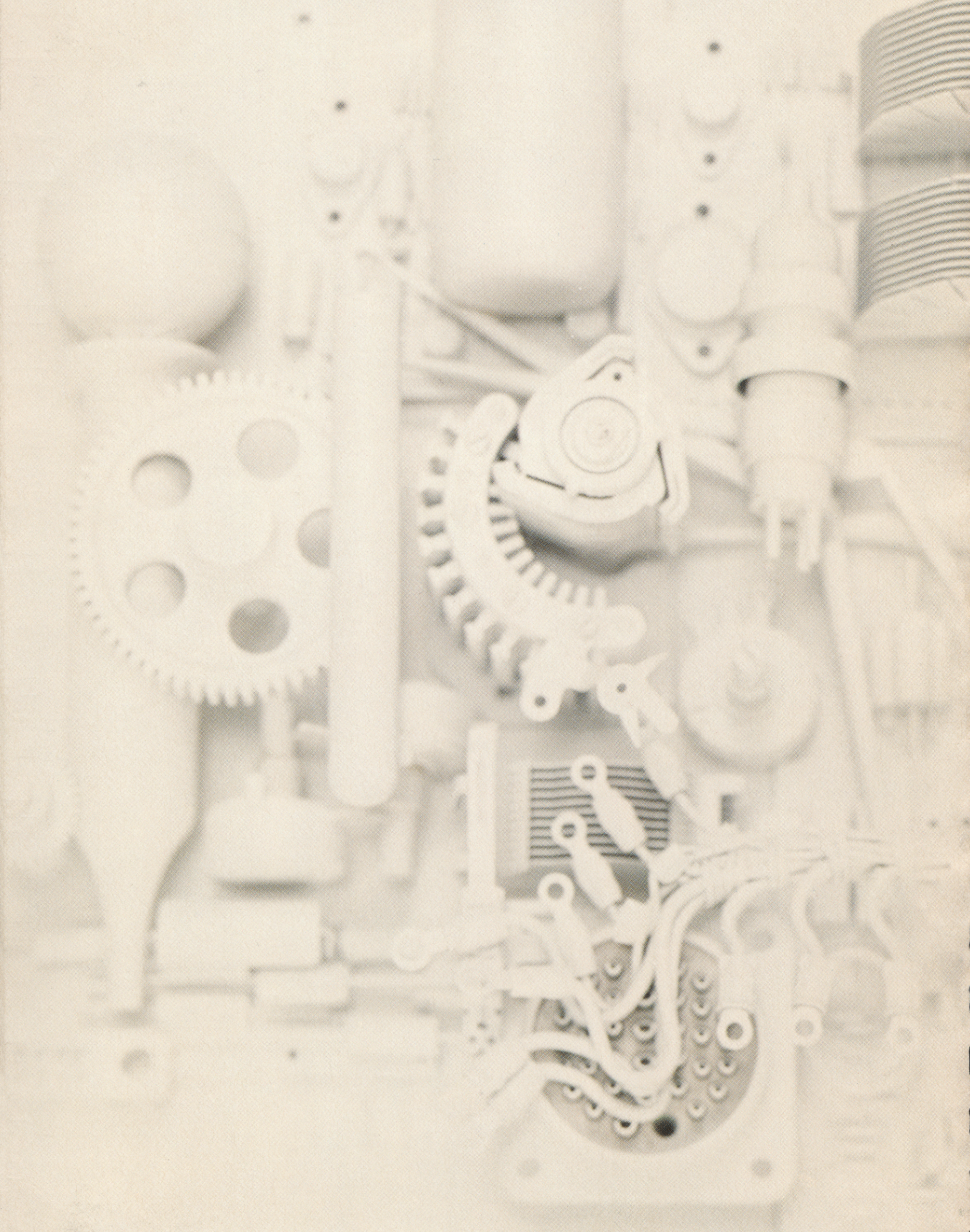

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