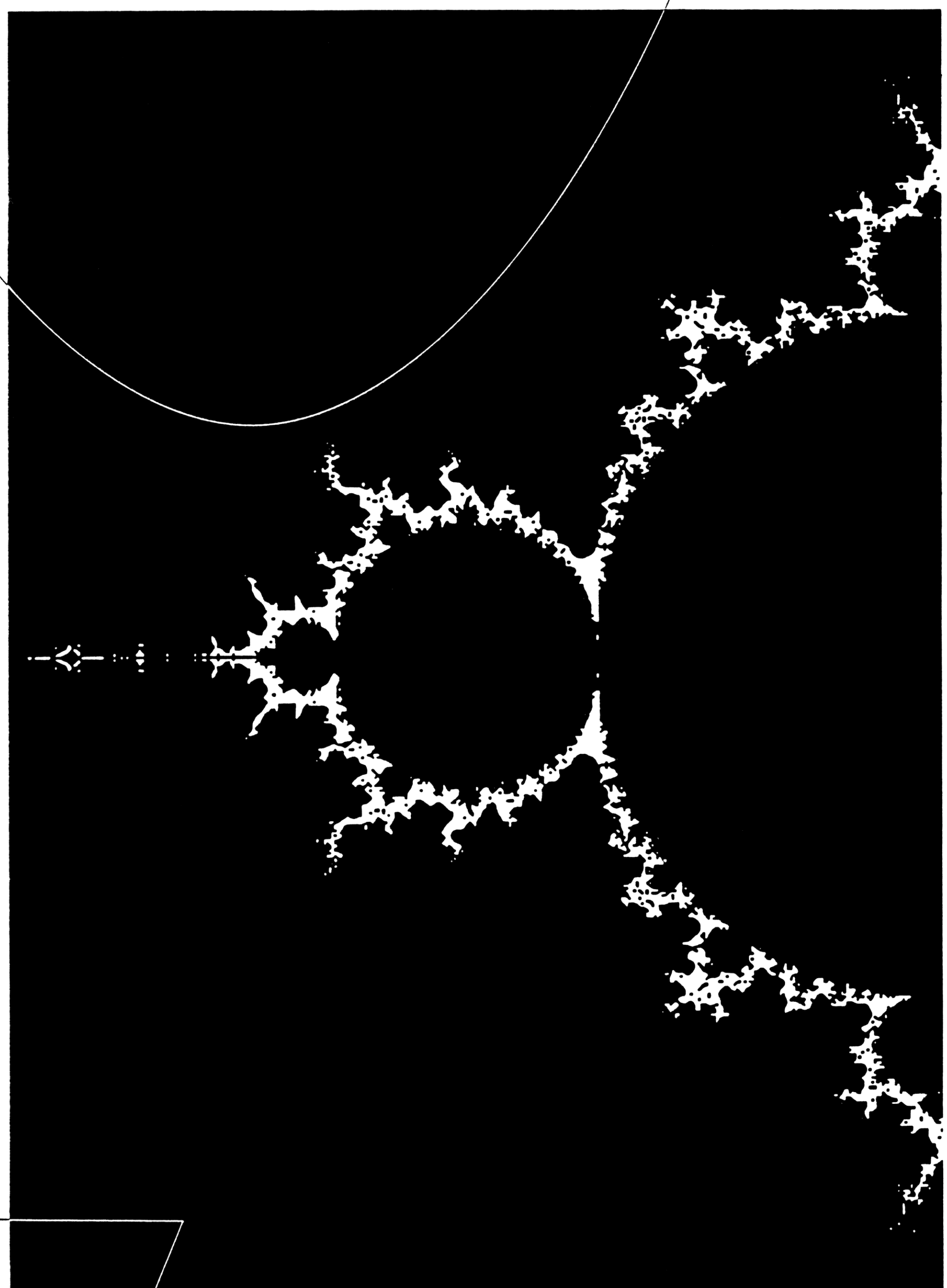
Yves Nievergelt

Fractals on Hewlett-Packard Supercalculators

Preliminary Edition



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Fractals

On Hewlett-Packard Supercalculators

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PREFACE

Introduction. Fractals are concrete objects — pictures — that students now see in books, on posters, and on computer, video, and television screens. Such pictures of fractals provide vivid illustrations of abstract mathematics, but they also require mathematics to be created and understood. Fortunately, the mathematics involved in creating and investigating fractals coincides with, and can help in understanding, the theory that students majoring in the mathematical sciences need to master in the introductory courses in complex analysis, real analysis, and topology: iterations, sequences, convergence, boundedness, neighborhood, continuity, compactness, homeomorphisms, measures, and so forth.

Purpose. To convince students of the close relationship between theory and practice, the present book demonstrates how the "abstract" topics taught in introductory courses in analysis and topology apply to "concrete" pictures of fractals. Besides presenting students with applications of mathematics at several levels, fractals offer several additional advantages: they are familiar to many different students; they demand no prerequisites from fields outside of mathematics; and they provide research projects with statements accessible to undergraduates. Reflecting the ubiquity of fractals, the present book may either serve as a main text in a seminar on fractals or on research for undergraduates, or it may serve as a supplement — as do the instructor's handouts on special topics — in such courses as complex analysis, real analysis, and topology.

Chapter 1 introduces the fractals called "Julia sets," and develops plotting algorithms based upon complex variables. The material includes a new bound for the size of quadratic Julia sets and for Mandelbrot's set, which involves only *basic* algebra and the triangle inequality with complex numbers. The new bound then leads to a justification of the Non-Attracting Fixed-Point Inverse Iteration Method that utilizes only elementary complex variables.

Chapter 2 begins with a simple construction of von Koch's fractal snowflake. The exposition then proceeds with an explanation of Hausdorff dimension, and with a new, *elementary* proof of the Hausdorff dimension of von Koch's snowflake, which utilizes only a counting argument and the basic topological concepts of compactness and of Cauchy sequences of continuous functions.

To accommodate readers with various mathematical backgrounds, each chapter begins with examples at the level of a beginner in the subject. For intermediate readers, each chapter then reviews the relevant theory from the core of complex analysis, real analysis, or topology, as needed. Following the theory, exploratory term projects allow students to explore lesser known aspects of fractals, at a challenging level involving computing experiments and theoretical conjectures, in a way similar to Gleick's account of Mitchell J. Feigenbaum's discovery of Feigenbaum's constant with an HP-65 pocket calculator. Finally, each chapter proposes research problems particular to the topic under consideration, which give students opportunities to investigate practical and theoretical extensions of the text and of their courses; the proposed problems may lead to publishable original research. Also, each chapter contains extensive solutions to all the exercises — with explanations and intermediate steps — as well as hints and suggested strategies for the term projects.

Computing. Throughout the text, exercises and term projects allow students to experiment with "supercalculators," which are pieces of software or hardware — for example, Hewlett-Packard Company's HP-28C, HP-28S, or HP-48SX hand-held calculators, or any computer equipped with adequate mathematical software — that link graphic, symbolic, and numerical capabilities to one another. Such experiments also demonstrate how mathematical theory and pictures of fractals reflect each other.

Please note that the programs listed herein are only working prototypes, which appear here only to illustrate the ideas explained in the text and to document how certain displays of fractals were obtained, without any warranty of any kind.

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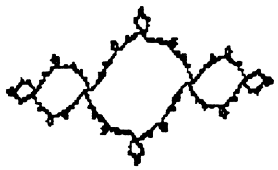
Cheney and Seattle, Washington, June 1990,
Yves Nievergelt.

CHAPTER 1



QUADRATIC JULIA SETS

in Complex Analysis

Summary. The present chapter reviews the theory necessary to define the concept of "Julia set" and to design an algorithm to plot Julia sets. The resulting pictures illustrate many concepts related to complex numbers, at levels from precalculus through complex analysis.

Prerequisites. The treatment of filled Julia sets, in the second section, demands only a working knowledge of the arithmetic of complex numbers, as reviewed in the first section. The presentation of Julia sets, in the fourth section, requires a familiarity with the contents of a standard introduction to complex variables (such topics are reviewed in the third section).

CONTENTS

0. INTRODUCTION

"Julia sets" are examples of mathematical objects informally called "fractals." Pictures of Julia sets, such as those in exhibit 1, are important, because they provide a graphic medium for communicating certain abstract but useful concepts from real and complex analysis. For instance, Julia sets illustrate the sensitivity to initial conditions of the convergence or divergence of some sequences of complex numbers. Such sensitivity forms a part of the informal concept of "chaos" and is sometimes called the "Butterfly Effect": weather is so sensitive to small perturbations that a butterfly flapping its wings in Tokyo may cause the weather later to change in Paris. This sensitivity renders long term weather forecasting practically impossible. For a historical account of the discovery of the Butterfly Effect by M.I.T.'s Edward Lorenz, see Gleick's *Chaos* (reference [38]).

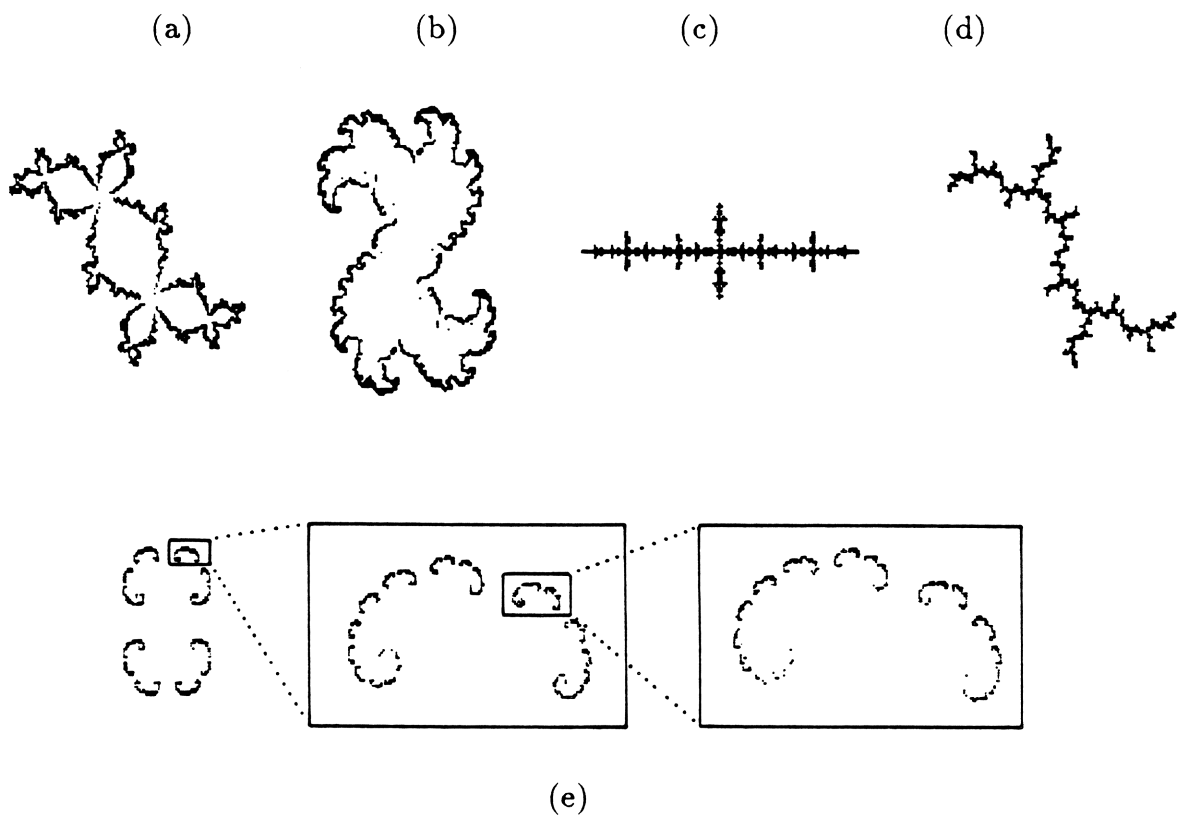


Exhibit 1. (a) – (d) Pictures of Julia sets. (e) Similarity under magnification.

There are advanced mathematical surveys of the theory of Julia sets, for instance, those by Barnsley [28], Blanchard [29], Devaney [30], Devaney and Keen [32], and Falconer [33], [34], with abundant examples of images of Julia sets, notably in the works of Mandelbrot [35] and of Peitgen *et al.* [36], [37]. In contrast, this chapter provides an introduction to Julia sets, beginning at the level of precalculus and ending at about the level of a rigorous introduction to complex analysis. Because a treatise about Julia sets would require a substantial amount of theory, this chapter aims at only two goals: a definition and illustrations of the concept of Julia set; and a rigorous — but simpler than otherwise available — development of an algorithm to plot Julia sets with computers or supercalculators.

1. COMPLEX GEOMETRY AND ALGEGBRA

The present section reviews those concepts from elementary complex analysis that form the mathematical basis of "filled Julia sets," at the level of precalculus and calculus.

1.1. The algebra of complex numbers

A definition in terms of the set $\mathbb{R}$ of all real numbers will demystify complex numbers.

Definition 1. A **complex number** is an ordered pair of real numbers, often denoted by $z = (x, y)$, with $x \in \mathbb{R}$ representing the **first coordinate** of z, and $y \in \mathbb{R}$ representing the **second coordinate** of z, as shown in figure 1. The set of all complex numbers is called the **complex plane** and denoted by the symbol $\mathbb{C}$ ("blackboard" or "special roman" C).

Thus, the set of complex numbers is the ordinary two-dimensional real plane: $\mathbb{C} = \mathbb{R}^2$.

EXAMPLE 1. $(1,0)$, $(0,1)$, $(-3/4, \sqrt{2})$, and $(\pi, -e)$ are four complex numbers.

Besides the representation of complex numbers with cartesian coordinates, there exists another representation, with polar coordinates.

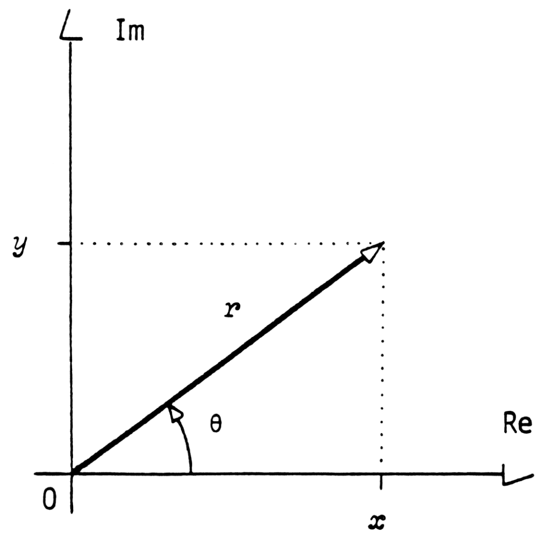


Figure 1. Cartesian coordinates, (x, y), and polar coordinates, (r, θ).

Definition 2. The **modulus** (also called the **magnitude**) of a complex number $z = (x, y)$ is the number $|z|$, also denoted by the letter r, and defined by

$$r = |z| = \sqrt{x^2 + y^2}.$$

The **principal argument** (or simply the **argument**) of a complex number $z = (x, y)$ is the number $\operatorname{Arg}(z)$, also denoted by θ (the Greek letter "theta"), and defined by

$$\theta = \operatorname{Arg}(z) = \begin{cases} -\operatorname{Arccos}(x/\sqrt{x^2 + y^2}) & \text{if } y < 0, \\ \operatorname{Arccos}(x/\sqrt{x^2 + y^2}) & \text{if } y \geq 0. \end{cases}$$

Thus, the modulus $r = |z|$ represents the distance from the complex number z to the origin, and the argument $\theta = \text{Arg}(z)$ represents the angle, in the range $-\pi < \theta \leq \pi$, formed by the complex number z and the positive first coordinate axis, as in figure 1.

EXAMPLE 2.

(2.1) The complex number $z = (1,1)$ has modulus $r = |(1,1)| = \sqrt{1^2+1^2} = \sqrt{2}$ and argument $\theta = \text{Arg}(1,1) = \text{Arccos}(1/\sqrt{1^2+1^2}) = \text{Arccos}(1/\sqrt{2}) = \pi/4$.

(2.2) The complex number $i = (0,1)$ has modulus $r = |(0,1)| = \sqrt{0^2+1^2} = 1$ and argument $\theta = \text{Arg}(0,1) = \text{Arccos}(0/\sqrt{0^2+1^2}) = \text{Arccos}(0) = \pi/2$.

(2.3) The complex number $w = (1, -\sqrt{3})$ has modulus $r = |(1,-\sqrt{3})| = \sqrt{1+3} = 2$ and argument $\theta = \text{Arg}(1, -\sqrt{3}) = \cdots = -\text{Arccos}(1/2) = -\pi/3$.

While the preceding definition and example proceeded from cartesian to polar coordinates, the converse operation converts polar coordinates back to cartesian coordinates by projecting a point $z = (x,y)$ onto each of the coordinate axes:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta).$$

Thus, with $r = |(x,y)|$ and $\theta = \text{Arg}(x,y)$, as shown in figure 1,

$$(x,y) = r \cdot (\cos(\theta), \sin(\theta)).$$

EXAMPLE 3. For the complex number with modulus $r = \sqrt{2}$ and principal argument $\theta = \pi/4$, we obtain $x = r \cdot \cos(\theta) = \sqrt{2} \cdot 1/\sqrt{2} = 1$ and $y = r \cdot \sin(\theta) = \sqrt{2} \cdot 1/\sqrt{2} = 1$.

The first advantage of complex numbers lies in their algebra, described in the following two definitions, which provides a concise notation for dealing with algebraic and geometric operations in the plane.

Definition 3. The **addition** of complex numbers maps two complex numbers, $w = (u,v)$ and $z = (x,y)$, to their **sum,** denoted by $w + z$ and defined by

$$w + z = (u,v) + (x,y) = (u+x, v+y).$$

Thus, the addition of complex numbers *is* the ordinary addition of points ("vectors") in the plane. Graphically, the sum $w + z = (u+x, v+y)$ lies at the fourth corner of the convex parallelogram with the other three corners at the points $w = (u,v)$, $z = (x,y)$, and the origin, $(0,0)$, as shown in figure 2a.

EXAMPLE 4. $(1,2) + (3,4) = (1+2, 3+4) = (3,7)$.

REMARK 1. (TRIANGLE INEQUALITY) Since the modulus of a complex number equals the ordinary Euclidean distance from that number to the origin, the modulus satisfies the same **triangle inequality** as the distance does: for all complex z and w,

$$|z + w| \leq |z| + |w|,$$

with equality ($|z+w| = |z| + |w|$) if, but only if, there exists a non-negative real number $\ell \geq 0$ such that either $z = \ell w$ or $w = \ell z$ (with $\ell z = \ell(x, y) = (\ell x, \ell y)$, as with the ordinary multiplication of a number by a vector in the plane). This means that $|z+w| = |z| + |w|$ if, but only if, z and w lie in the same direction from the origin.

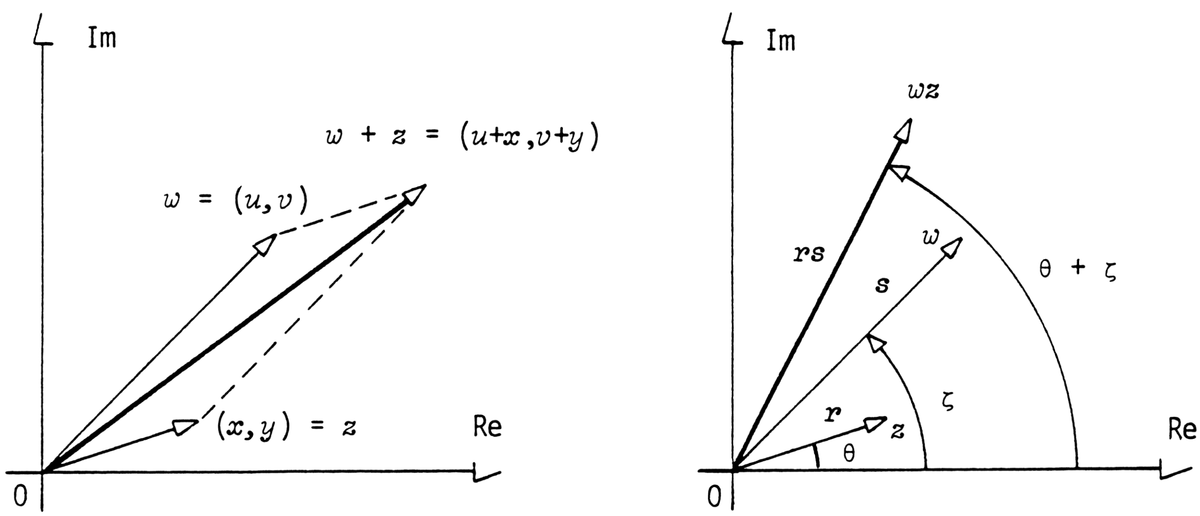
Definition 4. The **multiplication** of complex numbers maps two complex numbers, $w = (u, v)$ and $z = (x, y)$, to their **product,** denoted by $w \cdot z$, or $w \times z$ (which does not denote a cross product in the present context), or wz, with

$$wz = (u, v)(x, y) = (ux - vy, uy + vx).$$

EXAMPLE 5. $(1,2)(3,4) = ((1\times3)-(2\times4), (1\times4)+(2\times3)) = (3-8, 4+6) = (-5, 10)$.

Graphically, the multiplication of a complex number z by a complex number w rotates z by the argument of w and expands or contracts z by the modulus of w, as in figure 2b. In mathematical symbols,

$$\text{Arg}\,(wz) = \text{Arg}\,(w) + \text{Arg}\,(z),$$
$$|wz| = |w| \cdot |z|.$$



(a) (b)

Figure 2. Graphic interpretations of the complex addition and multiplication.

A verification of the graphical interpretation of the complex multiplication consists of expressing the multiplication not with cartesian coordinates (as in the definition) but with polar coordinates (as in the graphical interpretation). To this end, let $r = |z|$ and $\theta = \text{Arg}\,(z)$ denote the polar coordinates of z, and let $s = |w|$ and $\zeta = \text{Arg}\,(w)$ (ζ is the Greek letter "zeta") denote the polar coordinates of w:

$$x = r \cdot \cos(\theta) \quad \text{and} \quad y = r \cdot \sin(\theta),$$

$$u = s \cdot \cos(\zeta) \quad \text{and} \quad v = s \cdot \sin(\zeta).$$

Consequently,

$$wz = (u,v)(x,y) = (ux - vy, uy + vx) =$$

$$(s \cdot \cos(\zeta) \cdot r \cdot \cos(\theta) - s \cdot \sin(\zeta) \cdot r \cdot \sin(\theta)\ ,\ s \cdot \cos(\zeta) \cdot r \cdot \sin(\theta) + s \cdot \sin(\zeta) \cdot r \cdot \cos(\theta))$$

$$= rs(\cos(\zeta)\cos(\theta) - \sin(\zeta)\sin(\theta)\ ,\ \cos(\zeta)\sin(\theta) + \sin(\zeta)\cos(\theta))$$

$$= rs(\cos(\zeta + \theta), \sin(\zeta + \theta)).$$

Thus, the product $wz = (u,v)(x,y)$ is the complex number with modulus $|wz| = rs = |w| \cdot |z|$ and argument $\operatorname{Arg}(wz) = \zeta + \theta = \operatorname{Arg}(w) + \operatorname{Arg}(z)$.

EXAMPLE 6. With $z = (\sqrt{3}, 3)$ and $w = (\sqrt{3}, 1)$ the product becomes

$$zw = (\sqrt{3},3)(\sqrt{3},1) = (\sqrt{3}\sqrt{3} - 3 \times 1, \sqrt{3} \times 1 + 3 \times \sqrt{3}) = (0, 4\sqrt{3}).$$

Verify that $|zw| = |z| \cdot |w|$ and that $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$:
$|zw| = |(0, 4\sqrt{3})| = 4\sqrt{3} = 2\sqrt{3} \times 2 = |(\sqrt{3},3)| \cdot |(\sqrt{3},1)| = |z| \cdot |w|$, ✓
$\operatorname{Arg}(zw) = \operatorname{Arg}(0, 4\sqrt{3}) = \pi/2 = \pi/3 + \pi/6 = \operatorname{Arg}(\sqrt{3},3) + \operatorname{Arg}(\sqrt{3},1) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$. ✓

With its addition and its multiplication, the set $\mathbb{C}$ forms a **number field** (also abbreviated as a **field**), which means that the operations of addition and multiplication satisfy the properties listed in table 1, the proof of which follows through straightforward algebra by expanding and comparing both sides of each equation.

Table 1. The algebraic properties of the field of complex numbers, $\mathbb{C}$.
The following properties hold for all complex numbers (u,v), (x,y), and (p,q).

Associativity of +	$((u,v)+(x,y))+(p,q) = (u,v)+((x,y)+(p,q))$
Commutativity of +	$(u,v)+(x,y) = (x,y)+(u,v)$
Identity (0) for +	$(x,y)+(0,0) = (x,y) = (0,0)+(x,y)$
Inverse for +	$(x,y)+(-x,-y) = (0,0) = (-x,-y)+(x,y)$
Associativity of ×	$((u,v)(x,y))(p,q) = (u,v)((x,y)(p,q))$
Commutativity of ×	$(u,v)(x,y) = (x,y)(u,v)$
Identity (1) for ×	$(x,y)(1,0) = (x,y) = (1,0)(x,y)$
Inverse for ×	If $(x,y) \neq (0,0)$ then $(x,y)(x/(x^2+y^2), -y/(x^2+y^2)) = (1,0)$
Distributivity of × over +	$(u,v)((x,y)+(p,q)) = ((u,v)(x,y)) + ((u,v)(p,q))$

REMARK 2. (INCLUSION OF $\mathbb{R}$ INTO $\mathbb{C}$) The set of all complex numbers with second coordinate equal to zero, of the form $(x,0)$, satisfies all the algebraic rules of the real numbers with ordinary addition and multiplication. Thus,

$$(x,0) + (u,0) = (x+u, 0) \quad \text{and} \quad (x,0)(u,0) = (xu, 0).$$

Consequently, we often identify the set $\{(x,0) : x \in \mathbb{R}\} \subset \mathbb{C}$ with $\mathbb{R}$. For instance, we identify $(1,0)$ with 1 and $(0,0)$ with 0. With such an identification, $1 \times (p,q)$ means $(1,0) \times (p,q)$ and, similarly, $0 + (p,q)$ means $(0,0) + (p,q)$.

REMARK 3. (HISTORY) According to van der Waerden's *A History of Algebra* [39], pages 47–52, one of the first conceptions of complex numbers arose in attempts to solve polynomial equations in finance. At the time (in the fifteenth century), complex numbers were considered to be "imaginary," before the availability of any formal definition. For such historical reasons, the second coordinate y of a complex number $z = (x, y)$ is called the **imaginary part** (or the **complex part**) of z, abbreviated by $y = \mathrm{Im}(z)$, and the first coordinate, x, is called the **real part** of z, abbreviated by $x = \mathrm{Re}(z)$. Remember, however, that the "imaginary" part is also a "real" number.

EXAMPLE 7. For $z = (1, 2)$, the real part is $\mathrm{Re}(1, 2) = 1$ and the complex part is $\mathrm{Im}(1, 2) = 2$.

REMARK 4. (COMPLEX ARITHMETIC WITH SUPERCALCULATORS) The HP-28 and HP-48 supercalculators (as their predecessor, the HP-15C) can perform all the operations defined in the present subsection, as explained in table 2.

Table 2. Complex arithmetic with the HP-28 and HP-48.

Keys	Comments	Display
`(1,2)` [ENTER] `(3,4)` [+]	Add two complex numbers.	`(4,6)`
`(1,2)` [ENTER] `(3,4)` [×]	Multiply two complex numbers.	`(-5,10)`
`(1,2)` [ENTER] `(3,4)` [÷]	Divide two complex numbers.	`(.44,.08)`
`(3,4)` [ENTER] `ABS`	Compute the modulus.	`5`
`(-1,0)` [ENTER] `ARG`	Compute the principal argument.	`3.14159...`
`(1,2) RE`	Extract the real part.	`1`
`(1,2) IM`	Extract the imaginary part.	`2`
`(3,4) C→R`	Extract both coordinates.	`3` `4`
`5` [ENTER] `6 R→C`	Build a complex number.	`(5,6)`
`(1,1) R→P`	(HP-28) Convert to polar form.	`(1.4142...,.7853...)`
`(1,1)` [ENTER] `POLAR`	(HP-48) Convert to polar form.	`(1.4142...,∡.7853...)`

REMARK 5. (LOCATION OF COMMANDS) On the HP-28C&S, the commands `ABS`, `ARG`, `RE`, `IM`, `R→C`, `C→R`, and `R→P` are in the `CMPLX` menu.

On the HP-48SX, the commands `ABS`, `ARG`, `RE`, and `IM` are in the `PARTS` submenu of the `MTH` menu, the commands `R→C` and `C→R` are in the `OBJ` submenu of the `PRG` menu, and the command `POLAR` is available on the keyboard with the blue (right-hand) shift key.

To get arguments in radians, set your supercalculator in *Radians* mode by means of the `RAD` command in the `MODES` menu.

Exercises

Routine exercises

Exercise 1. Perform the following operations (which illustrate the properties listed in table 1) either by hand or with a supercalculator.

(1.1) $(0,0)+(2,3)=$
(1.2) $(2,3)+(0,0)=$
(1.3) $(1,0)\times(2,3)=$
(1.4) $(2,3)\times(1,0)=$
(1.5) $(2,3)+(4,5)=$
(1.6) $(4,5)+(2,3)=$
(1.7) $(2,3)\times(4,5)=$
(1.8) $(4,5)\times(2,3)=$
(1.9) $(2,3)+(-2,-3)=$
(1.10) $(-2,-3)+(2,3)=$
(1.11) $(2,3)\times({}^{2}\!/_{13},-{}^{3}\!/_{13})=$
(1.12) $({}^{2}\!/_{13},-{}^{3}\!/_{13})\times(2,3)=$
(1.13) $((2,3)+(4,5))+(6,7)=$
(1.14) $(2,3)+((4,5)+(6,7))=$
(1.15) $((2,3)(4,5))(6,7)=$
(1.16) $(2,3)((4,5)(6,7))=$
(1.17) $(2,3)((4,5)+(6,7))=$
(1.18) $((2,3)(4,5))+((2,3)(6,7))=$
(1.19) $(-1,0)\times(2,3)=$
(1.20) $(2,3)\times(-1,0)=$
(1.21) $(1,0)\times(1,0)=$
(1.22) $(0,1)\times(0,1)=$

The following exercises involve only straightforward computational proofs.

Exercise 2. For each complex number $z=(x,y)$ denote by $\bar{z}$ the complex number $\bar{z}=(x,-y)$, called the **complex conjugate** of z. Applying the definition of complex addition and multiplication, express the following three results in terms of x and y.

(2.1) $z+\bar{z}=(?,?)$ (2.2) $z-\bar{z}=(?,?)$ (2.3) $z\times\bar{z}=(?,?)$

Exercise 3. Prove that if a complex number z has modulus r and argument θ, then z^2 has modulus r^2 and argument 2θ. Thus, $|z^2|=|z|^2$ and $\mathrm{Arg}(z^2)=2\mathrm{Arg}(z)$.

Exercise 4. Prove the reverse triangle inequality: for all w and z in $\mathbb{C}$,

$$||z|-|w||\leq|z-w|.$$

Exercise 5. For the purpose of this exercise, let

$$\mathrm{sign}(y)=\begin{cases}1 & \text{if } y\geq 0,\\ -1 & \text{if } y<0.\end{cases}$$

Moreover, define a complex square root

$$\sqrt{\ }:\mathbb{C}\to\mathbb{C},\ z=(x,y)\mapsto\sqrt{z}=w=(u,v)$$

by the formulae [obtained by solving $(x,y)=(u,v)^2=(u^2-v^2,2uv)$ for u and v]

$$u=\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\quad\text{and}\quad v=\mathrm{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}.$$

Prove that $w^2 = z$, that is, $(u,v)(u,v) = (x,y)$.

Exercise 6. For each complex number $z \in \mathbb{C}$ denote by w any square root of z. In other words, let w represent any complex number such that $w^2 = z$.

(6.1) Prove that $-w$ is also a square root of z; thus, $(-w)^2 = z$.

(6.2) Prove that there exists no square root of z other than w and $-w$.

Exercise 7. Prove that if $z \in \mathbb{C}$ and if $w = \sqrt{z}$, with the complex square root defined in exercise 5, then $|w| = \sqrt{|z|}$ (the ordinary real square root) and $\operatorname{Arg}(w) = \operatorname{Arg}(z)/2$.

Exercise 8. Either by hand or with the square-root key on a supercalculator, compute the square roots of the following complex numbers:

(8.1) $z = -1$; (8.2) $z = -4$; (8.3) $z = i = (0,1)$.

Exercise 9. Prove the complex quadratic formula: for all complex coefficients (complex numbers) $a, b, c \in \mathbb{C}$ with $a \neq 0$, all the solutions of the equation

$$az^2 + bz + c = 0$$

are

$$z_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad z_- = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Exercise 10. Determine all the complex solutions of the equation $z^2 + 6z + 25 = 0$.

1.2. Sequences of complex numbers

Many applications of complex numbers involve not only a few complex numbers but infinite sequences of complex numbers, usually denoted by $(z_k)_{k=0}^{\infty}$, or $(w_j)_{j=1}^{\infty}$, or simply (z_k) for brevity. (Such notation as $\{z_k\}$ is also common for complex sequences, but we shall also need some notation for the singleton [set] consisting of only one number, z_k, and the *only* accepted notation for such a set is $\{z_k\}$; thus, the notation (z_k) for sequences avoids a confusion with sets.) A **sequence** of complex numbers is a function $s : \mathbb{N} \to \mathbb{C}$, with values usually denoted by z_k instead of the functional notation $s(k)$. The study of sequences of complex numbers begins with the question of whether a sequence remains bounded, diverges to infinity, converges to a limit, or none of these.

Definition 5. A subset $Z \subset \mathbb{C}$ is **bounded** if, but only if, there exists a non-negative real number B such that

$$|z| \leq B$$

for every element $z \in Z$. Of course, a subset $Z \subset \mathbb{C}$ is **unbounded** if, but only if, it is not bounded. Similarly, a sequence (z_k) in $\mathbb{C}$ is **bounded** if, but only if, the set of all its elements, $Z = \{z_k : k \in \mathbb{N}\}$, is bounded.

Definition 6. A sequence (z_k) in $\mathbb{C}$ **diverges to infinity** if, but only if,

$$\lim_{k \to \infty} |z_k| = \infty.$$

Since the limit involves only real numbers, because $|z_k| \in \mathbb{R}_+$, the limit means that for each real number $B \in \mathbb{R}$ there exists an index $K_B \in \mathbb{N}$ such that if $k > K_B$ then $|z_k| > B$. Informally, the definition means that we may arrange that all moduli $|z_k|$ be as large as we want, provided that we exclude the first K_B terms.

REMARK 6. An unbounded sequence need not diverge to infinity.

EXAMPLE 8. Let $z_k = k(1+(-1)^k)$ for each $k \in \mathbb{N}$. Then

$$z_k = k\left(1+(-1)^k\right) = \begin{cases} 2k & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd .} \end{cases}$$

Thus, (z_k) neither remains bounded nor diverges to infinity.

Definition 7. A sequence (z_k) in $\mathbb{C}$ **converges** (or **tends**) **to a limit** $z \in \mathbb{C}$ if, but only if, for each positive tolerance (number) t there exists an index K_t such that if $k > K_t$ then $|z_k - z| < t$.

Informally, the definition means that we may arrange that all the terms z_k be as close as we want to z, provided that we exclude the first K_t terms.

Proposition 1. *Every converging sequence remains bounded.*

Proof. Suppose that a sequence (z_k) in $\mathbb{C}$ converges to a limit $z \in \mathbb{C}$. The definition of convergence applied to $t = 1$ (or any other positive number) ensures the existence of an index K_1 such that if $K_1 < k$ then $|z_k - z| < 1$. For such indices $k > K_1$ the triangle inequality shows that

$$|z_k| = |z_k - z + z| \leq |z_k - z| + |z| < 1 + |z|.$$

For the other indices ($k \leq K_1$), of which there are only finitely many, let

$$B_0 = \max\left\{|z_k| : k \in \{0, \ldots, K_1\}\right\}.$$

(The notation max means the maximum element in the set following max; thus, B_0 represents the largest of $|z_0|$, $|z_1|$, ..., $|z_{K_1}|$.) Hence, $|z_k| \leq B_0$ for every $k \leq K_1$. Finally, let

$$B = \max\{B_0, 1 + |z|\},$$

which ensures that $|z_k| \leq B$ for every term z_k in the sequence (z_k). □

REMARK 7. A bounded sequence need not converge to any limit.

EXAMPLE 9. The sequence (z_k) defined by $z_k = (-1)^k$ remains bounded (it alternates between one and negative one) but it does not converge.

2. FILLED JULIA SETS

The "filled Julia set" associated with a complex function determines whether an initial point $z_0 \in \mathbb{C}$ gives rise to a bounded or an unbounded sequence (z_k) through iteration of the function under consideration, as explained in detail below. To avoid unnecessary technical difficulties and to focus on the nature of filled Julia sets, the present chapter restricts itself to those complex functions defined by quadratic polynomials.

2.1. The concept of filled Julia set

Notation. For each complex number $c \in \mathbb{C}$ let f_c denote the function

$$f_c : \mathbb{C} \to \mathbb{C}, \text{ with } f_c(z) = z^2 + c$$

EXAMPLE 10. If $c = 0$ then $f_0(z) = z^2 + 0 = z^2$. Similarly, if $c = i = (0, 1)$ then $f_i(z) = z^2 + i$.

Definition 8. For each natural number $n \in \mathbb{N}$, the **n-th iteration of** f_c is the function $f_c^{\circ n}$ defined inductively by

$$\begin{aligned}
&f_c^{\circ 0}(z) = z,\\
&f_c^{\circ 1}(z) = f_c(z) = z^2 + c,\\
&f_c^{\circ 2}(z) = f_c\left(f_c(z)\right) = f_c\left(z^2 + c\right) = \left(z^2 + c\right)^2 + c,\\
&f_c^{\circ 3}(z) = f_c\left(f_c\left(f_c(z)\right)\right) = \left((z^2 + c)^2 + c\right)^2 + c,\\
&\vdots\\
&f_c^{\circ n}(z) = f_c\left(f_c^{\circ n-1}(z)\right).
\end{aligned}$$

Next, for each complex number z_0, define a sequence (z_n) by $z_n = f_c^{\circ n}(z_0)$. Thus,

$$\begin{aligned}
&z_1 = f_c(z_0) = z_0^2 + c,\\
&z_2 = f_c^{\circ 2}(z_0) = f_c\left(f_c(z_0)\right) = z_1^2 + c = \left(z_0^2 + c\right)^2 + c,\\
&z_3 = z_2^2 + c,\\
&\vdots\\
&z_{n+1} = z_n^2 + c,\\
&\vdots
\end{aligned}$$

By induction, the formula $z_{n+1} = z_n^2 + c$ specifies the entire sequence (z_n). Remember that each complex number z_n in the sequence depends not only upon n but also upon c and z_0, but the notation omits such references for simplification.

EXAMPLE 11. For $c = -1$ the polynomial f_c takes the values $f_{-1}(z) = z^2 - 1$. If $z_0 = 1$ then the sequence (z_n) becomes $1, 0, -1, 0, -1, 0, -1, \ldots$, because

$$z_0 = 1, \quad z_1 = z_0^2 + c = 1^2 - 1 = 0, \quad z_2 = z_1^2 + c = 0^2 - 1 = -1, \quad z_3 = z_2^2 + c = (-1)^2 - 1 = 0,$$

and so forth. Thus, with $z_0 = 1$, the sequence (z_n) alternates between 0 and -1 forever; in particular, the sequence (z_n) remains bounded. By contrast, if $z_0 = 2$ then

$$z_0 = 2, \quad z_1 = z_0^2 + c = 2^2 - 1 = 3, \quad z_2 = z_1^2 + c = 3^2 - 1 = 8, \quad z_3 = z_2^2 + c = 8^2 - 1 = 63,$$

and so forth: with $z_0 = 2$, the sequence (z_n) diverges to infinity (as proved below).

Definition 9. The **filled Julia set** of f_c is the set K_c of all complex numbers z_0 for which the sequence $(z_n) = (f_c^{\circ n}(z_0))$ remains bounded.

EXAMPLE 12. For the particular value $c = 0$, the quadratic polynomial $f_c = f_0$ is the squaring function, $f_0 : \mathbb{C} \to \mathbb{C}$, with $f_0(z) = z^2$. Consequently,

$$\begin{aligned}
z_1 &= f_0(z_0) = z_0^2, \\
z_2 &= f_0(z_1) = z_1^2 = \left(z_0^2\right)^2 = z_0^4, \\
z_3 &= f_0(z_2) = z_2^2 = \left(z_4\right)^2 = z_0^8, \\
&\vdots \\
z_n &= (z_0)^{(2^n)}, \\
&\vdots
\end{aligned}$$

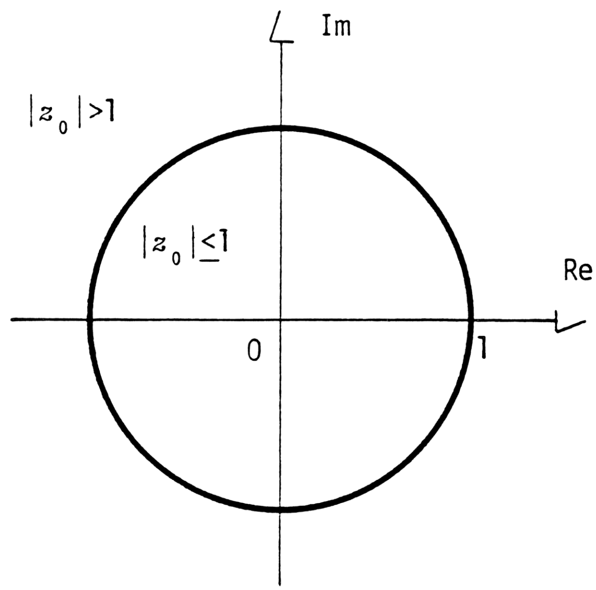


Figure 3. The sequence $(z_n) = (f_c^{\circ n}(z_0))$ diverges to infinity if $|z_0| > 1$ and it remains bounded if $|z_0| \leq 1$.

Thus, $|z_n| = \left|(z_0)^{(2^n)}\right| = |z_0|^{(2^n)}$, from which two cases emerge:

$$\begin{cases}
\text{If } |z_0| > 1 \text{ then } |z_n| = |z_0|^{(2^n)} \text{ diverges to infinity.} \\
\text{If } |z_0| \leq 1 \text{ then } |z_n| = |z_0|^{(2^n)} \leq 1^{(2^n)} = 1 \text{ remains bounded.}
\end{cases}$$

The preceding considerations prove that the filled Julia set K_0 of the squaring function $f_0 : \mathbb{C} \to \mathbb{C}$, $f_0(z) = z^2$ consists of all points at a distance of at most one from the origin, as in figure 3:

$$K_0 = \{z_0 : z_0 \in \mathbb{C} \text{ and } |z_0| \leq 1\}.$$

The example of the squaring function, with $c = 0$, required only basic inequalities with moduli to determine its filled Julia sets; unfortunately, all values of c other than $c = 0$ and $c = -2$ (examined in example 21 in the fourth section) require substantial analysis for determining the nature of the filled Julia set. Fortunately, a few partial results that involve only some algebra and inequalities provide some information about the location and size of the filled Julia set. For instance, the most elementary such result guarantees that each filled Julia set K_c is bounded.

Theorem 1. *For each* $c \in \mathbb{C}$ *let* $R_c = \max\{2, |c|\}$ *(the larger of* 2 *and* $|c|$*). Then the filled Julia set* K_c *of the quadratic polynomial* f_c *is bounded by* R_c*, in the sense that if* $z_0 \in K_c$ *then* $|z_0| \leq R_c$*. Thus, no point in* K_c *may have a magnitude greater than* R_c*.*

Proof. (The present proof lacks elegance, but it may inspire you for exercise 12, at the end of this section, and for the project on Mandelbrot's set, at the end of the chapter.) Suppose first that $|c| \leq 2$; in this case, $R_c = \max\{2, |c|\} = 2$. To show that K_c is bounded, consider a point $z_0 \in \mathbb{C}$ such that $|z_0| > R_c$ and prove that z_0 gives rise to an unbounded sequence, as follows. Set $d = |z_0| - 2 > 0$, so that $|z_0| = 2 + d$, and estimate $|z_1|$:

$$|z_1| = |z_0^2 + c| \geq \big||z_0|^2 - |c|\big| = \big|(2+d)^2 - |c|\big)$$
$$= \big|4 + 2d + d^2 - |c|\big| = \big|2 + 2d + d^2 + (2 - |c|)\big| > 2 + 2d.$$

Thus, if $|z_0| = 2 + d$, then $|z_1| > 2 + 2d$. By induction, the same calculation shows that $|z_n| > 2 + 2^n d$, which diverges to infinity.

If $|c| > 2$ then $R_c = \max\{2, |c|\} = |c|$ but a similar argument leads to the same conclusion. If $|z_0| > R_c = |c|$, set $d = |z_0| - |c|$, so that $|z_0| = |c| + d$. Hence,

$$|z_1| = \big|z_0^2 + c\big| \geq \big||z_0|^2 - |c|\big| = \big|(|c| + d)^2 - |c|\big|$$
$$= \big||c|^2 + 2d|c| + d^2 - |c|\big| = \big||c|(|c| - 1) + 2d|c| + d^2\big| > |c|(2 - 1) + 4d = |c| + 4d.$$

Thus, if $|z_0| = |c| + d$ then $|z_1| > |c| + 4d$. By induction, the same calculation shows that $|z_n| > |c| + 4^n d$, which diverges to infinity. Consequently, if $|z_0| > R_c$ then $z_0 \notin K_c$, and, by contraposition, if $z_0 \in K_c$ then $|z_0| \leq R_c$. □

Notice that the theorem asserts only that *if* $z_0 \in K_c$ *then* $|z_0| \leq R_c$; it does not assert the converse, because there may exist points z_0 such that $|z_0| \leq R_c$ but $z_0 \notin K_c$: not all complex numbers with a magnitude of at most R_c need be in the filled Julia set. In other words, the filled Julia set K_c may be a *proper* subset of the disc with radius R_c and center at the origin.

REMARK 8. (A BETTER BOUND) The preceding theorem yields the standard estimate found in texts, $R_c = \max\{2, |c|\}$ (see the chapter by Bodil Branner, "The Mandelbrot Set," in the book edited by Devaney and Keen [32], page 80). Such an estimate need not be optimal, in the sense that the filled Julia set K_c may be smaller than the bound R_c. For instance, exercise 12 at the end of the present subsection shows that

$$r_c = \frac{1 + \sqrt{1 + 4|c|}}{2}$$

provides a better bound for K_c, because $r_c \leq R_c$ and, still, if $z_0 \in K_c$ then $|z_0| \leq r_c$.

The availability of a bound for filled quadratic Julia sets suggests a crude algorithm for determining whether a point $z_0 \in \mathbb{C}$ lies in K_c.

Direct Iteration Method for filled quadratic Julia sets. For each $c \in \mathbb{C}$, the following algorithm generates points in the filled Julia set K_c of the quadratic polynomial $f_c : \mathbb{C} \to \mathbb{C}$ with $f_c(z) = z^2 + c$.

Step 1. Choose a maximum number of iterations $N \in \mathbb{N}$, for instance, $N = 12$, depending upon time and accuracy (see Strang's text [40], page 510).

Step 2. Choose a bound for the filled Julia set K_c, for instance

$$r_c = \frac{1 + \sqrt{1 + 4|c|}}{2}.$$

Step 3. Color white all points $z_0 \in \mathbb{C}$ such that $|z_0| > r_c$: such points do not belong to K_c.

Step 4. Test each point z_0 such that $|z_0| \leq r_c$ to determine whether $z_0 \in K_c$, as follows. Start computing elements of the sequence

$$z_1 = z_0^2 + c, \quad z_2 = z_1^2 + c, \quad z_3 = z_2^2 + c, \quad \ldots, \quad z_n = z_{n-1}^2 + c, \quad \ldots$$

and stop if either of the following conditions holds.

(∞) If at some stage $n \in \{1, \ldots, N\}$ the sequence returns an element z_n such that $|z_n| > r_c$, then the sequence diverges to infinity (see exercise 12.2); consequently, color the initial z_0 white. By symmetry (see exercise 11), also color $-z_0$ white. Stop the iterations and select another z_0.

(K_c?) If $n = N$ (the maximum allowed number of iterations) and if $|z_n| \leq r_c$ for all the computed elements $z_0, \ldots, z_N$ then we do not yet know whether the sequence diverges or remains bounded; nevertheless, color the initial z_0 black (it may belong to the filled Julia set K_c). Also color $-z_0$ black.

Exhibit 2 shows implementations of the Direct Iteration Algorithm on the HP-28(C or S) and HP-48, with an example.

Despite its simplicity, the algorithm just presented suffers from the disadvantage that while it can assert that an initial point z_0 gives rise to a diverging sequence, it cannot assert that a point z_0 belongs to K_c. Thus, it colors all points z_0 for which it cannot make a decision. Another algorithm, presented in the next section, provides a remedy for this disadvantage by producing only points in K_c.

Filled Julia sets on the HP-28C&S

```
« CLLCD
FOR y
 r NEG r
 FOR x x y
  R→C 1 12
  START
   SQ c +
  NEXT
  IF ABS r ≤
   THEN x y
   R→C PIXEL
  END h
 STEP h
STEP LCD→
»         ENTER
   'scan' STO
-------------
« c ABS 4 *
1 + √ 1 + 2
/ 'r' STO r
32 / 'h' STO
h 137 * 2 /
's' STO
s r R→C PMAX
s NEG 0 R→C
PMIN
»         ENTER
  'setup' STO
-------------
« setup 0 r
scan s h NEG
R→C PMAX
s NEG r h +
NEG R→C PMIN
r NEG h NEG
scan
»         ENTER
   'fill' STO
```

TUTORIAL

```
.25  'c' STO
 USER  fill
```

Comments on the programs for filled Julia sets

Subroutine `setup` computes a bound for the filled Julia set,

$$r = r_c = \frac{1 + \sqrt{1 + 4|c|}}{2},$$

and adjusts the screen so that it covers the square $[-r, r] \times [-r, r]$, hence also the filled Julia set; h equals the width of each pixel.

`ENTER` and `STO`re in '`setup`'

For the HP-28C&S only, subroutine `scan` examines each pixel in either the upper or the lower half of the square $[-r, r] \times [-r, r]$; `scan` and `fill` perform the same task as `fill` does on the HP-48SX.

`ENTER` and `STO`re in '`scan`'

Program `fill` tests each pixel $z_0 = (x, y)$ in the square $[-r, r] \times [-r, r]$, by iterating

$$f_c(z) = z^2 + c$$

twelve times within the loop

```
START
    SQ c +
NEXT
```

If $|z_{12}| \leq r_c$ then `fill` colors the pixels at z_0 and at $-z_0$ in black.

`ENTER` and `STO`re in '`fill`'

Tutorial:
`STO`re a complex number in '`c`'
execute `fill`
in the `USER` or `VAR` menu.

Exhibit 2.

Filled Julia sets on the HP-48SX

```
« c ABS 4 *
1 + √ 1 + 2
/ 'r' STO r
64 / 'h' STO
ERASE r 128
/ 131 * r
R→C DUP NEG
SWAP PDIM #
131d # 128d
PDIM
»        ENTER
 'setup' STO
------------
« setup 0 r
FOR y
r NEG r
 FOR x x y
  R→C 1 12
  START
   SQ c +
  NEXT
  IF ABS r ≤
   THEN x y
   R→C DUP
   NEG PIXON
   PIXON
  END h
 STEP h
STEP
»        ENTER
  'fill' STO
```

TUTORIAL

```
.25  'c' STO
 VAR fill
```

Exercises

Exercise 11. Prove that every quadratic filled Julia set is symmetric with respect to the origin.

Exercise 12. For each complex number $c \in \mathbb{C}$ let

$$r_c = \frac{1 + \sqrt{1 + 4|c|}}{2}.$$

(12.1) Prove that if $|z_0| > r_c$ then (z_n) diverges to infinity.

(12.2) Prove that regardless of $|z_0|$, if there exists an element z_k such that $|z_k| > r_c$ for some index $k \in \mathbb{N}$ then (z_n) diverges to infinity.

(12.3) Prove that $r_c \leq \max\{2, |c|\}$.

(12.4) Prove that if c is real and non-positive ($c \in]-\infty, 0]$) then r_c provides an optimal bound, in the sense that K_c contains two diametrically opposite points each at distance r_c from the origin.

Exercise 13. Prove that $(z_n) = (f_c^{\circ n}(z_0))$ either remains bounded or diverges to infinity. (In contrast with other complex sequences, [for instance, the sequence in example 8], the sequences (z_n) that arise as iterations of the quadratic functions f_c cannot be unbounded without diverging to infinity.)

2.2. The Fixed-Point Inverse Iteration Method

The Direct Iteration Algorithm demonstrated in the preceding subsection suffered from the disadvantage of not recognizing points in the filled Julia set. Another algorithm, demonstrated in the present subsection, generates points only in the filled Julia set. A further improvement of the new algorithm, explained in the next sections, will generate points only on the "edge" of the filled Julia set, which produces more reliable pictures. Both algorithms rely on the concept of "fixed point."

Definition 10. A **fixed point** of a function $f : \mathbb{C} \to \mathbb{C}$ is a point z_* such that $f(z_*) = z_*$.

EXAMPLE 13. Consider the function $f_{1/4} : \mathbb{C} \to \mathbb{C}$ with $f_{1/4}(z) = z^2 + {}^1\!/_4$. To determine whether $f_{1/4}$ has any fixed point, and, if so, to find such fixed points, solve the equation $f_{1/4}(z) = z$, which, in this particular example, yields

$$\begin{aligned} f_{1/4}(z) &= z, \\ z^2 + {}^1\!/_4 &= z, \\ z^2 - z + {}^1\!/_4 &= 0, \end{aligned}$$

$$z = \frac{-(-1) \pm \sqrt{1^2 - 4 \cdot 1 \cdot {}^1\!/_4}}{2 \cdot 1} = \frac{1 \pm 0}{2} = \frac{1}{2}.$$

Hence, the function $f_{1/4}$ has one fixed point, $z_* = {}^1\!/_2$. To verify this result, observe that $f_{1/4}(z_*) = f_{1/4}({}^1\!/_2) = ({}^1\!/_2)^2 + {}^1\!/_4 = {}^1\!/_4 + {}^1\!/_4 = {}^1\!/_2 = z_*$; thus, $f_{1/4}({}^1\!/_2) = {}^1\!/_2$, ✓

EXAMPLE 14. For each complex number $c \neq {}^1\!/_4$, the quadratic polynomial

$$f_c : \mathbb{C} \to \mathbb{C},\ f_c(z) = z^2 + c$$

has two distinct fixed points,

$$z_+ = \frac{1 + \sqrt{1 - 4c}}{2} \quad \text{and} \quad z_- = \frac{1 - \sqrt{1 - 4c}}{2}.$$

To verify this assertion, substitute z_+ and z_- for z and check that $f_c(z_+) = z_+$ and $f_c(z_-) = z_-$, or apply the quadratic formula, which shows that

$$\begin{array}{lrcl} \text{if} & f_c(z) & = & z, \\ \text{then} & z^2 + c & = & z, \\ \text{hence} & z^2 - z + c & = & 0, \end{array}$$

and

$$z = \frac{-(-1) \pm \sqrt{1^2 - 4 \cdot 1 \cdot c}}{2 \cdot 1} = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

Finally, since $c \neq {}^1\!/_4$, it follows that $1 - 4c \neq 0$ and that the two fixed points z_+ and z_- are distinct ($z_+ \neq z_-$). □

Every fixed point z_* of a function f_c lies in the filled Julia set K_c, because the sequence of iterations (z_n) starting at $z_0 = z_*$ remains at z_* forever: $z_1 = f(z_0) = f(z_*) = z_*$, $z_2 = f(z_1) = f(z_*) = z_*$, and so forth, $z_n = z_*$ for every z_n. Therefore, (z_n) remains bounded. Moreover, fixed points play a crucial rôle in the context of Julia sets, partly because they provide infinitely many points in the filled Julia set.

Proposition 2. *If z_* is a fixed point of the quadratic polynomial f_c then all its preimages under $f_c^{\circ n}$ (all the complex numbers z_0 for which there exists an integer $n \in \mathbb{N}$ such that $z_n = f_c^{\circ n}(z_0) = z_*$) belong to the filled Julia set.*

Proof. Suppose that $z_n = f_c^{\circ n}(z_0) = z_*$. Since z_* is a fixed point of f_c,

$$z_{n+1} = f_c(z_n) = f_c(z_*) = z_*,\ z_{n+2} = f_c(z_{n+1}) = f_c(z_*) = z_*,\ \ldots$$

and all terms beyond z_n equal z_*. Thus, (z_k) remains bounded, hence $z_0 \in K_c$. □

The preceding proposition suggests an algorithm to plot filled Julia sets.

Fixed-Point Inverse Iteration Method. For each $c \in \mathbb{C}$ plot points in the filled Julia set as follows.

Step 1. Determine a fixed point of f_c: choose either of the two fixed points

$$z_* = z_\pm = \frac{1 \pm \sqrt{1-4c}}{2}.$$

Step 2. Compute a preimage z_{-1} of the fixed point z_* by solving

$$\begin{aligned} f_c(z_{-1}) &= z_*, \\ z_{-1}^2 + c &= z_*; \end{aligned}$$

thus, chose either square root,

$$z_{-1} = \pm\sqrt{z_* - c},$$

and plot the point z_{-1}.

Step 3. Compute a preimage of the point just obtained by solving

$$\begin{aligned} f_c(z_{-2}) &= z_{-1}, \\ z_{-2}^2 + c &= z_{-1}, \end{aligned}$$

which gives

$$z_{-2} = \pm\sqrt{z_{-1} - c},$$

and plot z_{-2}.

Continue in this fashion until enough points appear on the plot.

The experiments in the exercises reveal that all the preimages of a fixed point belong to the filled Julia set, but that they may fail to yield a satisfactory picture of the whole set: the preimages may stay confined in a subset so small that they do not oultine a representational picture of the filled Julia set. A slight variant of the inverse iteration method remedies this problem, but an explanation of that improved method requires concepts from topology, reviewed in the next section. For this reason, if you want to try a program at this point, then you may try the Non-Attracting Fixed-Point Inverse Iteration Method, as demonstrated in exhibit 4, page 33, in the fourth section.

EXAMPLE 15. Suppose that $c = 1/4$.

Step 1. Determine the fixed points by solving $f_{1/4}(z_*) = z_*$, which gives

$$z_*^2 + 1/4 = z_*,$$

from which it follows that $f_{1/4}$ has only one fixed point, $z_* = 1/2$. Plot z_*.

Step 2. Find a preimage of the fixed point by solving

$$\begin{aligned} f_{1/4}(z_{-1}) &= z_*, \\ z_{-1}^2 + 1/4 &= 1/2, \end{aligned}$$

$$z_{-1} = \pm\sqrt{1/2 - 1/4} = \pm\sqrt{1/4} = \pm 1/2.$$

Notice that one of the two solutions, $1/2$, coincides with the preceding point, z_*. Consequently, select the other solution, $z_{-1} = -1/2$, and plot z_1.

Step 3. Compute a preimage of z_{-1} by solving

$$\begin{aligned} f_{1/4}(z_{-2}) &= z_{-1}, \\ z_{-2}^2 + 1/4 &= -1/2, \end{aligned}$$

$$z_{-2} = \pm\sqrt{-1/2 - 1/4} = \pm\sqrt{-1}\sqrt{3/4} = \pm i\sqrt{3}/2.$$

Choose either solution randomly, for instance, $z_{-2} = i\sqrt{3}/2$, and plot it.

Step 4. Compute a preimage z_{-3} of z_{-2}, for instance

$$z_{-3} = -\sqrt{z_{-2} - c} = -\sqrt{i\sqrt{3}/2 - 1/4},$$

and plot it, and so forth.

Exhibit 3 displays a picture generated by the Fixed-Point Inverse Iteration Method implemented on the HP-48SX (see exhibit 4, page 33). Observe that all the points appeared to lie on the "edge" of the filled Julia set, a phenomenon explained in the fourth section.

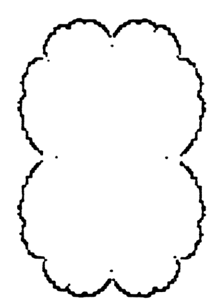


Exhibit 3.

Exercises

Experiments and conjectures

Exercise 14. The present exercise investigates the filled Julia set K_{-2}.

(14.1) Determine the two fixed points of $f_{-2} : \mathbb{C} \to \mathbb{C}$, $f_{-2}(z) = z^2 - 2$.

(14.2) For each fixed point, compute the first three preimages, z_{-1}, z_{-2}, and z_{-3}, as demonstrated in example 15.

(14.3) Prove that if $-2 \leq z_{-n} \leq 2$ then $-2 \leq z_{-(n+1)} \leq 2$ (with $z_{-(n+1)}$ representing any solution of $f_{-2}(z_{-(n+1)}) = z_{-n}$). Conclude that if the Fixed-Point Inverse Iteration Method starts from either fixed point of f_{-2}, then all the points that it generates lie in the interval $[-2, 2]$.

We shall prove later that the filled Julia set K_{-2} is indeed the closed interval $[-2, 2]$, from -2 through 2. However, the following two exercises illustrate how the choice of the fixed point may influence the quality of the picture produced by the Fixed-Point Inverse Iteration Method. These exercises also warn against quick unverified conjectures.

Exercise 15. Let $c = 0$ and consider the squaring function, with $f_0(z) = z^2$.

(15.1) Determine the two fixed points of the squaring function, f_0.

(15.2) For each fixed point of f_0, compute the first few preimages as explained in example 15 with one modification: at each step, keep *both* preimages, $\sqrt{z_{-n} - c}$ and $-\sqrt{z_{-n} - c}$. Compare the plot resulting from one fixed point with the plot resulting from the other fixed point.

Exercise 16. With $c = -3/4$ let $f_{-3/4} : \mathbb{C} \to \mathbb{C}$, $f_{-3/4}(z) = z^2 - 3/4$.

(16.1) Verify that $z_* = -1/2$ is a fixed point of $f_{-3/4}$, and also determine the other fixed point.

(16.2) Start from the fixed point, $z_* = -1/2$, and apply the Fixed-Point Inverse Iteration Method as explained in example 15, with one modification: select the ***positive*** square root at each step, so that

$$z_0 = -1/2,\ z_{-1} = +\sqrt{-1/2 - (-3/4)} = 1/2,\ z_{-2} = +\sqrt{z_{-1} - (-3/4)},\ \ldots$$

To what limit z does the sequence of preimages (z_{-n}) appear to converge?

(16.3) Prove your conjecture (prove that $\lim_{n\to\infty}(z_{-n}) = z$).

(16.4) Start from the other fixed point and plot the resulting sequence (z_{-n}).

3. COMPLEX ANALYSIS AND TOPOLOGY

3.1. The topology of the complex numbers

The concept of limit applies not only to sequences, which are functions from $\mathbb{N}$ to $\mathbb{C}$, but also to functions from $\mathbb{C}$ to $\mathbb{C}$.

Definition 11. Let $f : D \to \mathbb{C}$ be a function defined on some subset D of the complex plane. Also, let $z_0 \in D$ represent a point in the domain, and let $w_0 \in \mathbb{C}$. Then $f(z)$ **tends to** w_0 **as** z **tends to** z_0 if, but only if, for each positive tolerance $t \in \mathbb{R}_+^*$ there exists a positive distance (number) $d \in \mathbb{R}_+^*$ such that the following condition holds:

If $|z - z_0| < d$, then $z \in D$ and $|f(z) - w_0| < t$.

The situation just described is abbreviated by the notation

$$\lim_{z \to z_0} f(z) = w_0.$$

Informally, the definition means that (1) the domain contains a disc with positive radius and centered at z_0, and (2) we may arrange that $f(z)$ lie as close as we want to w_0, specifically, within any small distance t, provided that we impose upon z the condition that it lie sufficiently close to z_0, specifically, within some distance d. The concept of limit of complex functions yields the concept of continuity.

Definition 12. A function $f : D \to \mathbb{C}$ is **continuous** at a point $z_0 \in D \subset \mathbb{C}$ if, but only if,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

The function f is **continuous on a subset** $S \subset D$ if, but only if, the function f is continuous at each point $z_0 \in S$.

EXAMPLE 16. Let c denote any complex number and consider the function

$$f_c : \mathbb{C} \to \mathbb{C}, \; f_c(z) = z^2 + c.$$

Such complex functions will give rise to fractal Julia sets in the next sections. The function f_c is continuous at each $z_0 \in \mathbb{C}$, as the following argument shows. Firstly, the difference $f_c(z) - f_c(z_0)$ factors,

$$f_c(z) - f_c(z_0) = \left(z^2 + c\right) - \left(z_0^2 + c\right) = z^2 - z_0^2 = (z - z_0)(z + z_0). \tag{1}$$

Secondly, restrict the argument to those points $z \in \mathbb{C}$ at distance less than one from z_0; in other words, assume that $|z - z_0| < 1$. For such points z, the triangle inequality gives

$$|z| = |z - z_0 + z_0| \leq |z - z_0| + |z_0| < 1 + |z_0|,$$

and, consequently,

$$|z + z_0| \leq |z| + |z_0| < (1 + |z_0|) + |z_0| = 1 + 2|z_0|. \tag{2}$$

Substituting this result into the factorization (1) of $f(z) - f(z_0)$ yields

$$|f(z) - f(z_0)| = |z - z_0| \cdot |z + z_0| < |z - z_0| \cdot (1 + 2|z_0|). \tag{3}$$

Next, for each positive tolerance (real number) $t > 0$ let

$$d = \min\left\{1, \frac{t}{1 + 2|z_0|}\right\}$$

(the smaller of 1 and $t/(1 + 2|z_0|)$). If $|z - z_0| < d$ then $|z - z_0| < d \leq 1$ and the estimate (2) holds, and, consequently, (3) holds. Therefore,

$$\begin{aligned} |f(z) - f(z_0)| = |z - z_0| \cdot |z + z_0| < |z - z_0| \cdot (1 + 2|z_0|) < d(1 + 2|z_0|) \\ \leq \frac{t}{1 + 2|z_0|}(1 + 2|z_0|) = t, \end{aligned}$$

which means that f_c is continuous at z_0. □

The concepts of convergence, limit, and continuity may also be expressed in terms of "open sets," as demonstrated below.

Definition 13. The **open disc** with radius r and center at z is the set of all complex numbers at distance less than r from z, a set denoted by $D(z, r)$; thus,

$$D(z, r) = \{w : w \in \mathbb{C} \text{ and } |w - z| < r\}.$$

The **unit disc** is the set $D(0, 1)$, which consists of all complex numbers at distance less than one from the origin.

Besides open discs, which do not include the circle surrounding them, "closed discs," which include the circle surrounding them, also arise in the context of Julia sets.

Definition 14. The **closed disc** with radius r and center z is the set of all complex numbers at distance at most r from z, and denoted by $\overline{D(z, r)}$; thus,

$$\overline{D(z, r)} = \{w : w \in \mathbb{C} \text{ and } |w - z| \leq r\}.$$

Definition 15. A subset $U \subset \mathbb{C}$ is **open** if, but only if, for each point $z \in U$ there exists a positive real number r such that $D(z, r) \subset U$.

Thus, a subset is open if, but only if, it contains an open disc around each of its points.

EXAMPLE 17. Every open disc $D(z, r)$ is an open set. To verify this assertion, let w represent any point in $D(z, r)$; thus, $|w - z| < r$. Next, let $s = (r - |w - z|)/2$,

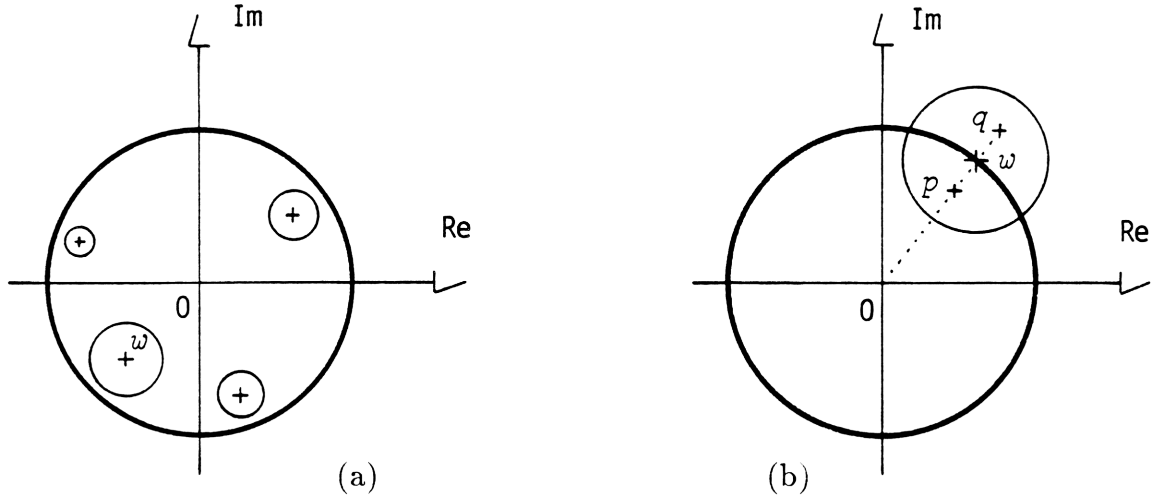


Figure 4. (a) Open discs are open sets. (b) The boundary of a disc is the circle surrounding it.

which represents half the distance from w to the circle surrounding the disc $D(z,r)$. If a complex number $q \in \mathbb{C}$ belongs to the open disc $D(w,s)$, then $|q-w| < s$, and, by the triangle inequality,

$$|q-z| = |q-w+w-z| \leq |q-w|+|w-z| < (r-|w-z|)/2+|w-z| = r-|w-z|/2 < r.$$

Hence, every point q in $D(w,s)$ also lies in the disc $D(z,r)$, which means that $D(w,s) \subset D(z,r)$, as in figure 4a. Therefore, the open disc $D(z,r)$ is an open set. □

The concepts of "open disc" and "open set" form the basis for the concept of "boundary."

Definition 16. The **boundary** of a set $S \subset \mathbb{C}$ is denoted by ∂S and consists of all points $z \in \mathbb{C}$ such that for each positive radius r the open disc $D(z,r)$ contains at least one point in S and at least one point not in S.

Informally stated, the definition means that a point w lies on the boundary of a set S if, but only if, there exist points in S and points outside S as close as we want to w.

EXAMPLE 18. Let $S = \overline{D(0,1)}$ represent the closed unit disc; its boundary $\partial\overline{D(0,1)}$ consists of the **unit circle,** which surrounds it and has radius 1 and center at the origin, 0, as in figure 4b.

To verify this assertion, consider any point w on that circle, at distance 1 from the origin; thus, $|w| = |w-0| = 1$. For each positive radius r consider the open disc $D(w,r)$ with radius r and center at the same point w. Then $D(w,r)$ contains a point p in S and a point q not in S, because we may find such points p and q along the half-line from the origin through w, for instance, $p = (1-[r/2])w$, for which $|p| < 1$, and $q = (1+[r/2])w$, for which $|q| > 1$, as shown in figure 4b

Consequently, for each positive radius r, the disc $D(w,r)$ contains a point p in $\overline{D(0,1)}$ and a point q not in $\overline{D(0,1)}$. Therefore, w lies on the boundary of the unit disc. Conversely, every point on the boundary lies on the unit circle. □

The following proposition describes the relationship between open sets and continuity.

Proposition 3. *A function* $f : D \to \mathbb{C}$ *is continuous at a point* $z \in D \subset \mathbb{C}$ *if, but only if, for each open set* V *containing* $f(z)$ *there exists an open set* U *such that* $z \in U \subset D$ *and* $f(U) \subset V$. *In particular, if the domain* D *is itself open, then the function* f *is continuous on* D *if, but only if, for each open set* $V \subset \mathbb{C}$ *the inverse image* $f^{-1}(V) = \{z : z \in D \text{ and } f(z) \in V\}$ *is open.*

Proof. Suppose that for each open set V containing $f(z)$ there exists an open set U such that $z \in U \subset D$ and such that $f(U) \subset V$. For each positive real number $t > 0$, consider the open set $V = D(f(z), t)$ (the open disc with radius t and center at $f(z)$). Since the corresponding set U described in the hypothesis is open, there exists a positive real number $c > 0$ such that U contains a disc $D(z, c)$. Consequently, again by the hypothesis, $f(D(z,c)) \subset f(U) \subset V = D(f(z), t)$, which means that if $|w - z| < c$ then $w \in D$ and $|f(w) - f(z)| < t$. Thus, f is continuous at z.

Conversely, suppose that f is continuous at a point $z \in D$. The definition of continuity then asserts that for each open set V containing $f(z)$, which contains an open disc of the form $D(f(z), t)$, there exists a positive real number $c > 0$ such that if $|w - z| < c$ then $w \in D$; consequently, $D(z, c) \subset D$, and $|f(w) - f(z)| < t$, and hence $f(D(z,c)) \subset D(f(z), t) \subset V$. Therefore, the set $U = D(z, c)$ satisfies the requirements of the propositions. □

The following proposition describes the relationship between continuous functions and converging sequences.

Proposition 4. *A function* $f : D \to \mathbb{C}$ *is continuous at a point* $z \in D \subset \mathbb{C}$ *if, but only if, for each sequence* (z_k) *in* $\mathbb{C}$ *converging to* z, *the sequence* $(f(z_k))$ *converges to* $f(z)$.

Proof. Suppose that f is continuous at $z \in D$, and consider a sequence (z_k) converging to z. To show that $(f(z_k))$ converges to $f(z)$, let $t > 0$ represent a positive real number. By continuity of f at z, there exists a positive real number $c > 0$ such that $D(z, c) \subset D$ and $f(D(z,c)) \subset D(f(z), t)$. By convergence of the sequence (z_k) to z, there exists an index $K \in \mathbb{N}$ such that if $k > K$ then $|z_k - z| < c$, which means that $z_k \in D(z, c)$; consequently, $f(z_k) \in f(D(z,c)) \subset D(f(z), t)$, which means that $|f(z_k) - f(z)| < t$. Therefore, the sequence $f(z_k)$ converges to $f(z)$.

For the converse, proceed by contraposition: assume that f is not continuous at $z \in D$. Then there exists a positive number $t > 0$ such that for each positive number $c > 0$ there exists a complex number z_c with $|z_c - z| < c$ but $|f(z_c) - f(z)| \geq t$ or $z_c \notin D$. By induction on $k \in \mathbb{N}$ and with $c_k = 1/(k+1)$, the sequence (z_{c_k}) converges to z but $f(z_{c_k})$ does not converge to $f(z)$ (some of the terms $f(z_{c_k})$ may also fail to exist). □

REMARK 9. (This remark serves only to explain the title of the present section.) The collection $\mathcal{T}$ of all open subsets of the complex plane,

$$\mathcal{T} = \{U : U \subset \mathbb{C} \text{ and } U \text{ is open}\},$$

satisfies the following properties, which make $\mathcal{T}$ a **topology** on $\mathbb{C}$:

(1) The empty set, $\emptyset$, and the entire plane, $\mathbb{C}$, both belong to the collection: $\emptyset \in \mathcal{T}$ and $\mathbb{C} \in \mathcal{T}$.
(2) For each family (set) $\mathcal{F} \subset \mathcal{T}$ of open sets, the union of all the open sets in the family is again an open set: $\bigcup \mathcal{F} \in \mathcal{T}$.
(3) For each *finite* family $\mathcal{G} \subset \mathcal{T}$ of open sets, the intersection of all the (finitely many) open sets in the family is again an open set: $\bigcap \mathcal{G} \in \mathcal{T}$.

3.2. Theorems from complex analysis

Most concepts familiar from calculus have their counterparts in complex functions, for instance, the derivative.

Definition 17. The **derivative at a point** $z \in D$ of a function $f : D \subset \mathbb{C} \to \mathbb{C}$ is the following limit (if it exists), denoted by $f'(z)$:

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

The function f is called **complex differentiable** if, but only if, its derivative $f'(z)$ exists at every point $z \in D$.

Notice that the definition of the derivative of a complex function coincides formally with that of the ordinary derivative of a real function. The complex derivative differs from the real derivative only by the fact that all values in the limit (z, $f(z)$, h, and $f(z+h)$) may be complex; in particular, this means that the complex increment h may tend to zero in any fashion in the complex plane. Due to the similarities between complex and real algebra, and between the topologies on the complex plane and the real line, most formulae from ordinary calculus also hold in complex calculus.

EXAMPLE 19. Consider the function $f_c : \mathbb{C} \to \mathbb{C}$ with $f_c(z) = z^2 + c$. Then

$$f_c'(z) = \lim_{h \to 0} \frac{f_c(z+h) - f_c(z)}{h} = \lim_{h \to 0} \frac{[(z+h)^2 + c] - [z^2 + c]}{h}$$

$$= \lim_{h \to 0} \frac{z^2 + 2hz + h^2 - z^2}{h} = \lim_{h \to 0} (2z + h) = 2z.$$

We shall take the following particular cases of theorems from complex analysis for granted. Their proofs require at least a short course in complex analysis, but their statements may be easily understood and are indispensable to the study of Julia sets.

Theorem 2. (Cauchy's Inequality) *Consider a complex differentiable function* $f : D \subset \mathbb{C} \to \mathbb{C}$ *and a closed disc* $\overline{D(w,r)} \subset D$ *contained in the domain. Also, let* $M = \max\{|f(z)| : |z - w| = r\}$ *denote the maximum value of* $|f(z)|$ *over all points* z *on the circle with center* w *and radius* r. *Then*

$$|f'(w)| \leq \frac{M}{r}.$$

Theorem 3. **(Open Mapping Theorem)** *Let* $f : \mathbb{C} \to \mathbb{C}$ *represent a* non-constant *function defined and complex differentiable on the whole plane. Then for each open subset* $U \subset \mathbb{C}$ *the image* $f(U)$ *is also open.*

Routine exercise

Exercise 17. Prove the following assertion.
At a fixed point z_**, the following equality holds:* $\left(f_c^{\circ n}\right)'(z_*) = \left(f_c'(z_*)\right)^n$.

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4. PROPERTIES OF QUADRATIC JULIA SETS

Complex analysis and topology provide indispensable tools to describe the nature of "Julia sets," for not only the properties but also the definition of Julia sets rely on topology.

4.1. Two examples of Julia sets

The topological concepts reviewed in the preceding section provide the language necessary to define the nature of Julia sets, which are the boundaries of filled Julia sets.

Definition 18. The **Julia set** of a quadratic polynomial f_c with $f_c(z) = z^2 + c$ is the boundary ∂K_c of the filled Julia set K_c, and it is usually denoted by J_c.

EXAMPLE 20. Recall, from example 12, on page 12, that the filled Julia set K_0 of the squaring function f_0 ($f_0(z) = z^2$) is the closed unit disc,

$$K_0 = \overline{D(0,1)} = \{z : z \in \mathbb{C} \text{ and } |z| \leq 1\}.$$

Recall also, from example 18, on page 23, that the boundary of the closed unit disc is the unit circle; therefore, the Julia set of the squaring function is the unit circle:

$$J_0 = \partial K_0 = \{w : w \in \mathbb{C} \text{ and } |w| = 1\}.$$

EXAMPLE 21. Consider the quadratic polynomial $f_{-2} : \mathbb{C} \to \mathbb{C}$, with $f_{-2}(z) = z^2 - 2$. As conjectured in exercise 14, the filled Julia set K_{-2} of f_{-2} is the segment $[-2, 2]$, which coincides with its own boundary. Thus, $J_{-2} = [-2, 2]$. To prove this assertion, use a composition with the function

$$g : \mathbb{C} \setminus [-2,2] \to \mathbb{C},\ g(z) = \frac{z + \sqrt{z-2}\sqrt{z+2}}{2}.$$

(The notation $\mathbb{C} \setminus [-2,2]$ stands for the complement of the interval $[-2,2]$; thus, the set $\mathbb{C} \setminus [-2,2]$ looks like a plane with a slit along $[-2,2]$). As explained below, the function g maps $\mathbb{C} \setminus [-2,2]$ onto the complement of the closed unit disc K_0, in other words, onto $\{z : z \in \mathbb{C} \text{ and } |z| > 1\}$, as shown in figure 5. The function g has an inverse function,

$$g^{-1} : \{w : w \in \mathbb{C} \text{ and } |w| > 1\} \to \mathbb{C} \setminus [-2,2],\ g^{-1}(w) = w + \frac{1}{w}.$$

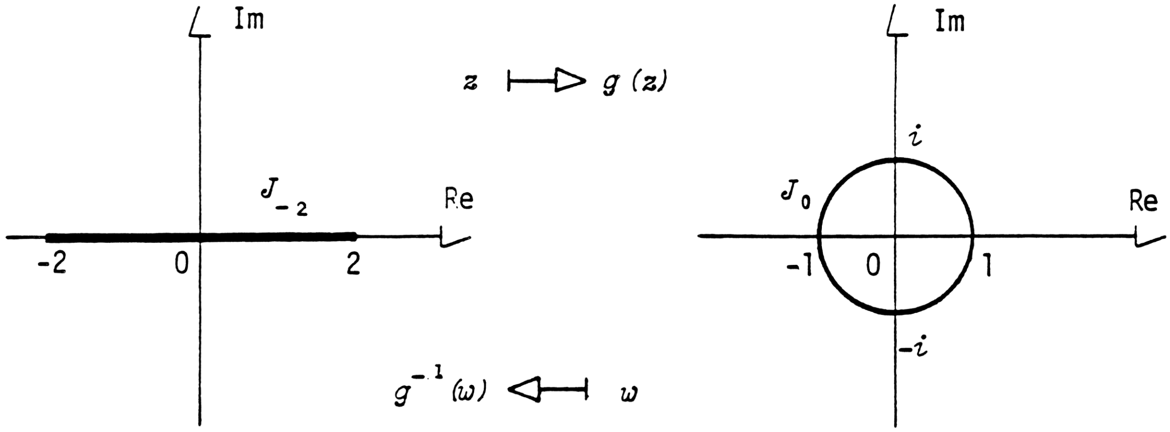


Figure 5. The function g maps the region outside the interval $[-2,2]$ onto the region outside the closed unit disc.

(The formula $w+1/w$ also extends to the complex numbers w for which $|w|=1$, and it explains why g maps $\mathbb{C}\setminus[-2,2]$ onto the complement of the unit disc. If $w=(x,y)$ lies on the unit circle, then $|w|=1$, and $1/w=\overline{w}$, because $w\overline{w}=(x,y)(x,-y)=x^2+y^2=|w|^2=1$; moreover, $|x|=\sqrt{x^2}\leq\sqrt{x^2+y^2}=|w|=1$, and, consequently, $|w+1/w|=|w+\overline{w}|=|(x,y)+(x,-y)|=2|x|\leq 2$. Therefore, if w lies on the unit circle, then $|w+1/w|$ lies on the interval $[-2,2]$.)

Straightforward algebra shows that $g^{-1}\circ f_0\circ g=f_{-2}$, which means that $g^{-1}(f_0(g(z)))=f_{-2}(z)$ for every $z\in\mathbb{C}\setminus[-2,2]$. Consequently, $f_{-2}^{\circ n}=\left(g^{-1}\circ f_0\circ g\right)^{\circ n}=g^{-1}\circ f_0^{\circ n}\circ g$ (because intermediate compositions $g^{-1}\circ g$ cancel out). Therefore, $(z_n)=(f_{-2}^{\circ n}(z_0))=\left(g^{-1}\left(f_0^{\circ n}(g(z_0))\right)\right)$ diverges if, but only if, $(w_n)=(f_0^{\circ n}(g(z_0)))$ does, which means that the complement of the filled Julia set of the squaring function, $\mathbb{C}\setminus K_0$, is the image under g of that of f_{-2}. Hence, $\mathbb{C}\setminus K_0=g(\mathbb{C}\setminus K_{-2})$ and $\mathbb{C}\setminus K_{-2}=g^{-1}(\mathbb{C}\setminus K_0)=\mathbb{C}\setminus[-2,2]$. Finally, $J_{-2}=\partial K_{-2}=\partial[-2,2]=[-2,2]$ in $\mathbb{C}$.

4.2. Invariance of Julia sets under their quadratic polynomials

Except for the particular values $c=0$ and $c=-2$, examined in the preceding subsection, the investigation of the Julia set of f_c for values of c other than 0 or -2 requires a more substantial analysis for *each* value of c separately. Therefore, instead of investigating any particular Julia set, we shall focus on a few general properties of Julia sets that rely only upon the analysis and topology reviewed in the preceding section. The first property of quadratic Julia sets examined here concerns their invariance under their associated polynomial f_c. That is, each function f_c maps the associated filled Julia set K_c onto itself.

Proposition 5. *For each complex constant* $c\in\mathbb{C}$, *the filled Julia set* K_c *is invariant under the function* f_c; *that is,* $f_c(K_c)=K_c$.

Proof. For each initial point $z_0\in K_c$ in the filled Julia set, the sequence of iterations

$$z_0,\ z_1=f_c(z_0),\ z_2=f_c(z_1),\ z_3=f_c(z_2),\ \ldots$$

remains bounded. Consequently, so does the sequence starting at z_1,

$$z_1,\ z_2 = f_c(z_1),\ z_3 = f_c(z_2),\ z_4 = f_c(z_3),\ \ldots.$$

Therefore, $z_1 = f(z_0)$ also lies in the filled Julia set K_c, which proves that $f_c(K_c) \subset K_c$.

For the converse inclusion, select either preimage of z_0, for instance, $z_{-1} = \sqrt{z_0 - c}$, so that $f_c(z_{-1}) = z_0$; then the sequence that starts from z_{-1} also remains bounded:

$$z_{-1},\ z_0 = f_c(z_{-1}),\ z_1 = f_c(z_0),\ z_2 = f(z_1),\ \ldots.$$

Consequently, the preimage z_{-1} also belongs to the filled Julia set K_c, which shows that $K_c \subset f_c(K_c)$ (because $z_0 = f_c(z_{-1})$). □

Not only does each filled Julia set K_c remain invariant under the function f_c, but so does its boundary $\partial K_c = J_c$ (the Julia set itself), and its complement, $\mathbb{C} \setminus K_c$, which has a special name that reflects its properties.

Definition 19. The **basin of attraction of infinity** of a quadratic polynomial f_c is the set, denoted by $A_{\infty,c}$, which consists of all complex numbers $z_0 \in \mathbb{C}$ that give rise to an unbounded sequence. Thus, $A_{\infty,c} = \mathbb{C} \setminus K_c$.

The term "basin of attraction of infinity" arises from the fact that, *for the particular sequences* $(z_n) = (f_c^{\circ n}(z_0))$ *considered here,* if a point z_0 does not belong to the filled Julia set, then the unbounded sequence that it initiates diverges to infinity (recall exercise 13). Informally, infinity attracts all sequences that start at any point $z_0 \in A_{\infty,c}$. The following sequence of theoretical exercises shows that each of the sets $A_{\infty,c}$, K_c, and J_c is invariant under mappings by f_c.

Theoretical exercises

Exercise 18. Prove the following assertion.
The basin of attraction of infinity, $A_{\infty,c}$, *is an open set.*

Exercise 19. Prove the following assertion.
The basin of attraction of infinity is invariant under f_c. *In other words,* f_c *maps* $A_{\infty,c}$ *onto itself; thus,* $f_c(A_{\infty,c}) = A_{\infty,c} = f_c^{-1}(A_{\infty,c})$.

Exercise 20. Demonstrate that $f_c(J_c) \subset J_c$ and that $f_c^{-1}(J_c) \subset J_c$.

Exercise 21. Denote by $K_c^{\circ} = K_c \setminus \partial K_c$ the **interior** of the filled Julia set, which consists of all points $z \in K_c$ that do not lie on its boundary (all points $z \in K_c$ for which there exists a positive radius $r > 0$ such that $D(z,r) \subset K_c$). Prove that $f_c(K_c^{\circ}) = K_c^{\circ} = f_c^{-1}(K_c^{\circ})$.

Exercise 22. Apply the preceding exercises to prove the following assertion.
For each complex constant $c \in \mathbb{C}$ *the Julia set* $J_c = \partial K_c$ *is invariant under the associated quadratic polynomial* f_c *and its inverse; in mathematical formulae,* $f_c(J_c) = J_c = f_c^{-1}(J_c)$.

4.3. The Non-Attracting Fixed-Point Inverse Iteration Method

The property of quadratic Julia sets examined here concerns the presence on the boundary J_c of at least one of the two fixed points of f_c, where the derivative $f'(z_*)$ has a modulus of at least one. From such a fixed point, the Inverse Iteration Method generates only points on the Julia set J_c.

Definition 20. A **non-attracting** fixed point of a function $f : \mathbb{C} \to \mathbb{C}$ is a fixed point z_* where $|f'(z_*)| \geq 1$. A non-attracting fixed point z_* is called **repelling** if $|f'(z_*)| > 1$ and **parabolic** (or **neutral**) if $|f'(z_*)| = 1$. In contrast, at an **attracting** fixed point, $|f'(z_*)| < 1$. (A fixed point where $f'(z_*) = 0$ is also called **superattracting,** but we shall not need so fine a classification.)

The terminology for the classification of fixed points arises from the behavior of iterations beginning with an initial point near a fixed point. If z_0 lies near an attracting fixed point z_*, then the sequence of iterations $(z_n) = (f^{\circ n}(z_0))$ converges to z_*. In contrast, if the initial point z_0 lies near a repelling fixed point z_*, then the sequence (z_n) may (but need not) diverge away from z_*.

EXAMPLE 22. With $c = 0$ the squaring function f_0, with $f_0(z) = z^2$, has two fixed points, 0 and 1, since $f_0(0) = 0^2 = 0$ and $f_0(1) = 1^2 = 1$. The first fixed point is attracting, whereas the second fixed point is repelling, as the following arguments confirm.

(22.1) At $z = 0$, $f_0(0) = 0$ and $|f'(0)| = |2 \times 0| = 0 < 1$.

(22.2) At $z = 1$, $f_0(1) = 1$ and $|f'(1)| = |2 \times 1| = 2 > 1$.

Moreover, from examples 12 and 20, recall that the filled Julia set K_0 consists of the closed unit disc, $K_0 = \overline{D(0,1)}$, and that its boundary, $\partial K_0 = J_0$, is the unit circle $J_0 = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}$. The attracting fixed point $0 \in K_0$ belongs to the filled Julia set, whereas the repelling fixed point $1 \in J_0$ lies on the Julia set.

EXAMPLE 23. With $c = 1/4$ the function $f_{1/4} : \mathbb{C} \to \mathbb{C}$, $f_{1/4}(z) = z^2 + 1/4$ has only one fixed point, at $z_* = 1/2$, as shown in example 13, on page 16. This fixed point is parabolic, because $|f'_{1/4}(1/2)| = |2 \times 1/2| = 1$. This particular parabolic fixed point lies on the Julia set $J_{1/4} = \partial K_{1/4}$, as the following proposition shows.

Proposition 6. *The parabolic fixed point $z_* = 1/2$ lies on the Julia set $J_{1/4}$.*

Proof. Recall that proposition 2, page 17, (with $n = 0$) established that every fixed point lies in the filled Julia set; consequently, $1/2 \in K_{1/4}$. To show that $1/2$ lies on the boundary, it suffices to prove that each open disc $D(1/2, r)$ contains a point z_0 not in $K_{1/4}$. To this end, consider the point $z_0 = 1/2 + [r/2]$, which initiates the sequence

$$z_0 = 1/2 + [r/2],$$

$$z_1 = z_0^2 + c = (1/2 + [r/2])^2 + 1/4 = 1/2 + (r/2)(1 + r/2),,$$

$$\vdots$$

Comparing the distances from $1/2$ to z_0 and z_1, observe that

$$\frac{z_1 - z_*}{z_0 - z_*} = \frac{1/2 + (r/2)(1 + r/2) - 1/2}{1/2 + [r/2] - 1/2} = \frac{(r/2)(1 + r/2)}{r/2} = 1 + r/2 > 1.$$

Hence, induction shows that $|z_n - z_0| > (1 + r/2)^n$, which diverges to infinity. □

In contrast to the situation with $c = 1/4$, for every $c \neq 1/4$ the quadratic polynomial f_c has at least one repelling fixed point, and every such fixed point belongs to the boundary, as the following two propositions show.

Proposition 7. *If $c \neq 1/4$ then f_c has a repelling fixed point.*

Proof. Recall from example 14 that for $c \neq 1/4$ the quadratic polynomial f_c has two distinct fixed points, expressed by the quadratic formula

$$z_{\pm} = \frac{1 \pm \sqrt{1-4c}}{2}.$$

At the fixed points,

$$f_c'(z_{\pm}) = 2z_{\pm} = 1 \pm \sqrt{1-4c}.$$

If either fixed point, for instance $z_+ = (1 + \sqrt{1-4c})/2$, is not repelling, then $|f_c'(z_+)| = |1 + \sqrt{1-4c}| \leq 1$. For the rest of the argument, recall from definition 1 and remark 3 that

$$|z| = |(x, y)| = \sqrt{x^2 + y^2} = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}.$$

In particular, if $y = \operatorname{Im}(z) \neq 0$ then $|z| > |x|$. Thus, $\operatorname{Re}(\sqrt{1-4c}) < 0$: since $c \neq 1/4$, it follows that $1 - 4c \neq 0$, and if $\operatorname{Re}(\sqrt{1-4c})$ were zero then $\operatorname{Im}(\sqrt{1-4c})$ would not be zero, and the Pythagorean theorem would show that $|1 + \sqrt{1-4c}| > 1$, contrary to the initial assumptions. With $\operatorname{Re}(\sqrt{1-4c}) < 0$ it follows that $\operatorname{Re}(1 + \sqrt{1-4c}) < 1$, which means that $1 + \sqrt{1-4c}$ lies in the left-hand half-plane bounded on its right by the vertical line through $x = 1$. Consequently, at the other fixed point, $\operatorname{Re}(-\sqrt{1-4c}) = -\operatorname{Re}(\sqrt{1-4c}) > 0$ and $\operatorname{Re}(1 - \sqrt{1-4c}) > 1$. Therefore, $|1 - \sqrt{1-4c}| \geq |\operatorname{Re}(1 - \sqrt{1-4c})| > 1$, which means that the other fixed point, z_-, is repelling. □

Proposition 8. *Every repelling fixed point belongs to the Julia set.*

Proof. By contraposition, suppose that a fixed point z_* does not belong to the Julia set $J_c = \partial K_c$. Since every fixed point lies in the filled Julia set K_c, it follows that there exists an open disc $D(z_*, r) \subset K_c$. (If there were no such disc, then, by definition of the boundary, z_* would lie on the boundary J_c.) Also, since the complement $A_{\infty,c} = \mathbb{C} \setminus K_c$ is an open set, it follows that $\overline{D(z_*, r)} \subset K_c$ because no point on the boundary of the disc $D(z_*, r)$ may lie in $A_{\infty,c}$. Since the filled Julia set remains invariant under the function f_c, it follows that $f_c^{\circ n}(K_c) \subset K_c \subset \overline{D(0, M)}$, and hence it follows from Cauchy's inequality and exercise 17 that

$$|f_c'(z_*)|^n = |(f_c^{\circ n})'(z_*)| \leq \frac{M}{r}.$$

Thus, the powers $(f_c'(z_*))^n$ remain bounded, which has the necessary consequence that $|f_c'(z_*)| \leq 1$. Therefore, z_* is not a repelling fixed point. □

Corollary 1. *All the preimages of every repelling fixed point of f_c belong to the Julia set J_c.*

Proof. Every repelling fixed point of f_c belongs to the Julia set J_c, and all preimages of all points on the Julia set remain on the Julia set ($f_c^{-1}(J_c) = J_c$) by exercise 22. □

Though all the preimages of a repelling fixed point belong to the Julia set, there might not exist enough of them to outline the Julia set. Fortunately, the following proposition guarantees that every non-attracting fixed point has infinitely many distinct preimages.

Proposition 9. *For each complex constant $c \in \mathbb{C}$ every non-attracting fixed point z_* of f_c has infinitely many distinct preimages.*

Proof. Observe that if the preimage $f_c^{-1}(\{z_*\})$ of a fixed point z_* consists of only one point, namely $f_c^{-1}(\{z_*\}) = \{z_*\}$, then z_* is a double root of the equation $z^2 + c = z_*$, which has the necessary consequence that $f_c'(z_*) = 0$, and, consequently, $2z_* = 0$. Therefore, z_* is an attracting fixed point, which contradicts the assumption.

Thus, a non-attracting fixed point has at least two distinct preimages, which we denote by z_* itself and $z^\dagger$. Also, denote by Z the set of all preimages of z_*; thus,

$$Z = \bigcup_{n=1}^{\infty} \left(f_c^{\circ n}\right)^{-1}(\{z_*\}).$$

To obtain a contradiction, assume that Z contains only finitely many preimages, and let N represent the smallest number of backward iterations necessary to produce all the preimages:

$$N = \min\left\{m : m \in \mathbb{N} \text{ and } Z = \bigcup_{n=1}^{m} \left(f_c^{\circ n}\right)^{-1}(\{z_*\})\right\}.$$

Notice that $N \geq 2$ because $f_c^{-1}(\{z_*\}) = \{z_*, z^\dagger\}$ contains a preimage $z^\dagger$ different from the fixed point z_*, and because $f_c^{-1}(\{z^\dagger\})$ contains neither z_* nor $z^\dagger$, since $f_c(z_*) = z_* \neq z^\dagger$ and $f_c(z^\dagger) = z_* \neq z^\dagger$. Next, let

$$z_0 \in \left(f_c^{\circ N}\right)^{-1}(\{z_*\}) \setminus \bigcup_{n=1}^{N-1} \left(f_c^{\circ n}\right)^{-1}(\{z_*\})$$

represent a point obtained after exactly N backward iterations of f_c. Thus, $f_c^{\circ N}(z_0) = z_*$ but $f_c^{\circ n}(z_0) \neq z_*$ for every $n < N$. Moreover, select any preimage of z_0, and denote it by $z_{-1} \in f_c^{-1}(\{z_0\})$. Then z_{-1} is itself a preimage of z_*, because

$$f_c^{\circ N+1}(z_{-1}) = f_c^{\circ N}\left(f_c(z_{-1})\right) = f_c^{\circ N}(z_0) = z_*.$$

Consequently, there exists an integer $m \in \{2, \ldots, N\}$ such that $z_{-1} \in \left(f_c^{\circ m}\right)^{-1}(\{z_*\})$, and, therefore,

$$z_* = f_c^{\circ m}(z_{-1}) = f_c^{\circ m-1}\left(f_c(z_{-1})\right) = f_c^{\circ m-1}(z_0),$$

which has the necessary consequence that $z_0 \in \left(f_c^{\circ m-1}\right)^{-1}(\{z_*\})$, contradicting the hypothesis on z_0. □

The foregoing results guarantee that if the Fixed-Point Inverse Iteration Method starts from a repelling fixed point, or from the parabolic fixed point if $c = 1/4$, and if it retains all the preimages that it computes, then it generates infinitely many points, all of which belong to the Julia set. In practice, however, many implementations retain only one preimage z_{-n} at each stage and then select the next preimage $z_{-(n+1)} = \pm\sqrt{z_{-n} - c}$ through a random choice of the sign $\pm$, as in the program in exhibit 4. Yet such a random choice does not ensure (but only makes it improbable) that the algorithm will not remain trapped in a loop through finitely many preimages. A method for selecting the sign $\pm$ at each stage so that the algorithm generates a sequence of infinitely many distinct preimages constitutes the topic of the second research problem in this chapter.

Non-Attracting Fixed-Point Inverse Iteration Method. The following modification of the Fixed-Point Inverse Iteration Method generates only points on the Julia set.

Step 1'. Compute $1+\sqrt{1-4c}$. If $|1+\sqrt{1-4c}| > 1$, select the fixed point

$$z_* = z_+ = \frac{1+\sqrt{1-4c}}{2}.$$

Otherwise select the other fixed point, which is

$$z_* = z_- = \frac{1-\sqrt{1-4c}}{2}.$$

Next, proceed exactly as in the Fixed-Point Inverse Iteration Method, selecting either square root randomly at each step.

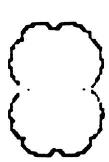
REMARK 10. There exists a theorem stronger than proposition 9 (which guaranteed that non-attracting fixed points have infinitely many preimages). Specifically, the set Z of all preimages of a non-attracting fixed point z_* on the Julia set, as described in proposition 9, is *dense* in the Julia set. Informally, this means that there exist preimages z_{-n} of z_* as close as we want to any point $z \in J_c$ on the Julia set. Mathematically, this means that for each point $z \in J_c$ on the Julia set, and for each open disc $D(z,r)$ about z, there exists a preimage $z_{-n} \in f_c^{\circ n^{-1}}(z_*)$ in $D(z,r) \cap J_c$. For a proof of this assertion, which relies upon Montel's theorem about normal families of complex analytic functions, consult Linda Keen's chapter, "Julia Sets," page 62, in Devaney and Keen's book [32].

REMARK 11. Though the Non-Attracting Fixed-Point Inverse Iteration Method may generate an infinite and dense set of points on Julia sets, comparisons with pictures obtained through different algorithms reveal that such points are not evenly distributed on the Julia set. As a consequence, the resulting plots show many points in some parts of the Julia sets, and very few in other parts, which yields a poor resolution of the finer details of Julia sets. For other algorithms, consult the books by Peitgen *et al.* [36], [37].

Julia sets on the HP-28C&S

```
« c ABS 4 *
1 + √ 1 + 2
/ 'r' STO r
32 / 'h' STO
h 137 * 2 /
's' STO
s r R→C PMAX
s NEG 0 R→C
PMIN » ENTER
 'setup' STO
-------------
« 1 c 4 * -
√ 1 + (1,0)
* DUP
IF ABS 1 <
 THEN 2 -
 NEG
END 2 / DUP
PIXEL» ENTER
 'fixed' STO
-------------
« CLLCD
1 2000
START c - √
 (1,0) RAND
 IF .5 <
  THEN NEG
 END * DUP
 PIXEL
NEXT DROP
LCD→ » ENTER
  'back' STO
-------------
« setup
fixed back s
h NEG R→C
PMAX s NEG r
h + NEG R→C
PMIN fixed
back » ENTER
 'Julia' STO
```

```
.25 'c' STO
 USER Julia
```



Comments on the programs for Julia sets

Subroutine **setup** computes a bound for the filled Julia set,

$$r = r_c = \frac{1+\sqrt{1+4|c|}}{2},$$

and adjusts the screen so that it covers the square $[-r,r]\times[-r,r]$, hence also the filled Julia set; h equals the width of each pixel.

ENTER and STOre in 'setup'

Subroutine `fixed` calculates a non-attracting fixed point; for the numerator, if $|1+\sqrt{1-4c}| < 1$ then $2-(1+\sqrt{1-4c}) = 1-\sqrt{1-4c}$ and $|1-\sqrt{1-4c}| > 1$ if $c \neq 1/4$ (as explained in the text).

ENTER and STOre in 'fixed'

For the HP-28C&S only, subroutine back iterates the computation of preimages,

$$f^{-1}(\{z\}) \ni \pm\sqrt{z-c},$$

with the commands c $-\sqrt{}$, and with a random choice of sign.

ENTER and STOre in 'back'

The main program, Julia, computes and plots preimages of $f_c(z) = z^2 + c$ (alone on the HP-48SX, with back on the HP-28C&S).

ENTER and STOre in 'Julia'

Tutorial:

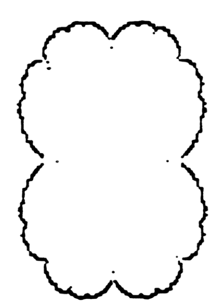
STOre a complex number in 'c'
execute Julia
in the USER or VAR menu.

Exhibit 4.

Julia sets on the HP-48SX

```
« c ABS 4 *
1 + √ 1 + 2
/ 'r' STO r
64 / 'h' STO
ERASE r 128
/ 131 * r
R→C DUP NEG
SWAP PDIM #
131d # 128d
PDIM » ENTER
 'setup' STO
-------------
« 1 c 4 * -
√ 1 + (1,0)
* DUP
IF ABS 1 ≤
 THEN 2 -
 NEG
END 2 / DUP
PIXON» ENTER
 'fixed' STO
-------------
« setup
fixed 1 2000
START c - √
 (1,0) RAND
 IF .5 <
  THEN NEG
 END * DUP
 DUP NEG
 PIXON PIXON
NEXT DROP
»     ENTER
 'Julia' STO
```

```
.25 'c' STO
 VAR Julia
```



Exercises

Routine exercises

Exercise 23. Verify that $f_{-3/4} : \mathbb{C} \to \mathbb{C}$, $f_{-3/4}(z) = z^2 - 3/4$ has a parabolic fixed point at $z_- = -1/2$ and a repelling fixed point at $z_+ = 3/2$.

Exercise 24. Identify the nature (repelling, parabolic, or attracting) of each fixed point of the function $f_{-2} : \mathbb{C} \to \mathbb{C}$, with $f_{-2}(z) = z^2 - 2$.

Theoretical exercises

Exercise 25. As the preceding examples and exercises reveal, not all quadratic polynomials of the type $f_c(z) = z^2 + c$ have a parabolic fixed point. In fact, f_c has a parabolic fixed point if, but only if, the constant c lies on a particular curve in the complex plane. Identify that curve. (Write the two equations that mean that f_c has a parabolic fixed point. Then transform the resulting equations into one polar equation, and identify the polar equation with that of a standard curve.)

Exercise 26. Prove that an attracting fixed point z_* of f_c cannot lie on the Julia set, but belongs to the interior K_c° of the filled Julia set. Moreover, if z_0 lies close enough to z_* then (z_n) tends to z_*. (The proof may require a complex line integral.)

Experiments and conjectures

Exercise 27. Try the Non-Attracting Fixed-Point Inverse Iteration Method and generate pictures of the Julia sets J_c corresponding to the following constants:

(27.1) $c = (-0.12256117, 0.74586177)$ This constant c satisfies the equation $c^3 + 2c^2 + c + 1 = 0$ (which yields theoretical properties of the Julia set J_c) and J_c is called "Douady's rabbit;" see Devaney's *An Introduction to Chaotic Dynamical Systems*, 2nd ed., [30], plate 3.

(27.2) $c = (0.360284, 0.100376)$ J_c is called "the dragon;" see *loc. cit.*, plate 1.

(27.3) $c = -1.54368901269$ This constant c satisfies the equation $c^4 + 4c^3 + 6c^2 + 6c + 4 = 0$, and J_c is Devaney's dendrite; see *loc. cit.*, pages 293–295.

(27.4) $c = i = (0, 1)$ The Julia set J_i is called "the dendrite."

(27.5) $c = 2/5$ The resulting Julia set $J_{2/5}$ consists of many disconnected pieces.

The following two exercises investigate the results of the Fixed-Point Inverse Iteration Method with a predetermined sequence of signs to select either square root at each step.

Exercise 28. Considering the squaring function, with $f_0(z) = z^2$, start from the fixed point $z_+ = 1$, select $z_{-1} = -\sqrt{z_+} = -1$, and, henceforth, always choose the square root described in exercise 5 (which coincides with the square-root key on supercalculators):

$$z_0 = z_+ = 1, \quad z_{-1} = -1, \quad z_{-2} = \sqrt{-1} = i = (0, 1), \quad z_{-3} = \sqrt{(0, 1)} = (1/\sqrt{2}, 1/\sqrt{2}), \; \ldots$$

Compute and plot several additional preimages, and describe the result.

Exercise 29. With the function $f_{1/4} : \mathbb{C} \to \mathbb{C}$, $f_{1/4}(z) = z^2 + 1/4$, start from its unique fixed point, $z_* = 1/2$, and *alternate* the choice of the sign ($-$ or $+$), which yields

$$\begin{aligned}
&z_0 = z_* = 1/2,\\
&z_{-1} = -\sqrt{z_0 - c} = -\sqrt{1/2 - 1/4} = -\sqrt{1/4} = -1/2,\\
&z_{-2} = +\sqrt{z_{-1} - c} = +\sqrt{-1/2 - 1/4} = +\sqrt{-3/4} = i\sqrt{3}/2,\\
&z_{-3} = -\sqrt{z_{-2} - c} = -\sqrt{i\sqrt{3}/2 - 1/4} = -\sqrt{(-1/4, \sqrt{3}/2)}\\
&\quad = (-0.570695986873\ldots, -0.758744956776\ldots),\\
&\vdots
\end{aligned}$$

Perform about two dozen iterations and describe the result.

Exercise 30. How do Julia sets relate to the Butterfly Effect in chaos? Answer this question with a short essay based upon the properties of Julia sets developed in the present chapter and upon the description of the Butterfly Effect in the introduction. (Of course, you may also consult other references, but doing so is not required.)

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5. EXPLORATORY TERM PROJECTS

The present section proposes two projects that may be attempted in parallel with the study of this chapter (hence the phrase "term projects"). Since the results involve only elementary complex variables and are known to specialists, the projects do not qualify as research. However, some of the details are not readily available in many sources, which means that you may have to search the literature for some results and derive others yourself (hence the adjective "exploratory").

5.1. Julia sets of general quadratic polynomials

Project 1. Instead of the particular type of quadratic polynomial considered so far, f_c with $f_c(z) = z^2 + c$, consider *every* quadratic polynomial f, for which

$$f(w) = pw^2 + qw + j,$$

with complex coefficients p, q, and j, such that $p \neq 0$.

How do the Julia sets of *all* quadratic polynomials f relate to the Julia sets of the particular type J_c for f_c ?

To arrive at a conjecture (a tentative answer), you may either modify the algorithms given in this chapter and then plot many Julia sets for various p, q, and j, or you may proceed theoretically as in example 21, on page 26. In either case, however, you then ought to prove your conjecture (or revise it and start again).

5.2. A better algorithm for Mandelbrot's set

Project 2. Instead of fixing $c \in \mathbb{C}$ and examining what sequences (z_n) emerge from each point z_0, fix $z_0 = 0$ and for each $c \in \mathbb{C}$ consider the sequence $(w_n) = (f_c^{\circ n}(0))$:

$$0, \quad f_c(0) = 0^2 + c = c, \quad f_c(c) = c^2 + c, \quad f_c(c^2 + c) = (c^2 + c)^2 + c, \quad \ldots, f_c^{\circ n}(0), \ldots$$

The **Mandelbrot set** is the set, denoted by $\mathcal{M}$, that consists of all complex numbers $c \in \mathbb{C}$ for which the sequence $(w_n) = (f_c^{\circ n}(0))$ remains bounded.

(1) Verify that $-2 \in \mathcal{M}$.

(2) Prove that if $c \in \mathcal{M}$ then $|c| \leq 2$.

(3) Prove that Mandelbrot's set is symmetric with respect to the horizontal axis.

(4) Modify and test the "Direct Iteration Method" to plot the Mandelbrot set $\mathcal{M}$, in a fashion similar to that for filled Julia sets.

(5) Prove that your algorithm is better than the ones found in the literature.

6. RESEARCH PROBLEMS

The two research problems proposed here concern two problems already mentioned in the text and investigated in the exercises. As is the case for all mathematical *research* problems, the literature does not appear to offer any solution to these problems, and nobody knows in advance what methods may lead to solutions.

6.1. An optimal bound for quadratic Julia sets

As proved in the text, no point $z \in K_c$ in the filled Julia set of the quadratic polynomial defined by $f_c(z) = z^2 + c$ may have a modulus greater than $R_c = \max\{2, |c|\}$. Yet the bound R_c is unnecessarily large for most Julia sets, because, as proved in the exercises, no point $z \in K_c$ may have a modulus greater than $r_c = (1 + \sqrt{1 + 4|c|})/2 \leq R_c$. Nevertheless, the better bound r_c may also be unnecessarily large. Find a better estimate of the size of the filled Julia set.

6.2. An infinite sequence of preimages

Recall that all the infinitely many preimages of every repelling fixed point — or of the parabolic fixed point if $c = 1/4$ — of the quadratic polynomial f_c lie on the Julia set itself (the boundary of the filled Julia set), and that therefore, keeping all the preimages at each step of the Fixed-Point Inverse Iteration Method generates infinitely many points that outline the Julia sets J_c. Yet also recall, from exercises 15 and 16, that plotting only one preimage at each step may trap the Fixed-Point Inverse Iteration Method in a loop through only very few points scattered on the Julia set without providing any satisfactory picture of J_c. Find a way to select either square root ($+$ or $-$) at each step so that the Fixed-Point Inverse Iteration Method does not generate any point more than once. Prove that your algorithm satisfies the requirement just stated.

7. SOLUTIONS TO ALL THE EXERCISES

7.1. Solutions to the exercises

Exercise 1.

(1.1) $(0,0)+(2,3)=(2,3)$
(1.2) $(2,3)+(0,0)=(2,3)$
(1.3) $(1,0)\times(2,3)=(2,3)$
(1.4) $(2,3)\times(1,0)=(2,3)$
(1.5) $(2,3)+(4,5)=(6,8)$
(1.6) $(4,5)+(2,3)=(6,8)$
(1.7) $(2,3)\times(4,5)=(-7,22)$
(1.8) $(4,5)\times(2,3)=(-7,22)$
(1.9) $(2,3)+(-2,-3)=(0,0)$
(1.10) $(-2,-3)+(2,3)=(0,0)$
(1.11) $(2,3)(2/13,-3/13)=(1,0)$
(1.12) $(2/13,-3/13)(2,3)=(1,0)$
(1.13) $((2,3)+(4,5))+(6,7)=(12,15)$
(1.14) $(2,3)+((4,5)+(6,7))=(12,15)$
(1.15) $((2,3)(4,5))(6,7)=(-196,83)$
(1.16) $(2,3)((4,5)(6,7))=(-196,83)$
(1.17) $(2,3)((4,5)+(6,7))=(-16,54)$
(1.18) $((2,3)(4,5))+((2,3)(6,7))=(-16,54)$
(1.19) $(-1,0)\times(2,3)=(-2,-3)$
(1.20) $(2,3)\times(-1,0)=(-2,-3)$
(1.21) $(1,0)\times(1,0)=(1,0)$
(1.22) $(0,1)\times(0,1)=(-1,0)$

Exercise 2. (2.1) $z+\bar{z}=2\mathrm{Re}(z)=2x$. (2.2) $z-\bar{z}=2\mathrm{Im}(z)=2y$. (2.3) $z\times\bar{z}=|z|^2=x^2+y^2$.

Exercise 3. $|z^2|=|zz|=|z|\cdot|z|=|z|^2$. $\mathrm{Arg}\,(zz)=\mathrm{Arg}\,(z)+\mathrm{Arg}\,(z)=2\mathrm{Arg}\,(z)$.

Exercise 4. $|z|=|z-w+w|\leq|z-w|+|w|$, hence $|z|-|w|\leq|z-w|$. If $|z|\geq|w|$ then $||z|-|w||=|z|-|w|\leq|z-w|$. If $|z|\leq|w|$ then permute the rôles of z and w.

Exercise 5. Observe that if $w=(u,v)$ then $w^2=(u,v)(u,v)=(u^2-v^2,uv+vu)=(u^2-v^2,2uv)$; thus,

$$w^2=(u,v)^2=\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\ ,\ \mathrm{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)^2$$

with

$$=u^2-v^2=\left(\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\right)^2-\left(\mathrm{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}\right)^2$$

and

$$2uv=2\sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}\mathrm{sign}(y)\sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}.$$

Hence,

$$w^2 = \left(\frac{x+\sqrt{x^2+y^2}}{2} - \frac{-x+\sqrt{x^2+y^2}}{2}, 2\text{sign}(y)\sqrt{\frac{x^2+y^2-x^2}{4}} \right) = (x,y) = z.$$

Exercise 6. (6.1) $(-w)^2 = w^2 = z$. (6.2) All square roots $q \in \mathbb{C}$ of z satisfy the quadratic equation $q^2 - z = 0$, and such a polynomial equation, of degree two in q, cannot have more than two roots.

Exercise 7. Apply the result of exercise 3: since $w^2 = z$, it follows that $|z| = |w^2| = |w|^2$, hence $|w| = \sqrt{|z|}$, and $\text{Arg}\,(z) = \text{Arg}\,(w^2) = 2\text{Arg}\,(w)$, hence $\text{Arg}\,(w) = \text{Arg}\,(z)/2$.

Exercise 8. (8.1) $\sqrt{-1} = \pm(0,1) = \pm i$. (8.2) $\sqrt{-4} = \pm(0,2) = \pm 2i$. (8.3) $\sqrt{(0,1)} = \pm(1/\sqrt{2}, 1/\sqrt{2})$.

Exercise 9. Complete the square, *verbatim* as in precalculus:

$$0 = az^2 + bz + c = a\left(z^2 + \frac{b}{a}z\right) + c = a\left(\left(z + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c.$$

Hence, dividing both sides by a, rearranging terms, and taking square roots yield

$$z + \frac{b}{2a} = \pm\sqrt{\frac{1}{a}\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Exercise 10. $z_+ = (-3,4)$ and $z_- = (-3,-4)$; $z = (-6 \pm \sqrt{6^2 - 4\cdot 25})/2$.

Exercise 11. If $z_0 \in K_c$ then $-z_0 \in K_c$ because $z_1 = z_0^2 + c = (-z_0)^2 + c$ and the sequences starting at z_0 and $-z_0$ coincide from z_1 and beyond.

Exercise 12. Set $d = |z_0| - r_c > 0$, so that $|z_0| = r_c + d$. Then

$$|z_0|^2 = \left(\frac{1+\sqrt{1+4|c|}}{2} + d\right)^2 = \cdots = \frac{1+\sqrt{1+4|c|}}{2} + d\left(1+\sqrt{1+4|c|}\right) + d^2 + |c|.$$

Use the expression just found and the reverse triangle inequality to obtain

$$|z_1| = |z_0^2 - c| \geq ||z_0|^2 - |c|| > \frac{1+\sqrt{1+4|c|}}{2} + d(1+\sqrt{1+4|c|}).$$

To complete the induction, assume that

$$|z_n| > \frac{1+\sqrt{1+4|c|}}{2} + d(1+\sqrt{1+4|c|})^n$$

holds for some integer $n \in \mathbb{N}^*$, and repeat the preceding argument with d replaced by $d(1+\sqrt{1+4|c|})^n$, which leads to the corresponding inequality for $n+1$. Therefore, if $|z_0| > (1+\sqrt{1+4|c|})/2$ then (z_n) diverges to infinity.

(12.2) Start the sequence at z_k instead of at z_0.

(12.3) If $|c| < 2$ then $r_c = (1+\sqrt{1+4|c|})/2 < (1+\sqrt{1+4\cdot 2})/2 = 2$. If $c = 2$ then $r_c = 2 = |c|$. If $|c| > 2$ then $|c| - 1 > 1$, hence $(2|c|-1)^2 = 4|c|^2 - 4|c| + 1 = 4|c|(|c|-1)+1 > 4|c|+1$; consequently, $r_c = (1+\sqrt{1+4|c|})/2 < (1+\sqrt{(2|c|-1)^2})/2 = |c|$. Therefore, $r_c \leq \max\{2, |c|\}$ for every complex constant c.

(12.4) If $c \in]-\infty, 0]$ then $-c = |c|$ and $1 - 4c = 1 + 4|c|$. Consequently, $z_+ = (1+\sqrt{1-4c})/2 = (1+\sqrt{1+4|c|})/2 = r_c$, which shows that one of the fixed points, z_+, lies at distance r_c from the origin, but that each fixed point belongs to the filled Julia set, and so does its opposite $-z_+$, by the preceding exercise.

Exercise 13. If (z_n) is unbounded then there exists an index $N \in \mathbb{N}$ such that $|z_N| > r_c = (1+\sqrt{1+4|c|})/2$, whence (z_n) diverges to infinity.

Exercise 14.

(14.1) $z_+ = 2$ and $z_- = -1$: solve $f_{-2}(z) = z$, $z^2 - 2 = z$, $z^2 - z - 2 = 0$, $z = (1 \pm \sqrt{1-4(-2)})/2$.

(14.2) From $z_+ = 2$, $z_{-1} = -\sqrt{2-(-2)} = -2$, $z_{-2} = \sqrt{-2-(-2)} = 0$, $z_{-3} = \pm\sqrt{0-(-2)} = \pm\sqrt{2}$.

From $z_- = -1$, $z_{-1} = \sqrt{-1-(-2)} = 1$, $z_{-2} = \pm\sqrt{1-(-2)} = \pm\sqrt{3}$, $z_{-3} = \pm\sqrt{\sqrt{3}+2}$.

(14.3) If $-2 \leq z_{-n} \leq 2$ then $|z_{-(n+1)}| = \sqrt{z-(-2)} \leq \sqrt{2-(-2)} = 2$.

Exercise 15.

(15.1) $z_+ = 1$ and $z_- = 0$: solve $f_0(z) = z$, $z^2 = z$, $z(z-1) = 0$.

(15.2) From $z_- = 0$, the inverse iteration method remains trapped at zero: if $z_{-n} = 0$ then $z_{-(n+1)} = \sqrt{0+0} = 0$. From $z_+ = 1$, the inverse iteration method generates preimages z_{-n} that are all distinct and all lie on the circle with radius 1 and center at 0.

Exercise 16.

(16.1) $z_+ = 3/2$ and $z_- = -1/2$: solve $f_{-3/4}(z) = z$, $z^2 - 3/4 = z$, $z^2 - z - 3/4 = 0$.

(16.2) $z_- = -1/2$, $z_{-1} = 1/2$, $z_{-2} = \sqrt{5}/2$, $z_{-3} = (\sqrt{3+2\sqrt{5}})/2$, $z_{-4} = 1.454\ldots$, $z_{-5} = 1.484\ldots$, $z_{-6} = 1.494\ldots$, $z_{-7} = 1.498\ldots$; thus, the sequence appears to converge to 1.5.

(16.3) Since $z_{-(n+1)} = \sqrt{z_{-n} + 3/4}$, consider the function $g : [1/4, 3/2] \to [1/4, 3/2]$, $g(x) = \sqrt{x + 3/4}$. Since $g'(x) = 1/(2\sqrt{x + 3/4})$, it follows that $|g'(x)| \leq |g'(1/4)| = 1/2 < 1$ for every $x \in [1/4, 3/2]$. By the real Mean Value Theorem, if $z_n \in [1/4, 3/2]$ then there exists a real number $x \in [1/4, 3/2]$ such that

$$z_{-(n+1)} - 3/2 = g(z_{-n}) - g(3/2) = (z_{-n} - 3/2) \cdot g'(x).$$

Consequently, $|z_{-(n+1)} - 3/2| \leq |z_{-n} - 3/2|/2$, and, therefore, the sequence of preimages (z_{-n}) converges to $z = 3/2$.

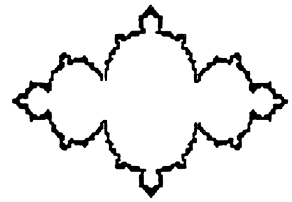


Exhibit 5.

(16.4) From $z_+ = 3/2$, the sequence of preimages (z_{-n}) traces exhibit 5.

Exercise 17. *Proof.* Apply the chain rule, $(f \circ g)' = (f' \circ g)g'$:

$$\left(f_c^{\circ 2}\right)'(z_*) = f_c'\left(f_c(z_*)\right) f_c'(z_*) = f_c'(z_*) f_c'(z_*) = \left(f_c'(z_*)\right)^2.$$

Then apply induction on n. □

Exercise 18. *Proof.* The results of exercise 12 establish that a point z_0 initiates a divergent sequence if, but only if, there exists an index $n \in \mathbb{N}$ such that $|z_n| > r_c$. This means that some term z_n lies outside the closed disc with radius $r_c = (1 + \sqrt{1 + 4|c|})/2$; thus, $z_n \in \mathbb{C} \setminus \overline{D(0, r_c)}$. Since $z_n = f_c^{\circ n}(z_0)$, this also means that

$$z_0 = \left(f_c^{\circ n}\right)^{-1}(z_n) \in \left(f_c^{\circ n}\right)^{-1}\left(\mathbb{C} \setminus \overline{D(0, r_c)}\right).$$

Consequently,

$$A_{\infty,c} = \bigcup_{n=0}^{\infty} \left(f_c^{\circ n}\right)^{-1}\left(\mathbb{C} \setminus \overline{D(0, r_c)}\right).$$

Since $\mathbb{C} \setminus \overline{D(0, r_c)}$ is an open set, and since $f_c^{\circ n}$ is continuous, it follows that each preimage $\left(f_c^{\circ n}\right)^{-1}\left(\mathbb{C} \setminus \overline{D(0, r_c)}\right)$ is open. As a union of open sets, $A_{\infty,c}$ is open. □

Exercise 19. *Proof.* To show that $f_c(A_{\infty,c}) \subset A_{\infty,c}$, suppose that $z_0 \in A_{\infty,c}$. Then (z_n) diverges to infinity, which means that the sequence that starts at $z_1 = f(z_0)$ also diverges to infinity, which also means that $z_1 \in A_{\infty,c}$. Therefore, $f_c(A_{\infty,c}) \subset A_{\infty,c}$, and, equivalently, $A_{\infty,c} \subset f_c^{-1}(A_{\infty,c})$; thus, $f_c(A_{\infty,c}) \subset A_{\infty,c} \subset f_c^{-1}(A_{\infty,c})$.

For the reverse inclusions, select either preimage of z_0, for instance $z_{-1} = \sqrt{z_0 - c}$, so that $f_c(z_{-1}) = z_0$. If $z_0 \in A_{\infty,c}$ then the sequence (z_n) diverges to infinity, and so does the sequence that starts from z_{-1} since both sequences have the same terms with indices shifted by one. Consequently, $z_{-1} \in A_{\infty,c}$, which means that $A_{\infty,c} \subset f_c(A_{\infty,c})$. Since the same argument holds for all preimages of $z_0 \in A_{\infty,c}$, it follows that $f_c^{-1}(A_{\infty,c}) \subset A_{\infty,c}$. □

Exercise 20. If $z_0 \in f_c(J_c)$ then there exists a point $z_{-1} \in J_c$ such that $f_c(z_{-1}) = z_0$. By continuity of f_c, for each open disc $D(z_0, r)$ the set $f_c^{-1}(D(r_0, r))$ is open and contains z_{-1}; hence it contains an open disc $D(z_{-1}, s) \subset f_c^{-1}(D(z_0, r))$. Since $z_{-1} \in J_c = \partial K_c$, it follows that $D(z_{-1}, s)$ contains a point $p \in K_c$ and a point $q \in A_{\infty,c}$. Consequently, $f_c(p) \in f_c(K_c) \cap f_c(D(z_{-1}, s)) = K_c \cap D(z_0, r)$ and $f_c(q) \in f_c(A_{\infty,c}) \cap f_c(D(z_{-1}, s)) = A_{\infty,c} \cap D(z_0, r)$. Therefore, $z_0 \in \partial K_c$.

Moreover, the preceding argument also shows that if $z_{-1} \in J_c$ then $f_c(z_{-1}) \in J_c$, which proves that $J_c \subset f_c^{-1}(J_c)$.

Exercise 21. By proposition 5, $f_c(K_c^\circ) \subset f_c(K_c) \subset K_c$, and, by the Open Mapping Theorem, $f_c(K_c^\circ)$ is an open set. Hence, $f_c(K_c^\circ) \subset K_c^\circ$. If also follows that $K_c^\circ \subset f_c^{-1}(K_c^\circ)$. Similarly, $f_c^{-1}(K_c^\circ) \subset f_{c1}(K_c) \subset K_c$ by proposition 5, and $f_c^{-1}(K_c^\circ)$ is open by continuity of f_c, hence $f_c^{-1}(K_c^\circ) \subset K_c^\circ$.

Moreover, if $z_0 \in K_c^\circ$ then there exist preimages of z_0. For each such preimage, $z_{-1} \notin A_{\infty,c}$ by exercise 19, and $z_{-1} \notin \partial K_c = J_c$ by exercise 20. Therefore, $z_{-1} \in \mathbb{C} \setminus (A_{\infty,c} \cup J_c) = K_c^\circ$ and $K_c^\circ \subset f_c(K_c^\circ)$.

Exercise 22. If $z_0 \in J_c$ then there exist two preimages z_+ and z_- (which need not be distinct from each other) such that $f_c(z_+) = z_0 = f_c(z_-)$, provided by the quadratic formula (or the Fundamental Theorem of Algebra). If z_1 denotes any preimage of z_0 (thus, $z_{-1} \in \{z_+, z_-\}$), then $z_{-1} \notin K_c^\circ$ by exercise 21, and $z_{-1} \notin A_{\infty,c}$ by exercise 19. Therefore, $z_{-1} \in \mathbb{C} \setminus (K_c^\circ \cup A_{\infty,c}) = \partial K_c = J_c$, and $z_0 = f_c(z_{-1}) \in f_c(J_c)$, hence $J_c \subset f_c(J_c)$.

In addition, the preceding argument also shows that if $z_{-1} \in f_c^{-1}(J_c)$ then $z_{-1} \in J_c$, which proves that $f_c^{-1}(J_c) \subset J_c$.

Finally, the results just obtained in the present exercise, and those from exercises 20, establish that $f_c(J_c) = J_c = f_c^{-1}(J_c)$.

Exercise 23. $|f'_{-3/4}(-1/2) = |2 \times (-1/2)| = |-1| = 1$ and $|f'_{-3/4}(3/2)| = |2 \times 3/2| = 3 > 1$.

Exercise 24. Both are repelling. $|f'_{-2}(-1)| = |2 \times (-1)| = 2 > 1$ and $|f'_{-2}(2)| = |2 \times 2| = 4 > 1$.

Exercise 25.

Solution with conformal mappings. $|1 \pm \sqrt{1-4c}| = 1$ if, but only if, $1 \pm \sqrt{1-4c}$ lies on the unit circle, which means that $\pm\sqrt{1-4c}$ lies on the circle C with radius 1 and center at $(1, 0)$. Hence, the squaring function maps $\pm\sqrt{1-4c}$ to $1-4c$ but it also maps the circle C onto the *cardioid* with polar equation $r = 1 + \cos(\theta)$ (see Spiegel's Shaum's Outline [14], chapter 8, page 210, mapping C-2). A rotation by one-half turn (a multiplication by -1) maps $1-4c$ to $4c-1$, then a translation to the right by 1 gives $4c$, and finally a compression by a factor of $1/4$ yields c and the cardioid in exhibit 6.

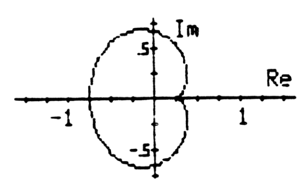


Exhibit 6.

Solution with algebra in polar coordinates. $1 \pm \sqrt{1-4c} = (\cos\theta, \sin\theta) = e^{i\theta}$. Thus,

$$1 - 4c = (e^{i\theta} - 1)^2 = e^{i2\theta} - 2e^{i\theta} + 1 = (\cos(2\theta) - 2\cos\theta + 1\ ,\ \sin(2\theta) - 2\sin\theta)$$

$$= \left(2(\cos\theta)^2 - 2\cos\theta\ ,\ 2\sin\theta\cos\theta - 2\sin\theta\right) = 2(\cos\theta - 1)(\cos\theta\ ,\ \sin\theta)$$

which has modulus $r(\theta) = 2(\cos\theta - 1)$ and argument $\mathrm{Arg}(\cos\theta, \sin\theta) = \theta$. Therefore, $1 - 4c$ lies on the cardioid with polar equation $r = 2(\cos\theta - 1)$.

Exercise 26. If z_* is an attracting fixed point of f_c then $|f_c'(z_*)| < 1$. By continuity of f_c', there exists an open disc $D(z_*, r)$ such that $|f_c'(z_0)| < 1/2 + |f_c'(z_*)|/2 < 1$ for each $z_0 \in D(z_*, r)$. With an integral along the straight line segment from z_* to z_0, the Fundamental Theorem of Complex Calculus yields

$$|z_1 - z_*| = |f_c(z_0) - f_c(z_*)| = \left|\int_{z_*}^{z_0} f_c'(z)dz\right|$$

$$\leq |z_0 - z_*| \sup\{|f_c'(z)| : z \in D(z_*, r)\} \leq |z_0 - z_*| \cdot (1 + |f_c'(z_*)|)/2 < |z_0 - z_*|.$$

Hence, induction shows that (z_n) converges to z_*; in particular, $z_0 \in K_c$.

Exercise 27.

(27.1) See exhibit 1a, in the introduction to this chapter.

(27.2) See exhibit 1b.

(27.3) See exhibit 1c.

(27.4) See exhibit 1d.

(27.5) See exhibit 1e.

Exercise 28. The sequence of preimages clusters near the points -1 and 1.

Exercise 29. The sequence gets trapped and alternates between two points.

Exercise 30. The Butterfly Effect consists of the great sensitivity of changes in the weather due to small initial perturbations; such sensitivity may give us the impression of chaos, because we do not know what small initial perturbations cause the large changes that we observe. Similarly, sequences $(z_n) = (f_c^{\circ n}(z_0))$ of iterations of a function f_c may exhibit great sensitivity to the initial point, z_0, especially if z_0 lies on the Julia set J_c: every perturbation that keeps z_0 in the filled Julia set K_c keeps the sequence bounded, whereas any perturbation of z_0 into the complement $\mathbb{C} \setminus K_c$ of the filled Julia set — even one too small to be measured — causes the sequence to diverge to infinity.

7.2. Hints for the term projects

Project 1. The present line of thoughts expands on a comment in Falconer's *Fractal Geometry*, [34], page 204, and shows that the study of the filled Julia set of each complex quadratic polynomial reduces to that of an associated polynomial of the type f_c. To this end, let p, q, and j denote three complex coefficients, with $p \neq 0$, and let

$$f : \mathbb{C} \to \mathbb{C}, \; f(w) = pw^2 + qw + j.$$

(1) Determine complex numbers a, b, and c, in terms of p, q, and j, such that for each $w \in \mathbb{C}$

$$pw^2 + qw + j = \left((aw + b)^2 + c - b\right)/a.$$

(2) Consider the function

$$h : \mathbb{C} \to \mathbb{C}, \; h(w) = aw + b.$$

Solve $h(w) = z$ for w to obtain a formula for the inverse function $h^{-1} : \mathbb{C} \to \mathbb{C}$, $h^{-1}(z) = ?$ Verify that $h\left(h^{-1}(z)\right) = z$ and $h^{-1}\left(h(w)\right) = w$.

(3) Prove that $f = h^{-1} \circ f_c \circ h$; thus, $f(w) = h^{-1}\left(f_c(h(w))\right)$ for every $w \in \mathbb{C}$.

(4) Deduce that $(w_n) = (f^{\circ n}(w_0))$ remains bounded if, but only if, $(z_n) = (f_c^{\circ n}(h(w_0)))$ remains bounded.

(5) Conclude that the filled Julia set of f is the image under h^{-1} of the filled Julia set of f_c. Multiplication by a rotates the Julia set by $\operatorname{Arg}(a)$ and expands (if $|a| > 1$) or contracts (if $|a| < 1$) it by a factor of $|a|$. Adding b translates the Julia set by b.

Project 2.

(1) With $c = -2$ the sequence becomes 0, $f_{-2}(0) = 0^2 - 2 = -2$, $f_{-2}(-2) = (-2)^2 - 2 = 2$, $f_{-2}(2) = 2^2 - 2 = 2$, and thereafter the sequence stays at 2 forever; in particular, it remains bounded, and thus $-2 \in \mathcal{M}$.

(2) Suppose that $|c| > 2$ and set $d = |c| - 2 > 0$, so that $|c| = 2 + d$. Then $w_0 = 0$, $w_1 = f_c(0) = 0^2 + c = c$, $w_2 = f_c(w_1) = c^2 + c$, and

$$|w_2| = |c^2+c| = |c(c+1)| = |c|\cdot|c+1| \geq |c|\cdot\big||c|-1\big| = |c|\cdot|(2+d)-1| = |c|(1+d) > |c|+2d.$$

Suppose that $|w_n| > |c| + 2 \cdot 4^{n-2}d$, which holds for $n = 2$. Then

$$|w_{n+1}| = |w_n^2 + c| \geq \big||w_n|^2 - |c|\big| > \left|\left(|c| + 2 \cdot 4^{n-2}d\right)^2 - |c|\right|$$

$$= \left||c|^2 + 2|c|2 \cdot 4^{n-2}d + (2 \cdot 4^{n-2}d)^2 - |c|\right|$$

$$> |c|^2 - |c| + 2 \cdot 2 \cdot 2 \cdot 4^{n-2}d > |c| + 2 \cdot 4^{n-1}d.$$

Therefore, $|w_n| > |c| + 2 \cdot 4^{n-2}d$, which diverges to infinity. This proves that if $|c| > 2$ then $c \notin \mathcal{M}$.

(3) Show that if $c \in \mathcal{M}$, then $\bar{c} \in \mathcal{M}$.

(4) Since $c \notin \mathcal{M}$ if $|c| > 2$, you need only test those complex c for which $|c| \leq 2$.

Second, for a popular algorithm, prove the following proposition.

If $|w_n| > \max\{2, |c|\}$ then (w_n) diverges to infinity.

Proof. Case 1: $|c| \leq 2$. If $|c| \leq 2$ then $\max\{2, |c|\} = 2$. Set $d = |w_n| - \max\{2, |c|\} = |w_n| - 2 > 0$, so that $|w_n| = 2 + d$. Then

$$|w_{n+1}| = |w_n^2 + c| \geq ||w_n|^2 - |c|| = (2+d)^2 - |c| = 4 + 2d + d^2 - |c|$$
$$= 2 + 2d + d^2 + (2 - |c|) \geq 2 + 2d.$$

Henceforth, $|w_{n+k}|$ diverges to infinity as $2 + 2^k d$ does.

Case 2: $|c| > 2$. If $|c| > 2$ then $\max\{2, |c|\} = |c|$. Set $d = |w_n| - \max\{2, |c|\} = |w_n| - |c| > 0$, so that $|w_n| = |c| + d$. Then

$$|w_{n+1}| = |w_n^2 + c| \geq ||w_n|^2 - |c|| = (|c| + d)^2 - |c| = |c|^2 + 2d|c| + d^2 - |c|$$
$$= |c|(|c| - 1) + 2d|c| + d^2 \geq |c| + 4d.$$

Thereafter, $|w_{n+k}|$ diverges to infinity as $|c| + 4^k d$ does. □

Third, choose a maximum number of iterations, $N \in \mathbb{N}$. For each complex constant c in the closed disc $\overline{D(0,2)}$ (or in the square $[-2,2] \times [-2,2]$), compute the first N points of the sequence $w_0 = 0$, $w_1 = 0^2 + c$, $w_2 = c^2 + c$, $\ldots$, $w_N = w_{N-1}^2 + c$. If $|w_N| > \max\{2, |c|\}$ then $c \notin \mathcal{M}$, by the proposition just proved; color c in white.

If $|w_N| \leq \max\{2, |c|\}$ then the results obtained so far do not guarantee that $c \in \mathcal{M}$. Nevertheless, color c (and $\bar{c}$, by symmetry) in black; the accuracy of the resulting picture of Mandelbrot's set depends upon the number of iterations.

(5) For a better algorithm, prove the preceding proposition with $r_c = (1 + \sqrt{1 + 4|c|})/2$ instead of $R_c = \max\{2, |c|\}$. Exhibit 8 displays working prototypes of programs based upon such an algorithm, which produced the pictures in exhibit 7.

Mandelbrot's set on the HP-28C&S

Tutorial for plotting Mandelbrot's set

(1) Type, `ENTER`, and `STO`re all the routines in exhibit 8

(2) Execute the command `Benoit`.

(3) Note that the HP-28 plots the upper and lower halves separately.

Mandelbrot's set on the HP-48SX

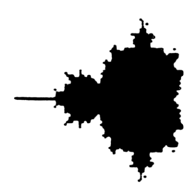


Exhibit 7.

For a different program, see William C. Wickes' book [41], pages 337–339.

```
« 1 'k' STO
WHILE DUP
 ABS rc ≤
 k 12 ≤ AND
REPEAT
 SQ OVER +
 'k' 1 STO+
END
IF ABS rc ≤
 THEN PIXEL
 ELSE DROP
END » ENTER
 'tests' STO
-------------
« CLLCD
FOR y 4 y SQ
 - √ DUP NEG
 SWAP
 FOR x x y
  R→C DUP
  DUP ABS 4
  * 1 + √ 1
  + 2 /
  'rc' STO
  tests side
 STEP side
STEP LCD→
»        ENTER
 'sweep' STO
-------------
« 137 32 /
's' STO s 2
R→C PMAX s
NEG 0 R→C
PMIN .0625
'side' STO
0 2 » ENTER
     'up' STO
-------------
« side NEG s
OVER R→C
PMAX s NEG
-2 side -
R→C PMIN -2
SWAP » ENTER
  'down' STO
-------------
« up sweep
down sweep
»        ENTER
'Benoit' STO
```

Comments on the programs for Mandelbrot's set

Subroutine **tests** examines a pixel $c = (x, y)$, starting from $z_0 = (0,0)$ and iterating $f_c(z) = z^2 + c$ at most twelve times; if $|z_k| > r_c$ at any iteration, then the subroutine leaves $c = (x, y)$ white, otherwise it colors c in black.

ENTER and STOre in 'tests'

For the HP-48SX only, **scene** adjusts the screen so that it covers the square $[-2,2] \times [-2,2]$, which contains Mandelbrot's set.

ENTER and STOre in 'scene'

For the HP-28C&S only, **up** adjusts the screen so that it covers the upper half of the square.

ENTER and STOre in 'up'

For the HP-28C&S only, **down** adjusts the screen so that it covers the lower half of the square.

ENTER and STOre in 'down'

For the HP-28C&S only, **sweep** scans each pixel $c = (x, y)$ in $D(0, 2)$, which contains Mandelbrot's set; **sweep** computes $r_c = (1 + \sqrt{1 + 4|c|})/2$ for each c and calls **tests**.

ENTER and STOre in 'sweep'

The main program, **Benoit**, manages the other subroutines; on the HP-48SX, it also performs the tasks that **sweep** does on the HP-28C&S.

ENTER and STOre in 'Benoit'

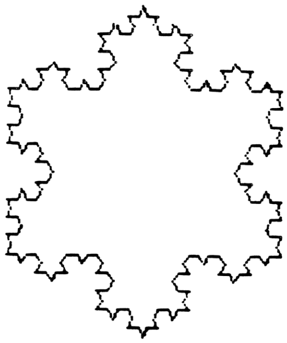
Tutorial: execute Benoit
← **on the HP-28C&S,**
on the HP-48SX →

Exhibit 8.

```
« 1 'k' STO
WHILE DUP
 ABS rc ≤
 k 12 ≤ AND
REPEAT
  SQ OVER +
  'k' 1 STO+
END
IF ABS rc ≤
 THEN DUP
  CONJ PIXON
  PIXON
 ELSE DROP
END »  ENTER
 'tests' STO
-------------
« 131 64 / 2
R→C DUP NEG
SWAP PDIM #
131d # 128d
PDIM .03125
'side' STO
»       ENTER
 'scene' STO
-------------
« scene 0 2
FOR y
 4 y SQ - √
 DUP NEG
 SWAP
 FOR x x y
  R→C DUP
  DUP ABS 4
  * 1 + √ 1
  + 2 /
  'rc' STO
  tests side
 STEP side
STEP » ENTER
'Benoit' STO
```

TUTORIAL
VAR Benoit

CHAPTER 2



FRACTALS

in Real Analysis and Topology

Summary. The first section presents the construction of von Koch's snowflake and shows that it is topologically equivalent to a circle but with infinite length. The second section explains the concepts of "Hausdorff dimension" and "fractal." The present chapter also demonstrates how the description and the explanation of the result of a simple construction (by hand or with a computer) require a substantial abstract theory.

Prerequisites. The construction and the program in the first subsection have no formal prerequisites beyond a working knowledge of precalculus. The subsequent explanations of von Koch's snowflake and its fractal nature require the prior or concurrent study of basic topology and real analysis.

CONTENTS

0. INTRODUCTION

Benoit B. Mandelbrot coined the word "fractal" in 1975 to name such sets as a circle, a segment, all Julia sets (described in the preceding chapter), and von Koch's snowflake (pictured in figure 1 and examined in the present chapter).

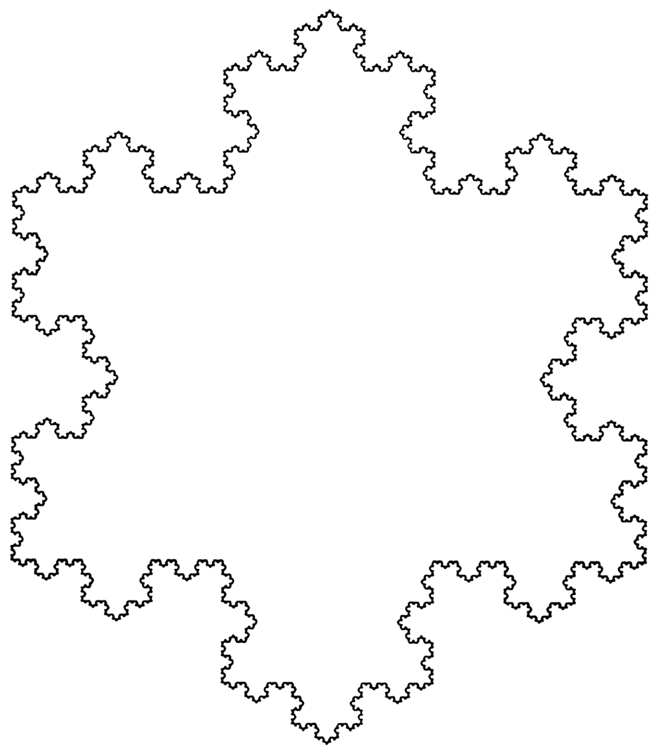


Figure 1. An example of a fractal: von Koch's snowflake.

Despite its popularity among some mathematicians and scientists, the word "fractal" has eluded (and does not need) an accurate definition, perhaps because, though intended to designate sets, the word "fractal" refers not to any type of set but to one *aspect* of all sets in the plane and in other spaces: their "Hausdorff dimension" (a concept defined in the second section). Thus, studying fractals means studying the Hausdorff dimension of any sets. To demonstrate the need for the Hausdorff dimension, the first section constructs a particular set, von Koch's snowflake, which has infinite length (a measure of size with one dimension) and null area (a measure of size with two dimensions). Consequently, any meaningful "measure" of the "size" of such a set must occur with a "fractional dimension," between one and two.

1. VON KOCH'S SNOWFLAKE

The present section constructs von Koch's snowflake and investigates its topological properties, which reveal that the snowflake is a simple closed curve, topologically equivalent to a circle, but with infinite length.

1.1. Construction and plot of von Koch's snowflake

The construction of von Koch's snowflake starts from an **oriented segment** AB in the plane. Informally, an oriented segment AB is a segment of straight line *from* a point A, designated as the *first* point, *to* a point B, designated as the *second* point. Thus, the oriented segment AB, oriented from A to B, differs from the oriented segment BA, oriented from B to A. (Formally, the oriented segment AB is equivalent to the ordered pair (A, B), which differs from the ordered pair (B, A) if $A \neq B$.)

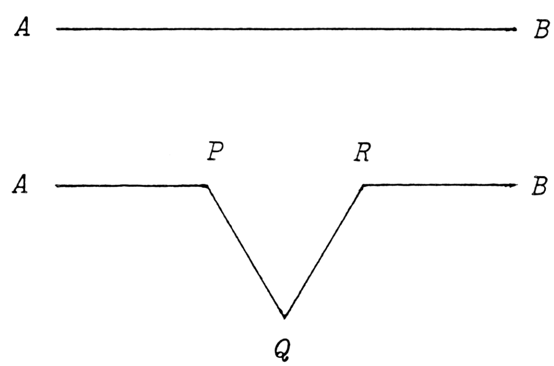


Figure 2. The basic step of von Koch's construction.

From the oriented segment AB, the construction of von Koch's snowflake proceeds inductively, through a sequence of steps. Basic to the whole construction, the first step transforms an initial oriented segment $E_0 = AB$ into an **oriented polygonal path** $E_1 = APQRB$, formally equivalent to the sequence of points (A, P, Q, R, B), as shown in figure 2, and consisting of four new oriented segments, AP, PQ, QR, and RB, each of a length equal to one third that of AB. To specify on which side of the initial segment AB the new vertex Q must lie, adopt the convention that the point Q must be on the "right-hand side" of the oriented segment AB; mathematically, this means that the ordered basis $(\vec{AQ}, \vec{AB})$ is right-handed, or, equivalently, that the matrix with two rows and two columns $(\vec{AQ}, \vec{AB})$ has a positive determinant. Since all four new segments have equal lengths, the convention also means that to construct PQ it suffices to rotate AP by $-\pi/3$ and then to translate A to P. (Also, applied to the three sides of a triangle to produce von Koch's snowflake, the orientation just described agrees with the orientation of the boundary as defined in algebraic topology.)

After the first step has transformed the initial oriented segment $E_0 = AB$ into the oriented polygonal path $E_1 = APQRB$, which forms the first stage of von Koch's

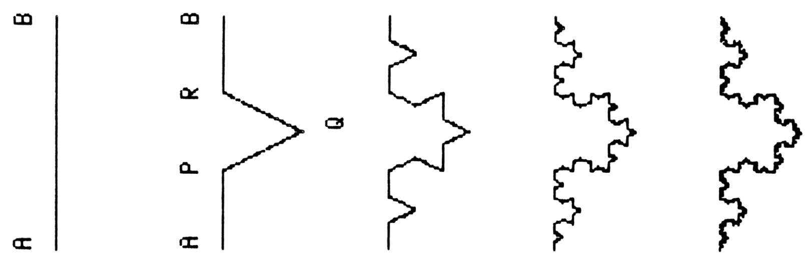
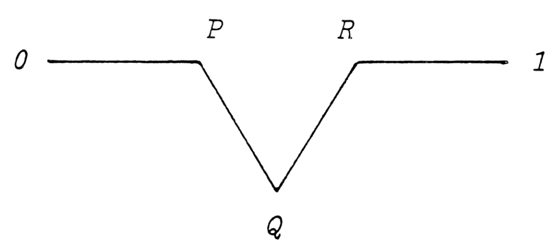


Exhibit 1. The first five stages of von Koch's construction, plotted by the HP-28S.

construction, the second step applies the same transformation to each of the four segments AP, PQ, QR, and RB. This produces an oriented polygonal path E_2 (the second stage of the construction) with four times four (sixteen) segments, each with length one third that of the preceding ones, in effect one ninth that of AB, and so forth. In general, the k-th step produces an oriented polygonal path E_k (the k-th stage) with 4^k segments, each with length $(1/3)^k$ that of AB, as in exhibit 1. Also, all the $3 \times 4^{k-1}$ new vertices (new corners) produced at the k-th step appear on the same side of the polygonal path as the previous ones, thanks to the convention about the orientation adopted in the description of the first step. The mathematical "limit" of von Koch's construction yields one side of von Koch's snowflake, denoted by E_∞. Exhibit 2 lists a program that continues to execute such steps until all the generated vertices exhaust the available memory; exhibit 1 displays the consecutive results of such steps. To explain how such a program may work (for any type of computer or supercalculator), the following exercises establish some quantitative properties of the construction.

Routine exercises

Exercise 1. Consider the unit segment, $E_0 = [0,1] \subset \mathbf{R}^2$, oriented from $0 = (0,0)$ to $1 = (1,0)$. From that interval, the first step of von Koch's construction produces a polygonal path $E_1 = 0PQR1$. Calculate the coordinates of the vertices P, Q, and R.



Exercise 2. Instead of the specific segment $[0,1]$, consider any oriented segment AB and assume as given the coordinates of its endpoints, $A = (x,y)$ and $B = (u,v)$. Calculate the coordinates of the new vertices, P, Q, and R, in terms of x, y, u, and v. You may express your results with coordinates, with vectors and matrices, or with complex numbers.

von Koch's snowflake on the HP-28C&S

```
« 'A' STO 'B' STO B A -
3 / 'C' STO B B C - A C
turn * - A C + A 1 4
START line DEPTH ROLLD
NEXT »
      ENTER  'Koch'  STO
------------------------
« CLLCD 1 DEPTH 2 -
IFERR
 START Koch
 NEXT
THEN CLEAR 30 SF
ELSE PRLCD CR CR
END DEPTH ROLLD
      ENTER  'pass'  STO
------------------------
« -1 3 √ INV R→C 1.5 *
'turn' STO 3 √ 2 * DUP
32 * 'side' STO INV DUP
137 * 64 / DUP 0 R→C
PMAX SWAP R→C NEG PMIN
(.5,0) (-.5,0) »
       ENTER  'set'  STO
------------------------
« set CLLCD line 30 CF
  WHILE 30 FC?
  REPEAT pass
  END CLEAR »
     ENTER  'flake'  STO
------------------------
« DUP2 DUP2 - DUP ABS
side * CEIL DUP ROT ROT
/ 'dt' STO 1 SWAP
START DUP PIXEL dt +
NEXT DROP2 »
      ENTER  'line'  STO
```

TUTORIAL: USER flake

Comments on the programs for von Koch's snowflake

With points A on level 1 and B on level 2, this subroutine leaves A, P, Q, R, B on the stack, with
$R = B - (B - A)/3$,
$Q = A - (-1, 1/\sqrt{3})(B - A)/2$,
$P = A + (B - A)/3$.

ENTER and STOre in 'Koch'

With an oriented polygonal path (complex numbers) on the stack, this subroutine performs one elementary step of von Koch's construction once for each pair of consecutive vertices, thus leaving the next entire stage on the stack.

ENTER and STOre in 'pass'

This subroutine computes turn $= (-3, \sqrt{3})/2$ (thus turn $*(A - B) = (Q - A)$), the number of pixels per unit length side, and the dimensions of the screen; it leaves the initial stage on the stack.

ENTER and STOre in 'set'

This is the main program; it iterates von Koch's construction.

ENTER and STOre in 'flake'

Only for the HP-28C&S, this subroutine draws the segment joining two points on the stack.

ENTER and STOre in 'line'

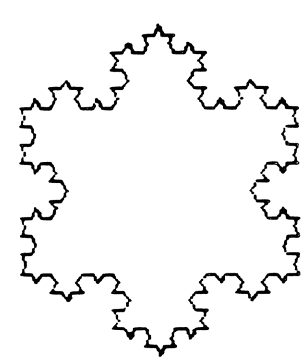
Tutorial: execute flake in the USER or VAR menu.

Exhibit 2.

von Koch's snowflake on the HP-48SX

```
« 'A' STO 'B' STO B A -
3 / 'C' STO B B C - A C
turn * - A C + A 1 4
START
DUP2 LINE DEPTH ROLLD
NEXT »
      ENTER  'Koch'  STO
------------------------
« ERASE 1 DEPTH 2 -
IFERR
 START Koch
 NEXT
THEN CLEAR 30 SF
ELSE PICT RCL 'snow' STO
END DEPTH ROLLD »
      ENTER  'pass'  STO
------------------------
« -1 3 √ INV R→C 1.5 *
'turn' STO 1 3 √ INV R→C
NEG 1 3 √ R→C PDIM #
131d # 131d PDIM (-1,0)
0 3 √ R→C (1,0) (-1,0) »
       ENTER  'set'  STO
------------------------
« set 30 CF
  WHILE 30 FC?
  REPEAT pass
  END CLEAR »
     ENTER  'flake'  STO
```

TUTORIAL: VAR flake



1.2. Cauchy sequences of continuous functions

The construction of von Koch's snowflake presented in the preceding subsection proceeds with a sequence of elementary steps, but the sequence contains infinitely many such steps, which means that the final object — von Koch's snowflake — is a limit. Therefore, the analysis of such sets as von Koch's snowflake requires standard facts about limits of sequences of functions, which will be reviewed in the present subsection.

Definition 1. A **curve** in the plane is a continuous function $f : [0,1] \to \mathbb{R}^2$, $t \mapsto f(t) = (X(t), Y(t))$, from the closed unit segment into the plane. (The image $f([0,1]) \subset \mathbb{R}^2$ traced by f corresponds to the intuitive concept of curve.)

EXAMPLE 1. The function $s : [0,1] \to \mathbb{R}^2$ with $s(t) = (\cos(2\pi t), \sin(2\pi t))$ traces the unit circle, with center at the origin and radius equal to one.

EXAMPLE 2. The function $f_0 : [0,1] \to \mathbb{R}^2$ with $f_0(t) = (t, 0)$ traces the unit segment $[0,1] \times \{0\}$ on the first coordinate axis.

Exercise 3. Write a formula, $f_1(t) = (X(t), Y(t))$, for the curve $0PQR1$ corresponding to the first stage of the construction of von Koch's snowflake (with four segments).

Definition 2. For each pair of real numbers $a < b$ the notation $C^0([a,b], \mathbb{R}^2)$ represents the linear space of all continuous functions from $[a,b]$ into $\mathbb{R}^2$, endowed with the **maximum norm,** denoted by $\| \|_\infty$ and defined for each function $f \in C^0([a,b], \mathbb{R}^2)$ by

$$\|f\|_\infty = \max\{\|f(t)\| : t \in [a,b]\},$$

with $\| \|$ denoting the usual Euclidean norm on $\mathbb{R}^2$. Thus, $\|f\|_\infty$ represents the greatest magnitude that the vector $f(t) = (X(t), Y(t))$ reaches as t traces $[a,b]$.

REMARK 1. The notation $\| \|_\infty$ for the maximum norm comes from a standard exercise in real analysis, which pertains to the p-norms defined by $\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}$ and which shows that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$. (See the references by Folland, [22], #7, p. 179, and Rudin, [20], #4, p. 71.)

Definition 3. A sequence (f_n) of functions in $C^0([a,b], \mathbb{R}^2)$ **converges** to a function $f \in C^0([a,b], \mathbb{R}^2)$ if, but only if, $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$.

Informally, the definition of "convergence" means that the sequences of vectors $(f_n(t)) = (X_n(t), Y_n(t))$ converge to $f(t) = (X(t), Y(t))$ at a rate independent of t.

Definition 4. A **Cauchy sequence** (f_n) of functions in $C^0([a,b], \mathbb{R}^2)$ is a sequence such that for each positive real number $\varepsilon > 0$ there exists an integer N_ε such that if $n > N_\varepsilon$ and if $m > N_\varepsilon$ then $\|f_n - f_m\|_\infty < \varepsilon$.

Informally, the functions in a Cauchy sequence get as close as we want to one another, but they need not converge to a limit, except in certain particular spaces of functions, for instance, in $C^0([a,b], \mathbb{R}^2)$.

Theorem 1. *Each Cauchy sequence in* $C^0([a,b],\mathbf{R}^2)$ *converges to a limit.*

Proof. Firstly, observe that for each number $t \in [a,b]$ the sequence of vectors $(f_n(t))$ also forms a Cauchy sequence, in $\mathbf{R}^2$, because $\|f_n(t) - f_m(t)\| \leq \|f_n - f_m\| < \varepsilon$ if $n, m > N_\varepsilon$. Consequently, by completeness of the real numbers, each sequence $(f_n(t))$ converges to some limit, which we denote by $f(t)$. That is, $f(t) = \lim_{n\to\infty} f_n(t)$.

Secondly, the limit function $f : [a,b] \to \mathbf{R}^2$, $f(t) = \lim_{n\to\infty} f_n(t)$ is continuous: if $r, s \in [a,b]$, then the triangle inequality gives

$$\|f(r) - f(s)\| = \|f(r) - f_n(r) + f_n(r) - f_n(s) + f_n(s) - f(s)\|$$

$$\leq \|f(r) - f_n(r)\| + \|f_n(r) - f_n(s)\| + \|f_n(s) - f(s)\|.$$

Since $(f_n(r))$ and $(f_n(s))$ converge to $f(r)$ and $f(s)$ respectively, for every $\varepsilon > 0$ there exists an index $N_{\varepsilon/3}$ such that if $n > N_{\varepsilon/3}$ then $\|f(r) - f_n(r)\| < \varepsilon/3$ and $\|f(s) - f_n(s)\| < \varepsilon/3$. Moreover, since each f_n is continuous on the closed and bounded, hence compact, interval $[a,b]$, it follows that each f_n is *uniformly* continuous, which means that there exists a positive real number $\delta_{\varepsilon/3}$ such that if $r, s \in [a,b]$ and if $|r - s| < \delta_{\varepsilon/3}$ then $\|f_n(r) - f_n(s)\| < \varepsilon/3$. Therefore, if $|r - s| < \delta_{\varepsilon/3}$ then

$$\|f(r) - f(s)\| \leq \|f(r) - f_n(r)\| + \|f_n(r) - f_n(s)\| + \|f_n(s) - f(s)\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which means that f is (uniformly) continuous on $[a,b]$; thus, $f \in C^0([a,b],\mathbf{R}^2)$.

Thirdly, the sequence (f_n) converges to its limit f with respect to the maximum norm: for each $\varepsilon > 0$ and for each $t \in [a,b]$ there exists an index $N_{t,\varepsilon/2}$ such that if $m > N_{t,\varepsilon/2}$ then $\|f(t) - f_m(t)\| < \varepsilon/2$, because $(f_m(t))$ converges to $f(t)$. Also, there exists an index $N_{\varepsilon/2}$ such that if $n, m > N_{\varepsilon/2}$ then $\|f_n - f_m\| < \varepsilon/2$, because (f_n) is a Cauchy sequence. Consequently, if $m > N_{t,\varepsilon/2}$ and if $n > N_{\varepsilon/2}$ then

$$\|f(t) - f_n(t)\| \leq \|f(t) - f_m(t)\| + \|f_m(t) - f_n(t)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since n does not depend upon t, this inequality means that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$. □

Exercises

The following exercises establish the topological properties of von Koch's snowflake. To this end, let $f_n : [0,1] \to \mathbf{R}^2$ denote the n-th stage of the construction of von Koch's snowflake. For example, $f_0(t) = (t, 0)$ represents the initial stage, which traces the unit segment, and f_1 represents the first stage, with four segments, and so forth.

Firstly, von Koch's snowflake is a continuous curve.

Exercise 4. Prove that $\|f_{n+1} - f_n\|_\infty \leq 1/(3^n 2\sqrt{3})$. Deduce that if $0 \leq m < n$ then $\|f_n - f_m\|_\infty \leq ({}^1\!/\!_3)^m$. Conclude that the sequence (f_n) converges uniformly to a continuous function $f : [0,1] \to \mathbf{R}^2$. Its image, $E_\infty = f([0,1])$, forms one side of the snowflake.

Secondly, von Koch's complete snowflake — constructed from each of the three sides of an equilateral triangle — is homeomorphic to a circle.

Exercise 5. By induction, prove that at each stage of the construction the function f_n is **accretive** (or **expansive**), which means that if $r, s \in [a, b]$ then $\|f_n(r) - f_n(s)\| \geq |r - s|$. Deduce that the limit, f, is also accretive ($\|f(r) - f(s)\| \geq |r - s|$), and conclude that f is injective: if $r \neq s$ then $f(r) \neq f(s)$. Finally, prove that each side of the snowflake is topologically homeomorphic to the unit segment, $[0, 1]$, which means that the inverse function f^{-1} is also continuous on $f([0, 1])$. Thus, the snowflake has topological dimension 1. To show that the entire snowflake is homeomorphic to a circle, observe that the three sides intersect one another only at their endpoints.

Thirdly, von Koch's snowflake has infinite length.

Exercise 6. Prove that the length of the image of f_{n+1} equals $4/3$ that of f_n. Then find a double sequence of points $\left((t_{n,j})_{j=0}^{j=n}\right)_{n=0}^{\infty}$, which you may consider as a sequence $(t_{n,j})_{j=0}^{j=n}$ for each stage $n \in \mathbb{N}$, such that for each n

$$0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n-1} < t_{n,n} = 1$$

and such that

$$\lim_{n \to \infty} \max \{|t_{n,j} - t_{n,j-1}| : j \in \{1, \ldots, n\}\} = 0,$$

$$\lim_{n \to \infty} \sum_{j=1}^{n} \|f(t_{n,j}) - f(t_{n,j-1})\| = \infty.$$

This means that for each positive length $L > 0$ there exists a polygonal path with vertices $f(t_{n,j})$ on the snowflake and with length greater than L. (The curve f — the snowflake — is then called **non-rectifiable.**)

Lastly, von Koch's snowflake (which consists only of the "edge") has null area.

Exercise 7. Prove that the image $E_\infty = f([0, 1])$ has null area. (Observe that the image consists exclusively of the curve f, which traces only the boundary of the flake.)

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2. HAUSDORFF DIMENSION AND FRACTALS

The existence of sets such as von Koch's snowflake, which are closed and bounded curves covering a null area, like a circle, but with infinite length, unlike a circle, reveals the inadequacy of the two standard concepts for measuring the size of planar sets, length and area: for von Koch's snowflake, the one-dimensional measure, length, is infinite, while the two-dimensional measure, area, is null. Neither dimension provides a satisfactory measure of the size of such a set. This demonstrates the need for a concept of non-integral dimensions and for a method of measuring size in such non-integral dimensions, that is, a measure between length and area. To develop such a method, the first two subsections, below, review the necessary concepts from analysis and topology, and the third subsection demonstrates by examples how one might determine the dimension of a set.

2.1. The Gamma function and the volume of balls

Just as any concept of measure requires a unit, the Hausdorff measure presented here relies upon the closed **unit balls** in $\mathbf{R}^n$, denoted by $\overline{B(0,1)}$, which consist of all the points at Euclidean distance at most one from the origin. The following concepts pertain to the n-dimensional volume of the unit ball in $\mathbf{R}^n$, which constitutes the unit for measuring the size of other sets.

Definition 5. The **gamma function** is the function $\Gamma :]0,\infty[\to \mathbf{R}$ defined by

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx.$$

EXAMPLE 3. For example,

$$\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx = -e^{-x}|_0^\infty = -0-(-1) = 1.$$

From integration by parts follows the recursion formula $\Gamma(t+1) = t\Gamma(t)$ for every $t > 0$:

$$\Gamma(t+1) = \int_0^\infty x^{t+1-1}e^{-x}dx = x^t(-e^{-x})|_0^\infty - \int_0^\infty tx^{t-1}(-e^{-x})dx = 0 + t\Gamma(t).$$

From the recursion formula and $\Gamma(1) = 1$ follow

$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1\cdot 1 = 1$,
$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 = 2$,
$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2 = 6$, and so forth,
$\Gamma(n) = \Gamma(n+[n-1]) = (n-1)\Gamma(n-1) = (n-1)!$ for each positive integer n.

The calculation of values of Γ for non-integral arguments involves more substantial analysis. For instance, multiples of one-half require the following integral.

Lemma 1.

$$\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}.$$

Proof. For details, see Robert Weinstock's note [9], which presents an elementary proof along the following line of thought. Consider the function f $[0,\infty[\to \mathbf{R}$, with

$$f(x) = \int_0^1 \frac{e^{-x(1+t^2)}}{1+t^2}dt.$$

Notice that $f(0) = \text{Arctan}(1) = \pi/4$, that $\lim_{x\to\infty} f(x) = 0$, and that

$$f'(x) = -\int_0^1 e^{-x(1+t^2)}dt = -\frac{e^{-x}}{\sqrt{x}}\int_0^{\sqrt{x}} e^{-u^2}du$$

(after the change of variable $u = t\sqrt{x}$). Then define $g : \mathbf{R} \to \mathbf{R}$ by

$$g(z) = \int_0^z e^{-u^2} du$$

and observe that

$$-\frac{\pi}{4} = \lim_{x\to\infty} (f(x) - f(0)) = \int_0^\infty f'(x)dx = -\int_0^\infty \frac{e^{-x}}{\sqrt{x}} g(\sqrt{x})dx$$

$$= -2\int_0^\infty g'(z)g(z)dz = 0 - \left(\int_0^\infty e^{-u^2} du\right)^2 . \square$$

REMARK 2. The integral just established is the basis of the "normal" Gaussian distribution in probability and statistics. Another proof proceeds from cartesian to polar coordinates in a double integral:

$$\left(\int_0^\infty e^{-u^2} du\right)^2 = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta = \frac{\pi}{2}\int_0^\infty \frac{-e^{-s}}{2} ds = \frac{\pi}{4}.$$

The alternate proof just sketched is an exercise in many calculus texts, but a rigorous proof of the necessary ingredients usually appears only in analysis texts.

EXAMPLE 4. For $t = 1/2$, setting $u = \sqrt{x}$, $x = u^2$, and $dx = 2u\, du$ gives

$$\Gamma(1/2) = \int_0^\infty x^{1-1/2} e^{-x} dx = \int_0^\infty u^{-1} e^{-u^2} 2u du = 2\int_0^\infty e^{-u^2} du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Hence, the recursion formula gives

$\Gamma(3/2) = \Gamma(1 + 1/2) = (1/2)\Gamma(1/2) = \sqrt{\pi}/2$,

$\Gamma(5/2) = \Gamma(3/2 + 1) = (3/2)\Gamma(3/2) = 3\sqrt{\pi}/4$, and so forth.

Exercise 8. Calculate $\Gamma(3.5)$.

Definition 6. For each positive real number d let $\alpha_d = \Gamma(1/2)^d/\Gamma(1 + d/2)$.

EXAMPLE 5.

$\alpha_0 = \Gamma(1/2)^0/\Gamma(1 + 0/2) = \sqrt{\pi}^0/\Gamma(1) = 1/1 = 1$,

$\alpha_1 = \Gamma(1/2)^1/\Gamma(1 + 1/2) = \sqrt{\pi}/(\sqrt{\pi}/2) = 2$,

$\alpha_2 = \Gamma(1/2)^2/\Gamma(1 + 2/2) = (\sqrt{\pi})^2/1 = \pi$,

$\alpha_3 = \Gamma(1/2)^3/\Gamma(1 + 3/2) = (\sqrt{\pi})^3/(3\sqrt{\pi}/4) = 4\pi/3$.

Notice that

$\alpha_0 = 1$ is the number of point(s) in the singleton $\{0\}$,

$\alpha_1 = 2$ is the length of $[-1, 1]$,

$\alpha_2 = \pi$ is the area of the unit disc, and that

$\alpha_3 = 4\pi/3$ is the volume of the unit ball in $\mathbf{R}^3$.

A theorem from analysis also guarantees that α_d equals the d-dimensional volume of the unit ball in $\mathbf{R}^d$ for each integral d (see Folland's *Real Analysis* text, [22], pages 76–77). For non-integral values of d, the interpretation of α_d as the volume of a ball no longer holds, but the numbers α_d will provide a unit of volume for all dimensions, consistent with that for integral dimensions.

REMARK 3. (SUPERCALCULATORS) Such supercalculators as the HP-28, the HP-48, and their predecessor the HP-15C, have built-in commands (called FACT or !, by analogy to factorials of positive integers) to compute the Gamma function, as demonstrated in table 1.

Table 1. Computation of the Gamma function with the HP-28(C&S) and HP-48

Keys	Comments	Display
.5 [red] [REAL] [FACT]	(HP-28) Compute $\Gamma(0.5)$.	.886226925453
.5 [MTH] [PROB] [!]	(HP-48) Compute $\Gamma(0.5)$.	.886226925453
[red] π [shift] [NUM] [$\sqrt{x}$] 2 [$\div$]	Verify with $\sqrt{\pi}/2$.	.886226925455

2.2. Definition of Hausdorff dimension and measure

The determination of the dimension and size of a set consists essentially of covering the set with countably many smaller sets, each with positive diameter at most δ, and then adding not necessarily their length, δ, or area, $\pi(\delta/2)^2$, or volume, $4/3\pi(\delta/2)^3$, but other powers, δ^d. The analysis of the method just described reveals that for each set there exists one (exactly one) exponent $d \geq 0$ that yields a measure between zero and infinity. That exponent will be called the "Hausdorff dimension" of the set. A detailed exposition of these concepts requires the definitions in this subsection.

The following definitions review the foundational concepts related to the construction of the real numbers (see Spivak's *Calculus* text, [7], chapter 28).

Definition 7. (Maximum and minimum) Consider a subset S of the real line. If there exists a number $M \in \mathbf{R}$ such that both $M \in S$ and $s \leq M$ for every $s \in S$, then M is called the **maximum** of S, and is denoted by $\max(S)$. (If no such number M exists, then S has no maximum.)

Similarly, if there exists a number $m \in \mathbf{R}$ such that both $m \in S$ and $m \leq s$ for every $s \in S$, then m is called the **minimum** of S, and is denoted by $\min(S)$. (If no such number m exists, then S has no minimum.)

EXAMPLE 6. The closed unit segment, $S = [0, 1]$, has a maximum, $M = 1$, and a minimum, $m = 0$.

Definition 8. (Upper and lower bounds) Let S represent a subset of the real line. An **upper bound** for S is a number $B \in \mathbf{R}$ such that $s \leq B$ for every $s \in S$. Similarly, a **lower bound** for S is a number $b \in \mathbf{R}$ such that $b \leq s$ for every $s \in S$. (Some sets have no upper or lower bound.)

EXAMPLE 7. The open half line $S =]-\infty, 0[$ has no lower bound but many upper bounds, for instance, $B = .007$, $B = 1$, $B = \sqrt{2}$, $B = e$, $B = \pi$, $B = 37$, and $B = 1{,}000{,}000$ are all upper bounds for $]-\infty, 0[$.

The following theorem from analysis distinguishes the real numbers from the rational numbers (for a proof, see Spivak's text, [7], page 556).

Theorem 2. *If a set $S \subset \mathbb{R}$ has an upper bound, then the set $U \subset \mathbb{R}$ of all upper bounds for S has a minimum, $\min(U) \in U$. Similarly, if a set $S \subset \mathbb{R}$ has a lower bound, then the set $L \subset \mathbb{R}$ of all lower bounds of S has a maximum, $\max(L) \in L$.*

Definition 9. (Supremum and infimum) If a set $S \subset \mathbb{R}$ has an upper bound, then the **supremum** of S is the minimum upper bound for S, and it is denoted by $\sup(S)$ or $\text{l.u.b.}(S)$ (for *l*east *u*pper *b*ound). If not, then we abbreviate the statement "S has no upper bound" by $\sup(S) = \infty$.

Similarly, if S has a lower bound, then the **infimum** of S is the maximum lower bound for S, and it is denoted by $\inf(S)$ or $\text{g.l.b.}(S)$ (for *g*reatest *l*ower *b*ound). The notation $\inf(S) = -\infty$ indicates that S has no lower bound.

The supremum and the infimum differ from the maximum and the minimum by the sole additional requirement that the maximum and the minimum of a set must belong to that set, whereas the supremum and the infimum need not belong to that set.

EXAMPLE 8. The open unit segment, $]0,1[= \{x : x \in \mathbb{R} \text{ and } 0 < x < 1\}$, has a supremum, $\sup]0,1[= 1 = \min[1,\infty[$, and an infimum, $\inf]0,1[= 0 = \max]-\infty, 0]$. Yet $]0,1[$ has neither a maximum nor a minimum, because $1 \notin]0,1[$ and $0 \notin]0,1[$.

Exercise 9. Prove that if $S \subset T \subset \mathbb{R}$, then $\sup(S) \leq \sup(T)$ and $\inf(S) \geq \inf(T)$.

The following definitions establish the concept of non-integral dimensions. (See also Morgan's introductory text [26] and Federer's reference [25].)

Definition 10. (Diameter) For each non-empty $S \subset \mathbb{R}^n$, the **diameter** of S is the "extended" real number (finite or infinite, according to the definition 9) denoted by $\text{diam}(S)$ and defined by

$$\text{diam}(S) = \sup\{\|\vec{\mathbf{x}} - \vec{\mathbf{y}}\| : \vec{\mathbf{x}}, \vec{\mathbf{y}} \in S\}.$$

By convention, $\text{diam}(\emptyset) = 0$.

Definition 11. A **cover** of a subset $E \subset \mathbb{R}^n$ is a family $\mathcal{C}$ of subsets of $\mathbb{R}^n$, the union of which contains E; thus, $E \subset \cup\mathcal{C}$. A **countable cover** of E is a cover of E that consists of countably many subsets of $\mathbb{R}^n$, of the type $\mathcal{C} = \{S_j : j \in \mathbb{N}\}$ with each $S_j \subset \mathbb{R}^n$ and $E \subset \cup_{j=0}^{\infty} S_j$. (The definition allows some of the sets S_j to be empty, so that a countable cover may contain either infinitely or finitely many non-empty sets.)

The smaller the sets S_j in a cover $\mathcal{C}$ of a set, the better the cover approximates the size of that set.

Definition 12. For each $E \subset \mathbf{R}^n$, for each non-negative real number d, and for each positive real number δ, consider all countable covers $\mathcal{C} = \{S_j : j \in \mathbf{N}\}$ of E where every S_j has diameter at most δ, and define the number

$$\mathcal{H}_{d,\delta}(E) = \inf \left\{ \alpha_d \sum_{j=0}^{\infty} \left(\frac{\operatorname{diam}(S_j)}{2} \right)^d : E \subset \bigcup_{j=0}^{\infty} S_j \text{ and } \operatorname{diam}(S_j) \leq \delta \text{ for each } j \in \mathbf{N} \right\},$$

which provides a preliminary estimate of the size of the set E in dimension d.

Observe that if we decrease δ then we allow fewer covers (only those with smaller diameters), and, consequently, by exercise 9, the infimum increases. Consequently, $\mathcal{H}_{d,\delta}(E)$ tends to a limit or diverges to infinity as δ decreases.

Definition 13. **(Hausdorff measure)** For each set $E \subset \mathbf{R}^n$ and for each non-negative real number $d \geq 0$, the ***d*-dimensional Hausdorff measure** of E is the extended (finite or infinite) real number denoted by $\mathcal{H}_d(E) \in \mathbf{R} \cup \{\infty\}$ and defined by

$$\mathcal{H}_d(E) = \lim_{\delta \to 0} \mathcal{H}_{d,\delta}(E).$$

Exercise 10. From the definition of the Hausdorff measure, prove that if $0 \leq p < q < \infty$ and if $\mathcal{H}_p(E) < \infty$, then $\mathcal{H}_q(E) = 0$. Thus, if the Hausdorff measure of a set is a finite number in dimension p, then it equals zero in all higher dimensions $q > p$.

Exercise 11. Prove that if $0 \leq p < q < \infty$ and if $\mathcal{H}_q(E) < \infty$, then $\mathcal{H}_p(E) = \infty$. Thus, if the Hausdorff measure of a set is a finite number in dimension q, then it is infinite in all lower dimensions.

The preceding two exercises reveal that if we measure the size of a set in various dimensions, then we obtain zero for all higher dimensions and infinity for all lower dimensions, and there can exist at most one dimension between both extremes.

Definition 14. For each set $E \subset \mathbf{R}^n$ define the **Hausdorff dimension** of E as the number denoted by $h(E)$ and expressed by

$$h(E) = \inf \{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) < \infty\}.$$

Exercise 12. From the definition of the Hausdorff dimension and from the results of exercises 10 and 11, prove the following alternate expressions for the Hausdorff dimension:

$$h(E) = \inf \{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = 0\},$$

$$h(E) = \sup \{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) > 0\},$$

$$h(E) = \sup \{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = \infty\}.$$

Exercise 13. Prove that if $0 \leq s < \infty$ and if $0 < \mathcal{H}_s(E) < \infty$ then $s = h(E)$. (Such a set E is then called an s-**set.**)

The practical significance of the result of the foregoing exercise is that if we determine a real number s such that the s-dimensional Hausdorff measure of E is strictly positive (but not infinite), then we have also found the Hausdorff dimension s of E. In view of exercises 10 and 11, it also follows that each s-set has a positive finite Hausdorff measure only in dimension $d = s$, because $\mathcal{H}_d(E) = \infty$ for $d < s$ and $\mathcal{H}_d(E) = 0$ for $d > s$. Such a result provides a practical method for calculating the Hausdorff dimension of certain sets, as demonstrated in the following subsection.

2.3. Examples of Hausdorff dimensions and measures

This subsection demonstrates by examples how to calculate the Hausdorff dimension and the Hausdorff measure of a few familiar sets in space.

EXAMPLE 9. (FINITE SET OF POINTS) Let $E = \{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k\}$ consist of k distinct points in $\mathbf{R}^n$. Further, let $r = \min\{\|\vec{\mathbf{x}}_i - \vec{\mathbf{x}}_j\| : i \neq j \text{ and } i, j \in \{1, \ldots, k\}\}$, which represents the minimum distance between two distinct points in E; thus, if $\vec{\mathbf{x}}_i \neq \vec{\mathbf{x}}_j$ then $\|\vec{\mathbf{x}}_i - \vec{\mathbf{x}}_j\| \geq r$. For each non-negative dimension d and for each positive diameter $\delta \in]0, r[$, we may cover the set $E = \{\vec{\mathbf{x}}_1, \ldots, \vec{\mathbf{x}}_k\}$ with k disjoint balls, $\mathcal{C} = \{B(\vec{\mathbf{x}}_1, \delta/2), \ldots, B(\vec{\mathbf{x}}_k, \delta/2)\}$, centered at each point $\vec{\mathbf{x}}_i$, with radius $\delta/2$. For such a cover $\mathcal{C}$,

$$\alpha_d \sum_{j=1}^{k} \left(\frac{\operatorname{diam} B(\vec{\mathbf{x}}_j, \delta/2)}{2}\right)^d = \alpha_d \sum_{j=1}^{k} \left(\frac{\delta}{2}\right)^d = \alpha_d \cdot k \cdot \frac{\delta^d}{2^d}.$$

Consequently, $\mathcal{H}_{d,\delta}(E) \leq \delta^d k \alpha_d / 2^d$, since $\mathcal{H}_{d,\delta}(E)$ represents the infimum of all such sums. Therefore,

$$0 \leq \mathcal{H}_d(E) = \lim_{\delta \to 0} \mathcal{H}_{d,\delta}(E) \leq (k\alpha_d/2^d) \lim_{\delta \to 0} \delta^d = \begin{cases} k\alpha_d/2^d = k & \text{if } d = 0, \\ 0 & \text{if } d > 0. \end{cases}$$

Hence, by definition of the Hausdorff dimension, E has Hausdorff dimension $h(E) = 0$.

To calculate the 0-dimensional Hausdorff measure of E, observe that we never need more than k sets to cover E, because one set S_j for each point $\vec{\mathbf{x}}_j$ in E suffices. Consequently, since each countable cover $\mathcal{C} = \{S_j : j \in \mathbb{N}\} \supset E$ contains some set S_j containing $\vec{\mathbf{x}}_j$, we need only retain k sets $S_1, \ldots S_k$ and we may discard all the others. Moreover,

$$\sum_{j=1}^{k} \left(\frac{\operatorname{diam}(S_j)}{2}\right)^d \leq \sum_{j=0}^{\infty} \left(\frac{\operatorname{diam}(S_j)}{2}\right)^d,$$

which means that the infimum over such finite subcovers $\{S_1, \ldots, S_k\}$ does not exceed (and, consequently, by exercise 9, equals) the infimum over all countable covers. However, with $d = 0$, all such finite covers yield the same measure, because

$$\sum_{j=1}^{k} \left(\frac{\operatorname{diam}(S_j)}{2}\right)^0 = \sum_{j=1}^{k} 1 = k.$$

Consequently,

$$\mathcal{H}_{0,\delta}(E) = \inf\left\{\alpha_0 \sum_{j=1}^{k}\left(\frac{\operatorname{diam}(S_j)}{2}\right)^0 : E \subset \bigcup_{j=1}^{n} S_j,\ \operatorname{diam}(S_j) \le \delta\right\} = \inf\{k\} = k.$$

Finally,

$$\mathcal{H}_0(E) = \lim_{\delta\to 0}\mathcal{H}_{0,\delta}(E) = \lim_{\delta\to 0} k = k.$$

Therefore, a set with k points has Hausdorff dimension zero and 0*-dimensional Hausdorff measure equal to k.*

Exercise 14. Determine the Hausdorff dimension, d, and the d-dimensional Hausdorff measure, of a *countable* set of distinct points $E = \{\vec{x_j} : j \in \mathbb{N}\} \subset \mathbb{R}^n$.

REMARK 4. Notice that the Hausdorff dimension of $\{\vec{x_1}, \ldots, \vec{x_k}\}$ equals 0 regardless of the relative positions of the points $\vec{x_1}, \ldots, \vec{x_k}$. In particular, this means that the Hausdorff dimension of a set $E \subset \mathbb{R}^n$, like its topological dimension, does not pertain to the global position of E in $\mathbb{R}^n$ but to an intrinsic and local characteristic of the set E itself. For instance, for each point $\vec{x_j}$ in $E = \{\vec{x_1}, \ldots, \vec{x_k}\}$ the open ball $B(\vec{x_j}, r)$ contains only one point from E, namely the point $\vec{x_j}$, and in that ball the set E looks like one single point, which has dimension 0. (In contrast, the dimension of the space occupied by the set E relates to the concept of *embedding* of E into $\mathbb{R}^n$.)

EXAMPLE 10. (STRAIGHT LINE SEGMENT) Let $E = [0, \ell]$ represent a closed and bounded straight line segment of length ℓ in $\mathbb{R}^n$, from the origin 0 to a point ℓ on the first coordinate axis. For each positive diameter $\delta > 0$ cover the segment $[0, \ell]$ by $n_\delta = \lfloor \ell/\delta \rfloor + 1$ sets S_j, each of diameter at most δ (recall that the "floor" notation, $\lfloor \ell/\delta \rfloor$, represents the largest integer that does not exceed ℓ/δ). For instance, cover $[0, \ell]$ by $\lfloor \ell/\delta \rfloor$ consecutive intervals $[k\delta, (k+1)\delta]$ of length equal to δ, followed by one interval $[\lfloor \ell/\delta \rfloor \delta, \ell]$ of length at most δ. For such a covering,

$$\mathcal{H}_{d,\delta}([0,\ell]) \le \alpha_d \sum_{j=1}^{n_\delta}\left(\frac{\operatorname{diam}(S_j)}{2}\right)^d \le \alpha_d\left(\lfloor \ell/\delta \rfloor + 1\right)\left(\frac{\delta}{2}\right)^d = \alpha_d\left(\frac{\ell}{2^d}\delta^{d-1} + \frac{\delta^d}{2^d}\right).$$

As the diameter δ tends to zero, notice that

$$(1)\quad \mathcal{H}_d([0,\ell]) = \lim_{\delta\to 0}\mathcal{H}_{d,\delta}([0,\ell]) \le \alpha_d \lim_{\delta\to 0}\left(\frac{\ell}{2^d}\delta^{d-1} + \frac{\delta^d}{2^d}\right) = \begin{cases} \infty & \text{if } d < 1,\\ \alpha_1 \ell/2 = \ell & \text{if } d = 1,\\ 0 & \text{if } d > 1.\end{cases}$$

Consequently, $\mathcal{H}_d([0, \ell]) = 0$ for every $d > 1$, and it follows that $h([0, \ell]) \le 1$.

To prove the converse inequality, consider *any* cover $\mathcal{C} = \{S_j : j \in \mathbb{N}\}$ by sets S_j each of diameter at most δ. Since the intersections $I_j = S_j \cap [0, \ell] \subset [0, \ell]$ lie on the interval $[0, \ell]$, we may replace each set S_j by I_j and still cover $[0, \ell]$. Further, for each positive real ε we may replace each set I_j by an open interval $J_j =]a_j, b_j[\supset I_j$ that contains I_j and such that $\operatorname{diam}(I_j) \leq \operatorname{diam}(J_j) < \operatorname{diam}(I_j) + \varepsilon/2^j$. Moreover, since the closed and bounded interval $[0, \ell]$ is compact in $\mathbf{R}^n$, there exists then a *finite* subcover of $[0, \ell]$ by finitely many of the open intervals J_j, for instance, by $\{J_1, \ldots, J_k\}$ (after a rearrangement to avoid double indices). If $k = 1$ then a single interval $J_1 =]a_1, b_1[$ covers $[0, \ell]$, and $b_1 - a_1 > \ell$. Next, proceed by induction: suppose that for some integer $m \in \mathbf{N}$, the inequality $\sum_{j=1}^{m}(b_j - a_j) > b - 0 = b$ holds for every cover of every closed and bounded interval $[0, b]$ by $k = m$ open intervals, and consider a cover of $[0, \ell]$ by $k = m + 1$ open intervals $]a_1, b_1[, \ldots,]a_{m+1}, b_{m+1}[$ (which need not lie in this order on $[0, \ell]$). Also, let $a_i = \max\{a_j : j \in \{1, \ldots, m+1\}\}$ denote the maximum left-hand endpoint among all the intervals that contain ℓ, and remove from the cover that interval $J_i =]a_i, b_i[$. Then the remaining m intervals J_j, for $j \neq i$, cover the subinterval $[0, a_i] \subset [0, \ell]$ because $[0, a_i] \cap]a_i, \ell] = \emptyset$. Thus,

$$\sum_{j=1}^{m+1}(b_j - a_j) = \sum_{\substack{j=1 \\ j \neq i}}^{m+1}(b_j - a_j) + (b_i - a_i) \geq (a_i - 0) + (b_i - a_i) = b_i > \ell,$$

which completes the induction. Consequently, with $d = 1$ and $\alpha_1 = 2$, and for each $\varepsilon > 0$,

$$\mathcal{H}_{1,\delta}([0, \ell]) \geq \alpha_1 \sum_{j=0}^{\infty} \left(\frac{\operatorname{diam}(S_j)}{2}\right)^1 \geq 2 \sum_{j=1}^{k} \frac{\operatorname{diam}(J_j) - \varepsilon/2^j}{2} \geq \ell - \varepsilon. \tag{2}$$

Letting ε tend to zero gives $\mathcal{H}_{1,\delta}([0, \ell]) \geq \ell$, and then letting δ tend to zero yields $\mathcal{H}_1([0, \ell]) \geq \ell$. With $\mathcal{H}_1([0, \ell]) \leq \alpha_1 \ell/2 = \ell$ in (1), the two inequalities just obtained give $\mathcal{H}_1([0, \ell]) = 1$, which, by exercise 13, yields the conclusion:

A closed and bounded segment $[0, \ell]$ *of length* ℓ *has Hausdorff dimension* $d = h([0, \ell]) = 1$ *and one-dimensional Hausdorff measure equal to its length:* $\mathcal{H}_1([0, \ell]) = \ell$.

REMARK 5. More generally than the preceding examples might suggest, the Hausdorff measure coincides with the ordinary measure for measurable sets in integral dimensions. Differentiable curves — for example, circles — have Hausdorff dimension 1, and their one-dimensional Hausdorff measure equals their length; similarly, differentiable surfaces — for example, spheres — have Hausdorff dimension 2, and their two-dimensional Hausdorff measure equals their area. (See Morgan's text [26], pages 14–17, or Federer's [25], Theorem 2.10.35, page 197.) Moreover, as indicated in remark 4, differentiable curves have dimension 1 and surfaces have dimension 2 regardless of their position or twisting in $\mathbf{R}^n$. Thus, the Hausdorff dimension and the Hausdorff measure provide generalizations of the intuitive concepts of integral dimensions and of size.

EXAMPLE 11. (VON KOCH'S SNOWFLAKE) Cover von Koch's snowflake, E_∞, with the same triangles used to prove that it has null area, in exercise 7. At the k-th stage E_k of the construction, 4^k triangles $S_1, \ldots S_{4^k}$ each of diameter $(1/3)^k$ cover the snowflake. Consequently, with $\delta_k = (1/3)^k$,

$$\mathcal{H}_{d,\delta_k} \leq \alpha_d \sum_{j=1}^{4^k} \left(\frac{(1/3)^k}{2}\right)^d = \alpha_d 4^k \left((1/3)^k/2\right)^d = \frac{\alpha_d}{2^d}\left(\frac{4}{3^d}\right)^k.$$

Observe that

$$\lim_{k\to\infty}\left(\frac{4}{3^d}\right)^k = \begin{cases} \infty & \text{if } 4/3^d > 1, \text{ which means that } d < \ln 4/\ln 3, \\ 1 & \text{if } 4/3^d = 1, \text{ which means that } d = \ln 4/\ln 3, \\ 0 & \text{if } 4/3^d < 1, \text{ which means that } d > \ln 4/\ln 3. \end{cases}$$

Consequently, $\mathcal{H}_d(E) = 0$ for every $d > \ln 4/\ln 3$, from which it follows that $h(E) \leq \ln 4/\ln 3$. Of course, the result just obtained suggests that $h(E) = \ln 4/\ln 3$, but this does not yet prove equality, because it merely leads to $\mathcal{H}_d(E) \leq \infty$, not necessarily $\mathcal{H}_d(E) = \infty$, for every $d < \ln 4/\ln 3$.

The remainder of the proof follows the pattern of the treatment of Cantor's set in Falconer's *Fractal Geometry*, [34], page 32. To prove that the Hausdorff dimension of von Koch's snowflake equals $s = \ln 4/\ln 3$, it suffices to prove that it has a positive s-dimensional Hausdorff measure (see exercise 13). To this end, consider *any* cover $\mathcal{C} = \{S_j : j \in \mathbb{N}\}$ of the snowflake E_∞, with non-empty sets S_j each of diameter δ_j at most δ, with $0 < \delta \leq 1$. For each positive real number ε, and for each (non-empty) set S_j, select any point $\vec{x_j} \in S_j$ and embed S_j into an open ball $B_j = B(\vec{x_j}, r_j)$, centered at $\vec{x_j}$ with radius $r_j = \delta_j + \varepsilon$. (Such radius r_j exceeds the minimum radius, $\delta_j/\sqrt{3} + \varepsilon$, necessary for the ball B_j to enclose the set S_j. See the first research project, below.) By compactness of the snowflake (the continuous image $f([0,1])$ of the compact unit segment), there exists a cover of E_∞ by finitely many of the balls just constructed, $B_1, \ldots, B_m$. For each such ball B_j, there exists one integer k_j such that

(1) $$(1/3)^{k_j+1} \leq \text{diam}B_j < (1/3)^{k_j}.$$

Consequently, the ball B_j contains at most one vertex of the k_j-th stage E_{k_j}, because all vertices of that stage lie at least $(1/3)^{k_j}$ apart from one another. At each subsequent stage E_ℓ, with $\ell \leq k_j$, the ball B_j contains at most $2 \times 4^{\ell - k_j}$ vertices of E_ℓ, because B_j contains at most two adjacent segments of length $(1/3)^{k_j}$ in E_{k_j}. To relate the number of vertices in B_j to the diameter of B_j, recall that $s = \ln 4/\ln 3$, so that $3^s = 4$; consequently,

(2) $$4^{\ell-k_j} = 4^\ell 4^{-k_j} = 4^\ell (3^s)^{-k_j} = 4^\ell (3^{-(k_j+1)+1})^s = 4^\ell 3^s (3^{-(k_j+1)})^s \leq 4^\ell 3^s (\text{diam}B_j)^s$$

by virtue of the first inequality in (1). Thus, each ball $B_j \in \{B_1, \ldots, B_m\}$ contains at most $2 \times 4^\ell 3^s (\text{diam}B_j)^s$ vertices of E_ℓ. Since the balls cover E_∞, and since all the vertices of the ℓ-th stage E_ℓ remain through the subsequent stages and belong to E_∞, it follows that the m balls $B_1, \ldots, B_m$ contain the $4^\ell + 1$ vertices of E_ℓ. Therefore, by inequality (2), and with an inequality allowing for duplication (some of the vertices may lie in more than one ball),

$$4^\ell + 1 \leq \sum_{j=1}^m \text{number of vertices in } B_j \leq \sum_{j=1}^m 2 \times 4^\ell 3^s (\text{diam}B_j)^s.$$

Hence, dividing by $2 \times 4^\ell 3^s$ and rearranging terms gives

$$(3^{-s} + 4^{-(\ell+1)})/2 \leq \sum_{j=1}^m (\text{diam}B_j)^s = \sum_{j=1}^m (2\text{diam}S_j + 2\varepsilon)^s = 2^s \sum_{j=1}^m (\text{diam}S_j + \varepsilon)^s.$$

Moreover, dividing by 2^s twice, letting ε tend to zero and ℓ tend to infinity yields

$$4^{-(s+1)}/2 = 3^{-s}2^{-s}2^{-s}/2 \leq \sum_{j=1}^{m} \left(\frac{\mathrm{diam} S_j}{2}\right)^s \leq \sum_{j=0}^{\infty} \left(\frac{\mathrm{diam} S_j}{2}\right)^s .$$

Therefore, taking the infimum of all such sums for all covers shows that $\mathcal{H}_s(E_\infty) \geq \alpha_s 4^{-(s+1)}/2 > 0$: von Koch's snowflake has positive s-dimensional Hausdorff measure with $s = \ln 4/\ln 3$, which has the necessary consequence that it has Hausdorff dimension $\ln 4/\ln 3 = 1.261859507\ldots$.

Exercise 15. Identify the intersection of von Koch's snowflake and the initial segment $[0,1]$ (it coincides with a set famous in analysis and topology). Then calculate its Hausdorff dimension.

2.4. Fractal coastlines versus planimeters

In suggesting real examples of fractals, most texts liken them to coastlines, implying that coastlines may have a non-integral Hausdorff dimension; in particular, this means that such coastlines have infinite lengths.

Exercise 16. Prove that if a set E (which represents a coastline) has a Hausdorff dimension $h(E)$ strictly greater than one, then E has infinite length.

Yet note that a coastline with infinite length would prevent geographers from using Amsler's planimeter to measure areas on maps, for the following reason. The planimeter applies Green's Theorem in the plane, which states that for each planar open set $\Omega \subset \mathbb{R}^2$ with a simple, closed, and differentiable boundary $\partial\Omega$, and for all differentiable functions $P, Q : \overline{\Omega} \to \mathbb{R}$, the following two integrals agree:

$$\int\int_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy = \oint_{\partial\Omega} (P dx + Q dy) .$$

We may think of Green's Theorem as the Fundamental Theorem of Calculus with two variables. In particular, for $P(x,y) = -y$ and $Q(x,y) = x$, Green's Theorem yields the **Area Formula,**

$$\mathrm{Area}(\Omega) = \int\int_{\Omega} dx \wedge dy = \frac{1}{2}\int\int_{\Omega} (1-(-1))\, dx \wedge dy = \frac{1}{2}\int\int_{\Omega} \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dx \wedge dy$$
$$= \frac{1}{2}\oint_{\partial\Omega} (-y dx + x dy) .$$

Based on the Area Formula, Amsler's planimeter consists of an articulated two-arm instrument, with one end fixed and the other tracing the coastline on a map, and with the pivot between both arms recording the line integral in the Area Formula. (For details, consult, for example, the reference by Ronald W. Gatterdam [5].)

Exercise 17. Suppose that Ω represents the map of a country which is bordered by a fractal coastline $\partial\Omega$ with Hausdorff dimension strictly between one and two ($1 < h(\partial\Omega) < 2$), and, by the preceding exercise, infinite length. Explain why a geographer may still estimate the area of the country with the planimeter. Would tracing an infinitely long coastline not require an infinite amount of time?

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3. EXPLORATORY TERM PROJECTS

The present section proposes two projects that may be attempted in parallel with the study of this chapter (hence the phrase "term projects"). Since the results involve only elementary complex variables and are known to specialists, the projects do not qualify as research. However, some of the details are not readily available in many sources, which means that you may have to search the literature for some results and derive others yourself (hence the adjective "exploratory").

3.1. Other fractals of von Koch's type

Project 1. Von Koch's construction generalizes to other fractals. For instance, exhibit 3 shows how to transform an oriented segment AB into an oriented polygonal path $APQRSTUVB$ by inserting seven new vertices, which form eight new edges, each of length equal to one quarter that of the initial segment. One of the new vertices, S, lies at the midpoint of AB, with no kink in the path there. (The fractal obtained by iteration of the particular construction shown in exhibit 3 is called a quadric curve of von Koch's type.)

(1) To understand von Koch's construction better, write and test a program to produce consecutive stages of von Koch's snowflake on a different type of supercalculator or computer.

(2) Modify the programs for von Koch's snowflake to experiment with fractals of von Koch's type but of your own design.

(3) Conjecture the value of the Hausdorff dimension for the fractals that you create.

(4) In appropriate cases, generalize the proof of the Hausdorff dimension of von Koch's snowflake (in example 11) to your fractals.

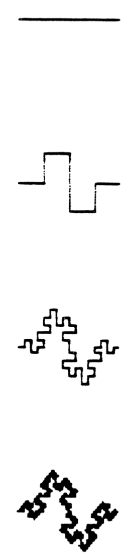

Exhibit 3.

3.2. The smallest ball containing a bounded set

Project 2. Recall that the calculation of the Hausdorff dimension of von Koch's snowflake involved embedding the bounded sets of the cover into balls with radii slightly larger than necessary (see example 11, pages 61 – 63).

For each diameter $d \geq 0$ and for each integer $n \geq 2$, determine the optimal radius $R \geq 0$ such that for every set $S \subset \mathbf{R}^n$ with diameter d, the set S lies in a ball with radius at most R. Note that according to the requirements just stated, the radius R may depend upon d and n but not upon S.

4. RESEARCH PROBLEM

4.1. The Hausdorff dimension of Julia sets

The research problem suggested here focuses on the Hausdorff dimension and measure of quadratic Julia sets, which were presented in the preceding chapter. For the following background, see Falconer's *Fractal Geometry,* [34], pages 208–218.

For "small" values of the complex constant $c \in \mathbb{C}$, the Julia set J_c of the quadratic polynomial f_c defined by $f_c(z) = z^2 + c$ has a Hausdorff dimension approximated by

$$d_c = h(J_c) \approx 1 + \frac{1}{4\ln 2}|c|^2 + \cdots.$$

Stated more accurately, this means that there exist a positive radius $r > 0$ and a power series of the type $\sum_{n=0}^{\infty} a_n |c|^n$, with real coefficients $a_n \in \mathbb{R}$, which converges to the Hausdorff dimension, $d_c = h(J_c) = \sum_{n=0}^{\infty} a_n |c|^n$, for every constant $c \in \mathbb{C}$ such that $|c| < r$. Moreover, $a_0 = 1$, $a_1 = 0$, and $a_2 = 1/(4\ln 2)$. Also, for $|c| < r$, the d_c-dimensional Hausdorff measure of the Julia set J_c satisfies the inequalities $0 < \mathcal{H}_{d_c}(J_c) < \infty$.

EXAMPLE 12. If $c = 0$ then $|c| = |0| = 0 < r$ and $d_0 = h(J_0) = 1 + 0 + 0 + \cdots = 1$. Recall that for $c = 0$ the Julia set J_0 of the squaring function is the unit circle, which, indeed, has topological dimension and Hausdorff dimension 1; thus, $h(J_0) = 1$ and $\mathcal{H}_1(J_0) = 2\pi$ (the circumference of the unit circle).

REMARK 6. If $|c| < 1/4$ then the Julia set J_c is a simple closed curve (topologically homeomorphic to a circle), like von Koch's snowflake. Also as for von Koch's snowflake, if $0 < |c| < r$ then $1 < 1 + |c|^2/(4\ln 2) + \cdots < 2$ and the Julia set J_c has a non-integral Hausdorff dimension between 1 and 2.

For "large" values of the complex constant c, more accurately, for $|c| > (5 + 2\sqrt{6})/4$, the Hausdorff dimension d_c of the Julia set J_c satisfies the following two inequalities:

$$\frac{2\ln 2}{\ln(4[|c| + \sqrt{2|c|}])} \leq d_c \leq \frac{2\ln 2}{\ln(4[|c| - \sqrt{2|c|}])}.$$

Problem. Provide a more accurate estimate of the Hausdorff dimension d_c of the quadratic Julia set J_c and of its d_c-dimensional Hausdorff measure.

5. SOLUTIONS TO ALL THE EXERCISES

5.1. Solutions to the exercises

Exercise 1. $P = (1/3, 0)$, $R = (2/3, 0)$, and $Q = (1/2, -1/3\sqrt{3}/2)$.

Exercise 2. $P = A + 1/3(B - A)$, $R = A + 2/3(B - A)$, and $Q = P + e^{-i\pi/3}(P - A) = P + (1/2, -\sqrt{3}/2)(P - A)$.

Exercise 3.

$$f(t) = \begin{cases} (t, 0) & \text{if } 0 \le t \le 1/3, \\ (1/3, 0) + e^{-i\pi/3}(t - 1/3, 0) = ((t/2) + 1/6, -(t - 1/3)\sqrt{3}/2) & \text{if } 1/3 \le t \le 2/3, \\ (t, 0) & \text{if } 2/3 \le t \le 1. \end{cases}$$

Exercise 4. At the n-th stage of von Koch's construction, the polygonal path has sides of length $(1/3)^n$ and differs the most from the next stage at the midpoint of each side, by a distance of $(1/3)^{n+1}\sqrt{3}/2 = 1/(3^n 2\sqrt{3})$. By the formula for the geometric series,

$$\|f_n - f_m\|_\infty \le \sum_{j=1}^{n-m} \|f_{m+j} - f_{m+j-1}\|_\infty \le \sum_{j=1}^{n-m} \frac{1}{2\sqrt{3}} 3^{-(m+j-1)}$$

$$= \frac{3^{-m}}{2\sqrt{3}} \sum_{\ell=0}^{n-m-1} (1/3)^\ell = \frac{3^{-m}}{2\sqrt{3}} \frac{1 - (1/3)^{n-m}}{1 - 1/3} \le \frac{3^{-(m-1)}}{4\sqrt{3}} \le 3^{-m}.$$

Consequently, (f_n) forms a Cauchy sequence, which converges to a continuous limit, f.

Exercise 5. Observe that the assertion holds for $n = 0$, in which case $f_0(t) = (t, 0)$ and $\|f_0(r) - f_0(s)\| = \|(r, 0) - (s, 0)\| = |r - s|$. If the assertion holds for some index $n = k \in \mathbf{N}$, then $\|f_{k+1}(r) - f_{k+1}(s)\| \ge \|f_k(r) - f_k(s)\| \ge |r - s|$, because the additional vertices introduced by f_{k+1} increase distances further. Consequently, by continuity of the norm, $\|f(r) - f(s)\| = \|\lim_{n\to\infty}(f_n(r) - f_n(s))\| = \lim_{n\to\infty} \|f_n(r) - f_n(s)\| \ge |r - s|$. Thus, if $r \ne s$ then $\|f(r) - f(s)\| \ge |r - s| > 0$ and $f(r) \ne f(s)$. Finally, observe that $f : [0, 1] \to f([0, 1]) \subset \mathbf{R}^2$ is continuous and injective on its compact domain, $[0, 1]$, and that its image is a Hausdorff space, which has the necessary logical consequence (a standard theorem in topology) that its inverse, $f^{-1} : f([0, 1]) \to [0, 1]$, is also continuous.

Exercise 6. For each stage of von Koch's construction choose for $(t_{n,j})_{j=0}^{j=n}$ all the $n + 1$ vertices present at that stage. Since there are then 4^n sides each of length $(1/3)^n$, it follows that $|t_{n,j} - t_{n,j-1}| = (1/3)^n$ tends to zero, as required, and that $\sum_{j=1}^{n} \|f(t_{n,j}) - f(t_{n,j-1})\| = (4/3)^n$ diverges to infinity.

Exercise 7. Cover the n-th stage of von Koch's construction, f_n, with 4^n isoceles triangles each of length $(1/3)^{n-1}$ and height $(1/3)^n\sqrt{3}/2$, so that they also cover all the subsequent stages, and, consequently, also the limit. The limit — the snowflake itself — has an area that does not exceed the total area covered by such triangles, which equals

$$4^n \times 1/2 \times (1/3)^{n-1} \times (1/3)^n\sqrt{3}/2 = \cdots = 3\sqrt{3}4^{n-1}/3^{2(n-1)} = 3\sqrt{3}(4/9)^{n-1},$$

and which tends to zero. Therefore, the area of the snowflake melts away to zero.

Exercise 8. $\Gamma(3.5) = \Gamma(7/2) = \Gamma(5/2 + 1) = 5/2\Gamma(5/2) = 5/2 \cdot 3\sqrt{\pi}/4 = 15/8\sqrt{\pi}$.

Exercise 9. If $s \in S \subset T$ then $s \in T$ and, consequently, $\inf(T) \leq s \leq \sup(T)$. Thus, $\inf(T)$ is a lower bound for S and $\sup(T)$ is an upper bound for S, and, therefore, $\inf(T) \leq \inf(S)$ and $\sup(S) \leq \sup(T)$.

Exercise 10. If $0 \leq p < q < \infty$ and $\mathcal{H}_p(E) < \infty$, factor δ^{q-p} out of $\mathcal{H}_{q,\delta}$, which gives

$$\mathcal{H}_{q,\delta} = \inf\left\{\alpha_q \sum_{j=0}^{\infty}\left(\frac{\operatorname{diam}(S_j)}{2}\right)^q : E \subset \bigcup_{j=0}^{\infty} \text{ and } \operatorname{diam}(S_j) \leq \delta \text{ foreach } j\right\} \leq$$

$$\delta^{q-p}\inf\left\{\alpha_q \sum_{j=0}^{\infty}\left(\frac{\operatorname{diam}(S_j)}{2}\right)^p : E \subset \bigcup_{j=0}^{\infty} \text{ and } \operatorname{diam}(S_j) \leq \delta \text{ foreach } j\right\} = \delta^{q-p}\mathcal{H}_{p,\delta}(E).$$

Consequently, as δ tends to zero,

$$\mathcal{H}_q(E) = \lim_{\delta \to 0} \mathcal{H}_{q,\delta}(E) \leq \lim_{\delta \to 0} \delta^{q-p}\mathcal{H}_{p,\delta} = 0\mathcal{H}_p(E) = 0.$$

Exercise 11. Proceed as in the previous exercise, but factor δ^{p-q} out of $\mathcal{H}_{p,\delta}(E)$ and observe that δ^{p-q} diverges to infinity as δ tends to zero, because $p - q < 0$.

Exercise 12. Firstly, recall the definition of the Hausdorff dimension:

$$h(E) = \inf\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) < \infty\}.$$

To prove the first alternate expression, consider any $d > h(E)$ and let p denote any number such that $0 \leq p < d < \infty$, for example, $p = (h(E) + d)/2$. Since $h(E) < p$ the definition of $h(E)$ shows that $\mathcal{H}_p(E) < \infty$. Then exercise 10, with $d = q$, shows that $\mathcal{H}_d(E) = 0$, which thus holds for every $d > h(E)$. Yet $\mathcal{H}_d(E) = \infty$ for every $d < h(E)$, again by definition of $h(E)$. Therefore, $h(E) = \inf\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = 0\}$.

For the second and third expressions, observe again that $\mathcal{H}_d(E) = \infty$ for every $d < h(E)$ and that $\mathcal{H}_d(E) = 0$ for every $d > h(E)$, by the argument just presented for the first expression. Consequently, $h(E) = \sup\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) > 0\}$, and $h(E) = \sup\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = 0\}$.

Exercise 13. Suppose that $0 \leq s < \infty$ and that $0 < \mathcal{H}_s(E) < \infty$. By exercise 10, it follows that $\mathcal{H}_d(E) = 0$ for every $d > s$. Also, by exercise 11, it follows that $\mathcal{H}_d(E) = \infty$ for every $d < s$. Consequently, $s = \inf\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = 0\} = h(E)$, by definition of the Hausdorff dimension.

Exercise 14. Proceed as with finitely many points, but cover each point $\vec{\mathbf{x}}_j \in E$ with a ball $S_j = B(\vec{\mathbf{x}}_j, \delta/2^j)$ (which may also contain other points $\vec{\mathbf{x}}_i \in E$). Then observe that

$$\sum_{j=0}^{\infty}\left(\frac{\operatorname{diam}(S_j)}{2}\right)^d = \sum_{j=0}^{\infty}\left(\frac{\delta}{2^{j+1}}\right)^d = \delta^d \sum_{j=0}^{\infty}\left(\frac{1}{2^d}\right)^{j+1} = \begin{cases} \delta^d 2^d/(1 - 2^d) & \text{if } d > 0, \\ \infty & \text{if } d = 0. \end{cases}$$

Consequently, $h(E) = 0$ and then $\mathcal{H}_0(E) = \sum_{j=0}^{\infty} 1 = \infty$. *Therefore, a countable set of (distinct) points has Hausdorff dimension 0 and infinite 0-dimensional Hausdorff measure.*

Exercise 15. From the initial stage of von Koch's construction, $E_0 = [0,1]$, the first stage replaces the middle third by the polygonal path PQR, thus leaving the two extreme thirds on the unit segment, $E_1 = [0, 1/3] \cup [2/3, 1]$. Then the second stage in turn replaces the middle thirds of both intervals in E_1 by polygonal paths, leaving on the unit segment the four subintervals $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and so forth, which yields **Cantor's set.** As proved in Falconer's *Fractal Geometry*, [34], pages 31–32, Cantor's set has Hausdorff dimension $d = \ln 2/\ln 3 = 0.630\,929\,753\,57\ldots$.

Exercise 16. If E has Hausdorff dimension $h(E) > 1$ then exercise 12, third expression, $h(E) = \sup\{d : d \in [0, \infty[\text{ and } \mathcal{H}_d(E) = \infty\}$, shows that if $d < h(E)$ then $\mathcal{H}_d(E) = \infty$. In particular, since $1 < h(E)$, it follows that the length of E equals $\mathcal{H}_1(E) = \infty$.

Exercise 17. If the coastline $\partial\Omega$ is a continuous closed curve, of the type $f : [0,1] \to \mathbb{R}^2$, then a standard theorem of analysis guarantees that for each positive real number ε there exists a differentiable function $f_\varepsilon\ [0,1] \to \mathbb{R}^2$ such that $f_\varepsilon(0) = f_\varepsilon(1)$ (which ensures that f_ε also traces a closed curve) and $\|f - f_\varepsilon\|_\infty < \varepsilon$. Therefore, a geographer may let the planimeter follow the rectified coastline f_ε and obtain an estimate as close as desired to the area of the country.

5.2. Hints for the term projects

Project 1. Exhibit 4, on the following page, shows one way of plotting von Koch's snowflake with *Mathematica*.

Returning to Hewlett-Packard supercalculators, modify subroutine `Koch` so that it calculates and inserts the desired points on the stack, for instance B, $B - (B - A)/4$, $B - \{(B - A)/4\}(1,1)$, $B - \{(B - A)/4\}(2,1)$, $(A + B)/2$, $(A + B)/2 + (B - A)(0,1)/4$, $(A + B)/2 + (B - A)(-1,1)/4$, $A + (B - A)/4$, and A. Then let `Koch` iterate the loop between `START` and `NEXT` eight times instead of four.

The proof of example 11 generalizes to constructions that do not intersect themselves and with an equilateral path at each stage.

Project 2. In the plane $\mathbb{R}^2$, if a set S has diameter d, then there exists one smallest closed disc B containing S, called the **circumscribed** disc, and that disc has radius r with $d/2 \leq r \leq d/\sqrt{3}$; moreover, there exist planar sets $S \subset \mathbb{R}^2$ with diameter d such that $r = d/\sqrt{3}$, which means that $R = d/\sqrt{3}$ (see Yaglom and Boltyanskiĭ's *Convex Figures*, [27], exercise 3, pages 105–106, and exercise 6-1, pages 213–215).

For $\mathbb{R}^n$, prove that $d/2 \leq r \leq d/\sqrt{2(n+1)/n}$, and that there exist sets $S \subset \mathbb{R}^n$ with diameter d such that $r = d/\sqrt{2(n+1)/n}$, which means that $R = d/\sqrt{2(n+1)/n}$, a result suggested by Kit Hanes.

```
flake:=(        (* Mathematica program for von Koch's snowflake *)
  u = N[Exp[-Pi*I/3]];                    (* rotates by -1/6 turn *)
  vertices={0,1,Conjugate[u],0};          (* isoceles triangle *)
  u = 1 + u;                                     (* translates *)
  stages = 5;                 (* maximum number of iterations *)
  Do[ stage, {stages} ];                (* performs iterations *)
  exhibit)                              (* exhibits the result *)

stage:=(       (* "stage" goes from one stage to the next stage *)
  m = 1; n = 1;              (* next vertex and side to process *)
  sides = Length[vertices]-1;      (* current number of sides *)
  points = 4*sides + 1;         (* current number of vertices *)
  new = Table[0, {j,points}];      (* array for new vertices *)
  Do[ step, {sides} ];             (* one step for each side *)
  vertices = new)                    (* retains new vertices *)

step := (        (*"step" does an elementary step on one segment*)
  a = vertices[[n]];               (* old left-hand endpoint *)
  b = vertices[[n+1]];             (* old right-hand endpoint *)
  s = (b-a)/3;                    (* new third of old segment *)
  m = 4*n - 3;                    (* index of next new vertex *)
  new[[m]] = a;                (* old first point stays fixed *)
  new[[m+1]] = a + s;            (* first new vertex at third *)
  new[[m+2]] = a + u*s;          (* second new vertex rotated *)
  new[[m+3]] = b - s;        (* third new vertex at 2nd third *)
  new[[m+4]] = b;               (* old last point stays fixed *)
  m = m + 4; n = n + 1)      (* next vertex & side to process *)

exhibit:=(        (* "exhibit" displays the latest stage built *)
  n = Length[vertices];                 (* number of vertices *)
  image =              (* converts complex numbers to graphics *)
  Table[{Re[vertices[[i]]],Im[vertices[[i]]]},{i,n}];
  ListPlot[image, Axes -> None, AspectRatio -> Automatic,
  PlotJoined -> True, PlotStyle -> Thickness[.001] ] )

flake    (* execute "flake"; reduce the picture to fit the page *)
```

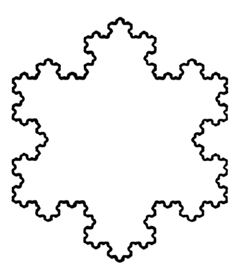

Exhibit 4. Program in *Mathematica* for plotting von Koch's snowflake, based upon the same algorithm as that for the HP-28 and HP-48 in exhibit 2.

REFERENCES

Precalculus texts with complex numbers

[1] Thomas W. Hungerford and Richard Mercer, *Precalculus Mathematics,* 2nd ed., Saunders College Publishing, Philadelphia, PA, 1985. ISBN 0-03-000843-3. QA39.2.H857. 512'.1.

[2] Warren L. Ruud and Terry L. Shell, *Prelude to Calculus,* Wadsworth, Belmont, CA, 1990. ISBN 0-534-10290-5. QA39.2.R88 1990. 510–dc20.

References for topics in calculus and its applications

[3] Robert Ellis and Denny Gulick, *Calculus With Analytic Geometry,* 4th ed., Harcourt Brace Jovanovich, San Diego, CA, 1990. ISBN 0-15-505687-5.

[4] Ross L. Finney and George B. Thomas, Jr., *Calculus,* Addison-Wesley, Reading, MA, 1990. ISBN 0-201-19343-4. QA303.F44 1989. 515–dc20.

[5] Ronald W. Gatterdam, "The planimeter as an example of Green's Theorem," *The American Mathematical Monthly,* Vol. 88, No. 9 (November 1981), p. 701–704.

[6] Abe Mizrahi and Michael Sullivan, *Calculus and Analytic Geometry,* 3rd ed., Wadsworth, Belmont, CA, 1990. ISBN 0-534-11646-9. QA303.M6878 1990. 515.'15–dc20.

[7] Michael Spivak, *Calculus,* Publish or Perish, Wilmington, DE, 1980. ISBN 0-914098-77-2.

[8] James Stewart, *Calculus,* Brooks/Cole, Pacific Grove, CA, 1987. ISBN 0-534-06690-9. QA303.S8825 1986. 515.'15.

[9] Robert Weinstock, "Elementary Evaluations of $\int_0^\infty e^{-x^2}dx$, $\int_0^\infty \cos x^2 dx$, and $\int_0^\infty \sin x^2 dx$," *American Mathematical Monthly,* Vol. 97, No. 1 (January 1990), p. 39–42.

Calculus text with complex analysis and Julia sets

[10] Robert Ellis and Denny Gulick, *Calculus With Analytic Geometry,* 4th ed., Harcourt Brace Jovanovich, San Diego, CA, 1990. ISBN 0-15-505687-5.

Undergraduate texts in complex analysis

[11] Ralph Philip Boas, *Invitation to Complex Analysis,* Random House/Birkhauser, New York, 1987. ISBN 0-394-35076-6. QA 331.B644 1987. 515.9.

[12] Ruel V. Churchill and James Ward Brown, *Complex Variables and Applications,* 5th ed., McGraw-Hill, New York, 1990. ISBN 0-07-010905-2.

[13] Stephen D. Fisher, *Complex Variables,* Wadsworth & Brooks/Cole, Belmont, CA, 1986. ISBN 0-534-06168-0. QA331.F589 1986.

[14] Murray R. Spiegel, *Schaum's Outline of Theory and Problems of Complex Variables,* McGraw-Hill, New York, 1964. ISBN 09-060230-1.

Undergraduate texts in topology and real analysis

[15] K. G. Binmore, *The Foundations of Analysis: A Straightforward Introduction, Book 2: Topological Ideas,* Cambridge University Press, 1981. ISBN 0-521-29930-6.

[16] Colin Clark, *Elementary Mathematical Analysis,* 2nd ed., Wadsworth Publishers of Canada, Belmont, CA, 1982. ISBN 0-534-98018-X. QA300.C56 1982.

[17] Benjamin T. Sims, *Fundamentals of Topology,* Macmillan, New York, 1976. ISBN 0-02-410640-2. QA611.S495.

Graduate texts in complex analysis

[18] Lars Valerian Ahlfors, *Complex Analysis,* 3rd ed., McGraw-Hill, New York, 1979. ISBN 0-07-000657-1. QA331.A45 1979. 515'.93.

[19] Raghavan Narasimhan, *Complex Analysis in One Variable,* Birkhauser. Boston, 1985. ISBN 0-8176-3237-9. QA331.N27 1985. 515.9.

[20] Walter Rudin, *Real and Complex Analysis,* 3rd ed., McGraw-Hill, New York, 1987. ISBN 0-07054234-1. QA300.R82 1987. 515.

Graduate texts in topology and real analysis

[21] James Dugundji, *Topology,* Allyn and Bacon, Boston, 1966. ISBN 0-205-00271-4.

[22] Gerald B. Folland, *Real Analysis: Modern Techniques and Applications,* Wiley, New York, 1984. ISBN 0-471-80958-6. QA300.F67 1984.

[23] Edwin Hewitt and Karl Stromberg, *Real and Abstract Analysis,* Springer-Verlag, Berlin, 1965.

[24] John L. Kelley, *General Topology,* Springer-Verlag, New York, 1975. ISBN 0-387-90125-6. QA611.K4 1975. 514'.3.

Texts in geometric measure theory

[25] Herbert Federer, *Geometric Measure Theory,* Springer-Verlag, New York, 1969.

[26] Frank Morgan, *Geometric Measure Theory: A Beginner's Guide,* Academic Press, San Diego, CA, 1988. ISBN 0-12-506855-7. QA312.M67 1987. 515.4'2.

Text on convexity

[27] I. N. Yaglom and V. G. Boltyanskiǐ, *Convex Figures* (English translation of the 1951 Russian edition), Holt, Rinehart and Winston, New York, 1961.

Books and articles about Julia sets and fractals

[28] Michael Barnsley, *Fractals Everywhere,* Academic Press, San Diego, 1988. ISBN 0-12-079062-9. QA614.86B37 1988. 516–dc19.

[29] Paul Blanchard, "Complex Analytic Dynamics on the Riemann Sphere," *Bulletin of the American Mathematical Society (New Series),* Vol. 11, No. 1 (July 1984), p. 85–141.

[30] Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems,* 2nd ed., Addison-Wesley, Redwood City, CA, 1989. ISBN 0-201-13046-7. QA614.D486.

[31] Robert L. Devaney, *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics,* Addison-Wesley, Menlo Park, CA, 1990. ISBN 0-201-23288-X.

[32] Robert L. Devaney and Linda Keen, Editors, *Chaos and Fractals: The Mathematics Behind the Computer Graphics,* American Mathematical Society, Providence, RI, 1989. ISBN 0-8218-0137-6. T385.C454 1989. 006.6–dc20.

[33] Kenneth J. Falconer, *The Geometry of Fractal Sets,* Cambridge University Press, 1985. ISBN 0-521-33705-4. QA447. 515.7'3.

[34] Kenneth J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications,* Wiley, Chichester, 1990. ISBN 0-471-92287-0. QA614.86.F35 1990. 514'.74–dc20.

[35] Benoit Mandelbrot, *The Fractal Geometry of Nature,* Updated and Augmented, W. H. Freeman, New York, 1983. ISBN 0-7167-1186-9. QA447.M357 1982. 516'.15.

[36] Heinz-Otto Peitgen and Peter H. Richter, *The Beauty of Fractals,* Springer-Verlag, New York, 1986. ISBN 0-387-15851-0. QA447.P45 1986.

[37] Heinz-Otto Peitgen and Dietmar Saupe, ed., *The Science of Fractal Images,* Springer-Verlag, New York, 1988. ISBN 0-387-96608-0. QA614.86.S35 1988. 516–dc19.

Historical references

[38] James Gleick, *Chaos: making a new science,* Viking, New York, 1987. ISBN 0-670-81178-5. Q172.5.C45G54 1987. 003.

[39] Bartel Leenert van der Waerden, *A History of Algebra: From al-Khwārizmī to Emmy Noether,* Springer-Verlag, New York, 1985. ISBN 0-387-13610-X. QA151.W343 1985. 512'.009.

Applied reference

[40] Gilbert Strang, *Introduction to Applied Mathematics,* Wellesley-Cambridge Press, Wellesley, MA, 1986. ISBN 0-9614088-0-4. QA37.2.S87 1986. 510.

Reference with a program for Mandelbrot's set on the HP-28S

[41] William C. Wickes, *HP-28 Insights: Principles and Programming of the HP-28C/S,* Larken Publications, Corvallis, OR, 1988.

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