DISCOVERING CALCULUS

WITH THE

HP-28 AND THE HP-48

ROBERT T. SMITH

AND

ROLAND B. MINTON

Finding the Commands on the HP-28S

Explanation: Directory path and page are given in parentheses. For example, CLLCD (Plot-4) means that the command CLLCD is on the fourth page of the Plot directory. To find CLLCD, press [PLOT] [NEXT] [NEXT] [NEXT] (or [PLOT] [PREV]).

ACOS (Trig)
ASIN (Trig)
ATAN (Trig)
CENTR (Plot-2)
CLLCD (Plot-4)
COS (Trig)
COSH (Logs-2)
CRDIR (Memory)
C→R (Complx)
DEG (Mode)
DEPTH (Stack-2)
DGTIZ (Plot-4)
DO (Branch-3)
DRAX (Plot-4)
DRAW (Plot)
DRWΣ (Plot-3)
DROP2 (Stack)
DROPN, DUPN (Stack-2)
DUP, DUP2 (Stack)
ELSE (Branch)
END (Branch)
EXP (Logs)
FACT (Real)
FIX (Mode)
FOR (Branch-2)
HOME (Memory)
IF (Branch)
IFTE (Branch-2)
ISOL (Solv)
LN (Logs)
LCD→, →LCD (String)
MENU (Memory)
NEXT (Branch-2)
ORDER (Memory)
PIXEL (Plot-4)
PMAX (Plot)
PMIN (Plot)
PPAR (Plot-2)
P→R (Trig-2)
RAD (Mode)
RCEQ (Plot, Solv)
ROLLD (Stack-2)
R→C (Complx)
R→P (Trig-2)
SIN (Trig)
SINH (Logs-2)
SOLVR (Solv)
START (Branch-2)
STD (Mode)
STEQ (Plot Solv)
TAN (Trig)
TAYLR (Algebra-2)
THEN (Branch)
UNTIL (Branch-3)
WHILE (Branch-3)
*H (Plot-2)
*W (Plot-2)
Σ ÷ (Stat)

All other HP-28S commands used in this book are on the keyboard.

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Robert T. Smith
Millersville University of Pennsylvania

Roland B. Minton
Roanoke College

McGraw-Hill, Inc.
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To the women in my life:
my daughter, Katherine,
my wife, Pam, and
my mother, Anne

R.T.S.

To my parents, Paul and Mosse;
for my children, Kelly and Greg;
with my wife, Jan

R.B.M.

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Preface

Calculus and Calculators

Over the last ten years, microcomputer technology has revolutionized business and industry. Similarly, the wide accessibility of computing power is changing the way in which mathematics is learned and applied. We would estimate that the HP-28S could perform 95 percent of the calculations on a typical calculus exam. However, the point of learning the calculus is not simply to generate correct answers to problems. Indeed, the purpose is to learn how to think mathematically. In this book, we show how the graphical, symbolic and numerical capabilities of the HP-28 and HP-48 families of calculators can help students think about and understand mathematics in new ways.

As we complete this book, the HP-28S is barely three years old. Within the last year, it has been joined by its cousin, the HP-48SX, and most recently by the HP-48S. These graphing calculators are incredibly powerful, each with a tremendous amount of built-in memory and a sophisticated programming language. Indeed, with the ability to perform symbolic and graphical as well as numerical manipulations, these machines might be more correctly referred to as hand-held computers. Used properly, the HP-28S/48SX/48S can be a valuable tool in exploring the concepts of calculus.

What you find in this book is the result of our experiences in the classroom as well as in computing in general. Our discussion does not focus on the HP-28S/48SX/48S. Rather, we concentrate our efforts on the main concepts of the calculus, using the calculator as a tool. We have therefore emphasized only those features of the HP-28S/48SX/48S that aid the study of calculus. There are many other features of these machines which we give only passing mention to or ignore altogether. For those interested in a more thorough discussion of the calculators' features, we suggest the excellent book HP-28 Insights by William Wickes (see the bibliography starting on page 267).

Using This Book

As its size alone should indicate, this book is certainly not a complete calculus text. The reader should have a standard calculus text as well, for although much of our discussion is self-contained, we make frequent reference to material found in such a text. Our choice of topics reflects those most often found in the first two semesters of calculus which we feel most benefit from the introduction of the HP-28S/48SX/48S. We also provide more realistic problems as well as the means of solving them.

You can use this book in a variety of ways. You will find, as our own students have, that you can sit down with this book and your calculator and learn calculus with the fresh approach of a new technology. Work through the book to gain that new perspective, use our exercises to supplement those of your standard calculus text, or – and this is our preferred choice – do both. No experience with graphing calculators is needed to use this text. We start from scratch in Chapter 1. You need only have the desire to learn some new mathematics using a slightly different approach.

We strongly recommend that you work carefully through Chapter 1 before going on to the later chapters. The material on graphing found there is prerequisite for almost everything that follows. This is particularly true if you are using the HP-28S, whose graphics are more difficult to deal with than the HP-48SX.

The HP-28S and the HP-48SX are in some ways very similar machines and we most often discuss their use making no distinction between the two. As far as the programming goes, the two machines are almost identical. In those few instances where there are differences, we have tried to explain them carefully and when there is any chance for confusion, we have discussed the two machines separately. The most noticeable difference is in the handling of graphics and so, in Chapter 1, we have provided easy-to-use programs which allow the HP-28S to mimic some of the graphics functions of the HP-48SX. We make no distinction at all between the HP-48SX and the HP-48S, since the two are identical, except for the expandability of the HP-48SX. For the purposes of this text, then, all references to the HP-48SX apply equally to the HP-48S.

We have tried to write programs that are as easy to follow as possible. By doing so, we sometimes sacrifice a certain level of computational efficiency or sophistication, but we hope that we have better enabled you to understand the mathematics. We have included many examples throughout the text, as well as a wide variety of exercises. Of particular note are the "Exploratory Exercises" found at the end of every section. These provide good experience in tackling some extended, open-ended problems. We have also provided numerous tips on using the HP-28S/48SX/48S more efficiently.

For your convenience, when we refer to a special key (or a "softkey" located in one of the many menus), we put that key in a box. For instance, $\boxed{\texttt{ENTER}}$ indicates that you should press the ENTER key, rather than type the letters E-N-T-E-R. Likewise, $\boxed{\texttt{CRDIR}}$ indicates that you should press the CRDIR softkey located in the Memory menu. In order to help you find the commands in the numerous menus, we have included a listing inside the front cover of the book (for the HP-28S) and inside the back cover (for the HP-48SX/48S).

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Robert T. Smith
Lancaster, Pennsylvania

Roland B. Minton
Salem, Virginia

About the Authors

Robert T. Smith is Associate Professor of mathematics at Millersville University of Pennsylvania, where he has taught since 1987. Prior to that, he was on the faculty at Virginia Polytechnic Institute and State University. He earned his Ph.D. in mathematics from the University of Delaware in 1982.

Dr. Smith's research interests are in the application of mathematics to problems in engineering and the physical sciences. He has published a number of research articles on the applications of partial differential equations as well as on computational problems in tomography. He is a member of the American Mathematical Society, the Mathematical Association of America and the Society for Applied and Industrial Mathematics.

Professor Smith lives in Lancaster, Pennsylvania, with his wife Pam and daughter Katherine. He is an avid volleyball and softball player and a longtime Philadelphia Phillies fan.

Roland B. Minton is Associate Professor of mathematics at Roanoke College, where he has taught since 1986. Prior to that, he was on the faculty at Virginia Polytechnic Institute and State University. He earned his Ph.D. in mathematics from Clemson University in 1982.

Dr. Minton's research interests are in stochastic systems theory and sports science. His publications include journal articles and co-authorship of a technical monograph in control theory and parameter estimation, as well as articles on the teaching of mathematics. He is a member of the American Mathematical Society, the Mathematical Association of America and the Consortium for Mathematics and Its Applications.

Professor Minton lives in Salem, Virginia, with his wife Jan, daughter Kelly and son Greg. He enjoys playing golf and tennis and collecting Greg Minton baseball cards. A love of rock and roll and the Marx Brothers helps him understand his parental responsibilities.

CHAPTER 1

Overview of the HP-28 and HP-48

1.1 Introduction

Calculus is a very broad and tremendously deep study with many and varied applications. As the name implies, it is filled with calculation. You will compute velocities, areas, volumes, etc. The skills that you learn in calculus are basic tools for studying (and yes, practicing) engineering, physics, chemistry, economics and many other diverse fields. But, calculus is more than just necessary background work for the sciences. It is a fascinating subject in its own right, where geometry and algebra come together into a powerful problem solver. The mathematical theory developed here will allow you to make connections between seemingly unrelated real world problems, ultimately leading to a deeper understanding of the world in which we live.

These are some pretty strong statements that we've made. Indeed, what we have described above is the ideal. Unfortunately, the ideal and the reality are not always quite the same. The often messy details of algebra and computation can sometimes obscure the calculus that's behind them. We cannot, nor would we want to, get rid of all the algebra and computation involved in studying calculus. Indeed, these things are necessary. However, we also don't want you to get so lost in the details that you miss the larger picture.

By giving you easy access to fast calculations and graphics, your HP-28S/48SX can help you to discover important relationships. Because of its speed and easy interface, you can ask those "What if I did this..." questions, without a large investment of time. In short, you can experiment. You can focus on the calculus questions, and leave many of the details to your HP-28S/48SX assistant.

As is the case throughout the book, most of what follows is applicable to users of both the HP-28S and the newer HP-48SX graphing supercalculators. Except for the handling of the graphics and the layout of keys and location of programs in subdirectories, there are few differences in the functionality of the two machines that affect their use here. In the instances where there are differences, we first give the HP-28S keystrokes and displays and make comments regarding the modifications needed for the HP-48SX. We also end the section with a set of notes for HP-48SX users.

ARITHMETIC ON THE HP-28S/48SX

The first features that you are likely to notice on the HP-28S/48SX are the Reverse Polish Notation (RPN) and the stack. When you first turned your machine on, you probably tried to add or multiply two numbers. If you tried to enter

$$3+5$$

the calculator beeped at you before you could even type the 5. Most calculators use algebraic notation, where expressions are entered in the same way as they are written (as above). This is not so for RPN machines such as the HP-28S/48SX. For instance, to compute 3+5, you would enter

$$3\ \boxed{\text{ENTER}}\ 5\ + \qquad \text{OR} \qquad 3\ \boxed{\text{SPACE}}\ 5\ +$$

To compute $3 - 5/21$, press

$$3\ \boxed{\text{ENTER}}\ 5\ \boxed{\text{ENTER}}\ 21\ /\ -$$

At first, if you are used to algebraic calculators, this seems rather awkward, but with practice, RPN will become quite natural. In fact, you'll find in the examples below that for longer calculations, RPN has significant advantages over algebraic notation.

Next, try computing $4(2+3/\sqrt{11})$. Note that the square root symbol on the HP-28S is printed in red above the $-$ key, so that you need to press the red (shift) key followed by the $-$ key to get $\sqrt{x}$.

There are a number of ways to compute $4(2+3/\sqrt{11})$, depending on how much pencil-and-paper work you do first and how you choose to use the stack. One sequence is

$$3\ \boxed{\text{ENTER}}\ 11\ \boxed{\text{ENTER}}\ \boxed{\sqrt{x}}\ /\ 2\ \boxed{\text{ENTER}}\ +\ 4\ \boxed{\text{ENTER}}\ *$$

The result is 11.6181361349.

NOTE: Throughout the text we will denote multiplication by * and division by /. You should note that although the keys are marked by "×" and "÷," respectively, the operations are displayed as "*" and "/" on the screen. Further, we will denote exponentiation by $\wedge$ (e.g., x^3 will correspond to "X $\wedge$ 3"). This usage is consistent with the displays of both the HP-28S and the HP-48SX and the labeling of the HP-28S key. The corresponding HP-48SX key is labeled y^x.

You can think of the calculator's stack as a (almost endless) scratchpad on which values are stored. Each line of the stack is numbered, with the bottom line labeled line 1. Although you can see only 3 or 4 lines of the stack at any one time, there may be many, many numbers stored on lines which are not presently visible. For the moment, think of the stack as a tremendously long list of numbers. We will discuss the uses and manipulation of numbers on the stack shortly.

Try computing $6\sqrt{5/3}+7\sqrt[3]{5/8}$. To do this on an algebraic calculator, you would need to do several side calculations, recording the results of each by storing them in a memory or by writing them down for use in completing the calculation. On the HP-28S/48SX, this can be done with great ease, seamlessly. For instance, we can break the calculation up into three parts:

(1) 5 $\boxed{\text{ENTER}}$ 3 / $\boxed{\sqrt{x}}$ 6 *

(2) 5 $\boxed{\text{ENTER}}$ 8 / 1 $\boxed{\text{ENTER}}$ 3 / $\wedge$ 7 *

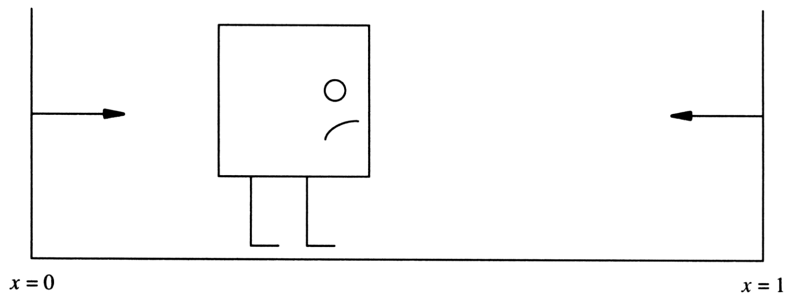
(3) Press + to add the results of the calculations in (1) and (2)

The result of this calculation is 13.7308825058.

The following examples will introduce you to some of the most commonly used features of the HP-28S/48SX, while exploring some interesting situations.

Example 1. The Stack

Consider the case of the trapped video game robot. A robot starts at one end of a corridor ($x = 0$) and moves toward the other end ($x = 1$). At the same time, the walls at the ends of the corridor begin closing in, moving at half the robot's speed.



The robot runs into a wall at $x = 2/3$ (why?) then immediately turns and goes the other way. Since the walls are now separated by a distance of 1/3, the next collision occurs after the robot has gone (2/3)(1/3) back to the left. The second collision, then, is at $x = 2/3 - 2/9$. The walls are now separated by a distance of 1/9. (Why?) The third collision will be at $x = 2/3 - 2/9 + 2/27$, and so on. Where exactly will the robot be trapped?

We have analyzed the problem enough to see the pattern of where the collisions occur. Let's use the HP-28S/48SX to crunch some numbers. First, compute 2/3 (press 2 ENTER 3 /). Then compute 2/9 (press 2 ENTER 9 /) and subtract the two values (press −). Then add 2/27 (press 2 ENTER 27 / +), and continue in this way to generate the table below.

Key Sequence	Result
2 ENTER 3 /	.666666666667
2 ENTER 9 / −	.444444444445
2 ENTER 27 / +	.518518518519
2 ENTER 81 / −	.493827160494
2 ENTER 243 / +	.502057613169
2 ENTER 729 / −	.499314128944

It looks like we're *homing in* on .5. We'll examine this idea of "homing in" (called a *limit*) very carefully in Chapter 2. We'll look further into limits involving sums in Chapter 6. For now, let's take .5 as an educated guess and rethink the problem somewhat. Is .5 a reasonable solution? The answer is "yes." Since the walls are traveling at the same speed, they will meet in the middle at $x = .5$, with the robot trapped between them. ∎

You should notice several things about the preceding problem before going on to Example 2. We worked this problem in three stages: basic analysis, calculations to estimate the solution and an evaluation of the estimate. You should follow this process whenever possible. A numerical answer is of little value without understanding its meaning.

Example 2. The Solve Menu

In Example 1, we were able to guess what turned out to be the precise solution by computing several collision points and recognizing the pattern. We will often search for an answer by repeatedly computing values of a function. For instance, we know from algebra that for

$$f(x) = \frac{1}{x^2 + 120x - 22}$$

$f(x)$ gets steadily smaller as x gets larger (for $x > 1$). Let's find a "smallness threshold," e.g., find the smallest positive integer n such that $f(n) < .001$. We'll start by computing $f(2)$ and then we'll see how the HP-28S/48SX can make our task easier. The keystroke sequence

$$2 \ \boxed{x^2} \ 120 \ \boxed{\text{ENTER}} \ 2 * + 22 - \boxed{1/x}$$

gives us $f(2) = .0045045045$, which is not small enough.

It looks like it will take a lot of typing to generate all of the function values we need. Fortunately, the HP-28S/48SX has some features to minimize this work. We can put the expression for $f(x)$ onto the stack by enclosing the expression in single quote marks. The ' tells the HP-28S/48SX to delay execution of the commands. Type

$$\text{'} \, 1 \,/\, (\, X \wedge 2 + 120 * X - 22 \,) \, \text{'} \ \boxed{\text{ENTER}}$$

Note that the parentheses are *not* optional here. (Why not?)

Next, activate the Solve menu by pressing $\boxed{\texttt{SOLV}}$ (or $\boxed{\texttt{SOLVE}}$ on the HP-48SX). The top row of keys are now "soft keys" whose function is described by the labels appearing on the screen directly above them. The soft key labeled $\boxed{\texttt{STEQ}}$ (for "store equation") will take the expression on line 1 of the stack and store it for later use in a variable named EQ. The soft key labeled $\boxed{\texttt{RCEQ}}$ (for "recall equation" on the HP-28S; use $\boxed{\hookrightarrow}$ $\boxed{\texttt{STEQ}}$ on the HP-48SX) will recall the expression stored in EQ and return it to line 1 of the stack. Since we presently want to store our function, we press $\boxed{\texttt{STEQ}}$. Notice that the expression has now been removed from the stack.

Now, activate the Solver menu (press the $\boxed{\texttt{SOLVR}}$ soft key). To evaluate $f(2)$, first set $x = 2$ by pressing 2 followed by the soft key $\boxed{\texttt{X}}$. (Note that pressing the usual X key will not have the same effect.) The top of the screen should show $\boxed{\texttt{X: 2}}$ Now press the soft key $\boxed{\texttt{EXPR=}}$. The value $f(2) = .0045045045$ should be returned to the stack. Try computing $f(3)$: press 3 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$ and we get $f(3) = .00288184$. Continue by computing $f(4)$, $f(5)$, and so on, until you have $f(n) < .001$. You should get $f(8) = .00099800 < .001$. The entire sequence of calculations follows.

Key Sequence	Result (to 8 places)
2 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00450450
3 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00288184
4 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00210970
5 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00165837
6 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00136240
7 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00115340
8 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$	.00099800

■

We should mention at this point how to correct an expression that has been mistyped. Rather than retype the entire expression, you should use the $\boxed{\texttt{EDIT}}$ command, built into the HP-28S/48SX. Suppose that, instead of 'X∧2+120*X−22', you had accidently typed

' X ∧ 2 + 122 * X − 22'

If this expression is on line 1 of the stack, you can edit it by pressing $\boxed{\texttt{EDIT}}$ (located above the $\boxed{\texttt{ENTER}}$ key on the HP-28S and above the $\boxed{\texttt{+/-}}$ key on the HP-48SX). The four arrow keys (in the top row of keys on the HP-28S) will now move a blinking cursor through the expression. If you are using an HP-28S, pressing any key replaces the current character with the one pressed. (This is called *replace mode*.) On The HP-48SX, you must first press the soft key $\boxed{\texttt{INS}}$ to enter replace mode. Try this now by moving the cursor over to the second 2 in 122 and pressing 0. Press $\boxed{\texttt{ENTER}}$ and the original expression appears corrected on line 1 of the stack.

Suppose, instead, that the coefficient of X should be 1200 instead of 120. Press $\boxed{\texttt{EDIT}}$ and move the cursor to the spot where you want to insert the extra 0. On the HP-28S, first press the $\boxed{\texttt{INS}}$ key (in the top row of keys; this puts the editor in insert mode) and then press 0 to insert the extra 0. Again, pressing $\boxed{\texttt{ENTER}}$ returns the edited expression to line 1 of the stack.

Finally, suppose that you had wanted x^3 instead of x^2. Move the cursor over to the 2 and press the $\boxed{\texttt{DEL}}$ key to delete the 2 and then replace it with a 3.

NOTE: On the HP-48SX, the editor is initially automatically in *insert mode* and you must press the $\boxed{\texttt{INS}}$ soft key to switch back and forth between insert and replace mode.

As you edit more and more complicated expressions, you will find the need to switch back and forth between insert and replace mode while in the process of editing a single expression. This will become routine in a short time.

Example 3. The Stack Menu

At this point, if you've been following along with the calculations, your stack should contain a number of entries, not all of which are visible. Before clearing the stack, let's experiment with some of the commands which allow you to manipulate items on the stack. Activate the Stack menu by pressing $\boxed{\texttt{STACK}}$ on the HP-28S (it is a red label located above the G; on the HP-48SX, press $\boxed{\texttt{PRG}}$ and then $\boxed{\texttt{STK}}$). Press $\boxed{\texttt{DUP}}$ [located on the second page of the HP-48SX Stack menu (press $\boxed{\texttt{NEXT}}$ to get the next page)]: the .001 on line 1 is duplicated. Now, press $\boxed{\texttt{DROP}}$ (or the $\boxed{\leftarrow}$ key on the HP-48SX; in both cases this is not a soft key): the entry on line 1 of the stack is removed.

In the Stack menu, $\boxed{\texttt{DUP2}}$ and $\boxed{\texttt{DROP2}}$ work like $\boxed{\texttt{DUP}}$ and $\boxed{\texttt{DROP}}$, but operate on 2 lines of the stack at the same time. Try these now. If you haven't already done so, go to the second page of the Stack menu by pressing $\boxed{\texttt{NEXT}}$. Now, enter 3 $\boxed{\texttt{DUPN}}$ Although you may not be able to see it yet, something did happen. Press $\boxed{\texttt{VIEW}\uparrow}$ and the screen displays lines 2, 3, and 4. Press $\boxed{\texttt{VIEW}\uparrow}$ twice more and note that the entries on lines 4-6 are duplicates of those on lines 1-3. (On the HP-48SX, the display will show the first 4 lines of the stack and you use the up/down arrow keys to move around the stack.) That is, 3 `DUPN` copied the first 3 stack lines onto the next 3 lines.

Enter 3 `ROLLD` . The value on line 1 is rolled to line 3, with the values in lines 2 and 3 rolled down to lines 1 and 2, respectively. Try 3 `ROLL` . The value on line 3 is rolled to line 1, with the values on lines 1 and 2 rolled up. Finally, press `SWAP` The values on lines 1 and 2 are interchanged. We encourage you to discover the functions of the other stack commands on your own. When you are ready to move on, press `CLEAR` (or `CLR` on the HP-48SX) to clear the entire stack.

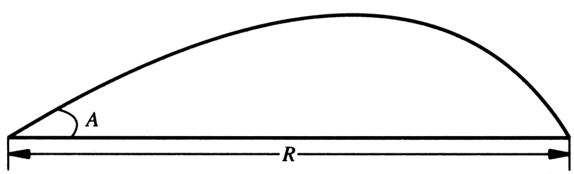
We summarize below the most frequently used stack commands. (The entries in parentheses indicate the corresponding HP-48SX commands.)

Command	Result
STACK (STK)	Activate the stack menu
DUP	Copy line 1
n DUPN	Copy first n lines
DROP	Delete line 1
n DROPN	Delete first n lines
VIEW↑ (△)	Move viewing window up
VIEW↓ (▽)	Move viewing window down
n ROLL	Move line n to line 1
n ROLLD	Move line 1 to line n
SWAP	Swap lines 1 and 2

■

In Example 2, we used the Solver to simplify repeated function evaluation. We now illustrate the use of the Solver for a more complicated example.

Example 4. Parameters and the Solver Menu



Suppose that a ball is thrown from ground level with initial speed S ft/sec and initial angle A above the horizontal. If air effects (such as lift and drag) are ignored, an equation giving the horizontal range in feet is

$$R = \frac{S^2 \sin(2A)}{32}$$

If a ball is thrown at the initial speed of 100 mph at an angle of 30°, how far will it go? First, make sure that your calculator is in degrees mode: type D E G [ENTER] or press [MODE] and then press the [DEG] soft key (on the HP-48SX, press [MODES] and [DEG] is located on the third page of the Modes menu). Type

' S ∧ 2 * [SIN] 2 * A) / 32 ' [ENTER]

Note that the sine function is located in the Trig menu on the HP-28S. Activate the Solve menu (press [SOLV] or [SOLVE]) and store the expression (press [STEQ]). Activate the Solvr menu (press the soft key [SOLVR]), and you will see soft keys for both variables, [S] and [A] , as well as one for [EXPR=] . Before we store a value in S, we need to convert 100 mph into feet per second. This is done by multiplying by 5280 (feet per mile) and dividing by 3600 (seconds per hour). Press

100 [ENTER] 5280 * 3600 /

or use [CONVERT] (see the HP manual). You should have 146.666 on line 1 of the stack. Press the soft key [S] . Then enter 30 and press the soft key [A] . Press [EXPR=] to obtain the range: $R = 582$ ft. How much farther would the ball go if it were thrown at an angle of 40° with the same initial speed? Press 40 [A] [EXPR=] and we find that the ball would go 662 ft. Press – (this subtracts the value on line 1 from the value on line 2) to see that the difference is about 80 ft.

Now, suppose that an outfielder needs to throw a ball 300 ft, and that he/she can throw with an initial speed of 100 mph. What is the best angle of release? In this case, best would mean the smallest angle which gets the ball to its destination. (Why?) From our work above, we can conclude that 30° is too high. From the following calculations, we conclude that the best angle would seem to be slightly greater than 13°.

Key Sequence	Result (rounded)
20 [A] [EXPR=]	432.10
10 [A] [EXPR=]	229.91
15 [A] [EXPR=]	336.11
13 [A] [EXPR=]	294.68
14 [A] [EXPR=]	315.59

■

Note that we found an answer to this problem through a simple process of trial-and-error. In section 1.3, we will show a more direct way of solving an equation while getting a more precise answer. For now, let's note how painless the trial-and-error process was with the help of the Solver.

Example 5. User-Defined Functions

The HP-28S/48SX is a remarkably flexible machine. With the large number of built-in programs and the expansive 32K of user memory (i.e., 32,000 bytes or about as much as some early personal computers), you can customize your machine with ease, without doing any complicated programming. Throughout the rest of this section, we will be giving you ideas on the best way to customize your HP-28S/48SX for use in calculus.

First, press [USER] (or [VAR] on the HP-48SX) to activate the User menu. You should see soft keys displayed for A, S, X and EQ, which are the variables we have used in our previous examples, as well as keys for any other variables which you've intentionally or unintentionally stored. Press [A] and you get the last angle that you tried in Example 4. The HP-28S/48SX, then, stores whatever variables and routines you create as you solve problems. We can take advantage of this storage to save commonly used variables and functions, for repeated use.

Let's start by creating a user-defined function for Example 2. All HP-28S/48SX programs begin with the delimiter [≪] , and we use the symbol [→] (in red above the U on the HP-28S and in blue above the 0 on the HP-48SX) to create a *local variable.* Local variables allow us to create highly readable programs without requiring an extra variable in our User menu. As an example, enter the following program.

≪ → X ' 1 / (X ∧ 2 + 120 * X − 24) ' ≫

A brief explanation of each of the components of the program follows.

Program Step	Explanation
$\ll$	Begin the program.
$\rightarrow$ X	Take the value from line 1 of the stack and store it in the local variable X.
' 1 / (X ∧ 2 + 120 * X − 24) '	Evaluate the given algebraic expression.
$\gg$	End the program.
ENTER	Place the program on the stack.
'F' STO	Store the program under the name F in the User or Var menu.

NOTE: The HP-28S/48SX is very sensitive to the placement of blank spaces. For example, in the above program, there must be a blank space between the → and the X. Throughout this text, be very observant as to the inclusion or the lack of blank spaces. This is very important!

Following the first ' is the function which we examined in Example 2. Notice that F has become the first soft key in your User menu. We can now evaluate F without entering the Solver menu. Press 2 and then the soft key F and $f(2)$ is computed, as before. Notice that the value of the local variable X in the program does not affect the value of the (external) variable X in your menu: press the soft key X (it is not 2!). The function F is now available to use at any time, in the context of Example 2 or as part of another program (as we will do in Example 6).

NOTE: If you no longer have use for a variable, then you can easily remove it from memory. For example, type 'A' PURGE to delete A (PURGE is located above the 4 on the HP-28S and above the DEL key on the HP-48SX).

PROGRAM NOTE: If your program does not work for some reason, and returns an error or simply keeps running and won't stop, you need only press the ON key to return the stack to normal. If you do have a problem, you should first take a look at the program. For the program above, this can be done by pressing 'F'

$\boxed{\texttt{RCL}}$ (recall). $\boxed{\texttt{RCL}}$ is located above the $\boxed{\texttt{STO}}$ key on both the HP-28S and the HP-48SX. Doing this will return a copy of the program to line 1 of the stack. If the program contains an error, you can correct it by using the $\boxed{\texttt{EDIT}}$ command, discussed previously. When you are through editing, you will need to use $\boxed{\texttt{STO}}$ to store the newly edited expression. An alternative to this is the $\boxed{\texttt{VISIT}}$ command (located above the ' key on the HP-28S and above the $\boxed{\texttt{+/-}}$ key on the HP-48SX). Pressing 'F' $\boxed{\texttt{VISIT}}$ will recall the expression stored in the variable F to the stack and automatically enter edit mode. When $\boxed{\texttt{ENTER}}$ is pressed at the end of the edit, the edited expression is automatically stored back in the user variable from which it was taken. This is faster and sometimes more convenient than using the three separate commands $\boxed{\texttt{RCL}}$, $\boxed{\texttt{EDIT}}$ and $\boxed{\texttt{STO}}$ to accomplish the same thing. ■

The HP-28S/48SX accommodates many different levels of programming, from the crude to the very sophisticated. Since this is not intended as a text on the programming of the machine, but rather, in its use in learning calculus, we have included mostly very short and easily understood programs. We suggest some more sophisticated programs in the exercises for those who are interested in this aspect. The interested reader is also referred to the excellent book <u>HP-28 Insights: Principles and Programming of the HP-28C/S</u> , written by William Wickes (see the bibliography at the back of the book). Our aim throughout the book will be to provide the student with useful programs that are easy to use, easy to understand and as simple as possible. Therefore, at times we have avoided what might be a more efficient program for the sake of choosing something which is simpler.

Example 6. Programming

As a first example, we will write a *one-step* program for Example 2. An automated version of this program will be discussed in the exercises.

Recalling Example 2, we would like to push one button and have the HP-28S/48SX calculate the value of $f(n)$ for the next value of n. The key to many HP-28S/48SX programs is having an understanding of how you want to make use of the stack. In the present case, we'll keep the current value of n on line 2 and the value of $f(n)$ on line 1.

Let's first run through the procedure manually. Enter 2 and $f(2)$ on the stack: press

2 [ENTER] 2 [F]

where again, [F] refers to a soft key in the User menu. Before computing the next value of f, we want to remove $f(2)$ from the stack, and so, press [DROP] . Next, to increase the 2 on line 1 to 3, press 1 +. We can now evaluate $f(3)$ by pressing [F] although we'll have $f(3)$ on the stack, but not 3! So we need to first duplicate 3 (press [DUP] or [ENTER]) before pressing [F] .

We are now ready to write a program. When typing the program, you should press the [DROP] key instead of typing the 4 letters in the word DROP. Also, get [DUP] from the Stack menu. Enter the following program:

≪ [DROP] 1 + [DUP] F ≫

Program Step	Explanation
≪ [DROP]	Begin the program and drop the entry on line 1 of the stack.
1 +	Add 1 to the value on line 1.
[DUP]	Copy the value on line 1.
F ≫ [ENTER]	Compute the value of F at the value on line 1. Place the program on the stack.
'PG' [STO]	Store the program under the name PG.

Since the program needs two values on the stack (one to be dropped and one to add 1 to) we need to initialize the stack: type 2 [ENTER] 2 [F] . Now, press [PG] several times, and observe how the values of n and $f(n)$ (located on lines 2 and 1 of the display) change. ■

Note that the HP-28S/48SX has no difficulty in referencing our user-defined function F, and that we wrote this program essentially by including all the same keystrokes we would have used if we were performing all the steps manually. If you want to change the function F (as we ask you to in the exercises) you may do so without making any changes to PG. The HP-28S/48SX encourages this type of structured programming: breaking up larger tasks into small independent tasks.

HP-48SX NOTES

For the most part, the changes in instructions noted in this section for HP-48SX users have been notational. As a reminder, we summarize the differences in the keystrokes here:

HP-28S	HP-48SX
SPACE	SPC
$\wedge$	y^x
CHS	+/−
RCEQ	↪STEQ
USER	VAR
DROP	←

Most of the menus included in the HP-28S are also included in the HP-48SX, although some are found deeper in the directory structure. For example, the HP-48SX does not have direct access to the Stack menu. Instead, Stk is a menu which can only be accessed by first pressing [PRG] . The [△] key acts the same as [VIEW↑] on the HP-28S, but also brings up a menu of several useful stack commands, including [ROLL] , [ROLLD] , [DUPN] and [DROPN] .

The HP-48SX is easily switched back and forth between degrees and radians mode by pressing [RAD] (above the 1). In radians mode, you will see the message RAD in the top left corner of the screen. Press [RAD] again, the message disappears, and you are back in degrees mode.

The trig functions sine, cosine and tangent, the exponential function e^x and the natural logarithm function $\ln x$ are located on the keyboard of the HP-48SX (e^x and $\ln x$ are above the [$1/x$] key), while they are accessed from Trig and Logs menus on the HP-28S.

The factorial is the ! key located in the Prob menu which is accessed by pressing [MTH] first.

Exercises 1.1

In exercises 1-9, compute the indicated values. If you are familiar with exponential functions, compute the values in exercises 10-12, using the EXP soft key in the Logs menu (or the e^x key on the HP-48SX).

1. $4+5/12$	2. $4/(3-1/6)$	3. $7-4(1+2/3)$
4. $4^2+2/7$	5. $(4-4/9)^2$	6. 13.5^3-12^2
7. $(3/7+1)^{1/2}$	8. $(4.2*3.1+1)^{1/3}$	9. $(3.4*4.1+1/3)^{1.2}$
10. e^4+1	11. $(e^2+e^{-2})/2$	12. $e^{-3}+\ln(3)$

In exercises 13-18, compute the first 6 terms as in Example 1. If possible, guess what number the sum is "homing in" on. NOTE: in exercises 16 and 18, you may want to use FACT in the Real menu (or ! in the Prob menu on the HP-48SX).

13. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

14. $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots$

15. $\frac{1}{1(2)}+\frac{1}{2(3)}+\frac{1}{3(4)}+\ldots$

16. $\frac{1}{2}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots$

17. $4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\ldots$

18. $1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

19. Repeat Example 1 for $\pi/6-\frac{(\pi/6)^3}{3!}+\frac{(\pi/6)^5}{5!}-\frac{(\pi/6)^7}{7!}+\ldots$ To make this as easy as possible, store $\pi/6$ in a variable P. Note that π is in red above the period (or above the SPC key on the HP-48SX) and pressing π gives you the *symbol* π. To get a decimal approximation of π, press →NUM .

In exercises 20-21, we will discover the relationship between the quadratic formula and our work in Example 2.

20. Note that $\frac{1}{x^2+120x+22}<\frac{1}{1000}$ if $x^2+120x+22>1000$ or $x^2+120x-978>0$. Use the quadratic formula to solve this equation. You should get two solutions. Use your calculator to find a decimal approximation of the larger solution. How does this relate to our answer in Example 2?

21. Use the quadratic formula as in exercise 20 to determine the smallest integer $n>0$ such that $f(n)<1/1000$ for $f(x)=2/(x^2+10x+100)$.

22. In this exercise, we assume that you have stored the function in exercise 21 using STEQ (if not, do so now). Enter the Solve menu. On the HP-28S, press

RCEQ and then EDIT ; on the HP-48SX, press EDEQ . The function $2/(x^2 + 10x + 100)$ should be on the command line and a flashing cursor positioned at the first character. Use the arrow keys to move the cursor and change 100 to 80. On the HP-28S, press ENTER STEQ ; on the HP-48SX, press ENTER . Rework exercise 21 with this new function.

23. As in exercise 22, edit the current EQ, this time changing 80 to C. To set C=40, press 40 'C' STO . Rework exercise 21 using C=40 and C=120.

24. Press USER (or VAR on the HP-48SX) and find the user-defined function F we used in Example 5. Press ' F ' VISIT and this function will be displayed for editing. Change the power on x from 2 to 3 and rework Example 6. Note that you do not need to change program PG.

25. Using VISIT as in exercise 24, change the power on x in F from 3 to N. Set N=4 by pressing 4 'N' STO . Then rework Example 6.

26. You have undoubtedly noticed that the results of certain calculations have been displayed in scientific notation. For instance, compute 2/27. The displayed answer of 7.40740740741E-2 is shorthand for $7.40740740741 \times 10^{-2}$ or .0740740740741. You have some control over the calculator's display through the FIX command located in the Modes menu. Press 6 FIX and see how the displayed number changes. Also press 2 FIX . If you do not need 12 digits of accuracy or simply do not want the screen cluttered with 12 digits, you can use FIX at your convenience. Press STD to return to the standard 12-digit display.

27. As with most calculators, the HP-28S/48SX has a $1/x$ key which will compute $1/x$ for whatever x is on the display. This is an example where *all* calculators use RPN. Unlike other calculators, though, the $1/x$ key on the HP-28S/48SX will find the multiplicative inverse of expressions. To discover this, press ' X ∧ 2 - 22 ' ENTER and then press $1/x$. The INV stands for inverse, and the expression on the stack is completely equivalent to $\dfrac{1}{x^2 - 22}$. If you press x^2 this expression is squared, and if you press COS the expression on the stack is equivalent to $\cos[1/(x^2 - 22)^2]$. In many cases, you can save keystrokes using this nice feature of the HP-28S/48SX.

28. Type in the program ≪ 0 DO 1 + DUP F UNTIL .001 < END DUP F ≫ and store this in the user variable PG2. Compare PG2 to PG in Example 6. To rework Example 6 (or exercise 25, if you have changed F) simply press PG2

Which program, PG or PG2, do you like better? We will typically present programs like PG to avoid *infinite loops* [PG2 would never stop if $f(x) > .001$ for all x unless we interrupted it by pressing $\boxed{\texttt{ON}}$] and to gain the educational value of watching processes step by step.

29. When going to a bank, have you ever thought that your line moved slower than the other lines? William Feller in his classic <u>An Introduction to Probability Theory and Its Applications</u>, volume 2, devotes a section to "The Persistence of Bad Luck." Feller derives the formula $f(n) = \dfrac{1}{n(n+1)}$ for the probability that you will wait longer than the next $n-1$ people choosing a line (under various assumptions such as everyone receives the same service and the lines never thin out). In exercise 15, you were asked to add up 6 of these probabilities. The total probability is always 1 (100%). Did you guess 1 as the limiting value in exercise 15?

30. In the situation of exercise 29, what is the average number of people that you would wait longer than? This is computed with the *expected value* formula $f(1) + 2f(2) + 3f(3) + 4f(4) + \ldots$ Show that this sum keeps getting larger without ever homing in on any number. Talk about bad luck picking lines!

31. The so-called "dining room problem" is another probability problem whose solution can be found in Feller's <u>An Introduction to Probability Theory and Its Applications</u>, 3rd edition, volume 1. Suppose that n people sit down to eat and then discover that there are name tags at each seat. What is the probability that *nobody* sat down at the right place? The formula is $P = \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{1}{n!}$ (if n is even) which you were asked to compute for $n = 6$ in exercise 16. What number did you guess the sum homes in on? If the number of people for dinner becomes larger, do you think it becomes more or less likely that someone will accidentally sit in the right place? Compare your answer in exercise 16 with $e^{-1} = .367879441171\ldots.$

32. An ancient riddle is attributed to Zeno, a Greek philosopher of the fourth century B.C. A version of *Zeno's paradox* starts with Achilles 1 meter behind his rival in a race, but gaining at the rate of 1 meter/sec. Clearly, Achilles catches up in 1 sec. But, argues Zeno, before he catches up he must cut the distance to .5 meters (this takes .5 sec), and then he must cut the distance to .25 meters (this takes .25 sec), then to .125 meters, and so on. Therefore, it would seem Achilles can never catch up! The sum in exercise 13 gives the

amount of time it takes to complete this seemingly endless process. Did you get 1 sec?

33. You may have already heard about the dangers of the population explosion. The following dramatic warning is adapted from the article "Doomsday: Friday 13 November A.D. 2026" by Foerster, Mora and Amiot in Science, volume 132, November 1960, pp. 1291-1295. Create, as in Example 5, a user-defined function P given by

$$P(x) = x + .005x^{2.01}$$

If x is the current population, $P(x)$ is a prediction of next year's population. Start by entering 3.049 (the population in 1960 was about 3.049 billion). Then press the soft key $\boxed{\text{P}}$. You should get about 3.096, which is an estimate of the population in 1961. Press $\boxed{\text{P}}$ again: 3.144 billion is an estimate of the 1962 population. Press $\boxed{\text{P}}$ repeatedly and compare the equation's estimates with the actual populations shown. Then project ahead to the year 2035. Frightening, isn't it?

YEAR	POPULATION (in billions)
1970	3.721
1980	4.473
1985	4.865

EXPLORATORY EXERCISE

Introduction

In this exercise, we will take an extended look at a question that is *open-ended*; that is, there is no single correct answer or even a single correct strategy for finding a solution. The HP-28S/48SX makes it easy to try out ideas. This is when mathematics becomes fun! The general question to be addressed is: what is the optimal angle at which to punt a football? Mathematicians view words like "optimal" or "best" with suspicion, and a large part of this problem is to decide what should be meant by "optimal." We will assume that a ball kicked with initial speed S and initial angle A will cover a horizontal distance of $R = \dfrac{S^2\sin(2A)}{50}$ where the 50 is intended to take into account gravity and air resistance. The ball will be in the air for $T = \dfrac{S\sin(A)}{16}$ seconds, during which time the punt coverage team will be able to run $25T + 30$ ft from the punter.

Problems

We want to compare the distances covered by the punt and the punt coverage team. Recall that we can evaluate two expressions at the same time in the Solver. We will take advantage of this feature by entering the equation

$$' \ S \wedge 2 * \mathrm{SIN}(\ 2 * A\ )\ /\ 50 = 25 * S * \mathrm{SIN}(\ A\ )\ /\ 16 + 30\ '$$

and storing it (press $\boxed{\text{STEQ}}$). Now enter the Solver. We will use degree measures for angles so make sure you are in degrees mode. Enter 90 for S (press 90 $\boxed{\text{S}}$) and 60 for A (60 $\boxed{\text{A}}$). Press $\boxed{\text{LEFT=}}$ and $\boxed{\text{RT=}}$ (or $\boxed{\text{EXPR=}}$ on the HP- 48SX). The left side of the equation is the distance covered by the punt (in this case, about 140 ft) and the right side of the equation is the distance covered by the coverage team (in this case, about 147 ft). Since the coverage team can cover more ground than the punt, it is reasonable to assume that there would be no punt return. Now, try an angle of 45°. The punt goes 162 ft and the coverage team goes 130 ft. How far would this punt be returned? A simple rule would be to split the difference: the punt returner would make it back to the 146-ft mark. From the punt team's perspective, this is better than the 140-ft result we got from an angle of 60°. Your job is to develop a rule for deciding where the punt returner is tackled (preferably more sophisticated than the one given above) and then maximize the net distance.

Further Study

There are numerous follow-up questions for this problem. For example, we used an initial speed of 90 ft/sec in the problem above. Does the optimal angle change if you change S? Compare your results to others' in your class, and have fun!

1.2 Graphing Capabilities

The expression "a picture is worth a thousand words" is particularly true in mathematics. For complicated problems, a graph is often useful to communicate the statement as well as the solution of a problem. In mathematics, graphs also provide simple summaries of the important properties of a function. For instance, the graph displayed in Figure 1.1 (which is taken from the screen of an HP-48SX) shows a function which has a minimum value of approximately $y = 0$, located at about $x = 0$. We would also anticipate that the function would continue to get larger to the left and to the right of the screen displayed.

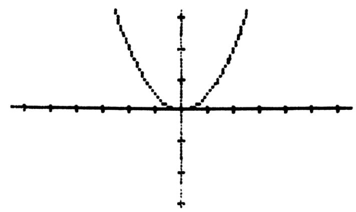


FIGURE 1.1

The HP-28S/48SX lets us immediately see what the graph of a function looks like. We can then use calculus skills to precisely label the important points on the graph, and to precisely determine the behavior of the function in regions not displayed by the graph. This all sounds great, but there is a significant caution. Your HP-28S/48SX does not draw graphs. All that it does is plot points, albeit lots and lots of points. You should not be so dazzled by the speed and comparative ease of the graphics that you lose sight of this limitation. The graphs produced by the HP-28S/48SX are essentially a bunch of (not necessarily connected) dots on the screen. Even so, because of the large number of points being plotted, the graphs are very useful and we will exploit them.

Because of the differences in the graphics on the HP-28S and HP-48SX, we present separate sections for the two machines. The HP-28S section has more discussion, and HP-48SX users are urged to read this section before working through the HP-48SX section. HP-28S users may wish to read through the HP-48SX section for comparison purposes.

GRAPHING ON THE HP-28S

We begin our discussion by examining the graphics features built into the HP-28S. While these are very powerful, they are not very easy for beginning users to handle. In the latter half of our presentation, we provide the reader with a collection of graphing "utility" programs, which can easily be keyed in in a few minutes and which provide for a simplified handling of the graphics. We urge the reader to pay particular attention to this material, as we will use these programs throughout the remainder of the text.

The graphing commands of the HP-28S are located in the Plot menu. Press

$\boxed{\text{PLOT}}$. The first page of the menu contains the soft keys

$\boxed{\text{STEQ}}$ $\boxed{\text{RCEQ}}$ $\boxed{\text{PMIN}}$ $\boxed{\text{PMAX}}$ $\boxed{\text{INDEP}}$ $\boxed{\text{DRAW}}$

You should recognize $\boxed{\text{STEQ}}$ and $\boxed{\text{RCEQ}}$ from our work in section 1.1 with the Solve menu. The $\boxed{\text{STEQ}}$ (store equation) and $\boxed{\text{RCEQ}}$ (recall equation) soft keys are provided in both menus for the sake of convenience. In the examples to follow, we will learn the uses of many of the other soft keys in the Plot menu (which has 4 pages). We will close this section with some graphing utilities to automate some of the more common graphing sequences. In most cases, these utilities are already available as built-in functions on the HP-48SX.

WARNING: Before drawing a graph of a function with the variable X, make sure that you have purged the value of X from the current directory: press 'X' $\boxed{\text{PURGE}}$ ($\boxed{\text{PURGE}}$ is located above the 4 on the HP-28S and above the $\boxed{\text{DEL}}$ key on the HP-48SX). Failure to do so may cause a problem with drawing the graph.

Example 1. Draw

So, what are we waiting for? Let's graph a function! Store the function $f(x) = x^2 - 1$. That is, enter

' X ∧ 2 − 1 ' $\boxed{\text{STEQ}}$

Now, press $\boxed{\text{DRAW}}$. At first, nothing appears on the screen; points are being "plotted" off the screen. Eventually, a nice parabola appears. The tick marks on the x-axis and y-axis have *default values* of 1 unit each, so that the parabola seems to bottom out at $y = -1$ and cross the x-axis at $x = -1$ and $x = 1$ (see Figure 1.2).

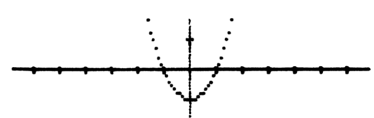


FIGURE 1.2

While the picture is displayed on the screen, the calculator is in *interactive plot mode*. The four keys marked $\boxed{\triangle}$, $\boxed{\triangledown}$, $\boxed{\triangleleft}$ and $\boxed{\triangleright}$ (in the top row of keys) will move a small cross-hair (the cursor) around the graphics window. For example, press △ 7 times and you will see the cursor moving up the y-axis. Now, press ▷ 13 times and the cursor will be on top of one of the points of the parabola plotted by the HP-28S. What point is this? Press and hold the ◁▽▷ key and the coordinates of the location of the cursor will be displayed for as long as you hold the key. An alternative is to press INS . When you do this, the screen may appear to blink, but otherwise nothing seems to happen. To see what this has done, press ON to leave interactive plot mode. The graph is erased and replaced by the stack, which now has the point (1.3,.7) on line 1. This is the point we had moved the cursor to. But, is this actually a point on the parabola? Compute $1.3^2 - 1 = .69$ to discover that (1.3,.7) actually lies just above the parabola. ■

You should realize that the graphs produced by the HP-28S are not perfect representations. They quickly provide us with some idea of what the true graph looks like, but we will need to use the power of calculus to obtain precision in the areas where we want it.

Example 2. CENTR, *W, and *H

Here, we examine the graph of $f(x) = x^2 - 1$ from several different perspectives. First, we will translate our viewing window (i.e., that small portion of the xy-plane which the calculator is currently displaying) with the centering command.

Enter (1.3,.7) on line 1 of the stack. Then go to the second page of the Plot menu (press NEXT) and press CENTR . Next, press PREV (above NEXT) to return to page 1 of the Plot menu. Press DRAW and you get a new graph. The cursor is always placed at the center of the screen, which is now located at the point (1.3,.7) specified in the CENTR command. In this view, the bottom of the parabola has been cut off (see Figure 1.3).

We can change the vertical scale to recover sight of the bottom of the graph. Press ON NEXT (to return to page 2 of the Plot menu). Pressing the soft key *H multiplies the height of the viewing window by whatever factor is on line 1 of the stack. Press 2 *H and redraw the graph (press PREV DRAW).

The parabola has seemingly spread out (see Figure 1.4), and we can see more of it. The vertical tick marks now represent 2 units (the bottom of the parabola at $y = -1$ is half a tick mark down from the x-axis). In effect, we have "zoomed out" our viewing window.

FIGURE 1.3

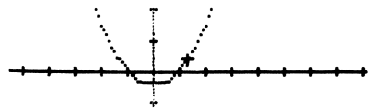


FIGURE 1.4

In the same way that [*H] controls the vertical scale, the [*W] (multiply width) soft key controls the horizontal scale. To discover its effect, press

[ON] [NEXT] 2 [*W] [PREV] [DRAW]

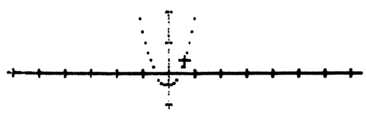


FIGURE 1.5

The parabola now looks similar to what we saw in Figure 1.2, although there are fewer points plotted (see Figure 1.5). The tick marks now represent 2 units both horizontally and vertically. We have again "zoomed out" our view. You should observe that zooming is a powerful graphing tool, but our current implementation is somewhat awkward. An automatic zoom-out utility will be given later to greatly simplify this procedure. ■

Example 3. PMIN and PMAX

We saw two different ways to zoom out in Example 2. We will now see how to zoom in on a specific part of a graph. Starting from the screen in Figure 1.5, move the cursor 6 steps below the origin and 6 steps to the left of the origin and press [INS] . Next, move the cursor 6 steps above the origin and 10 steps to the right of the origin, and press [INS] again. Press [ON] and you should see the points (-1.1,-1.3) and (2.1,1.1) on lines 2 and 1 of the stack, respectively. Finally, press the soft keys [PMAX] , [PMIN] , and [DRAW] . We obtain the graph pictured in Figure 1.6.

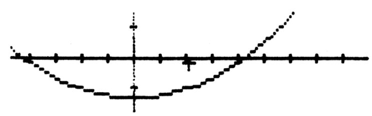


FIGURE 1.6

$\boxed{\texttt{PMAX}}$ has set the upper right corner of the screen to be the point (2.1,1.1) and $\boxed{\texttt{PMIN}}$ has set the lower left corner of the screen to be (-1.1,-1.3). We have in effect zoomed in on that portion of the graph lying in the rectangular window with the corners specified in the $\boxed{\texttt{PMAX}}$ and $\boxed{\texttt{PMIN}}$ commands. In the process, we have made it difficult to recognize that we are looking at the graph of a parabola. In general, the more that we zoom in on a certain portion of a graph, the harder it becomes to see the overall (global) behavior of the function. As we will see in Example 4, perspective is an important aspect of correctly interpreting graphs. ∎

Example 4. Asymptotes

Vertical and horizontal asymptotes are among the most recognizable features of a graph. They also provide for many students the first example of the interplay between graphs and equation solving. In this example, we will look at the graph of $f(x) = \dfrac{x-1}{x^2-x-2}$. You should recall that setting the numerator equal to 0 ($x - 1 = 0$) gives you the x- intercept of the graph, while setting the denominator equal to 0 ($x^2 - x - 2 = 0$) gives you the location of the vertical asymptotes, provided all common factors have already been cancelled. In addition, you may have seen the idea of letting x become arbitrarily large and identifying the result with a horizontal asymptote. In this case, as x becomes very large, $f(x)$ approaches 0 and the line $y = 0$ (i.e. the x-axis) is a horizontal asymptote. Let's look at this on the HP-28S. Enter

' (X − 1) / (X ∧ 2 − X − 2) ' $\boxed{\texttt{STEQ}}$ $\boxed{\texttt{DRAW}}$

and you should see the graph in Figure 1.7. This does not match our expectations at all! With some imagination, you might be able to visualize the vertical asymptote $x = 2$, but the horizontal asymptote is simply not there. The problem is that we still have the screen set up with the viewing window of Example 3. We can get back to the default viewing window of Example 2 by entering

[ON] [NEXT] ' [PPAR] ' [PURGE]

This removes the variable PPAR from the User menu. (PPAR is a variable storing the parameters for the current viewing window.) Now, redraw the graph by pressing [PREV] [DRAW] . What we see in Figure 1.8 is more like what we expected.

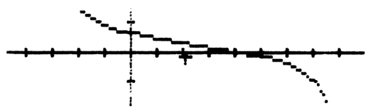


FIGURE 1.7

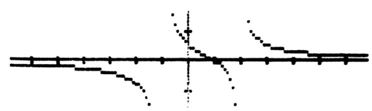


FIGURE 1.8

The graph appears to have vertical asymptotes at $x = -1$ and $x = 2$, an x-intercept at $x = 1$, and becomes horizontal to the left and to the right. So, we needed to zoom out from Figure 1.7 to see all the features we anticipated from our analysis. What would happen if we zoomed out further? Try

[ON] [NEXT] 10 [*H] [PREV] [DRAW]

Very few points show up. Actually, a lot of points get plotted, but the y-values are so small compared to the scale (10 units per tick mark) that the points all get plotted on the x-axis (see Figure 1.9).

Also, try the sequence

[ON] [NEXT] ' [PPAR] ' [PURGE] 10 [*W] [PREV] [DRAW]

Again, most of the plotted points are placed on the x-axis (see Figure 1.10). This time, it appears as if there is only one vertical asymptote (at $x = 0$). The significance of Figures 1.7-1.10 is that the appearance of a graph is highly dependent on the scale you choose. It is essential to verify any features you see on a graph with calculus, because appearances can be deceiving, very deceiving. ■

THE GRAPHICS ENVIRONMENT

You should now be familiar enough with the HP-28S graphics commands to be concerned with the ease of their use. Notice that we have repeated certain multi-key

FIGURE 1.9

FIGURE 1.10

sequences several times, and that it is easy to automate such sequences with short programs. We will do this presently. However, each program that we write will occupy a slot in the User menu, and you have already seen how aggravating multi-page menus can be. The solution is to group our programs into separate directories, much as is done on a computer. Below, we suggest several graphics programs (many of which are already built into the HP-48SX) and a directory structure.

We will start by creating a directory called PLOTR. Simply enter

' P L O T R ' [CRDIR]

[[CRDIR] (create directory) is located in the Memory menu.] Note that [PLOTR] is now the first entry in the User menu. Press the [PLOTR] soft key and all the menu entries disappear. PLOTR is essentially a new shelf that we have built to store programs and as of now we have not put any programs on this shelf.

You should enter all of the following programs. Note that [→LCD] and [LCD→] are soft keys found in the String menu. If you type these in manually, you must be sure that there is no space between the → and LCD. Note that most of the programs can be entered with just a few keystrokes. Except for variable names, all commands can be entered with a single keystroke.

The first program which we offer will exit the current directory and return the user to the HOME directory with a single keystroke. This facilitates easy movement in and out of your new PLOTR directory. Enter

≪ [HOME] ≫ [ENTER] ' Q U I T ' [STO]

Next, we give a program that will draw a graph in the usual way, but which will also store the graph so that it can be recalled at some later point, without the need to redraw the entire graph.

≪ [CLLCD] [DRAW] [LCD→] 'PICT' [STO] [DGTIZ] ≫

Program Step	Explanation
≪ $\boxed{\text{CLLCD}}$	Clear the LCD display.
$\boxed{\text{DRAW}}$	Draw the graph of the function stored in the variable EQ.
$\boxed{\text{LCD}\rightarrow}$	Place a coded image of the display on line 1 of the stack.
'PICT' $\boxed{\text{STO}}$	Store the image in a variable named PICT.
$\boxed{\text{DGTIZ}}$ ≫ $\boxed{\text{ENTER}}$	Turn on the interactive graphics mode, end the program and place the program on line 1 of the stack.
'PLOT' $\boxed{\text{STO}}$	Store the program under the name PLOT in the current directory.

Rather than use PLOT immediately, we offer the following program as a convenience that you can use each time you want to graph a function for the first time. It will read an expression (function, equation, etc.) from line 1 of the stack, store it in the variable EQ and draw a graph using the PLOT graphing routine above. We use PLOT instead of DRAW since PLOT will also save a digitized image of the display for easy retrieval.

≪ $\boxed{\text{STEQ}}$ $\boxed{\text{PLOT}}$ ≫

Program Step	Explanation
≪ $\boxed{\text{STEQ}}$	Store the expression on line 1 of the stack in the variable EQ in the current directory.
$\boxed{\text{PLOT}}$ ≫ $\boxed{\text{ENTER}}$	Draw the graph of EQ, end the program and place the program on line 1 of the stack.
'NEWF' $\boxed{\text{STO}}$	Store the program under the name NEWF in the current directory.

Before continuing to enter new programs, test your new programs out. Press

'SIN(X)' [ENTER] [NEWF] . If you have your calculator in radians mode, you should see a nice graph of $y = \sin(x)$. As always, to exit the graphics mode, press [ON] .

The next program will recall a stored display image to the screen with one keystroke.

≪ PICT [→LCD] [DGTIZ] ≫

Program Step	Explanation
≪ PICT [→LCD]	Return the screen image stored in the variable PICT to the display.
[DGTIZ] ≫ [ENTER]	Turn on the interactive graphics mode, end the program and place the program on line 1 of the stack.
'GRAPH' [STO]	Store the program under the name GRAPH in the current directory.

Press [GRAPH] now to recall the last graph to the screen.

Our next program is the first of two automatic zoom programs. This first one will zoom out a fixed amount at the touch of a button.

≪ 1.25 [*W] 4 [*H] [PLOT] ≫

Program Step	Explanation
≪ 1.25 [*W]	Multiply the scale on the x-axis by a factor of 1.25.
4 [*H]	Multiply the scale on the y-axis by a factor of 4.
[PLOT] ≫ [ENTER]	Draw the graph of EQ, end the program and place it on line 1 of the stack.
'ZOOM' [STO]	Store the program under the name ZOOM in the current directory.

Press [ZOOM] now and you will produce a graph that is "zoomed out" from the last one.

The second of our two zooming programs will read two points from the stack (usually, these will be points that were digitized from a graph) and draw a new graph, using these points as the upper right and lower left corners, respectively, of the new display window. This is perhaps the most useful of all our graphing utilities.

≪ [PMAX] [PMIN] [PLOT] ≫

Program Step	Explanation
≪ [PMAX]	Read the point from line 1 of the stack and make it the upper right of the next graphics window.
[PMIN]	Read the point from line 2 of the stack and make it the lower left of the next graphics window.
[PLOT] ≫ [ENTER]	Draw a new graph using the new parameters and the PLOT utility.
'ZBOX' [STO]	Store the program under the name ZBOX in the current directory.

At this time, you should try out your new program. First press [GRAPH] to return the last graph to the display. To zoom in on a particular portion of this graph, move the cursor (recall that this is controlled by the arrow keys in the first row on the keyboard) to the lower left corner of a rectangular portion of the graph in which you are interested. Pressing [INS] will digitize the point. Then, move the cursor to the upper right corner of the rectangle of interest and digitize that point. Press [ON] to exit the graphics mode and you should see the coordinates of the two digitized points on the stack. Press [ZBOX] and a new graph will be drawn, using the two digitized points as the corners of the new display window.

Our next program redefines the center of the display window to be the point on line 1 of the stack. There is no difference between this and the [CENTR] command in the Plot menu, except that this will be conveniently located in the PLOTR directory and that the graph is then drawn using the PLOT utility, thereby storing a coded image of the graphics display for easy access.

≪ [CENTR] [PLOT] ≫

Program Step	Explanation
≪ [CENTR]	Take the point on line 1 of the stack and make it the center of the next graphics window.
[PLOT] ≫ [ENTER]	Draw the graph with the new center using PLOT.
'CENTER' [STO]	Store the program under the name CENTER.

Note that only the first 5 letters of the name will appear in the menu.

To try your new program, first press [GRAPH] to return the last graph to the display. Move the cursor to some point of interest (say an intercept) and digitize the point. Press [ON] and then [CENTE] to draw a new graph with the center of the graphics window at the indicated point.

The following program will read an expression from line 1 of the stack and then draw its graph superimposed over the last graph drawn and stored in PICT. The new graph is, in turn, stored in PICT and hence, by repeating the process, we may superimpose the graphs of any number of expressions. You should note, however, that due to the small size of the HP-28S display, the display starts to get quite cluttered after 2 or 3 graphs.

≪ [CLLCD] [STEQ] PICT [→LCD] [DRAW] [LCD→] 'PICT' [STO] [DGTIZ] ≫

Program Step	Explanation
≪ [CLLCD]	Clear the LCD display.
[STEQ]	Store the expression on line 1 of the stack in the variable EQ.
PICT [→LCD]	Restore the graph in PICT to the LCD display.
[DRAW]	Draw the graph of EQ overtop of the current display.
[LCD→] 'PICT' [STO]	Store the new display in the variable PICT.
[DGTIZ] ≫ [ENTER]	Activate the interactive graphics.
'OVERD' [STO]	Store the program under the name OVERD in the current directory.

Before we test out the new program, we give one final graphics utility. This program will reset the window parameters to their default settings. You should always run this program before drawing the graph of any new expression. If it has not been run, you will see the variable PPAR (containing all of the current window parameters) as the first menu item in your PLOTR directory and the program NEWF will be hidden on the second page of the menu. Enter

≪ 'PPAR' [PURGE] ≫

and then press [ENTER] and 'RESET' [STO] .

Press [RESET] now to reset the graphics window. To redraw the current function with the default window, simply press [PLOT] . Exit the graphics mode and press 'COS(X)' [ENTER] [OVERD] . This will draw the graph of $y = \cos(x)$ overtop of the last graph drawn.

Finally, we give a program which will automatically put the menu entries in your PLOTR directory in a convenient order. The program includes a list of the affected menu entries and the HP-28S command [ORDER] , which will rearrange the menu to match the order in the list. Any menu items not listed are automatically placed at the end of the menu. Enter

≪ { [RESET] [CENTER] [ZOOM] [ZBOX] [GRAPH] [NEWF]
[OVERD] [PLOT] [QUIT] } [ORDER] ≫

and press [ENTER] and 'ORDR' [STO] .

Press [ORDR] now and your PLOTR directory will be ready to use.

That was a lot of typing! Let's get a quick reward for that work by working some examples using our new utilities.

Example 5. Oblique Asymptotes

A change of scale is useful for identifying a third type of asymptote: the oblique (or slant) asymptote. For example, for $f(x) = \dfrac{2x^3}{x^2 - 1}$, by solving $x^2 - 1 = 0$, we know that there should be vertical asymptotes at $x = -1$ and $x = 1$. But what happens to $f(x)$ as x becomes large? Perhaps the graph will give us a hint. First, graph the function with the default parameters: press

[RESET] ' 2 * X ∧ 3 / (X ∧ 2 − 1) ' [ENTER] [NEWF]

The vertical asymptotes at $x = -1$ and $x = 1$ are visible, but there are no clues about the behavior of $f(x)$ for $x < -1$ or $x > 1$ (see Figure 1.11).

Let's zoom out: press [ON] [ZOOM] . You should see the graph from about $x = -2.5$ to $x = 2.75$. Press [ON] [ZOOM] again and we obtain the graph in Figure 1.12.

FIGURE 1.11

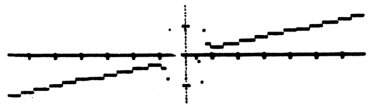


FIGURE 1.12

It is now hard to visualize the vertical asymptotes, but the graph seems to straighten out to the left and to the right. Long division can help to explain what is happening. Since $\dfrac{2x^3}{x^2-1}$ can be rewritten as $2x + \dfrac{2x}{x^2-1}$, for large values of x the difference between $f(x)$ and $2x$ becomes very small. What we see in Figure 1.12 looks like the graph of $y = 2x$, for x large. To verify this, draw $y = 2x$ on top of the graph in Figure 1.12 by pressing [ON] '2*X' [OVERD] . ■

It should be clear at this point that there is more to graphing than plotting points or pushing buttons on a calculator. A change of scale can dramatically alter the appearance of a graph, and the proper choice of scale depends on what information is being sought.

Example 6. Intersections of Graphs

There are many instances when we would be interested in finding the point(s) of intersection of two graphs. For example, one of the applications of calculus which we will see in Chapter 5 is the problem of finding the area between two curves. To do so, it will be necessary to know where the curves intersect. The HP-28S can draw several graphs simultaneously to help us locate any points of intersection.

For example, where do $y = x^4$ and $y = 2x + 3$ intersect? Worded differently, what are the solutions of the equation $x^4 = 2x + 3$? We can graph $y = x^4$ and $y = 2x + 3$ simultaneously by typing

' X ∧ 4 = 2 * X + 3 ' [ENTER] [RESET] [NEWF]

What does the HP-28S do here? It graphs both sides of the equation separately. In Figure 1.13, we see an apparent intersection near $x = -1$, as well as some evidence that the graphs will intersect a second time for $x > 1$.

Note that for $x = -1$, $x^4 = 2x + 3 = 1$, so that we have in fact found an intersection. We'll need to look further to see if there are any others. Zoom out (press [ON] [ZOOM]).

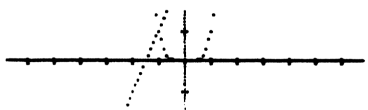


FIGURE 1.13

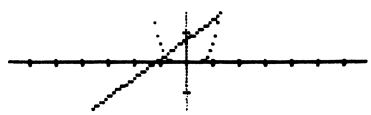


FIGURE 1.14

There appears to be a second intersection between $x = 1$ and $x = 2$ (see Figure 1.14; remember that ZOOM multiplies the x-scale by 1.25). To get a better approximation, move the cursor just below and to the left of the intersection and press [INS] . Next, move the cursor just above and to the right of the intersection and press [INS] . Then press [ON] [ZBOX] . You should get a screen similar to that in Figure 1.15.

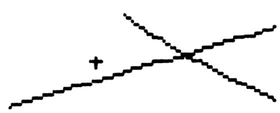


FIGURE 1.15

Move the cursor to the apparent location of the intersection, press [INS] [ON] and read off your approximation. We got (1.576,6.14); yours may differ slightly. Recall that if you want to look at the graph again, just press [GRAPH] and it is instantly reproduced. ∎

Example 7. Estimating Zeros

We could have approached Example 6 differently by looking for zeros of $x^4 - 2x - 3$, since $x^4 - 2x - 3 = 0$ is equivalent to $x^4 = 2x + 3$. We should caution that rewriting an equation can alter the type of graph that is called for. Recall that from algebra, we know that a fourth-order equation like $x^4 - 2x - 3 = 0$ can have at most 4 solutions.

Draw the graph of $y = x^4 - 2x - 3$ with the default parameters. Press

$\boxed{\texttt{ON}}$ $\boxed{\texttt{RESET}}$ ' X ∧ 4 − 2 * X − 3 ' $\boxed{\texttt{ENTER}}$ $\boxed{\texttt{NEWF}}$

As in Figure 1.13, you can see an apparent intersection at $x = -1$ (see Figure 1.16).

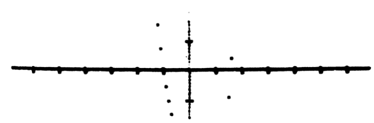


FIGURE 1.16

This time, there is also strong evidence that there is a solution between $x = 1$ and $x = 2$. Specifically, there are two points plotted, one with $y < 0$ followed by one with $y > 0$. We will discuss later the Intermediate Value Theorem which tells us that for this function the graph must cross the x-axis somewhere between the two points. That is, there is a zero somewhere in between. We can use ZBOX to zoom in and get a good estimate of the zero. Are there more than 2 zeros? We do not have any evidence indicating that there is a third or fourth zero, but we will need some calculus to decide for certain whether or not there are any more zeros. ∎

GRAPHING WITH THE HP-48SX

The graphing commands for the HP-48SX are accessed from the Plot menu. Press $\boxed{\texttt{PLOT}}$ (which is above the 8 in orange). This menu contains the entries

PLOTR PTYPE NEW EDEQ STEQ CAT

Note that PLOTR and PTYPE are displayed with bars on top of the P's, indicating that they are themselves directories: press either one and a new menu will appear. As in the Solve menu, the [STEQ] (store equation) key will take an expression from line 1 of the stack and store it in the variable EQ, in the current directory. We can recall the equation stored in EQ by pressing the blue (right shift) key followed by [STEQ] .

As an alternative to storing the expression to be plotted in the variable EQ, the HP-48SX will also allow us to store the expression in any other user variable, using the command [NEW] . The user will be prompted to supply a name for the expression and the expression will be stored in both the designated name and EQ. The most significant advantage to using [NEW] is that it will allow us to easily superimpose the graphs of several functions. For this reason, we will use [NEW] instead of [STEQ] for plotting.

Example 8. Draw

To draw the graph of $f(x) = x^2 - 1$, first enter the expression on the stack:

' X ∧ 2 − 1 ' [ENTER]

We then suggest the following sequence of instructions.

[PLOT] [NEW] [PLOTR] [RESET] [DRAW] [LABEL]

Note that the [RESET] command is located on the second page of the Plotr menu. Press [NEXT] to go to that page and [PREV] to return to the first page. The graph produced by this is seen in Figure 1.17.

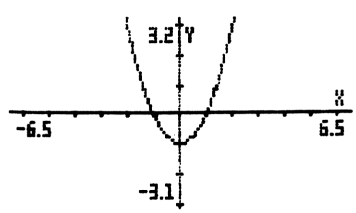


FIGURE 1.17

Keystroke	Explanation
PLOT	Activate the Plot main menu.
NEW	Store the expression on line 1 of the stack under the name supplied by the user at the prompt and EQ.
PLOTR	Activate the Plotr menu. The HP-48SX tells you that it has stored this equation, and gives you the range of x and y values it is expecting to display.
RESET	Reset the graphics window parameters to default values and erase previous graphs.
DRAW	Draw the graph of the expression in EQ.
LABEL	Turn on the x- and y-axis labels.

The four arrow keys will move the cursor around the screen. Press △ seven times, then press ▷ until the cursor covers one of the points on the parabola. Press COORD and the HP-48SX will delete the menu and display the point (1.3,.7) at the bottom of the screen. You can get the menu back by pressing any one of the soft keys. To exit the display mode, press ON . ■

NOTE: By pressing the GRAPH key (located above the ◁ key), you can at any time instantly restore the last graph to the display. This eliminates the need to redraw a graph each time it may be needed and hence saves considerable time.

Example 9. CENT and ZOOM

With the cursor still located at (1.3,.7), press CENT and the graph is redrawn with (1.3,.7) at the center of the display window (see Figure 1.18).

To change the scale, press ZOOM . You must select one of several options. You can scale the x- or y-axis separately (press X or Y , respectively) or together by the same factor (press XY). You can also scale the x-axis and have the calculator automatically scale the y-axis so that all of the graph fits on the display (press XAUTO). Because the y-scale factor is unknown, we urge caution in the use of this feature. This time, press X . You are prompted for a scale factor: try 2.

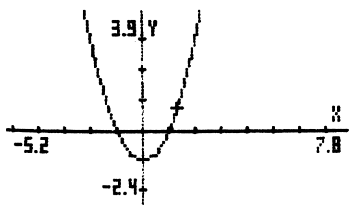


FIGURE 1.18

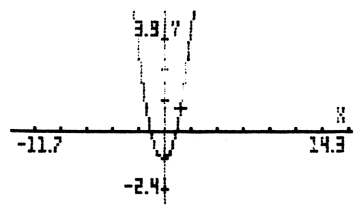


FIGURE 1.19

The parabola seems to be much thinner in Figure 1.19. Press [LABEL] and verify that the tick marks on the x-axis now represent 2 units each, while the tick marks on the y-axis still represent 1 unit each. You should experiment some with the X, Y, XY and XAUTO options in ZOOM. Notice how the appearance of the parabola changes dramatically as you zoom back and forth. ∎

Example 10. ZBOX

Although the ZOOM feature allows us to zoom out (by entering a factor larger than 1) or zoom in (by entering a factor less than 1), in most situations we will want to use a different feature for zooming in. First, get back the original graph of the parabola which we examined in the last two examples. Press [RESET] (to reset the graphics window parameters and erase the previous graph) and then press [DRAW] (to redraw the graph).

Move the cursor to the point (-.8,-1) (press [COORD] to see the coordinates of the location of the cursor). Press the [×] (multiplication) key. When you move the cursor away, an × is left behind marking the spot. Now move the cursor to (2,1.5). Press any one of the soft keys to recover the Plotr menu, and then press [ZBOX] . This will zoom in on the rectangular box whose extreme corners are marked by the × and by the present location of the cursor [here, the corners are (-.8,-1) and (2,1.5), respectively] (see Figure 1.20).

Press [LABEL] to verify that the lower left corner of the screen is (-.8,-.1) and that the upper right corner is (2,1.5).

Notice that, using ZBOX, we can with great ease zoom in on any specific portion of the graphics window of interest. ∎

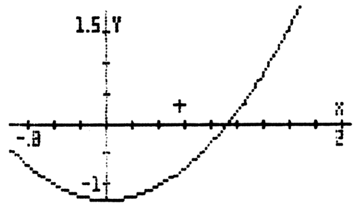


FIGURE 1.20

Example 11. Asymptotes

To graph $f(x) = \dfrac{x-1}{x^2-x-2}$, enter

' (X − 1) / (X ∧ 2 − X − 2) ' [ENTER]

This will place the expression on line 1 of the stack. Next, enter the sequence

[PLOT] [NEW] [PLOTR] [RESET] [DRAW]

Recall that you will be prompted for a name after you enter the [NEW] command. This should produce the graph pictured in Figure 1.21.

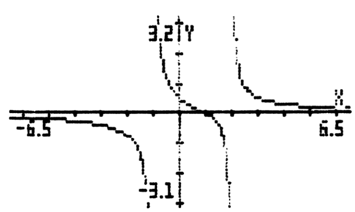


FIGURE 1.21

Notice that the vertical asymptotes at $x = 2$ and at $x = -1$ and the horizontal asymptote at $y = 0$ are clearly visible. ■

Example 12. Oblique Asymptotes

Draw a graph of $f(x) = \dfrac{2x^3}{x^2 - 1}$. To do this, enter the following (note that the parentheses are not optional).

' 2 * X ∧ 3 / (X ∧ 2 − 1) '

PLOT NEW PLOTR RESET DRAW

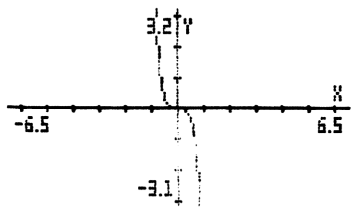


FIGURE 1.22

We can see only that part of the graph from $x = -1$ to $x = 1$, in Figure 1.22. Outside of this interval, the y-values appear to have run off the display. To see more of the graph, zoom out in the y-direction. Press

ZOOM Y 4 ENTER

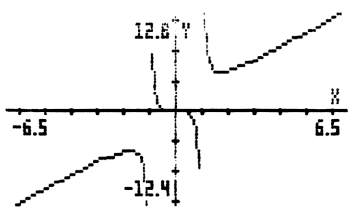


FIGURE 1.23

It should be apparent from Figure 1.23 that the graph straightens out to the left and to the right and approaches the oblique asymptote $y = 2x$ (see the discussion of Example 5 earlier in this section). Try zooming out further. Press

ZOOM XY 4 ENTER

In Figure 1.24, what we see is essentially the graph of $y = 2x$, with a few blips

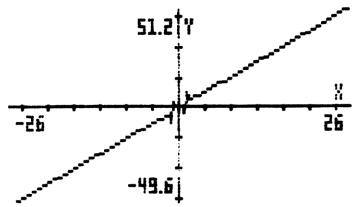


FIGURE 1.24

near the origin. Verify that this is true by drawing $y = 2x$ on top of the graph in Figure 1.24. Press

[ON] ' 2 * X ' [ENTER] [NEW] [PLOTR] [DRAW]

We want to emphasize the importance of scaling. What you see in a graph is highly dependent on what scale is being used. You should therefore look at every graph from several different perspectives in order to fully appreciate its features.

It should be noted that the HP-48SX has an automatic scaling function, AUTO, in the sense that for a given interval of x-values, the calculator will adjust the y-scale so that the entire graph will fit on the screen. AUTO is used in place of DRAW. Try it out on this function. Press

[RESET] [AUTO]

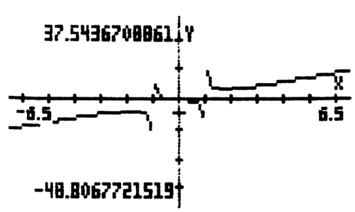


FIGURE 1.25

Compare the results in Figure 1.25 with Figures 1.22-1.24. Notice that the oblique asymptotes are clearly illustrated and that the two vertical asymptotes are at least suggested. However, since we have not manually adjusted the scale, it is not clear just what we are looking at. For this reason, we suggest that you use DRAW and manually zoom in and out, where necessary, mindful of how you are adjusting the scale. ■

Example 13. Intersections of Graphs

To simultaneously graph $y = x^4$ and $y = 2x + 3$, press

' X ∧ 4 = 2 * X + 3 ' [PLOT] [NEW] [PLOTR] [RESET] [DRAW]

We obtain the graph in Figure 1.26. To zoom in on the visible intersection, we move the cursor to (-1.2,.8) and press [×] . Then move the cursor to (-.8,1.2) and press [ZBOX] . (Of course, you could also use any other choice of two points defining the extreme corners of a rectangle enclosing the intersection.) We obtain the graph in Figure 1.27.

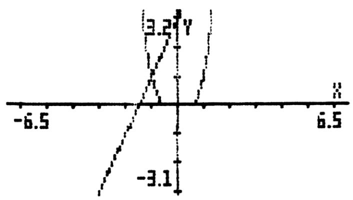


FIGURE 1.26

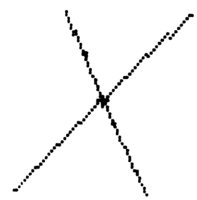


FIGURE 1.27

Move the cursor to the apparent intersection and we get the approximate location $(-1, 1.00317)$. Notice that Figure 1.26 suggests that there may be another intersection, located to the right of the y-axis. To look for a second intersection, first redraw the original graph and then zoom out. Enter

[RESET] [DRAW] [ZOOM] [Y] 4 [ENTER]

and we get Figure 1.28, which clearly indicates a second intersection to the right of the y-axis. We can now use ZBOX as above to zero in on that point. ■

NOTE: The FCN menu (the [FCN] key is in the menu that appears below all graphs) provides an easier way to estimate intersections. We will discuss its use in detail in section 1.3.

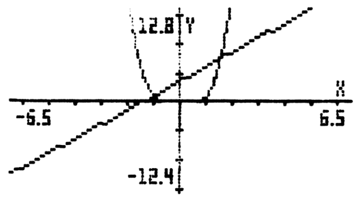


FIGURE 1.28

Exercises 1.2

In exercises 1-10, graph the given function and use ZOOM, ZBOX and CENTER to determine how many zeros the function has. Note that in exercises 9-10, x is in radians.

1. $f(x) = x^3 + 4x^2 + x - 1$
2. $f(x) = x^3 - 5x^2 + 2x - 1$
3. $f(x) = x^3 - 3x^2 + 2x + 1$
4. $f(x) = x^4 + 3x^3 - x + 1$
5. $f(x) = x^4 - 5x^2 + 7$
6. $f(x) = x^4 + 2x^2 + 1$
7. $f(x) = x^5 - 3x^3 + 2x^2 - 1$
8. $f(x) = x^6 + 4x^5 + 2x - 1$
9. $f(x) = x^2 + 4\cos x - 1$
10. $f(x) = x^3 + x \sin x$

11. Based on your graphs in exercises 1-8, describe what the graph of an nth-order polynomial looks like. Make up several polynomials of your own to help see the patterns.

12. Based on your answers to exercises 1-8, how many zeros would you expect an nth-order polynomial to have? Make up several polynomials of your own to test your answer.

In exercises 13-18, graph the function and find all asymptotes (vertical, horizontal and oblique).

13. $f(x) = \dfrac{2x}{x^2 + 2x - 3}$
14. $f(x) = \dfrac{3x - 1}{x^2 - 4}$
15. $f(x) = \dfrac{2x^2 + 1}{x^2 + x + 1}$
16. $f(x) = \dfrac{x^3 + 4}{x^2 - 4}$
17. $f(x) = \dfrac{x^3 + 2x + 1}{x^2 + 4}$
18. $f(x) = \dfrac{x^3 - 4}{2x^3 + 7}$

In exercises 19-24, graph both sides of the equation simultaneously and use ZBOX to estimate the points of intersection of the graphs.

19. $\sqrt{x^3+2} = 4x+1$

20. $x^4 - 3 = x + 2$

21. $\cos x = x$

22. $\sin 2x = x^2 - 1$

23. $e^{-x} = x$

24. $e^x = x^2 - 1$

In exercises 25-28, use ZOOM to determine if there is an oblique asymptote. If there is not, what general shape does the graph assume? Check your answers with long division.

25. $f(x) = \dfrac{2x^3+2x}{x^2+x+1}$

26. $f(x) = \dfrac{3x^4-3x^2-1}{x^2+2x+1}$

27. $f(x) = \dfrac{2x^3-x+1}{x+2}$

28. $f(x) = \dfrac{3x^4-3x^2-1}{x+2}$

29. Based on your answers to exercises 25-28, state a rule for what the general shape of $f(x) = \dfrac{p(x)}{q(x)}$ assumes if $p(x)$ is a polynomial of degree n and $q(x)$ is a polynomial of degree m.

30. Graph the circle $x^2 + y^2 = 4$ by graphing the equation $\sqrt{4-x^2} = -\sqrt{4-x^2}$. What happens if you try ZOOM?

31. Graph the ellipse $x^2 + 4y^2 = 4$. What happens if you change the horizontal scale by a factor of 2 (2 $\boxed{\texttt{*W}}$ on the HP- 28S and $\boxed{\texttt{ZOOM}}$ $\boxed{\texttt{X}}$ 2 $\boxed{\texttt{ENTER}}$ on the HP-48SX)? What does this tell you about the relationship between a circle and an ellipse?

EXPLORATORY EXERCISE

Introduction

Although we will devote most of our energies in calculus finding exact solutions of problems, in real life the problems are often too difficult to solve completely. For instance, flying to the moon requires constant recalculation of trajectories as the spaceship gets slightly off-path. In such situations, it is common to replace a difficult problem (computing the *entire* flight path) with a simpler one (computing the next few miles of the flight path). We will look at an important approximation in this exercise.

Problems

First graph $y = x^2 - 2x + 1$, and move the cursor to the point (0,1). Then move the cursor 2 pixels to the left and 2 pixels down and mark this point (use $\boxed{\texttt{INS}}$ on the HP-28S or $\boxed{\times}$ on the HP-48SX). From there, move the cursor up 4 pixels and to the right 4 pixels and use ZBOX (press $\boxed{\texttt{INS}}$ $\boxed{\texttt{ON}}$ $\boxed{\texttt{ZBOX}}$ on the HP-28S or press $\boxed{\texttt{ZBOX}}$ on the HP-48SX). The graph should appear to be a straight line with (0,1) at the center of the screen. Use the cursor to find a second point on this curve [we already know (0,1) is a point] and find the equation of the line connecting (0,1) and the second point.

Now remove the graph and press $\boxed{\texttt{RESET}}$ to get back to the default graphing parameters. Draw the line found above on top of this graph (use OVERD on the HP-28S). The line and parabola share several pixels and are graphed with adjacent pixels in some areas. For which values of x are the two graphs close to each other? In this region, we can say that the line approximates the parabola well.

Repeat the above process with the following functions and points.

$f(x) = x^3$	at (0,0)
$f(x) = \sin x$	at (0,0)
$f(x) = \cos x$	at (0,1)
$f(x) = \sqrt{x+1}$	at (0,1)
$f(x) = x^2 - 1$	at (2,1)

Further Study

You will see variations of this problem under several names as you progress through your mathematics courses. As presented here, the line found is referred to as the *linear approximation* to the curve. Later, we will call it the *tangent line* to the curve (this may change your idea of what a tangent line is for curves like $\sin x$). The slope of the line will be called the *derivative* of the function at the point. The line is also an example of a *Taylor polynomial* which we will discuss in Chapter 6. For now, you can use the $\boxed{\texttt{TAYLR}}$ command on your calculator to check your answers. Enter the function and then press

' X ' $\boxed{\texttt{ENTER}}$ 1 $\boxed{\texttt{ENTER}}$ $\boxed{\texttt{ALGBRA}}$ $\boxed{\texttt{TAYLR}}$

and you will get the exact equation of the desired line [for $f(x) = x^2 - 2x + 1$ you should get $y = 1 - 2x$]. This will work for all but the last example, where we focus on a nonzero x. Finally, in terms of the introduction to this problem, you have seen that the linear approximation stays close to the actual curve for a small interval. If we follow the line for a short interval and then calculate the linear approximation at the next point, we will remain fairly close to the curve. This is the basis for a technique called *Euler's method* which is discussed in Chapter 3.

1.3 Using the Solver

In several examples in the two preceding sections, we found approximate solutions to problems through a process of trial-and-error. The speed of the HP-28S/48SX made this practical for the problems we encountered, but it is important to realize that most real world problems are more complicated and thereby more difficult to solve. A study of calculus provides us with some very effective tools for solving many such problems. These methods often will eventually lead us to solve an equation(s). For example, we may need to find an x (a *root* or *zero*) for which $f(x) = 0$.

While a given equation may precisely define a solution to the problem at hand, we still must be able to find the solution of the equation. For instance, the equation $x^3 - x^2 - 2x + 2 = 0$ has 3 solutions. One of these, $x = 1$, is easily found and easily checked (just plug in $x = 1$). Two more solutions can be found graphically. We can approximate these (try this graphically) by $x = 1.4$ and $x = -1.4$. With persistence, we can further refine our estimate of these zeros to be $x = 1.414$ and $x = -1.414$. (Do these digits look familiar?) In this case, of course, we can factor, to obtain

$$x^3 - x^2 - 2x + 2 = (x - 1)(x^2 - 2)$$

We then see that the zeros are $x = 1$ and $x = \pm\sqrt{2}$.

There are several important points to be made here. First, notice that not all answers are integers! Further, when the solutions are not integers, we often have to be content with an approximation with a fixed number of decimal places of accuracy. In many cases, there is no way to find an exact solution.

The HP-28S/48SX gives us several options for finding good approximations to solutions. In this section, we will discuss the use of the Solver menu. The topic is important enough that we return to it in Chapter 4, where we will examine, in detail, several methods for efficiently finding accurate approximations.

The examples in this section can be completed on the HP-48SX with some minor key changes which we detail in each example, as needed. In addition, some special features of the HP-48SX significantly simplify our problem-solving procedure. These are discussed in the HP-48SX Notes at the end of this section.

Example 1. The Solver

Let's return to the equation $x^3 - x^2 - 2x + 2 = 0$, and see how our calculators can be better used. We will start by drawing a graph. Press

' X ∧ 3 − X ∧ 2 − 2 * X + 2 ' [ENTER]
[RESET] [NEWF]

All 3 zeros are visible. (Figure 1.29 shows the HP-28S screen. On the HP-48SX, press [NEW] [PLOTR] [RESET] [DRAW] .) Now, move the cursor to the left until (on the HP-28S) you cover up the point just above the intersection with the x-axis.

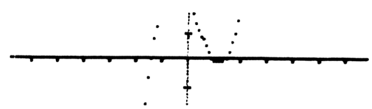


FIGURE 1.29

Press [INS] [ON] to find the estimate (-1.4,0) on line 1 of the stack. (On the HP-48SX, simply move the cursor to the left until it covers the apparent point of intersection with the x-axis and press [ENTER] [ON] . This will return the estimated point to line 1 of the stack.) If we were to continue graphically, we would use ZBOX to zoom in and get a more precise estimate. Instead, activate the Solver menu: press [SOLV] (or [SOLVE] on the HP-48SX). Note that we have already stored the equation we are trying to solve in the variable EQ ([NEWF] or [NEW] did that for us automatically) and so we press [SOLVR] and then the soft key [X] . This stores the point (−1.4,0) in the variable X. (The HP-28S/48SX actually is storing both coordinates of the point, but in this setting the calculator only uses the first coordinate, −1.4.)

To solve for the root, simply press the red key followed by the soft key [X] . (On the HP-48SX, press [↩] and the soft key [X] .) The HP-28S/48SX tells you that it is solving for x and then displays Figure 1.30.

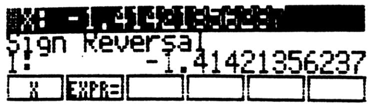


FIGURE 1.30

The "Sign Reversal" message that comes along with the answer indicates that the calculator has not been able to find an answer that it thinks is accurate to its maximum number of digits (12). ("Zero" is a more positive message.) Instead, the machine has found what it thinks is an approximation valid to about 11 digits of accuracy and is warning you that you ought to carefully examine the answer before you accept it. Let's see how close the approximation is (recall that the exact roots are 1 and $\pm\sqrt{2}$). Press

$$2 \; \boxed{\sqrt{x}} \; \boxed{\text{CHS}}$$

All of the digits displayed are correct! (Recall that for the HP-48SX, you use the $\boxed{\pm}$ key in place of $\boxed{\text{CHS}}$.) As we will see, the Solver is not always this accurate. You should always plug the approximation into the equation to get an idea of the accuracy. This is quite easy to do. Simply press $\boxed{\text{EXPR=}}$. Even if we did not know the precise answer, we would know that our approximation gives a very small function value. (Users of the HP-48SX should refer to the Notes at the end of the section for an alternative approach.)

You may wonder why we bothered drawing a graph. As we will see in our discussion in Chapter 4, it is important to start with an initial approximation close to the solution that you want. In the preceding, the point $(-1.4,0)$ told the calculator where to start looking for a solution. Try the sequence

$$0 \; \boxed{\text{X}} \; \boxed{\text{RED}} \; \boxed{\text{X}}$$

(or 0 $\boxed{\text{X}}$ $\boxed{\hookleftarrow}$ $\boxed{\text{X}}$ on the HP-48SX) and the calculator will output that $x = 1$ is a zero. That is, with the initial guess $x = 0$, the calculator finds the solution $x = 1$. Try the initial guesses $x = .1, .2, \ldots$ until you get the third solution $x \approx 1.4$. Especially notice what happens with the initial guesses $x = .8$ and $x = 1.2$. ■

Example 2. Solutions of Equations

In Example 6 of section 1.2, we used graphics to look for intersections of the graphs of $y = x^4$ and $y = 2x + 3$. To find the points of intersection more precisely, we are led to solve $x^4 = 2x + 3$. In section 2, we graphically found two solutions, one at $x = -1$ (check that this is an exact solution!) and the other near $x = 1.5$. To get greater precision for the second solution, we can use the Solver. First, store the equation. Press

$$\text{' X} \wedge 4 = 2 * \text{X} + 3 \text{ '} \ \boxed{\texttt{STEQ}} \ \boxed{\texttt{SOLVR}}$$

Start with an initial guess of $x = 1.5$: press 1.5 $\boxed{\texttt{X}}$. Then solve for x by pressing $\boxed{\texttt{RED}}$ $\boxed{\texttt{X}}$ ($\boxed{\hookleftarrow}$ $\boxed{\texttt{X}}$ on the HP-48SX). We get the "Sign Reversal" message and $x = 1.574743....$ We can get an idea of the accuracy of this solution by evaluating the left and right sides of the equation for this value of x: press $\boxed{\texttt{LEFT=}}$ and $\boxed{\texttt{RT=}}$. (On the HP-48SX, pressing the $\boxed{\texttt{EXPR=}}$ key will output values labeled "LEFT" and "RIGHT," corresponding to the values of the left and right sides of the equation, respectively.) The values do not match precisely, but for most routine purposes (i.e., if you are not planning to use this to put a person on the moon) we can probably be content with the accuracy. ■

The flexibility of the HP-28S/48SX is to be marveled at. The same sequence of steps helps us to approximate zeros of functions or solutions of more general equations. As we see in Example 3, with some trickery the Solver will also give us the maximum and minimum of a piece of a graph.

Example 3. Finding Maxima and Minima

The extremes of functions are typically of special interest to us. For example, in industry, you want to maximize production while minimizing costs. As a more concrete illustration, look at the graph of $f(x) = x^3 + 4x^2 + 3x + 3$. Enter

$$\text{' X} \wedge 3 + 4 * \text{X} \wedge 2 + 3 * \text{X} + 3 \text{ '} \ \boxed{\texttt{ENTER}}$$

Then press $\boxed{\texttt{RESET}}$ $\boxed{\texttt{NEWF}}$. After the initial plot is drawn, press $\boxed{\texttt{ON}}$ $\boxed{\texttt{ZOOM}}$ to zoom out sufficiently to see the behavior of the function. (On the HP-48SX, press

$$\boxed{\texttt{NEW}} \ \boxed{\texttt{PLOTR}} \ \boxed{\texttt{RESET}} \ \boxed{\texttt{DRAW}}$$

Then press $\boxed{\texttt{ZOOM}}$ $\boxed{\texttt{Y}}$ 4 to zoom out sufficiently in the y direction to see the behavior of the function.) The graph rises to a peak, drops down to a trough, then rises again. (See Figure 1.31 for the HP-28S graph.)

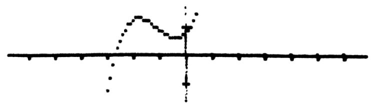


FIGURE 1.31

The peak is called a *relative maximum* and the trough is called a *relative minimum*. We would like to find the coordinates of these special points. First, move the cursor up to the apparent location of the relative minimum and press [INS] [ON] (on the HP-48SX, press [ENTER] [ON]). You should now have $(-.4, 2.4)$ or some nearby point on line 1 of the stack. Press

[SOLV] [SOLVR] [X] [RED] [X]

(On the HP-48SX, press [SOLVE] [SOLVR] [X] [↩] [X] .) We got $x = -.45141626064$ (your answer may differ slightly) which is labelled an "Extremum." Since *extremum* is the generic term for maximum or minimum, this would seem to be the point that we are looking for. We will need some calculus to evaluate the accuracy of this answer and we will look at this question further in Chapter 4.

Next, let's try to locate the relative maximum. To get back to the graph, simply press

[USER] [GRAPH]

(On the HP-48SX, simply press [GRAPH] .) Move the cursor to the apparent location of the relative maximum, press [INS] [ON] and we have an estimate of $(-2.4, 5.2)$. Repeat the sequence

[SOLV] [SOLVR] [X] [RED] [X]

and we get the zero $x = -3.37442376321$. On our graph, (from the HP-28S) there were actually 2 points at the peak. If you use the other point, $(-2.2, 5.2)$, the Solver produces the extremum at $x = -.4514$!

An experiment should clarify what the HP-28S/48SX is doing. Graph $y = x^3 + 4x^2 + 3x - 5$. Note that this is just our original function with 8 subtracted.

You can press

$\boxed{\text{RCEQ}}$ 8 − $\boxed{\text{USER}}$ $\boxed{\text{NEWF}}$

to graph this function. On the HP-48SX, press

$\boxed{\hookrightarrow}$ $\boxed{\text{STEQ}}$ 8 − $\boxed{\text{NEW}}$ $\boxed{\text{PLOTR}}$ $\boxed{\text{RESET}}$ $\boxed{\text{DRAW}}$

Move the cursor to the apparent location of the relative maximum and digitize the point (press $\boxed{\text{INS}}$ $\boxed{\text{ON}}$ or $\boxed{\text{ENTER}}$ $\boxed{\text{ON}}$ on the HP-48SX) to get $(-2.2, -2.8)$ on the stack. Now use the Solver to estimate the maximum. Press

$\boxed{\text{SOLV}}$ $\boxed{\text{SOLVR}}$ $\boxed{\text{X}}$ $\boxed{\text{RED}}$ $\boxed{\text{X}}$

to find the extremum at $x = -2.2152$. Note that if you give the Solver an initial guess of $(-2.4, -2.8)$ you get sent to a zero.

Can you tell what's happening? First of all, the behavior of the Solver depends very much on the initial guess provided. Sometimes you will have to change your initial guesses in order to get the information you want. In general, though, the Solver is going to move along the graph from the initial point *toward the x-axis* and find an extremum or zero, whichever comes first. Thus, it may find a relative minimum above the x-axis or a relative maximum below the x-axis.

If the extremum you are looking for is in the "wrong place," you can translate the graph (for instance, by subtracting 8, as we did above). With some effort, though, we can quickly obtain accurate approximations of many important points on a graph. ■

Example 4. Solving for One Variable in an Equation

In Example 4 of section 1.1, we used the Solver to help estimate the smallest angle at which a ball could be thrown to reach a certain distance. We will now use the equation-solving capabilities of the HP-28S/48SX to more precisely solve that problem.

Recall that the equation $R = S^2 \sin(2A)/32$ gives the range in feet of a ball released at angle A with initial speed S ft/s. Press

' R = S ∧ 2 * $\boxed{\text{SIN}}$ (2 * A) / 32 ' $\boxed{\text{STEQ}}$

and activate the Solvr menu. We want a throw to go 300 ft, so store 300 in R (press 300 $\boxed{\text{R}}$). For an initial speed of 100 mph = 146.66 ft/s, store 146.66 in S.

Now, solve for A (press [RED] [A] or [⟵] [A]). The solution in degrees is given as 13.2539615945. ■

HP-48SX NOTES

Example 1 can be easily worked on the HP-48SX without leaving the graphics display. With the cursor at $(-1.4, 0)$, press [FCN] [ROOT] (located in the menu at the bottom of the display). As before, we get the "Sign Reversal" message and the approximate root is displayed, but this time they are displayed at the bottom of the screen without exiting the Plotr menu and while the graph is still displayed above. Simple, isn't it? If we now exit Plotr (press [ON]), the zero estimate is on the stack and we can use the Solver to test the accuracy as before.

As an alternative to the method given in Example 2, we can use the command [ISEC] (intersection). Graph $x^4 = 2x + 3$ as before and then move the cursor close to a point of intersection. Then, press [FCN] [ISEC] (again located in the menu at the bottom of the screen). As with Example 1, we can find points of intersection without leaving the Plotr menu and without erasing the graph. This is particularly useful when we are looking for more than one point of intersection. Again, when you exit the graphics display, the point(s) of intersection will be listed on the stack.

We can also approximate relative extrema from the Plotr menu. Move the cursor to a point near the suspected extremum and press [FCN] [EXTRM] . The approximate relative extremum and any advisory messages are displayed at the bottom of the display while the graph is still displayed above. You should experiment with different cursor positions (i.e., initial guesses) and see which point the calculator identifies.

Exercises 1.3

In exercises 1-8, use the Solver to find all zeros. In exercises 1-4, we indicate how many zeros there are.

1. $x^4 + x^2 - 6 \quad (2)$
2. $x^3 + 2x^2 - 6x - 12 \quad (3)$
3. $x^4 + 3x^3 - x + 1 \quad (2)$
4. $x^6 + 4x^5 + 2x - 1 \quad (2)$
5. $x^4 + x^3 - 5x - 5$
6. $4x^5 + 8x - 1$
7. $\sin x - x^2 + 1$
8. $\sin 2x - 3x + 1$

In exercises 9-10, factor the function in the indicated exercises to find exact representations of the zeros.

9. exercise 1

10. exercise 2

In exercises 11-14, find all solutions of the equations. HINT: you may want to rewrite the equation before solving.

11. $x^3 + 20 = 10x^2 + 2x$

12. $x^3 + 9x^2 = 3x + 27$

13. $\sqrt{x^2 + 4} = x^2 + 2$

14. $(x^2 - 1)^{2/3} = 2x + 1$

In exercises 15-18, find all extrema of the functions.

15. $x^3 - 3x^2 + x - 4$

16. $x^3 + 2x^2 - 2x + 1$

17. $x^4 - 7x + 2$

18. $x^4 + x^2 + 2$

In exercises 19-22, assume the showroom is to be x ft by y ft, and write down the perimeter of the required walls in terms of x and y. Since $xy = 200$ (why?) you can replace y with $200/x$ and find the desired minimum. The calculator is unusually sensitive to initial guesses in these problems, so make sure your answer is reasonable.

19. A store needs to build a showroom with 200 ft^2 of floor space. If the cost of building the showroom is proportional to the perimeter of the room (why is this reasonable?), find the dimensions of the room that minimize cost.

20. In the showroom of exercise 19, suppose that one side of the room does not need to be walled in. Find the dimensions that minimize the cost.

21. In the showroom of exercise 19, suppose that two facing walls require 3-ft openings for doors. Find the dimensions that minimize the cost.

22. In the showroom of exercise 20, suppose that two facing walls require 3-ft openings for doors. Find the dimensions that minimize the cost.

23. Use the Solver to find the zero of $f(x) = x^{16}$ (set $x = 1$ to start). Draw the graph of $y = x^{16}$ and try to explain why it takes so long to get the obvious answer of $x = 0$. We will discuss this type of problem in Chapter 4.

24. Find the intersections of the ellipse $\dfrac{x^2}{9} + y^2 = 1$ and the parabola $y = x^2 - 2$. HINT: graph the ellipse by graphing the equation $\sqrt{1 - x^2/9} = -\sqrt{1 - x^2/9}$ and then overdraw the parabola.

25. Instead of using the Solver as we did in Example 4, we could have solved for the angle A using the inverse trig function arcsine. In this case, $A = \dfrac{1}{2}\arcsin\left(\dfrac{32R}{S^2}\right)$. The HP-28S/48SX evaluates the arcsine with the built-in

function ASIN (in the Trig menu of the HP-28S and in orange above $\boxed{\texttt{SIN}}$ on the HP-48SX). The HP-28S/48SX solves this equation symbolically using the $\boxed{\texttt{ISOL}}$ command in the Algebra menu. Press ' R = S ∧ 2 * $\boxed{\texttt{SIN}}$ (2 * A) / 32 ' $\boxed{\texttt{ENTER}}$ and then press ' A ' $\boxed{\texttt{ENTER}}$ $\boxed{\texttt{ISOL}}$ to solve for A. Along with the terms in our answer above, the calculator shows a term '(−1)∧n1+π*n1'. If you replace n1 with 0, you get our answer from above. What do you get with n1=1? n1=2? Explain why these are also correct answers (HINT: the sine function is periodic).

26. It is important to realize that the $\boxed{\texttt{ISOL}}$ command is very limited. Try to use $\boxed{\texttt{ISOL}}$ to solve for X in the equation $X + X^2 = 1$. On the HP-28S you get a response ($X = 1 - X^2$) which is correct but not very informative. The HP-48SX clearly indicates that it is unable to help.

EXPLORATORY EXERCISE

Introduction

The equation-solving techniques discussed in this section apply to one equation for one unknown quantity. It is at least as important in applications to be able to solve *several* equations for several unknowns. The HP-28S/48SX is equipped with *matrix* operations to solve such systems of equations. We will explore an example of a system of equations involving the maximum speed of a car on a circular (unbanked) road. From physics, we learn that $\frac{F}{m} = \frac{v^2}{r}$ where F is the friction force, m is the mass of the car, v is the speed of the car, and r is the radius of the circular path. To simplify calculations we will assume that $\frac{F}{m} = 100$ so that the speed of the car is given by $v = 10\sqrt{r}$. That is, speed is determined by the radius.

We will assume that we have the coordinates of 3 points on the car's path (in units of feet) which we will use to find the 3 unknowns (a, b and r) in the general equation of a circle of a circle $(x-a)^2+(y-b)^2 = r^2$. Suppose the 3 points are (0,0), (600,600) and (520,300). To find the equation of the circle through these points, we plug them into the general equation of the circle and solve for a, b and r. With $x = y = 0$, the equation becomes

$$a^2 + b^2 = r^2$$

With $x = y = 600$ the equation of the circle becomes

$$(600 - a)^2 + (600 - b)^2 = r^2$$

which we can rewrite as $600^2 - 2a600 + 600^2 - 2b600 = 0$ or after simplifying

$$1200a + 1200b = 720,000$$

Similarly, with $x = 520$ and $y = 300$ we get $520^2 - 2a520 + 300^2 - 2b300 = 0$ or

$$1040a + 600b = 360,400$$

This is the *system of equations* we want to solve.

Problems

We will solve these equations using a powerful technique of matrix operations. Essentially, we just copy down the numbers in the two displayed equations above into *matrix form* . We get

$$\begin{pmatrix} 1200 & 1200 \\ 1040 & 600 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 720,000 \\ 360,400 \end{pmatrix}$$

To solve for a and b we multiply by the *inverse* of the matrix on the left. As the name implies, multiplication by the inverse is in many ways analogous to division, and we use the / key on the HP-28S/48SX. To enter a matrix use the bracket keys [and] and press

[720000 $\boxed{\text{SPACE}}$ 360400] $\boxed{\text{ENTER}}$
[[1200 $\boxed{\text{SPACE}}$ 1200] [1040 $\boxed{\text{SPACE}}$ 600]] $\boxed{\text{ENTER}}$

Now press / . The first number displayed in the brackets is the value for a (about $a = .91$) and the second number is the value for b (about $b = 599$). With these values $r = \sqrt{a^2 + b^2} \approx 599$ ft and $v = 10\sqrt{r} \approx 245$ ft/sec.

How much faster could the car go if it *cut the corner*? Plot the points (0,0), (600,600) and (520,300) and compare the circle through these points to the circle through (0,0), (600,600) and (485,300). Repeat the above procedure to find the speed through the second set of points. What happens to the speed as you cut off more of the corner? Find the speed for the circle through (0,0), (600,600) and each of the following points: (400,300), (350,300) and (300,300).

Further Study

The study of matrix theory is an important part of a sophomore level course in *linear algebra.* You will see pieces of linear algebra throughout calculus, particularly when you get to three-dimensional calculus.

CHAPTER 2

Numerical Computation of Limits

2.1 Conjecturing the Value of a Limit

Recall that when we say

$$\lim_{x \to a} f(x) = L$$

we mean that the function f is defined everywhere in an open interval (c, d) containing a (except possibly at a itself), and that as x gets closer and closer to a, $f(x)$ will get closer and closer to the number L (called the *limit* of f as x approaches a). We will make this intuitive notion more precise in section 2.3. For the moment, this description will do quite nicely.

You might ask why such a big deal is made about limits, anyway. After all, don't you just "plug" the value $x = a$ into the function to compute the limit? Of course, you do, for limit problems like

$$\lim_{x \to 3}(x^2 - 3x + 2) = 2$$

In fact, for any polynomial p

$$\lim_{x \to a} p(x) = p(a)$$

Unfortunately, limits are not always so easy to compute. As we will see in Example 1, we can sometimes evaluate limits without plugging into the function.

Example 1. The Limit of a Rational Function

Suppose that we next consider $\lim_{x\to 3} \frac{x^2-9}{x-3}$. We can't substitute $x = 3$, since this would result in division by zero. Could this perhaps mean that the limit does not exist? We can use the HP-28S/48SX to examine what happens to this function as x gets closer and closer to 3. First, we draw a graph of the function. Enter

' (X ∧ 2 − 9) / (X − 3) ' ENTER

and press NEWF in the PLOTR subdirectory on the HP-28S and similarly on the HP-48SX. This will produce a graph of the function which does not show the behavior near the point of interest, $x = 3$. (See Figure 2.1.)

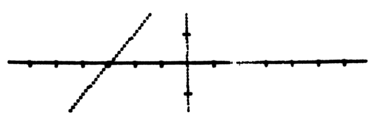


FIGURE 2.1

One way to obtain the graph near the value that we're interested in is to translate the center of the display. In the present example, move the cursor to the right to the point (3,0) and then move the cursor up to the top of the screen and digitize that point by pressing INS . Press ON to return to the menu and CENTE to draw a new graph with the digitized point at the center of the display. (With the HP-48SX you do not need to remove the graph to use the center command.) Repeating this process of translating the center several times will yield a graph depicting the behavior near $x = 3$. (Note that the center must only be translated once with the HP-48SX, due to the larger display, and that this can be done without leaving graphics mode by moving the cursor to the new center and simply pressing the $\boxed{\texttt{CENT}}$ soft key displayed below the graph.) We can observe several things. First, the graph appears to be a straight line (see Figure 2.2).

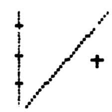


FIGURE 2.2

Second, it should look like the function has a value around 6 when x is near 3, although the point at $x = 3$ may be missing, depending on exactly how the new center has been chosen. (Why would the point be missing from the graph?) We can use the Solver to generate a table of values of $f(x)$ for x close to 3. All that you need to do is to enter the Solver menu (press $\boxed{\texttt{SOLV}}$ $\boxed{\texttt{SOLVR}}$), enter an x value, press the soft key $\boxed{\texttt{X}}$ and the soft key $\boxed{\texttt{EXPR=}}$. The value of the current function (already stored in the variable EQ) is computed and put on line 1 on the stack. The following table is then easily generated.

x	$f(x)$	x	$f(x)$
2.9	5.9	3.1	6.1
2.99	5.99	3.01	6.01
2.999	5.999	3.001	6.001
2.9999	5.9999	3.0001	6.0001
2.99999	5.99999	3.00001	6.00001

Notice that we have taken values of x approaching 3 both from above ($x > 3$) and from below ($x < 3$). (Make sure that you try entering 3 for x. What happens?) In this way, we can see if the values of $f(x)$ seem to be approaching the same value as x approaches 3 from above and from below. If not, we would expect that the limit does not exist.

Both the graphical and the numerical evidence point to the conclusion that the limit is 6. While this is indeed the correct value for the limit (we'll see why in a moment), it is not fair to say that we conclude that the limit is 6. We're really making more of a guess as to the value of the limit (although it's certainly an educated guess). The phrase that best describes what we're doing here is that we are making a *conjecture.* You have probably already noticed by now that the numerator of the function factors. We then have:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6$$

where the last limit is computed by substituting $x = 3$ (since it's the limit of a polynomial). We note that the cancellation in the above is valid since in the limit as x approaches 3, x gets arbitrarily close to 3, but $x \neq 3$, so that $(x-3) \neq 0$. ■

If you already have some amount of experience in computing limits, you probably knew the answer before we even started. But, what this example should still convey to you is a process for conjecturing the value of a limit of any function which is undefined at the point in question. This is especially useful for functions which are difficult to compute by hand. (It's worth noting here that the majority of functions we run into in applications fall into this category.)

A somewhat more challenging problem is the following.

Example 2. A Limit of a Product That Is Not the Product of the Limits

Consider $\lim_{x\to 0} x\sin(1/x)$. Again, we cannot resolve the limit by plugging in $x = 0$, since the function is not defined at $x = 0$. As with all the limit problems we'll face, we first draw a graph to get some idea of the behavior of the function near $x = 0$. (See Figure 2.3 for the initial HP-28S graph. *Make sure that your calculator is set to radians mode.*)

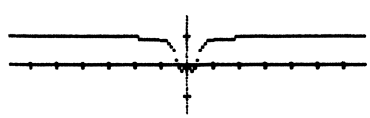


FIGURE 2.3

Although for this example the initial graph shows the behavior near $x = 0$, there is insufficient detail to make any serious guess as to the value of the limit.

Using the $\boxed{\text{ZBOX}}$ command, we can zoom in on the behavior near the origin (see Figures 2.4a and 2.4b).

FIGURE 2.4a

FIGURE 2.4b

It appears that, although the function oscillates faster and faster as x approaches 0, the function values are getting smaller and smaller in absolute value. That is, although oscillating wildly, the function seems to be approaching 0 as $x \to 0$. Using the Solver utility, you can quickly generate a table of values, such as the following.

x	$f(x)$	x	$f(x)$
.1	-5.4×10^{-2}	$-.1$	-5.4×10^{-2}
.01	-5.1×10^{-3}	$-.01$	-5.1×10^{-3}
.001	8.3×10^{-4}	$-.001$	8.3×10^{-4}
.0001	-3.1×10^{-5}	$-.0001$	-3.1×10^{-5}
1×10^{-5}	3.6×10^{-7}	-1×10^{-5}	3.6×10^{-7}
1×10^{-7}	4.2×10^{-8}	-1×10^{-7}	4.2×10^{-8}
1×10^{-9}	5.5×10^{-10}	-1×10^{-9}	5.5×10^{-10}

From both the graphical and the numerical evidence, we would conjecture that

$$\lim_{x \to 0} x \sin(1/x) = 0$$

We can verify that this conjecture is true by using the following theorem. ■

Theorem 2.1 (Pinching Theorem) Given a function f, if we can find functions g and h such that for all x in some open interval containing a (except possibly for $x = a$), $g(x) \le f(x) \le h(x)$ and if $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

This theorem is referred to in most calculus texts as the Pinching Theorem, the Sandwich Theorem, the Squeeze or Squeeze-Play Theorem or something to that

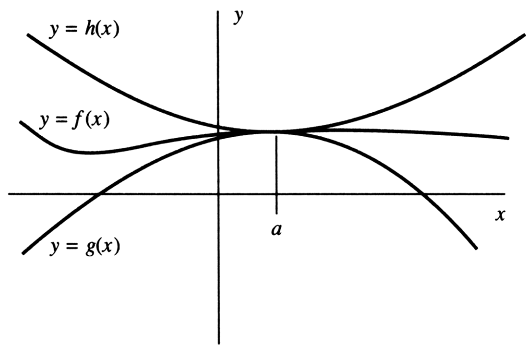


FIGURE 2.5

effect. Figure 2.5 gives a graphical interpretation of the Pinching Theorem.

While the theorem is fairly easy to understand (just think in terms of Figure 2.5), to use this in practice to find limits we must dream up appropriate functions g and h. This can at times be quite a challenge. You should note, however, that for a problem where we have already made a conjecture for the value of the limit, we have a leg up on this process.

Returning to Example 2, we had conjectured that

$$\lim_{x \to 0} x \sin(1/x) = 0$$

To use the Pinching Theorem here, we need to find two functions g and h such that

$$g(x) \leq x \sin(1/x) \leq h(x)$$

for all x in some open interval containing $x = 0$, except possibly at $x = 0$, and where

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = 0$$

In order to do this, then, we will certainly have to make use of some knowledge of the sine function. One of the simplest known facts about this function is that $-1 \leq \sin(t) \leq 1$, for all t. For the case at hand we can see that $-1 \leq \sin(1/x) \leq 1$ for all x, except $x = 0$. But, we are interested in $x \sin(1/x)$. If we multiply the above inequality through by x, we get

$$-x \leq x \sin(1/x) \leq x \quad (x > 0)$$

$$x \leq x \sin(1/x) \leq -x \quad (x < 0)$$

Why do we get different inequalities for $x > 0$ and $x < 0$? Recall that multiplication of an inequality by a number results in the inequality reversing when that number is negative. We can summarize the above two inequalities as follows:

$$-|x| \le x \sin(1/x) \le |x| \quad (x \neq 0)$$

You may observe this graphically by superimposing the graphs of the function $f(x) = x\sin(1/x)$ and the equation $|x| = -|x|$ (this equation is plotted so that we see on the screen the graphs of both $y = |x|$ and $y = -|x|$). This can be done by using the OVERDRAW utility discussed in Chapter 1 for the HP-28S, or by drawing the second graph without resetting the display or erasing the last graph on the HP-48SX (see Figure 2.6 for the HP-48SX graph).

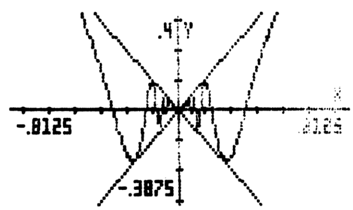


FIGURE 2.6

Clearly, $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$. From the Pinching Theorem, then we also have that

$$\lim_{x\to 0} x\sin(1/x) = 0$$

as we had conjectured. That was quite a long process, wasn't it? If you haven't already done so, you should ask now whether it was worth it or not. After all, from the graphical and numerical evidence, we had a pretty good idea that the limit was 0. Why was it necessary to *prove* that the limit was indeed 0? The answer is that the more complicated the problem is, the less helpful and accurate intuition is. Computations and graphs can also be deceiving, as we'll see in section 2.2. The only way to be certain of the value of a given limit is to prove it.

At this point, you should recognize that there are really two somewhat different reasons for conjecturing the value of a limit. The obvious reason is that we'd like to find an approximate answer to a problem to which we cannot seem to find (at least immediately) an exact answer by hand. An equally important reason is that we would like to gain sufficient insight into a limit problem that we might discover the precise value. This second reason is a bit more difficult to get your hands on, but is a very important reason for using a graphing calculator in exploring limit problems. We'll pursue this further in later examples and in the exercises at the end of the section.

The following is a particularly useful limit to know. It arises in the derivation of the derivative of $\sin x$.

Example 3. Limit of a Quotient That Is Not the Quotient of the Limits

Consider $\lim_{x \to 0} \frac{\sin x}{x}$. The initial graph produced by the HP-28S/48SX seems to indicate that the function approaches 1 as x approaches 0 (see Figure 2.7 for the initial HP-28S graph; again, make sure that your calculator is set to radians mode).

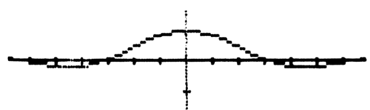


FIGURE 2.7

Using the Solver, we produce the following table of values.

x	$f(x)$	x	$f(x)$
.1	.998334166468	−.1	.998334166468
.01	.999983333417	−.01	.999983333417
.001	.999999833333	−.001	.999999833333
.0001	.999999998333	−.0001	.999999998333
.00001	.999999999983	−.00001	.999999999983

Certainly, we could compute more values, but this, together with Figure 2.7, would seem to be convincing evidence to warrant the testing of the conjecture:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

■

We refer the reader to the derivation of this limit found in any standard calculus text. There, as a preliminary step to finding the derivatives of the sine and cosine functions, the above limit is found, usually using a complicated geometric argument and the Pinching Theorem. We should mention that such a complex proof is constructed only after one has obtained experimentally, as above, a conjectured value. Unfortunately, most standard calculus texts make no mention at all of how anyone got the idea for whatever it is that is going to be proved. Consequently, many students of calculus look at such theorems as wholly unmotivated, formal exercises. We wish to help change that perception.

Example 4. A Limit with a Non-Integer Value

Consider $\lim_{x \to 0} \dfrac{x}{\sin 3x}$. The initial graph drawn by the HP-28S (Figure 2.8a) suggests that the limit is around .3. If we use the $\boxed{\texttt{ZBOX}}$ command to zoom in on where the graph seems to cross the y-axis (Figure 2.8b), then we might think that the value of the limit is around .33.

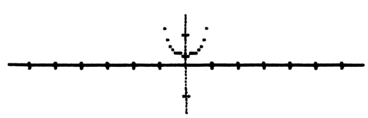


FIGURE 2.8a

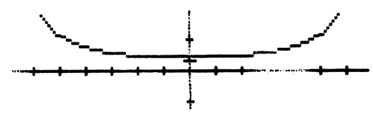


FIGURE 2.8b

Using the Solver, we construct the following table:

x	$f(x)$	x	$f(x)$
.1	.338386336183	$-.1$	.338386336183
.01	.333383338584	$-.01$	.333383338584
.001	.333333833334	$-.001$	.333333833334
.0001	.333333338333	$-.0001$	.333333338333

and so on. This graphical and numerical evidence is perhaps less convincing than in earlier examples. One reason might be that the values do not seem to be approaching a whole number value. If this were a homework problem, would this set off an internal alarm? (There must be something wrong. The answer looks too messy.)

Students often learn a very subtle lesson from solving textbook problems, all of whose answers are whole numbers. That is, we come to expect nice looking answers. When we don't get them, we start looking for the mistake. The unfortunate reality is that in real world applications of mathematics, we only very rarely run across problems which have whole number answers. Thus, we need to be practiced at solving more than just the usual unrealistic, but nice, problems.

From the preceding evidence, there are two reasonable ways in which we might solve the limit problem. First, from the evidence, we might say that the value of the limit is *approximately* equal to .333333338333. Second, in light of the expectation of a nice answer, we could leap to the conjecture that:

$$\lim_{x\to 0} \frac{x}{\sin 3x} = \frac{1}{3}$$

Of course, when it's possible to make and prove a conjecture, this is always preferable. We can prove the preceding conjecture, as follows. It should occur to the reader that the current problem is similar to Example 3. Here though, we have $\sin(3x)$ instead of $\sin(x)$. It sounds like a change of variable might be in order. Let $u = 3x$. Then $x = u/3$ and as x tends to 0, u tends to 0, also. We get

$$\lim_{x\to 0} \frac{x}{\sin 3x} = \lim_{u\to 0} \frac{u/3}{\sin u} = \frac{1}{3} \lim_{u\to 0} \frac{u}{\sin u} = \frac{1}{3}$$

as conjectured, where we have used the fact, established in the last example, that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

■

Example 5. A Limit with a Non-Rational Value

Consider $\lim_{x\to 0} \dfrac{x}{\sin \pi x}$. From the initial graph (Figure 2.9a) and the graph obtained from zooming in (using $\boxed{\text{ZBOX}}$) on the section of the y-axis in question (Figure 2.9b), we get the idea that the limit is somewhere around .3.

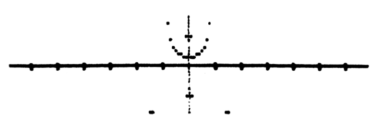


FIGURE 2.9a

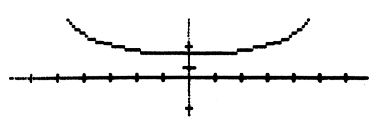


FIGURE 2.9b

Using the Solver, we generate the table

x	$f(x)$	x	$f(x)$
.1	.32360679775	$-.1$	.32360679775
.01	.318362252091	$-.01$	.318362252091
.001	.318310409783	$-.001$	.318310409783
.0001	.31830989142	$-.0001$	.31830989142
.00001	.318309886236	$-.00001$	.318309886236
.000001	.318309886185	$-.000001$	.318309886185

and so on. (Note that if the symbolic constant π is used in the function definition, you will need to add the keystroke $\boxed{\to\text{NUM}}$ to evaluate the symbolic value returned to the stack by the Solver.) As someone trained to look for whole number answers, you might be at great pains to arrive at a meaningful conjecture. For the moment, we need to be satisfied with the suggestion that the limit is approximately .3183098862. In the exercises, we shall see how to arrive at a meaningful conjecture for this problem. Looking back at Example 4 might give you an idea as to how this might be accomplished. ■

Example 6. A Limit of a Sum Where Neither Limit Exists

Find $\lim_{x\to 2} f(x)$ where $f(x) = \dfrac{(x-1)^{1/2}}{x^2-4} - \dfrac{1}{x^2-4}$. First note that the limits of the individual terms do not exist. The elementary rule that

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

only applies when all three of the limits exist. In this example, this is not the case and we must explore further to see what the limit might be.

The initial graph (see Figure 2.10a for an HP-28S graph) does not yield much insight. However, if we use the $\boxed{\text{ZBOX}}$ command to zoom in on the part of the graph near $x = 2$ (Figure 2.10b), we see that the function appears to be approaching a value somewhere around .123.

Using the Solver, we generate the table of values:

x	$f(x)$	x	$f(x)$
1.9	.13158129	2.1	.119040597
1.99	.12562814	2.01	.12437811
1.999	.12506253	2.001	.12493753
1.9999	.12500625	2.0001	.12499375
1.99999	.1250007	2.00001	.1249994
1.999999	.125	2.000001	.125
1.9999999	.125	2.0000001	.125

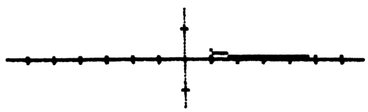


FIGURE 2.10a

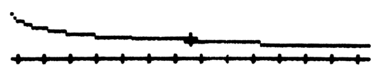


FIGURE 2.10b

We are led to make the conjecture

$$\lim_{x\to 2} \frac{(x-1)^{1/2}}{x^2-4} - \frac{1}{x^2-4} = .125$$

It is left as an exercise to show that the conjecture is correct. This requires some elementary algebraic manipulation. ∎

Many functions are not nearly so well behaved as those in the preceding examples. We need to be able to recognize when a function blows up at a point, as well as when it approaches a finite limit.

Example 7. A Function Whose Graph Has a Vertical Asymptote

Consider $\lim\limits_{x\to 0} \dfrac{1}{\sin x}$. Any graph of the function (see Figures 2.11a and 2.11b, for instance, for the initial HP-28S and HP-48SX graphs, respectively) seems to indicate that the function values go off the scale in the positive direction as $x \to 0$ ($x > 0$) and go off the scale in the negative direction as $x \to 0$ ($x < 0$).

Using the Solver, we generate the table ($f(x)$ rounded off):

x	$f(x)$	x	$f(x)$
.1	10	$-.1$	-10
.01	100	$-.01$	-100
.001	1000	$-.001$	-1000
.0001	10000	$-.0001$	-10000
.00001	100000	$-.00001$	-100000
.000001	1000000	$-.000001$	-1000000

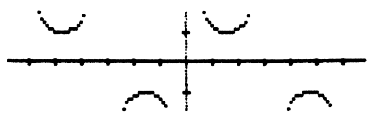


FIGURE 2.11a

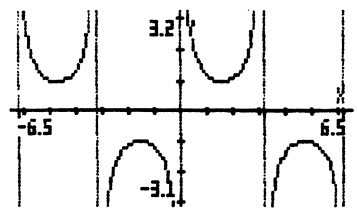


FIGURE 2.11b

and so on. Since the function does not seem to be approaching any fixed value as $x \to 0$, we are led to conjecture that the limit does not exist. To be more specific, since the function grows larger and larger, in absolute value, without bound, as x gets close to 0, we conjecture that

$$\lim_{x\to 0+} \frac{1}{\sin x} = +\infty \qquad \text{and} \qquad \lim_{x\to 0-} \frac{1}{\sin x} = -\infty$$

where by this we mean that the limit does not exist, but more specifically, it doesn't exist because the absolute values of the function values are growing large, without bound. Recall that, in this case, the graph is said to have a *vertical asymptote* at $x = 0$. Verifying these conjectures takes a bit more work than our earlier examples and we omit this. ∎

Now that we have explored limits which tend to ∞, the seasoned student of calculus might guess (or possibly conjecture) that we would next examine the limiting behavior of functions as x tends to infinity. We begin this with an obvious first example.

Example 8. A Function Whose Graph Has a Horizontal Asymptote

Consider $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$. The graphs produced by the HP-28S/48SX (Figures 2.12a and 2.12b show the initial HP-28S and HP-48SX graphs, respectively) seem to indicate that the function approaches the x-axis as x gets larger and larger. (Recall that such a line is called a *horizontal asymptote.*) Figures 2.12a and 2.12b show the horizontal asymptote $y = 0$.

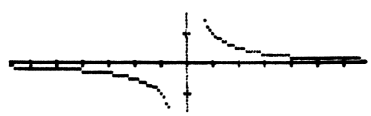


FIGURE 2.12a

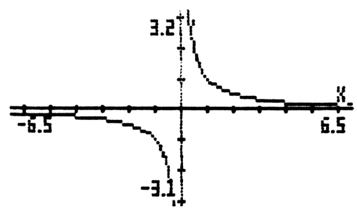


FIGURE 2.12b

A table of values is easily constructed (even by hand).

x	$f(x)$	x	$f(x)$
10	.1	−10	−.1
100	.01	−100	−.01
1000	.001	−1000	−.001
10000	.0001	−10000	−.0001
100000	.00001	−100000	−.00001

It should now be intuitively quite clear that as x gets larger and larger in absolute value, $1/x$ will get closer and closer to 0. We then have that

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

It is now easy to conclude (see the exercises) that, for any positive integer power k,

$$\lim_{x\to\infty} \frac{1}{x^k} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^k} = 0$$

We can use these two limits to solve a large class of limit problems. ∎

Example 9. A Limit of the Form ∞/∞

Consider $\lim\limits_{x\to\infty}\dfrac{x^2+5x-7}{3x^2+4x+9}$. This is a limit of a quotient, but the limits in the numerator and the denominator are both infinite. At first glance, then, this limit has the indeterminate form ∞/∞. Repeatedly using the $\boxed{\texttt{CENTE}}$ command to translate the center of the display to the furthest point to the right on the x-axis, we produce the graphs in Figures 2.13a-2.13c using the HP-28S.

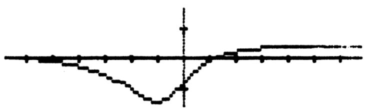


FIGURE 2.13a

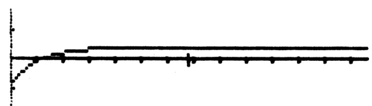


FIGURE 2.13b

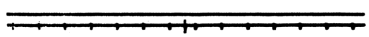


FIGURE 2.13c

The displays seem to suggest that the function tends to a limit of about .3 as x tends to ∞. As usual, a table of values is revealing.

x	$f(x)$
10	.40974212
10	.40974212
100	.34506232
1000	.3345506
10000	.33345551
100000	.33334556
1000000	.33333456
10000000	.33333346

From this evidence, we might conjecture that

$$\lim_{x\to\infty} \frac{x^2+5x-7}{3x^2+4x+9} = \frac{1}{3}$$

It's not hard at all to verify this conjecture. A rule of thumb for dealing with limits of rational functions (i.e., quotients of polynomials) as $x \to \infty$ or $x \to -\infty$ is to divide the numerator and denominator of the fraction by the highest power of x which appears in the denominator. For the present example, this means that we should divide top and bottom by x^2. We get

$$\lim_{x\to\infty} \frac{(x^2+5x-7)/x^2}{(3x^2+4x+9)/x^2} = \lim_{x\to\infty} \frac{1+5/x-7/x^2}{3+4/x+9/x^2} = \frac{1}{3}$$

as conjectured. ■

Example 10. A Limit Involving a Square Root

Consider $\lim\limits_{x\to-\infty} \dfrac{(x^2-7)^{1/2}}{2x+1}$. Note that, like the last example, this is of the indeterminate form ∞/∞. Figures 2.14a and 2.14b show the HP-28S graphs of this function, where we have used the $\boxed{\text{CENTE}}$ command to translate the center of the display to the furthest left point on the x-axis, in order to try to observe the limiting behavior.

FIGURE 2.14a

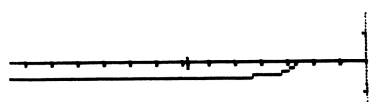


FIGURE 2.14b

From the graphs, it appears that the function has a horizontal asymptote of $y = -.5$, as x tends to $-\infty$. More convincing yet is the table:

x	$f(x)$
-10	$-.50756057$
-100	$-.50233665$
-1000	$-.50024837$
-10000	$-.500025$
-100000	$-.5000025$
-1000000	$-.50000025$
-10000000	$-.500000025$

It is now reasonable to conjecture that

$$\lim_{x\to-\infty} \frac{(x^2-7)^{1/2}}{2x+1} = -\frac{1}{2}$$

We leave it as an exercise to show that this conjecture is true. One need only apply the rule of thumb described in the last example, but very carefully. (HINT: Divide numerator and denominator by x and recall that $\sqrt{x^2} = |x|$.) ∎

We want to emphasize the interplay between the graphics, the numerical computation and the analysis of the conjecture. You might be tempted to forget about the graphs in practice. After all, just how much information have we obtained from them, anyway? It's true that the graphs which a graphing calculator generates are far too crude most times to be able to obtain even a reasonably accurate guess as to the value of a limit. So why bother with them at all? This is easy to answer. The graph gives us some intuition, an expectation of what a reasonable answer might be. If the numbers generated suggest a limit consistent with what we expect from the graphs, then we can be comfortable with our approximation or with our conjecture. However, if the limiting value suggested by a table is far out of line with our expectation, then we have serious cause for concern that one or both of the numbers or the graphs are misleading. In this case, the problem requires further analysis.

As an extreme example of what can go wrong with mindlessly computing a limit from a table of values alone, we offer the following.

Example 11. A Limit Requiring the Use of Computation and Graphics

Consider $\lim\limits_{x\to\infty} (x-\pi)\cos(\pi x)$. There is nothing particularly unusual about this function. We can easily construct the following table of values by using the

Solver. (Recall that we'll need to add the extra keystroke $\boxed{\to\texttt{NUM}}$ to take care of the evaluation of the symbolic constant π.)

x	$f(x)$
10	7
100	97
1000	997
10000	9997
100000	99997
1000000	999997
10000000	9999997

From this table, we are led to the conjecture:

$$\lim_{x\to\infty} (x-\pi)\cos \pi x = \infty$$

On the other hand, if we use a different set of x values for our table, still tending to infinity, we get:

x	$f(x)$
9	−6
99	−96
999	−996
9999	−9996
99999	−99996
999999	−999996
9999999	−9999996

This set of values might lead one to the seemingly reasonable conjecture that

$$\lim_{x\to\infty} (x-\pi)\cos \pi x = -\infty$$

Both of these conjectures cannot be correct! In fact, neither of them is correct. If we had taken the time to first examine the graph of the function, we might have noticed that the function exhibits a great deal of oscillation as $x \to \infty$. (See Figures 2.15a-2.15d for HP-28S graphs; 2.15a is the initial graph; 2.15b was produced by translating the center of the display over to the extreme right end of the x-axis by using the $\boxed{\texttt{CENTE}}$ command; 2.15c and 2.15d are produced by repeatedly using $\boxed{\texttt{ZOOM}}$ to zoom out.) Having then constructed either or, better yet, both of the above tables, we could correctly conjecture that the limit does not exist, but does not tend to $+\infty$ or to $-\infty$.

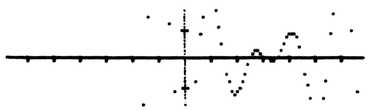


FIGURE 2.15a

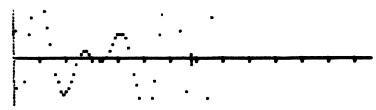


FIGURE 2.15b

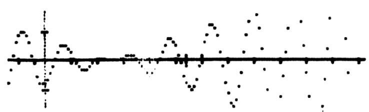


FIGURE 2.15c

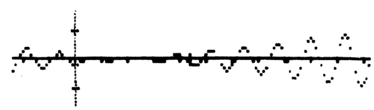


FIGURE 2.15d

■

Exercises 2.1

In exercises 1-6, use graphics and the Solver to conjecture the value of the limit, or conjecture that the limit does not exist.

1. $\displaystyle\lim_{x\to 1}\frac{x-1}{x^2-4x+3}$

2. $\displaystyle\lim_{x\to 0}\frac{x+1}{x^2+2x+3}$

3. $\displaystyle\lim_{x\to 0}\frac{x^2+x}{\sqrt{x^3+2x^2}}$

4. $\displaystyle\lim_{x\to 1}\frac{x^2-x}{\sqrt{x^3-x^2-x+1}}$

5. $\displaystyle\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}$

6. $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x}$

In exercises 7-12, conjecture the value of the limit. Then verify your conjecture by factoring or using the Pinching Theorem.

7. $\displaystyle\lim_{x\to 2}\frac{x^2+2x-8}{x^2-4}$

8. $\displaystyle\lim_{x\to 1}\frac{x^3-x^2}{x^2+x-2}$

9. $\displaystyle\lim_{x\to 0}x^2\sin(1/x)$

10. $\displaystyle\lim_{x\to 0}x\cos(1/x)$

11. $\lim\limits_{x\to 0} \dfrac{x}{x^2+4}$

12. $\lim\limits_{x\to 0} x\sqrt{x^2+4}$

In exercises 13-16, you are asked to discover some general limit rules.

13. Conjecture the value of each limit (assume $c \neq 0$). $\lim\limits_{x\to 0} \dfrac{x}{\sin 4x}$, $\lim\limits_{x\to 0} \dfrac{x}{\sin \pi x}$, $\lim\limits_{x\to 0} \dfrac{x}{\sin cx}$. Verify your conjectures as in Example 4.

14. Conjecture the values of the limits or conjecture that they do not exist: $\lim\limits_{x\to 0} x^2$, $\lim\limits_{x\to 0} x^{1/2}$, $\lim\limits_{x\to 0} x^{-2}$. State a rule giving the various cases for evaluating $\lim\limits_{x\to 0} x^k$.

15. Conjecture the values of the limits or conjecture that they do not exist: $\lim\limits_{x\to \infty} x^2$, $\lim\limits_{x\to \infty} x^{1/2}$, $\lim\limits_{x\to \infty} x^{-2}$. State a rule giving the various cases for evaluating $\lim\limits_{x\to \infty} x^k$.

16. Conjecture the values of the limits or conjecture that they do not exist: $\lim\limits_{x\to \infty} \dfrac{e^x}{x^k}$, $\lim\limits_{x\to \infty} \dfrac{\ln x}{x^k}$ for various values of k. What does this tell you about the relative "size" of logarithms, polynomials and exponential functions?

In exercises 17-22, conjecture the value of the limit or conjecture that the limit does not exist.

17. $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x^2+1}$

18. $\lim\limits_{x\to 1} \dfrac{\sqrt{x+3}-2x}{x^2+2x-3}$

19. $\lim\limits_{x\to 1} \dfrac{x^3-1}{\sqrt{x}-1}$

20. $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{(x-1)^2}$

21. $\lim\limits_{x\to 0} \dfrac{\sin x^2}{x^2}$

22. $\lim\limits_{x\to 0} \dfrac{\cos x}{x}$

EXPLORATORY EXERCISE

Introduction

What is the top speed of a human being? In the 1988 Olympics, it was reported that Ben Johnson reached a peak speed of 24 mph before crossing the finish line first in the 100-meter dash. If the track is well marked, a VCR with frame-by-frame advance can be used to determine speed. Many video cameras record at 30 frames per second so that by counting frames we can measure time.

Problems

Suppose that we collect the following data for a runner. Using the formula

$$\text{Average speed} = \frac{\text{Distance}}{\text{Time}}$$

estimate the peak speed of the runner. For instance, for the entire 100 meters the average speed is 10 meter/sec. But this is not peak speed because between the 50- and 60-meter marks the average speed is $10/.95 = 10.5$ meter/sec. To convert to more familiar units of mph, simply divide by .447.

meters	seconds	meters	seconds
30	3.2	62	6.266
40	4.2	64	6.466
50	5.166	70	7.066
56	5.766	80	8.0
58	5.933	90	9.0
60	6.1	100	10.0

Why are all the times multiples of .033? How much does this affect the accuracy of your estimate of peak speed? To improve our estimate we would either need a better video camera or a formula relating distance and time. The second half of this problem explores the latter (unrealistic) situation.

Suppose that the function $f(t) = 3t^2$ represents the distance covered in t seconds. For instance, after $t = 2$ seconds the runner has gone $f(2) = 12$ meters. The average speed between 1 and 2 seconds is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{12 - 3}{2 - 1} = 9\,\text{meter/sec}$$

What is the instantaneous speed at $t = 2$? We can get a better estimate than 9 meter/sec by computing the average speed between $t = 1.5$ and $t = 2$. Better still, compute the average speed between $t = 1.9$ and $t = 2$. Continue this process to estimate the speed at $t = 2$.

Further Study

The instantaneous rate of change, which you are asked to find above, is called the *derivative* and is an integral part of most calculus-based applications. We will give a more in-depth treatment of the derivative in Chapters 3 and 4.

2.2 Loss of Significance Errors

When conjecturing the values of limits by using the evidence obtained from the HP-28S/48SX (or any other computational device, for that matter), we must always keep in mind that the numbers (and consequently also the graphs) obtained thereby are only approximate. Most of the time, this will cause us no serious trouble. The HP-28S/48SX carries out calculations to a very high degree of precision. Sometimes, however, the results of round-off errors in calculations are disastrous. In this section, we shall examine how and when these loss of significance errors occur. We'll also look at how to recognize these sometimes difficult to find computational errors and how to deal with the occurrence for a limited number of cases.

Example 1. A Flawed Calculation

Consider the limit

$$\lim_{x\to\infty} x[(x^2+4)^{1/2}-x]$$

Following the procedure worked out in the previous section, we draw several graphs to try to get a rough idea of the behavior of the function as x tends to infinity (see Figures 2.16a and 2.16b for the HP-28S graphs).

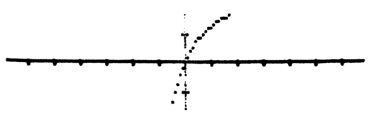


FIGURE 2.16a

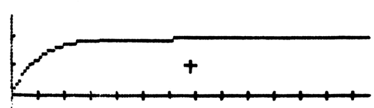


FIGURE 2.16b

From the graphs, it seems that the function remains just about constant at around 1.95 as x goes further and further out to the right. (The graphs pictured above were produced by first performing a ZOOM on the original graph and then by repeatedly translating the center over to the right edge of the preceding window.) Using the Solver, we obtain the following table:

x	$f(x)$
10	1.9804
100	1.9998
1000	2
10000	2
100000	2
1000000	0
10000000	0

The last two values in the table should come as quite a surprise. These are inconsistent with what we expected from examination of the graphs. These are also surprising since from $x = 10$ to $x = 100000$, the corresponding function values seem to be homing in on 2. Then, all of a sudden, the values jump down to 0.

Could it be that we simply did not look at the graph for sufficiently large values of x? Certainly, this is always a possibility, since we're only drawing a large enough piece of the graph to try to get an idea of the limiting behavior as $x \to \infty$. However, this is not the case here. This is an example of what is called a *loss of significance error*. The preceding table suggests that we look more carefully at what happens to the function between $x = 100,000$ and $x = 1,000,000$. We get the following from the Solver:

x	$f(x)$
100000	2
500000	2
750000	2.25
900000	1.8
950000	1.9
990000	1.98
999900	1.9998
999999	1.999998
1000000	0.0

This should strike you as rather strange. It would seem that the function is well-behaved, slowly approaching 2, as x approaches 1,000,000, but something unusual occurs at $x = 1,000,000$. ■

The reason for the unusual behavior witnessed in the last example boils down to how the HP-28S/48SX stores real numbers internally. Without getting into the conversion of decimal (base 10) real numbers into binary (base 2) and vice versa, let it suffice to say that the calculator stores real numbers in scientific notation. Thus, the real number 1234567 is stored as 1.234567×10^6. The part of the number in front of the power of 10 is called the *mantissa* and the power of 10 is called the *exponent.* (So, here the mantissa is 1.234567 and the exponent is 6.)

Since no computing device has infinite memory, there are limitations on the size of the numbers which can be used by such a machine, as well as limits on the number of digits in the mantissa which are held internally by the machine. In the case of the HP-28S/48SX, real numbers are permitted to have a mantissa between 1 and 9.99999999999 and an exponent between -499 and 499. It should be noted that, while this represents a high degree of accuracy especially for a hand-held computing device, it still presents very real problems with accuracy in certain computations.

Essentially, this suggests that real numbers are represented internally only to the first 12 significant digits (finite precision). Again, this is more than sufficient for most computations, but will present an occasional problem. We will examine here what the consequences of such limited accuracy may be on the computation of limits. First, we look at several simple examples.

Example 2. Representation of Real Numbers in Finite Precision

1/3 is stored internally as $3.33333333333 \times 10^{-1}$.
2/3 is stored internally as $6.66666666667 \times 10^{-1}$. ∎

In Example 3, we will see what happens if we subtract two numbers which differ in the 13th significant digit.

Example 3. Arithmetic in Finite Precision

Notice that

$$1.000000000002 \times 10^{15} - 1.000000000001 \times 10^{15} = .000000000001 \times 10^{15} = 1000$$

However, if the above operation is carried out on a machine with a 12-digit mantissa, both numbers are represented by 1.0×10^{15} and consequently the difference between the two values is computed as 0. (Try this now on your HP-28S/48SX.) Similarly,

$$1.000000000006 \times 10^{18} - 1.000000000004 \times 10^{18} = .000000000002 \times 10^{18} = 2,000,000$$

In this case, if the calculation is carried out on a machine with a 12-digit mantissa, the first number is represented by $1.00000000001 \times 10^{18}$ and the second number as 1.0×10^{18} due to the limited accuracy and rounding. The difference between the two values is then computed as $.00000000001 \times 10^{18}$ or 1.0×10^7 or 10,000,000. Once again, this is a serious error. ∎

In both of the preceding computations, we have a computed value which is grossly inaccurate, caused by a subtraction of numbers whose significant digits were very close to one another. This type of error is referred to as a *loss of significant digits error* or simply a *loss of significance error.* These are subtle, but often disastrous, computational errors. Returning now to Example 1, we'll see that it was this type of computational error which caused the values after $x = 1,000,000$ to be trashed.

Recall that the function under consideration was

$$f(x) = x[(x^2 + 4)^{1/2} - x]$$

Let's follow the computation for $x = 1,000,000$ one step at a time, as the calculator carries it out. First compute $(x^2 + 4)^{1/2}$ $(x = 1.0 \times 10^6)$:

$$\begin{aligned}(x^2 + 4)^{1/2} &= [(1 \times 10^6)^2 + 4]^{1/2} = [1 \times 10^{12} + 4]^{1/2} \\ &= [1.000000000004 \times 10^{12}]^{1/2} \\ &= [1.0 \times 10^{12}]^{1/2} \\ &= 1.0 \times 10^6\end{aligned}$$

Recall that $1.000000000004 \times 10^{12}$ is rounded off to 1.0×10^{12} because the calculator only carries 12 digits. Thus, the calculator gives

$$f(1 \times 10^6) = (1 \times 10^6)[1.0 \times 10^6 - 1.0 \times 10^6] = 0$$

Note that the real culprit here is not the rounding of 1.000000000004 to 1, but the fact that this was followed by a subtraction. Additionally, notice that this is not a problem confined to the numerical computation of limits, but a problem common to numerical computation, in general.

RULE OF THUMB: If at all possible, avoid subtractions, in order to avoid loss of significance errors. This can sometimes be accomplished by performing some algebraic manipulation of the function.

Returning once again to Example 1, notice that we may avoid the subtraction which seems to have been the cause of our problems, although in the process, we will complicate the expression for the function. Notice that

$$\begin{aligned} f(x) &= x[(x^2+4)^{1/2} - x] \\ &= x[(x^2+4)^{1/2} - x]\frac{(x^2+4)^{1/2}+x}{(x^2+4)^{1/2}+x} \\ &= \frac{x[(x^2+4)-x^2]}{(x^2+4)^{1/2}+x} = \frac{4x}{(x^2+4)^{1/2}+x} \end{aligned}$$

where, for $x > 0$, the last expression has no subtraction and hence also no loss of significance error. If we plot this last representation of the function, we get the same graphs as for the original representation of the function (Figures 2.16a - 2.16b). We now compute the table of values:

x	$f(x)$
10	1.9804
100	1.9998
1000	1.999998
10000	1.99999998
100000	1.9999999998
1000000	2.0
10000000	2.0
100000000	2.0

This is more like the kind of progression of values which we had seen in earlier examples. From this evidence, together with a graph of the function over a large interval, we might reasonably conjecture that

$$\lim_{x\to\infty} x[(x^2+4)^{1/2} - x] = 2$$

At this point, you should have a fairly good idea of how a loss of significance error can occur. In the next several examples, we shall pursue this a bit further.

Example 4. An Error Where Subtraction Is Not Explicitly Indicated

Consider $\lim\limits_{x\to-\infty} x[(x^2+4)^{1/2} + x]$. At first glance, you might think that since there's no subtraction explicitly indicated, there will be no loss of significance error.

Upon closer examination, however, notice that since x is tending to minus infinity, an addition of two numbers of opposite sign (i.e., a subtraction) is taking place inside the brackets. Again, we can see the same peculiar behavior as that evident in Example 1. From Figures 2.17a and 2.17b (from the HP-28S, obtained by using the $\boxed{\texttt{ZOOM}}$ command once and then translating the center to the furthest point to the left on the x-axis) it appears that the function tends to a value around -1.95 as $x \to -\infty$.

FIGURE 2.17a

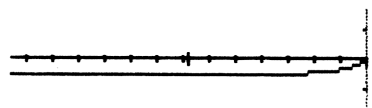


FIGURE 2.17b

We obtain the following table from the Solver:

x	$f(x)$
-10	-1.9804
-100	-1.9998
-1000	-2.0
-10000	-2.0
-100000	-2.0
-1000000	0.0
-10000000	0.0

Again, the sudden change in values should appear suspicious. Upon closer examination, it should be clear that we do indeed have a loss of significance error. The remedy, as it was for Example 1, is to rewrite the expression.

$$\begin{aligned} f(x) &= x[(x^2+4)^{1/2}+x] \\ &= x[(x^2+4)^{1/2}+x]\frac{(x^2+4)^{1/2}-x}{(x^2+4)^{1/2}-x} \\ &= \frac{x[(x^2+4)-x^2]}{(x^2+4)^{1/2}-x} = \frac{4x}{(x^2+4)^{1/2}-x} \end{aligned}$$

Here, again, the last expression has no subtraction (for $x < 0$) and hence should have no loss of significance error. Using this last expression, we construct the table:

x	$f(x)$
-10	-1.9804
-100	-1.9998
-1000	-1.999998
-10000	-1.99999998
-100000	-1.9999999998
-1000000	-2.0
-10000000	-2.0
-100000000	-2.0

Once again, the algebraic manipulation has eliminated the subtraction and, hence, also has eliminated the loss of significance error. We can now conjecture that

$$\lim_{x \to -\infty} x[(x^2+4)^{1/2}+x] = -2$$

■

You will have noticed that the loss of significance errors in each of the last two examples occurred when the x value reached about 1,000,000 in absolute value. Unfortunately, these errors do not occur only when dealing with numbers fairly large in absolute value. They can occur any time two nearly equal numbers are subtracted (or two numbers of nearly equal absolute values but opposite signs are added).

Example 5. A Loss of Significance Error Near $x = 0$

Consider $\lim_{x \to 0} \dfrac{1-\cos(x)}{x^2}$. Using the $\boxed{\text{ZBOX}}$ command twice to zoom in on the behavior of the function near $x = 0$, we obtain the HP-28S graphs in Figures 2.18a and 2.18b.

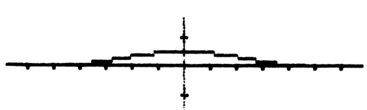


FIGURE 2.18a

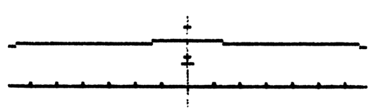


FIGURE 2.18b

From the graphs, it appears that the function is approaching a value around .49 or .5, as $x \to 0$. Using the Solver, we obtain the table:

x	$f(x)$	x	$f(x)$
.1	0.4996	$-.1$	0.4996
.01	0.499996	$-.01$	0.499996
.001	0.5	$-.001$	0.5
.0001	0.5	$-.0001$	0.5
.00001	0.5	$-.00001$	0.5
.000001	0.0	$-.000001$	0.0
.0000001	0.0	$-.0000001$	0.0

Notice that the values in the table take a sudden jump from .5 to 0.0. From the graphs in Figures 2.18a - 2.18b, we might reasonably expect a value near .5. This indicates that there may be a loss of significance error in the computation of $f(x)$. In this case, the value of $\cos(x)$ is very nearly equal to 1 when x is nearly 0. The subtraction of 1 and $\cos(x)$ (two very nearly equal values) then causes the error.

We should note that, while from the first 5 entries in either column of the table and the graphs, we might reasonably conjecture that the value of the limit is .5 (in fact, this is correct) there is a broader question here. How can we reliably compute values of $f(x)$ for x close to 0? From the preceding, it should be clear that we cannot use the given representation of the function. Consider the following algebraic manipulation.

$$\begin{aligned} f(x) &= \frac{1-\cos x}{x^2} = \frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} \\ &= \frac{1-(\cos x)^2}{x^2(1+\cos x)} = \frac{\sin^2 x}{x^2(1+\cos x)} \end{aligned}$$

Notice that, as in the previous examples, this last expression has no subtraction and hence, also will have no loss of significance error. We can now construct the table:

x	$f(x)$	x	$f(x)$
.1	.49958	$-.1$	.49958
.01	.499996	$-.01$	.499996
.001	.49999996	$-.001$	.49999996
.0001	.4999999996	$-.0001$	.4999999996
.00001	.499999999995	$-.00001$	.499999999995
.000001	.5	$-.000001$	.5
.0000001	.5	$-.0000001$	.5

Once again, we have eliminated the loss of significance error by performing an algebraic manipulation. We should be able to use the last expression above for accurate calculation of the values of $f(x)$ and we can, as well, make the conjecture:

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$$

■

We must point out that the method used to avoid the loss of significance error in the last three examples is not one which will work for all problems, or for most problems, for that matter. Again, these errors are often hard to find and somewhat tricky to fix and a complete exposition of these is best deferred to a course in Numerical Analysis. Our main point in discussing these here is to make the student aware of them at an early point, since they will invariably be encountered in calculation. It is useful to be able to recognize when these errors occur and to know how to fix them in at least a limited number of cases.

Exercises 2.2

In exercises 1-6, conjecture the value of the limit. For what value of x does a loss of significance error appear?

1. $\lim_{x\to\infty} x[\sqrt{4x^2+1}-2x]$
2. $\lim_{x\to-\infty} x[\sqrt{4x^2+1}+2x]$
3. $\lim_{x\to\infty} \sqrt{x}[\sqrt{x+4}-\sqrt{x+2}]$
4. $\lim_{x\to\infty} x^2[\sqrt{x^4+8}-x^2]$
5. $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$
6. $\lim_{x\to 0} \dfrac{1-\cos x^2}{x^4}$

In exercises 7-10, rework the indicated exercise after rewriting the function to reduce loss of significance error.

7. exercise 1
8. exercise 2
9. exercise 3
10. exercise 4

In exercises 11-13 (as well as the Exploratory Exercise) you will see what effect a small numerical error can have.

11. Compare $\lim_{x\to 1} \dfrac{x^2+x-2}{x-1}$ and $\lim_{x\to 1} \dfrac{x^2+x-2.01}{x-1}$.
12. Compare $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$ and $\lim_{x\to 2} \dfrac{x-2}{x^2-4.01}$.

13. Compare $\sin \pi x$ and $\sin 3.14x$ for $x = 1$ (radian), $x = 10$, $x = 100$, and $x = 1000$.
14. Compute $(1.6 \times 10^{499})^2$ on the calculator and by hand. In what ways does 10^{500} correspond to ∞ on the calculator?
15. Show that on the calculator $1.6 \times 10^{499} - .8 \times 10^{499} \neq (1.6 - .8) \times 10^{499}$. Explain why they are different.

EXPLORATORY EXERCISE

Introduction

The theory of *chaos* has been called the most interesting scientific development of the last 25 years. One of the fundamental principles of chaos is that small changes in numbers (such as those due to loss of significance errors) can have substantial effects on calculations. Physically, this concept is dramatized by the "Butterfly Effect." This states that the air stirred by a butterfly in China can create or disperse a hurricane in the South Atlantic two days later. Below we look at a basic example of chaos.

Problems

Set up a user-defined function called CHAOS:

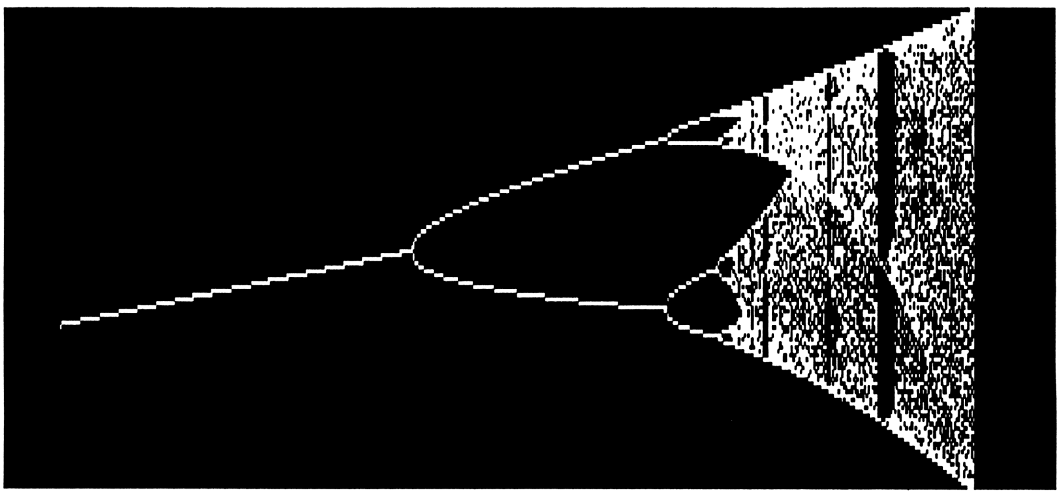
$$\ll \rightarrow X\ 'X * (C - X)'\ EVAL \gg$$

Start with C=2 (press 2 'C' $\boxed{\texttt{STO}}$), put .5 on the stack and press $\boxed{\texttt{CHAOS}}$ several times. After a few times, the number 1 will be returned every time: we are "stuck" at 1. Now try C=2.5, put .5 on the stack and evaluate CHAOS until you get stuck at 1.5. This seems pretty tame: a small change in C produced a small change in output. But, try C=3.2 (again starting with .5 on the stack). This time, you will get stuck alternating between 2 different numbers. A small change in C produced a substantially different behavior.

Verify the following statements: with C=3.48 you get stuck on a 4-number cycle. With C=3.555 you get an 8-number cycle. With C=3.565 you get a 16-number cycle. With C=3.569 you get a 32-number cycle. With C=3.5697 you get a 64-number cycle. With C=3.57 you get chaos (no repetitions).

This information is summarized in the diagram below. For instance, the point (2,1) signifies that with C=2 you get stuck at 1; the point (2.5,1.5) signifies that with C=2.5 you get stuck at 1.5. The diagram divides into 2 branches and then 4 branches and then 8 branches and so on as the process develops 2-number cycles, 4-number cycles, 8-number cycles and so on. The places where the branches occur are called *bifurcation points.* Try to find them.

What happens with C=3.5? Since C=3.48 produces a 4-number cycle and C=3.555 produces an 8-number cycle, C=3.5 is a surprise! Try to identify where C=3.5 is on the diagram.



Further Study

Chaos is a very young scientific field, but there are several books out which are very well written and enjoyable. Chaos by James Gleick is the most general (it spent several weeks on the best seller list). The work we did above is the basis for current attempts to understand turbulence (see Nonlinear Dynamics and Chaos by Thompson and Stewart).

2.3 Exploring the Definition of Limit

We have now spent many pages discussing various aspects of the computation of limits. This may seem just a bit odd, in that we have never actually defined what a limit is. Oh, sure, we have an *idea* of what a limit is, but that's all. In section 2.1, we reminded the student of the intuitive notion of a limit. Again, we say that

$$\lim_{x \to a} f(x) = L$$

if $f(x)$ gets closer and closer to L when x gets closer and closer to a.

We have so far been quite happy with this vague, although intuitive, description. For many purposes, this notion is certainly sufficient. However, this needs to be made more precise. In doing so, we will begin to see how mathematical analysis (that branch of mathematics of which the calculus is the most elementary study) works.

Studying more advanced mathematics without an understanding of the precise definition of limit is somewhat like studying brain surgery without bothering with all that background work in chemistry and biology. A brain surgeon certainly doesn't need these things to perform his or her job, but neither of the authors would consent to surgery by one who did not have a thorough understanding of these areas. Why not? In biology and medicine, it has only been through a careful examination of the microscopic world that a deeper understanding of our own macroscopic world has been achieved and good surgeons need to understand what they are doing and why they are doing it. Mere technical proficiency is simply not enough. Likewise, in mathematical analysis, it is only through an understanding of the microscopic behavior of functions (here, the precise definition of limit) that real understanding of the mathematics will come about.

We begin by careful examination of an elementary example.

Example 1. Analysis of a Limit of a Polynomial

Within the framework of our intuitive notion of limit, it's easy to believe that

$$\lim_{x \to 3} (2x + 5) = 11$$

The function is a polynomial and we have already observed that the limit of any polynomial is found by simply plugging in the value for x. But, why is that? What is it that this statement is trying to communicate? You might answer that as x gets closer and closer to 3, the quantity $(2x+5)$ will get closer and closer to 11. But, this is rather vague. What do we mean when we say closer and closer?

A simple way to think about this is to say that we should be able to make $(2x+5)$ as close as we might like to 11, simply by making x sufficiently close to 3. So, suppose that we want $(2x+5)$ to be within, say, 1/2 of 11. Mathematically, this means that

$$-1/2 < (2x+5) - 11 < 1/2$$

or adding 11,

$$11 - 1/2 < 2x + 5 < 11 + 1/2$$

For what values of x can we guarantee that this will be true? From the graph of $f(x) = (2x+5)$, we can read off an answer. Since we are only interested in the behavior of the function near $x = 3$ and since we want the function values to lie between 10.5 and 11.5, we zoom in on the graphics window with extreme corners (2,10.5) and (4,11.5). (On the HP-28S, put these two points on the stack and press $\boxed{\texttt{ZBOX}}$. On the HP-48SX, enter 2 and 4 on the stack and press $\boxed{\texttt{XRNG}}$ and enter 10.5 and 11.5 on the stack and press $\boxed{\texttt{YRNG}}$ and $\boxed{\texttt{DRAW}}$.) See Figure 2.19 for the HP- 48SX graph. From the graph, it looks like for x between about 2.76 and 3.25, the graph stays on the screen, i.e. $f(x)$ stays between 10.5 and 11.5.

FIGURE 2.19

In this case, however, why not just solve the above inequality for x? We have

$$21/2 < 2x + 5 < 23/2$$

and subtracting 5,

$$11/2 < 2x < 13/2$$

so that

$$11/4 < x < 13/4$$

In particular, notice that this says that

$$3 - 1/4 < x < 3 + 1/4$$

More generally, just how close would you like $(2x + 5)$ to be to 11? Pick some arbitrary distance and call it ϵ (*epsilon*, where $\epsilon > 0$). What range of values of x will guarantee that $(2x+5)$ is within a distance ϵ of 11? Figure 2.20 gives a graphic solution of this problem.

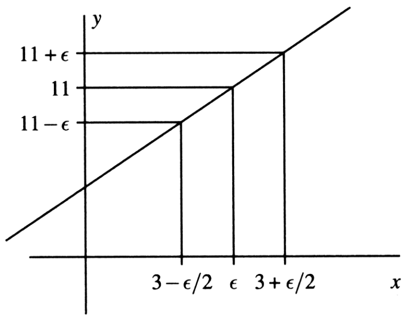


FIGURE 2.20

Again, we require that

$$-\epsilon < (2x + 5) - 11 < \epsilon$$

$$11 - \epsilon < 2x + 5 < 11 + \epsilon$$

Solving the inequality for x, we get

$$6 - \epsilon < 2x < 6 + \epsilon$$

$$3 - \epsilon/2 < x < 3 + \epsilon/2$$

$$-\epsilon/2 < x - 3 < \epsilon/2$$

We summarize this by saying that if we want $(2x + 5)$ to be within ϵ of 11, then x must be within $\epsilon/2$ of 3, i.e.,

$$|(2x + 5) - 11| < \epsilon \qquad \text{whenever} \qquad |x - 3| < \epsilon/2$$

■

Next, we consider this more precise notion of limit for an example where the function is undefined at the point in question.

Example 2. Analysis of a Limit of a Rational Function

Consider $\lim_{x \to 2} \frac{3x^2 - 12}{x - 2} = 12$. As in Example 1, we would like to know how close x must be to 2, in order to guarantee that the function is within some arbitrary distance ϵ ($\epsilon > 0$) of 12. (See Figure 2.21 for a graphical representation of this problem.)

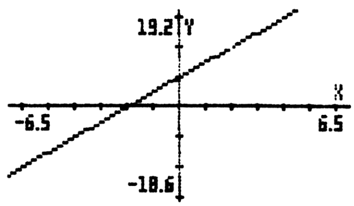


FIGURE 2.21

Notice that, in this case, the function is undefined at $x = 2$ and so we look for a number δ ($\delta > 0$) such that if

$$0 < |x - 2| < \delta$$

then

$$\left| \frac{3x^2 - 12}{x - 2} - 12 \right| < \epsilon$$

(Note that we have $|x - 2| > 0$, so that $x \neq 2$, since the function is not defined at $x = 2$.) This last inequality corresponds to

$$12 - \epsilon < \frac{3x^2 - 12}{x - 2} < 12 + \epsilon$$

Since $x \neq 2$, we can factor and cancel, yielding

$$12 - \epsilon < \frac{3(x - 2)(x + 2)}{x - 2} < 12 + \epsilon$$

$$12 - \epsilon < 3(x+2) < 12 + \epsilon$$

Solving this inequality for x, we get:

$$2 - \epsilon/3 < x < 2 + \epsilon/3$$

i.e.,

$$-\epsilon/3 < x - 2 < \epsilon/3$$

We can now see that choosing $\delta = \epsilon/3$ will do the job. That is, if

$$0 < |x - 2| < \epsilon/3$$

then

$$\left|\frac{3x^2 - 12}{x - 2} - 12\right| < \epsilon$$

as desired. This should give you a sufficiently clear idea of the process to make a general definition. ■

Definition (Precise Definition of Limit) For a function f defined in some open interval including a (except possibly at a itself), we say $\lim\limits_{x \to a} f(x) = L$ if given any number $\epsilon > 0$, there is another number $\delta > 0$ such that whenever $0 < |x - a| < \delta$ we guarantee that $|f(x) - L| < \epsilon$.

We want to emphasize that this formal definition is not a completely new idea, but is simply a more precise formulation of the intuitive notion of limit which we have been using since the start of our discussion of limits. The difference is that we want to use this definition to carefully prove the conjectured value of a limit.

We have seen how to use the graphics and the computing power of the HP-28S/48SX to arrive at a conjecture for the value of a limit. But, how can we make use of the calculator in working with the above definition to prove that a limit has a certain value? In short, we cannot. However, it is of some value to explore the definition using the calculator. While we won't prove any theorems, we might gain some insight into what the definition is saying and how these δ's relate to the ϵ's.

First, as an illustration, let's look at a limit for which we can explicitly compute δ in terms of ϵ.

Example 3. Analysis of a Limit of a Quadratic Polynomial

Consider $\lim_{x\to 2}(x^2-1) = 3$. Given $\epsilon > 0$, we seek a $\delta > 0$ such that if $0 < |x-2| < \delta$, then

$$|(x^2-1)-3| < \epsilon$$

We can turn the problem around somewhat by assuming that for some $\delta > 0$, $0 < |x-2| < \delta$. Then, we have

$$|(x^2-1)-3| = |x^2-4| = |x+2|\cdot|x-2|$$

If we further assume that x lies in the interval [1,3] (we are only interested in what happens near $x = 2$, anyway), we get that $|x+2| \leq 5$, from which it follows that

$$|x^2-4| = |x+2|\cdot|x-2| \leq 5|x-2|$$

We now require that

$$|(x^2-1)-3| < 5|x-2| < \epsilon$$

This occurs if and only if

$$|x-2| < \epsilon/5$$

Thus, we can choose $\delta = \epsilon/5$ (as long as $\epsilon \leq 5$, so that x also stays in the interval [1,3]).

We should point out here that for most problems, finding the δ corresponding to a given ϵ is a very difficult task to accomplish algebraically. However, we can use the graphics of the HP-28S/48SX to gain insight into the relationship between δ and ϵ. First, we illustrate this for the present example.

Consider the choice $\epsilon = 1/2$. In this case, we are interested in what x-values near $x = 2$ will guarantee that the function values stay between $(3 - 1/2)$ and $(3 + 1/2)$, i.e., between 2.5 and 3.5. In this case, we set the x-range to be [1,3] and the y-range to be [2.5,3.5]. [On the HP-28S, enter the extreme corners of the window, (1,2.5) and (3,3.5) on the stack and press ZBOX . On the HP-48SX, enter 1 and 3 on the stack and press XRNG and enter 2.5 and 3.5 on the stack and press YRNG and DRAW .] See Figure 2.22 for the HP-48SX graph. From the graph, you can observe that by keeping the x-values in the interval (1.9,2.1), the graph will stay on the screen (i.e., the y-values will stay in the desired interval). Notice that this set of x-values corresponds to those obtained from our previous analysis. (There, we had $|x-1| < \delta$, where $\delta = \epsilon/5$, so that when $\epsilon = 1/2$, $\delta = 1/10$.) ■

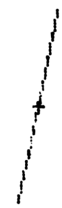


FIGURE 2.22

We point out that an exploration of the definition of limit such as that exhibited in Example 3 is of most interest for problems where it is not obvious what the relationship between δ and ϵ might be. This is the case in Example 4.

Example 4. Exploring the Definition of Limit Where δ Is Unknown

Consider $\lim_{x\to 1} \cos(\pi x/2) = 0$. We would all certainly like to believe that this limit is correct. After all, $\cos(\pi/2) = 0$ and so this seems only fair. Also, any graph of $y = \cos(\pi x/2)$ will suggest that this should be true. Proving this is yet another matter. Given an $\epsilon > 0$, we look for a $\delta > 0$ such that

$$|\cos(\pi x/2) - 0| < \epsilon$$

whenever $0 < |x-1| < \delta$. Finding δ in terms of an arbitrary ϵ is not easy. (Try this and see what we mean!) We can, however, use the graphics utilities of the HP-28S/48SX to experimentally find values of δ which seem to work for some selected values of ϵ.

Let's start with the value $\epsilon = 1/2$. This means that we would like to know if we can find a $\delta > 0$ so that $0 < |x-1| < \delta$ guarantees that

$$0 - 1/2 < \cos(\pi x/2) < 0 + 1/2$$

or

$$-1/2 < \cos(\pi x/2) < 1/2$$

Here, we are interested in what range of x-values (near $x = 1$) will guarantee that the function values will stay between $-1/2$ and $1/2$. Set the x-range to be [0,2] and the y-range to be $[-.5, .5]$. [On the HP-28S, enter the extreme corners of the display, $(0, -.5)$ and $(2, .5)$ on the stack and press $\boxed{\texttt{ZBOX}}$. On the HP-48SX, enter 0 and 2 on the stack and press $\boxed{\texttt{XRNG}}$ and enter $-.5$ and $.5$ on the stack and press $\boxed{\texttt{YRNG}}$ and $\boxed{\texttt{DRAW}}$.] See Figure 2.23a for the HP-28S graph. Notice from the graph that if x is between about .68 and 1.32, the graph will stay on the screen, i.e., the y-values will fall in the desired range. This says that δ is approximately .32, here.

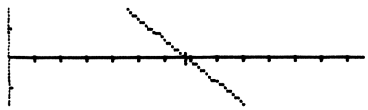


FIGURE 2.23a

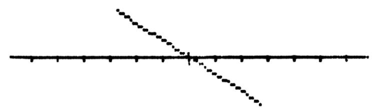


FIGURE 2.23b

Next, we try $\epsilon = .25$. First, zoom in on the part of the graph of interest. Figure 2.23b was produced by entering the points $(.6, -.25)$ and $(1.4, .25)$ on the stack and pressing $\boxed{\texttt{ZBOX}}$. You can do the same on the HP-48SX by using the $\boxed{\texttt{XRNG}}$ and $\boxed{\texttt{YRNG}}$ commands, as above. Here, we want

$$-.25 < \cos(\pi x/2) < .25$$

In this case, notice that if x is between about .85 and 1.15, the graph stays on the screen and, hence, the y-values fall in the desired range. This gives us an approximate value for δ of .15.

Repeat this process for a few even smaller values of ϵ. We can continue this indefinitely. This is, of course, the whole idea of the definition of limit. Again, while finding the δ 's for a few ϵ 's will not *prove* a conjecture as to the value of a limit, it should serve to illustrate the idea, as well as to provide evidence that our conjecture is correct. In the exercises, we will explore this idea further, both using the power of the HP-28S/48SX and by solving some problems by hand. ∎

Exercises 2.3

In exercises 1-8, graphically find values of δ corresponding to $\epsilon = .1$ and $\epsilon = .05$.

1. $\lim\limits_{x \to 0} 3x + 5 = 5$
2. $\lim\limits_{x \to 2} 2x + 1 = 5$
3. $\lim\limits_{x \to 0} x^2 + 1 = 1$
4. $\lim\limits_{x \to -1} 2x^2 + 3 = 5$
5. $\lim\limits_{x \to 1} \sqrt{x + 3} = 2$
6. $\lim\limits_{x \to 2} \sqrt{x^3 + 1} = 3$

7. $\lim_{x\to 1} \frac{x^2+3}{x} = 4$

8. $\lim_{x\to 0} \cos x = 1$

In exercises 9-11 the function has the form $f(x) = kx$ for some constant k. Verify that $\delta = \epsilon/|k|$ works for $\lim_{x\to 0} f(x) = 0$.

9. $f(x) = 3x$

10. $f(x) = -2x$

11. $f(x) = x/2$

12. Rework exercises 9-11 for $\lim_{x\to 1} f(x)$. Would $\delta = \epsilon/|k|$ work for any c in $\lim_{x\to c} f(x)$?

In exercises 13-16, verify graphically that the limits do not exist. Explain why there is no δ that works for $\epsilon = .1$. NOTE: to graph the function in exercise 13, press

≪→ X ' IFTE(X>1,X∧2,X+1) ' ≫

and then proceed as for any other function using NEWF on the HP-28S or NEW on the HP-48SX.

13. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 & x > 1 \end{cases}$

14. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} x^3-4 & x \le 2 \\ x^2+1 & x > 2 \end{cases}$

15. $\lim_{x\to 1} \frac{4x}{(x-1)^2}$

16. $\lim_{x\to 0} \frac{x^2+x}{\sqrt{x^3+4x^2}}$

17. State precisely what it means for a limit to not exist.

18. State precisely what is meant by $\lim_{x\to c} f(x) = \infty$. Use $\lim_{x\to 0} \frac{1}{x^2}$ as an example to guide your thinking.

19. A manufacturer of steel balls signs a contract to produce 2-lb balls. The customer allows a deviation of at most .02 lb (1%). The radius and weight are related by $W = \frac{r^3}{4}$ so that the radius is supposed to be 2 cm. How much can the radius deviate from 2 cm if the weight is to stay within the customer's specifications?

20. If the manufacturer in exercise 19 receives special orders with reduced tolerances of .01 lb and .005 lb, how much does the radius tolerance need to be reduced to?

EXPLORATORY EXERCISE

Introduction

In exercises 9-12, you found a general formula for δ in terms of ϵ for linear functions. Here, we will discover a similar formula for quadratic functions. In particular, we want to find a constant k such that $\delta = \epsilon/k$ works for $\lim_{x \to 0} ax^2 + bx + c = c$.

Problems

We will look at some examples before trying to generalize. For $f(x) = x^2 + 3x + 2$ zoom in on (0,2) until the graph appears to be straight. Find two points on this curve and compute the slope m between these two points. Show that $k = m$ does not work but k slightly larger than m *does* work. Repeat this process for $f(x) = 4x^2 + 3x + 2$ and $f(x) = x^2 + x + 2$. Conjecture the slope of $f(x) = ax^2 + bx + c$ at $(0, c)$ and then conjecture a solution to our original problem.

Further Study

The work you have done above will be useful in several ways as you progress through calculus. You have probably discovered how to compute the *derivative* (slope) of a quadratic function. You have also come very close to proving an important result which states that if a function is differentiable at $x = a$ then it must be continuous at $x = a$.

CHAPTER 3

Differentiation

3.1 Construction of Tangent Lines

We are all quite familiar with the notion of a tangent line to a circle. This is a line which intersects the circle in exactly one point. Unfortunately, this idea does not generalize to all curves. Recall that we define the tangent line to the curve $y = f(x)$ at the point $P(x_0, y_0)$ by considering a sequence of lines joining P with nearby points Q (these are called *secant lines*). As Q gets closer and closer to P, the lines are approaching the tangent line (see Figure 3.1).

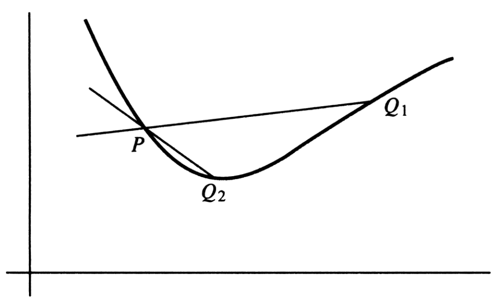


FIGURE 3.1

Notice that since we're already specifying the *point* of tangency, all we need to do to define the tangent line is to find its slope. We will do this shortly. But first, we want to come at this problem from a slightly different direction.

What is it that the tangent line is telling us about the graph of a function?

One way to think about this is as follows. If you are walking along the graph of a function, the direction in which you are facing at any given point is along the tangent line. Further, if we zoom in enough on the graph, it should look fairly straight. The straight line that we then observe is an approximation to the tangent line. That should ring some bells. Your HP-28S/48SX is ideal for drawing graphs and zooming in on various points of interest.

Example 1. Using Graphics to Find an Approximate Tangent Line

Consider $f(x) = \frac{1}{2}x^3 - 1$, near $x = 1$. We will explore this idea of zooming in on the point of interest, $(1, -.5)$, until the graph looks fairly straight. We start with a graph of $y = f(x)$ using the default graphing parameters (press $\boxed{\text{RESET}}$). The initial HP-28S graph is shown in Figure 3.2.

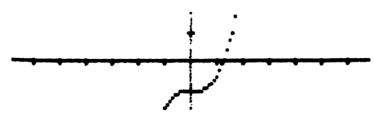


FIGURE 3.2

If you move the cursor over to the point $(1, -.5)$, you'll note that the graph does not look very straight nearby.

However, if we use the $\boxed{\text{ZBOX}}$ command (on either the HP-28S or 48SX) we can see that the more that we zoom in on the vicinity of the point in question, the straighter the graph appears to be (see Figures 3.3a and 3.3b for successively zoomed HP-28S graphs).

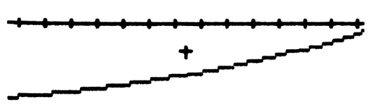


FIGURE 3.3a

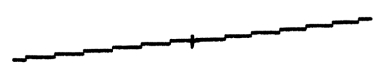


FIGURE 3.3b

The endpoints of the portion of the graph appearing in Figure 3.3b are approximately $(.941176470585, -.581685744016)$ and $(1.05882352941, -.404786680541)$ but yours may differ somewhat. To find the slope of the line joining these two points, first put the points on the stack (move the cursor over to the point and press INS on the HP-28S; on the HP-48SX, press the ENTER key). Next, press ON and −. The HP-28S/48SX will return the point whose coordinates are the differences of the respective coordinates of the original points. Then press C→R [located in the Complx menu on the HP-28S and in the second page of the Obj menu (press PRG OBJ NEXT) on the HP-48SX]. This places the x and y coordinates of the point on lines 2 and 1 of the stack, respectively. Press SWAP and then / to compute the slope. In the present case, we get a slope of 1.50364203952. What does this slope represent? Well, it would seem to be an approximation to the slope of the tangent line.

To observe what the line through $(1, -.5)$ with slope 1.50364203952 looks like together with the original graph, plot them simultaneously, as follows:

1. Press RESET to reset the graphics window parameters to their default values
2. Press EQ EDIT (or EDEQ on the HP-48SX) to return the current equation to the command line and prepare to edit it
3. Move the cursor to the 2nd quote mark, insert: = 1.50364203952 * (X−1) −.5 and press ENTER .

Press NEWF on the HP-28S or NEW on the HP-48SX and draw the graph of the equation. (Recall that the two expressions on either side of the equation will be graphed simultaneously.) See Figure 3.4 for the HP-48SX graph of this. It is not clear whether or not this is the tangent line that we are seeking. Actually, it's a secant line, and acts as only a crude approximation to the tangent line.

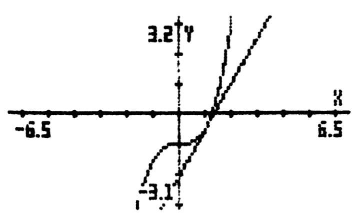


FIGURE 3.4

While we might be able to improve our approximation by zooming in on the point of tangency even further, there is a better way to do this. You should keep in mind that the graphs produced by the HP-28S/48SX are at best fairly rough representations of the actual graphs. Thus, there's a built-in limit to how good an approximation of slope can be if it is derived from such an imperfect graph. ∎

In general, for a function $f(x)$, if we want the slope of the tangent line to $y = f(x)$ at the point corresponding to $x = a$, then we should graphically zoom in on the vicinity of the point of tangency enough that the graph appears to be a straight line. We again want to make the point that appearances can be deceiving. However, the basic idea here is correct. We can accomplish the same thing algebraically without the inaccuracies inherent in the reliance on graphics alone.

Pick a point on the curve nearby the point of tangency, $(a, f(a))$, say $(a + h, f(a+h))$, for some small value h. If h is small enough [i.e., if $(a+h, f(a+h))$ is close enough to $(a, f(a))$] then the portion of the graph between these two points should appear fairly straight. In this case, the slope of the secant line joining these two points will approximate the slope of the tangent line. From the usual formula for slope, the slope of this secant line is

$$m_{\text{sec}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Example 2. Computing the Slope of a Secant Line

Again, for $f(x) = \frac{1}{2}x^3 - 1$, compute the slope of the secant line joining the points corresponding to $x = 1$ and $x = 1.1$. We get

$$m_{\text{sec}} = \frac{f(1.1) - f(1)}{.1} = \frac{-.3345 - (-.5)}{.1} = 1.655$$

A better approximation to the slope of the tangent line might be obtained if we find the slope of the secant line joining $(1, -.5)$ and an even closer point, say the point corresponding to $x = 1.01$. We find

$$m_{\text{sec}} = \frac{f(1.01) - f(1)}{.01} = \frac{-.4848495 - (-.5)}{.01} = 1.51505$$

∎

There are several things to notice here. First, the slopes of the secant lines computed above are reasonably close to the approximation to the slope of the tangent line found graphically in Example 1. Second, it is reasonable to expect that the closer that we choose the second point to the point of tangency, the closer the slope of the corresponding secant line should be to the slope of the tangent line. Rather than compute a long sequence of slopes manually, we suggest the following HP-28S/48SX program.

PROGRAM TIP: Return to the HOME directory (press [QUIT] if you are in a subdirectory) and create a new subdirectory: Enter ' TANG ' and press [CRDIR] . Enter the new subdirectory by pressing [TANG] in the User menu. Then store the usual QUIT program: type ≪ [HOME] ≫ [ENTER] ' Q U I T ' [STO] . Finally, enter the program:

≪→ H ' (F (X0 + H) − F (X0)) / H ' ≫

Program Step	Explanation
≪ → H	Store the value on line 1 of the stack in the local variable H.
'(F(X0+H) − F(X0))/H'	Compute the slope of the secant line joining the points corresponding to x = X0 and x = X0+H and return this value to the stack.
≫ ENTER	End the program and put it on line 1 of the stack.
'MSEC' STO	Store the program under the name MSEC in the current directory.

Before running this program, you will need to create two other entries in your current directory: one for the function F and one for the x coordinate of the point of tangency, X0. For the present example, we can use:

≪→ X ' .5 * X ∧ 3 − 1 ' ≫ [ENTER] ' F ' [STO]

1 [ENTER] ' X0 ' [STO]

Entering a value for H on the stack and pressing [MSEC] will compute the slope of the secant line joining the points on the graph of $y = F(x)$ corresponding to the x values X0 and X0+H. For example, using H = .1, we get the slope 1.655, as above. Having this program, however, allows us to easily compute values of MSEC for a sequence of values of H getting closer and closer to 0. In this way, we can observe the limiting behavior of the slopes of the secant lines and from this conjecture the value of the slope of the tangent line.

H	MSEC	H	MSEC
1.0	3.5	−1.0	0.5
0.1	1.655	−0.1	1.355
0.01	1.51505	−0.01	1.48505
0.001	1.5015005	−0.001	1.4985005
0.0001	1.50015	−0.0001	1.49985
0.00001	1.500015	−0.00001	1.499985
0.000001	1.5	−0.000001	1.499998
0.0000001	1.5	−0.0000001	1.5

From this table, we see that the slopes of the secant lines seem to be getting closer and closer to 1.5 as H gets closer and closer to 0. Intuitively, then, it seems reasonable to conjecture that the slope of the tangent line is 1.5. To check that this conjecture is consistent with our geometric intuition about tangent lines, draw the graphs of $y = F(x)$ together with the line through $(1, -.5)$ with slope 1.5. Type

' .5 * X ∧ 3 − 1 = 1.5 * (X − 1) − .5 ' [ENTER]

and draw the graph using the default graphics window parameters (press [RESET] first). The resulting HP-28S graph is shown in Figure 3.5.

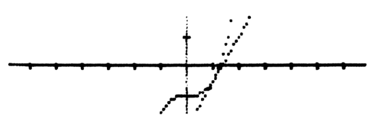


FIGURE 3.5

In the figure, it looks like this is indeed the tangent line that we're looking for.

WARNING: If you take the value of H to be too small, the calculation of m_{sec} may be subject to a loss of significance error, as discussed in Chapter 2. For example, here a value of H $=$ .00000000001 yields $m_{\text{sec}} = 1.5$, while H $=$ .000000000001 produces the value $m_{\text{sec}} = 0.0$.

Let's briefly review Example 2 to see how we might more precisely define the notion of tangent line. In the preceding, we computed the slopes of secant lines for a sequence of points getting closer and closer to the point of tangency. We observed that the limiting value of these slopes should be the slope of the tangent line. In general, we have:

Definition The slope of the tangent line to $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$m_{\tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided the limit exists.

It should be stressed that when conjecturing the value of such limits numerically (e.g., with the HP-28S/48SX) you should always compare the conjectured value with what you expect from the graph (by zooming in on the behavior near the point of tangency, as in Example 1) and further check that the line through the point of tangency with the conjectured slope looks like the tangent line when plotted simultaneously with the graph of the function.

Example 3. Conjecturing the Value of the Slope of the Tangent Line

Find the slope of the tangent line to the graph of $f(x) = x \sin(\pi x)$ at $x = 2$. First, we draw the graph of the function and zoom in repeatedly, until the graph appears fairly straight (see Figures 3.6a and 3.6b for the appropriate HP-48SX graphs).

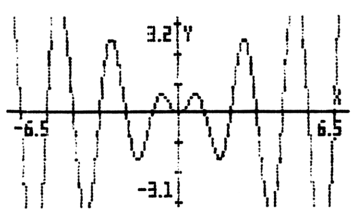


FIGURE 3.6a

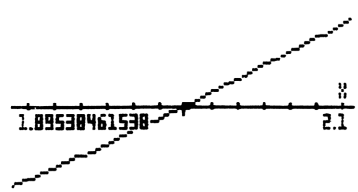


FIGURE 3.6b

In Figure 3.6b, the graph appears fairly straight and if you compute the slope of the line joining two points on this segment of the graph, you should get a value around 6.2. (Try this now for yourself!)

Next, use the program MSEC to compute a sequence of values for m_{sec}. Note that you'll first need to store a new program for the function F and a new value for the x-coordinate of the point of tangency, X0. Here, we enter:

2 [ENTER] ' X0 ' [STO]

and

≪→ X ' X * [SIN] (π * X) ' [→NUM] ≫ [ENTER] ' F ' [STO]

Notice that we have added the [→NUM] key at the end of the program so that the HP-28S/48SX will compute numerical instead of symbolic values for F. This is necessary because with the use of the symbolic constant π in the function, the calculator will automatically compute symbolic values.

Using MSEC, we generate the following table of values.

H	MSEC	H	MSEC
.1	6.48935688189	−.1	5.87132289315
.01	6.31356257552	−.01	6.25074105736
.001	6.28631655205	−.001	6.28003337711
.0001	6.28349945592	−.0001	6.28287113739
.00001	6.28321741491	−.00001	6.28315458303
.000001	6.28318314158	−.000001	6.2831768584
.0000001	6.28320031416	−.0000001	6.28319968584

From the table, while the values may not look particularly nice, they seem to be getting closer and closer to a number around 6.2832. As further evidence, this is close to the value we expected from our use of the graphics. We should note that using even smaller values of H may not lead to progressively better approximations to the slope of the tangent line. Again, remember that these computations are highly subject to loss of significance errors. These results may be improved by using some of the hints found in section 2.2.

As a final test of the sensibility of our answer, we draw the graph of $y = f(x)$ with the suspected tangent line superimposed on it. That is, we plot the graph of the equation

' X * SIN(π * X) = 6.2832 * (X − 2) '

See Figure 3.7a for the HP-48SX graph with the default graphics parameters and

Figure 3.7b for a zoomed-in graph. In both cases, the line drawn looks very much like the tangent line, as expected.

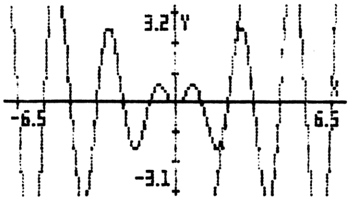


FIGURE 3.7a

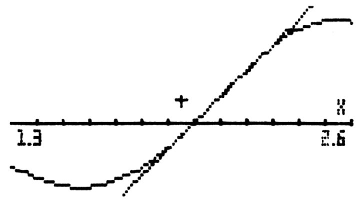


FIGURE 3.7b ■

In this section, we have explored the notion of tangent line to the graph of a function and have seen how to compute approximations to the slope of the tangent line at a given point. We've also seen how to test our conjectured approximate slopes by using the graphics power of the HP-28S/48SX. In section 3.2, we will examine a notion related to the slope of the tangent line and then see a more efficient way of computing approximations to these values.

Exercises 3.1

In exercises 1-8, use ZBOX to estimate the slope of the tangent line at the given point.

1. $f(x) = x^2 - 1$, $x = 1$
2. $f(x) = x^2 - 1$, $x = 2$
3. $f(x) = x^3 - x$, $x = 0$
4. $f(x) = x^3 - x$, $x = 1$
5. $f(x) = \sqrt{x^2 + 1}$, $x = 0$
6. $f(x) = \sqrt{x^2 + 1}$, $x = 1$
7. $f(x) = \sin x$, $x = 0$
8. $f(x) = \sin x$, $x = \pi$

In exercises 9-16, use MSEC to estimate the slope of the tangent line at the given point and compare to exercises 1-8.

9. $f(x) = x^2 - 1$, $x = 1$
10. $f(x) = x^2 - 1$, $x = 2$
11. $f(x) = x^3 - x$, $x = 0$
12. $f(x) = x^3 - x$, $x = 1$
13. $f(x) = \sqrt{x^2 + 1}$, $x = 0$
14. $f(x) = \sqrt{x^2 + 1}$, $x = 1$
15. $f(x) = \sin x$, $x = 0$
16. $f(x) = \sin x$, $x = \pi$

In exercises 17-18, use the graph and MSEC to determine that the slope of the tangent line does not exist.

17. $f(x) = |x|$, $x = 0$
18. $f(x) = (x^2)^{1/3}$, $x = 0$

In exercises 19-22, compute the slope of the tangent line by hand and compare your answers to those obtained in exercises 9-12.

19. $f(x) = x^2 - 1$, $x = 1$
20. $f(x) = x^2 - 1$, $x = 2$
21. $f(x) = x^3 - x$, $x = 0$
22. $f(x) = x^3 - x$, $x = 1$

In exercises 23-26, use the graph and MSEC to estimate the slope of the tangent line at $x = 0$, if it exists. HINT: Use IFTE ("if-then-else") to enter the functions. In exercise 23, type ≪ → X ' IFTE (X<0,X∧2−1,X∧3−1) ' ≫.

23. $f(x) = \begin{cases} x^2 - 1 & x < 0 \\ x^3 - 1 & x \geq 0 \end{cases}$

24. $f(x) = \begin{cases} x^2 - x & x < 0 \\ x^3 - x & x \geq 0 \end{cases}$

25. $f(x) = \begin{cases} x/2 & x < 0 \\ x/4 & x \geq 0 \end{cases}$

26. $f(x) = \begin{cases} x^2 - 1 & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$

EXPLORATORY EXERCISE

Introduction

Progress in mathematics is often made through exploration with an eye towards finding patterns. It turns out that slopes of tangent lines are easily computed from the original function. That is, there are nice patterns for us to discover below.

Problems

Use MSEC to estimate the slope of the tangent line to $\sin x$ at x=0,.2,.4,.6,.8,..., 3.0 and save the slopes in the following format. After getting a slope of 1 at $x = 0$, press [0,1] $\boxed{\text{ENTER}}$. This is a *vector* notation which the HP- 28S/48SX uses for statistical data. When you have all 16 vectors on the stack, press $\boxed{\Sigma+}$ (in the Stat menu) 16 times to store the data. Then plot the data (use $\boxed{\text{DRW}\Sigma}$ in the Plot menu of the HP-28S or $\boxed{\text{SCATR}}$ in the Stat menu of the HP-48SX). Does this curve look familiar? By correctly identifying this curve you will find an easy rule for finding slopes of tangent lines to $y = \sin x$. Repeat the above for $y = \cos x$. Finally, estimate the slope of the tangent line to $y = e^x$ at x=−1.5, −1.3, −1.1, ..., 1.5. Then plot the data and identify the curve to find an easy rule for finding slopes of tangent lines to $y = e^x$.

Further Study

The rules found above are three of the basic *derivative* rules which you will use throughout the rest of your mathematics career. Rigorous derivations of these rules can be found in your calculus book.

3.2 Numerical Differentiation

There are two main concepts from which the notion of derivative follows. One is the notion of tangent line which we examined in section 3.1. The second concept is the notion of *velocity* .

We first want to briefly explore what is meant by the term velocity. Think of what you mean when you use the word. More importantly, ask yourself how you would compute velocity if asked. The first thought that comes to mind is probably the formula $distance = rate \times time$ which you studied in high school algebra. If you want to know the rate (the velocity) then you need only divide the distance by the time. Simple, isn't it? It's also not generally correct, as we'll see.

For example, suppose that you are pulled over on the highway by a police officer. The officer steps up to your car and asks that dreaded question,"Do you know how fast you were going?" As a good student of mathematics, you might be tempted to answer something like,"Well, I've been driving for 3 years, 2 months, 7 days, 5 hours and about 45 minutes. I've kept very careful records and I can tell you that I've driven exactly 45,259.7 miles in that much time. Therefore, according to my calculations, I was going only

$$\frac{45,259.7\,\text{miles}}{27,917.75\,\text{hours}} = 1.62118\,\text{mph}$$

Of course, the authors do not recommend that you use this argument if you're pulled over. To be sure, it's ridiculous and most police officers would be reaching for their handcuffs (or a straight jacket) by the time you finished explaining your calculation. But, *why* is this wrong? Certainly there's nothing wrong with the formula or the calculations. Well, the police officer might reasonably argue that during this 3-year period, you were not even in a car most of the time and, hence, the results are invalid.

Suppose that you think quick and substitute the following argument instead, "Officer, I left my house at exactly 6:17 pm tonight and by the time you pulled me over at 6:43 pm, I had driven exactly 17 miles. Now, that says that I was going only

$$\frac{17\,\text{miles}}{26\,\text{minutes}} \times \frac{60\,\text{minutes}}{1\,\text{hour}} = 39.23077\,\text{mph}$$

well under the posted 45-mph speed limit."

What's wrong this time? Even assuming that you did not stop at all during the entire trip, this argument is unconvincing. It should be clear that, although this is a much better estimate of your velocity than the 1.6 mph arrived at previously, you are still computing the velocity using too long of a time period. It's not hard to realize that since cars speed up and slow down (and can do so very quickly) we must compute the velocity using a much smaller time period. Well now, how small is small enough?

It's time for some answers. What we've been computing is what is usually referred to as *average velocity* . What we are really interested in (as well as what the police officer is interested in) is *instantaneous velocity* , the velocity at an instant in time. We must now see how this can be computed.

Example 1. Instantaneous Velocity

Suppose that you are driving in a straight line and that the distance that you've traveled at time t (measured in minutes) is given by the function

$$s(t) = \frac{1}{2}t^2 - \frac{1}{12}t^3 \qquad 0 \le t \le 4$$

Find the instantaneous velocity at $t = 2$ minutes. Here, we assume that $s(t)$ gives values measured in miles. As a starting point, we might compute the average velocity during the time interval [0,2]. We get

$$\begin{aligned}\text{Average velocity} &= \frac{s(2) - s(0)}{2 - 0} = \frac{1.33333333333 - 0}{2} \\ &= .666666666665\,\text{miles/minute} \\ &= 39.9999999999\,\text{mph}\end{aligned}$$

You might expect that we could improve our estimate by averaging over a smaller time interval. For example, on the interval [1,2], we get

$$\begin{aligned}\text{Average velocity} &= \frac{s(2)-s(1)}{2-1}\\ &= 1.3333333333 - .416666666667\\ &= .916666666663\,\text{miles/minute}\\ &= 54.9999999998\,\text{mph}\end{aligned}$$

Of course, we can continue this process indefinitely. The smaller we make the time interval, the better the approximation of the velocity should be. In general, on the time interval $[t_0 - h, t_0]$, the average velocity is given by

$$v_{\text{ave}} = \frac{s(t_0) - s(t_0 - h)}{t_0 - (t_0 - h)} = \frac{s(t_0) - s(t_0 - h)}{h}$$

Notice the similarity of this to the formula developed in section 3.1 for the slope of a secant line. We can use a program similar to MSEC to compute this value. In the TANG directory, enter the following program.

≪→ H '(S(T0) – S(T0 – H)) / H' ≫

Program Step	Explanation
≪ → H	Take the value on line 1 of the stack and store it in the local variable H.
'(S(T0)–S(T0–H))/H'	Compute the average velocity on the interval [T0–H,T0] and return this to the stack.
≫ ENTER	End the program and enter it on the stack.
'VEL' STO	Store this in the current directory under the name VEL

Before you run this program, you will need to enter a value for the time T0 and a program for the distance function S in the current directory. For the present example, enter 2 ' T0 ' $\boxed{\text{STO}}$ and use the program:

≪→ X ' .5 * X ∧ 2 − 1 / 12 * X ∧ 3 ' ≫ [ENTER] ' S ' [STO]

Entering a value for H on line 1 of the stack and pressing [VEL] will return the average velocity on the desired interval to line 1 of the stack. We can thus easily compute average velocities for a sequence of values of H, as in the following table.

H	AVE VEL	H	AVE VEL
1.0	0.916666666663	−1.0	0.91666666667
0.1	0.9991666666	−0.1	0.9991666667
0.01	0.999991666	−0.01	0.999991667
0.001	0.99999991	−0.001	0.99999992
0.0001	1.0	−0.0001	1.0
0.00001	1.0	−0.00001	1.0
0.000001	1.0	−0.000001	1.0

Notice from the table that as we make the time interval over which we're averaging smaller and smaller, the average velocity seems to be getting closer and closer to 1 mile/minute (60 mph). This limiting value is what we mean by instantaneous velocity. We can now give a definition of velocity. ■

Definition If the position of an object traveling in a straight line at time t is given by the function $s(t)$ [i.e., $s(t)$ gives the location on a number line of the object], then the *instantaneous velocity* of the object at time t_0 is given by

$$\lim_{h\to 0} \frac{s(t_0) - s(t_0 - h)}{h}$$

or equivalently, by

$$\lim_{h\to 0} \frac{s(t_0 + h) - s(t_0)}{h}$$

Notice that this is precisely the same as the definition of the slope of the tangent line, except that, here, the variable is t instead of x. In particular, this says that the numerical computation of velocities will also be highly subject to loss of significance errors. Thus, taking H too small may result in a gross error in the computed average velocity and, hence, also in the conjectured value of the instantaneous velocity.

As we have noted, the limit in the preceding definition arises naturally in several different contexts. Actually, this limit is so common that we give it a name.

Definition The *derivative* of a function f is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The function f' is defined for every x for which the limit exists. If f' is defined at x_0, f is called *differentiable* at x_0.

You should note the relationship between the derivative function and tangent lines and velocities. The slope of the tangent line at the point $(x_0, f(x_0))$ is the value of the derivative function at $x = x_0, f'(x_0)$. Likewise, if $s(t)$ represents distance traveled along a straight line, then the velocity at time $t = t_0$ is $s'(t_0)$.

Your calculus text will spend a great deal of time developing rules for computing the derivatives of various common functions. These are extremely important, but it is not our intention to reproduce all of this material here. We refer the student to any standard calculus text for this discussion.

We would like to point out at this time one of the truly impressive features of the HP-28S/48SX. Both of these calculators have the ability to compute derivatives of many common functions symbolically. There are several different contexts in which this can be done. We shall briefly discuss these here and refer the reader to his/her calculator manuals or to the excellent book by Wickes for a more complete treatment.

To compute the derivative of a function, simply enter the expression defining the function onto the stack. Then, put the variable with respect to which you are differentiating on line 1 of the stack (so that the function is now on line 2). Pressing the $\boxed{\texttt{d/dx}}$ key on the HP-28S or the $\boxed{\partial}$ key on the HP-48SX will return the symbolic derivative of most common functions to line 1 of the stack.

Example 2. Computing a Derivative Using the HP-28S/48SX

To find the derivative of $f(x) = x \sin(3x)$, first enter 'X' $\boxed{\text{PURGE}}$ and

' X * $\boxed{\text{SIN}}$ (3 * X) ' $\boxed{\text{ENTER}}$

Then put the variable on the stack. Press: ' X ' $\boxed{\text{ENTER}}$. Finally, pressing the derivative key ($\boxed{\texttt{d/dx}}$ or $\boxed{\partial}$) will return the derivative function to the stack:

'SIN(3 * X) + X * (COS(3 * X) * 3)'

■

EXCEPTION: If there is a value stored in a variable X (in general, if there is a value stored in whatever variable you are differentiating with respect to) in the current directory or any parent directory (i.e., any directory of which the current directory is a subdirectory; for example, the HOME directory is the parent directory of the TANG subdirectory), then the calculator will return the value of the derivative function at that point, instead of the symbolic derivative.

In experimenting with the differentiation routines of the calculators, you will find that they are programmed with the derivatives of polynomials, exponentials and logarithms and the trigonometric functions sine, cosine and tangent. The machine also "knows" some standard rules of differentiation, notably the product rule and the chain rule (as seen in Example 2). Thus, you can compute a wide variety of derivatives symbolically with your HP-28S/48SX.

The HP-28S/48SX also has a mode of operation in which you can watch the calculation of a derivative proceed step-by-step, following through as each product rule, chain rule, etc., is performed.

Example 3. Step-by-Step Computation of Derivatives

To find the derivative of $f(x) = x^2 \sin(3x)$, enter:

' ∂ X (X ∧ 2 * [SIN] (3 * X)) '

Note that the ∂ symbol is obtained on the HP-28S by pressing the [d/dx] key.

Pressing the [EVAL] key performs one step of the differentiation process. The successive steps here are:

' ∂X (X∧2) * SIN(3*X) + X∧2 * ∂X (SIN(3*X))'
(perform the product rule)

' ∂X (X) * 2 * X * SIN(3*X) + X∧2 * (COS(3*X) * ∂X (3*X))'
[compute the derivative of x^2 and use the chain rule to differentiate $\sin(3x)$]

' 2 * X * SIN(3*X) + X∧2 * (COS(3*X) * (3 * ∂X (X))'
[simplify the derivative in the first term and use the chain rule to differentiate (3x)]

' 2 * X * SIN(3*X) + X∧2 * (COS(3*X) * 3)'
(simplify the derivative in the second term)

Note that the calculation is completed when the last '∂X(X)' is evaluated. You can use this process to follow through the calculation of more complicated derivatives. ■

We wish to point out that, while this symbolic differentiation capability is quite a nice feature of these machines (and sets them apart from every other calculator on the market today), it is not their most significant feature, at least as far as *learning* calculus is concerned. The graphing and programmability of the machines are far more significant to us as regards the present chapter.

Finally, we would like to examine a method for getting improved approximations to the values of derivatives at a given point (i.e., that is improved over the computations done in section 3.1 and earlier in this section). You should wonder why we would want to approximate something which we can compute exactly (symbolically) either by hand or by using the differentiation routines of the HP-28S/48SX. The explanation is that in practice we have trouble computing derivatives symbolically. Very often, all we know about a function is a collection of data: measurements of the value of a function at various points. It may also happen that a function is simply too complicated to make symbolic computation of the derivative practical.

For example, if you wish to study the movement of a planet, you are not handed a formula giving its position at a given time. As with many real world problems, you make observations of its position at a number of specific times and then try to determine the velocity from these discrete observations.

We could get very involved in a discussion of numerical differentiation, but such a discussion is best left to a text in numerical analysis (see for instance, Conte and deBoor, Elementary Numerical Analysis , 3rd edition). Instead, we give one reasonably good way of computing derivatives numerically. Recall that in order to approximate the value of a derivative at a point,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

we have previously computed values of the difference quotient

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

for values of h (both positive and negative) close to zero. For $h > 0$, this is called a *forward difference*, and for $h < 0$, this is a *backward difference* .

An alternative approach is to approximate $f'(x_0)$ by the *centered difference*

$$\frac{f(x_0+h)-f(x_0-h)}{2h}$$

for small values of $h>0$. For reasons beyond the scope of this introductory discussion, it turns out that this centered difference is generally a better approximation to $f'(x_0)$ than either the forward or backward difference. We caution the reader that the centered differences are also highly subject to loss of significance errors as h gets close to zero.

Example 4. Centered Difference Approximation to a Derivative

Approximate the value of $f'(x)$ at $x = 2$, for $f(x) = x\sin(\pi x)$. First, note that you can easily modify your MSEC or VEL programs to compute centered differences [e.g., in MSEC by changing the F(X0) to F(X0−H) and then dividing by 2*H instead of H]. You will also need to enter new values for X0 and the function F in your current directory. We obtain the following table of values, where the DIFF column lists the values of the centered differences for the corresponding values of H.

H	DIFF
.1	6.1803398875
.01	6.28215181645
.001	6.2831749646
.0001	6.28318529665
.00001	6.28318599895
.000001	6.28318
.0000001	6.2832
.00000001	6.284
.000000001	6.28

We can compute the exact value of $f'(2)$, as follows. From the product rule and the chain rule,

$$f'(x) = \sin(\pi x) + \pi x\cos(\pi x)$$

Thus, $f'(2) = 2\pi = 6.28318530718$. Notice that this is very nearly the value computed in the table with H = .0001. The later values in the table get progressively worse, although for the values of H displayed, they tend to stay "in the ballpark" of the exact value. Contrast this with the table of forward and backward differences (the MSEC values for $h>0$ and $h<0$, respectively) computed for this function in Example 3 of section 3.1. You should note that the centered difference values are more accurate for each given value of H. This is generally the case and, hence, it is usually better to use a centered difference approximation to a derivative than to use either a forward or backward difference approximation. However, we can see that the centered differences are still affected by loss of significance errors, as we had expected. ∎

Unfortunately, there is no simple way of eliminating the loss of significance errors in these computations. Such a topic is found in a course in numerical analysis. Of course, for simple functions, you can always (and should) compute the derivatives exactly by hand (or using your HP-28S/48SX). We also remind the reader that one can use the graphics discussed in section 3.1 as a check on the reasonableness of a centered difference approximation to a derivative.

Exercises 3.2

In exercises 1-6, use VEL to estimate the instantaneous velocity at the given time.

1. $s(t) = \dfrac{2t^3}{t^2+1}$, $t_0 = 0$
2. $s(t) = \dfrac{2t^3}{t^2+1}$, $t_0 = 2$
3. $s(t) = \dfrac{2t^3}{t^2+1}$, $t_0 = 10$
4. $s(t) = \dfrac{t}{\sqrt{t^2+2}}$, $t_0 = 0$
5. $s(t) = \dfrac{t}{\sqrt{t^2+2}}$, $t_0 = 2$
6. $s(t) = \dfrac{t}{\sqrt{t^2+2}}$, $t_0 = 10$

In exercises 7-12, use the HP-28S/48SX to calculate the derivative symbolically.

7. $f(x) = x^2 \sin x$
8. $f(x) = x \sin x^2$
9. $f(x) = \dfrac{x}{x^2+2}$
10. $f(x) = \dfrac{1-x^2}{x^2-1}$
11. $f(x) = (x \sin x)^2$
12. $f(x) = \sqrt{x^2+4}$

In exercises 13-20, compare the centered difference, backward difference and forward difference at the given x_0 for h=.1, .01 and .001.

13. $f(x) = x \cos x$, $x_0 = 0$
14. $f(x) = x \cos x$, $x_0 = 2$
15. $f(x) = \sqrt{x^2+1}$, $x_0 = 0$
16. $f(x) = \sqrt{x^2+1}$, $x_0 = 2$
17. $f(x) = x^2 e^{-x^2}$, $x_0 = 0$
18. $f(x) = x^2 e^{-x^2}$, $x_0 = 1$
19. $f(x) = \dfrac{\cos x}{x^2+1}$, $x_0 = 0$
20. $f(x) = \dfrac{\cos x}{x^2+1}$, $x_0 = 1$

In exercises 21-22, compute centered differences with h=.1, .01 and .001 for the given function at $x_0 = 0$. In exercises 17-18 of section 3.1, you found that the derivative does not exist at $x_0 = 0$. What evidence does the centered difference give?

21. $f(x) = |x|$

22. $f(x) = (x^2)^{1/3}$

23. With your calculator in degrees mode, compute the derivative of $\sin x$. Explain why there is a factor of $\pi/180$.

24. In radians mode, get the calculator's derivative of $f(x) = \tan x$. Which is correct, the calculator's answer or $f'(x) = \sec^2 x$?

In exercises 25-28, use the given data to estimate $f'(0)$.

25.

x:	-1	$-.6$	$-.2$	0.2	0.6	1
$f(x)$:	10	3.5	0.3	0.5	4.0	10

26.

x:	-1	$-.6$	$-.2$	0.2	0.6	1
$f(x)$:	2	1	.25	$-.15$	$-.25$	0

27.

x:	$-.3$	$-.2$	$-.1$	0	0.1	0.2	0.3
$f(x)$:	$-.2$	$-.15$	$-.1$	0	0.1	.25	0.3

28.

x:	$-.3$	$-.2$	$-.1$	0	0.1	0.2	0.3
$f(x)$:	2.0	1.4	1.1	1	1.2	1.5	2.2

29. Suppose a car has an average speed of 60 mph in a 65-mph speed limit zone. As discussed in the text, the validity of the average speed depends on the length of the time inteval. For instance, a 60-mph average over 1 hour does not prove that the driver never exceeded the 65-mph speed limit. Suppose the car cannot speed up or slow down more than 1 mph in 1 second. For how long of a time interval does the 60-mph average speed guarantee that the driver did not break the speed limit?

EXPLORATORY EXERCISE

Introduction

There are several ways of describing the characteristics of the graph of a function. In calculus, we typically use the properties of increasing or decreasing and concave up or down. As you look from left to right, if the graph goes up the function is *increasing* (for example, $y = x$). If the graph goes down the function is *decreasing* (for example, $y = -x$). If the graph curves up (like $y = x^2$) it is *concave up.* If the graph curves down (like $y = -x^2$) it is *concave down.*

Problems

In this exercise, we will discover the relationship between the values of f' and f'' and the properties of the graph of f. Start by simultaneously graphing $f(x) = x^3 - 3x^2 + 1$ and $f'(x) = 3x^2 - 6x$. How does the sign ($+$ or $-$) of f' relate to the properties (increasing/ decreasing, concave up/down) of the graph of f? What do the zeros of f' correspond to? Repeat this for $f(x) = \sin x$ $[f'(x) = \cos x]$. Do any of your interpretations change for $f(x) = x^3$ $[f'(x) = 3x^2]$? Write down the relationship between f' and f in as much detail as possible. Now compare the graphs of $f(x) = x^3 - 3x^2 + 1$ and the second derivative $f''(x) = 6x - 6$. How does the sign ($+$ or $-$) of f'' relate to the graph of f? What does the zero of f'' correspond to? Repeat this for $f(x) = \sin x$ and $f(x) = x^3$. Write down the relationship between f'' and f in as much detail as possible.

Further Study

You have discovered the basic components of what are known as the First Derivative Test and Second Derivative Test, which you will see shortly in calculus.

3.3 Tangent Line Approximations

How does your calculator "know" that $\sin(1.2345678) = .944005695311$? Think about it. We all understand the processes of addition, subtraction, multiplication and division and are quite capable of performing even lengthy hand computations involving these operations (if we are really pressed to do so). Still, we very often use our calculators as a convenience. They save us the time and effort required for doing hand computations. This is not the case for calculation of values of the trigonometric functions, exponentials, logarithms and even for computing fractional powers of real numbers. For these computations, we usually use our calculators because we know of no other way of finding these values. But, how does the calculator do it?

In this section, we would like to take a first step (although a very small step) toward understanding how certain kinds of approximations are made. We want to make the point early on that the technique which we will develop here is not terribly accurate. To be perfectly honest, we can describe these approximations as crude, at best. In fact, the values computed with the built-in functions of your HP-28S/48SX (or those of any other scientific calculator for that matter) will almost always be far better than those which we will develop here.

Why, then, would we be interested in developing such an inferior method of approximation? There are in fact, several reasons. First, the methods used internally in calculators are too complicated to discuss at this point, while the method we develop here will serve as an introduction to such approximations. Second, the method introduced here will guide us to the approximate solution of the more complicated problems found in the next section, problems which *cannot* be solved by the mere push of a button.

For a given function f, suppose that we need to find the value of f at the point x_1, where $f(x_1)$ is unknown. For example, cos(.5) is unknown, although we could use a calculator to approximate it (at least the authors don't know the value without using their calculators). The basic idea here is to find a value of x *near* x_1, say $x = x_0$, such that we already know the value of $f(x_0)$ exactly. If f is differentiable at x_0, draw in the tangent line to the graph of $y = f(x)$ at $x = x_0$ (see Figure 3.8).

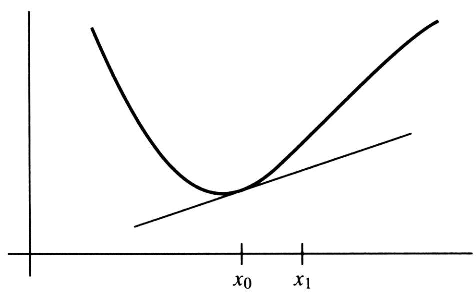


FIGURE 3.8

Keep in mind that the tangent line will "hug" the curve near the point of tangency (at $x = x_0$). This says that if x_0 is *close* to x_1, then the tangent line should still be close to the curve $y = f(x)$, at $x = x_1$. Examine Figure 3.8 to see that the y-values corresponding to $x = x_1$ on the curve and on the tangent line seem to be fairly close. To implement this idea, we need only find an expression for the tangent line. Since the slope of the tangent line is $f'(x_0)$ and the line passes through the point $(x_0, f(x_0))$, the equation is

$$\frac{y - f(x_0)}{x - x_0} = m_{\tan} = f'(x_0)$$

or

$$y = f(x_0) + f'(x_0)(x - x_0)$$

The y-coordinate of the point on the line corresponding to $x = x_1$ is then

$$y_1 = f(x_0) + f'(x_0)(x_1 - x_0)$$

To summarize, we are making the approximation

$$f(x_1) \approx f(x_0) + f'(x_0)(x_1 - x_0)$$

This is called the *tangent line* or *differential approximation* to $f(x_1)$.

Example 1. Tangent Line Approximation

Approximate the value of cos(.5). First, note that you can already get a highly accurate approximation from your HP-28S/48SX: $\cos(.5) = .87758256189$. (Before doing this, make certain that your calculator is set to *radians* mode.)

The first step is to find the number closest to $x = .5$ where we know the value of the cosine exactly. You should quickly realize that the closest such value is $x = \pi/6$. We know that $\cos(\pi/6) = \sqrt{3}/2$. Recall that for $f(x) = \cos(x)$, we have $f'(x) = -\sin(x)$. Thus, our approximation will be

$$\begin{aligned} \cos(.5) &\approx \cos(\pi/6) - \sin(\pi/6)(.5 - \pi/6) \\ &= \sqrt{3}/2 - .5(.5 - \pi/6) = .877824791584 \end{aligned}$$

Note that this is only a rough approximation to what we know to be the correct value, .87758256189. ∎

Once again, we emphasize that this method routinely produces only mediocre approximations. What the reader should gain from this exposition (since the method itself is, in the present context, of minimal value) is an appreciation of how the method was developed. This will also serve as an introduction to the problems of section 3.4, for which more direct methods may not be available.

Exercises 3.3

In exercises 1-12, compute the tangent line to $f(x)$ at $x = x_0$ and use it to approximate $f(x_1)$ for the given values of x_1. Compare to the exact values.

1. $f(x) = x^2$, $x_0 = 1$; $x_1 = -1, 0, 2, 3$
2. $f(x) = x^3$, $x_0 = 1$; $x_1 = -1, 0, 2, 3$
3. $f(x) = \sin x$, $x_0 = 0$; $x_1 = -\pi/4, -\pi/6, \pi/6, \pi/4$
4. $f(x) = \sin x^2$, $x_0 = 0$; $x_1 = -\sqrt{\pi/4}, -\sqrt{\pi/6}, \sqrt{\pi/6}, \sqrt{\pi/4}$
5. $f(x) = \sqrt{x+1}$, $x_0 = 0$; $x_1 = 1, 2, 3, 4$

6. $f(x) = \sqrt{x+1}$, $x_0 = 3$; x_1=1, 2, 4, 5
7. $f(x) = (x^2+1)^{1/3}$, $x_0 = 0$; x_1=.5, 1, 1.5, 2
8. $f(x) = (x^2+4)^{1/3}$, $x_0 = 2$; x_1=1, 1.5, 2.5, 3
9. $f(x) = \cos x$, $x_0 = 0$; x_1=$-\pi/4$, $-\pi/6$, $\pi/6$, $\pi/4$
10. $f(x) = \cos x$, $x_0 = \pi/2$; x_1=$\pi/4$, $\pi/3$, $2\pi/3$, π
11. $f(x) = e^x$, $x_0 = 0$; x_1=-2, -1, 1, 2
12. $f(x) = e^x$, $x_0 = 2$; x_1=0, 1, 3, 4

In exercises 13-18, find a tangent line approximation for the given value as was done in Example 1.

13. $\cos(2)$
14. $\cos(1.5)$
15. $\sqrt{4.2}$
16. $\sqrt[3]{8.4}$
17. $\tan(1)$
18. $\sin(.1)$

19. Suppose a person weighs P lb at sea level. At x ft above sea level, the person will weigh $W = \dfrac{PR^2}{(R+x)^2}$ lb, where $R \approx 21{,}120{,}000$ ft is the radius of the earth. Compute the tangent line approximation to W at $x_0 = 0$. In many applications, weight is considered to be constant. Why is this a reasonable assumption for applications near the surface of the earth?

EXPLORATORY EXERCISE

Introduction

We saw graphically that the tangent line gives a good approximation of a graph near the point of tangency. We can approach the idea of approximating a graph in a different way. Suppose we want a linear approximation of $f(x)$ near $x = x_0$. Since a line is determined by a point and the slope, the most we can demand is for the line to pass through $(x_0, f(x_0))$ and have slope $f'(x_0)$. The line with these properties is the tangent line $y = f'(x_0)(x - x_0) + f(x_0)$. We extend this idea below.

Problems

What is the best quadratic approximation of $f(x)$ near x_0? To simplify matters, take $x_0 = 0$. Quadratic functions have the form $Q(x) = ax^2 + bx + c$. Since there are 3 constants, we can make 3 demands. As above, we want $Q(x)$ to pass through $(0, f(0))$ and have slope $f'(0)$ at $x = 0$. Show that this means $Q(0) = f(0)$ and $Q'(0) = f'(0)$. Our last demand is $Q''(0) = f''(0)$ (graphically, this forces Q to curve in the same direction as f). Show that these requirements are satisfied with

$Q(x) = \frac{1}{2}f''(0)x^2 + f'(0)x + f(0)$. Repeat exercises 3, 5, 7 and 11 using $Q(x)$. Compare the accuracies of $Q(x)$ and the tangent line.

Further Study

$Q(x)$ is also known as the *second degree Taylor polynomial* of $f(x)$ centered at $x = 0$. By looking at higher-degree polynomials and demanding that higher-order derivatives match, you can derive higher-degree Taylor polynomials. We will study these in section 6.3.

3.4 Euler's Method

Many important phenomena in science and engineering are modeled by *differential equations.* In short, a differential equation is any equation involving the derivative or rate of change of an unknown function. For example,

$$y' - 2xy = x^2 \sin x$$

$$\frac{dy}{dx} + \cos y = (x^3 - 7x)\tan(\frac{x}{2})$$

$$2y(y')^3 - 3\cos x \sin y = 2x^2 - 3x + 7$$

are all differential equations, for the unknown function y as a function of x. Since these all involve only first derivatives, they are called *first-order equations.* The objective in solving these is to find a function y (a *solution*) which, when substituted into the equation, produces an identity (i.e., which satisfies the equation).

Example 1. Radioactive Decay

Physicists and chemists have long observed that radioactive substances decay at a rate directly proportional to the amount present. That is, if $Q(t)$ represents the amount of a certain radioactive substance present at time t, the rate of change of the amount with respect to time is

$$Q'(t) = kQ(t)$$

where k is the constant of proportionality (the *decay constant* which is known for any given radioactive substance). Note that, for any constant A, if

$$Q(t) = Ae^{kt}$$

then

$$Q'(t) = kAe^{kt} = kQ(t)$$

That is, $Q(t) = Ae^{kt}$ is a solution of the differential equation. In fact, since this function is a solution for every choice of the constant A, we have found infinitely many solutions, a different one for each value of A. An HP-48SX graph of a number of these is shown, for the case $k = -1$, in Figure 3.9.

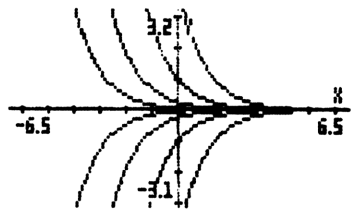


FIGURE 3.9

■

How are we to distinguish among this infinite number of solutions? If the question were left up to a 6 year old, he or she would select one by *pointing* to the one they wanted, i.e., by placing their finger on a specific point, say (x_0, y_0), of the desired curve. This is exactly what is done in practice. We specify that the solution curve we are looking for should pass through the point (x_0, y_0). That is, if the solution is $y = y(x)$, we require that

$$y(x_0) = y_0$$

This is called an *initial condition* since, in applications, x_0 often represents an initial time. A differential equation together with an initial condition is called an *initial value problem.*

For the solution of Example 1, if Q_0 is the amount of the radioactive substance present at time $t = 0$, we have the initial condition

$$Q(0) = Q_0$$

from which it follows that

$$Q_0 = Q(0) = Ae^0 = A$$

Thus, the solution of the initial value problem (i.e., the amount of the substance present at time t) is

$$Q(t) = Q_0e^{kt}$$

While we were able to guess the solution of the differential equation in Example 1, most problems are not so simple. There are many techniques available for solving various classes of first-order differential equations, but the study of these is best left to a concentrated course on differential equations. (There are many excellent references available; see, for example, Elementary Differential Equations and Boundary Value Problems, 4th edition, Boyce and DiPrima.) There are still many problems that are not easily solved directly. For such problems, we need to approximate the solution numerically. This is the purpose of the remainder of our discussion in this section.

Using the ideas developed in the last section, we can see how to approximate the solution of a first-order initial value problem. Consider the general case:

$$y' = f(x, y) \qquad\qquad y(x_0) = y_0$$

Suppose that the solution is $y = \phi(x)$. Then the differential equation gives us the slope of the tangent line to the graph of $y = \phi(x)$ at any given point (x, y) on the graph. In particular, from the initial condition, (x_0, y_0) is a point on the graph of the solution (you might think of this as the starting point). The slope of the tangent line at (x_0, y_0) is then given by the differential equation as $f(x_0, y_0)$. In Figure 3.10, you will see a typical solution curve, together with the tangent line at the initial point (x_0, y_0).

Note that if x_1 is close to x_0, then we might be able to use the tangent line to approximate the value of the solution at x_1.

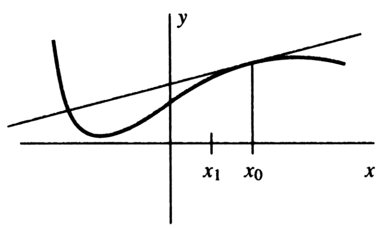


FIGURE 3.10

The equation of the tangent line is

$$m_{\tan} = \phi'(x_0) = f(x_0, y_0) = \frac{y - y_0}{x - x_0}$$

Like the tangent line approximation developed in section 3.3, if we follow the tangent line to the point corresponding to $x = x_1$, then the y-value at that point (call it y_1)

should approximate the value of the solution there. That is,

$$\phi(x_1) \approx y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

We now have an approximation to the value of the solution at some $x = x_1$. However, we cannot just carelessly use this formula to find approximate values of the solution at *any* point. Again, you can clearly see from Figure 3.10 that such an approximation should only be valid when x_1 is close to x_0. Fortunately, there is another way to think about this problem.

You should note that, unlike the tangent line approximations we sought in the last section, when we look for the solution of an initial value problem, we are looking for a function, or at least for the value of that function at a number of points. In practice, we seek the solution of such a problem on an interval $[a, b]$, where a is usually the initial value x_0. When we look for approximate solutions numerically, we usually look for approximate values of the solution function at a finite number of points. For the sake of simplicity, we choose equally spaced points, starting at $x_0 : x_1, x_2, x_3, \ldots,$ where

$$|x_{j+1} - x_j| = h \qquad\qquad j = 0, 1, 2, \ldots$$

and h is the *step size.* (See Figure 3.11 for an illustration of this partition of the interval.)

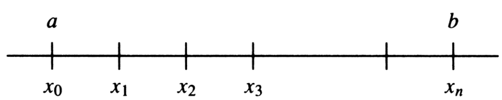


FIGURE 3.11

We use the tangent line approximation at x_1, i.e.,

$$\begin{aligned}\phi(x_1) \approx y_1 &= y_0 + f(x_0, y_0)(x_1 - x_0)\\ &= y_0 + hf(x_0, y_0)\end{aligned}$$

Next, we want to find an approximation to $\phi(x_2)$. Certainly, if we knew the equation of the tangent line at the point $(x_1, \phi(x_1))$, then we could proceed as above by following this line to the point corresponding to $x = x_2$. Of course, we don't even know the point $(x_1, \phi(x_1))$, although we do have an approximation for it. Notice that from the tangent line approximation,

$$\begin{aligned}\phi(x_2) &\approx \phi(x_1) + f(x_1, \phi(x_1))(x_2 - x_1)\\ &\approx y_1 + hf(x_1, y_1)\end{aligned}$$

where we have made the further approximation that the slope of the tangent line, $f(x_1, \phi(x_1))$, is close to $f(x_1, y_1)$. [This seems reasonable if y_1 is close to $\phi(x_1)$.] Continuing with this process, to get the approximation at the next point, we follow the approximate tangent line at the approximate point (x_2, y_2) to the point corresponding to $x = x_3$ and so on. We have the sequence of approximate values,

$$\phi(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n) \qquad\qquad n = 0, 1, 2, \ldots$$

This is called *Euler's method* for approximating the solution to an initial value problem. See Figure 3.12 for an illustration of an exact solution versus the approximate solution derived from Euler's method (the broken line graph represents the approximate solution).

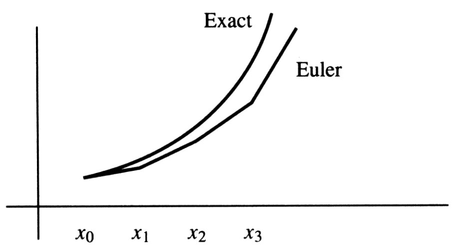


FIGURE 3.12

Example 2. Euler's Method for an Initial Value Problem

Consider the very simple initial value problem

$$y' = 2\,\frac{y}{x} \qquad\qquad y(1) = 2$$

First, we note that the exact solution of this problem is $y = \phi(x) = 2x^2$. [Verify this for yourself by differentiating $\phi(x)$ and plugging it into the equation and initial condition. We mention the exact solution here only so that we have some basis for comparison with our approximate solution.] If we did not know the solution (which is more commonly the case), we could try to approximate it using the Euler's method approximation given above.

Starting at the initial point (1,2), and using a step size of $h = .1$, we get

$$\begin{aligned}
\phi(1.1) \approx y_1 &= y_0 + hf(x_0, y_0) \\
&= y_0 + h\frac{2y_0}{x_0} = 2 + .1\frac{4}{1} = 2.4 \\
\phi(1.2) \approx y_2 &= y_1 + h\frac{2y_1}{x_1} = 2.4 + .1(4.8/1.1) \\
&= 2.4 + .436363636364 = 2.83636363636 \\
\phi(1.3) \approx y_3 &= y_2 + h\frac{2y_2}{x_2} = 3.30909090909
\end{aligned}$$

and so on. Notice that these calculations are very repetitive. The following program will help to compute further values. ■

PROGRAM TIP: Return to the HOME directory (press [QUIT] if you are in a subdirectory). We suggest that, as you have done before, you set up a separate subdirectory for these programs called DIFFQ. Enter 'DIFFQ' and press [CRDIR] . Enter the new subdirectory by pressing [DIFFQ] in the USER directory. As your first entry in the new directory, include the program ≪ [HOME] ≫ under the name QUIT to allow for easy return to the root directory.

We suggest the following program.

≪ [DUP] [C→R] → X Y 'X + H + (Y + H * F(X,Y)) * i' [→NUM] ≫

Program Step	Explanation
≪ [DUP]	Copy the object on line 1 of the stack.
[C→R] → X Y	Remove the x- and y-coordinates of the point on line 1 of the stack and store them in the local variables X and Y.
'X+H+(Y+H * F(X,Y)) * i' [→NUM]	Compute the new X and Y values and store them as the coordinates of a point which which is returned to line 1 of the stack.
≫ ENTER	End the program and enter it on the stack.
'EULER' STO	Store the program under the name EULER in the current directory.

Here, the "i" is the imaginary number $\sqrt{-1}$. Be careful to use the lower-case i. [On the HP-28S, press $\boxed{\text{LC}}$ (to turn on the lower-case letters) and enter i. Be sure to reset the keyboard to upper case by pressing $\boxed{\text{LC}}$ again. On the HP-48SX, press the leftshift key before pressing the I.]

In order to run this program, you'll need to store a value for the variable H in the current directory. In this case, press: .1 $\boxed{\text{ENTER}}$ ' H ' $\boxed{\text{STO}}$.

You will also need a program for the function F(X,Y). Again, for the present problem, use

≪→ X Y ' 2 * Y / X ' ≫ $\boxed{\text{ENTER}}$ ' F ' $\boxed{\text{STO}}$

Finally, the program requires you to place the initial point (x_0, y_0) on line 1 of the stack. Pressing the $\boxed{\text{EULER}}$ key will then compute the approximate solution at $x = x_1 = x_0 + h$ and return the point (x_1, y_1) to line 1 of the stack. Continuing to press $\boxed{\text{EULER}}$ will compute approximations at further points. Using $h = .1$, we constructed the following table. We have also listed the value of the exact solution, since this was available to us. This provides us with some comparison for determining the accuracy.

X	Approximate	Exact
1.1	2.4	2.42
1.2	2.83636363636	2.88
1.3	3.30909090909	3.38
1.4	3.81818181818	3.92
1.5	4.36363636363	4.5
1.6	4.94545454545	5.12
1.7	5.56363636363	5.78
1.8	6.21818181818	6.48
1.9	6.90909090909	7.22
2.0	7.63636363636	8.0

Note that the approximate solution given here leaves much to be desired. In particular, notice that the further x gets away from the initial value of 1.0, the worse the approximation tends to get. This is characteristic of Euler's method. While we can improve the results somewhat by taking smaller values of h, we cannot make h too small without facing the effects of loss of significance errors. Notice, too, that the smaller the value of h is, the larger the number of steps will be that are required to reach a given x-value. This is illustrated in the following table. Here, we compare the number of steps and the accuracy of the approximation at $x = 2.0$ for several values of h, again for Example 2.

H	Approximation	Error	Steps
.1	7.63636363636	.36363636364	10
.05	7.80952380952	.19047619048	20
.025	7.90243902425	.09756097575	40
.0125	7.95061728389	.04938271611	80
.00625	7.97515527948	.02484472052	160

Here, the error listed is the absolute value of the difference between the exact solution and the approximate solution at $x = 2.0$.

Notice that as the step size decreases and the amount of effort increases (i.e., the number of steps increases), the accuracy of our approximation improves, but not dramatically. (Try this for yourself and see how quickly you get tired). We could certainly write an automated program, but this will not correct the fundamental inefficiency of this method. The trouble with Euler's method is that it takes too small a step size and, hence, too many calculations to obtain a reasonable approximation. In the exercises, we will explore a related but somewhat more efficient method for approximating the solutions of differential equations. For the moment, we should be content that we've developed a method which provides some minimal accuracy for solving problems which have no other apparent means of solution.

Exercises 3.4

In exercises 1-4, show that $\phi(x)$ is a solution of the differential equation for any c.

1. $\phi(x) = cx^2, \quad xy' - 2y = 0$
2. $\phi(x) = ce^{2x}, \quad y' - 2y = 0$
3. $\phi(x) = x^2 + c, \quad y'' + y' = 2x + 2$
4. $\phi(x) = c\sin x, \quad y'' + y = 0$

In exercises 5-8, find the value of c such that ϕ satisfies the given initial condition.

5. $\phi(x) = cx^2, \quad y(1) = 3$
6. $\phi(x) = ce^{2x}, \quad y(0) = 5$
7. $\phi(x) = x^2 + c, \quad y(1) = 3$
8. $\phi(x) = c\sin x, \quad y(0) = 2$

In exercises 9-16, use EULER with h=.1 to estimate the solution at $x = 1$.

9. $y' = 2xy, \quad y(0) = 1$

10. $y' = \dfrac{x+1}{y}, \quad y(0) = 1$

11. $y' = 3x - y, \quad y(0) = 2$

12. $y' = y^2 - x, \quad y(0) = 2$

13. $y' = (x+y)^2, \quad y(0) = 1$

14. $y' = x^2 + y^2, \quad y(0) = 1$

15. $(x+1)y' = y + 3, \quad y(0) = 1$

16. $(x^2+2)y' = y^2, \quad y(0) = 1$

In exercises 17-20, repeat the given exercise with h=.05.

17. exercise 9

18. exercise 10

19. exercise 11

20. exercise 12

In exercises 21-24, use the exponential decay formula $Q(t) = Q_0 e^{kt}$.

21. A radioactive substance has decay rate $k = -.2\,\text{hour}^{-1}$. If 2 grams of the substance is present initially when will only half the original amount remain? NOTE: this is called the *half-life* of the substance.

22. In exercise 21, when will one-fourth the original amount remain? When will one-eighth remain? Explain what is meant by the statement "the half-life is independent of the initial amount."

23. If there are initially 2 grams of a substance present and the half-life is 1 hour, how much of the substance is left after 1 hour? 2 hours? 3 hours? 4 hours?

24. A common form of fossil dating is based on radioactive decay. Carbon-14 is found in living organisms and decays exponentially after death. The half-life is 5568 years. If a fossil is found to have 10% of its carbon-14 remaining, how old is the fossil? HINT: assume that there was 1 gram present at the time of death (t=0).

25. Write an automated program to do Euler's method. HINT: given h, determine how many steps are needed and loop (using the FOR NEXT structure) through Euler's method that many times.

26. In this exercise, we will use an improvement of Euler's method called (cleverly enough) the *improved Euler's method.* Using the same format as Euler's method, we will compute $y_{n+1} = y_n + hf_n$ where f_n is the slope of a line which approximates the actual slope of the solution. Instead of using $f_n = f(x_n, y_n)$ as in Euler's method, this time we will have f_n be the average of the slopes $f(x_n, y_n)$ and $f(x_{n+1}, y_{n+1})$. Why would you expect this to be more accurate than Euler's method? Explain why $f_n \approx .5[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$. We

then get the formula

$$y_{n+1} = y_n + \frac{f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))}{2} h$$

Rework exercise 9 using the improved Euler's method, and compare both approximations to the exact solution $y = e^{x^2}$.

EXPLORATORY EXERCISE

Introduction

Among all the detective movies made, nobody has ever filmed the following. A murder has been committed, the ace detective examines the scene of the crime, punches a few buttons on his calculator and announces the time of death. The HP-28S/48SX may not make it in Hollywood, but we will use it below to play detective. We will use Newton's Law of Cooling, which states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object and its environment. If $T(t)$ is the temperature at time t, then $T'(t) = k[T(t) - E]$ where E is the temperature of the environment and k is a constant.

Problems

At 6:00 we discover a secret agent bound (*chains bound*) and murdered. Next to him is a martini which got shaken before he could stir it. Room temperature is 70°, and the martini warms from 60° to 61° in the 2 minutes from 6:00 to 6:02. If the secret agent's martinis are always served at 40°, what is the time of death? There are two estimates to be made, both of which we can use EULER for. First set $T(0) = 60$ and (by trial-and-error) determine k such that $T(2) = 61$. Then set $T(0) = 40$ and determine t such that $T(t) = 60$. The time of death is t minutes before 6:00.

Further Study

In a course on differential equations, you will learn to find exact solutions to this and other problems arising from basic physical principles. Most calculus books also include a chapter on differential equations.

CHAPTER 4

Applications of Differentiation

4.1 Rootfinding Methods

Most students remember how much time was spent in their high school algebra class answering the question: for what values of x is

$$f(x) = ax^2 + bx + c = 0$$

The values (called *zeros* or *roots*) are, of course, found by using the now familiar quadratic formula. But, what if f is not a quadratic polynomial? You might well hope that either the question is irrelevant or that no one will ever ask you to solve such a problem. Unfortunately, the question *is* relevant and such questions need to be answered wherever calculus is applied, from engineering and the physical sciences to economics and business applications.

Example 1. A Hard Rootfinding Problem

In the vibration of an elastic string, under certain conditions, the natural frequencies (i.e., one of the characteristics that physicists use to describe the vibration) are solutions, x, of the equation

$$f(x) = \tan(x) - x = 0$$

Notice that this is not an algebraic problem. That is, we cannot use the usual rules of algebra to solve for the value(s) of x. However, you can see from a simple graph (we'll do this later) that there are indeed values of x which satisfy the equation (in fact, there are infinitely many of them). But how can we find these values? In short, we can't, at least not exactly. The best that we can do in this case is to find approximate solutions. ∎

So, what we have is a compelling problem with no apparent means of solution. In this section, we will discuss several simple methods which can be employed with a programmable calculator and show how to use the graphics and programming features of the HP-28S/48SX to implement these methods effectively.

Recall that we had briefly discussed this problem in Chapter 1, when we showed how to use the Solver to find the x-intercepts of functions whose graphs we were looking at. Well, what else is there to discuss then? There's no denying that the built-in Solver is quite capable. It can be used to quickly and accurately solve a wide variety of rootfinding problems. But, that is not enough. The Solver is a *black box*, where the inner workings are unknown to the user. While this is fine if our only interest is to find solutions to problems, the whole thrust of our discussion here is to find *understanding*. Further, the Solver does *not* always work and the advisory message is not always to be trusted.

Example 2. A Faulty Answer from the Solver

Use the Solver to try to find a root of $f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2$. First, draw a graph to see that there seems to be a root near $x = 1.1$. So, we use the Solver with the initial guess 1.1. (Store the function and enter the Solver. Enter 1.1 and press the soft key $\boxed{\text{X}}$ followed by $\boxed{\text{RED}}$ $\boxed{\text{X}}$ on the HP-28S, or followed by $\boxed{\hookleftarrow}$ $\boxed{\text{X}}$ on the HP- 48SX.) The Solver returns the approximate root 1.00013574976, with the advisory message "ZERO." This indicates that the Solver thinks that it has found a root exactly. This is not particularly surprising until we observe that $f(x)$ factors:

$$f(x) = x^4 - 5x^3 + 9x^2 - 7x + 2 = (x - 1)^3(x - 2)$$

Thus, the only two roots are clearly $x = 1$ and $x = 2$. While $x = 1.00013574976$ may not seem a poor approximation of 1.0 if you are using this to aim the throw of a baseball from center field to home plate, this is not particularly precise if you are instead aiming a spacecraft at the moon. This stands in sharp contrast to the confident advisory message "ZERO" displayed by the Solver. The moral of the story, of course, is that you should develop a healthy amount of skepticism for the results of the Solver (for finding roots or extrema, at least) or for the results of any other "black box" method, where the technique and the intermediate calculations are unknown. ■

In this chapter, we will examine how roots are found. We'll discuss the various strengths and weaknesses of the methods we develop and also look at problems whose roots are not easily found and where the Solver as well as our other methods can get fooled. Finally, we'll see how to follow the output of a rootfinding method to see when it is going astray.

THE METHOD OF BISECTIONS

Recall that the graph of a continuous function can be drawn without lifting the pencil from the paper. The following (a consequence of the Intermediate Value Theorem) should then be fairly evident.

Theorem 4.1 Suppose that f is a continuous function on the interval $[a, b]$, and that $f(a) \cdot f(b) < 0$. [i.e., $f(a)$ and $f(b)$ have opposite signs]. Then, for some number c in (a, b), $f(c) = 0$.

This simple result is the basis for the most elementary method of numerical rootfinding. Given that f has opposite signs at a and b, we might guess that a root could be halfway in between a and b, i.e., at

$$c = \frac{1}{2}(a + b)$$

If not, then a root must be in at least one of the intervals (a, c) or (c, b). To check if a root might be in (a, c), compare $f(a)$ and $f(c)$ for a change of sign. If there's no change, then there must be a root in (b, c). (Note that we say "a" root, as opposed to "the" root, since there may well be more than one root in the interval.) We then proceed to look at the value of f in the middle of the new interval and so on. The following algorithm, called the *Method of Bisections*, is thus generated.

Step	Explanation
1. Check that $f(a) \cdot f(b) < 0$.	Look for a sign change to make certain there's a root in $[a, b]$.
2. Let $c = \frac{1}{2}(a+b)$.	Find the midpoint of $[a, b]$.
3. If $f(c) = 0$, stop.	Stop if you find a root.
4. If $f(a) \cdot f(c) < 0$ replace b by c and go to step 2.	Check $[a, c]$ for a sign change. If there is one, look for a root in the interval $[a, c]$.
5. Otherwise, replace a by c and go to step 2.	If there's no sign change in $[a, c]$ then there's a a root in $[c, b]$. Look for a root there.

Of course, in practice, the stopping condition, $f(c) = 0$, is only rarely realized. Usually, c is considered to be an acceptable approximation if $f(c)$ is "small enough" in absolute value (how small is small enough is determined by the need for accuracy in the particular problem). Step 3 in the algorithm is then replaced by:

3a. If $|f(c)| <$TOL, stop

where TOL is some acceptable tolerance. Alternatively, we may be satisfied in knowing that a root is in the interval (a, b), where $(b - a)$ is sufficiently small. In this latter case, we replace step 3 with the step

3b. If $(b - a) <$TOL , stop

where, again, TOL is some acceptable tolerance.

We shall first illustrate the Bisections algorithm with an example done by hand and then proceed to an HP-28S/48SX program for the method.

Example 3. The Method of Bisections - Step by Step

Find a root of $f(x) = x^5 - 5x^3 + 3$ in the interval [0,1]. First note that $f(0) = 3$ and $f(1) = -1$. Since f is continuous (*all* polynomials are continuous!) and f has opposite signs at $x = 0$ and $x = 1$, then, by Theorem 4.1, there must be a root in the interval (0,1). For this particular problem, we'll agree to accept a solution accurate to 2 decimal places (i.e., we'll stop computing new values when $|b - c| < .01$).

Set $c = \frac{1}{2}(a+b) = .5$; $f(.5) = 2.4 > 0$. Since $f(1) < 0$, this means that there's a root between .5 and 1.

Set $c = \frac{1}{2}(.5+1) = .75; f(.75) = 1.1 > 0$. (There's a root between .75 and 1.)

Set $c = \frac{1}{2}(.75+1) = .875$; $f(.875) = .16 > 0$. (A root is between .875 and 1.)

Set $c = \frac{1}{2}(.875+1) = .9375$; $f(.9375) = -.4 < 0$. (A root is between .875 and .9375.)

Set $c = \frac{1}{2}(.875+.9375) = .90625$; $f(.90625) = -.11 < 0$.

Set $c = \frac{1}{2}(.875+.90625) = .890625$; $f(.890625) = .03 > 0$.

Note that although $f(.890625)$ is not 0, there must be a root in the interval (.890625,.90625). Thus, if we only need 2-digit accuracy, we can declare that the midpoint of that interval,

$$c = \frac{1}{2}(.890625 + .90625) = .8984375$$

is an approximate root, since no number in the interval (.890625,.90625) can be more than

$$b - c = .90625 - .8984375 = .008$$

away from the center, c. (Why is that?) ∎

If greater accuracy is desired, then one must simply continue this process further. The student will no doubt agree that this is a tedious procedure, at least when performed by hand. But, who needs to do computations by hand, or even manually, when we have a powerful machine like the HP-28S/48SX at our disposal? We suggest the following HP-28S/48SX program to perform the Bisections algorithm.

PROGRAM TIP: We recommend that you create a subdirectory of the main User directory (or the Vars directory on the HP-48SX) called ROOTS and store the various rootfinding programs in this subdirectory, just as you stored your graphics programs in the subdirectory PLOTR in Chapter 1 (if you are using the HP-28S). You can do this by entering 'ROOTS' `CRDIR` while in the Home directory. This will help to keep your Home directory from getting too cluttered. To enter the ROOTS directory, simply press the `ROOTS` key in the User (or Vars) directory. To exit the subdirectory, type HOME, or include the program: ≪ HOME≫ under the name QUIT in the subdirectory. Pressing the [QUIT] key will then return you to the HOME directory.

First, define and store the function F whose roots are being sought by typing

≪→ X ' expression ' ≫ ' F ' [STO]

This may give you a new entry, F, in your current directory. When the [F] soft key is pressed, the value on line 1 of the stack is read and the value of the function F at that point is computed and returned to the stack.

Now, enter the Bisections program:

≪→ A B ≪ '.5*(A+B)' [EVAL] [DUP] → C
≪ [IF] 'F(A)*F(C)' [EVAL] 0 < [THEN] A C [ELSE] C B [END] ≫≫≫

Program Step	Explanation
≪→ A B	Store the values on lines 1 and 2 of the stack in local variables A and B.
≪ '.5*(A+B)' [EVAL]	Compute the midpoint of A and B.
[DUP] → C	Duplicate the value on line 1 of the stack and store one copy of this in the local variable C.
≪ [IF] 'F(A)*F(C)' [EVAL] 0 <	Test for a sign change between A and C.
[THEN] A C	If there is a sign change on [A,C] place A and C on the stack.
[ELSE] C B	If there was no sign change on [A,C] place C and B on the stack.
[END] ≫≫≫	End the conditional IF statement and the program.
'BIS' STO	Store the program in the current directory under the name BIS.

To run the above program, place appropriate values for A and B on the stack, with A on line 2 and B on line 1. [Recall that the method requires that $A < B$

and $f(A) \cdot f(B) < 0$.] Pressing the $\boxed{\texttt{BIS}}$ key in the current directory will cause the program to execute. The program will return the value of C to line 3 of the stack. On lines 1 and 2, you will find the endpoints of an interval containing a root.

At this point, you should run the program given above for the example already done by hand. The following are the keystrokes needed to run the preceding program for this example.

$$\ll \rightarrow \text{X}\ '\text{X} \wedge 5 - 5 * \text{X} \wedge 3 + 3\text{'} \gg \text{'F'}\ \boxed{\texttt{STO}}$$

Enter the endpoints of an interval containing a root on the stack. For the present problem, such an interval is [0,1]. Press 0 $\boxed{\texttt{ENTER}}$ 1 $\boxed{\texttt{ENTER}}$. Press $\boxed{\texttt{BIS}}$ to execute the program.

Executing $\boxed{\texttt{BIS}}$ once returns the values .5, .5 and 1 to the first three lines of the stack. The last two values correspond to the endpoints of the current interval known to contain a root. Executing $\boxed{\texttt{BIS}}$ repeatedly will produce more and more refined intervals containing a root. Repeating the program 6 more times yields the interval [.890625,.8984375]. The width of the interval is .0078125. Thus, if we are looking for an approximation to a root valid to 2 decimal places, the midpoint of this interval, .89453125, will suffice. If more accuracy is needed, we can continue to execute $\boxed{\texttt{BIS}}$ until we obtain an interval whose width is less than the specified tolerance. (Just subtract the last 2 values on the stack to check. If the interval is not sufficiently small, you can return the endpoints of the last interval to the stack by pressing $\boxed{\texttt{UNDO}}$ on the HP-28S or by pressing $\boxed{\hookleftarrow}$ $\boxed{\texttt{STACK}}$ on the HP-48SX.) The midpoint of this last interval will serve as the approximate root: press + 2 /. Actually, an interval whose width is twice the tolerance will do. (Why?)

If you continue pressing $\boxed{\texttt{BIS}}$ for the above problem, you will arrive at the approximate root .89382746211, valid to the limits of the machine's accuracy. Pay particular attention to the values of C displayed on line 3 of the stack. Watching these values, you can see how the method of Bisections homes in on a root. If you've been counting, it should have taken 39 applications of $\boxed{\texttt{BIS}}$ (and perhaps a minute of your time) to arrive at this value. This does not seem too bad. After all, we are obtaining an approximate root valid to about 12 decimal places. Nonetheless, this is one of the most significant drawbacks to using Bisections in practice. Relative to other methods (several of which we will discuss later in this chapter), it is very slow. Of further concern is that Bisections can only be used when we can find values A and B between which f changes sign. Unfortunately, this cannot always be done.

Example 4. A Problem to Which Bisections Cannot Be Applied

Consider $f(x) = x^2$. Note that $f(x) > 0$, except for $x = 0$, and, hence, one could not use Bisections to locate the root $(x = 0)$. ■

NEWTON'S METHOD

The discerning reader will undoubtedly wonder why another rootfinding method is necessary. First, as noted above, Bisections is a relatively slow method and does not utilize the power of calculus. More generally, good problem solvers need an assortment of mathematical tools, just as carpenters need a variety of tools to perform their jobs.

A generally faster method for approximating roots of a function, called *Newton's method*, works in the following way. Using some graphical or numerical evidence, make a guess as to where a root is located. Specifically, guess that an approximate root of a given function f is located at some value x_0. Assuming that f is differentiable at x_0, we can draw in the tangent line to $y = f(x)$ at $x = x_0$ (see Figure 4.1).

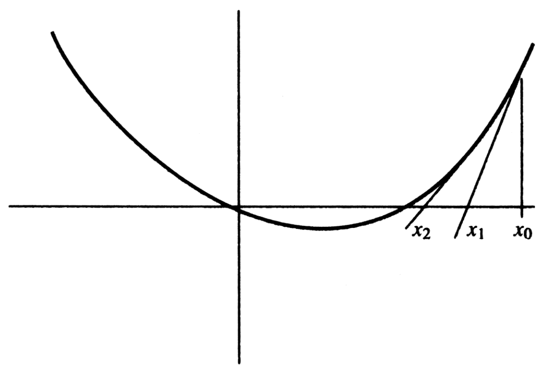


FIGURE 4.1

The slope of the tangent line is $f'(x_0)$ and hence its equation is

$$m_{\tan} = f'(x_0) = \frac{y - f(x_0)}{x - x_0}$$

Now, follow the tangent line to the point where it intersects the x-axis. In the figure, this appears to be closer to the root than x_0. Call the x-coordinate of this point x_1. Since this is the point of intersection with the x-axis, this corresponds to the point on the line where $y = 0$. From the above equation for $f'(x_0)$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Rather than be satisfied with this modest improvement over our original guess, repeat the above procedure, replacing x_0 with x_1, to obtain a new (and, hopefully, further improved) approximation x_2. Continue this process until no further progress is made by taking successive steps. This generates a sequence of approximations,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots$$

which is the general form of Newton's method. Let us return to Example 3 and compare the performance of Bisections with this new method.

Example 5. Newton's Method

Find a root of $f(x) = x^5 - 5x^3 + 3$, using Newton's method with an initial guess of $x_0 = 1$.

First, notice that $f'(x) = 5x^4 - 15x^2$. Newton's method then becomes

$$x_{n+1} = x_n - \frac{x_n^5 - 5x_n^3 + 3}{5x_n^4 - 15x_n^2}$$

Using $x_0 = 1$, we obtain the following table. [Be sure to check these results yourself. You can do this easily by entering the function $x - f(x)/f'(x)$ into the Solver.]

n	x_n	$f(x_n)$
1	.9	−.05451
2	.893854219516	−.00023526873
3	.893827462609	−.00000000448
4	.893827462099	0
5	.893827462099	0

Notice that the last two steps are identical, at least in the 12 decimal places displayed (about the most precision one can expect from the HP-28S/48SX). Since the value of the function is also reported as 0, you should begin to suspect that we have obtained a good approximation to a root. You should also notice that Newton's method produced an approximation in fewer steps and with less effort than the method of Bisections. Finally, we should observe that a good time to stop computing new approximations would seem to be when two successive approximations are sufficiently close, (i.e., $|x_{n+1} - x_n| <$TOL for some acceptable tolerance TOL) as is the case for x_4 and x_5 in the table. ■

CAUTION: In practice, even a reported function value close to zero does not guarantee that the approximation is close to an actual root. We will examine this further in the exercises and in section 4.2.

We now give an HP-28S/48SX program to perform Newton's method (before entering this, be sure that you are in your ROOTS subdirectory, which was discussed with the material on Bisections).

First, define the function F whose roots are being sought:

$$\ll\rightarrow \text{X ' expression '} \gg \text{' F '} \boxed{\texttt{STO}}$$

Second, define the derivative function, DF.

$$\ll\rightarrow \text{X ' expression '} \gg \text{' DF '} \boxed{\texttt{STO}}$$

Now enter the Newton's method program:

$$\ll \boxed{\texttt{DUP}} \rightarrow \text{X 'X-F(X)/DF(X)'} \gg$$

Program Step	Explanation
$\ll \boxed{\texttt{DUP}} \rightarrow$ X	Store the value on line 1 of the stack in the local variable X.
' X – F (X) / DF (X) '	Compute the Newton step and place the value on line 1 of the stack.
≫ ENTER ' NEWT ' STO	End the program. Store the program in the current directory under the name NEWT.

Pressing the $\boxed{\texttt{NEWT}}$ key in the current directory with the initial guess on line 1 of the stack will execute the program. Each time $\boxed{\texttt{NEWT}}$ is pressed, the next Newton step (x_{n+1}) is computed and placed on line 1 of the stack.

PROGRAM NOTE: Although the HP-28S/48SX has the facility to compute derivatives of most common functions, it is inefficient to have the machine do so in each step of a Newton's method procedure. Therefore, we have instead instructed the user to store the derivative in a user-defined function, DF. Of course, you could always use the HP-28S/48SX to find the derivative and then store that derivative in a program of the form given above.

Now, key in the program and test it on the example given above. The following keystrokes will execute the program for the preceding example.

$$\ll\rightarrow \text{X ' X} \wedge 5 - 5 * \text{X} \wedge 3 + 3\text{ '} \gg \text{' F '}\ \boxed{\texttt{STO}}$$

$$\ll\rightarrow \text{X ' }5 * \text{X} \wedge 4 - 15 * \text{X} \wedge 2\text{ '} \gg \text{' DF '}\ \boxed{\texttt{STO}}$$

$$1\ \boxed{\texttt{ENTER}}$$

$$\boxed{\texttt{NEWT}}$$

The successive approximations are displayed on the stack each time the program is executed. Continue until the last two displayed values are within the desired tolerance. Pay particular attention to how the approximation improves at each step of the process and compare this with the behavior of the Bisections method.

CAUTION: Since Newton's method is not guaranteed to work, it's a good idea to further test the validity of the results by computing the value of the function F at the suspected approximate root. This is accomplished by simply pressing the $\boxed{\text{F}}$ softkey in the current directory.

Notice that even for a very small tolerance, say TOL = .00000000001, the Newton's method program for this problem takes only a few steps. Compare this with the number of steps required for the Bisections method for this problem. This is one of the main advantages of Newton's method. It will usually take many fewer steps than the method of Bisections to achieve the same tolerance. For complicated problems, there can be a substantial difference between the two methods. Thus, Newton's method is typically favored over the method of Bisections. In automated programs (i.e., ones which will run automatically until a specified tolerance is reached), Bisections programs typically take much longer to run than programs for Newton's method. (Can you think of a time when Bisections would be the method of choice? Hint: Newton's method requires one to compute a derivative.)

Now that we've praised the benefits of Newton' method, we want to be perfectly honest with you. Newton's method does have a negative side. It does not always work. In practice, the initial guess must be chosen fairly close to a root in order that the successive approximations home in on a root. Just how close to the root it must be chosen, though, varies from problem to problem. The answer to this question would be of little use anyway, since, in practice, we do not know where the root is. (If we did, then we wouldn't be employing an approximation method like Newton's method!)

Example 6. An Initial Guess that Does Not Work

Consider the function

$$f(x) = x^3 - 3x^2 + x - 1$$

Notice that there is a root somewhere in the interval (1,3). (Why is that?) Using Newton's method with initial guess $x_0 = 1$, we get $x_1 = 0$ and $x_2 = 1$ and so on. The values alternate back and forth between 0 and 1, neither one of which is a root. Try the example for yourself. If we instead start with a slightly better initial guess, say $x_0 = 2$, Newton's method will converge quickly to the value 2.7692923542.

■

There are other bad things that can happen with Newton's method. In the next example, the successive values will wander away from the only root and tend toward minus infinity.

Example 7. Newton's Method Wanders Away from the Root

Consider the function

$$f(x) = \frac{(x-1)^2}{x^2+1}$$

with $x_0 = -2$. Obviously, f has only one root, at $x = 1$. Using our Newton's method program, we get the results:

n	x_n
1	−9.5
2	−65.9
3	−2302
4	−26541301
5	−3.5 E12
6	−6.2 E24

Notice that the last two values are very large in absolute value and are getting rather close to the outer reaches of the accuracy of the HP-28S/48SX. You should learn quickly to be skeptical of any reported approximate root that is so large in absolute value.

If we use an improved initial guess of $x_0 = -1$, the program immediately returns an "infinite result" error, caused by an attempted division by 0. (Why?) Finally, with an even better initial guess of $x_0 = 0$, Newton's method will converge to the root, but uncharacteristically slowly. (Try this. We'll look further into this type of behavior later.) ∎

On the whole, Newton's method should be viewed as a very useful and accurate method for finding approximate roots, when used with a bit of caution. In the exercises, we will demonstrate more cases where the method fails to yield an acceptable answer, as well as a number of typical examples where things work just fine.

USE OF GRAPHICS FOR DETERMINING INITIAL GUESSES

There is one important question which we have so far avoided: How does one come up with the initial guess(es) needed for either the method of Bisections or for Newton's method? One suggestion might be to randomly guess. However, we have already seen that the method of Bisections requires good input to obtain good answers in a reasonable length of time and that Newton's method may require a good initial guess just to work at all. We therefore require a more sensible approach to finding initial guesses. The graphics capabilities of modern graphing calculators and of the HP-28S/48SX, in particular, are well suited for this purpose.

Example 8. Using Graphics to Find Initial Guesses

Find the roots of $f(x) = x^4 - 2x^3 + x^2 + 4x - 6$. To get an idea of where the root(s) may be located, we may use the HP-28S PLOTR routines developed in Chapter 1 or use the built-in PLOT routines on the HP-48SX. In either case, these plotting functions are exceptionally easy to use. First, enter the equation to be plotted onto the stack:

$$\text{'} X \wedge 4 - 2 * X \wedge 3 + X \wedge 2 + 4 * X - 6 \text{'} \ \boxed{\texttt{ENTER}}$$

For the HP-28S, enter the PLOTR subdirectory and press $\boxed{\texttt{RESET}}$ to reset the parameters for the window. Press $\boxed{\texttt{NEWF}}$ (This stores the function under the name EQ in the current directory and then draws a plot using the default window parameters.) If you have followed the instructions for this example, what you see displayed are a few dots and the x- and y-axes (see Figure 4.2). This is typical of HP-28S displays. Because of the size limitations of the display, one very often will only see a few scattered dots, with the rest of the curve left to the imagination.

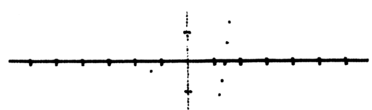


FIGURE 4.2

With some care, you can learn how to use these displays to obtain initial guesses for our rootfinding schemes.

To plot the graph with the HP-48SX, enter the Plot menu and press $\boxed{\texttt{NEW}}$. (The machine will prompt you for a name and then save the function under this name as well as under the name EQ in the current directory.) Next, press $\boxed{\texttt{PLOTR}}$ to enter the Plotr directory. Press $\boxed{\texttt{RESET}}$ (found on the second page of the directory menu) to clear the graphics display of the last graph and to reset the graphics window parameters. Finally, pressing $\boxed{\texttt{DRAW}}$ will produce a graph using the default window parameters (see Figure 4.3). Because of the larger size and higher resolution of the HP-48SX display, we can see considerably more detail than in the corresponding HP-28S graph.

Returning to the plot that has been generated by the HP-28S, you should be

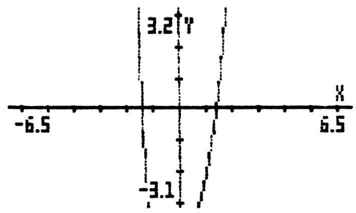


FIGURE 4.3

able to see that there appears to be a root between 1 and 2. (You can see this even better in the HP-48SX graph.) We could certainly return to the Bisections program with $a = 1$ and $b = 2$, or, for that matter, use a guess of $x_0 = 1$ in our Newton's method program. However, recall that the HP-28S/48SX can easily digitize points in a plot. The cursor (now located at the origin in the plot) can be moved about the display using the 4 arrow keys (located in the top row of the keyboard on the HP-28S). Moving the cursor to what looks to be the location of a root, we can digitize that point (i.e., return its coordinates to line 1 of the stack) by pressing the INS key on the HP-28S or ENTER on the HP-48SX. Remove the graph by pressing the ON key. You will see the coordinates of the point that you digitized on the stack. In our case, we obtained (1.4,0).

Now, use the x-coordinate of the digitized point as the initial guess for Newton's method, i.e., let $x_0 = 1.4$. Newton's method quickly produces the approximate root $x = 1.41421356237$, where $f(x) = -.00000000003$. Of course, we could also use the Bisections algorithm although, in practice, this is somewhat slower. Finally, return to the plot to see if there are any other roots which we might find. (Press GRAPH to return the last plot to the screen.) Notice that in the HP-28S plot (Figure 4.2) there is one point drawn close to the x-axis and to the left of $x = -1$. Digitize this point (again by moving the cursor using the arrow keys and pressing INS). Returning to the Plot menu, we find the point $(-1.4, -.3)$ displayed. (In the corresponding HP-48SX plot, we can clearly see the existence of a root near $x = -1.4$.) Using $x_0 = -1.4$ as our initial guess in Newton's method, we quickly obtain the approximate root $x = -1.41421356237$, where $f(x) = -.00000000008$. (Do these roots look familiar?) ■

REMARK: There are special difficulties which users of the HP-28S will encounter due to the small size and relatively low resolution of the screen. These can all be overcome by adjusting the size of the graphics window using the programs in the PLOTR subdirectory discussed in Chapter 1. The following example will exhibit some of the difficulties. Users of the HP-48SX are encouraged to work through the example simply as practice in rootfinding, even though there are no particular problems with their display of the graph.

Example 9. Special Problems with HP-28S Graphics

Consider $f(x) = x^6 - 3x^4 + x^2 - 3$. Using the NEWF routine, after resetting the graphics parameters, the HP-28S will plot 2 dots (see Figure 4.4a). (The HP-48SX does considerably better – see Figure 4.4b.) Where is the rest of the graph? As with the last example, the size limitations of the display are such that only a piece of the graph will be plotted; in this case, 2 points. We can improve this situation somewhat if we use the graphing utilities discussed in Chapter 1.

FIGURE 4.4a

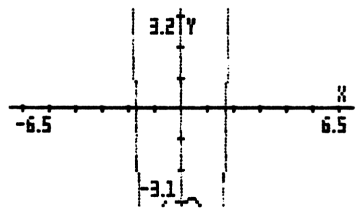


FIGURE 4.4b

By using the $\boxed{\texttt{ZBOX}}$ command, we can zoom in on the behavior of the function near either of the displayed points. Start by moving the cursor to a point just slightly to the left and below of one of the displayed points, using the arrow keys. Digitize this point by pressing the $\boxed{\texttt{INS}}$ key. Then, move the cursor to a point just to the right of the point and just above the x-axis (above the axis, since we're looking for roots, ultimately). Digitize this point and then return to the Plotr menu. (Press the $\boxed{\texttt{ON}}$ key.) The digitized points are displayed on the stack. In our case, we have $(1.3, -1.3)$ and $(2.2, .5)$. Press $\boxed{\texttt{ZBOX}}$ to zoom in on the graphics window with the digitized points as corners. The displayed graph (Figure 4.5) is still quite incomplete, but shows a sequence of points, with several on either side of the x-axis, indicating that there must be a root somewhere nearby, by the Intermediate Value Theorem.

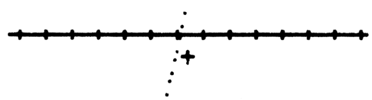


FIGURE 4.5

NOTE: Although the graph would seem to indicate that a root is located near $x = 1.735$, we should not accept this as an approximate root. [Simply compute $f(1.735)$ to see why this is unacceptable.] However, if we use this as the initial guess for Newton's method, we get the approximate root $x = 1.73205080757$, where $f(x) = 0$.

Having found one root, it still remains for you to see if there are any other roots. We will turn our attention to this in section 4.2. For the moment, we shall be content with locating a single root. ∎

There are other problems in locating roots of functions caused by the limitations of the HP-28S display. The graphics utilities discussed in Chapter 1 are sufficient to deal with almost any situation which will arise in searching for roots.

THE SECANT METHOD

We have seen that Newton's method can be a very useful tool for approximating roots of functions. We have also pointed out several significant limitations. First, the initial guess, x_0, must be chosen sufficiently close to the root (and we never know when a given guess is sufficiently close). Second, the method requires us to compute a derivative (whether it is done by hand or using the symbolic differentiation routines of the HP-28S/48SX). The latter requirement can be quite restrictive: the function may not be differentiable or the derivative computation may be prohibitively complicated. If this is the case, we could always use Bisections. However, Bisections is exceedingly slow and can be used only if we can find numbers A and B for which $f(A)$ and $f(B)$ have opposite signs. We present now a method which has most of the advantages of Newton's method, but does not require us to compute a derivative.

Given two initial guesses, x_0 and x_1 (not necessarily bracketing a root), draw the secant line joining the two points. The slope of this line is

$$m_{\text{sec}} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The equation of this secant line is then

$$\frac{y - f(x_1)}{x - x_1} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Much as with Newton's method, we follow the line to where it crosses the x-axis (i.e., where $y = 0$; see Figure 4.6). Call the x-coordinate of this new point x_2. We get

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

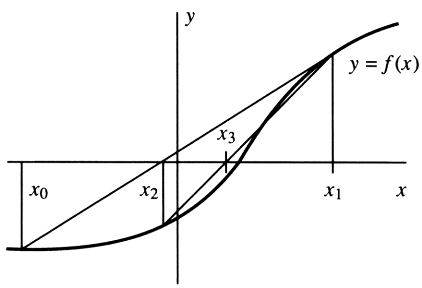


FIGURE 4.6

In Figure 4.6, x_2 appears to be closer to the root than either x_0 or x_1. We can repeat the procedure over and over, each time using the latest two values to compute an improved approximation. We get:

$$x_{n+2} = x_{n+1} - f(x_{n+1})\frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)} \qquad n = 0, 1, 2, \ldots$$

This is known as the *Secant method*. Note the similarity with Newton's method. In this case, we start with two initial guesses and then approximate the slope of the tangent line at x_{n+1} by the slope of the secant line joining the points corresponding to x_n and x_{n+1},

$$f'(x_{n+1}) \approx \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

In practice, the Secant method will converge almost as fast as Newton's method. Its main advantage over Newton's method is that it does not require the computation of a derivative. For some problems, this is a decisive advantage. We now return to an earlier example, to compare our new method with Bisections and with Newton's method.

Example 10. Secant Method

Find a root of $f(x) = x^5 - 5x^3 + 3$, using the Secant method with initial guesses $x_0 = 1$ and $x_1 = 0$.

Using the HP-28S/48SX, we obtain the following results (be certain to check these yourself):

n	x_n	$f(x_n)$
2	0.75	1.13
3	1.21	−3.17
4	0.8685	0.2185
5	0.88999	0.0336
6	0.89389907	−0.00063
7	0.893827265517	0.0000017
8	0.89382746209	0.00000000009
9	0.8938274621	0.0

We ceased computing new approximations when we ran across an x-value for which the reported function value is zero. Notice that the method takes a few more steps than Newton's method (8 steps compared to the 4 or 5 steps of Newton's method required to obtain the same accuracy). This somewhat slower convergence is typical of the Secant method. ∎

We now suggest an HP-28S/48SX program for the Secant method. First, define the function F whose roots are being sought.

≪→ X ' expression ' ≫ ' F ' $\boxed{\text{STO}}$

Then, enter the Secant method program.

≪ $\boxed{\texttt{DUP2}}$ → X Y 'Y−F(Y)*(Y−X)/(F(Y)−F(X))'≫

Program Step	Explanation
≪ $\boxed{\texttt{DUP2}}$ → X Y	Store the values on lines 1 and 2 of the stack in the local variables X and Y.
'Y-F(Y)*(Y−X) /(F(Y)−F(X))'	Compute the secant step and place the value on line 1 of the stack.
≫ ENTER	End the program.
' SCNT ' STO	Store the program in the current directory under the name SCNT.

Pressing the $\boxed{\texttt{SCNT}}$ key in the current directory with the initial guesses x_0 and x_1 on lines 2 and 1 of the stack, respectively, will execute the program. Each time that the program is run, the next secant approximation, (x_{n+1}), is computed and put on line 1 of the stack. As with Newton's method, new approximations should continue to be computed until two successive values are within the prescribed tolerance.

As with the Newton's method program, this Secant method program will usually only require a few steps, even for a very small value of the tolerance. Take a few minutes now to key in the program and to test it out by computing the values in the last table. (Be sure to store the program in the ROOTS subdirectory created earlier.) Using the same function F as for the Bisections and the Newton's methods programs, one need only enter initial guesses and then execute the program.

1 $\boxed{\texttt{ENTER}}$ (Enter an initial value on the stack.)

0 $\boxed{\texttt{ENTER}}$ (Enter an initial value on the stack.)

$\boxed{\texttt{SCNT}}$ (Execute the program.)

Continue to execute the program until successive values displayed on the stack are within the acceptable tolerance. Be sure to pay particular attention to how the approximation is improving at each step and compare this with the behavior of Newton's method and the method of Bisections.

The only advantage of the Secant method over Newton's method is that it does not require us to compute a derivative. Both methods may fail to work for a given problem. Much like Newton's method, the Secant method requires initial guesses that are sufficiently close to a root in order to guarantee convergence to that root. In practice, we can use the graphical techniques described earlier to arrive at these guesses.

We have now presented three different methods for approximating roots and given HP-28S/48SX programs for each one. Each has advantages and disadvantages and we have pointed these out, where possible. With the three methods given and the hints presented for finding initial guess(es) using the HP-28S/48SX's PLOTR routines, the student is now armed with all of the tools necessary for locating roots, with one exception. We have only discussed how to find *a* root of a function – not *all* the roots of a particular function. We will examine this question in the next section. By working carefully through the exercises, you will gain an appreciation for the various methods presented, as well as learn some of the shortcomings of each one. In this way, you will be prepared to deal effectively with a wide variety of rootfinding problems.

Exercises 4.1

In exercises 1-4, use Bisections, Newton's method and the Secant method to find approximate roots of the given function in the indicated interval. Use a tolerance of .0001. Compare the rates of convergence.

1. $x^3 - 4x^2 - 8x - 2$, [-2,-1]
2. $x^3 + 2x^2 - 49x - 8$, [-1,0]
3. $-x^6 + 4x^4 - 2x^3 + 8x + 2$, [2,3]
4. $x^4 - 7x^3 - 15x^2 - 10x - 1410$, [10,11]

In exercises 5-8, rework the indicated exercise by finding a root *outside* the given interval.

5. exercise 1
6. exercise 2
7. exercise 3
8. exercise 4

In exercises 9-10, use a rootfinding method to approximate the given root.

9. $\sqrt{3}$ (solve $x^2 - 3 = 0$)
10. $\sqrt[3]{2}$ (solve $x^3 - 2 = 0$)

In exercises 11-14, rewrite the given equation in the form $f(x) = 0$ and use a rootfinding method to approximate a solution in the interval.

11. $\sqrt{x^2+1} = x^3 - 3x - 1$, [2,3]
12. $x^2 - 5 = \dfrac{\sqrt{x^2+1}}{x+7}$, [2,3]

13. $\cos x = x$, $[0, \pi/2]$

14. $\sin x = x^2 - 1$, $[-\pi/2, 0]$

In exercises 15-19, the indicated method fails in spite of the fact that there is a root in the indicated interval. Explain why the method fails, and explain how the root can be found.

15. $4x^3 - 7x^2 + 1$, [0,1], Newton's method with $x_0 = 1$.
16. $4x^3 + 12x^2 - 15x + 4 = 0$, [0,1], Bisections with $a = 0$ and $b = 1$.
17. $x^3 + 4x^2 - 19x + 15 = 0$, [1,2], Secant method with $a = 1$ and $b = 2$.
18. $\dfrac{4x^2 - 8x + 1}{4x^2 - 3x - 7} = 0$, [1,2], Newton's method with $x_0 = 1.5$.
19. $\dfrac{x^3 + x^2 - 2x - 2}{x^2 + 3x - 4} = 0$, [0,2], Bisections with $a = 0$ and $b = 2$.
20. For $f(x) = 2x - \dfrac{400}{x}$, we have $f(-1) \cdot f(2) < 0$. What happens if we attempt to run Bisections with $a = -1$ and $b = 2$? Is there actually a root in $[-1, 2]$? Try using the Solver with an initial guess of 15. What does the message "SIGN REVERSAL" mean?

In exercises 21-24, show that $x = 1$ is the only root and compare the rates of convergence of Newton's method with $x_0 = 0$.

21. $x^3 - x^2 + 4x - 4 = 0$

22. $x^4 - 2x^3 + 2x^2 - 2x + 1 = 0$

23. $x^4 - 3x^3 + 4x^2 - 3x + 1 = 0$

24. $x^3 - 2x^2 + 2x - 1 = 0$

25. When light passes from one medium to another, it refracts according to Snell's Law $\dfrac{v_1}{v_2} = \dfrac{\sin\theta_1}{\sin\theta_2}$ where v_i is the velocity of light in the ith medium and θ_i is the angle from the vertical. In the figure on page 153, a person is looking at an underwater object. Using

$$\sin\theta_1 = \frac{x}{\sqrt{25 + x^2}} \qquad \sin\theta_2 = \frac{4 - x}{\sqrt{64 + (4 - x)^2}}$$

and $v_2 = .75v_1$, find x using the Secant method (why would this be simpler than Newton's method?). Also, find d, which is how far off the person's perception of the object is.

26. A tennis serve hit from a height of 8 feet at an angle of θ below the horizontal will be successful (neglecting spin) if θ satisfies $t_1 < \cos\theta < t_2$ where

$$8t_1^2 - 39t_1\sqrt{1 - t_1^2} = .95 \qquad 8t_2^2 - 60t_2\sqrt{1 - t_2^2} = 2.25$$

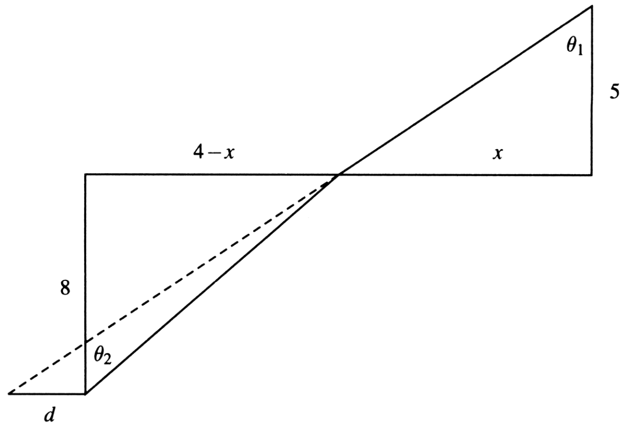
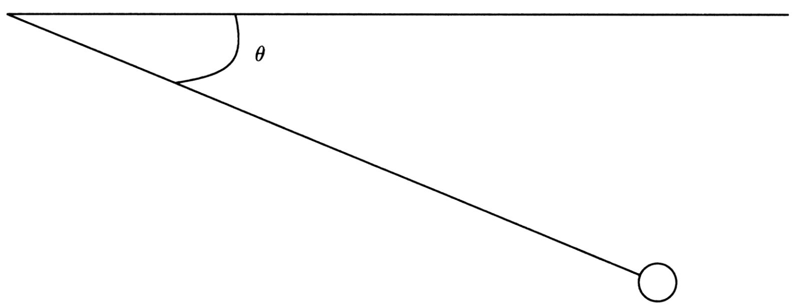


FIGURE FOR EXERCISE 25

Find θ_1 and θ_2 such that $\theta_1 < \theta < \theta_2$. For more details on tennis, see Tennis Science for Tennis Players by Howard Brody.



27. Store the following program in EQ: ≪ ' X ' F $\boxed{\text{EVAL}}$ ' N ' 1 $\boxed{\text{STO+}}$ ≫ . You will find $\boxed{\text{STO+}}$ in the Store menu on the HP-28S and the Variable Arithmetic menu (press $\boxed{\hookrightarrow}$ $\boxed{\text{MEMORY}}$) on the HP-48SX. When the calculator evaluates EQ, this program will in turn evaluate F and add 1 to the variable N. Repeat exercises 1-4 in the following way. First, store the function in F as a user-defined function. Then set N=0 and use the Solver to find the indicated root. Now evaluate N. This tells you how many times the Solver used EQ. Compare this rate of convergence to the rates found in exercises 1-4.

28. Based on the discussion in this section, what do you think the "SIGN REVERSAL" and "ZERO" messages from the Solver actually mean?

EXPLORATORY EXERCISE

Introduction

So far we have been finding one root at a time. In section 4.2, we will investigate how to find *all* of the roots. In this exercise, we will examine the behavior of Newton's method in a case of known multiple roots: $x(x-1)(x-2) = x^3 - 3x^2 + 2x = 0$. Clearly, the roots are 0, 1 and 2.

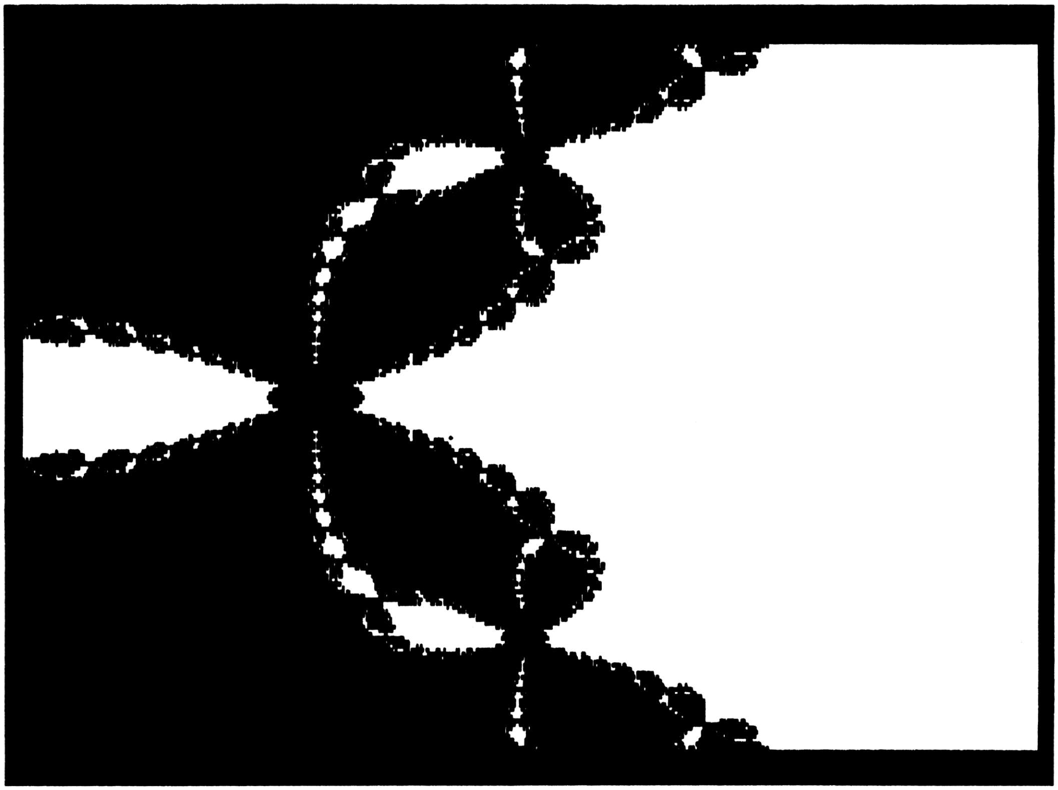
Problems

Try Newton's method with $x_0 = .1$, $x_0 = 1.1$ and $x_0 = 2.1$. All of these values are close to a root, and certainly nothing unusual happens. But, try $x_0 = .54$, $x_0 = .55$ and $x_0 = .56$. Are you surprised by the results? Examine the graph of $y = x^3 - 3x^2 + 2x$ and try to explain what happened.

If your curiosity has been piqued, then the next step is natural. We want to describe all starting values x_0 such that Newton's method converges to 0. This is called the *basin of attraction* of 0. Actually, we want to find the basins of attraction for all 3 roots. Start by determining which root Newton's method converges to from $x_0 = 0, .01, .02, ..., .99, 1.0$. You will want to write a program to do this. We suggest writing a general program for taking 100 steps between $x = A$ and $x = B$. (Such a program is given in the back of the book, but try writing one yourself!) Most of the basin boundaries are well defined, but there is some confusion between $x = .5$ and $x = .6$ (as we have already seen). It is reasonable to believe that all we need to do is magnify [.5,.6]. That is, run Newton's method with $x_0 = .5, .501, .502, ..., .599, .6$. Again, some clear boundaries emerge, but the picture is not sharp between .55 and .56. If you then magnify [.55,.56], you will find erratic behavior in [.552,.553]. Continue this process and you will always see confusion in 1 out of 10 subintervals. Can we ever accurately determine the basin boundaries?

Further Study

We have seen a simple formula (Newton's method) produce very complicated behavior. This is a dominant characteristic of the exciting mathematical field of chaos and fractals. The basin boundaries we investigated above are fractals which are similar to Cantor sets (see Fractals Everywhere by Michael Barnsley for details). For functions of 2 variables (or a complex variable) basins of attractions are often quite beautiful. The picture below is generated from Newton's method for the root $z = 1$ of the complex variable equation $z^3 - 1 = 0$.



4.2 Multiple Roots

In the last section, we presented several different methods for approximating roots of a function. One question which we did not answer there was how to find *all* of the roots of a given function. The more basic question is to determine just *how many* roots a given function has. Unfortunately, the general theory surrounding this question is rather incomplete. We shall examine some examples and give some hints here. Another question which plagues the numerical approximation of roots is: what happens when a function has a root of multiplicity greater than 1 at a given point [e.g., $f(x) = (x-2)^3$ has a root of multiplicity 3 at $x = 2$]? We will see that in this case, the speed and accuracy of both Newton's method and the Secant method will be reduced considerably, while Bisections may fail to work at all.

HOW MANY ROOTS ARE ENOUGH?

Obviously, many functions have more than one root. A reasonable question may be: How many roots are there? Unfortunately, there is no easy answer for this. Even for the familiar, relatively simple case of polynomial functions, the theory is inadequate. We do have:

Theorem 4.2 A polynomial p_n of degree n has at most n roots.

This says, for example, that a polynomial of degree 5 has at most 5 roots. Recalling that complex roots of a polynomial must come in conjugate pairs (i.e., if $a + b \cdot i$ is a root, then $a - b \cdot i$ is also a root), we see that polynomials of odd degree must have at least one real root. Hence, a polynomial of degree 5 could have 1,2,3,4 or 5 distinct real roots. For instance,

$$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5) \qquad \text{(5 roots)}$$
$$f(x) = (x-1)^2(x-2)(x-3)(x-4) \qquad \text{(4 roots)}$$
$$f(x) = (x-1)^3(x-2)(x-3) \qquad \text{(3 roots)}$$
$$f(x) = (x-1)^4(x-2) \qquad \text{(2 roots)}$$
$$f(x) = (x-1)^5 \qquad \text{(1 root)}$$

So, even for the familiar and relatively simple case of polynomials, we may not know how many roots there are, without actually factoring the polynomial (which we can do in only a small number of cases, in practice). For more general functions, the answer is even less clear. However, we can use the graphics capabilities of the HP-28S/48SX to help answer the question.

Example 1. Using Graphics to Determine the Number of Roots

Consider $f(x) = \sin x - x^2 + 1$. If we use the usual graphics routine to graph $y = f(x)$, we can clearly see two roots (see Figure 4.7a for an HP-28S graph): one in the interval (-1,0), the other in the interval (1,2). But, are there any roots that we don't see? That is, are there any roots outside of those displayed in the current graphics window? We can resolve this question easily for this particular example by plotting the graphs of $y = \sin x$ and $y = x^2 - 1$ simultaneously . Why? Well, notice that the equation

$$f(x) = \sin x - x^2 + 1 = 0$$

is equivalent to

$$\sin x = x^2 - 1$$

i.e., roots of f correspond to intersections of the two graphs $y = \sin x$ and $y = x^2 - 1$.

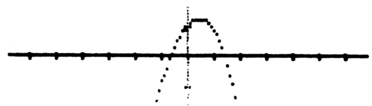


FIGURE 4.7a

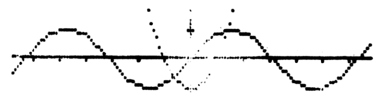


FIGURE 4.7b

The HP-28S plot of the two superimposed graphs (Figure 4.7b) clearly shows the two points of intersection. Since we know that the graph of $y = x^2 - 1$ is a parabola opening up and since the graph of $y = \sin x$ oscillates back and forth between -1 and 1, we can easily infer from the plot that there are no other points of intersection. Of course, for this simple graph, we might as easily have drawn the graphs freehand. However, having done this using the HP-28S/48SX, we have the added advantage that we can digitize initial guesses for the roots and use these in one of our root-finding schemes.

In the present case, the digitized points of intersection are $(-.6, -.5)$ and $(1.4, 1)$. Using the x-coordinates of these points as initial guesses for Newton's method, we get the approximate roots:

$$\begin{aligned} x &= -0.636732650 \qquad \text{where } f(x) = .000000000001 \\ x &= 1.409624004 \qquad \text{where } f(x) = .00000000001 \end{aligned}$$

From the foregoing discussion, these are seen to be the only roots of f. ∎

We hasten to add that the preceding example, while not typical of all rootfinding problems, is of a type often encountered in applications. The suggestion that you rewrite $f(x) = 0$ as $g(x) = h(x)$ (for some appropriate selection of h and g) and draw superimposed graphs of g and h will help in a number of situations. This is particularly useful when the function f has both a periodic term (in the foregoing case, $\sin x$) and a term which is not oscillatory (in this case, $-x^2 + 1$).

A good test of your rootfinding skills is to solve the following problem.

Example 2. A Hard Rootfinding Problem Solved

Find values of x for which $f(x) = \tan x - x = 0$. Certainly, there is no way of solving the problem algebraically, although it's clear that $x = 0$ is a root. Are there any others? The initial graph provided by the HP-28S (Figure 4.8a) gives us a rather confusing picture. The HP-48SX does somewhat better (see Figure 4.8b). In the HP-28S graph, there are a cluster of points around the origin (where we already know that there is a root) and four single points, with two on either side of the origin.

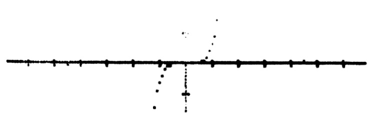


FIGURE 4.8a

FIGURE 4.8b

Notice that two of these points are very close to the x-axis. What is the behavior of the function near these points? At this stage, you might be led by the relative simplicity of the functions to draw freehand the superimposed graphs of $y = x$ and $y = \tan x$ (see Figure 4.9).

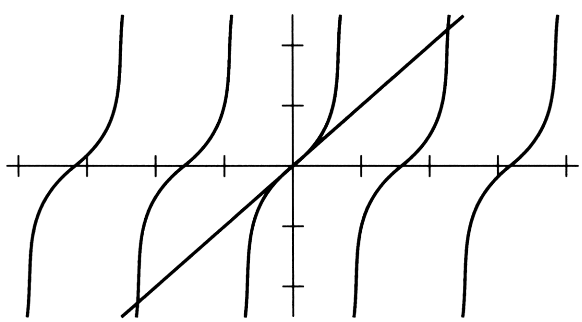


FIGURE 4.9

From Figure 4.9, we can see that there are infinitely many roots, since the tangent is periodic and blows up to infinity at $x = \pi/2,\ 3\pi/2,\ 5\pi/2,\ldots$. We will concern ourselves here only with the case $x > 0$. As can be seen from the graph, the points of intersection are fairly close to the points $x = \pi/2,\ 3\pi/2,\ 5\pi/2,\ldots$. In fact, the larger x gets, the closer the points of intersection (i.e., the roots of f) get to the points $x = \pi/2$, $3\pi/2$, $5\pi/2$, ... While it is of some significance to notice this, this does not help us find the roots. We cannot even use these values as initial guesses for any of our methods, since $\tan x$ is undefined at all of these points. However, we can use the less adequate graph provided by the HP-28S (or the better graph provided by the HP-48SX) together with the digitizing feature of the interactive graphics routines to obtain acceptable initial guesses.

As we have before, we use the arrow keys to move the cursor over to the point near the x-axis for $x > 0$. Digitizing the opposing corners of a box roughly centered at that point, we get $(4, -.5)$ and $(5, .5)$. Using the `ZBOX` command, we zoom in on this box (see Figure 4.10a). We now see clearly that there is a root near $x = 4.5$. Using $x_0 = 4.5$ in our Newton's method program, we obtain the approximate root $x = 4.49340945791$.

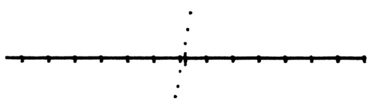


FIGURE 4.10a

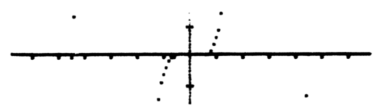


FIGURE 4.10b

Before looking for the next positive root, reset the display parameters (press `RESET`) and press `PLOT` (or `DRAW` on the HP-48SX) to obtain the original graph (Figure 4.10b). Seeing no other apparent locations of roots in the current window, move the cursor over to the extreme right edge of the display and digitize the point, (6.8,0). Pressing `CENTE` will translate the center of the current graphics window to the digitized point. You should see several single points plotted to the right of the root just found (Figure 4.11a). Digitize the opposing corners of a box including the first of these and the nearby section of the x-axis. In Figure 4.11b, we see the result of the `ZBOX` command with corners $(7, -1.4)$ and $(9, .6)$. In this latest plot, we still only see two points and these are both on the same side of the x-axis. We need to zoom in some more. Digitizing the points $(7.64705882353, -.303225806452)$ and (7.79411764706,.341935483871), and using the `ZBOX` command, we obtain the plot in Figure 4.11c, which clearly shows the existence of a root. If we digitize its apparent location, we get that it's around $x = 7.725$.

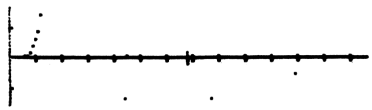


FIGURE 4.11a

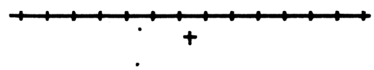


FIGURE 4.11b

Figure 4.11c shows the second positive root of $\tan x - x$.

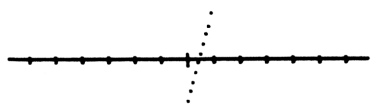


FIGURE 4.11c

Using the x-coordinate as an initial guess for Newton's method, we get the approximate root $x = 7.72525183694$. Continuing in this fashion, alternately translating the center of the display and then zooming in on the x-axis to see more points, we can locate as many roots of f as needed. The first 5 found with our Newton's method program are

x_n	$f(x_n)$
4.4934094579	0.000000000002
7.7252518369	0.000000000014
10.9041216594	−0.000000000034
14.0661939128	−0.000000000062
17.2207552719	−0.000000000091

At this point, you should verify the results in the preceding table and find the next largest positive root. Notice that one can see from Figure 4.9 about where the roots should be. However, we need to use the digitized points from the HP-28S/48SX plots to obtain acceptable initial guesses for Newton's method. ∎

What you should now recognize is the interplay between the theory, computation and graphics. The theory, by itself, is insufficient for finding roots or even for determining how many roots there may be. On the other hand, as we have seen, we cannot go around blindly stuffing guesses into Newton's method in the hope of finding a root, let alone all the roots. We should emphasize that, in solving practical problems, we need to take care to use all of the information at our disposal: theory, computation, freehand graphs and numerically generated graphs, such as those produced by the HP-28S/48SX.

MULTIPLE ROOTS

You may recall seeing the following definition.

Definition A function f is said to have a *root of multiplicity* n at $x = a$ if we can write

$$f(x) = (x - a)^n g(x)$$

where $\lim_{x \to a} g(x)$ exists and is nonzero.

Example 3. A Function with a Multiple Root

The function $f(x) = (x-2)^3(x^2+1)$ has a root of multiplicity 3 at $x = 2$. ■

All of the rootfinding methods which we have described encounter difficulties when we try to locate a root which has a multiplicity greater than 1.

Example 4. An Example where Newton's Method Is Very Slow

Locate a root of $f(x) = (x-3)^2$. First note that f has only one root, a root of multiplicity 2 at $x = 3$. Also note that Bisections cannot be used to locate this root, since $f(x) \geq 0$ for all x [i.e., $f(x)$ is never negative]. We apply Newton's method, using the initial guess $x_0 = 2$. The method works, but is unusually slow, as seen in the following table.

n	x_n	$x_n - x_{n-1}$
1	2.5	.5
2	2.75	.25
3	2.875	.125
4	2.9375	.0625
5	2.96875	.03125
32	2.99999999977	.000000000023
33	2.99999999988	.000000000011
34	2.99999999994	.000000000006

From the preceding table, we can see that, for the problem at hand, Newton's method took many more steps than it usually does.

Looking down the last column in the table, we can see that each successive value is about half the preceding value. This means that at each step of Newton's method, we move only about half as far (hopefully towards a root) as in the preceding step. This is very slow for Newton's method and is more like the convergence of the method of Bisections. The Secant method performs equally poorly for this problem. (Try this.) ■

REMARK: One might want to blame the poor performance of Newton's method on making a poor choice of the initial guess. However, a better initial guess will not improve the situation. This behavior is typical of convergence to a root whose multiplicity is greater than 1. The interested reader is referred to more advanced texts on the subject of numerical analysis for a more complete exposition (e.g., Burden and Faires, Numerical Analysis , 4th edition).

Unfortunately, slow convergence is not the only problem which we face when there are roots of multiplicity greater than 1. Consider Example 5.

Example 5. An Example where Newton's Method Has Poor Accuracy

Find the roots of $f(x) = x^4 - 5x^3 + 6x^2 + 4x - 8$. Using the graphics routines of the HP-28S/48SX, we find that there is a root near $x = 2.15$ (see Figure 4.12).

Using this as an initial guess in Newton's method yields the following table.

n	x_n	$f(x_n)$
1	2.10078125	0.003174025
2	2.06754755331	0.00094540994
3	2.04519576511	0.00028113074
15	2.00036287019	0.000000000016
16	2.00022789465	0.0
17	2.00022789465	0.0

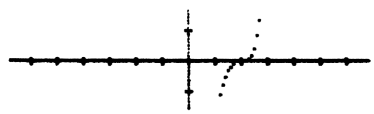


FIGURE 4.12

This would seem to be a rather unremarkable example. Certainly, Newton's method took many more steps than usual, but we've already seen that slow convergence can be caused by a root of multiplicity greater than 1. Quite naturally, then, we make the conclusion that $x = 2.00022789465$ is an approximate root (maybe even a root of multiplicity greater than 1). After all, Newton's method converged to this value and further steps yield no further progress. If this isn't enough evidence, the value of f at $x = 2.00022789465$ is reported to be 0.0. How much more evidence do we want, anyway?

At this point, it might be useful to notice that the given polynomial factors:

$$f(x) = x^4 - 5x^3 + 6x^2 + 4x - 8 = (x+1)(x-2)^3$$

Thus, the only roots are $x = -1$ and $x = 2$, with the latter being of multiplicity 3. Our suspected approximate root of $x = 2.00022789465$ is then seen to be accurate only to the 1st three decimal places. This is poor performance at best.

So, what went wrong with this application of Newton's method? Without getting into too many details, we can explain this as follows. If x is "close" to 2, $(x-2)$ will be close to zero, but the factor $(x-2)^3$ will be *very* close to zero. In the present case, for $x = 2.00022789465$, $(x-2) = .00022789465$ and $(x-2)^3 = 1.18E - 11$.

■

What, then, is the moral of this story? Should we learn that in the case of roots of multiplicity greater than 1, we should demand a somewhat smaller tolerance? Certainly, we cannot expect to get better than $f(x_n) = 0$ and $x_n - x_{n+1} = 0$. Perhaps we should be wary of all rootfinding problems involving multiple roots. Of course, we would need to know when the root that we are seeking has a multiplicity greater than 1. In practice, this can only be done by observing the slow convergence of our rootfinding scheme.

We should beware of problems with inordinately slow convergence. This generally suggests a multiple root and that spells trouble. We need to realize, too, that because of the limited accuracy of the HP-28S/48SX (usually about 12 digits) we will not, in many cases, be able to obtain even moderate accuracy in the root.

At this point, it might be interesting to see how well the built-in HP-28S/48SX Solver does on this problem. With the initial guess of 2.15 (enter 2.15, press the soft key $\boxed{\text{X}}$ and $\boxed{\text{RED}}$ $\boxed{\text{X}}$ on the HP-28S or $\boxed{\text{X}}$ $\boxed{\hookleftarrow}$ $\boxed{\text{X}}$ on the HP-48SX) the Solver gives the approximate root $x = 2.0002711078$, with the advisory message "ZERO" indicating that the machine thinks that it has found a root exactly. Of course, this answer is a bit worse than the already unacceptable answer found by our Newton's method program.

Notice that there is a further concern with using the Solver to solve such problems. Since we cannot observe the calculation in progress (as we can with our Newton's method program) we have no idea when something may be wrong. On the contrary, the advisory message "ZERO" and the fact that the function value at the reported root is 0 serve to convince us that everything is just fine and that we have just found an accurate approximation to a root. For this reason, we caution against using the Solver alone to find roots. When there are multiple roots, the Solver (and hence also the user) can be easily fooled into making an incorrect conclusion. We suggest that you use the Newton's method and Secant method programs given in this chapter and pay close attention to the progress of the calculation and not just to the final answer.

There are some things which can be done to improve the situation. However, a complete treatment of these methods is beyond the scope of the present work. We will give some hints in the exercises, but for a thorough treatment, the interested reader is referred to a text on numerical analysis. The real lesson for us here is to learn *caution* in solving for roots numerically. In practice, you must use a great deal of care, especially when a root of multiplicity greater than 1 is detected.

Exercises 4.2

In exercises 1-6, rewrite the equation in the form $f(x) = g(x)$ and use graphics to determine how many roots there are.

1. $\cos x^2 + x = 0$
2. $\sin x^2 + x^3 - 2x^2 + 1 = 0$
3. $x^8 + 3x^6 + 4x^2 - 4 = 0$
4. $(x^2 - 1)^{2/3} + 3x - 1 = 0$
5. $e^{-x} + x^2 - 1 = 0$
6. $3e^{-x}\cos(x - 1) - x^2 + 2x - 2 = 0$

In exercises 7-8, there are an infinite number of roots. Use graphics and a rootfinder to determine the three smallest positive roots.

7. $\sec x - x = 0$
8. $e^{-x} = \tan x$

In exercises 9-12, use graphics and a rootfinder to determine *all* the roots. Based on the rate of convergence, which roots do you suspect are multiple roots?

9. $x^4 - 12x^2 + 32 = 0$
10. $x^5 - x^4 - 10x^3 + 10x^2 + 25x - 25 = 0$
11. $x^5 - 10x^4 - 2x^3 + 20x^2 - 3x + 30 = 0$
12. $x^4 + 2x^3 - 6x^2 - 14x - 7 = 0$

In exercises 13-14, $x = 1$ and $x = 2$ are roots. In exercises 15-16, $x = 0$ is a root. Based on the convergence of Newton's method, determine which are multiple roots.

13. $x^4 - 2x^3 - 3x^2 + 8x - 4 = 0$
14. $x^4 - 7x^3 + 18x^2 - 20x + 8 = 0$
15. $x \sin x = 0$
16. $x(\cos x - 1) = 0$

17. In this exercise, we look at an alternative stopping rule for the Newton's method algorithm. Let $f(x) = x^4 - x^3 - 3x^2 + 5x - 2 = (x - 1)^3(x + 2)$.
    (a) Execute Newton's method with $x_0 = 1.5$. Stop when $|x_{n+1} - x_n| < .0001$.
    (b) Repeat part (a) but stop when $|f(x_n)| < .0001$.
    (c) Compare the number of steps executed and the accuracy.
18. All of our examples so far have had relatively small roots. Special problems may occur if a root is large. Consider $f(x) = (x - 400)^2(x + 1) = x^3 - 799x^2 + 159200x + 160000$.
    (a) Execute Newton's method with $x_0 = 300$. Stop when $|x_{n+1} - x_n| < .0001$.
    (b) Repeat part (a) but stop when $|x_{n+1} - x_n| < .0001|x_{n+1}|$.
    (c) Compare the number of steps executed and the accuracy. Under what circumstances might criterion (b) be more appropriate than criterion (a)?

In exercise 19, we will see one way to speed up the convergence of Newton's method in the case of a multiple root. Use this method to solve exercises 20-22.

19. Show that if f is a polynomial with a root c of multiplicity n then c is a root of

multiplicity 1 of f/f'. In this case, Newton's method would converge rapidly to c if f were replaced by f/f'. HINT: $f(x) = (x-c)^n g(x)$.

20. $x^4 - 5x^3 + 6x^2 + 4x - 8$ with $x_0 = 2.15$ (see Example 5).

21. $x^4 - x^3 - 3x^2 + 3x - 6$, $x_0 = 1.5$

22. $x^3 - x^2 - x + 1$, $x_0 = 1.2$

EXPLORATORY EXERCISE

Introduction

We have seen that Newton's method exhibits slow convergence to roots of multiplicity 2 or more. In exercises 19-22, we saw a messy way of speeding up the rate of convergence. Yet, we have never precisely said what we mean by *rate of convergence.* The stopping criterion we have used in our rootfinding methods is the difference $|x_{n+1} - x_n|$ between successive approximations. We use the quantity $\Delta_k = x_k - x_{k-1}$ to define the rate of convergence.

Problems

Start by running Newton's method with $x_0 = 1.5$ on the following examples while computing $\dfrac{\Delta_{k+1}}{\Delta_k}$ after steps 2, 3, ...

(a) $(x-1)(x+2)^3 = x^4 + 5x^3 + 6x^2 - 4x - 8$

(b) $(x-1)^2(x+2)^2 = x^4 + 2x^3 - 3x^2 - 4x + 4$

(c) $(x-1)^3(x+2) = x^4 - x^3 - 3x^2 + 5x - 2$

(d) $(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$

Conjecture a value for $r = \lim_{k\to\infty} \dfrac{\Delta_{k+1}}{\Delta_k}$ in cases (a)-(d). If r exists and is nonzero the method is said to *converge linearly.* Based on your calculations, formulate a hypothesis relating r to the multiplicity of the root. According to your hypothesis, what happens to the rate of convergence as the multiplicity of the root increases? Use Newton's method and your hypothesis to guess the multiplicity of the root $x = 0$ in the following cases.

(e) $x \sin x$ (f) $x \sin x^2$ (g) $x(x^x - 1)$

Further Study

The rate of convergence is a standard topic in numerical analysis. Introductory texts such as Numerical Analysis, 2nd edition, by Johnson and Riess have nice treatments of this topic.

4.3 Extrema and Applications

Everywhere we turn in business and industry today, we find someone asking questions like: "What's the least amount of time sufficient for completing this job?" or "What's the most money we can make on this investment?" or "What's the least amount of material that need be used to fabricate this device?" These questions are examples of what are called *optimization problems*, specifically *maximum/minimum* problems. In this section, we discuss some practical aspects of solving these problems. We start by pointing out that most such problems encountered in a typical calculus textbook, of necessity, have solutions which are roots of perhaps a quadratic or, at worst, a cubic polynomial. Most often, the solutions turn out to be integers.

As you might guess, it is rarely the case in real world problems that we would be so fortunate as to be presented with a quadratic polynomial with integer roots. Yet, it is not surprising that our textbook problems are so limited. To be sure, our facility for finding roots by pencil-and-paper methods is confined (with few exceptions) to low degree polynomials.

We should recognize, too, that there are few among us who do not instantly frown when the solution of a (textbook) problem starts to involve numbers other than integers. Most of us, unfortunately, have been trained to do just that. At the same time, it is exactly this type of problem (messy, user-hostile ones) which we will be facing when we apply calculus to almost any real world problem. As users of calculus, we must come to the point where we expect to get messy-looking answers and are surprised at the odd instance when we get an integer answer. The power of the HP-28S/48SX is ideally suited for dealing with these problems, using the rootfinding skills developed in the last two sections.

REVIEW OF ABSOLUTE EXTREMA

Recall the following definitions and theorems from elementary calculus.

Definition For x_0 in $[a, b]$, we call $f(x_0)$ the *absolute maximum* of f on the interval $[a, b]$ if $f(x_0) \geq f(x)$ for all x in $[a, b]$. $f(x_0)$ is the *absolute minimum* of f on $[a, b]$ if $f(x_0) \leq f(x)$ for all x in $[a, b]$. In either case, we call $f(x_0)$ an *absolute extremum.*

It should be fairly evident why someone would be interested in finding extrema.

Simple examples are everywhere: Business managers are interested in maximizing profits, while minimizing costs. Engineers are interested in maximizing the amount of energy which can be obtained from a barrel of oil and in minimizing the amount of material required to manufacture a given item. Similar examples abound in every branch of science, engineering, business and economics.

The first question that we might ask regarding the mathematics is whether every function has absolute extrema. The answer is no, but we do have the following.

Theorem 4.3 (Extreme Value Theorem) Suppose that f is a continuous function on the closed interval $[a, b]$. Then, f has both an absolute maximum and an absolute minimum on $[a, b]$.

Certainly, it is very comforting to know that for a broad class of functions (the set of continuous functions), there will always be absolute extrema on a closed interval. But, how do we find what those values are? In Chapter 1, we drew a graph of the function and tried to read from the graph what appear to be the extrema. Naturally, this is just a bit too crude, even if the graph is computer generated (such as by those produced by the HP-28S/48SX). There is a much more precise way to examine these problems. We can use the rootfinding methods developed in sections 4.2 and 4.3 or the Solver (discussed in section 1.3) to find approximations of extrema. First, we need the following definition.

Definition A number x_0 in the domain of a function f is called a *critical value* of f if $f'(x_0) = 0$ or if $f'(x_0)$ is undefined.

We can now state the main tool used for locating absolute extrema.

Theorem 4.4 Suppose that f is a continuous function on the closed interval $[a, b]$. Then, if $f(c)$ is an absolute extremum of f on $[a, b]$, c must be an endpoint (a or b) or a critical value.

NOTE: This result says that if we want to find the absolute extrema of a continuous function on a closed interval $[a, b]$, then we need only locate all of the critical values in (a, b) and simply *compare* the value of f at each of the endpoints and at each of the critical values. The largest of these numbers will be the absolute maximum; the smallest will be the absolute minimum. (That at least sounds easy, doesn't it?)

Example 1. Finding Extrema of a Polynomial

Find the absolute extrema of $f(x) = x^3 - 3x^2 - 9x + 7$ on the interval $[-2, 5]$. Here, $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1)$. Thus, the critical values are the roots of $f'(x)$ ($x = 3$ and $x = -1$). Since $f'(x)$ is a polynomial, it is defined everywhere and, hence, the only critical values of f are roots of f'. (The authors, of course, realize that this is another cooked-up textbook problem with integer roots, but it will serve as a good illustration of the procedure before we turn to the more messy and realistic problems to follow.)

Now, we compare the value of f at the endpoints and at the critical values to determine the relative extrema:

$$f(-2) = 5 \quad f(5) = 12 \quad f(-1) = 12 \quad \text{and} \quad f(3) = -20$$

Obviously, the absolute maximum of f on $[-2, 5]$ is 12 (this occurs at both $x = -1$ and $x = 5$) and the absolute minimum is -20 (this occurs at $x = 3$). ■

Example 2. Extrema of a Function with a Fractional Power

Find the absolute extrema of $f(x) = (x^2 - 4)^{2/3}$, on the interval $[-1, 3]$. Here,

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3}(2x) = \frac{4x}{3(x^2 - 4)^{1/3}}$$

In this case $f'(x) = 0$ only for $x = 0$. Further, $f'(x)$ is undefined whenever the denominator is zero, i.e., for $x = -2$ and $x = 2$ (both of which are in the domain of f). However, $x = -2$ is not in the interval under consideration. So, we need only compare:

$$\begin{aligned} f(-1) &= (-3)^{2/3} = 2.08008382305 \\ f(3) &= (5)^{2/3} = 2.92401773821 \\ f(0) &= (-4)^{2/3} = 2.51984209979 \\ f(2) &= (0)^{2/3} = 0 \end{aligned}$$

Clearly, the absolute maximum is $f(3) = 2.92401773821$, while the absolute minimum is $f(2) = 0$. Check these results, yourself. In doing so, you will notice that the HP-28S/48SX (like most calculators) has a problem when asked to compute certain fractional powers of negative numbers. In these cases, the user must compute the indicated power of the absolute value of the given number and then manually adjust the sign. ∎

Now that we have reviewed the procedure for locating absolute extrema, the only remaining questions are computational ones (i.e., How do we actually find the critical values?). We will deal with these questions next.

NUMERICAL SOLUTION OF EXTREMA PROBLEMS

How often in practice does one run into an extrema problem where the critical values are integer roots of a quadratic polynomial? If we were to answer "occasionally" we might still be guilty of exaggeration. Real world problems are rarely very pleasant and almost always require some computing to solve. The HP-28S/48SX is very well suited for solving many such problems. We exhibit here some examples that are typical of the type of problems encountered in applications. Pay particular attention to the interplay between the graphing, the analysis and the computation. As we'll see, no one of these three tools is sufficient for solving extrema problems in practice, but together they form a powerful combination.

Example 3. An Extrema Problem with Ugly Numbers

Find the absolute extrema of $f(x) = x^4 + 3x^3 - 5x^2 - 2x + 10$ on the interval $[-4, 2]$. It is always best to first get a rough idea of where the extrema might be from a graph of the function f. In the initial graph produced by the HP-28S, we see only a couple of dots graphed (see Figure 4.13a). Essentially, we need to compress the height of the graph, in order to fit it on the display. Using the ZOOM command will help out somewhat. More of the graph is shown, but still not enough (Figure 4.13b).

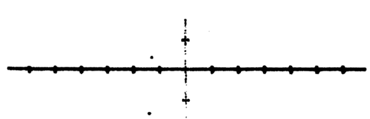


FIGURE 4.13a

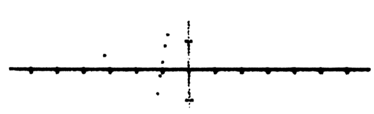


FIGURE 4.13b

Two more applications of $\boxed{\texttt{ZOOM}}$ will show much more detail, but because ZOOM zooms out in both the x- and y-directions simultaneously, the x-axis is now too compressed (Figure 4.13c). If we now use $\boxed{\texttt{ZBOX}}$ to zoom in on the part of the graph in which we are interested, we get the graph shown in Figure 4.13d [where the corners of the graphics window were defined by $(-4.5, -13.5)$ and $(2.25, 14.4)$]. Unfortunately, not all of the graph fits in this window, although we could not have known that from Figure 4.13c. One final $\boxed{\texttt{ZOOM}}$ will produce a fairly continuous curve on the interval in question (Figure 4.13e). On the HP-48SX, producing a useful graph is easier. You can enter the maximum and minimum x-values of interest, here -4.5 and 2.25, on the stack and press $\boxed{\texttt{XRNG}}$ (x-range) and $\boxed{\texttt{AUTO}}$. The y-values are then automatically scaled to fit on the screen and the resulting plot is drawn.

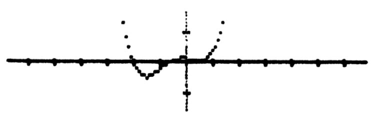


FIGURE 4.13c

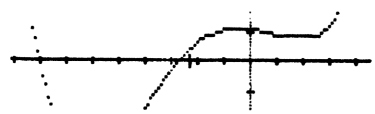


FIGURE 4.13d

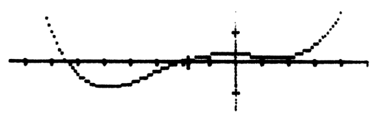


FIGURE 4.13e

Although we do not completely trust the picture, it suggests that the absolute maximum is at the endpoint $x = 2$, and the absolute minimum is at a relative minimum, around $x = -3$. To verify these conclusions and to make the values more precise, we must find the critical values and compare the values of the function at the endpoints and the critical values.

Notice that $f'(x) = 4x^3 + 9x^2 - 10x - 2$. This has no obvious factorization (at least not one obvious to the authors) and we must rely on numerical approximation of the roots. Note that since this is a cubic polynomial, there are no more than 3 roots. First, use the graphics routines to graph $f'(x)$ to see where any roots might be. (Don't forget to reset the graphing parameters first: press $\boxed{\texttt{RESET}}$. You might also use OVERD to superimpose the graphs of f and f', on the HP-28S. You can accomplish the same thing on the HP-48SX by drawing f' after drawing f, without first pressing RESET .) Zooming out from the initial graph, we obtain the graph in Figure 4.14a. Next, we use the ZBOX command, with corners defined as $(-4.05, -4.5)$ and (2.1,4.8) to produce the graph in Figure 4.14b. Although the graph lacks some detail, one should clearly be able to see that there are three roots: one around -3 and two between -1 and 1. Since we know that there can be at most three roots, there is no reason to produce a more detailed graph.

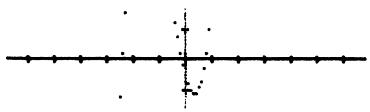


FIGURE 4.14a

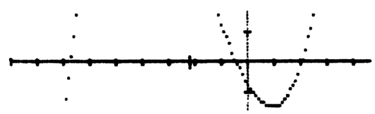


FIGURE 4.14b

Now we digitize the apparent locations of the roots. We obtain guesses of -3.0, $-.16$ and .92. Using our Newton's method program [looking for roots of $f'(x)$] with these initial guesses, we obtain the following results:

x_0	Approximate Root x	$f'(x)$
-3.0	-3.02241785918	0.0000000008
-0.16	-0.174672345714	0.0
0.92	0.947090204888	0.00000000002

Notice that since $f'(x)$ is cubic and we have located three roots, there are no others left to find. Finally, we need only compare the values of f at the endpoints and at the above three critical values. Obviously, the absolute maximum occurs at $x = 2$, and the absolute minimum occurs at $x = -3.02241785918$, both as expected from the graph of f.

$$f(-4) = 2$$
$$f(2) = 26$$
$$f(-3.02241785918) = -29.0112598806$$
$$f(-.174672345714) = 10.1817354516$$
$$f(.947090204888) = 6.97405567901$$

We should note that in this example the critical values are roots of a cubic equation which you probably cannot see how to factor. There is, of course, a formula for finding the roots of a cubic equation, although it is rather cumbersome to use. We chose instead to approximate the roots numerically. This is more instructive, of course, since in general there will be no formulas for finding the roots exactly. You should also note that, in this example, we knew in advance that there could be no more than 3 critical values. In general, we will have no idea of how many critical values to expect. We will therefore need to carefully search for critical values. ■

Example 4 is typical of the general situation.

Example 4. Extrema of a Non-Polynomial Function

Consider $f(x) = \cos x^2 + x^3 - 2x$ on the interval $[-1, 2]$. From the initial plot of $f(x)$ produced by the HP-28S, you can see most of the graph between $x = -1$ and $x = 2$ (see Figure 4.15a). One application of $\boxed{\texttt{ZOOM}}$ and you should have a pretty good idea of the behavior of the function on the interval of interest (see Figure 4.15b). An even better graph can be obtained by using the $\boxed{\texttt{ZBOX}}$ command to zoom in on the portion of the graph of interest. In Figure 4.15c the corners are $(-1, -1.8)$ and (2,4.2).

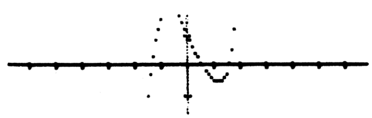


FIGURE 4.15a

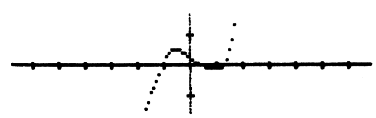


FIGURE 4.15b

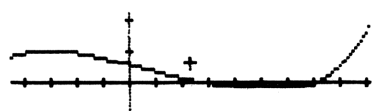


FIGURE 4.15c

The graph seems to indicate that the absolute maximum occurs at $x = 2$, while the absolute minimum occurs around $x = 1.2$. Also, there would seem to be a relative maximum around $x = -.7$. [Recall that f has a *relative maximum* at $x = c$ if and only if $f(c) \geq f(x)$, for all x in some open interval (a, b) containing c.] So, we should expect 2 critical values in the interval $(-1, 2)$.

Next, to make sure that we find all of the critical values, draw a graph of $f'(x) = -2x\sin(x^2) + 3x^2 - 2$. From the initial graph drawn by the HP-28S (see Figure 4.16), we can get initial guesses for roots near $x = -.7$ and $x = 1.2$.

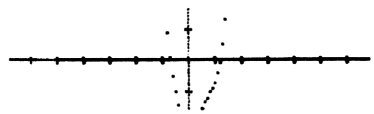


FIGURE 4.16

To have some confidence that there are no other roots on the interval $[-1, 2]$, try first using $\boxed{\texttt{ZOOM}}$ to zoom out so that all or most of the y-values are in the window. Then apply $\boxed{\texttt{ZBOX}}$ to zoom in on the portion of the x-axis of interest. Since the graph appears to be a relatively continuous curve with no roots other than the two already identified, we expect that there are no other roots on the interval $[-1, 2]$. Using our Newton's method program with these starting values, we get the following:

x_0	Approximate Root x	$f'(x)$
-0.7	-0.68092076714	0.0
1.2	1.21298116466	0.0

So, we have found approximations to 2 critical values, where we had only expected 2 critical values. Further, these 2 values are located near where we had expected to find extrema. Thus, we have no reason to search for any other critical values, in this case. It remains only to compare the values of the function f at the endpoints and at the critical values:

$$
\begin{aligned}
f(-1) &= 1.54030230587 \\
f(2) &= 3.34635637914 \\
f(-.68092076714) &= 1.94055525509 \\
f(1.21298116466) &= -.54196581576
\end{aligned}
$$

We can now read off the extrema. The absolute maximum occurs at $x = 2$ and the absolute minimum occurs at $x = 1.21298116466$, both as expected. You should not underestimate the importance of checking that the computed extrema correspond to those expected from the graph of $f(x)$. If they do not, then you should return to the graph and manipulate it so as to determine what you missed the first time (e.g., a missed critical value or an incorrect location for the relative extrema). ∎

By now, you should realize that we should be able to find the approximate locations for the absolute extrema of a continuous function, just by drawing a sufficiently detailed graph. However, to find the precise locations and to find the precise maximum and minimum values, we must rely on the results of our numerical rootfinding schemes and a comparison of function values. This process exhibits the interplay between the mathematical analysis, numerical computation and graphics typical of so many practical problems and which we have already seen in several other contexts. We provide one final example of such a problem, one where the graphics are not so easy to work with.

Example 5. Extrema of a Difficult Polynomial

Find the absolute extrema of $f(x) = x^6 - 7x^4 + 3x^3 - 5x + 1$ on the interval $[-1, 3]$. The initial graph produced by the HP-28S appears as only a few scattered dots (see Figure 4.17a). We first press $\boxed{\texttt{ZOOM}}$ to enlarge the range of y-values in the current window. This has improved the situation somewhat, at least between $x = -1$ and $x = 1$.

One can see a relative maximum around $x = -.5$, but the behavior on the interval [1,3] is still not clear. To remedy this, we press $\boxed{\texttt{ZOOM}}$ twice more. We can finally see where the graph bottoms out (see Figure 4.18). It appears that there is an absolute minimum around the point $(2, -32)$ and that the absolute maximum

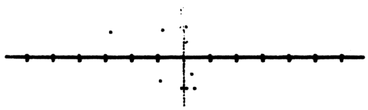


FIGURE 4.17a

FIGURE 4.17b

will occur at the endpoint, $x = 3$. Now that we have an idea of what we are looking for, we turn to finding the critical values.

FIGURE 4.18

Graphing the derivative, $f'(x) = 6x^5 - 28x^3 + 9x^2 - 5$, we look for zeros in the interval of interest, $(-1, 3)$. (Once again, it may help to draw superimposed graphs of f and f'.) Pay particular attention to locating any critical values near the suspected relative extrema seen in the graph of $f(x)$. The original graph provides few clues (see Figure 4.19a), so we start to zoom out, in the hope of locating some roots. Pressing $\boxed{\texttt{ZOOM}}$ once, we clearly see that there is a root near $x = -.5$ (see Figure 4.19b). Recall from the graph of f that we expected a relative maximum around $x = -.5$.

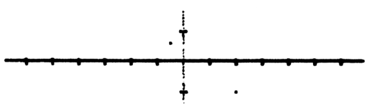


FIGURE 4.19a

FIGURE 4.19b

No other roots are apparent from the present display. If we press $\boxed{\texttt{ZOOM}}$ again, we can infer from the resulting graph (Figure 4.20) that there is another root located near $x = 2.0$ (recall from the graph of f that there seemed to be a relative minimum near $x = 2$). Finally, to be convinced that there are no roots on the interval (2,3), we need to zoom out several more times (see Figures 4.21a, 4.21b).

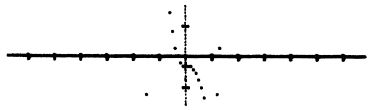


FIGURE 4.20

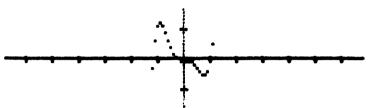


FIGURE 4.21a

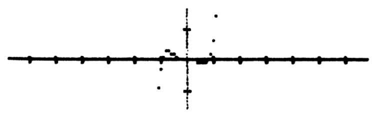


FIGURE 4.21b

So, from some routine work with the graphics, we have two guesses for critical values, $-.5$ and 2. Using our Newton's method program [for roots of $f'(x)$], with these initial guesses, we get the following results.

x_0	Approximate root x	$f'(x)$
-0.5	-0.479344270315	0.00000000001
2.0	2.00550138612	-0.00000000004

Finally, we compare the values of f at the critical values and at the endpoints:

$$f(-1) = -3$$

$$f(3) = 229$$

$$f(-.4979344270315) = 2.70887113843$$

$$f(2.00550138612) = -33.0027596493$$

We can now see that the absolute maximum occurs at $x = 3$, and the absolute minimum at $x = 2.00550138612$, as expected from the graph of f. ∎

Of course, the reason that we are interested in solving extrema problems is that they occur quite naturally in applications. We offer Example 6 as an illustration of the typical applied max/min problem, where the solution cannot be easily found through means of elementary algebra.

Example 6. Applied Maximum/Minimum

A city would like to build a new section of superhighway to link an existing bridge with another highway interchange, lying 8 miles east and 8 miles south of the bridge. Unfortunately, there is a 5-mile-wide stretch of marsh land which must be crossed (see Figure 4.22). Given that the highway costs 10 million dollars per mile to build over marsh and 7 million dollars per mile to build on dry land, how far to the east of the bridge should the highway be at the point where it crosses out of the marsh?

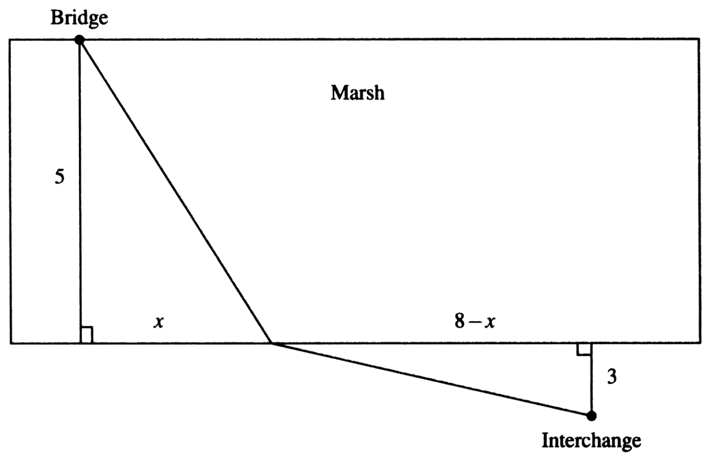


FIGURE 4.22

As with any applied max/min problem, you should first draw a picture and label the appropriate variables, as in Figure 4.22. Let x represent the distance in question (marked in Figure 4.22). Then the total cost of the project (in millions of dollars) is

$$\text{Cost} = 10 * \text{distance across marsh} + 7 * \text{distance across land}$$

In Figure 4.22, there are obviously two right triangles. Using the Pythagorean theorem, we find that the cost is given by

$$C(x) = 10(x^2 + 25)^{1/2} + 7[(8 - x)^2 + 9]^{1/2}$$

Store this function in the variable C (press 'C' $\boxed{\text{STO}}$). From the picture, it is easy to see that $0 < x < 8$.

So, we would like to find the minimum value of C on the interval [0,8]. First, to get an idea of where the minimum might be, we draw a graph of $y = C(x)$. The values of the function in the initial graph produced by the HP-28S are obviously all off the scale (no points at all are plotted). We then translate the center of the plot to the point (4,110). [We selected 110, since $C(0) \approx 110$.] A few points are plotted, but not enough. Using $\boxed{\texttt{ZOOM}}$ twice produces a fairly continuous curve on the interval [0,8]. (See Figure 4.23.) From this graph, we can see that the minimum value seems to be between $x = 2.75$ and $x = 4.5$.

FIGURE 4.23

To find the minimum value precisely, we will need to first find any critical values. The derivative of C is

$$C'(x) = 10x(x^2 + 25)^{-1/2} - 7(8 - x)[(8 - x)^2 + 9]^{-1/2}$$

Note that you can compute $C'(x)$ using the differentiation routines of the HP-28S/48SX. You should then store the expression for C' in a variable, say CD.

Notice that the only critical values occur where $C'(x) = 0$. (Why is that?) We now look for the roots of $C'(x)$. To get an idea of where these might be, we first draw a graph of $C'(x)$. From the initial HP-28S graph (Figure 4.24a), we can clearly see that there is a root around $x = 3.6$. Pressing $\boxed{\texttt{ZOOM}}$ twice, we are able to see a fairly continuous curve on the interval [0,8] which has only one zero. This graph is displayed in Figure 4.24b.

FIGURE 4.24a

FIGURE 4.24b

Note that we could use $x = 3.6$ as an initial guess for Newton's method, but this would require that we compute the derivative of $C'(x)$. While this is not a monumental task (you could always use the differentiation routines of the HP-28S/48SX), it is simpler to use the Secant method in this case. Using the Secant method program, with the initial guesses $x_0 = 3$ and $x_1 = 4$, we get the approximate root

$$x_n = 3.56005152031 \qquad \text{where} \qquad C'(x_n) = .00000000002$$

Finally, compare the value of $C(x)$ at the endpoints and at the critical value:

$$\begin{aligned} C(0) &= \$109,808,022.62 \\ C(8) &= \$115,339,811.32 \\ C(3.56005152031) &= \$98,888,374.49 \end{aligned}$$

Thus, if the roadway is built so that $x = 3.56...$, this will result in a savings of more than 10 million dollars over cutting straight across the marsh and a savings of more than 16 million dollars over cutting diagonally across the marsh. ■

The examples which we have given in this section together with the exercises to follow should give the student the necessary tools for solving a large variety of maximum/minimum problems found in applications. In solving such problems numerically, we urge caution, as always. You should be careful to check that the answer computed numerically corresponds to the solution expected from the graph of the function being maximized or minimized. If it does not, then further analysis is needed. Perhaps a refined graph of the function will shed some light on the problem or perhaps a critical value was missed in the graph of $f'(x)$. You should also check that the solution makes physical sense, if possible. All of these multiple checks reduce the likelihood of error. A good problem-solver must be on guard all the time, for there are many traps to fall into.

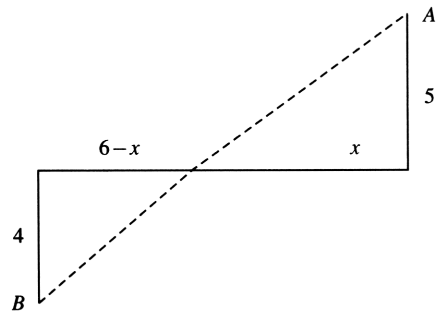
Exercises 4.3

In exercises 1-14, do the following: (a) use graphics to predict the maximum and minimum of the function on the interval; (b) use graphics and a rootfinding method to approximate the critical points; (c) find the maximum and minimum of the function on the interval.

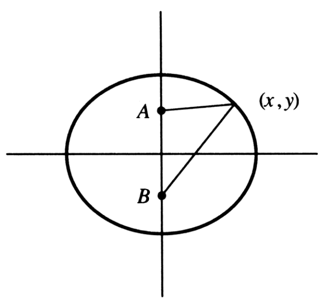
1. $x^3 - 6x^2 + 9x - 2$, [-2,2]
2. $x^3 - 6x^2 + 9x - 2$, [0,4]

3. $x^4 + 4x^3 - 6x^2 - 36x + 25$, $[-2, 2]$
4. $x^4 + 4x^3 - 6x^2 - 36x + 25$, $[-4, 0]$
5. $x^6 + 4x^4 - 3x^3 + 4x - 2$, $[-1, 3]$
6. $x^6 + 4x^4 - 3x^3 + 4x - 2$, $[-3, 0]$
7. $\sqrt{x^2+4} - \dfrac{x^2}{6} + 1$, $[-1, 3]$
8. $\sqrt{x^2+4} - \dfrac{x^2}{6} + 1$, $[-4, 1]$
9. $(x^2-1)^{2/3} - 2x + 1$, $[1, 3]$
10. $(x^2-1)^{2/3} - 2x + 1$. $[-2, 0]$
11. $x^2 \sin x - 2$, $[0, 4]$
12. $x^2 \sin x - 2$, $[-4, 0]$
13. $xe^{-x} + x^2$, $[0, 2]$
14. $e^{-x} + x^2$, $[-4, 4]$

15. Light travels at speed c in air and speed $.75c$ in water. Find x to minimize the time it takes light to get from point A in air to point B in water.



16. The points A$(0, 1)$ and B$(0, -1)$ are within the circle $x^2 + y^2 = 4$. Consider the path starting at point A, reflecting off the circle and finishing at point B. Find the points (x, y) on the right half of the circle (that is, $x \geq 0$) which minimize and maximize the reflecting distance from A to B. It is interesting to note that light can follow both paths (usually, light only follows paths of minimum time).



17. Washington needs to cross the Delaware to get to Trenton. Assume that the Delaware is 1 mile across and that Trenton is 2 miles inland and 10 miles downstream. If Washington's men can row at 3 mph and march at 4 mph, how far downstream should they row? How many minutes will they save compared to rowing straight across the river and then marching directly to Trenton?

18. The spinning of a clothes dryer causes it to vibrate as if being acted on by a downward force of f pounds, where $f = f_0 \sin wt$ for constants f_0 and w. Small springs and dampers may be used to reduce the vibrations. In studying the design of the machine (see Raven, Mathematics of Engineering Systems) an important quantity is $F = f_t/f_0$ where f_t is the amplitude of the force transmitted to the floor and f_0 is the amplitude of the vibrating force. This ratio has the form

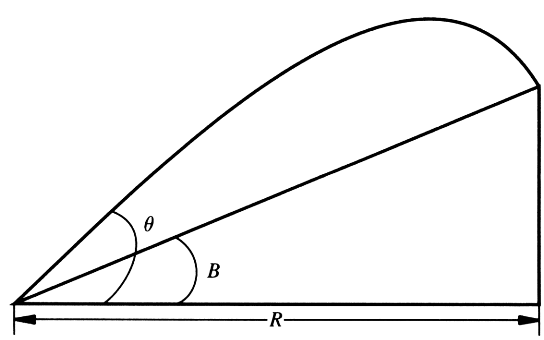
$$F = \sqrt{\frac{1+cb^2}{(1-b^2)^2+cb^2}}$$

where b is determined by the strength of the spring and c is determined by the amount of damping. For $c = .1$, find b to maximize F. For $c = .4$, find b to maximize F. (HINT: Find the maximum of F^2.) In practice, with $c = .1$ it is common to have $b = 4$. Explain why this differs from the value obtained above.

19. In sports where balls are thrown or hit, the ball often finishes at a different height than it starts at. Examples include a golf shot downhill and a basketball shot. In the diagram, a ball is released at an angle θ and finishes at an angle B above the horizontal (B can be negative for downhill trajectories). Neglecting air resistance and spin, the horizontal range is given by

$$R = \frac{2v_0^2\cos^2\theta}{g}(\tan\theta - \tan B)$$

where v_0 is the initial velocity of the ball and g is the gravitational constant. In the following cases, maximize R: (a) $B = 10$; (b) $B = 0$; (c) $B = -10$ (in degrees). Verify that $\theta = 45 + B/2$. [HINT: argue that you only need to maximize $\cos^2\theta(\tan\theta - \tan B)$]. This result and other uses of calculus in sports can be found in Sports Science by Brancazio.



20. A ball is thrown from $s = b$ to $s = a$ $(a < b)$ with initial speed v_0. Assuming that air resistance is proportional to speed, the time it takes the ball to reach $s = a$ is

$$T = -\frac{1}{c}\ln\left(1 - c\frac{b-a}{v_0}\right)$$

where c is a constant of proportionality. A baseball player is 300 feet from home plate and throws a ball directly towards home plate with an initial speed of 125 ft/sec. Another player stands x feet from home plate and has the option of letting the ball go by or catching it and, after a delay of .1 seconds, throwing the ball towards home plate with an initial speed of 125 ft/sec. Take $c = .1$ and find x to minimize the total time for the ball to reach home plate. What, if anything, changes if the delay is .2 seconds?

21. For the situation in exercise 20, for what length delay is it equally fast to have a relay and not have a relay? Why do you suppose it is considered important in baseball to have a relay option?

22. Repeat exercises 20 and 21 if the second player throws the ball with initial speed 100 ft/sec.

23. For a delay of .1 seconds, find the value of the initial speed of the second player's throw for which it is equally fast to have a relay and not have a relay.

24. Repeat exercise 20 if the second player throws the ball with an initial speed of 120 ft/sec.

EXPLORATORY EXERCISE

Introduction

The theory developed in this section gives us a definite answer about max/min problems involving a continuous function of one variable on a closed interval. As you might expect, not all applications of interest fit into this category. In this exercise, we develop a technique for solving a different type of max/min problem. We will analyze an old problem known as the "farmer problem." A farmer standing at $(-2, 0)$ needs to get water from a stream represented by $y = 6 - x$ and deliver the water to a cow at (2,0). From which point on the stream should the farmer get the water to minimize the total walking distance?

Problems

We first find the solution graphically. The set of points for which the total walking distance is d is given by the ellipse

$$\frac{x^2}{(d/2)^2} + \frac{y^2}{(d/2)^2 - 4} = 1$$

This is called the *level curve* of the distance variable d. Trying the value $d = 7$, graph the top half of the ellipse and $y = 6 - x$ simultaneously and convince yourself that the farmer will have to walk more than 7 units. Repeat the above with $d = 11$ and convince yourself that the farmer can walk less than 11 units.

With $d = 9$ the line $y = 6 - x$ just barely passes inside the ellipse. This tells us that the farmer can walk less than 9 units (why?). More importantly, we are now close to the best point on the stream. By decreasing d slightly, we should be able to find an ellipse that touches the stream at only one point. This point would be the solution of our problem. Estimate this point.

We can find this point analytically, too. Note that at the optimal point, the ellipse and line are tangent to each other, and hence have the same slope. The slope of the line is -1, so we know 3 requirements for the optimal point: (a) it is on the line; (b) it is on the ellipse; (c) at this point the slope of the ellipse is -1. Translating these into equations, we get

$$y = 6 - x \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2 - 4} = 1 \qquad \frac{x}{a^2} = \frac{y}{a^2 - 4}$$

where we have used $a = d/2$ for convenience. Find the optimal point.

Further Study

The geometry of the technique described above is the basis of a powerful result known as the Lagrange Multiplier Theorem. The theorem is normally stated in terms of a vector notation which simplifies calculations, but the principle is the same: at the optimal point, the level curve of the function to be optimized is tangent to the constraint curve (in the above example, the constraint curve is the stream $y = 6 - x$). This result is a fundamental part of the field of *calculus of variations*, which is typically a graduate course.

CHAPTER 5

Integration

5.1 Area and Riemann Sums

You are all familiar with the formulas for computing the area of rectangles, circles and triangles. We don't need to look very far to find good reasons for wanting to compute areas. For example, if you want to know how much grass seed you will need for your front yard, you'll need to find its area. The question of how to compute area is certainly much more profound than this example might make it seem, but this should serve to illustrate the point.

Most people's front yards are not perfect rectangles, circles or triangles. Does this mean that their yards don't have area? Certainly not, but the question remains as to how the area is to be computed. Notice that we've used the word *computed.* Areas are not measured directly, but rather are computed using some one-dimensional measurements and a formula or formulas.

What we need, then, is a more general description of area, one which can be used to find the area of almost any two-dimensional region imaginable. In this chapter, we will investigate the notion of the definite integral. At first, we will develop this as a tool for computing areas, but its usefulness goes far beyond this seemingly mundane question. It is, in fact, one of the central ideas of calculus. Our studies in this chapter will arm us with a powerful and flexible tool, which has applications in a wide variety of fields.

RIEMANN SUMS

We start our exploration of area by looking at a simple example on the HP-28S/48SX. First, graph the parabola $y = 2x - 2x^2$ using the default plotting parameters (i.e., first press $\boxed{\text{RESET}}$; see Figure 5.1 for the HP-28S graph). We would like to find the area of the region bounded by the x-axis and the graph. The region is clearly not a rectangle, a circle or a triangle, so we will look for an approximation of the area.

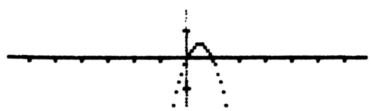


FIGURE 5.1

Since $2x - 2x^2 = 0$ if $x = 0$ or $x = 1$, the region extends from $x = 0$ to $x = 1$.

Although we often think of the calculator as plotting points, it actually colors in small squares on your screen called *pixels.* If you look closely enough at your calculator display, you will see a picture similar to the graph paper shown in Figure 5.2 (based on the HP-28S graph).

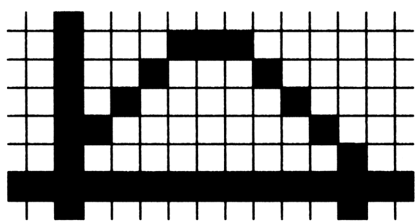


FIGURE 5.2

Each pixel represents a square of side .1 (on both the HP-28S and the HP-48SX). To estimate the area, then, we can add up the number of pixels in the region of interest and multiply the total by .01 (the area of each pixel). At $x = .1$, the graph is 2 pixels high, at $x = .2$ the graph is 3 pixels high and so on. If you are using an HP-28S, you should find

$$2 + 3 + 4 + 5 + 5 + 5 + 4 + 3 + 2 = 33$$

pixels, if you count the pixels representing the graph but not those representing the

x-axis. (Why don't we count both, or neither?) Our estimate of the area is then (33)(.01)=.33. If you are using an HP-48SX, you should find

$$2 + 3 + 4 + 5 + 5 + 5 + 4 + 3 + 2 + 1 = 34$$

pixels, leading to an area estimate of .34.

It can be shown that the exact area is 1/3 (we'll see how later), so it would seem that we have found a fairly good estimate. However, we have not yet developed a general procedure for computing area. First, we will need a way of systematically obtaining better and better estimates. (You should notice that for larger areas you would quickly tire of counting pixels.)

We can improve on this pixel-counting strategy by tracing along the curve with the cursor. Specifically, if the cursor is located at the point (.3,.4), then it is 4 pixels above the x-axis. Thus, the function values are related to the heights of the various columns of pixels. We exploit this in what follows.

We can think of our area approximation in terms of rectangles sitting on the x-axis and fitted-in under the graph, rather than in terms of pixels. Notice that in the HP-28S graph, at $x = .1$, the graph is 2 pixels high. The display then shows a rectangle of height .2 and width .1. Next to this, we see a rectangle of height .3 and width .1, and so on. Thus, Figure 5.3 is essentially Figure 5.2 with the rectangles shaded in.

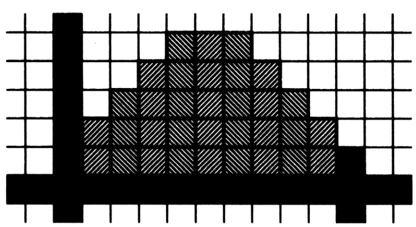


FIGURE 5.3

Our estimate of the area, then, is the sum of the areas of the rectangles:

$$(.2 + .3 + .4 + .5 + .5 + .5 + .4 + .3 + .2)(.1)$$

where we have factored out the common width of .1 (with corresponding results on the HP-48SX). Notice that we can rewrite this estimate as

$$(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9)(.1)$$

where the notation used here suggests the relationship between the function values and the heights of the rectangles.

You might wonder what would happen if we took more and more rectangles of increasingly small width (i.e., smaller pixels). Since this means a higher resolution picture, we should obtain a better approximation of the area. Indeed, by generalizing the preceding process, we get a definition which is very useful for computing areas. We start by dividing the interval $[a, b]$ into n subintervals of equal length $\Delta x = (b - a)/n$. (This is called a *regular partition* of the interval.) For each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, ..., n$, we choose any point c_i in $[x_{i-1}, x_i]$ (see Figure 5.4).

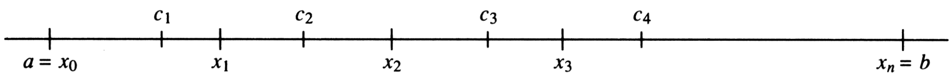


FIGURE 5.4

Definition The Riemann sum $R_n(f)$ of a function $f(x)$ corresponding to the above partition and the evaluation points $c_1, c_2, ..., c_n$ is

$$R_n(f) = [f(c_1) + f(c_2) + f(c_3) + ... + f(c_n)]\Delta x$$

Note that the value of a Riemann sum depends on the function, the choice of n, and the choice of the evaluation points.

Example 1. Computing Riemann Sums

Compute the Riemann sums with $n = 4$ and $n = 8$ for $f(x) = x^2$ on the interval $[1, 3]$, where for each $i = 1, 2, ..., n$, c_i is chosen to be the midpoint of the ith subinterval, $[x_{i-1}, x_i]$.

For $n = 4$, we find that $\Delta x = \dfrac{1}{2}$ and the subintervals that make up the partition are $\left[1, \dfrac{3}{2}\right]$, $\left[\dfrac{3}{2}, 2\right]$, $\left[2, \dfrac{5}{2}\right]$ and $\left[\dfrac{5}{2}, 3\right]$. The midpoints of the subintervals are then $c_1 = \dfrac{5}{4}$, $c_2 = \dfrac{7}{4}$, $c_3 = \dfrac{9}{4}$ and $c_4 = \dfrac{11}{4}$. Thus, we get

$$\begin{aligned} R_4(f) &= [f(5/4) + f(7/4) + f(9/4) + f(11/4)]\,\frac{1}{2} \\ &= [1.5625 + 3.0625 + 5.0625 + 7.5625]\frac{1}{2} \\ &= 8.625 \end{aligned}$$

Note that you can use the Solver to ease this calculation, as follows. First, store the function (enter ' X ∧ 2 ' STEQ) and then enter the Solver (press SOLVR). Next, compute 5/4 (press 5 ENTER 4 /), press the soft key X and then EXPR= to get $f(5/4) = 1.5625$. Similarly, compute the values $f(7/4) = 3.0625$, $f(9/4) = 5.0625$ and $f(11/4) = 7.5625$. Then add them up (press + three times) and, finally, divide by 2 (press 2 /).

For $n = 8$, you should verify that the Riemann sum is

$$R_8(f) = [f(9/8) + f(11/8) + f(13/8) + \dots + f(23/8)]\frac{1}{4} = 8.65625$$

■

In Figures 5.5 and 5.6, we see the rectangles corresponding to the Riemann sums R_4 and R_8, respectively. Based on these figures, we would expect R_8 to be the better approximation of the actual area under the curve. (Why is that?) In fact, you should convince yourself that the larger n is, the better the approximation R_n should be. We will use the following program to investigate this conjecture.

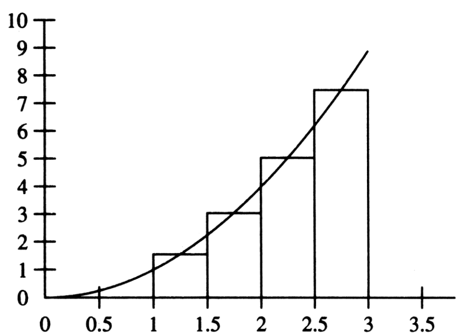


FIGURE 5.5

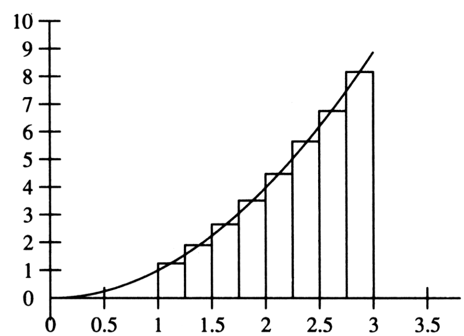


FIGURE 5.6

The program RIEM computes $R_N(F)$ for an integer N, an interval $[A, B]$ and a user-defined function F. The parameter R, $0 \le R \le 1$, lets us vary the evaluation points from the left endpoint ($R = 0$) to the right endpoint ($R = 1$), to anything in between ($0 < R < 1$). In Example 1 above, we used midpoint evaluations, which corresponds to the choice $R = .5$.

```
« '(B−A)/N' EVAL 'D' STO 0 1 N FOR I
'A+(I−1+R)*D' EVAL F + NEXT D * »
```

Program Step	Explanation
≪ '(B−A)/N' $\boxed{\texttt{EVAL}}$ 'D' $\boxed{\texttt{STO}}$	Compute Δx and store the value in the variable D.
0 $\boxed{\texttt{SPACE}}$	Put 0 on the stack to initialize the value of the sum.
1 N $\boxed{\texttt{FOR}}$ I	Start a loop.
'A+(I−1+R)*D' $\boxed{\texttt{EVAL}}$ F +	Add $F(c_i)$ to the sum.
$\boxed{\texttt{NEXT}}$	End the loop.
D * ≫	Multiply the sum by D and end the program.
ENTER 'RIEM' STO	Store the program under the name RIEM in the current directory.

To use the program, you must first store the values of the left and right endpoints of the interval $[a, b]$, in the variables A and B, respectively, the number of subintervals, n, in the variable N and a value for the parameter R, all in the current directory. You must also enter a program for the function F.

Example 2. Computing Riemann Sums with RIEM

Compute $R_8(f)$, $R_{25}(f)$, $R_{100}(f)$ and $R_{500}(f)$ for $f(x) = x^2$ on the interval $[0, 1]$ using left-hand ($R = 0$), midpoint ($R = .5$), and right-hand ($R = 1$) evaluations. To run RIEM, we first need to initialize the variables A, B, N, R and F. Here, we have

0 ' A ' $\boxed{\texttt{STO}}$
1 ' B ' $\boxed{\texttt{STO}}$
8 ' N ' $\boxed{\texttt{STO}}$
0 ' R ' $\boxed{\texttt{STO}}$
≪→ X ' X ∧ 2 ' ≫ $\boxed{\texttt{ENTER}}$ ' F ' $\boxed{\texttt{STO}}$

Now, press the soft key $\boxed{\texttt{RIEM}}$. Then change to $R = .5$ (press .5 'R' $\boxed{\texttt{STO}}$) and press $\boxed{\texttt{RIEM}}$ again. You should construct the following table of Riemann sums:

N	R=0	R=.5	R=1
8	.2734375	.33203125	.3984375
25	.3136	.3332	.3536
100	.32835	.333325	.33835
500	.332334	.333333	.334334

There are several observations that we can make from these results. First, the sums for $R = .5$ are in between the sums for $R = 0$ and $R = 1$. Since $y = x^2$ is an increasing function on $[0, 1]$, we can, in fact, conclude that the left-hand evaluations ($R = 0$) give the smallest function values and, hence, also the smallest Riemann sums. These sums are called *lower sums* . Further, the right-hand ($R = 1$) evaluations for an increasing function f give the largest Riemann sums. These sums are called *upper sums* . Finally, for any other choice of evaluation points, the Riemann sum for a given n will fall in between the corresponding lower and upper sums, in the case of an increasing function. Unfortunately, for many functions the lower and upper sums are not of practical help. The maximum and minimum values of the function on each subinterval are not necessarily at the endpoints of the subinterval and are often quite hard to find.

In the table of numbers given above, all three columns appear to be approaching 1/3, as N gets larger. It is possible to show by hand (you can find the details in most calculus books) that both the lower and upper sums approach 1/3. Then, by the Pinching Theorem, *all* Riemann sums approach 1/3 as $n \to \infty$. The following definition should now be meaningful. ■

Definition The *definite integral* of the function $f(x)$ over the interval $[a, b]$, which we denote by $\int_a^b f(x)\,dx$, is defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n(f)$$

if the limit exists and is the same for every choice of the evaluation points.

This definition may seem to be an abstract mathematical notion, but recall that it was motivated by the more concrete question of how to compute areas.

We started this section by estimating what we will now call $\int_0^1 (2x - 2x^2)\,dx$, i.e., the area between the parabola and the x-axis seen in Figure 5.1. You might wonder whether an integral always gives the area of some region. The next example shows that the answer is *no* . While we originally had in mind computing the area under the graph of a function f for which $f(x) > 0$, the integral of a function will make sense as long as the defining limit exists. In general, however, the integral can be thought of as representing something called *signed area.* We explore this in the next two examples.

Example 3. Integrals on the HP-28S/48SX and Signed Area

Estimate the value of the definite integral $\int_0^1 (x - 1)\,dx$ and compare your estimate to the area of the triangle bounded by the lines $x = 0$, $y = x - 1$, and the x-axis (see Figure 5.7).

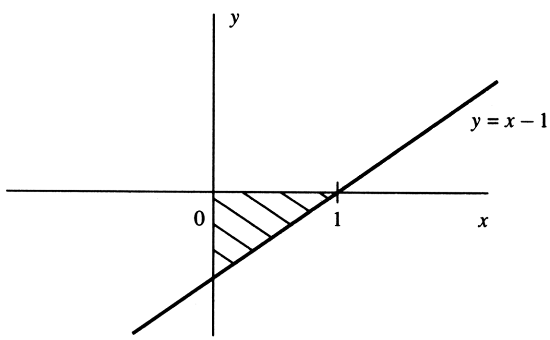


FIGURE 5.7

Here, we will estimate the value of the integral using the built-in routines of the HP-28S/48SX. The syntax is somewhat different on the two machines. On the HP-28S, enter

'X−1' $\boxed{\text{ENTER}}$ { X 0 1 } $\boxed{\text{ENTER}}$ .001 $\boxed{\text{ENTER}}$ $\int$

Keystroke	Explanation
'X−1' $\boxed{\text{ENTER}}$	Place the function on the stack.
{ X 0 1 } $\boxed{\text{ENTER}}$	List the variable and the limits of integration, A and B.
.001 $\boxed{\text{ENTER}}$	Enter the desired accuracy level.
$\int$	Return estimates of the value of the integral and the error to the stack.

On the HP-48SX, enter

' $\int$ 0 , 1 , X−1 , X $\boxed{\text{ENTER}}$ $\boxed{\to\text{NUM}}$

Keystroke	Explanation
' $\int$	Indicate that an integral is to be computed.
0 , 1 , X−1 , X $\boxed{\text{ENTER}}$	Enter the values of the limits of integration A and B, the function F and the variable of integration.
$\boxed{\to\text{NUM}}$	Return an estimate of the value of the integral to the stack.

The HP-28S will display two numbers, with the estimate of the value of the integral on line 2 of the stack. (On line 1, you will find an estimate of the absolute value of the maximum error in the approximation.) The estimate of the value of the integral is the only number displayed by the HP-48SX. (An estimate of the absolute value of the maximum error is stored under the variable name IERR in the current directory.) Here, we get an estimate of $-.5$. Notice that the area of the triangle in Figure 5.7 is $+.5$. The absolute value of the integral is correct. Here, the minus sign indicates that the area lies below the x-axis. This is an example of what we mean by *signed area.* ∎

NOTE: The HP-48SX also has a special $\boxed{\text{AREA}}$ command. To use this for Example 3, start by graphing $y = x - 1$. While the graph is still displayed, and with the cursor at $x = 0$, press $\times$ (the multiplication key), and move the cursor to $x = 1$. Then press FCN AREA . You need to be careful using this command, especially if you have zoomed in or out or if A or B is irrational. In these cases, the cursor may not be located *exactly* at $x = A$ or $x = B$ and, for this reason, the approximation may be slightly worse than you expect. Also note that if you only need, say, 3 digits of accuracy, then you can speed up the calculation by entering 3 FIX (FIX is located in the Modes menu). This fixes the number of decimal places at 3. If you do this, don't forget to reset the number of decimal places when you are through (press STD , again in the Modes menu).

Example 4. Sums of Signed Areas

Estimate $\int_0^1 (x^2 - x)\,dx$, $\int_1^2 (x^2 - x)\,dx$ and $\int_0^2 (x^2 - x)\,dx$ and interpret the integrals as signed areas. To obtain approximations with an error tolerance of .001, the keystrokes on the HP-28S are

'X∧2 – X' ENTER { X 0 1} ENTER .001 ENTER ∫
'X∧2 – X' ENTER { X 1 2} ENTER .001 ENTER ∫
'X∧2 – X' ENTER { X 0 2} ENTER .001 ENTER ∫

The keystrokes on the HP-48SX are:

'∫ 0,1,X∧2 – X,X ENTER →NUM
'∫ 1,2,X∧2 – X,X ENTER →NUM
'∫ 0,2,X∧2 – X,X ENTER →NUM

We obtain the estimates:

$$\int_0^1 (x^2 - x)\,dx \approx -.167$$

$$\int_1^2 (x^2 - x)\,dx \approx .833$$

$$\int_0^2 (x^2 - x)\,dx \approx .667$$

You might suspect that the exact values of the integrals are $-1/6$, $5/6$ and $2/3$, respectively. (It can be shown that these are in fact, correct. However, you should be careful not to jump to conclusions, as many integrals do not have rational values.) A quick sketch of the graph (see Figure 5.8) shows that $(x^2 - x) < 0$ on

(0,1) so that the area from $x = 0$ to $x = 1$ is $1/6$. Since $(x^2 - x) > 0$ on (1,2), the area from $x = 1$ to $x = 2$ is $5/6$. Now, the total area from $x = 0$ to $x = 2$ in Figure 5.8 is $1/6 + 5/6 = 1$. However,

$$\int_0^2 (x^2 - x)\,dx = -1/6 + 5/6 = 2/3$$

The integral adds up the signed areas, so that the proper interpretation of an integral requires a knowledge of where the function is positive and where it is negative. ∎

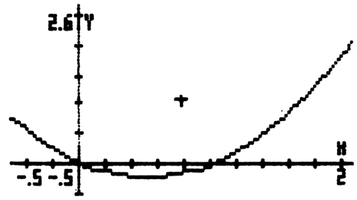


FIGURE 5.8

Note that Example 4 illustrates a general property of integrals, namely that for any c in $[a, b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

We will address one last fundamental question in this section. When does a definite integral exist (i.e., when does the limit defining an integral exist)? This turns out to be an important issue, because although a limited number of integrals can be computed exactly (we will see how to do this in the next section) *most* integrals cannot be computed exactly. We are usually forced to approximate the values of integrals using Riemann sums or some other computational method such as the routines built into the HP-28S/48SX (or those discussed in section 5.2). The problem with this approach is that, unless we have some way of knowing that a given integral exists, we will not know if our numerical approximation has any meaning (since numerical methods for almost any integral will produce *some* number, whether or not the integral actually exists). The question, then, is whether or not the "approximation" approximates anything meaningful. The following result gives a partial answer to this question.

Theorem 5.1 If f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx$ exists.

Notice that the theorem says nothing about what the possible effect of discontinuities might be. The bottom line is: for continuous functions, there are no problems with the existence of the integral. For discontinuous functions, we need to proceed with caution.

Example 5. Integrals of Discontinuous Functions

Investigate whether or not $\int_0^1 \frac{1}{x}\,dx$ and $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ exist. Since both integrands are discontinuous at $x = 0$, we do not know in advance whether or not either integral exists. Try evaluating them on the HP-28S/48SX with a 30-second time limit. That is, after 30 seconds press $\boxed{\texttt{ON}}$ to halt execution of the calculator's integration routine. (The HP28S/48SX routine will continue to refine its estimate until it is satisfied that it has found an adequate estimate of the value of the integral.) Neither computation is finished after 30 seconds. This slowness can mean either that the integral does not exist or that the integral is difficult to compute because of the discontinuity. The program RIEM can help us distinguish between the two cases. We get the following table of Riemann sums:

N	$f(x) = 1/x$	$f(x) = 1/\sqrt{x}$
16	4.736261	1.848856
64	6.122403	1.924392
256	7.508688	1.962194
1024	8.894981	1.981096

This gives us evidence that $\int_0^1 \frac{1}{x}\,dx$ may not exist (since the sums do not seem to be approaching a limit) and that $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ *does* exist (and equals approximately 2). Both of these conjectures, it can be shown, are correct. ∎

We close with a question that is probably as much philosophical as it is mathematical. Given that $\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$, is it reasonable to say that 2 is the area bounded by the graphs of $x = 0$, $y = 0$ and $y = \dfrac{1}{\sqrt{x}}$? (Draw the picture for yourself and think about this some. HINT: Can you draw the entire graph?)

Exercises 5.1

In exercises 1-2, count pixels to estimate the integrals.

1. $\displaystyle\int_0^1 x^2\,dx$

2. $\displaystyle\int_0^2 \sin x\,dx$

In exercises 3-6, use RIEM to compute lower sums and upper sums to show that they approach a common limit.

3. $\displaystyle\int_0^1 x^2\,dx$

4. $\displaystyle\int_0^1 x^3\,dx$

5. $\displaystyle\int_0^1 (1 - x^3)\,dx$

6. $\displaystyle\int_0^2 \sqrt{x}\,dx$

In exercises 7-10, the values of the integrals are integers. Use RIEM to discover these values.

7. $\displaystyle\int_0^2 (4x^3 - 7x)\,dx$

8. $\displaystyle\int_0^2 (2x^3 - 3x^2)\,dx$

9. $\displaystyle\int_1^2 3(x-1)^2\,dx$

10. $\displaystyle\int_0^8 3\sqrt{x+1}\,dx$

In exercises 11-18, estimate the areas of the indicated regions. Recall that the integral computes signed areas.

11. The region bounded by $y = x^4 - 1$ and $y = 0$.

12. The region bounded by $y = x^4 - 1$, $y = 0$, $x = 1$ and $x = 2$.

13. The region bounded by $y = x^2$, $y = 0$ and $x = 2$.

14. The region bounded by $y = x^2 - 2x$ and $y = 0$.

15. The region bounded by $y = \sqrt{x}$, $y = 0$ and $x = 3$.

16. The region bounded by $y = x^2$, $y = 0$, $x = -1$ and $x = 2$.

17. The region bounded by $y = \sin x$, $y = 0$, $x = 0$ and $x = 2\pi$.

18. The region bounded by $y = x^3$, $y = 0$, $x = -1$ and $x = 1$.

In exercises 19-22, determine whether or not the integral exists.

19. $\int_0^1 1/x^2\,dx$

20. $\int_0^1 x^{-2/3}\,dx$

21. $\int_2^4 \frac{1}{x-1}\,dx$

22. $\int_1^2 \frac{1}{x-1}\,dx$

23. Argue that $\int_0^2 f(x)\,dx = \int_0^2 g(x)\,dx$ in the case where $f(x) = \begin{cases} 2x, & x \le 1 \\ 3x^2, & x > 1 \end{cases}$ and $g(x) = \begin{cases} 2x, & x < 1 \\ 3x^2, & x \ge 1 \end{cases}$. HINT: what is the difference between f and g?

24. Estimate the integral in exercise 23.

25. Estimate $\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} x-1, & x < 1 \\ 4-x^2, & x \ge 1 \end{cases}$.

26. Investigate whether or not $\int_0^1 \sin(1/x)\,dx$ exists.

27. The Mean Value Theorem states that if f is differentiable on $[a,b]$ then there exists a number c in (a,b) such that $f'(c)[b-a] = f(b) - f(a)$. Determine c for $f(x) = x^3/3$ on the intervals [0,1], [1,2] and [2,3]. Using these c's as evaluation points, show that $R_3(f)$ equals the exact integral $\int_0^3 x^2\,dx = 9$. With the correct choice of c, then, it is possible for $R_n(f)$ to exactly equal the integral for any n.

28. For the integral $\int_0^3 x^2\,dx = 9$, show that there is a value of R for which program RIEM gives the exact integral. Carefully state the "mean value theorem" that this result illustrates. How does R compare to the c's found in exercise 27?

29. The following will make RIEM easier for you to use. Store the program $\ll$ { $\boxed{\texttt{STO}}$ A B N F } $\boxed{\texttt{MENU}}$ $\gg$ as RMENU and you may enter values for A, B, N and F as in the Solver menu.

EXPLORATORY EXERCISE

Introduction

We have made several references in this section to a method of computing integrals exactly. In this exercise, you will discover this method. The key is to look for a simple rule, so we will assume that $\int_0^c x^n\,dx = a\,c^b$ for some constants a and b which we will try to determine.

Problems

Compute $\int_0^1 x\,dx$ and $\int_0^2 x\,dx$ and determine a and b to match the conjecture $\int_0^c x\,dx = a\,c^b$ for $c = 1$ and $c = 2$. That is, estimate $\int_0^1 x\,dx$ and set the estimate equal to $a(1)^b$. Then, estimate $\int_0^2 x\,dx$ and set the estimate equal to $a(2)^b$. Use these two equations to solve for a and b. Now try $\int_0^3 x\,dx$ and $\int_0^4 x\,dx$ and see if the formula works for the a and b just found. Repeat the above procedure (that is, find new values for a and b) with $\int_0^c x^2\,dx$ and $\int_0^c x^3\,dx$. Now, look at your solutions. In the general formula $\int_0^c f(x)\,dx = g(c)$ what is the relationship between f and g? Test your conjecture on $\int_0^2 (x^2 - 2)\,dx$ and $\int_0^{\pi/2} \cos x\,dx$.

Further Study

You have several pieces of what is known as the Fundamental Theorem of Calculus. It only remains to extend your conjecture from this exercise to $\int_a^b f(x)\,dx$ for $a \neq 0$ and to prove the result (the standard proof uses the ideas of exercise 27).

5.2 Computation of Integrals

In section 5.1, we introduced the notion of the definite integral, defining it as a limit of Riemann sums. While this definition is theoretically quite important and provides us with a very straightforward definition of area, we must point out that integrals are in practice only rarely computed (or approximated) using Riemann sums. In this section, we look at more sophisticated techniques for approximating integrals, as well as a technique which will compute a limited number of integrals exactly.

THE FUNDAMENTAL THEOREM OF CALCULUS

The relationship between the integral and the derivative is a remarkable fact which brings unity to the seemingly disjoint studies of differential and integral calculus. You were asked to discover this relationship in an exploratory exercise in the previous section, and you can find a careful proof of the result in your regular calculus book. Its great significance is underscored by its name.

Theorem 5.2 (The Fundamental Theorem of Calculus) Suppose that f is continuous on the interval $[a, b]$. Let $F(x)$ be any function satisfying $F'(x) = f(x)$ for all x in $[a, b]$. (F is then called an *antiderivative* of f.) Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

To evaluate $\displaystyle\int_a^b f(x)\,dx$, then, we need only find an antiderivative of f (any one at all will do) and plug in the limits of integration (b and a).

This is a vitally important result because it gives us *exact* answers, and in some cases it is very easy to implement. Unfortunately, in many other cases it is quite difficult to find an antiderivative. Worse yet, many functions do not have antiderivatives which can be written in terms of the elementary functions with which we are all familiar. The HP-28S/48SX will provide at least some minimal help in this regard.

Example 1. Exact Integrals on the HP-28S/48SX

Compute $\displaystyle\int_0^1 (2x - 2x^2)\,dx$ using the Fundamental Theorem. We can, of course, easily do this without the help of the calculator. Notice that an antiderivative of $f(x) = 2x - 2x^2$ is $F(x) = x^2 - 2x^3/3$, so that the value of the integral is

$$F(1) - F(0) = (1 - 2/3) - (0 - 0) = 1/3$$

On the HP-28S, first find the antiderivative and then plug in the limits of integration: enter

'2*X−2*X∧2' $\boxed{\texttt{ENTER}}$ 'X' $\boxed{\texttt{ENTER}}$ 2 $\boxed{\texttt{ENTER}}$ ∫

$\boxed{\texttt{SOLV}}$ $\boxed{\texttt{STEQ}}$ $\boxed{\texttt{SOLVR}}$ 1 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$ 0 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$ −

Keystrokes	Explanation
' 2 * X − 2 * X ∧2 ' $\boxed{\texttt{ENTER}}$	Enter the function.
' X ' $\boxed{\texttt{ENTER}}$	Enter the variable.
2 $\boxed{\texttt{ENTER}}$	Enter the degree of the polynomial.
∫	Find an antiderivative.
$\boxed{\texttt{SOLV}}$ $\boxed{\texttt{STEQ}}$ $\boxed{\texttt{SOLVR}}$	Store the antiderivative F in the Solver
1 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$ 0 $\boxed{\texttt{X}}$ $\boxed{\texttt{EXPR=}}$ −	Compute $F(1) - F(0)$.

Note that the first three X's in this sequence refer to the letter X on the left keyboard, and the last 2 X's refer to the soft key $\boxed{\texttt{X}}$ in the Solver menu. The routine is more straightforward on the HP-48SX: enter

'∫0,1,2*X−2*X∧2,X $\boxed{\texttt{ENTER}}$ $\boxed{\texttt{EVAL}}$ $\boxed{\texttt{EVAL}}$

Keystrokes	Explanation
' ∫ 0,1,2*X−2*X∧2,X $\boxed{\texttt{ENTER}}$	Set up the integral.
$\boxed{\texttt{EVAL}}$	Try to find an antiderivative.
$\boxed{\texttt{EVAL}}$	Plug in the endpoints.

■

Although the HP-48SX can find antiderivatives for a much larger set of functions than the HP-28S (which can only integrate polynomials exactly), neither machine can perform important techniques such as integration by substitution or integration by parts. Owning an HP-28S/48SX is thus no substitute for learning the techniques of integration. (These are discussed in your regular calculus text.)

TRAPEZOID RULE

Due to our inability to find antiderivatives for many functions [try, for example, to find an antiderivative for $\sin(x^2)$, but don't try for too long] we need to supplement the Fundamental Theorem with effective approximation methods. We have already seen one such method, commonly called the *midpoint rule*, although we have not used that name before. The midpoint rule is a Riemann sum with evaluation points equal to the midpoints of the various subintervals (i.e., program RIEM with $R = .5$). As was the case for the rootfinding methods discussed in Chapter 4, it is important for us to have several methods available, so that we can balance accuracy and simplicity for a wide range of problems.

The *trapezoid rule*, which we describe below, is in many ways similar to the midpoint rule. Along with having a nice geometric interpretation, the trapezoid rule is significant because of its extension to a more powerful rule called Simpson's rule.

We first note that, in practice, we may not know the function which we're trying to integrate. That's right: we often will only know some *values* of a function at a collection of points. An algebraic representation of the function might not be available. For example, in experiments in the physical and biological sciences, it is usually the case that the only information available about a function comes from measurements made at a finite number of points.

Example 2. Estimating Area from Data with Trapezoids

Use the function values given in the following table to estimate the area bounded by the graphs of $x = 0$, $x = 1$, and the (unknown) function which generated the data.

x:	0	0.25	0.5	0.75	1.0
$f(x)$:	1	1.3	1.8	1.6	1.6

Conceptually, we have two tasks: first to conjecture a reasonable way to connect the given points, and then to estimate the area. The simplest way to connect the dots is with straight-line segments (at least, this is the way that the six-year-old daughter of one of the authors does it; see Figure 5.9). Notice that the region thus constructed is composed of 4 trapezoids. [Recall that the area of a trapezoid with sides h_1 and h_2 and base b is given by $b * (h_1 + h_2)/2$. This is the sum of the area of a rectangle plus the area of a triangle. (Why?)] The total area is then

$$.25\frac{f(0)+f(.25)}{2}+.25\frac{f(.25)+f(.5)}{2}+.25\frac{f(.5)+f(.75)}{2}+.25\frac{f(.75)+f(1)}{2}$$

$$=[f(0)+2f(.25)+2f(.5)+2f(.75)+f(1)]\frac{.25}{2}=1.425$$

■

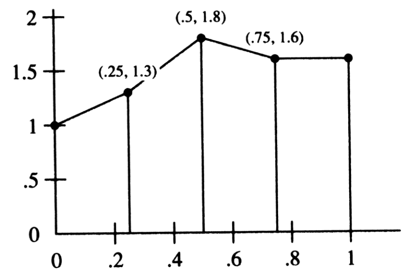


FIGURE 5.9

The trapezoid rule is a generalization of our work in Example 2. We first divide the interval $[a, b]$ into n equal pieces with endpoints $a = x_0 < x_1 < x_2 < \ldots < x_n = b$. The $(n+1)$-point trapezoid rule approximation of $\int_a^b f(x)\,dx$ is then:

$$T_n(f) = [f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]\,\frac{b-a}{2n}$$

In the exercises, you are asked to show that $T_n(f) = (A_n + B_n)/2$, where A_n is the Riemann sum from program RIEM with $R = 0$ and B_n is the Riemann sum with $R = 1$. This suggests the following simple (although not particularly efficient) program for implementing the trapezoid rule.

≪ 0 ' R ' $\boxed{\text{STO}}$ $\boxed{\text{RIEM}}$ 1 ' R ' $\boxed{\text{STO}}$ $\boxed{\text{RIEM}}$ + 2 / ≫

Program Step	Explanation
≪ 0 'R' $\boxed{\text{STO}}$ $\boxed{\text{RIEM}}$	Compute A_n.
1 'R' $\boxed{\text{STO}}$ $\boxed{\text{RIEM}}$	Compute B_n.
+ 2 / ≫	Compute $(A_n + B_n)/2$.
ENTER 'TRAP' STO	Store the program in the current directory under the name TRAP.

Example 3. Comparison of the Midpoint and Trapezoid Rules

For $\int_0^1 3x^2\,dx$, compare the midpoint and trapezoid rules with $n = 4$, $n = 8$ and $n = 16$ to the exact value of the integral. For the midpoint rule, we use RIEM with $R = .5$ and we use the new program TRAP for the trapezoid rule. Both programs require A, B, N and F to be initialized. Here, we have

$$0\ \text{'}\ A\ \text{'}\ \boxed{\text{STO}}\ 1\ \text{'}\ B\ \text{'}\ \boxed{\text{STO}}\ 4\ \text{'}\ N\ \text{'}\ \boxed{\text{STO}}$$
$$\ll\rightarrow X\ \text{'}\ 3 * X \wedge 2\ \text{'}\ \gg\ \boxed{\text{ENTER}}\ \text{'}\ F\ \text{'}\ \boxed{\text{STO}}$$

We get the following table of values.

N	Midpoint	Trapezoid
4	.984375	1.03125
8	.99609375	1.0078125
16	.99902343	1.001953125

Of course, from the Fundamental Theorem,

$$\int_0^1 3x^2\,dx = x^3\big|_0^1 = 1$$

exactly. The errors for the two rules here are fairly close, although the midpoint rule is slightly more accurate. This accuracy comparison is typical. The geometry of the two rules and the fact that $y = 3x^2$ is concave up on [0,1] should explain why the midpoint rule gives values that are too low and the trapezoid rule gives ones that are too high. (Draw a picture and think about this some.) ■

EXTRAPOLATION

Mathematicians always try to recognize, explain and take advantage of patterns. The approximations in Example 3 will reveal a pattern which we can take advantage of to obtain a better approximation.

Let's look more closely at the trapezoid rule approximations. $T_n(f)$ is getting smaller as n gets larger. From this pattern, we would expect the integral to be smaller than 1.001953125. In fact, we can be precise about how much smaller we expect the answer to be. Note that $T_4 - T_8 = .0234375$ and $T_8 - 1 = .0078125$. (Recall that the exact value of the integral is 1.) Also, $T_8 - T_{16} = .005859375$ and $T_{16} - 1 = .001953125$. Now for a surprise:

$$\frac{T_4 - T_8}{T_8 - 1} = \frac{T_8 - T_{16}}{T_{16} - 1} = 3.0$$

Both ratios are *exactly* equal to 3! In other words, if we represent the exact integral by I, we get

$$T_4 - T_8 = 3 * (T_8 - I) \quad \text{and} \quad T_8 - T_{16} = 3 * (T_{16} - I)$$

Solving this for I, we get $3I = 3T_8 + (T_8 - T_4)$ and $3I = 3T_{16} + (T_{16} - T_8)$. Finally, this leaves us with

$$I = T_8 + (T_8 - T_4)/3 \quad \text{and} \quad I = T_{16} + (T_{16} - T_8)/3$$

It turns out that the following result is true, in general: T_8 is about 3/4 of the way from T_4 to the exact integral I, and T_{16} is about 3/4 of the way from T_8 to the exact integral. That is,

$$I \approx T_8 + (T_8 - T_4)/3 \quad \text{and} \quad I \approx T_{16} + (T_{16} - T_8)/3$$

That's a pattern we can take advantage of! We make the following definition.

Definition The *Richardson extrapolation* of the trapezoid rule is $E_{2n} = T_{2n} + (T_{2n} - T_n)/3$.

For Example 3, $E_8 = T_8 + (T_8 - T_4)/3 = 1$ and $E_{16} = T_{16} + (T_{16} - T_8)/3 = 1$. Thus, in this example we have taken two trapezoid approximations and the Richardson extrapolation has led us directly to the exact value of the integral! In general, the extrapolation will not give the exact value of an integral, but it will greatly increase the accuracy of our approximation (usually much more so than by simply increasing n alone).

SIMPSON'S RULE

It turns out that the extrapolation of the trapezoid rule has a very simple form. Returning to Example 3, some messy but basic algebra gives us

$$\begin{aligned} E_8 = \frac{f(0)}{24} &+ 4\frac{f(.125)}{24} + 2\frac{f(.25)}{24} + 4\frac{f(.375)}{24} + 2\frac{f(.5)}{24} \\ &+ 4\frac{f(.625)}{24} + 2\frac{f(.75)}{24} + 4\frac{f(.875)}{24} + \frac{f(1)}{24} \end{aligned}$$

Notice the common factor of $1/24$ and the pattern that the coefficients follow: 1, 4, 2, 4, 2, 4, 2, 4, 1. This is an example of Simpson's rule, which has the general formula

$$S_n(f) = [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-1}) + f(x_n)]\frac{b-a}{3n}$$

where we have used the same notation as for the trapezoid rule. You should note here that the value of n must be even. (Why is that?)

Since Simpson's rule is an extrapolation of the trapezoid rule, it is, in general, much more accurate than either the trapezoid rule or the midpoint rule. In addition, Simpson's rule has a nice geometric interpretation. Recall that in Example 2 we connected 2 points at a time with line segments to form trapezoids. If we, instead, connect 3 points at a time with parabolas, we get Simpson's rule (although the algebra involved in showing this is quite messy). Since parabolas would give a smoother, perhaps more reasonable looking graph, we have a geometric explanation of the increased accuracy of Simpson's rule.

The following program is a simple (although like the program TRAP, not especially efficient) program for computing Simpson's rule approximations. It uses the program TRAP and requires you to initialize the same variables as TRAP does.

```
« TRAP DUP 'N/2' EVAL 'N' STO TRAP − 3 / + '2*N' EVAL 'N' STO »
```

Program Step	Explanation
≪ $\boxed{\text{TRAP}}$ $\boxed{\text{DUP}}$	Compute T_n and put a copy on the stack.
'N/2' $\boxed{\text{EVAL}}$ 'N' $\boxed{\text{STO}}$ $\boxed{\text{TRAP}}$	Compute $T_{n/2}$.
− 3 / +	Compute E_n.
'2*N' $\boxed{\text{EVAL}}$ 'N' $\boxed{\text{STO}}$ ≫	Restore the value of N.
ENTER 'SIMP' STO	Store the program under the name SIMP in the current directory.

Example 4. Simpson's Rule Approximations

Use Simpson's rule to estimate $\int_0^1 \sqrt{x^2+1}\,dx$. We will not reference an exact value, but rather try to decide what seems reasonable. Using SIMP, we generate the following approximations.

N	Simpson's Rule
4	2.95795560136
8	2.95788349721
16	2.95788557022

Since we expect Simpson's rule to be very accurate, (while it isn't always accurate, it often is) the agreement of the first 6 digits of S_8 and S_{16} lead us to conjecture that 2.95788 is a good approximation. ∎

With Simpson's rule, you have a generally very accurate numerical integration method. More importantly, the derivation of Simpson's rule will have acquainted you with some of the general concepts behind the numerical methods built into most computer integration software.

Exercises 5.2

In exercises 1-6 use the Fundamental Theorem as in Example 1 to compute the integral (the HP-28S cannot compute the antiderivative in exercises 5-6).

1. $\displaystyle\int_{-1}^{1} 5x^2\,dx$
2. $\displaystyle\int_{0}^{2} 2x^3\,dx$
3. $\displaystyle\int_{1}^{3} (x^4 - 2x^2 + 3x - 2)\,dx$
4. $\displaystyle\int_{2}^{5} (3x^3 - 2x^2 + 3)\,dx$
5. $\displaystyle\int_{0}^{2} \cos 3x\,dx$
6. $\displaystyle\int_{0}^{1} (2x+1)^5\,dx$

In exercises 7-10, use the trapezoid rule and Simpson's rule as in Example 2 to approximate $\displaystyle\int_0^1 f(x)\,dx$.

7.	x:	0	0.25	0.5	0.75	1				
	$f(x)$:	2	2.4	3.0	3.3	3.6				
8.	x:	0	0.25	0.5	0.75	1				
	$f(x)$:	3	2.1	2.7	3.4	4.2				
9.	x:	0	0.125	0.25	0.375	0.50	0.625	0.75	0.875	1
	$f(x)$:	1	1.3	1.5	1.6	1.6	2.0	2.4	2.9	3.5
10.	x:	0	0.125	0.25	0.375	0.50	0.625	0.75	0.875	1
	$f(x)$:	2	1.2	0.4	$-.5$	0	0.4	1.2	2.5	4.0

In exercises 11-14, compare the midpoint, trapezoid and Simpson's rule approximations for N=4, N=8 and N=16 to the exact value.

11. $\displaystyle\int_{0}^{2} (3x^2 - 1)\,dx$
12. $\displaystyle\int_{0}^{3} (3x^2 - 2x + 1)\,dx$
13. $\displaystyle\int_{1}^{2} (x-1)^4\,dx$
14. $\displaystyle\int_{0}^{1} 4/(x^2+1)\,dx \qquad (=\pi)$

In exercises 15-20, estimate the following integrals (4 digits accuracy):

15. $\displaystyle\int_{0}^{1} x\sqrt{x^3+1}\,dx$
16. $\displaystyle\int_{0}^{2} \sqrt{x^3+1}\,dx$
17. $\displaystyle\int_{0}^{2} 1/\sqrt{x^2+1}\,dx$
18. $\displaystyle\int_{0}^{2} 1/\sqrt{x^3+1}\,dx$
19. $\displaystyle\int_{0}^{2} \frac{2}{\sqrt{4-x^2}}\,dx$
20. $\displaystyle\int_{0}^{2} \frac{\sin x}{x}\,dx$

21. Show that $T_n = (A_n + B_n)/2$ for $n = 4$ and $n = 8$, as described in the discussion before program TRAP.

22. Use SIMP to try to estimate $\int_0^2 1/\cos x\, dx$. Graph $1/\cos x = \sec x$ to help your interpretation of the results.

In exercises 23-26, determine which rules (midpoint, trapezoid, Simpson's) give the exact integral for $n = 4$. Explain your results geometrically.

23. $\int_0^1 4x\, dx$

24. $\int_0^2 x^2\, dx$

25. $\int_0^2 4x^3\, dx$

26. $\int_0^1 5x^4\, dx$

27. As you might expect, Simpson's rule can be extrapolated in a similar way to our extrapolation of the trapezoid rule. Use $\int_0^1 5x^4\, dx$ and compute $\dfrac{S_4 - S_8}{S_8 - 1}$ and $\dfrac{S_8 - S_{16}}{S_{16} - 1}$. Derive a formula for E_{2n} in terms of S_{2n} and S_n.

28. Use your extrapolation formula from exercise 27 to improve your estimates from exercise 15 to 8 digits accuracy.

EXPLORATORY EXERCISE

Introduction

We have seen how simple extrapolation formulas can greatly improve the accuracy of our approximations. Specifically, we derived Simpson's rule $S_{2n} = T_{2n} + \dfrac{T_{2n} - T_n}{3}$. In exercise 27, you were asked to derive an extrapolation of Simpson's rule $E_{2n} = S_{2n} + \dfrac{S_{2n} - S_n}{15}$. We will continue to extrapolate our extrapolations in this exercise.

Problems

From the trapezoid calculations T_4, T_8, T_{16}, T_{32} and T_{64} we want to extrapolate as far as possible. We have two levels of extrapolation so far, as illustrated below.

$$\begin{array}{ccccccccc} T_4 & & T_8 & & T_{16} & & T_{32} & & T_{64} \\ & S_8 & & S_{16} & & S_{32} & & S_{64} & \\ & & E_{16} & & E_{32} & & E_{64} & & \end{array}$$

We want to extrapolate the E's to get two improved approximations (call them F_{32} and F_{64}) and then extrapolate to a "best" approximation (call it G_{64}). We will use $\int_0^2 x^{31}\,dx = 134217728$ to guide our thinking. First, fill in the above chart (compute the T's, S's and E's). We will guess the extrapolation formula for the F's. The formulas for the first two extrapolations differ only in a change of constants from 3 to 15. Since both 3 and 15 are 1 less than a power of 2 ($3 = 2^2 - 1$ and $15 = 2^4 - 1$) we look for the best k for the formula $F_{2n} = E_{2n} + \dfrac{E_{2n} - E_n}{2^k - 1}$. Which works best, $k = 4$, $k = 5$ or $k = 6$? Determine k and compute F_{32} and F_{64}. Then find the best m for the formula $G_{2n} = F_{2n} + \dfrac{F_{2n} - F_n}{2^m - 1}$.

Further Study

We relied on numerical evidence to find a sequence of extrapolation formulas. A good book on numerical analysis (see Numerical Analysis , 2nd edition, by Johnson and Riess) will show the theoretical explanation for the accuracy of these methods. The method derived above is typically called *Romberg integration.*

5.3 Applications of Integration

In the course of developing the notion of the definite integral, we have discovered three interpretations of integration. Our original motivation was the need for computing areas. We then defined the definite integral as a limit of Riemann sums. Finally, the Fundamental Theorem of Calculus related the definite integral to antiderivatives. Each of these interpretations is important in applications. We give a sampling of applications below.

We first look at applications based on the area interpretation of the integral.

Example 1. Area between Curves

Find the area between the graphs of $y = \cos x$ and $y = x^2 - 1$. Figure 5.10 shows the HP-28S graph of the region we are interested in.

In this region, $\cos x > x^2 - 1$ so that

$$\text{Area} = \int_a^b [\cos x - (x^2 - 1)]\,dx$$

FIGURE 5.10

where a and b are the x-values of the points of intersection of the curves. The points of intersection are not easily found, but can be estimated by using the Solver on the HP-28S and the HP-48SX (or the $\boxed{\text{FCN}}$ $\boxed{\text{ISECT}}$ command on the HP-48SX), or by using the methods developed in Chapter 4. You should verify that, approximately, $a = -1.1765$ and $b = 1.1765$. Since $F(x) = \sin x - x^3/3 + x$ is an antiderivative of the integrand, we have by the Fundamental Theorem that the area is

$$\sin(b) - b^3/3 + b - \sin(a) + a^3/3 - a \approx 3.114$$

To evaluate the above quantity, we can store the expression 'SIN(X)−X∧3/3+X' with $\boxed{\text{STEQ}}$ then use the Solver to plug in b and a, and subtract. ■

You may already be familiar with the notion of a normal probability distribution. For instance, standardized test scores are often normally distributed. That is, if you draw a graph of the frequency of various scores against the scores, the graph will look roughly bell-shaped. For ease of computation, statisticians often refer to the *standard normal distribution.* (This is the normal distribution where the mean or numerical average is 0 and where the standard deviation is 1.) This distribution is described by the graph of

$$y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(see Figure 5.11 for a graph of this function). The probability that a given score falls between two values a and b is then the area under the graph between $x = a$ and $x = b$. We discover an important property of this distribution in Example 2.

Example 2. An Application to Probability Theory

Given that the scores from a certain test are normally distributed, find the probability that a randomly selected score falls within 1 standard deviation of the

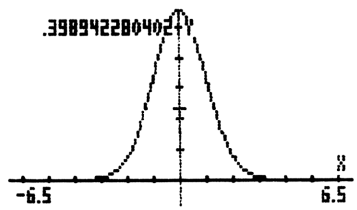


FIGURE 5.11

mean. The probability is the area underneath the above bell-shaped curve between $x = -1$ and $x = 1$,

$$\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Unfortunately, it turns out that there is no elementary antiderivative of $e^{-x^2/2}$. (Try to find one, but don't spend much time on it.) Therefore, we estimate the probability using Simpson's rule. We find that $S_8 = .6827$ and $S_{16} = .6827$, and hence conjecture that the probability is about 68.27% (i.e., roughly 68.27% of all scores will fall within 1 standard deviation of the mean). ■

You have experienced many situations where a force applied to an object changes the velocity of the object. For example, applying the brakes will decrease the velocity of a car. Two factors which determine how much the car slows down are the size of the force (how far you depress the brake pedal) and the length of time the force is applied. Physicists use the quantity called *impulse* to measure this effect. For a constant force F applied over a length of time t, the impulse is simply Ft. Newton's 2nd law of motion tells us that $Ft = m\Delta v$ where m is the mass of the object and Δv is the change in velocity of the object. However, forces are rarely constant, so the general form of impulse (which we denote by J) is needed:

$$J = \int_a^b F(t)\, dt$$

Here, a variable force $F(t)$ is applied from time $t = a$ to $t = b$. The general *impulse-momentum* equation is then $J = m\Delta v$.

Example 3. The Impulse of a Baseball

Suppose that a baseball, traveling at 130 ft/sec (about 90 mph), collides with a bat. A sensor on the ball records the force of the bat on the ball every .0001 seconds. The data (taken from The Physics of Baseball by R.K. Adair) is given below. What is the speed of the ball after the collision? If we ignore the energy lost due to friction (this aspect is discussed in the exercises), we may use the impulse-momentum equation: $J = m\Delta v$ where m is the mass of the ball (we'll use .01 slugs; regardless of what this name may remind you of, slugs is the correct unit of mass – as opposed to weight – in the English system), Δv is the change of velocity of the ball in feet/sec, and J is the impulse which equals the area under the graph of the force versus time. Then, $\Delta v = 100J$ and we can use Simpson's rule on the data to estimate the area J.

F (lb):	0	1250	4250	7500	9000	5500	1250	0	0
t (sec):	0	.0001	.0002	.0003	.0004	.0005	.0006	.0007	.0008

We have $n = 8$ (i.e., there are 9 data points) and

$$S_8 = [0 + 4(1250) + 2(4250) + 4(7500) + 2(9000)$$
$$+ 4(5500) + 2(1250) + 4(0) + 0]\frac{.0008}{24} \approx 2.866$$

Our estimate of the change in velocity is $100(2.866) = 286.6$ ft/sec. The ball then exits the collision traveling in the opposite direction at approximately $286.6 - 130 = 156.6$ ft/sec (107 mph). ■

Examples 4 and 5 use integrals to compute geometric quantities other than area, which applies to 2-dimensional regions. For a curve in 2 dimensions, the basic measure is length, and for 3-dimensional solids, one of the basic measures is volume. Formulas for the length of a curve and volume of a solid are derived in your regular calculus book. The format of these derivations is:

1. Divide the curve or solid into several pieces.
2. Approximate the measure of each piece and add the approximations.
3. Take the limit of the approximations as the number of pieces tends to infinity.

Such a limit of sums will typically be represented as an integral, as we did in the definition of definite integral.

Example 4. Arc Length of a Curve

Find the length of that portion of the parabola $y = -\dfrac{x^2}{500} + \dfrac{x}{2}$ which lies above the x-axis. The x-intercepts occur where $y = 0$, with $x = 0$ and $x = 250$. This parabola could represent the path of a thrown ball. We would normally say the throw is 250 feet long, which is the horizontal distance covered. The actual distance traveled by the ball is given by the length of curve formula

$$s = \int_0^{250} \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{250} \sqrt{1 + (-x/250 + 1/2)^2}\, dx$$

The built-in integration routine of the HP-28S/48SX gives an estimate of about 260 feet. We should emphasize that the length of curve formula only rarely produces an integral which can be computed exactly (by finding an antiderivative). The HP-28S/48SX can thus be invaluable in computing these integrals. ∎

Example 5. Volume of a Solid of Revolution

Find the volume of the solid obtained by rotating the region bounded by the graphs of $y = \sin x$ and $y = (x-1)^2$ about the x-axis. As in Example 1, we first graph the region (see Figure 5.12 for the HP-48SX graph) by graphing the equation $\sin x = (x-1)^2$, and then find the approximate points of intersection of the graphs.

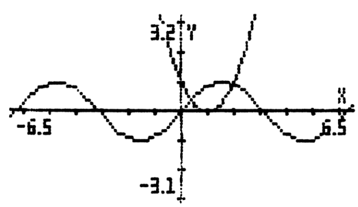


FIGURE 5.12

Using the Solver, we find that the x-coordinates of the points of intersection are approximately $a = .386237$ and $b = 1.961569$. The volume is then given by

$$\begin{aligned}\text{Volume} &= \pi \int_a^b [(f(x))^2 - (g(x))^2]\, dx \\ &= \pi \int_a^b [(\sin x)^2 - (x-1)^4]\, dx\end{aligned}$$

where we have used the *method of washers* to set up the integral. We now have several options. It is possible to find an antiderivative and evaluate the integral exactly. (Try this yourself to get the exact value.) On the calculator, the integral can be approximated as before, using SIMP. For a more elaborate use of your calculator, try on the HP-28S

$$\boxed{\text{RCEQ}}\ \boxed{x^2}\ \{\ \text{X}\ .386237\ 1.961569\ \}\ \boxed{\text{ENTER}}\ .001\ \boxed{\text{ENTER}}\ \int\ \pi\ *$$

On the HP-48SX, enter

$$.386237\ \boxed{\text{ENTER}}\ 1.961569\ \boxed{\text{ENTER}}\ \boxed{\hookrightarrow}\ \boxed{\text{STEQ}}\ \boxed{x^2}\ \text{'}\ \text{X}\ \boxed{\text{ENTER}}$$
$$\int\ \to\text{NUM}\ \pi\ *$$

The volume is then found to be approximately $.956\pi$. ■

Examples 6 and 7 utilize the relationship between the definite integral and the antiderivative. Since the derivative is the instantaneous rate of change of a quantity, the integral lets us work from a knowledge of the rate of change of a quantity back to a knowledge of the quantity itself.

Example 6. Computing Distance from Velocity

A runner moves with velocity $v(t) = 36t/\sqrt{t^2+2}$ ft/sec t seconds into a race. How far does she run in the first 10 seconds? Since velocity is the derivative of distance, we want $\displaystyle\int_0^{10} v(t)\,dt$. On the HP-28S, enter

$$\text{'}\ 36\ *\ \text{T}\ /\ \boxed{\sqrt{}}\ (\ \text{T}\ \wedge\ 2+2\ )\ \boxed{\text{ENTER}}$$
$$\{\ \text{T}\ 0\ 10\ \}\ \boxed{\text{ENTER}}\ .001\ \boxed{\text{ENTER}}\ \int$$

On the HP-48SX, enter

$$\text{'}\int 0\ ,\ 10\ ,\ 36\ *\ \text{T}\ /\ \boxed{\sqrt{}}\ (\ \text{T}\ \boxed{y^x}\ \ 2+2\ )\ ,\ \text{T}\ \boxed{\text{ENTER}}$$
$$\boxed{\to\text{NUM}}$$

We then find that the runner covers approximately 312 feet, or 104 yards, in the first 10 seconds. ■

Example 7. Computing Oil Flow

Suppose that the rate of flow of oil through a pipe is given by $f(t) = t(2 + \sin t)/(3+t)$ gallons per minute at time t (minutes). Find the amount of oil passing through the pipe in the first 15 minutes. If $A(t)$ is the number of gallons passing through the pipe in the first t minutes, then $A'(t) = f(t)$. Integrate both sides of this equation from $t = 0$ to $t = 15$:

$$\int_0^{15} A'(t)\,dt = \int_0^{15} f(t)\,dt$$

The Fundamental Theorem applies to the left side of this equation to yield $A(15) - A(0) = \displaystyle\int_0^{15} f(t)\,dt$. Finally, since $A(0) = 0$, we have $A(15) = \displaystyle\int_0^{15} f(t)\,dt$. Estimating the integral on the HP-28S/48SX using the built- in routines, we get $A(15) = 20$ gallons. ■

Exercises 5.3

In exercises 1-4, find the area between the curves.

1. $y = x^4$ and $y = 1 - x$
2. $y = x^4$ and $y = \cos x$
3. $y = 2x^2 - 1$ and $y = x + 1$
4. $y = \sin x$ and $y = -x^2$

In exercises 5-8, find the indicated probabilities.

5. A sample of a normal random variable is within 2 standard deviations of the mean (see Example 2).
6. Repeat exercise 5 for 3 standard deviations.
7. The lifetimes of some products are exponentially distributed. Compute the probability that a light bulb lasts less than 20 hours if the probability is given by $\displaystyle\int_0^{20} \frac{1}{10} e^{-x/10}\,dx$.
8. The probability that an electron is between a and b meters from its nucleus is modeled by $\displaystyle\frac{4}{a_0^3}\int_a^b r^2 e^{-2r/a_0}\,dr$ where a_0 is the Bohr radius, 5.29×10^{-11}m. Find this probability for $a = .5a_0$ and $b = a_0$.

In exercises 9-10, repeat Example 3 for the given data (which represent impact velocities of 89 mph and 58 mph, respectively).

9.	t:	0	.0001	.0002	.0003	.0004	.0005	.0006	.0007	.0008
	F:	0	1000	2100	4000	5000	5200	2500	1000	0
10.	t:	0	.0001	.0002	.0003	.0004	.0005	.0006	.0007	.0008
	F:	0	600	1200	2000	2500	3000	2500	1100	300

In exercises 11-12 the data represent a landowner's measurements in feet of the depth of a lot at 5-foot intervals. Estimate the area of the lot.

11.	x:	0	5	10	15	20	25	30	35	40
	y:	60	60	56	52	48	48	52	56	60
12.	x:	0	5	10	15	20	25	30	35	40
	y:	42	48	52	52	54	56	56	60	62

In exercises 13-16, estimate the length of the curve.

13. $y = -\frac{1}{30}x(x-50)$ on $[0, 50]$ (a 50-yard football punt).
14. $y = 10 + \cosh(x/30)$ on $[-20, 20]$ (the length of a telephone wire).
15. $y = \sin x$ on $[-\pi/6, \pi/6]$ (compare to a straight line).
16. $y = \sin x$ on $[0, 2\pi]$.

In exercises 17-20, find the volume of the solid described.

17. The region in exercise 1 rotated about the x-axis.
18. The region in exercise 2 rotated about the x-axis.
19. The region in exercise 3 rotated about $y = -1$.
20. The region in exercise 4 rotated about the y-axis.

In exercises 21-24, use the given speed to find the distance covered.

21. $s(t) = 36t/\sqrt{t^2+2}$ on [0,20] (compare to Example 6).
22. $s(t) = 40t/\sqrt{t^2+4}$ on [0,20] (compare to exercise 21).
23. $s(t) = -32t$ on [0,20] (a free fall without air resistance).
24. $s(t) = 130 - 130e^{-t/4}$ on [0,20] (fall with air resistance).

In exercises 25-26, we examine the energy lost to friction in the collision between a ball and striking object. During the collision the ball changes shape, first compressing and then expanding. If x represents displacement in inches and f represents force, the area under the curve $y = f(x)$ is proportional to the energy transferred. In each exercise, $f(x)$ gives the force during the compression of the ball and $g(x)$ gives the force during the expansion of the ball. Thus, the area between the curves $y = f(x)$ and $y = g(x)$ divided by the area under the curve $y = f(x)$ gives the percentage of energy lost due to friction. Compute the percentage. (See The Physics of Baseball by R. K. Adair for specific examples.)

25. $f(x) = 25,000x^2 + 10,000x$ and $g(x) = 50,000x^2$.
26. $f(x) = 5000x^2 + 3000x$ and $g(x) = 10,000x^2 + 1000x$.

In exercises 27-30, compute the moment of inertia $I = \frac{1}{3}\int_a^b x^2[f(x)-g(x)]\,dx$ where the region is bounded by $y = f(x)$ on top and $y = g(x)$ on bottom. The regions below are all crude models of baseball bats, and the moment of inertia is a measure of how hard the bat is to swing.

27. $f(x) = \frac{1}{2} + \frac{x}{30}, g(x) = -f(x), a = 0, b = 30.$
28. $f(x) = \frac{3}{5} + \frac{x}{30}, g(x) = -f(x), a = -3, b = 27.$
29. $f(x) = \frac{1}{2} + \frac{x}{30}, g(x) = -f(x), a = 0, b = 32.$

30. Same as exercise 27, but remove the rectangle from $y = -1/4$ to $y = 1/4$ and from $x = 27$ to $x = 30$.

31. If a lawn sprinkler sweeps back and forth at a constant rate, does it provide even coverage? Assume the angle of the sprinkler from the horizontal varies from $\pi/4$ to $3\pi/4$ at the constant rate $d\theta/dt = .2$ rad/sec. Also assume that the water flies $25\sin(2\theta)$ ft horizontally when the sprinkler is at the angle θ. Show that $\theta(t) = .2t + \pi/4$ and $\frac{dx}{dt} = 10\cos(.4t + \pi/2)$. Then compute and interpret $\int_0^1 \frac{dx}{dt}\,dt$, $\int_1^2 \frac{dx}{dt}\,dt$ and $\int_2^3 \frac{dx}{dt}\,dt$.

EXPLORATORY EXERCISE

Introduction

If we have points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the lines from A to the origin and B to the origin are perpendicular if $x_1x_2 + y_1y_2 = 0$, since slopes of perpendicular lines (in this case y_1/x_1 and y_2/x_2) multiply out to -1. The most familiar example is (1,0) and (0,1). These points lie on the x- and y-axes, respectively, which we use to represent all other points.

Problems

This exercise extends the idea of perpendicular lines to orthogonal functions. Since the graph of a function includes an infinite number of points, the generalization of $x_1x_2 + y_1y_2 = 0$ is an infinite sum equal to 0. Representing the infinite sum as an integral, we define functions f and g to be orthogonal on the interval [a,b] if $\int_a^b f(x)g(x)\,dx = 0$.

Show that the following pairs of functions are orthogonal on $[-1,1]$: (a) $\sin(\pi x)$ and $\cos(\pi x)$ (b) $\sin(\pi x)$ and $\sin(2\pi x)$ (c) $\cos(\pi x)$ and $\cos(2\pi x)$ (d) 1 and $\sin(\pi x)$. In fact, any two functions chosen from among $\sin(\pi x)$, $\sin(2\pi x)$, $\sin(3\pi x)$, ... , 1, $\cos(\pi x)$, $\cos(2\pi x)$, ... are orthogonal on $[-1,1]$. We say that these functions form an orthogonal set of functions on $[-1,1]$. Show that the set $\{\ 1,\ x,\ x^2,\ x^3,\ ...\ \}$ is not an orthogonal set of functions on $[-1,1]$.

Find constants c_1-c_7 (not all of which are zero) to make the following an orthogonal set of polynomial functions on $[-1,1]$: $\{\ 1,\ x,\ c_1x^2+c_2x+c_3,\ c_4x^3+c_5x^2+c_6x+c_7g\}$. HINT: If $f(x) = c_1x^2+c_2x+c_3$ and $g(x) = c_4x^3+c_5x^2+c_6x+c_7$, it is necessary that $\int_{-1}^{1} f(x)\,dx$, $\int_{-1}^{1} xf(x)\,dx$, $\int_{-1}^{1} g(x)\,dx$, $\int_{-1}^{1} xg(x)\,dx$ and $\int_{-1}^{1} x^2g(x)\,dx$ all equal 0. Compute these integrals exactly and find values of the constants (there will be more than one choice) to make all integrals equal zero.

Further Study

The purpose of orthogonal functions is the same as the perpendicular x- and y-axes: to make it easy to represent various functions in a common language. This turns out to be a powerful result in many applications (in case you were wondering what this was doing in an applications section). The Fourier series, discussed in section 6.3, is based on the orthogonal set of sines and cosines above. The orthogonal set of polynomials you found above are called Legendre polynomials. A general reference is <u>Orthogonal Transforms for Digital Signal Processing</u> by Ahmed and Rao.

5.4 Alternative Coordinate Systems

We have emphasized several times that good problem solvers require a variety of skills. For instance, although the built-in integration routine on the HP-28S/48SX is powerful, we have seen integrals for which a sequence of Riemann sums or Simpson's rule provides a better approximation. Similarly, our standard choice of writing y as a function of x is not appropriate for all integrals of interest. In this section, we will look at two alternatives to using the equation $y = f(x)$ to describe a curve. In some cases, merely treating x as a function of y simplifies calculations. A more fundamental change in perspective is provided by switching to the polar coordinate system.

CHANGE OF INDEPENDENT VARIABLE

In our development of the Riemann sum, we approximated areas by using rectangles of common width and variable height. In this scheme, we must keep careful track of the top and bottom of the region we are measuring to determine the heights of the rectangles. Finding the areas of regions without a well-defined top or bottom, such as in Examples 1 and 2, can be awkward.

Example 1. Two Integrals for One Area

Find the area of the region bounded by the graphs of $y = x$, $y = 2 - x$ and $y = 0$. The HP-28S graph of this region is shown in Figure 5.13. (This is produced by graphing the equation $x = 2 - x$.)

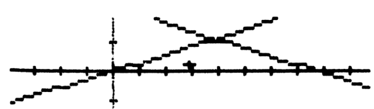


FIGURE 5.13

Geometrically, this problem is simple. The region is a triangle with area $(1/2)(2)(1) = 1$. However, if we wanted to use integrals to represent the area, then we would have to use two integrals to compute the area, since the region is bounded above by $y = x$ for $0 < x < 1$ and by $y = 2 - x$ for $1 < x < 2$. Specifically, we have that

$$\text{Area} = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

■

Example 2. Two Awkward Integrals for One Area

Sketch the region bounded by the parabolas $x = y^2$ and $x = 2 - y^2$ and represent its area with integrals. Recall from algebra that these parabolas open to the right and left, respectively. Since neither is a function of x, on the calculator you must graph the equations $\sqrt{x} = -\sqrt{x}$ and $\sqrt{2-x} = -\sqrt{2-x}$ using OVERD on the HP-28S (simply do not use ERASE or RESET between successive uses of DRAW on the HP-48SX). The HP-48SX graph is shown in Figure 5.14.

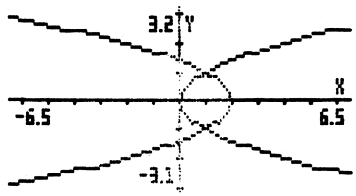


FIGURE 5.14

The region is bounded by the top and bottom of the curve $x = y^2$ for $0 < x < 1$ and by the top and bottom of the curve $x = 2 - y^2$ for $1 < x < 2$, so that

$$\text{Area} = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^2 [\sqrt{2-x} - (-\sqrt{2-x})]\,dx$$

■

In Examples 1 and 2, we had to use two integrals to find the area of one region. Further, in Example 2 we had to first solve the given equations for y in terms of x to find the functions to be integrated. The thought "there must be a better way" has probably occurred to you, and the form of Example 2 suggests a better way. Why not treat x as a function of y? The general area formula in this case becomes

$$\text{Area} = \int_a^b [g(y) - f(y)]\, dy$$

where $x = g(y)$ defines the right boundary of the region and $x = f(y)$ defines the left boundary of the region. In Figures 5.13 and 5.14, the left and right boundaries of the regions are well-defined, so that this approach should work well.

Example 3. Areas as Integrals with Respect to y

Compute the areas of Examples 1 and 2 in terms of integration with respect to y. In Example 1, the right boundary of the triangle is the line $y = 2 - x$, which we rewrite as $x = 2 - y$. The left boundary of the triangle is the line $x = y$. The figure extends from $y = 0$ to $y = 1$ (these are the solutions of $2 - y = y$). We then have

$$\text{Area} = \int_0^1 [(2 - y) - y]\, dy$$

In Example 2, the right boundary of the region is the curve $x = 2 - y^2$ and the left boundary of the region is the curve $x = y^2$. Since $2 - y^2 = y^2$ if $y = \pm 1$,

$$\text{Area} = \int_{-1}^1 [(2 - y^2) - y^2]\, dy$$

Both integrals are easy to compute by hand and equal 1 and 8/3, respectively. For the latter integral, on the HP-28S enter

' 2 − 2 * Y ∧ 2 ' $\boxed{\text{ENTER}}$
{ Y −1 1 } $\boxed{\text{ENTER}}$ .001 $\boxed{\text{ENTER}}$ ∫

On the HP-48SX enter

' ∫ −1 , 1 , 2 − 2 * Y ∧ 2 , Y $\boxed{\text{ENTER}}$ $\boxed{\to\text{NUM}}$

■

With this example, it becomes easier to appreciate the calculator's insistence on your identifying the variable of integration.

POLAR COORDINATES

The circle is one of the most important shapes occurring in nature and in mathematics. It is also one of the few geometrical objects for which we have an exact formula for area. It is ironic, then, that it is very difficult to compute the area of a circle by integrating. First recall that the circle of radius r centered at the point (a, b) has the equation

$$(x-a)^2 + (y-b)^2 = r^2$$

For the top half of a circle of radius 1 centered at the origin, we have $a = b = 0$ and $r = 1$, so the equation is

$$y = \sqrt{1-x^2}$$

Then, we get

$$\text{Area} = \int_{-1}^{1} \sqrt{1-x^2}\, dx$$

To compute this exactly, we need the sophisticated (and messy) technique of trigonometric substitution, which you will see in your study of techniques of integration. The news gets worse: it is difficult to even set up the integrals necessary to compute the area of a third of a circle. (Try it!)

The problem is with our approach. As we have defined it, integration is based on sums of areas of rectangles, so that we are trying to fit rectangular pegs into circular holes, so to speak. Of course, you can do this, but it's not pretty. Our first step in improving this situation is to introduce polar coordinates.

The standard rectangular coordinates locate points by measuring a horizontal distance x and vertical distance y from the origin. In polar coordinates, we locate the same point by measuring its distance r from the origin and the angle θ between the line from the origin to the point and the positive x-axis (see Figure 5.15).

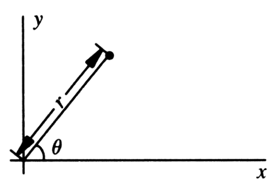


FIGURE 5.15

The angle θ is measured from the positive x-axis as usual: counterclockwise is positive, clockwise is negative, and radian measurement is preferred over degrees. From trigonometry, we get

$$r = \sqrt{x^2+y^2} \qquad x = r\cos\theta$$
$$\tan\theta = y/x \qquad y = r\sin\theta$$

The circle $x^2 + y^2 = 4$ then becomes $r^2 = 4$ or $r = 2$ (i.e., the set of all points whose distance from the origin is 2). Also, the equation $\theta = \pi/4$ describes the line $1 = \tan \pi/4 = \tan \theta = y/x$, or $y = x$.

The following result is needed to use polar coordinates to compute areas. For a region bounded by the graphs of $\theta = a$, $\theta = b$ and $r = f(\theta)$ as in Figure 5.16, we have

$$\text{Area} = \int_a^b \frac{1}{2} f^2(\theta)\, d\theta$$

■

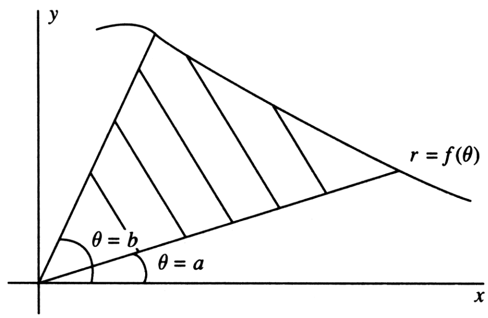


FIGURE 5.16

Example 4. Area of a Circle

Use polar coordinates to compute the area of the circle $r = 1$. In this case, $r = f(\theta) = 1$ and to get the full circle we need θ running from 0 to 2π. Then, we have that

$$\text{Area} = \int_0^{2\pi} (1/2)(1)^2\, d\theta = \pi$$

■

Example 5. Area of 1/8-Circle

Compute the area of the region bounded by the graphs of $y = 0$, $y = x$ and the upper portion of the circle $x^2 + y^2 = 4$. This is simply one-eighth of a circle, but the integration in rectangular coordinates is quite ugly. (Try to set it up.) In polar coordinates, the region is bounded by $\theta = 0$, $\theta = \pi/4$ and $r = 2$. We then have

$$\text{Area} = \int_0^{\pi/4} (1/2)(2)^2\, d\theta = \pi/2$$

■

Example 6. Area Bounded by a Circle and a Line

Compute the area of the portion of the circle $x^2 + y^2 = 9$ above the line $y = 1$ (see Figure 5.17). In polar coordinates, the equation of the circle is $r = 3$.

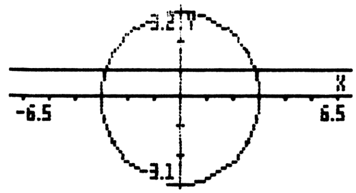


FIGURE 5.17

Since $y = r\sin\theta$, we change the equation of the line $y = 1$ into $r\sin\theta = 1$ or $r = 1/\sin\theta$. The intersections occur where $3 = 1/\sin\theta$ or $\sin\theta = 1/3$. Then θ goes from $a = \arcsin(1/3)$ to $b = \pi - \arcsin(1/3)$. Finally, we get

$$\begin{aligned} \text{Area} &= \int_a^b \frac{1}{2}[9 - 1/\sin^2\theta]\,d\theta \\ &= \int_a^b \frac{1}{2}[9 - \csc^2\theta]\,d\theta \\ &= \frac{9}{2}(b-a) + \frac{1}{2}[\cot(b) - \cot(a)] = 8.2502026 \end{aligned}$$

Note that since the HP-28S/48SX does not have the cosecant function built-in, you would need to type $1/\sin^2\theta$ instead of $\csc^2\theta$ to use the calculator to approximate the integral. ■

Along with providing a convenient way of computing various areas involving circles, polar coordinates can be used to graph curves that you have probably not yet seen in your studies. The HP-48SX provides a built-in utility for drawing graphs in polar coordinates. To help with drawing these curves on the HP-28S, we suggest the following graphing program.

≪ CLLCD DRAX FOR TH TH F TH R→C P→R PIXEL .05 STEP ≫

Keystrokes	Explanation
≪ [CLLCD] [DRAX]	Clear the screen and draw the x- and y-axes.
[FOR] TH	Start the loop.
TH F	Compute $f(\theta)$.
TH [R→C]	Form the point (r, θ).
[P→R] [PIXEL]	Convert the polar point (r, θ) to the rectangular point (x, y) and plot the point on the screen.
.05 [STEP] ≫	Increment θ and repeat the loop until the end of the program.
ENTER 'POLAR' STO	Store the program under the name POLAR in the current directory.

Example 7. Graphing a Spiral

Use the program POLAR on the HP-28S or the built-in utility on the HP-48SX to graph the spiral $r = \theta/6$ for $0 < \theta < 3\pi$. To run POLAR, we will need a user-defined function F. In this case, we enter

≪→ T ' T / 6 ' ≫ [ENTER] ' F ' [STO]

Next, enter the endpoints for θ on the stack (make sure your calculator is in radians mode) by pressing

0 [ENTER] [π] [→NUM] 3 *

Now run the program (press the soft key [POLAR]) and you should get the spiral seen in Figure 5.18.

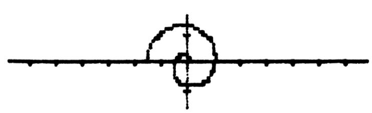


FIGURE 5.18

On the HP-48SX, plotting a graph in polar coordinates is a slight variation of the usual plotting procedure. First put the function to be plotted on line 1 of the stack. Here, we enter: 'X/6' $\boxed{\text{ENTER}}$. Press $\boxed{\text{NEW}}$ to store the function, giving the function a name when prompted. Next, press $\boxed{\text{PTYPE}}$ $\boxed{\text{POLAR}}$ to indicate that the equation should be graphed as a polar equation in the form $r = f(\theta)$. (Note that although we are using X as the variable on the calculator, the graph will be automatically plotted in polar coordinates, since we chose the POLAR plot type.) Press $\boxed{\text{PLOTR}}$ $\boxed{\text{RESET}}$ as usual and then enter a string containing the name of the independent variable and the lower and upper limits of the independent variable, in the format { 'X' lower upper} (the default setting is for the variable to be X and the lower and upper limits to be 0 and 2π). Here, use

$$\{\ \text{'X'}\ 0\ 3*\pi\ \}\ \boxed{\text{ENTER}}\ \boxed{\text{INDEP}}\ \boxed{\text{DRAW}}$$

This will produce the portion of the graph with values of θ between 0 and 3π (radians). ■

NOTE: After plotting a polar graph on the HP-48SX, you can graph any number of additional polar graphs without needing to re-enter the $\boxed{\text{PTYPE}}$ $\boxed{\text{POLAR}}$ command. The calculator will remain in polar graphing mode until you switch back to standard rectangular plotting mode. This is done by pressing $\boxed{\text{PTYPE}}$ $\boxed{\text{FUNC}}$ in either the Plot or Plotr menu.

Example 8. Area of One Leaf of a Rose

Find the area of one leaf of $r = 2\sin 3\theta$. On the HP-28S, use POLAR to graph the region described. First, redefine the current function F:

$$\ll\rightarrow \text{T ' } 2 * \boxed{\text{SIN}} \text{ (} 3 * \text{T) '} \gg \boxed{\text{ENTER}} \text{ ' F ' } \boxed{\text{STO}}$$

You might first try $0 \leq \theta \leq 2\pi$, since 2π is the period of $\sin x$. You should get the 3-leaf rose seen in Figure 5.19, but traced out twice. (The graph shown is from the HP-48SX.)

Then, it is reasonable to expect that one copy of the rose is traced out for $0 < \theta < \pi$ and one leaf is traced out for $0 < \theta < \pi/3$. We verify this by noting that a leaf starts and stops with $r = 0$, and $r = 0$ if and only if $\theta = 0, \pi/3, \ldots$. (Draw this

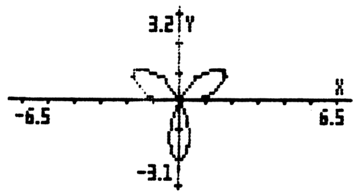


FIGURE 5.19

portion to see what we mean.) Thus,

$$\begin{aligned}\text{Area} &= \int_0^{\pi/3} \frac{1}{2}(2\sin 3\theta)^2\,d\theta \\ &= \int_0^{\pi/3} 2\sin^2 3\theta\,d\theta\end{aligned}$$

This can be evaluated using the trigonometric identity $2\sin^2 3\theta = 1 - \cos 6\theta$. Thus,

$$\text{Area} = [\theta - (1/6)\sin 6\theta]_0^{\pi/3} = \pi/3$$

■

With polar coordinates available, when you encounter a problem (particularly one involving circular geometry) you should now ask yourself whether it is more appropriate to attack the problem using rectangular coordinates (x, y) or using polar coordinates (r, θ).

Exercises 5.4

In exercises 1-6, find the areas of the regions bounded by the given curves using x as a function of y.

1. $y = x$, $y = 2 - x$ and $y = 0$
2. $y = x^2$, $y = 2 - x$, and $y = 0$
3. $x = y^2$ and $x = 8 - y^2$
4. $x = y^2$ and $x = 4$
5. $x^2 + 4y^2 = 4$ and $x = 0$ (right piece)
6. $x = 4 - y^2$ and $x = 0$

In exercises 7-10, find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

7. $y = x$, $y = 2 - x$ and $y = 0$

8. $x = y^2$ and $x = 2 - y^2$

9. $x = y^2$ and $x = 8 - y^2$

10. $y = x^2$, $y = 2 - x$ and $y = 0$

In exercises 11-18, sketch a graph of the polar function.

11. $r = \cos 2\theta$

12. $r = 1 + \sin\theta$

13. $r = 2\sin\theta$

14. $r = 2\cos\theta$

15. $r = 1 + 2\cos\theta$

16. $r = 1 + \cos\theta$

17. $r = 2\cos 3\theta$

18. $r^2 = 4\sin\theta$

In exercises 19-24, find the area of the polar region.

19. $r = \cos 2\theta$ (one leaf)

20. $r = 1 + \sin\theta$

21. $r = 2\sin\theta$

22. $r = 2\cos\theta$

23. $r = 1 + 2\cos\theta$ (inner loop)

24. $r = 1 + \cos\theta$

In exercises 25-30, find the area of the region bounded by the curves.

25. $x^2 + y^2 = 1$, $y = \dfrac{1}{\sqrt{3}}x$ and the positive x-axis

26. $x^2 + y^2 = 1$, $y = -x$ and the positive x-axis

27. $x^2 + y^2 = 9$ above $y = 2$

28. $x^2 + y^2 = 9$ above $y = -2$

29. $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$

30. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-2)^2 = 4$

31. Determine the number of leaves in the roses $r = 2\sin\theta$, $r = 2\sin 2\theta$, $r = 2\sin 3\theta$ and $r = 2\sin 4\theta$. Conjecture a rule for the number of leaves in $r = 2\sin n\theta$ for any positive integer n.

32. Determine the number of leaves in the roses $r = 2\sin\theta/2$, $r = 2\sin 3\theta/2$, $r = 2\sin 5\theta/2$ and $r = 2\sin 7\theta/2$. Conjecture a rule for the number of leaves in $r = 2\sin n\theta/2$ for any odd integer n.

EXPLORATORY EXERCISE

Introduction

Example 6 in the text and exercises 27-28 refer to the same problem which we call "Fletcher's oil problem." Suppose a cylindrical oil tank (circular cross-sections perpendicular to the ground) has an opening at the top. A measuring stick can be inserted to measure the height of the oil in the tank. If the cylinder has diameter 6 feet, what percentage of the tank is full when the oil has height h?

Problems

We ignore the length of the cylinder (why is this reasonable?) and state the problem as computing $A(h)/9\pi$ where $A(h)$ is the area of that portion of $x^2 + y^2 = 9$ below $y = h$. As in Example 6, if $0 < h < 3$, we compute the area *above* $y = h$ as $\int_a^b .5[9 - h^2 \csc^2 \theta]\, d\theta = 4.5(b - a) + h^2[\cot(b) - \cot(a)]$ where $a = \arcsin(h/3)$ and $b = \pi - \arcsin(h/3)$. The problem is, Fletcher the oil man does not want to calculate arcsines or cotangents. We can do two things for him. First, design a stick with marks at the appropriate heights to indicate $\frac{1}{8}$-tank left, $\frac{1}{4}$-tank left, $\frac{3}{8}$-tank left, etc. Second, come up with a simple rule of thumb to describe how the height relates to the amount of oil left.

Further Study

This problem does not directly relate to other areas of mathematics. However, we hope the reader will devote much further study to the art of making complicated mathematical results useful and easy to understand. Communication between mathematicians and users of mathematics is vitally important.

CHAPTER 6

Sequences and Series

6.1 Sequences

One of the underlying concepts of calculus is that we can often solve complicated problems by generating a sequence of approximations which tend to get closer and closer to the exact solution. We have already seen this idea applied to the study of limits, derivatives and integrals as well as to several rootfinding methods. In this section, we will formally introduce the mathematical notion of a *sequence* of real numbers. Our discussion will serve to unify much of our previous work as well as to lay the foundation for the remainder of the chapter.

Example 1. Conjecturing the Value of a Limit

Conjecture the value of the limit (if it exists) $\lim_{x\to 1} \dfrac{\sqrt{x+1}-2}{x-1}$. We can use the Solver to generate some function values.

x	1.1	1.01	1.001	1.0001
$f(x)$	.248457	.249844	.249984	.249998

Of course, we also need some function values for $x < 1$.

x	1.1	1.01	1.001	1.0001
$f(x)$	.251582	.250156	.250015	.250001

Based on this evidence, we would feel comfortable conjecturing that the limit equals .25. ■

Thinking through Example 1, it is natural to focus more on the values of $f(x)$ than on the corresponding x-values. In fact, as long as the x-values approach 1, it does not seem especially important which x's we use. However, we must closely examine the pattern of numbers .248457, .249844. .249984, .249998 and the pattern of numbers .251582, .250156, .250015, .250001. Even though the two sequences of numbers are quite different, they appear to have a common limit of .25.

In the definitions below, we extract the most important features of Example 1 and put them into a general framework. For a more complete discussion of sequences, we refer you to your calculus text.

Roughly speaking, a *sequence* of real numbers is a collection of values, each corresponding to a specific choice of an integer (the *index*). We typically use set notation to describe the sequence a_1, a_2, a_3, ..., writing $\{ a_1, a_2, a_3, ...\}$ or more simply $\{a_n\}_{n=1}^{\infty}$.

For example, if $a_n = \dfrac{1}{n^2}$ for $n = 1,2,3,...$, we have the sequence

$$1, \quad \frac{1}{4}, \quad \frac{1}{9}, \quad \frac{1}{16}, \quad \frac{1}{25}, \quad \frac{1}{36}, \quad \frac{1}{49}, \ldots$$

Note that as we go further and further out in the sequence (i.e., as the index n gets larger and larger), the terms of the sequence are getting closer and closer to 0. In this case, we say that the sequence has the limit 0 or that the sequence *converges to 0.*

Loosely speaking, then, a sequence has the limit L if a_n gets closer and closer to L as n gets larger and larger. In this case, we write

$$\lim_{n \to \infty} a_n = L$$

Let's look a little closer to see what this might mean. We should be able to make a_n as close as we like to L , simply by making n large enough. So, given a desired degree of closeness, say $\epsilon > 0$, we want to have $|a_n - L| < \epsilon$ for n sufficiently large. More precisely, this means that for every $\epsilon > 0$ there must be an integer N such that if $n > N$, $|a_n - L| < \epsilon$. We take this as our definition of convergence.

Compare this to the definition of the limit of a function given in section 2.3. There, we had that

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ we could find a $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. The main difference is the substitution of the condition that $n > N$ for $0 < |x - a| < \delta$. This difference comes from the fact that the sequence a_n is a function defined only for positive integers n and n is tending to infinity. This gives us the flexibility needed to precisely define limits of sequences coming from Newton's method, Riemann sums or any other process.

Example 2. Finding the Limit of a Sequence

Find the limit of the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$. We first observe that the numbers are all smaller than 1 and appear to be approaching 1, as $n \to \infty$. To prove $\lim\limits_{n \to \infty} a_n = 1$, we will use the formula $a_n = \frac{n}{n+1}$. From the definition, we need to find an N so that $n > N$ guarantees that

$$\left| \frac{n}{n+1} - 1 \right| < \epsilon$$

But $\left| \frac{n}{n+1} - 1 \right| = \frac{1}{n+1} < \epsilon$ if $1 < \epsilon(n+1)$. Solving for n, we get

$$n > \frac{1 - \epsilon}{\epsilon}$$

Thus, if N is any integer greater than $\frac{1-\epsilon}{\epsilon}$, then $|a_n - 1| < \epsilon$ whenever $n > N$, and we have proved that $\lim\limits_{n \to \infty} \frac{n}{n+1} = 1$.

■

Note the similarity between this proof and that for a limit problem we have already seen, namely $\lim\limits_{x \to \infty} \frac{x}{x+1} = 1$.

Obviously, our proof in Example 2 was dependent on our having a simple formula for a_n. In practice, it is not always possible to simply represent a_n. In that case, we are limited to observing the sequence and conjecturing a value for the limit. All of the examples in this section start with a formula for a_n, so that we may observe several different types of behavior in controlled situations.

Example 3. A Limit Involving a Trig Function

Find $\lim_{n\to\infty}\left(1+\frac{\cos \pi n}{n}\right)$, if it exists. We first examine several terms of the sequence and look for a pattern. We find

$$\begin{aligned} a_1 &= 1+\cos\pi = 0 \\ a_2 &= 1+\frac{\cos 2\pi}{2} = \frac{3}{2} \\ a_3 &= 1+\frac{\cos 3\pi}{3} = \frac{2}{3} \\ a_4 &= \frac{5}{4} \qquad a_5 = \frac{4}{5} \end{aligned}$$

and so on. On the basis of these few terms, you might guess that the sequence is approaching 1. Computing several more terms might serve to convince you of this. We would also be very comfortable conjecturing that

$$\lim_{x\to\infty}\left(1+\frac{\cos \pi x}{x}\right) = 1+0 = 1$$

We use the fact that $|\cos \pi n| = 1$ for every integer n to prove that 1 is the limit. Note that

$$|a_n - 1| = \left|\frac{\cos \pi n}{n}\right| = \frac{1}{n} < \epsilon$$

if $n > 1/\epsilon$. Thus, if N is any integer greater than $1/\epsilon$, then $|a_n - 1| < \epsilon$ whenever $n > N$. ∎

You may have wondered why we seemed to make a distinction between the limit problems $\lim_{x\to\infty}\left(1+\frac{\cos \pi x}{x}\right)$ and $\lim_{n\to\infty}\left(1+\frac{\cos \pi n}{n}\right)$. There is a subtle difference, since x may be any real number while n may only be an integer. Example 4 illustrates this distinction.

Example 4. A Tricky Limit

Find $\lim_{n\to\infty}(1+\cos 2\pi n)$, if it exists. First, look at $\lim_{x\to\infty}(1+\cos 2\pi x)$. We conclude that the limit does not exist, since $\cos 2\pi x$ oscillates between -1 and 1 and, hence, does not approach any limiting value. However, if we list the first few terms of the sequence we quickly get a different answer, since

$$a_1 = 1 + \cos 2\pi = 2$$
$$a_2 = 1 + \cos 4\pi = 2$$
$$a_3 = 1 + \cos 6\pi = 2$$

and so on. Clearly, $\lim_{n\to\infty}(1+\cos 2\pi n) = 2$ since $a_n = 2$ for all n and a formal proof is hardly necessary. ■

In Examples 5 and 6, we examine a pair of sequences which approach 0 as $n \to \infty$. These examples illustrate two important general rules which we will use in section 6.2. The proofs of these results are left for the exercises.

Example 5. Comparing Powers of n

Find $\lim_{n\to\infty} \dfrac{n^2}{n^{5/2}+3}$ if it exists. We look at several terms of the sequence (use the Solver to generate these values):

$$a_1 = .25$$
$$a_{10} = .31325$$
$$a_{100} = .09999$$
$$a_{1000} = .03162$$
$$a_{10000} = .00999$$
$$a_{100000} = .00316$$
$$a_{1000000} = .001$$

At this point, you might recognize a pattern and conjecture that the sequence is slowly approaching 0.

One of the difficulties in observing limits on the calculator is identifying when a sequence has stopped changing. In this example, $a_{98} = .10101$ and $a_{99} = .10050$, so it might be tempting to conjecture a limit of .1. It always helps to look at a few more terms of the sequence! It also helps to know some general rules to simplify the thought process. This example illustrates a relatively simple rule. That is, if $p(n)$ and $q(n)$ consist only of powers of n and the degree (i.e., the largest exponent) of q is larger than the degree of p, then $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$.

■

Example 6. Comparing Polynomials and Exponentials

Find $\lim_{n\to\infty} \frac{n^2}{e^n}$, if it exists. We compute

$$\begin{aligned} a_1 &= .36787 \\ a_{10} &= .004539 \\ a_{100} &= .000000 \\ a_{1000} &= .000000 \end{aligned}$$

It is not difficult to conjecture a limit of 0. Note that both n^2 and e^n tend to infinity as $n \to \infty$. Thus, this limit tells us that e^n must become large faster than n^2 does. In this case, we say that e^n *dominates* n^2, and the general rule is that exponentials with positive exponents will dominate polynomials. ■

Exercises 6.1

In exercises 1-4, find the limit of the sequence as in Example 2.

1. $\frac{3}{1}, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}, \dots, \frac{2n+1}{n}, \dots$

2. $\frac{2}{1}, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \dots, \frac{3n-1}{n}, \dots$

3. $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^n\frac{1}{n}, \dots$

4. $2, \frac{1}{2}, \frac{4}{3}, \frac{3}{4}, \dots, 1 - \frac{(-1)^n}{n}, \dots$

In exercises 5-10, find the limit of the sequence, if it exists. If the limit exists, find N in terms of ϵ.

5. $a_n = 1 + \frac{\sin(\pi n/2)}{n}$

6. $a_n = 1 + \frac{\cos(n)}{n}$

7. $a_n = 3 + \sin(\pi n)$

8. $a_n = 2 + \cos(\pi n/2)$

9. $a_n = 2 + \frac{(-1)^n}{n}$

10. $a_n = 3 + (-1)^n$

In exercises 11-18, inspect the sequence and conjecture a limit as $n \to \infty$.

11. $a_n = \dfrac{n^3}{n^4+5}$

12. $a_n = \dfrac{2n^2+4n}{n^3-6}$

13. $a_n = \dfrac{3n^3}{8n^2-4}$

14. $a_n = \dfrac{5\sqrt{n}}{n+2}$

15. $a_n = \dfrac{n^2}{e^n}$

16. $a_n = n^3 e^{-n/2}$

17. $a_n = \dfrac{e^n}{n^5+4n+2}$

18. $a_n = \dfrac{e^{-n/2}}{n^2+4n+1}$

In exercises 19-22, use the Solver (and FACT in the Real menu of the HP-28S or ! in the Prob menu of Math of the HP-48SX) to conjecture the limit of the sequence, if it exists.

19. $a_n = \dfrac{n^2}{n!}$

20. $a_n = \dfrac{e^n}{n!}$

21. $a_n = \dfrac{n!}{e^{3n}}$

22. $a_n = \dfrac{n^2 e^n}{n!}$

23. Based on your answers in exercises 19-22, which term is dominant, polynomials, exponentials or factorials?
24. Determine the limit of the sequence $\sin(1/x_n)$ for $x_n = 1/n\pi$ and for $x_n = 2/(4n+1)\pi$. What does this tell you about $\lim\limits_{x\to 0} \sin(1/x)$?
25. Prove that $\lim\limits_{n\to\infty} \dfrac{n^2}{n^{5/2}+3} = 0$. HINT: $\dfrac{n^2}{n^{5/2}+3} < n^{-1/2}$ for all n.
26. Prove that $\lim\limits_{n\to\infty} \dfrac{n^2}{e^n} = 0$. HINTS: $\dfrac{n^2}{e^n} < \epsilon$ if $n - 2\ln(n) > \ln(1/\epsilon)$.
27. Use the calculator to estimate the limits of $(1+1/n)^n$, $(1+2/n)^n$ and $(1+3/n)^n$. Compare to e, e^2 and e^3.

EXPLORATORY EXERCISE

Introduction

In applied mathematics, calculations are not typically restricted to real numbers. As you move beyond graphs (where complex numbers have not played a role for us) you will see more and more complex variables. A complex number z may be written as $z = a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. For instance, using the quadratic formula we get the solutions of $x^2 - 2x + 5 = 0$ to be $\dfrac{2 \pm \sqrt{4-20}}{2} = 1 + 2i$ and $1 - 2i$. Multiplication of complex numbers depends on the identity $i^2 = -1$.

Thus $(1+2i)^2 = 1+4i+4i^2 = 1+4i-4 = -3+4i$. A useful image is to associate the complex number $a+bi$ with the point (a,b). The HP-28S/48SX, in fact, does not distinguish between two-dimensional points and complex numbers. Also, the HP-28S/48SX uses the same syntax for complex arithmetic as for real arithmetic. In this exercise we will look at the sequence defined by $z_{n+1} = z_n^2 - c$, $z_0 = 0$, where $c = a + bi$ is a complex constant. The user-defined function FZ will help us track the sequence. Press ≪→ X ' X ∧ 2 − C ' ≫ ' FZ ' $\boxed{\text{STO}}$. This looks exactly like the notation we used for real functions, but it will work equally well for complex numbers.

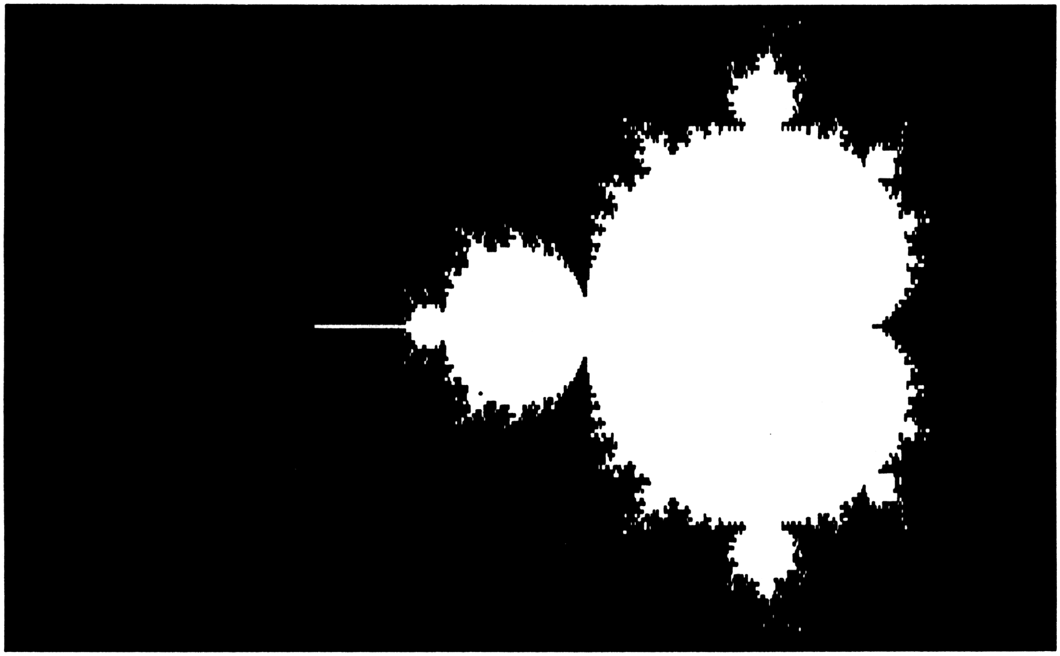
Problems

Determine the behavior of the sequence for $c = .5,\ 1,\ 1.2,\ 1.5,\ 3,\ .2+.2i,\ 1+.2i$, $1+i$ and other values. In general, which c's produce which behavior? To test c=.5, press .5 ' C ' $\boxed{\text{STO}}$ then 0 $\boxed{\text{ENTER}}$ and press $\boxed{\text{FZ}}$ several times. You should see the sequence converge to about -.3660. To test $c = .2+.2i$, press (.2,.2) ' C ' $\boxed{\text{STO}}$ then 0 $\boxed{\text{ENTER}}$ and press $\boxed{\text{FZ}}$ several times. The sequence converges to approximately $-.1864 - .1456i$. Before you jump to an incorrect conclusion try $c = 1+i$: the sequence blows up! You should also find values of c for which the sequence eventually alternates between 2 values, values of c for which the sequence alternates between 4 values, ... The various behaviors are summarized in the remarkable picture below (see HP-28 Insights by William Wickes) which shows some of the detail of what is known as the *Mandelbrot set*. The set is displayed using the following rule: if for $c = a+bi$ the sequence blows up, the point (a,b) is colored. The set is sometimes called the "snowman" because it looks like smaller and smaller balls stacked on top of each other (in our picture, the snowman has fallen down). It turns out that points within the same "ball" have the same behavior (for instance, converging to 1 number, or converging to 2 numbers,...). The set is infinitely complicated in the sense that if you zoom in on what appears to be the edge of the set, you reveal more detail and will find what appear to be miniature copies of the set itself!

Further Study

This exercise opens several doors. The Mandelbrot set is an example of a *fractal*, of which much has been written recently (see, for example, The Science of Fractal Images, ed. by Peitgen and Saupe, Springer-Verlag). The study of sequences such as $z_{n+1} = z_n^2 - c$ belongs to *dynamical systems theory* (see, e.g., An Introduction to Chaotic Dynamical Systems by R. Devaney).

THE MANDELBROT SET



6.2 Infinite Series

Among the many specific sequences which we have already observed in our exploration of calculus are Newton's method approximations and Riemann sums. Both are examples of a special type of sequence, called a *sequence of partial sums*, which we will examine more carefully in this section.

Recall the Newton's method formula for solving the equation $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots$$

One way to think of this is as

$$x_{n+1} = x_n + c_{n+1} \qquad n = 0, 1, 2, \ldots$$

Here, the new approximation x_{n+1} is the sum of the previous approximation, x_n, plus a so-called *correction term* c_{n+1}, where

$$c_{n+1} = -\frac{f(x_n)}{f'(x_n)}$$

Thus, we have that

$$\begin{aligned} x_1 &= x_0 + c_1 \\ x_2 &= x_1 + c_2 = x_0 + c_1 + c_2 \end{aligned}$$

and in general,

$$x_n = x_0 + c_1 + c_2 + c_3 + \ldots + c_n$$

We write this in *summation notation* as

$$x_n = x_0 + \sum_{i=1}^{n} c_i$$

If Newton's method succeeds in finding a root, x, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges and

$$x = \lim_{n\to\infty} x_n = x_0 + \lim_{n\to\infty} \sum_{i=1}^{n} c_i$$

which we write as

$$x = x_0 + \sum_{n=1}^{\infty} c_n$$

The last expression in this equation is called an *infinite series* .

We have the following definitions.

Definition For the sequence $\{a_n\}_{n=1}^{\infty}$, the Mth *partial sum* is

$$S_M = \sum_{n=1}^{M} a_n$$

The infinite series $\sum\limits_{n=1}^{\infty} a_n$ is said to *converge* if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges, in which case we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

If the sequence $\{S_n\}_{n=1}^{\infty}$ does not converge (*diverges*), we say that the infinite series $\sum_{n=1}^{\infty} a_n$ *diverges* .

Example 1. An Infinite Series from Newton's Method

Use Newton's method with an initial guess of $x_0 = 0$ to estimate a root of $x - e^{-x} = 0$. Using the program NEWT from Chapter 4, we get (to 6 decimal places)

$$\begin{aligned} x_1 &= 1 \\ x_2 &= .537883 \\ x_3 &= .578976 \\ x_4 &= .562737 \\ x_5 &= .568840 \\ &\cdot \\ &\cdot \\ x_{14} &= .567143 \\ x_{15} &= .567143 \end{aligned}$$

It appears that Newton's method has converged nicely to a root. We are led to believe this because the iterations get closer and closer together until eventually the first 6 digits do not change at all. Said a different way, the correction terms appear to tend to 0. We then expect that the sequence $\{x_n\}_{n=1}^{\infty}$ converges and, hence, that the infinite series $\sum_{n=1}^{\infty} c_n$ also converges, all because the sequence $\{c_n\}_{n=1}^{\infty}$ appears to converge to 0. ∎

It should be no surprise that we are going to use Example 1 to make what might seem to be a believable conjecture. That is, if $\lim_{n\to\infty} a_n = 0$ we might expect that the infinite series $\sum_{n=1}^{\infty} a_n$ will converge. We should emphasize that this is a reasonable expectation, and in fact is often used in casual investigations of series. However, as we will see in the next example, having $\lim_{n\to\infty} a_n = 0$ does *not* guarantee that the infinite series $\sum_{n=1}^{\infty} a_n$ converges. This is one reason that we have so cautiously talked of conjectures throughout the book, while emphasizing the need for double-checking answers. We do not need to look too far to find a counterexample to the preceding conjecture.

Example 2. The Harmonic Series

Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Because this series is so important, it is commonly referred to by its name, "harmonic series" (see section 6.3 for the meaning of this name). Clearly $\lim_{n\to\infty} \frac{1}{n} = 0$ and so by our ill-fated conjecture above we would expect that the series converges. In fact, it does seem to converge on the calculator, since on the calculator $1 + 10^{-13}$ "equals" 1. But, a clever argument shows that the series does, in fact, diverge. First note that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1$$

Also,

$$\frac{1}{5} + \frac{1}{6} + ... + \frac{1}{16} > 1$$

and

$$\frac{1}{17} + \frac{1}{18} + ... + \frac{1}{64} > 1$$

In fact, no matter how large n is, one can show that

$$\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{4n} > 1$$

Thus, the sum keeps getting larger and larger without bound and, hence, the series does not converge. ∎

The implication of Example 2 is that simply watching the iterations of Newton's method to see that they get closer together, as in Example 1, does not guarantee the convergence of the series. We would do well to check the conjectured answer $x = .567143$ of Example 1 by plugging it back into the function $f(x) = x - e^{-x}$.

Here, we get $f(.567143) = -.0000004$ which gives us further evidence that we have, in fact, found a good approximation to a root.

Our original conjecture should be revised as follows. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ *may or may not* converge. However, if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ does not converge (it diverges).

It turns out that determining whether a series converges or diverges is not that easy in practice. We need all of the tests for convergence found in your calculus book, as well as a mental catalog of significant series (such as the harmonic series) and their convergence or divergence properties. Because of loss of significance errors, the calculator is not a primary weapon in attacking infinite series. (In fact, the HP-28S/48SX will suggest to you that the harmonic series converges!) Below, we offer some ways in which the calculator *does* help us in our investigations.

The program SERIES computes the sum of 100 terms at a time of the series $\sum_{n=c}^{\infty} F(n)$, so that we may monitor changes in the value of the partial sums. The program requires a user-defined function F, a 0 on line 2 of the stack and the value of $c - 1$ on line 1 of the stack. On the HP-48SX, you may use the command Σ to compute partial sums. Even with this command, we emphasize the importance of computing a sequence of partial sums as we do in the examples.

≪ 100 + $\boxed{\text{DUP2}}$ $\boxed{\text{DUP}}$ 99 − $\boxed{\text{SWAP}}$ $\boxed{\text{FOR}}$ N N F + $\boxed{\text{NEXT}}$ $\boxed{\text{SWAP}}$ ≫

Program Step	Explanation
≪ 100 + $\boxed{\text{DUP2}}$	Set up the stack.
$\boxed{\text{DUP}}$ 99 − $\boxed{\text{SWAP}}$	Set up the loop limits.
$\boxed{\text{FOR}}$ N	Start the loop.
N F +	Add F(N) to the sum.
$\boxed{\text{NEXT}}$ $\boxed{\text{SWAP}}$ ≫	End the loop and the program.
ENTER 'SERIES' STO	Store the program under the name SERIES in the current directory.

You may have already come across geometric series in your regular calculus text. In short, a geometric series is a series for which each term is a constant multiple of the preceding term. That is, these are of the form

$$\sum_{n=0}^{\infty} a \cdot r^n$$

where r is called the *ratio* and a is the first term of the series. One remarkable property of geometric series is that we always know when they converge (when $|r| < 1$). Further, when they converge, they will converge to the value $\frac{a}{1-r}$. This is extraordinary in that we only rarely know the sum of a series. We can use the program SERIES (or Σ) to observe the convergence of geometric series.

Example 3. Geometric Series

Find the sum of the geometric series $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \ldots$. This is a geometric series since each term of the sum is a constant multiple ($r = 1/3$) of the previous term. We write the series as

$$\frac{3}{4}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \ldots\right] = \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$$

From the above formula, the sum is $\frac{3}{4}\left(\frac{1}{1-1/3}\right) = \frac{9}{8}$. We can observe the series converge using the SERIES program. First, you'll need to enter a program for the function F:

≪→ N ' .75 * (1 / 3) ∧ N ' ≫ $\boxed{\text{ENTER}}$ ' F ' $\boxed{\text{STO}}$

Since the sum starts at $n = 0$, we enter 0 on line 2 of the stack and $0 - 1 = -1$ on line 1 of the stack (press 0 $\boxed{\text{ENTER}}$ 1 $\boxed{\text{CHS}}$ $\boxed{\text{ENTER}}$). Then press the soft key $\boxed{\text{SERIE}}$. After a few seconds, you should get 1.125 on line 2 of the stack and 99 on line 1, indicating that $S_{99} = 1.125$. Since $1.125 = 9/8$, the series has already converged (within the limitations of the accuracy of the HP-28S/48SX). If you press $\boxed{\text{SERIE}}$ again, you will find $S_{199} = 1.125$ also. On the HP-48SX, store the sequence 'Σ(N=0, M, .75*(1/3)∧N)' in SUM, set M=99 and press $\boxed{\text{SUM}}$ $\boxed{\to\text{NUM}}$ to find S_{99}. Then set M=199 and press $\boxed{\text{SUM}}$ $\boxed{\to\text{NUM}}$ to find S_{199}. Of course, not all series will converge as quickly as this one did, but for a convergent series, the program SERIES (or Σ) helps us to conjecture its value. ∎

Example 4. *p*-Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is an example of another special type of series called a *p-series* . These are of the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. The present example is the case where $p = 2$. In your calculus text, you will find that *p*-series are convergent if $p > 1$ and divergent if $p \leq 1$. Although *p*-series are easy to identify as convergent or divergent (ours is convergent because $p = 2 > 1$), there are no easy ways to compute their sums. Our SERIES program can help us with this. First, enter a new program for the function F:

≪→ N ' 1 / N ∧ 2 ' ≫ [ENTER] ' F ' [STO]

Then, put 0 on line 2 and $1 - 1 = 0$ on line 1 of the stack (0 [ENTER] 0 [ENTER]). Now press [SERIE] several times to get the partial sums

$$S_{100} = 1.63498$$
$$S_{200} = 1.63994$$
$$S_{300} = 1.64160$$
$$S_{400} = 1.64243$$
$$S_{500} = 1.64293$$

On the HP-48SX, store 'Σ(N=1, M, 1/N∧2)' in SUM, set M=100 and press [SUM] [→NUM] , set M=200 and press [SUM] [→NUM] and so on. This is a slowly converging series, but it appears that the sum is about 1.64. (Actually, it has been shown that the sum is exactly $\pi^2/6 \approx 1.644934$.) ∎

One of the more useful techniques for determining whether a series converges or diverges is to somehow compare that series with a given series whose convergence or divergence is already known. The two main tools are the following tests.

Comparison Test Suppose that $0 < a_n < b_n$ for all n and that $\sum_{n=0}^{\infty} b_n$ converges. Then the series $\sum_{n=0}^{\infty} a_n$ also converges. Likewise, if $0 < b_n < a_n$ for all n and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges also.

Limit Comparison Test Suppose that $0 < a_n$, $0 < b_n$, for all n and that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then, if $L > 0$, the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ either both converge or both diverge.

Once you have some experience with these two tests, you will realize that the hardest part of implementing them is to find the right series with which to make a comparison. For the comparison test, it is often hard to see which way the inequality goes for a given prospective comparison series. Your HP-28S/48SX can be of some help in seeing how to use these tests.

Example 5. The Comparison Test

Determine whether or not the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3+n}$ converges. The key here is to notice that for large n, $\frac{\sqrt{n}}{3+n} \approx \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Further, the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since $p = 1/2 < 1$. We then conjecture that our series also diverges. However, $\frac{\sqrt{n}}{3+n} < \frac{1}{\sqrt{n}}$ for all n. Unfortunately, to use the Comparison Test, the inequality needs to go the other way! Don't give up; just compare to a different divergent series. For example, try $\sum_{n=1}^{\infty} \frac{1}{n}$. But, is $\frac{\sqrt{n}}{3+n} > \frac{1}{n}$? Your calculator will help. Press

' $\boxed{\sqrt{x}}$ X / (3 + X) = 1 / X ' $\boxed{\text{STEQ}}$

and then use the Solver to compare the two expressions. Try $x = 1$, $x = 2$, ... until the left-hand side is larger than the right-hand side. From the computations, you should notice that

$$\frac{\sqrt{n}}{3+n} > \frac{1}{n} \qquad \text{for} \qquad n > 4$$

Of course, you'll still need to prove that this inequality actually holds for all $n > 4$. At least the computations lead us to believe that this is true. For $n > 4$, we get

$$\frac{\sqrt{n}}{3+n} > \frac{\sqrt{n}}{n+n} > \frac{2}{2n} = \frac{1}{n}$$

Thus, by the Comparison Test, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3+n}$ diverges. We note that the Limit Comparison Test also works quite well for this series. (Try this!) ∎

Recall that an *alternating series* is a series where successive terms are alternately positive and negative. Determining when such a series converges is a fairly simple matter. You will find a version of the following in any standard calculus text.

Alternating Series Test For the alternating series, $\displaystyle\sum_{n=1}^{\infty}(-1)^n a_n$, if

(i) $a_0 > a_1 > a_2 > ... > a_n > a_{n+1} > ... > 0$ and (ii) $\displaystyle\lim_{n\to\infty} a_n = 0$

then the series converges.

Notice that this is very close to our initial conjecture about a convergence test for series.

Example 6. An Alternating Series

Estimate the sum of the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{3+n}$. First notice that the series converges, by the alternating series test. (Make sure that you check the details of this.) We can use our program to estimate the value of the sum. Using SERIES (or Σ), we get the partial sums

$$S_{100} = -.06321$$

$$S_{500} = -.08943$$

$$S_{1000} = -.09588$$

$$S_{2000} = -.10048$$

$$S_{3000} = -.10252$$

We conjecture that the limit of this slowly converging series is about $-.10$. We note that to modify SERIES to compute 1000 terms at a time, simply change 100 to 1000 and 99 to 999 in the program. ■

We conclude this section with two powerful tests of convergence. First, recall that the series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if the series of absolute values, $\sum_{n=1}^{\infty} |a_n|$, converges.

The Root Test Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. Then, if $L < 1$, the series converges absolutely. If $L > 1$, the series diverges. Finally, if $L = 1$, the test yields no information.

Example 7. The Root Test

Determine whether or not $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges. Note that because of the presence of the term 3^n, it is reasonable to try the Root Test. We need to compare $\lim_{n\to\infty} \sqrt[n]{n^3/3^n} = \lim_{n\to\infty} n^{3/n}/3$ to 1. But, is $\lim_{n\to\infty} n^{3/n} < 3$? We can use the Solver to discover the behavior of the sequence $n^{3/n}$. Store the function 'X∧(3/X)' (put the function on the stack and press $\boxed{\text{STEQ}}$) and use the Solver to generate the following values:

1.148 for X=100
1.0209 for X=1000
1.0027 for X=10000

and so on. It would seem reasonable to conjecture that $\lim_{n\to\infty} n^{3/n} < 3$, in which case the series converges by the Root Test. In fact, the limit in question is 1. (You can show this by taking the natural logarithm of the expression.) Now that we know that the series converges, we can use the program SERIES (or Σ) to compute some partial sums in an effort to approximate the sum. In this case, we get $S_{1000} = S_{2000} = 4.125$. ■

The Ratio Test may be the most frequently used in applications.

The Ratio Test Suppose that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $L > 1$, the series diverges. Finally, if $L = 1$, the test yields no information.

Example 8. The Ratio Test

Determine whether or not $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. We first note that series involving factorials can often be examined using the ratio test. Here, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{n!}{(n+1)!}\\ &= \lim_{n\to\infty} \frac{n!}{(n+1)n!}\\ &= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1\end{aligned}$$

The ratio test then says that the series converges absolutely. The next reasonable question to ask is, "What does the series converge to?" Ordinarily, this remains unknown. We will again use our program SERIES (or Σ) to answer the question. We get:

$$S_{100} = 2.71828182846$$

$$S_{200} = 2.71828182846$$

and so on. In fact, $S_{1000} = 2.71828182846$. Do you recognize this number? You should. It's the irrational number e (at least a 12-digit approximation of e). ∎

Series can be quite tricky to deal with in practice. However, armed with your HP-28S/48SX and a full array of convergence tests, you can readily discover when they converge and compute approximations to the values to which they converge.

Exercises 6.2

In exercises 1-18, estimate the sum of the infinite series, if it converges.

1. $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \ldots$

2. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

3. $\frac{2}{3}+1+\frac{3}{2}+\frac{9}{4}+\ldots$

4. $\frac{3}{4}-\frac{1}{4}+\frac{1}{12}-\frac{1}{36}+\ldots$

5. $-1+\frac{2}{3}-\frac{4}{9}+\frac{8}{27}\cdots$

6. $-2-\frac{2}{3}-\frac{2}{9}-\frac{2}{27}-\cdots$

7. $\sum_{n=1}^{\infty}\frac{1}{n^3}$

8. $\sum_{n=1}^{\infty}\frac{1}{n^{2/3}}$

9. $\sum_{n=1}^{\infty}(-1)^n\frac{3}{n}$

10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{n+1}$

11. $\sum_{n=2}^{\infty}\frac{4}{n^2}$

12. $\sum_{n=3}^{\infty}\frac{-2}{n^{1.2}}$

13. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$

14. $\sum_{n=1}^{\infty}\frac{\sin(\pi n/2)}{2n}$

15. $\sum_{n=1}^{\infty}\frac{2}{\sqrt{n}}$

16. $\sum_{n=2}^{\infty}\frac{4}{n\sqrt{n}}$

17. $\sum_{n=1}^{\infty}(-1)^n\frac{3n}{\sqrt{n^2+2}}$

18. $\sum_{n=1}^{\infty}(-1)^n\frac{5}{\sqrt{n^2+4}}$

In exercises 19-22, use the Comparison Test to determine whether or not the series converges.

19. $\sum_{n=1}^{\infty}\frac{2}{n^2-3}$

20. $\sum_{n=1}^{\infty}\frac{n+1}{n^3-5}$

21. $\sum_{n=1}^{\infty}\frac{3}{\sqrt{n}+2}$

22. $\sum_{n=1}^{\infty}\frac{n-1}{n^2+3}$

In exercises 23-26, use the Root Test or the Ratio Test to determine whether or not the series converges.

23. $\sum_{n=1}^{\infty}\frac{n^2}{6^n}$

24. $\sum_{n=1}^{\infty}\left(\frac{2}{3+n}\right)^n$

25. $\sum_{n=1}^{\infty}\frac{n2^n}{n^{2n}}$

26. $\sum_{n=1}^{\infty}\frac{n^2 2^n}{e^n}$

27. In exercise 26 of section 5.1, you were asked to conjecture whether or not $\int_0^1 \sin(1/x)\,dx$ exists. To argue that it does exist, we will use infinite series. Since $\sin(1/x) = 0$ if $x = 1/\pi,\ 1/2\pi,\ 1/3\pi,\ \ldots$ compute $\int_{1/\pi}^1 \sin(1/x)\,dx$,

$\int_{2/\pi}^{1/\pi} \sin(1/x)\,dx$, $\int_{3/\pi}^{2/\pi} \sin(1/x)\,dx$, ... Each individual integral exists but does the sum of the integrals exist? Compute the 3 integrals above to see if a pattern develops. Note that they alternate signs, so the infinite series converges if the sequence $a_n = \int_{1/n\pi}^{1/(n+1)\pi} \sin(1/x)\,dx$ tends to 0. Using $|\sin(1/x)| \leq 1$, show that $|a_n|$ tends to 0.

28. What are the odds of winning a deuce game in tennis? In this situation, a player wins the game by winning two points in a row. If each player wins one point, the deuce starts over. If player A wins 60% of the points, A wins both points with probability .36 and the points are split with probability .48. Of the 48% split, A wins both of the next two points with probability .36 and they split points again with probability .48. Argue that player A wins the game with probability $.36 + (.48)(.36) + (.48)^2(.36) + (.48)^3(.36) + ...$ and compute the sum.

29. A basketball player makes 90% of his free throws. How many would you expect to be made before the first miss? If the $(n+1)$st is the first miss, there are n made and then 1 missed, which occurs with probability $(.9)^n(.1) = p(n+1)$. The expected number of made free throws is $\sum_{n=0}^{\infty} np(n+1)$. Estimate the sum of this series.

EXPLORATORY EXERCISE

Introduction

Infinite series are useful in many situations. In exercise 27, we saw an infinite series of integrals. We will see in the next section that infinite series of functions are important. We take a quick look at such a series in this exercise.

Problems

Start by doing long division to get $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ...$ This equation only makes sense if the infinite series converges. Show that the series converges if $-1 < x < 1$ and diverges otherwise. Now integrate the equation term by term to get $\arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + ...$ Again, the equation only makes sense if the series converges. Show that the series converges if $-1 < x \leq 1$ and diverges otherwise. Finally, determine all x's for which the following series converge, and estimate the value of the series for three different x's: $\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n$ and $\sum_{n=1}^{\infty}\frac{x^n}{n!}$.

Further Study

It turns out that the above results are not unusual. *Power series* (series involving powers of x) have what is called a *radius of convergence.* If a power series converges on the interval $(0, r)$ then it also converges on $(-r, 0)$. The convergence at the endpoints $x = -r$ and $x = r$ may differ. Your calculus book has more information on power series.

6.3 Series Representations of Functions

Infinite series are often used to represent some quantity of interest. The partial sums are then used to approximate the sum of the series, providing increasingly accurate approximations as the number of terms summed increases. For instance, five iterations of Newton's method may give us a good approximation of the solution of an equation. The sixth iteration (i.e., adding one more term to the partial sum) will generally give a better approximation. Finally, the exact solution equals the limit of the Newton approximations (i.e., the sum of the series).

In this section, we will extend the notion of series representations from those for single numbers to those for functions. That is, given a function $y = f(x)$, we will approximate it as a sum of simpler functions (e.g., polynomials), with the property that the approximation improves as we add more terms to the sum. This may sound like an ambitious project, but there are numerous important applications based on such series representations of functions. For instance, we shall see how a music synthesizer uses a series of pure tones to imitate a particular musical instrument. We will discuss the two most prominent series representations, Taylor series and Fourier series.

TAYLOR SERIES

Although we have sometimes called them by other names, we have already seen some *Taylor polynomials.* For instance, what is the best straight-line approximation of $y = x^2 - 1$? If we are especially interested in maintaining accuracy near the point (1,0) we would choose the tangent line to the curve at (1,0), namely $y = 2x - 2$ (see Figure 6.1 for an HP-48SX graph). If we are more interested in accuracy near the point (2,3), we would choose the tangent line at (2,3) given by $y = 4x - 5$ (see Figure 6.2 for an HP-48SX graph). These tangent lines are examples of Taylor polynomials of degree 1.

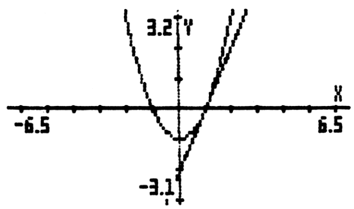


FIGURE 6.1

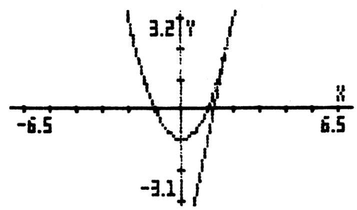


FIGURE 6.2

Since few things in life are linear (i.e., follow straight lines), we need to develop better approximations than tangent lines. So we ask more difficult questions. For a function $f(x)$, which quadratic (2nd-order polynomial) function best approximates it near some point? What is the best cubic approximation? What is the best 4th-order approximation? The following definition provides some answers.

Definition The *Taylor polynomial* of degree n centered about $x = a$ approximating the function $f(x)$ is

$$P_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots + c_n(x - a)^n$$

where the coefficients are given by $c_i = \dfrac{f^{(i)}(a)}{i!}, \quad i = 0, 1, 2, \dots, n.$

Example 1. Computing Taylor Polynomials

Compute the Taylor polynomials of degrees 1, 2 and 3 centered about $x = 0$ for $f(x) = e^x - 1$. First, note that

$$f'(x) = f''(x) = f'''(x) = \dots = f^{(n)}(x) = e^x$$

We then compute the coefficients $c_0 = f(0) = 0$, $c_1 = f'(0) = 1$, $c_2 = f''(0)/2 = 1/2$

and $c_3 = f'''(0)/3! = 1/6$. Thus,

$$P_1(x) = 0 + 1(x - 0) = x$$

Note that this is the tangent line at (0,0) as seen in Figure 6.3 [the HP-48SX display of $P_1(x)$ and $f(x)$]. Similarly,

$$P_2(x) = 0 + 1(x - 0) + \frac{1}{2}(x - 0)^2 = x + \frac{x^2}{2}$$

[see Figure 6.4 for the HP-48SX graph of $P_2(x)$ and $f(x)$]. Finally,

$$P_3(x) = 0 + 1(x - 0) + \frac{1}{2}(x - 0)^2 + \frac{1}{6}(x - 0)^3 = x + \frac{x^2}{2} + \frac{x^3}{6}$$

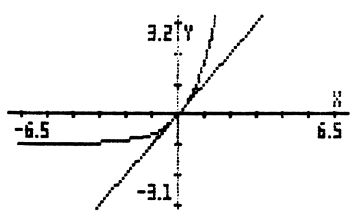


FIGURE 6.3

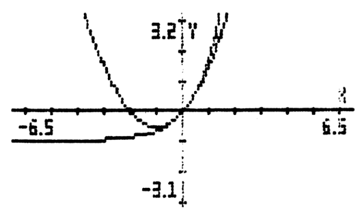


FIGURE 6.4

Figure 6.5 shows the HP-48SX graph of $P_3(x)$ and $f(x)$.

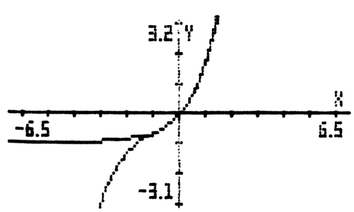


FIGURE 6.5

You should notice that $P_2(x) = P_1(x) + \frac{x^2}{2}$ and $P_3(x) = P_2(x) + \frac{x^3}{6}$. Taylor polynomials of higher order (i.e., higher degree) build on the Taylor polynomials of lower order. Also notice in Figures 6.3-6.5 that all three polynomials are close to the graph of $e^x - 1$ near $x = 0$. However, P_2 remains close for a wider domain of x's than P_1, and P_3 in turn remains close longer than P_2. That is, our approximations of $e^x - 1$ are improving as the degree of the approximating polynomial gets larger, just as we wanted! ■

Example 2. Taylor Polynomials for sin(x)

Repeat Example 1 for $f(x) = \sin(x)$. Note that $f'(x) = \cos(x), f''(x) = -\sin(x)$ and $f'''(x) = -\cos(x)$. We then compute the coefficients $c_0 = \sin 0 = 0$, $c_1 = \cos 0 = 1$, $c_2 = -\sin 0/2 = 0$ and $c_3 = -\cos 0/6 = -1/6$. Then $P_1(x) = x$ is the tangent line seen in Figure 6.6 [the HP-28S graph of $P_1(x)$ and $f(x)$]. We also have $P_2(x) = x$, since $c_2 = 0$. Finally, $P_3(x) = x - \dfrac{x^3}{6}$ [see Figure 6.7 for the HP-28S graph of $P_3(x)$ and $f(x)$]. Notice that here $P_2(x) = P_1(x)$. This points out that the Taylor polynomial P_n is actually a polynomial of degree *at most n*. You can think about the fact that $P_2 = P_1$ in the following way. You cannot draw a parabola that approximates $y = \sin x$ any better (at least near $x = 0$) than the straight line shown in Figure 6.6. (Try this for yourself.) ■

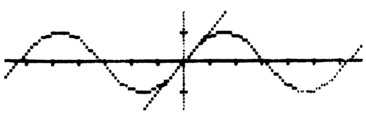


FIGURE 6.6

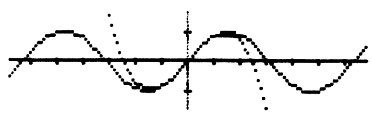


FIGURE 6.7

We should note that the HP-28S/48SX will automatically calculate Taylor polynomials centered about $x = 0$. To obtain the nth degree Taylor polynomial for $f(x)$ about $x = 0$, enter the function on the stack. On the next line, enter 'X' (or whatever other variable that you are using) and, finally, enter the desired degree on line 1 of the stack. Pressing the [TAYLR] soft key will return the desired Taylor polynomial to the stack. (Recall that TAYLR is located in the Algebra menu.)

For example, to get $P_3(x)$ for $f(x) = \sin x$, enter

'SIN(X)' $\boxed{\text{ENTER}}$
'X' $\boxed{\text{ENTER}}$
3 $\boxed{\text{ENTER}}$
$\boxed{\text{TAYLR}}$

The Taylor polynomial $P_3(x)$ is then returned to line 1 of the stack.

Taylor polynomials can be used to approximate all sorts of quantities. For example, in the exercises in Chapter 3, we saw how to use them to approximate the solution of a differential equation. Taylor polynomials can also be used to approximate the value of a definite integral.

Example 3. Estimating the Value of an Integral

Use the Taylor polynomial $P_3(x)$ found above to estimate $\int_0^1 \sin(x^2)\,dx$. Since $P_3(x)$ approximates $\sin x$, we write $\sin x \approx x - \frac{x^3}{6}$ and hence

$$\sin(x^2) \approx x^2 - \frac{(x^2)^3}{6} = x^2 - \frac{x^6}{6}$$

Integrating, we get

$$\int_0^1 \sin(x^2)\,dx \approx \int_0^1 \left(x^2 - \frac{x^6}{6}\right)\,dx = \frac{1}{3} - \frac{1}{42} = .30952$$

The exact value is .31027. ■

The HP-28S/48SX uses Taylor polynomials to approximate the value of definite integrals. This is especially evident on the HP-28S. If you ask it for an antiderivative of $\sin x$, and tell it that $\sin x$ is a polynomial of degree 3, it happily returns $\frac{x^2}{2} - \frac{x^4}{24}$, which is an antiderivative of $P_3(x)$.

Example 4. Higher-Order Taylor Polynomials

Compute $P_8(x)$ and $P_{16}(x)$ centered about $x = 0$ for $f(x) = \sin x$ and compare their graphs to that of $y = \sin x$. Enter

'SIN(X)' $\boxed{\text{ENTER}}$

$$\text{'X'} \ \boxed{\text{ENTER}}$$
$$8 \ \boxed{\text{ENTER}}$$
$$\boxed{\text{TAYLR}}$$

Then, graph the function by pressing $\boxed{\text{RESET}}$ $\boxed{\text{NEWF}}$ on the HP-28S or $\boxed{\text{NEW}}$ $\boxed{\text{RESET}}$ $\boxed{\text{DRAW}}$ on the HP-48SX. Finally, graph $y = \sin x$ on top of the graph of $P_8(x)$ (using OVERD on the HP-28S or by simply graphing $\sin x$ without first pressing $\boxed{\text{RESET}}$ on the HP-48SX). The HP-48SX graph is shown in Figure 6.8. A similar sequence produces the (HP-48SX) graphs of $P_{16}(x)$ and $\sin x$ shown in Figure 6.9.

Note that, to within the resolution of the calculator's graphics display, $P_{16}(x) = \sin x$ for all x in $[-6, 6]$. ∎

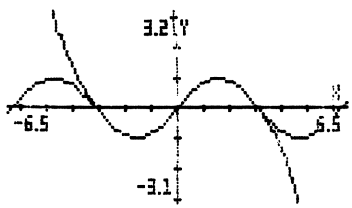


FIGURE 6.8

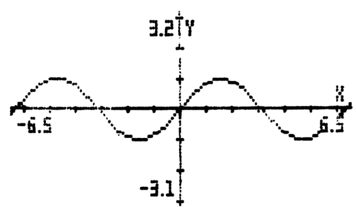


FIGURE 6.9

We have so far accomplished at least part of our goal: we can now find increasingly accurate polynomial approximations to a given function. The final question is: is the function given exactly by the limit of the sequence of approximations? The following result tells us when the answer is yes. First, we give a name to this limit.

Definition For a function $f(x)$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the *Taylor series expansion* for $f(x)$ about $x = a$.

Theorem 6.1 Suppose that the function f has derivatives of all orders (i.e., f', f'', f''', ... all exist) in some open interval containing $x = a$. If the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges for each x in the interval, then it converges to $f(x)$ for all x in the interval.

Thus, we may compute the Taylor series and then determine for which x's the series converges. If the series converges in an interval containing x, then the series must converge to $f(x)$. We do not have to worry about the series converging to the "wrong answer."

Example 5. Using Taylor Series to Find Limits

Recall that in Chapter 2, we discussed the problem of loss of significance errors in computing limits. Here, we use Taylor series to argue that

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Recall from Example 5 of section 2.2 that the Solver cannot accurately compute the values of the function for small values of x. As an alternative to using the Solver, we first compute the Taylor series $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ for $\cos x$. Identifying the coefficients as $a_n = \frac{(-1)^n x^{2n}}{(2n)!}$, we can use the Ratio Test to prove that the series converges for all x. It is then meaningful to write

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

It follows that

$$1 - \cos x = \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

and

$$\frac{1 - \cos x}{x^2} = \frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} + \ldots \quad (x \neq 0)$$

We then conclude that $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \left[\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} + \ldots \right] = \frac{1}{2}$.

■

The reader should beware that our calculations are "formal." That is, they look right, but we have not taken the care to show that each step in the derivation is legal and meaningful (for instance, for which x's does the series $\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} + \ldots$ converge?).

FOURIER SERIES

One of the benefits of Taylor series is that they enable us to calculate values such as $\sin(1)$ using only addition, subtraction, multiplication and division. This is the language of humans and our computers. However, nature's language is rarely arithmetic. Nature often comes to us in sine waves: sight and sound, for instance, are essentially wave phenomena. In applications involving waves, sines and cosines are typically simpler and more natural than polynomials. We are then led to solve the following problem. Given a function $f(x)$, represent f as a series of sines and cosines.

This may be a bit ambitious, so we start with a simpler problem. Suppose that we know in advance that a function is the sum of a few sines and cosines. For instance, suppose

$$f(x) = a_1 \cos \pi x + a_2 \cos 2\pi x + b_1 \sin \pi x + b_2 \sin 2\pi x$$

Given $f(x)$, how can we determine the constants a_1, a_2, b_1 and b_2? The solution may not be obvious, but we only need integration to understand it. First, multiply the above equation by $\cos \pi x$ and then integrate from -1 to 1. We get

$$\begin{aligned}\int_{-1}^{1} f(x) \cos \pi x \, dx &= a_1 \int_{-1}^{1} \cos^2 \pi x \, dx + a_2 \int_{-1}^{1} \cos 2\pi x \cos \pi x \, dx \\ &\quad + b_1 \int_{-1}^{1} \sin \pi x \cos \pi x \, dx + b_2 \int_{-1}^{1} \sin 2\pi x \cos \pi x \, dx\end{aligned}$$

This looks like a mess, but evaluate the integrals. You should find that the first integral on the right is 1 and the other 3 integrals are 0! Thus

$$\int_{-1}^{1} f(x) \cos \pi x \, dx = a_1$$

and we have solved for a_1. How can we find a_2? Multiply the original equation by

$\cos 2\pi x$ and integrate from -1 to 1. Here, we get

$$\int_{-1}^{1} \cos 2\pi x\,dx = a_1 \int_{-1}^{1} \cos \pi x \cos 2\pi x\,dx + a_2 \int_{-1}^{1} \cos^2 2\pi x\,dx$$
$$+ b_1 \int_{-1}^{1} \sin \pi x \cos 2\pi x\,dx + b_2 \int_{-1}^{1} \sin 2\pi x \cos 2\pi x\,dx$$

Again, all but one of the integrals on the right side are 0 and we get

$$\int_{-1}^{1} f(x) \cos 2\pi x\,dx = a_2$$

You should be able to supply the details behind the remaining formulas:

$$\int_{-1}^{1} f(x) \sin \pi x\,dx = b_1$$

$$\int_{-1}^{1} f(x) \sin 2\pi x\,dx = b_2$$

We now present the general result.

For a function f, define the *Fourier series* of f on the interval $[-L, L]$ by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

By essentially the same process as that illustrated above, we find that

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L)\,dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L)\,dx$$

The constants a_n, $n = 0, 1, 2, \ldots$ and b_n, $n = 1, 2, \ldots$ are called the *Fourier coefficients* of f. For details of this, see Churchill and Brown, Fourier Series and Boundary Value Problems.

Theorem 6.2 Suppose that f and f' are continuous on the interval $[-L, L]$ except for possibly a finite number of jump discontinuities. Then, the Fourier series for f on $[-L, L]$ converges to $f(x)$ at all points where f is continuous.

This is indeed a nice result! The function f does not even have to be continuous for the series to converge. However, Fourier series are often very slowly converging (i.e., it takes many terms to obtain a reasonable approximation), as we shall see in the examples to follow.

Example 6. Fourier Series

Compute the Fourier series for $f(x) = x^2$ on $[-1, 1]$ and graph the 4th and 8th partial sums of the series. We compute the coefficients $a_n = \int_{-1}^{1} x^2 \cos n\pi x \, dx$ and $b_n = \int_{-1}^{1} x^2 \sin n\pi x \, dx$ using integration by parts. We get

$$\begin{aligned} a_n &= [x^2 \frac{1}{n\pi} \sin n\pi x + 2x \frac{1}{n^2\pi^2} \cos n\pi x - 2\frac{1}{n^3\pi^3} \sin n\pi x]_{-1}^{1} \\ &= \frac{4(-1)^n}{n^2\pi^2} \qquad \text{if} \qquad n \neq 0 \\ b_n &= [-x^2 \frac{1}{n\pi} \cos n\pi x + 2x \frac{1}{n^2\pi^2} \sin n\pi x + 2\frac{1}{n^3\pi^3} \cos n\pi x]_{-1}^{1} = 0 \end{aligned}$$

Also, $a_0 = \int_{-1}^{1} x^2 \, dx = \frac{2}{3}$. The Fourier series for f is then

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos n\pi x = \frac{1}{3} + \frac{4}{\pi^2}[-\cos \pi x + \frac{1}{4} \cos 2\pi x - \frac{1}{9} \cos 3\pi x + \frac{1}{16} \cos 4\pi x + ...]$$

The HP-48SX graphs of $y = x^2$, together with the 4th and 8th partial sums, are shown in Figures 6.10 and 6.11, respectively.

Note that in both cases, the curves are essentially identical on the interval $[-1, 1]$. Although the curves are not particularly close outside of $[-1, 1]$, this is not a deficiency, since our only intention in finding the Fourier series expansion was to find an approximation valid on $[-1, 1]$. ∎

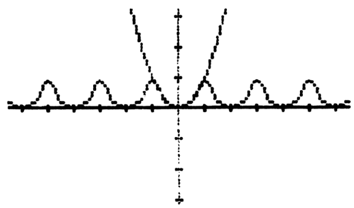


FIGURE 6.10

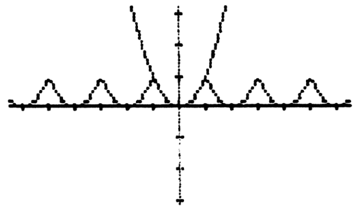


FIGURE 6.11

Example 7. Fourier Series

Repeat Example 6 for $f(x) = x$. This time,

$$a_n = \int_{-1}^{1} x \cos n\pi x \, dx = 0 \text{ for } n = 0, 1, 2, \ldots$$

$$\begin{aligned} b_n &= \int_{-1}^{1} x \sin n\pi x \, dx \\ &= [-x \frac{1}{n\pi} \cos n\pi x + \frac{1}{n^2\pi^2} \sin n\pi x]_{-1}^{1} \\ &= \frac{-2(-1)^n}{n\pi} \end{aligned}$$

The Fourier series is then

$$\frac{2}{\pi}[\sin \pi x - \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x - \frac{1}{4} \sin 4\pi x + \ldots]$$

The HP-48SX graphs of $y = x$, together with the 4th and 8th partial sums, are shown in Figures 6.12 and 6.13, respectively. ■

Figures 6.10-6.13 may surprise you. It really is possible to approximate parabolas and straight lines with sums of sines and cosines. The computation of the Fourier coefficients was not easy, but with practice, the symmetry tricks (e.g., in Example 7, a_n can be seen to be 0 because the integrand, $x \cos n\pi x$, is an odd function and the integration is over a symmetric interval) and integration by parts will become routine.

The series derived in Example 7 is very important in the design of music synthesizers. On an oscilloscope, each musical instrument has an identifying waveform

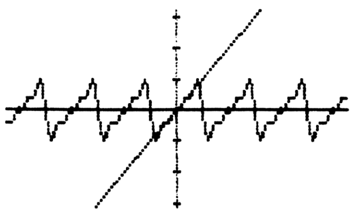


FIGURE 6.12

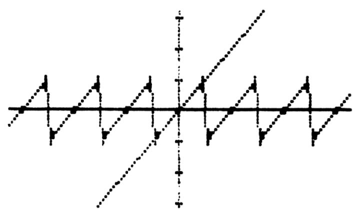
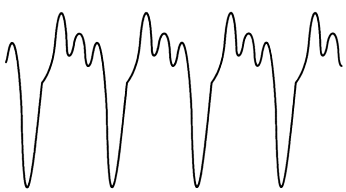


FIGURE 6.13

(see Figures 6.14 and 6.15 for the waveforms of a saxophone and clarinet, respectively; reprinted with permission from UMAP module 588, "Music and the Circular Functions," by Dorothea Bone). A pure tone is represented by a sine wave. By combining a small number of pure tones in the proper proportions, a music synthesizer approximates the sounds of various instruments.

How are the proper proportions determined? The answer is: by using Fourier series!



Saxophone G#
209 vibrations per second

FIGURE 6.14

Clarinet C
260 vibrations per second

FIGURE 6.15

The function $y = x$ generates one of the two basic non-pure waves built into synthesizers (the other, called a *square wave*, is discussed in the exercises). This wave is called a *sawtooth wave*. The proportions of the various sine terms (called the *harmonics*) are crucial. In absolute value (with $2/\pi$ factored out) the size of the nth Fourier coefficient in the expansion of $f(x) = x$ is $1/n$. In the language of music synthesizers, the "harmonic content" varies as $1/n$ (recall that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the harmonic series). By itself, the sawtooth wave represents an oboe-like sound, but it is easily modified (by varying the Fourier coefficients) to produce other familiar tones.

Exercises 6.3

In exercises 1-6, compute the Taylor polynomials of degrees 1, 2 and 3 of $f(x)$ centered at $x = a$ and sketch the graphs.

1. $\cos x$, $a = 0$
2. $x^4 - 1$, $a = 0$
3. $\sqrt{x+1}$, $a = 0$
4. $\dfrac{1}{x+1}$, $a = 0$
5. $\sin x$, $a = \pi$
6. $\ln x$, $a = 1$

In exercises 7-10, use the TAYLR command to compare the graphs of $P_8(x)$, $P_{16}(x)$ and $f(x)$. In each case, take $a = 0$.

7. $\cos x$
8. $\sqrt{x+1}$
9. $\dfrac{1}{x+1}$
10. $\tan x$

In exercises 11-14, use Taylor series to argue that the limits are correct.

11. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^2} = 0$
12. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}$
13. $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$
14. $\displaystyle\lim_{x\to 0} \frac{\ln x - (x-1)}{(x-1)^2} = -\frac{1}{2}$

In exercises 15-18, estimate $\displaystyle\int_0^1 f(x)\,dx$ using $P_3(x)$ for $\cos x$ (in exercises 15 and 17) or $\sqrt{x+1}$ (exercises 16 and 18).

15. $f(x) = \cos(x^2)$
16. $f(x) = \sqrt{x^2+1}$
17. $f(x) = \cos(x^3)$
18. $f(x) = \sqrt{x^3+1}$

In exercises 19-26, determine the first 4 terms of the Fourier series for $f(x)$ on $[-1, 1]$ and graph the 4th partial sum. HINT: in exercises 19-22, use Examples 6-7.

19. $f(x) = x - 1$
20. $f(x) = x^2 - 1$
21. $f(x) = 2x - 1$
22. $f(x) = 3x^2 - 1$
23. $f(x) = |x|$
24. $f(x) = 3\sin(2\pi x)$
25. $f(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0 \end{cases}$
26. $f(x) = \begin{cases} -1/2 & x \le -1/6 \\ \sin(\pi x) & -1/6 < x < 1/6 \\ 1/2 & x \ge 1/6 \end{cases}$
27. The Fourier series for the function in exercise 25 is the *square wave* which music synthesizers use. Describe the harmonic content of the square wave.
28. The Fourier series for the function in exercise 26 represents the *clipping* which a guitar amplifier does. The clipped function has nonzero harmonic content for all n, with a richer tone than a pure $\sin x$. Describe the harmonic content.

EXPLORATORY EXERCISE

Introduction

Fourier series is a part of the field of *Fourier analysis*, which is vital to many engineering applications. Fourier analysis includes Fourier transforms (you may have heard of the Fast Fourier Transform, or FFT) and various techniques for applying Fourier series to real world phenomena. We get an idea of how these techniques work below.

Problems

Given measurements of a signal (waveform), the goal is to construct the Fourier series of the signal function. We start with a simple version of the problem. Suppose the function has the form $f(x) = a_0/2 + a_1 \cos \pi x + a_2 \cos 2\pi x + b_1 \sin \pi x + b_2 \sin 2\pi x$ and we have the measurements $f(-1) = 0$, $f(-1/2) = 1$, $f(0) = 2$, $f(1/2) = 1$ and $f(1) = 0$. Plugging into f we get $f(-1) = a_0/2 - a_1 + a_2 = 0$, $f(-1/2) = a_0/2 - a_2 - b_1 = 1$, $f(0) = a_0/2 + a_1 + a_2 = 2$, $f(1/2) = a_0/2 - a_2 + b_1 = 1$ and $f(1) = a_0/2 - a_1 + a_2 = 0$. Note that b_2 never appears in an equation, and the $f(-1)$ and $f(1)$ equations are identical. We have 4 equations and 4 unknowns. Solve the equations [HINT: start by comparing the $f(1/2)$ and $f(-1/2)$ equations, then the $f(0)$ and $f(1)$ equations]. You should conclude that $f(x) = 1 + \cos \pi x$, and we have no information about b_2. To get b_2 we would need another function value. Thus, the number of measurements determines how many terms we can find in the Fourier series. Repeat the above for measurements $f(-1/2) = -1/2$, $f(0) = 0$, $f(1/2) = 1/2$ and $f(1) = 0$ and compare to the Fourier series in Example 7.

There is, fortunately, an easier way to determine the Fourier coefficients. Recall that $a_n = \int_{-1}^{1} f(x) \cos(n\pi x)\, dx$ and $b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\, dx$. From the function values at $x = -1/2$, 0, 1/2 and 1, we can estimate the integral. Which approximation rule gives the correct values of a_n and b_n in the above examples? Use this approximation rule to find the relevant constants given $f(-3/4) = -3/4$, $f(-1/2) = -1/2$, $f(-1/4) = -1/4$, $f(0) = 0$, $f(1/4) = 1/4$, $f(1/2) = 1/2$, $f(3/4) = 3/4$ and $f(1) = 0$. Again, compare to the series in Example 7.

Further Study

The general formulation of our work above is called the *inverse Fourier transform.* This can be found in numerous engineering mathematics books (see, for example, Orthogonal Transforms for Digital Signal Processing by Ahmed and Rao) although it is typically presented in terms of complex variables. For an enjoyable overview of several current applications of Fourier analysis, see Visualization by Friedhoff and Benzon, Abrams Publishers.

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Answers to Odd-Numbered Exercises

Section 1.1

1. 4.41666666667
3. .33333333332
5. 12.6419753087
7. 1.19522860933
9. 24.2901663962
11. 3.76219569108
13. .984375 $\to$ 1.0
15. .857142857143 $\to$ 1.0
17. 2.97604617604 $\to \pi$
19. .499999999999 $\to$ .5
21. $n = 39$
23. $n = 40(c = 40)$, $n = 39(c = 120)$
33. P(1970) = 3.594 billion, P(1980) = 4.376 billion, P(1985) = 4.909 billion, and P(2035) = 4.130×10^{53} billion

Section 1.2

1. 3 roots
3. 1 root
5. no roots
7. 3 roots
9. no roots
13. vertical asymptotes: $x = -3$, $x = 1$; horizontal asymptote: $y = 0$
15. no vertical asymptotes; horizontal asymptote: $y = 2$
17. no vertical asymptotes; oblique asymptote: $y = x$
19. (.10365,1.4146), (16.4817,66.9268)
21. $\pm$ (.7391,.7391)
23. (.5671,.5667)
25. oblique asymptote: $y = 2x - 2$
27. no oblique asymptote, looks like a parabola for $|x|$ large.
29. Similar to $y = x^{n-m}$ (true for $n > m$).
31. It looks circular.

Section 1.3

1. -1, 1.70997594668
3. -1, -2.8311772072
5. ± 1.41421356237
7. 1.409624004 , $-.636732650807$

Section 1.3 (cont.)

9. $(x^2-2)(x^2+3)$
11. 10, ± 1.41421356237
13. 0
15. max at .183505, min at 1.816496
17. min at 1.20507104
19. 14.1421-by-14.1421
21. same as exercise 19
23. $f(x)$ is nearly 0 for x between $-.8$ and .8.
25. $\frac{1}{2}[\pi - \arcsin(32R/S^2)]$, $\frac{1}{2}[2\pi + \arcsin(32R/S^2)]$

Section 2.1

1. $-1/2$
3. does not exist
5. 0
7. $3/2$
9. 0
11. 0
13. $1/4$; $1/\pi$; $1/c$
15. does not exist, does not exist, 0
17. 0
19. 6
21. 1

Section 2.2

1. .25; around $x = 50,001$
3. 1; around $x = 1 \times 10^{10}$
5. $1/6$; around $x = .0001$
11. 3; does not exist
13.

	$x=1$	$x=10$	$x=100$	$x=1000$
$\sin \pi x$	0	0	0	0
$\sin 3.14x$	.00159	$-.0159$	$-.159$	$-.9997$

15. 16×10^{499} is stored as $9.99999999999 \times 10^{499}$.

Section 2.3

1. $\epsilon = .1 : \delta = .0333$; $\epsilon = .05 : \delta = .01666$ (Your values for δ could be smaller.)
3. $\epsilon = .1 : \delta = .316227766017$; $\epsilon = .05 : \delta = .2236$
5. $\epsilon = .1 : \delta = .39$; $\epsilon = .05 : \delta = .1975$
7. $\epsilon = .1 : \delta = .0465$; $\epsilon = .05 : \delta = .0241$
19. .0066

Section 3.1

1. 2
3. -1
5. 0
7. 1
9. 2
11. -1
13. 0
15. 1
19. 2
21. -1
23. 0
25. does not exist

Section 3.2

1. 0.0
3. 2.01940986178
5. .136082763488
7. $2x \sin x + x^2 \cos x$
9. $1/(x^2+2) - 2x^2/(x^2+2)^2$
11. $2(\sin x + x \cos x)x \sin x$

13.

h	Forward	Backward	Centered
.1	.995004165278	.995004165278	.99500416528
.01	.999950000417	.999950000417	.999950000415
.001	.9999995	.9999995	.9999995

15.

h	Forward	Backward	Centered
.1	.0498756211	−.0498756211	0.0
.01	.004999875	−.004999875	0.0
.001	.0005	−.0005	0.0

17.

h	Forward	Backward	Centered
.1	.0990049833749	−.0990049833749	0.0
.01	.00999900005	−.00999900005	0.0
.001	.000999999	−.000999999	0.0

19.

h	Forward	Backward	Centered
.1	.990099009901	.990099009901	.9900990099
.01	.999900009999	.999900009999	.99990001
.001	.999999000001	.999999000001	.999999

21. All centered differences are 0.

23. d/dx $[\sin x°] =$ d/dx $[\sin(x\pi/180)] = (\pi/180)\cos(x\pi/180)$

25. .5

27. 1

29. 11 seconds

Section 3.3

1. $f(x)$	1	0	4	9
$T(x)$	−3	−1	3	5
3. $f(x)$	$-1/\sqrt{2}$	$-1/2$	$1/2$	$1/\sqrt{2}$
$T(x)$	$-\pi/4$	$-\pi/6$	$\pi/6$	$\pi/4$
5. $f(x)$	1.4142	1.7320	2	2.2360
$T(x)$	1.5	2.0	2.5	3.0
7. $f(x)$	1.0772	1.2599	1.4812	1.7099
$T(x)$	1	1	1	1
9. $f(x)$	.7071	.8660	.8660	.7071
$T(x)$	1	1	1	1
11. $f(x)$	.1353	.3678	2.7182	7.3890
$T(x)$	−1	0	2	3

13. −.41825

15. 2.05

17. 1.54326

19. $P - 2Px/R$

Section 3.4

5. 3.0
7. 2.0
9. 2.3346
11. 1.74339
13. 29.6923
15. 5
17. 2.51066
19. 1.7924
21. 3.4657 hr
23. 1, 1/2, 1/4, 1/8
25. ≪ '(B-A)/H' EVAL 'N' STO 1 N FOR I EULER NEXT ≫

Section 4.1

1. -1.2184; BIS-132 steps, NEWT-4 steps, SCNT-5 steps
3. 2.0238; BIS-13 steps, NEWT-5 steps, SCNT-5 steps
5. $-.2975$; BIS-13 steps, NEWT-3 steps, SCNT-7 steps
7. -1.10485; BIS-13 steps, NEWT-3 steps, SCNT-7 steps
9. 1.73205
11. 2.1381
13. .73908
15. Division by 0 on second step; try $x_0 = .5$: $x = .4362$.
17. Division by 0 on first step; try $a = 1.5$ and $b = 2$: $x = 1.870495$.
19. Division by 0 with $x = 1$; multiply by $x^2 + 3x - 4$: $x = 1.414$.
21. $(x-1)(x^2+4) = 0$ if $x = 1$; one Newton step.
23. $(x-1)^2(x^2-x+3) = 0$ if $x = 1$; 14 Newton steps with TOL=.0001
25. $x = 1.847$, $d = 8 \times 1.847/5 - 2.153 = .8022$
27. N=12 (exercise 1), N=14 (exercise 3)

Exploratory: ≪→ A B '.01*(B−A)' EVAL 'D' STO 0 100 FOR I 'A+D*I' EVAL DUP 1 10 FOR J NEWTO NEXT .5 + FLOOR R→C NEXT ≫ where NEWTO is program NEWT with the DUP removed.

Section 4.2

1. 1 root
3. 2 roots
5. 2 roots
7. 4.91718592529, 7.7241531924, 11.0859017288
9. -2.82842712474, -2.0, 2.0, 2.82842712475
11. -1.73205080757, 1.73205080757, 10.0
13. 1.0 (multiple), 2.0
15. 0.0 (multiple)
21. 2.2599210499

Section 4.3

1. $f(1) = 2.0$ (absolute max), $f(-2) = -52.0$ (absolute min)
3. $f(1.73205080757) = -25.57$ (absolute min),
   $f(-1.73205080757) = 57.57$ (absolute max)
5. $f(-.48018994) = -3.36$ (absolute min), $f(3) = 982$ (absolute max)

7. $f(0) = 3$ (absolute min), $f(2.2360679) = 3.16666$ (absolute max)

9. $f(1.2646054) = -.81844344$ (absolute max), $f(1) = f(3) = -1.0$ (absolute min)

11. $f(4) = -14.109$ (absolute min), $f(2.288929) = 1.945$ (absolute max)

13. $f(0) = 0$ (absolute min), $f(2) = 4.27$ (absolute max)

15. $x = 3.9144$

17. T(1.0739472) = 2.776 hours (absolute min), saves .1069 hours (6.4 minutes)

21. .1877 sec

23. 116 ft/sec ($x = 105$)

Section 5.1

1. .38
3. 1/3
5. 3/4
7. 2.0
9. 1.0
11. 1.6
13. 2.6666
15. 3.464
17. 4.0
19. does not exist
21. exists
25. 1.166666
27. .577350269189, 1.52752723165, 2.51661147842

Section 5.2

1. 10/3
3. 39.06666
5. −.093138499
7. 2.875 (Trapezoid Rule); 2.86667 (Simpson's Rule)
9. 1.94375 (Trapezoid Rule); 1.9458333 (Simpson's Rule)

11.

N	Midpoint	Trapezoid	Simpson's
4	5.875	6.25	6.0
8	5.96875	6.0625	6.0
16	5.9921875	6.015625	6.0

13.

N	Midpoint	Trapezoid	Simpson's
4	.189697265625	.220703125	.200520833333
8	.197402954101	.205200195312	.200032552083
16	.199349403381	.201301574707	.200002034505

15. .58826
17. 1.4436
19. π (use RIEM)
23. All 3 are exact.
25. Only Simpson's Rule is exact.
27. $E_{2n} = S_{2n} + (S_{2n} - S_n)/15$

Section 5.3

1. 1.8458
3. 2.9205
5. .9545
7. .86466
9. $J = 2.133$, $v_f = 57$ mph
11. 2160
13. 67.848
15. 1.44829 (vs. 1.44797)
17. 2.96867π
19. 2.3364π
21. 670.886
23. −6400

25. .2
27. 7500
29. 9466.31
31. −1.973475, −5.608858, −8.358723; length of ground covered

Section 5.4

1. 1
3. 21.333333333
5. π
7. $2\pi/3$
9. 16π
19. .7854
21. π
23. .5435
25 $\pi/12$
27. 3.09748
29. .5708
31. n (n odd) or $2n$ (n even)

Section 6.1

1. 2.0
3. 0.0
5. 1.0, $N > 1/\epsilon$
7. 3.0, $N = 1$
9. 2.0, $N > 1/\epsilon$
11. 0.0
13. does not exist
15. 0.0
17. does not exist
19. 0.0
21. does not exist
23. factorials

Section 6.2

1. 8/3
3. diverges
5. −3/5
7. 1.202
9. −2.08
11. 2.578
13. −.693
15. diverges
17. diverges
19. converges
21. diverges
23. converges
25. converges
29. 9

Section 6.3

1. $P_1(x) = 1; P_2(x) = P_3(x) = 1 - \frac{1}{2}x^2$
3. $P_1(x) = 1 + \frac{1}{2}x; P_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2; P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$
5. $P_1(x) = P_2(x) = -(x - \pi); P_3(x) = -(x - \pi) + \frac{1}{6}(x - \pi)^3$
15. .9
17. .92857
19. $-1 + \dfrac{2}{\pi}[\sin \pi x - \dfrac{1}{2} \sin 2\pi x + \dfrac{1}{3} \sin 3\pi x - \dfrac{1}{4} \sin 4\pi x + ...]$
21. $-1 + \dfrac{4}{\pi}[\sin \pi x - \dfrac{1}{2} \sin 2\pi x + \dfrac{1}{3} \sin 3\pi x - \dfrac{1}{4} \sin 4\pi x + ...]$
23. $\dfrac{1}{2} + \dfrac{-4}{\pi^2}[\cos \pi x + \dfrac{1}{9} \cos 3\pi x + \dfrac{1}{25} \cos 5\pi x + \dfrac{1}{49} \cos 7\pi x + ...]$
25. $\dfrac{4}{\pi}[\sin \pi x + \dfrac{1}{3} \sin 3\pi x + \dfrac{1}{5} \sin 5\pi x + \dfrac{1}{7} \sin 7\pi x + ...]$
27. The harmonic content varies as $1/n$ for odd n.

Index

Finding the Commands on the HP-48S and HP-48SX

Explanation: Directory path and page are given in parentheses. For example, REPEA (Prg/Brch-2) means that the command REPEA is on the second page of the Brch subdirectory of the Prg directory. To find REPEA, press PRG BRCH NEXT .

AREA (Plot/Plotr/Draw/Fen)
CENT (Plot/Plotr/Draw)
CLLCD (Prg/Dspl-4)
COORD (Plot/Plotr/Draw)
CRDIR (Memory)
C→R (Prg/Obj-2)
DEPTH (Prg/Stk)
DO (Prg/Brch)
DRAW (Plot/Plotr)
DRAX (Plot/Plotr-3)
DROP2, DRPN (Prg/Stk-2)
DUP, DUP2, DUPN (Prg/Stk-2)
EDEQ (Solve, Plot)
ELSE (Prg/Brch-3)
END (Prg/Brch-2)
ERASE (Plot/Plotr)
EXTR (Plot/Plotr/Draw/Fcn)
FCN (Plot/Plotr/Draw)
FIX (Modes)
FOR (Prg/Brch)
FUNC (Plot/Ptype)
IF (Prg/Brch)
IFTE (Prg/Brch-3)
ISECT (Plot/Plotr/Draw/Fcn)
LABEL (Plot/Plotr/Draw)
LCD→, →LCD (Prg/Dspl-4)
NEXT (Prg/Brch-2)
NEW (Plot)
ORDER (Memory)
PLOTR (Plot)
POLAR (Plot/Ptype)
PTYPE (Plot)
REPEA (Prg/Brch-2)
RESET (Plot/Plotr-2)
ROLL, ROLLD (Prg/Stk)
ROOT (Plot/Plotr/Draw/Fcn, Solve)
R→C (Prg/Obj-2)
SCATR (Stat-3)
SOLVR (Solve)
START (Prg/Brch)
STD (Modes)
STEQ (Plot, Solve)
TAYLR (Algebra)
THEN (Prg/Brch-2)
UNTIL (Prg/Brch-2)
WHILE (Prg/Brch)
XRNG (Plot/Plotr)
YRNG (Plot/Plotr)
ZBOX (Plot/Plotr/Draw)
ZOOM (Plot/Plotr/Draw)
*H (Plot/Plotr-3)
*W (Plot/Plotr-3)
< (Prg/Test-2)
> (Prg/Test-2)
Σ + (Stat)
! (Mth/Prob)

All other HP-48S/48SX commands used in this book are on the keyboard.

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