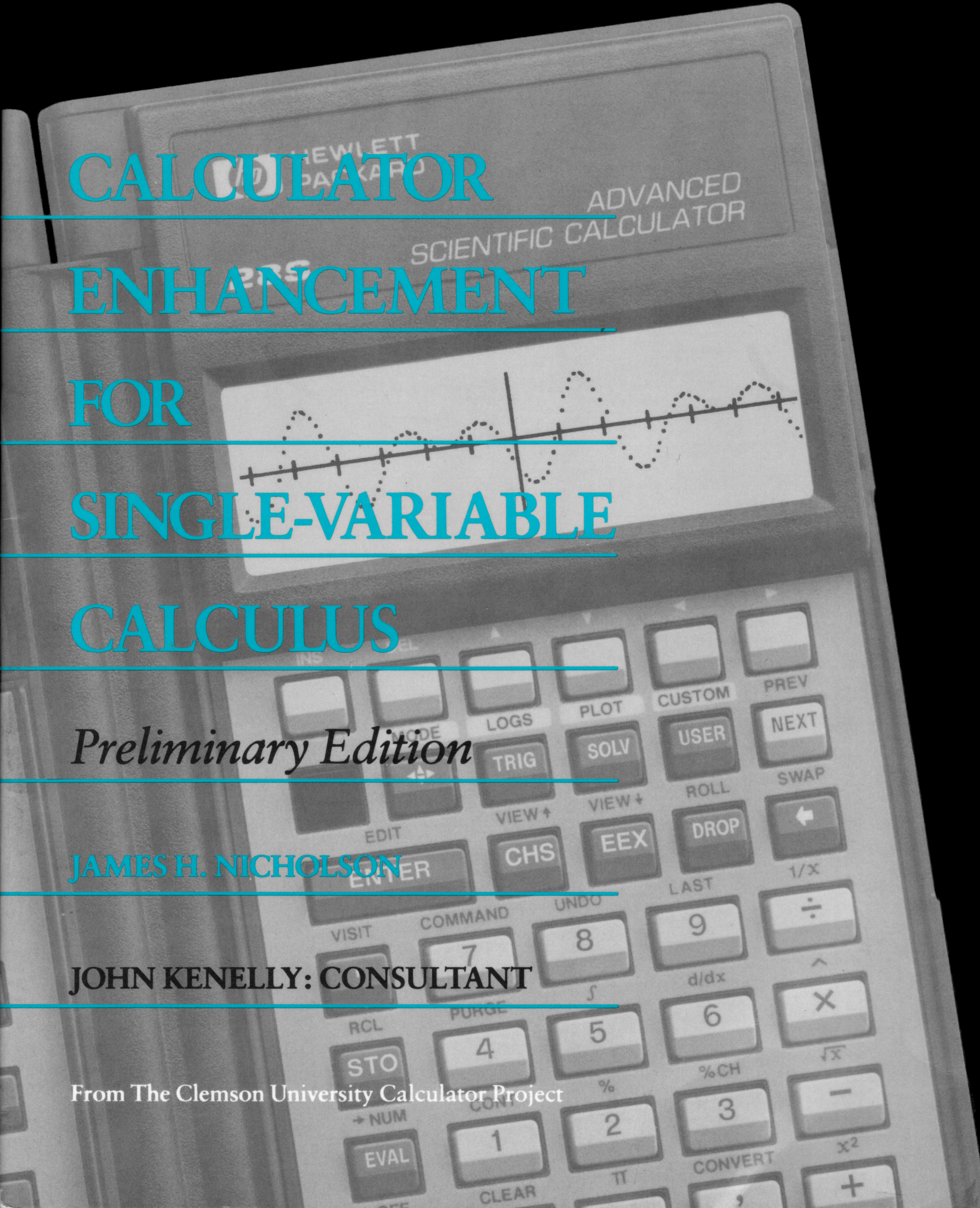

CALCULATOR ENHANCEMENT FOR SINGLE-VARIABLE CALCULUS

Preliminary Edition

JAMES H. NICHOLSON

JOHN KENELLY: CONSULTANT

From The Clemson University Calculator Project

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CALCULATOR ENHANCEMENT FOR SINGLE-VARIABLE CALCULUS

A Manual of Applications Using the HP-28S Calculator

Preliminary Edition

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HBJ

HARCOURT BRACE JOVANOVICH, PUBLISHERS
and its subsidiary, Academic Press

San Diego New York Chicago Austin Washington, D.C.
London Sydney Tokyo Toronto

ISBN: 0-15-505676-X

Library of Congress Catalog Card Number: 90-81983

Printed in the United States of America

PREFACE

This is the preliminary edition of a manual to be used as supplementary material for the single-variable calculus course. The students should have a graphing programmable calculator (Hewlett-Packard Model 28S) and use it without restriction throughout the course. Most of the material in this manual has been used in such a course at Clemson University for three semesters. Although the preliminary edition this manual deals specifically with the use of the HP-28S, it will be replaced by the first edition which will also include material for using the new HP-48SX calculator.

The material is organized in terms of the different aspects of calculator use and it is textbook independent. In fact, it has been used with two different calculus texts at Clemson. The order of presentation is roughly the order in which it has been presented to the classes. There are two sets of exercises at the end of most chapters. Set I is intended for beginning calculus students; Set II is intended for Calculus 2 students.

Use of a graphing calculator greatly enhances the geometric, graphical aspect of calculus. From the outset of the course, derivatives can be continually associated with graphs and differentiation becomes much more than merely a set of algebraic manipulations. Each derivative can be associated with a graph. When the calculator can also do symbol manipulation, find derivatives and solve equations, it becomes both notebook and scratchpad for the students. But the calculator does not organize problem-solving. The students must still find derivatives, their zeroes, and direct the work of the calculator. The calculator simply does some of the manipulation for them.

With graphs so readily available, students continually see the basic ideas of functions increasing or decreasing, and their graphs being concave up or concave down. But to find precisely where the major changes in these properties occur, i.e., the extreme points and inflection points, they must still use basic calculus tools to get more than graphical estimates.

With a graphing calculator, one basic element of the beginning calculus course is changed. Since graphs are readily available on the calculator, no problem can have a graph as its main result. Graphs must be coupled with the major concepts of the course. Likewise, the major concepts of the course can always be associated with graphs.

Programming skill is not required, but some students do fairly extensive programming on their own. Several sets of programs are distributed to the students which extend the graphing capabilities of the calculator and permit rapid calculation of such things as Riemann sums and the limit of convergent infinite series.

Acknowledgements

The author wishes to thank the Fund for the Improvement of Postsecondary Education (FIPSE) whose funding has made possible the calculator-enhanced courses at Clemson.

Last, but not least, thanks to his wife Meredith for her patience during the preparation of this manual.

PROLOGUE

(to the student)

This manual deals with the application of your HP-28S calculator to the tasks of calculus. Although the keystrokes are given for the operations we perform, this manual is not intended to be, and is not, a general reference on the use of the HP-28S. You should familiarize yourself with general calculator operations using the owners manual. In particular, familiarize yourself with the reverse Polish notation and the use of the HP-28S as a ***calculator***. With four stack levels visible, RPN can be very useful for organizing arithmetic operations.

In this manual calculator commands are given in **boldface**, e.g., **PLOT**. If they are single keystrokes, they are also boxed: $\boxed{\text{PLOT}}$. If the key for a command is a menu key, this is indicated by a superscript **M** to the right of the box: $\boxed{\text{DRAW}}^{\mathbf{M}}$. The red key for second functions has generally been omitted from the keystroke sequences since it is assumed that you know that, if a command is the second "red" function over a key, you must press the red key first to get the desired command. However, when using **SOLVR**, $\boxed{\text{RED}}$ $\boxed{\text{X}}^{\mathbf{M}}$ is the command to solve for X, not an indicated second function, so the $\boxed{\text{RED}}$ keystroke is given.

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CHAPTER 1
GRAPHING

INTRODUCTION

Calculus is a first course in the study of the behavior of functions. In the first two semesters of calculus the functions you study will be functions, with real-number values, that are defined on a set of real numbers. In the last semester of the course you will deal with functions, with real-number values, defined on subsets of the plane or three-dimensional space. Later on, vector valued functions will be studied.

Some of the most important aspects of functional behavior - maximum and minimum values, as well as values where the rate of change of the function has a maximum or a minimum - can be very effectively displayed by the graph of the function. In fact, one of the main goals in first-semester calculus is the presentation of efficient ways of accurately finding the coordinates of such points and thus an accurate graph of the function. The appearance in the last few years of graphics calculators and graphing programs for microcomputers has made readily available for calculus students electronic means for achieving a sketch of the graph of almost any function they will deal with in calculus.

Techniques for accurately finding the coordinates of peaks, low spots and other important points on the graph of a function involve several procedures on the HP-28S. Many of these will be discussed later in a chapter on Curve Sketching. In this chapter, some basic facts about the nature of graphs on the HP-28S and basic techniques for graphing functions on the calculator are presented.

THE NATURE OF GRAPHS ON THE HP-28S

The display screen on the HP-28S is a 137 by 32 rectangular grid of picture element dots called "pixels". When the $\boxed{\text{DRAW}}^{\text{M}}$ command is activated, the entire screen is used to display the graph of the expression stored in the storage area designated as EQ.

On the graphing screen, with the default plotting parameters, the pixels in the leftmost column have x coordinates -6.8 and the pixels in the rightmost column have x coordinates 6.8. The highest row of pixels has y coordinates 1.6 and the lowest row of pixels has y coordinates -1.5. Two adjacent columns of pixels have x coordinates differing by 0.1 unit and two adjacent rows of pixels have y coordinates differing by 0.1 unit. The grid marks on the coordinate axes are at the unit marks, i.e., 1, 2, 3 etc.

When you activate the $\boxed{\text{DRAW}}^{\text{M}}$ command for a function f stored in EQ, the HP-28S calculates values of $y = f(x)$ for each of the 137 values of x, 0.1 unit apart, from $x = -6.8$ to $x = 6.8$. If the size of the y coordinate puts the point on the display, i.e., $-1.5 \leq y \leq 1.6$, the appropriate pixel is energized and appears as a point of the graph. So the "graph" of a function on the HP-28S is actually a set of 137, or less, distinct points. When the HP-28S graphs an *equation* instead of a function, it graphs each side of the equation as a function, and so the graph of an equation consists of 274, or fewer, distinct points.

GRAPHING A FUNCTION OR EQUATION ON THE HP-28S

With a function or equation displayed at level 1 of the stack, press $\boxed{\text{PLOT}}$ to bring up the plotting menu, then $\boxed{\text{STEQ}}^{\text{M}}$ to store the expression in EQ and then $\boxed{\text{DRAW}}^{\text{M}}$ to plot the graph. When you want to return to the stack display screen from the graphing screen, press $\boxed{\text{ON}}$.

SETTING THE VIEWING WINDOW

Probably the most important factor in learning to create informative graphs on the HP-28S is learning to set the viewing window to display in appropriate proportions the part of the graph you wish to see. Unlike some graphing calculators and computer programs, the HP-28S does not automatically set the viewing window to accomodate the particular function being plotted. When you graph a function on the HP-28S, the plotting parameters are those used previously. That is, either the parameters set for the last graph or the default parameters set in the calculator at the factory and restored when PPAR is purged.

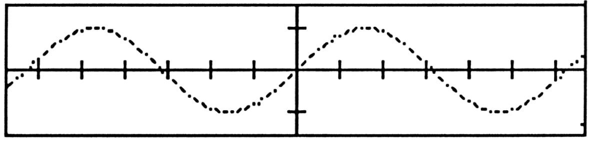
The default setting for the x coordinate range, -6.8 to 6.8, and the y coordinate range, -1.5 to 1.6, works very nicely for trigonometric functions of amplitude 1. The unit distances for x and for y are the same, so your visual intuition of slopes and areas is preserved.

EXAMPLE 1.

To illustrate the default viewing window settings and the graph of y = sin x, perform the following:

'PPAR [PURGE]
'[TRIG] [SIN]^{M} X [ENTER]
[PLOT] [STEQ]^{M} [DRAW]^{M}

(make sure that [MODE] is set in radians and (2π) appears in the upper right corner of the screen)



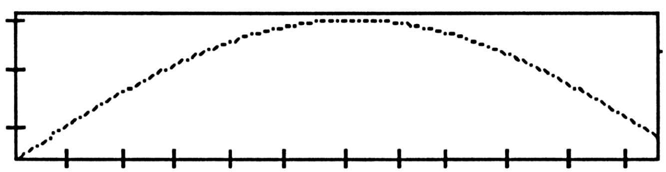
Unfortunately, for many polynomial functions, the default range for y is so limited that very few points of the graph will appear on the screen. But the parameters can easily be changed and they must be adjusted to best illustrate most functions.

If you know in advance the ranges you want for x and y for the graph of a given function, you can set up the desired viewing window by entering the new lower-left corner of the screen as $\boxed{\text{PMIN}}^{\text{M}}$ and the new upper-right corner of the screen as $\boxed{\text{PMAX}}^{\text{M}}$ The keys for these are on the first line of the $\boxed{\text{PLOT}}$ menu. That is, if you want (a,b) as the coodinates of the lower-left corner of the screen and (c,d) as the coordinates of the upper-right corner of the screen, enter (a,b) and press $\boxed{\text{PMIN}}^{\text{M}}$, then enter (c,d) and press $\boxed{\text{PMAX}}^{\text{M}}$. When you next execute $\boxed{\text{DRAW}}^{\text{M}}$, the graphing screen will have the corners you have chosen.

EXAMPLE 2.

To illustrate changes in the corners of the viewing window, complete the steps of Example 1, return to the stack display screen by pressing $\boxed{\text{ON}}$ and then the $\boxed{\text{PLOT}}^{\text{M}}$ menu. Enter the following:

(0,0 $\boxed{\text{PMIN}}^{\text{M}}$ **(3,1** $\boxed{\text{PMAX}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$



New lower-left and upper-right corners for the viewing screen can also be chosen with a graph displayed on the screen. This is handy for zooming in on part of a graph. With a graph displayed on the screen, use the cursor keys (the rightmost four keys on the menu line of white keys) to move the crosshairs to the point you want as your new lower-left corner. Then enter this point on the stack by pressing $\boxed{\text{INS}}$ (above the leftmost key on the menu line of keys). Now move the crosshairs to the point you want as the new upper-right corner and enter it on the stack by pressing $\boxed{\text{INS}}$. Return to the stack display screen by pressing $\boxed{\text{ON}}$. The coordinates of the two points you selected will be displayed on stack levels 1 and 2. Enter these points as $\boxed{\text{PMAX}}^{\text{M}}$ and $\boxed{\text{PMIN}}^{\text{M}}$. If you now press $\boxed{\text{DRAW}}^{\text{M}}$, the graphing screen will have the desired corners.

If you attempt to graph a function having a large range of y coordinates with the screen set for a small range of y coordinates, you will get very few points since most points of the graph are off of the screen. You can compress (or expand) the graph vertically by using the $\boxed{\text{* H}}^{\text{M}}$ key on the second line of the $\boxed{\text{PLOT}}$ menu. If you want to compress the graph vertically by a factor of n, $1 < n$, key in n and then press $\boxed{\text{* H}}^{\text{M}}$. When you return to the graphing screen, the mark on the y axis will then represent n units, if it was 1 unit before, or, in any case, n times what it represented before. Of course, if you enter n, $0 < n < 1$, into $\boxed{\text{* H}}^{\text{M}}$, you will expand the graph vertically.

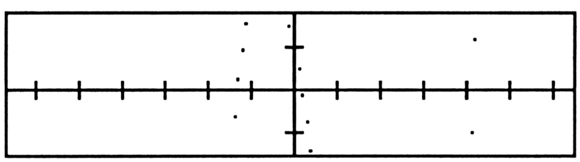
EXAMPLE 3.

To illustrate compressing a graph for more complete viewing, we will try to graph $y = x^3 - 3x^2 - 5x + 1$. Key in the following:

'PPAR $\boxed{\text{PURGE}}$

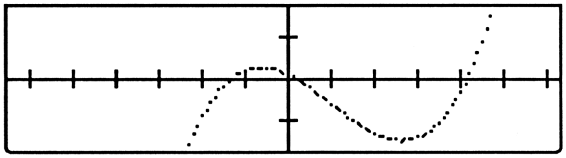
'x ^ 3 - 3 * x ^ 2 - 5 * x + 1 $\boxed{\text{ENTER}}$

$\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$



We get 11 isolated points, so enter:

$\boxed{\text{ON}}$ $\boxed{\text{NEXT}}$ 10 $\boxed{\text{*H}}^{\text{M}}$ $\boxed{\text{PREV}}$ $\boxed{\text{DRAW}}^{\text{M}}$



We compressed the height of the graph by a factor of 10. That is, the mark on the y axis in the final graph represents 10 units instead of 1 unit. The factor of 10 was determined by trial and error. A smaller factor did not give a good picture.

You can similarly expand or compress the graph horizontally by entering a positive factor into $\boxed{\text{*W}}^{\text{M}}$ (also on the second line of the $\boxed{\text{PLOT}}$ menu).

SOME THOUGHTS ON GRAPHS

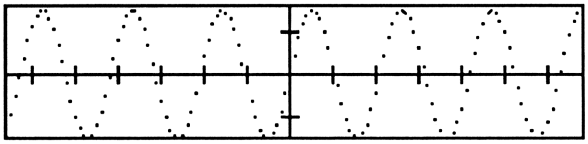
The graph of a function, like the function itself, is an abstraction. A "graph" sketched on paper or plotted on a calculator or computer screen is an attempt to physically reproduce this abstraction, or at least enough of it to discover some facts about the behavior of the function itself. All such attempts to physically reproduce graphs are subject to inherent limitations and inaccuracies.

A graph plotted on the HP-28S consists of discrete points. Some graphs such as sin x plotted with the default plotting parameters look rather "continuous" and smooth, while the graph of $\frac{3}{2}\sin(3x)$ begins to look rather disconnected or "dotty".

EXAMPLE 4.

Graph $f(x) = \frac{3}{2}\sin(3x)$.

Key in [PLOT] [NEXT] [PPAR]^M [PURGE]
' (3/2) * SIN(3*X [ENTER]
[PREV] [STEQ]^M [DRAW]^M

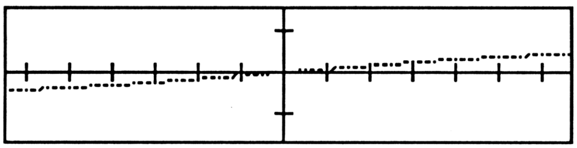


On the other hand, the graph of sin(x/6) with the default parameters shows a distinct "stairstep" characteristic since x-coordinates of adjacent columns of pixels differ only by .025 units and so the corresponding y coordinates are the same, correct to the nearest 0.1 unit; so parts of the "graph" appear to be horizontal line segments.

EXAMPLE 5.

Graph $f(x) = \frac{1}{2}\sin\left(\frac{x}{6}\right)$. If you have not changed the plotting parameters since Example 4:

Key in ' .5 * SIN(X/6) [ENTER] [STEQ]^M [DRAW]^M



The point here is that, while the graph of a function on the HP-28S can give valuable clues to the behavior of the function, the graph on the screen is not an accurate representation of the actual graph. All electronic reproductions of graphs are subject to similar limitations, although the graphs on a much larger computer monitor screen will often look smoother because of the greater resolution (many more pixels) and the fact that many computer graphing programs fill in the spaces between points with some sort of curve.

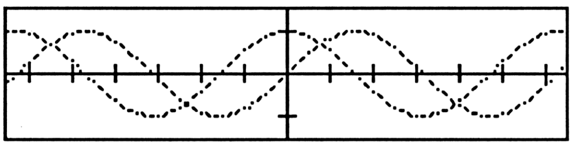
However, substantial information can be inferred from the graphs of functions on the HP-28S. The technology readily produces graphical images with enough accuracy for an overall analysis of the function without the use of much formal mathematical technique.

EXAMPLE 6.

In the default screen, i.e., **'PPAR** $\boxed{\text{PURGE}}$, graph sin x and cos x simultaneously. This is accomplished by entering: **'SIN(X) = COS(X)'** $\boxed{\text{ENTER}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$.

A quick set of key strokes would be:

X $\boxed{\text{SIN}}^{\text{M}}$ X $\boxed{\text{COS}}^{\text{M}}$ $\boxed{=}$ $\boxed{\text{ENTER}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$



Casual observation notes that the functions have the same behavior and the sine is just the cosine delayed by a small amount. A close estimate of the lag, which we know to be $\pi/2$, could be confirmed by an investigation of the coordinates on the sine curve of the first peak after the vertical coordinate axes. To do this, move the cursor to the peak and touch the $\boxed{\text{INS}}$ key, return to the text screen $\boxed{\text{ON}}$ and note the x coordinate -- the value should be close to $\pi/2 \approx 1.57$.

Return to the graph through the $\boxed{\text{DRAW}}^{\text{M}}$ key and note how the rise and fall in the curves are linked. Later on, in calculus, we will establish that one curve is the derivative of the other and then our current intuitive sense will be made formal and quantified. The intuitive sense is the feeling for the curves increasing and decreasing. Beyond that we sense that at different times the rate of change is itself changing, i.e., sometimes the increase is with an increasing rate of increase and sometimes with a decreasing rate of increase. You are observing cyclical behavior. It is a very important phenomena and it occurs frequently in stock markets, personal moods, wave action and all sorts of different activities. The understanding and ability to analyze the dynamic behavior of cyclical patterns is an important reason for studying calculus. Here, with with the graphing calculator, we are able to intuitively understand the some of the patterns and anticipate future results.

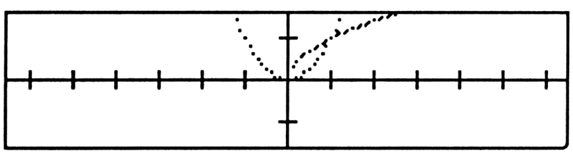
You can easily note when the curves cross and establish the coordinates of the points by the previous technique. We see that the x-values are $-3\pi/4$, $\pi/4$, $5\pi/4$, etc. But note how easily the graph gives the values in regions beyond the first quadrant. These values are typically memorized, with considerable difficulty, in high school trigonometry courses and we see that the graphing calculator gives us an easy means to accurately recall the information whenever it might be needed.

Likewise, important properties of a function and its inverse are easily noted on the graphing calculator:

EXAMPLE 7.

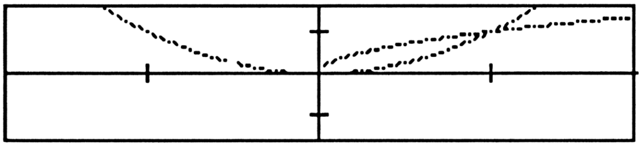
Examine the graphs of the x^2 and its inverse function $\sqrt{x}$ (the square root of x). First sketch the curves with the default viewing settings:

'PPAR [PURGE]
X [X²] X [√X] [=] [ENTER]
[PLOT] [STEQ]M [DRAW]M



Then adjust the horizontal scale by :

[ON] [4] [1/X] [PLOT] [NEXT] [*W] [PREV] [DRAW]M



Note that the square function and its inverse function, the square root, meet on the line y = x and the graphs are reflections across the line y = x. Are these properties true of any function and its inverse?

SAVING AND COMBINING GRAPHS ON THE HP- 28S :

With a graph displayed on the screen of the HP-28S, you can save the screen display as an "image string" by pressing the [DEL] key (over the second white menu key). When you return to the stack display, this string will appear on level 1 and you can then save it under a name of your choice. You can then recall it from the [USER] menu and, with the image string on level 1, reproduce the display it represents by executing [→LCD]M on the [STRING] menu (left keyboard) . You can also save the stack display screen as a string by executing [LCD→]M, and then reproduce it later with [→LCD]M.

EXAMPLE 8.

Graph y = sin x (see example 1)

With the graph showing, press [DEL], the second white key on the menu line. Now return to the stack display with [ON]. Save the image string on level 1: ' SN [STO]. To recall the screen saved, do [USER] [SN]M [STRING] [→LCD]M. Notice that the sine graph is not plotted again; the whole screen is reproduced at once.

You can combine two graphs whose strings you have displayed on levels 1 and 2 by executing the command [OR]M (on the second line of the [TEST] menu, over [O] , or just key in [O] [R] then [ENTER]). This is the logical operator "or" and will give you the combined string on level 1. Executing $\boxed{\rightarrow\text{LCD}}^{M}$ on the combined string will give you the union of the two graphs. In fact, in this way you can combine as many graphs as the resolution of the screen will allow you to use intelligently (or at least as many as the memory will bear; each image string takes 548 bytes to store).

EXAMPLE 9.

To graph y = sin x, y = cos x and y = tan x on the same axes:

You should have the image string for sin x stored under 'SN' from Example 8. Now do

' COS(X $\boxed{\text{ENTER}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{M}$ $\boxed{\text{DRAW}}^{M}$
$\boxed{\text{DEL}}$ $\boxed{\text{ON}}$ ' CS $\boxed{\text{STO}}$.

This saves the image string for cos x under 'CS'.

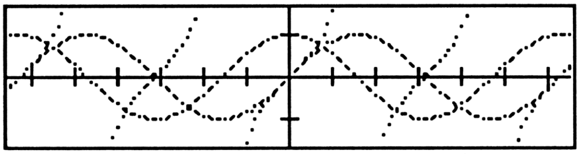
' TAN(X $\boxed{\text{ENTER}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{M}$ $\boxed{\text{DRAW}}^{M}$
$\boxed{\text{DEL}}$ $\boxed{\text{ON}}$ 'TN $\boxed{\text{STO}}$

This saves the image string for tan x under 'TN'.

Recall the strings with $\boxed{\text{USER}}$ $\boxed{\text{TN}}^{M}$ $\boxed{\text{CS}}^{M}$ $\boxed{\text{SN}}^{M}$.

Now combine the levels 1 and 2 strings with OR $\boxed{\text{ENTER}}$.

If we now converted this string to the screen it represents, we would have sin x and cos x superimposed. However, we will proceed and get all three graphs. To combine the string for tan x on level 2 with the combined string on level 1, again do OR $\boxed{\text{ENTER}}$. Now convert this new combined string to its screen with $\boxed{\rightarrow\text{LCD}}^{M}$. This shows all three combined.



We will discuss another method of combining graphs in a later section of this chapter.

SOME CONDITIONS UNDER WHICH THE GRAPHING PROGRAM WILL TERMINATE ON THE HP-28C (This section does not apply to the HP-28S.)

There are three fairly commonly encountered situations which occur in the calculation of the y coordinates for a graph which will cause the graphing program on an early version of the HP-28C to terminate. The calculator will automatically return to the stack display screen and show an error message.

If calculating y involves a division in which the denominator is 0 and the numerator is not 0, the HP-28C will abort the graphing routine and return to the stack display screen with the message "INFINITE RESULT" displayed. This situation is easily corrected by clearing the flag (#59) which causes the error message. Key in: **59 CF** $\boxed{\text{ENTER}}$. Then $\boxed{\text{DRAW}}^{\text{M}}$ again.

Two other situations are not so easily corrected, but the behavior of the HP-28C can be modified by the program on p. 303 of from William Wickes' book.[1] If, in calculating a value of y, a complex number is obtained, the graphing routine will shut down and the HP-28C will return to the stack display screen with the message "NON-REAL RESULT". If calculating y involves a division in which both numerator and denominator are 0, the graphing routine will shut down and the HP-28C will return to the stack display screen with the message "UNDEFINED RESULT".

---

[1]"HP-28 Insights", William C. Wickes, copyright 1988 by Larken Publications.

SOME USEFUL PROGRAMS TO FACILITATE GRAPHING

Create a subdirectory called GSHO by entering [MEMORY] 'GSHO [CRDIR]M . Recall it by [USER] [GSHO]M. Enter the following programs:

QUIT

« HOME [ENTER] 'QUIT [STO]

Input: none Effect: returns to HOME directory

It is a good practice to include a quit key at the end of the menu in each subdirectory.

NEWGRAPHS

```
« CLLCD DUP              | Clears screen, duplicates expression
   1 →LIST               | Forms a list with expression
   'EQS' STO             | Stores list in 'EQS'
STEQ DRAW                | Stores expression in 'EQ', draw
LCD→ 'SCR' STO           | Forms image-string, stores in 'SCR'
   DGTIZ »               | Activates cursor
 [ENTER] 'NEWGRAPHS [STO]
```

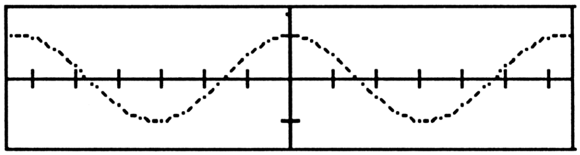
Input: expression Effect: graphs expression, stores graph

The program NEWGRAPHS graphs the expression on level 1, and thus automatically executes [PLOT] [STEQ]M and [DRAW]M. So it graphs a function on level 1 in one keystroke if you have the [NEWG]M key showing on the menu line. More important than saving keystrokes, however, is the fact that if you graph a function with NEWGRAPH, you then have the programs discussed below to use with your graph.

EXAMPLE 10.

Graph f(x) = cos x using **NEWGRAPHS.**

Key in **X** [TRIG] [COS] [USER] (and [GSHO]^{M} , if necessary) [NEWG]^{M}



GETG

« SCR →LCD DGTIZ » [ENTER] **'GETG** [STO]

Input: none	Effect: recalls last graph to screen

OVERDRAW

« DUP	I Duplicate expression
1 →LIST	I Form 1-element list
EQS +	I Combine with list in EQS
'EQS' STO	I Store comb. list in 'EQS'
STEQ	I Store expression in EQ
SCR →LCD	I Convert string in SCR to screen
DRAW	I Graph expression
LCD→ 'SCR' STO	I Form string for screen, store in SCR
DGTIZ »	I Activate cursor

[ENTER] **'OVERDRAW** [STO]

Input: expression	Effect: draws new graph over old

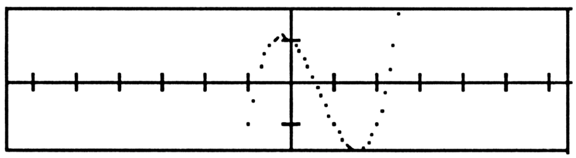
The program **OVERDRAW** draws the graph of the expression on level 1 over the screen of the last graph drawn with **NEWGRAPHS.** Its end effect, showing both graphs together on the screen, is the same as that achieved either by the method of combining image strings, or by graphing the equation equating the two functions. It involves fewer keystrokes than saving and combining strings; in fact, its program involves that technique as you can note above, Furthermore, it is sometimes desirable to see the second graph plotted over the first, rather than having them appear simultaneously. If you **OVERDRAW** the graph of a function onto a screen that already contains the graphs of two functions, then you get all three combined, and so on.

EXAMPLE 11.

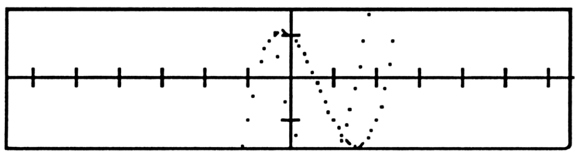
Use **NEWGRAPHS** to sketch the graph of $f(x) = x^3 - 2x^2 - x + 1$ and then use **OVERDRAW** to draw the graph of $g(x) = 3x^2 - 4x - 1$ over the graph of f.

Key in 'PPAR [PURGE]

' X ^ 3 - 2 * X ^ 2 - X + 1 [ENTER] [NEWG]M



[ON] ' 3 * X ^ 2 - 4 * X - 1 [ENTER] [OVERD]M



If you know about derivatives, you will note that g is the derivative of f. As g plots, you can see that, at the values of x where g crosses the x axis, f has a high or low point. What kind of point does f have at the values of x where g has a high or low point?

REDRAW

« CLLCD EQS SIZE 1 SWAP FOR I EQS I GET STEQ DRAW NEXT LCD→ 'SCR' STO DGTIZ »

$\boxed{\text{ENTER}}$ **' REDRAW** $\boxed{\text{STO}}$

Input: none | Effect: used in **BOX** and **ZOOM**

This is simply a utility program for use with **BOX** and **ZOOM** below.

BOX

« PMAX PMIN REDRAW » $\boxed{\text{ENTER}}$ **'BOX** $\boxed{\text{STO}}$

Input: coordinate pair, coordinate pair | Effect: gives a new graph with chosen corners

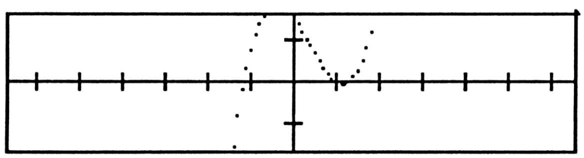
This program allows you to choose new lower left and upper right corners for the graphing screen, either by entering them on the stack or by choosing them from a graph with the cursor and entering them on the stack with $\boxed{\text{DEL}}$. The upper right corner must be the bottom point on the stack.

EXAMPLE 12.

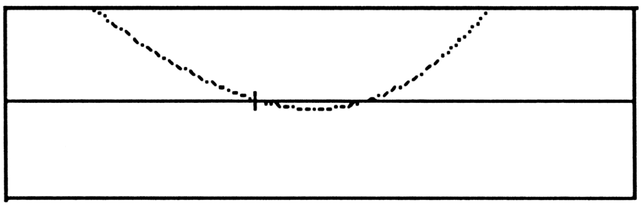
Graph $f(x) = x^3 - 1.14x^2 - 1.37x + 1.517$. Many problems in engineering and science do not have "nice" whole number coefficients. Once you have entered the function in the HP-28S, the "ugliness" of the coefficients is immaterial.

Key in **' PPAR** $\boxed{\text{PURGE}}$

' X ^ 3 - 1.14 * X ^ 2 - 1.37 * X + 1.517 $\boxed{\text{ENTER}}$ $\boxed{\text{NEWG}}$ M



It is not clear what the graph is doing around x = 1, where it seems to touch the x axis twice, so let's take a closer look at that part. Move the cursor to a point just left and below where f first hits the x axis and record its coordinates with $\boxed{\text{INS}}$. Now move the cursor to a point just right and above where the graph leaves the x axis and record its coordinates with $\boxed{\text{INS}}$. Now return to the stack display with $\boxed{\text{ON}}$. I got (.7, -.2) on level 2 and (1.7,.2) on level 1; your points may be slightly different. Now get a new screen with these points as lower left corner and upper right corner by pressing $\boxed{\text{BOX}}^{\text{M}}$.



It is clear from the new graph that the graph of f crosses the x axis at two points near x = 1 and lies below them in between. If you want a rough idea of the two x intercepts shown, you can the crosshairs to each of these points, record their coordinates with $\boxed{\text{INS}}$ and read these coordinates on the stack. We will learn later to easily find these intercepts with the 12-place accuracy of the 28S.

ZOOM

« *W *H CENTR REDRAW »

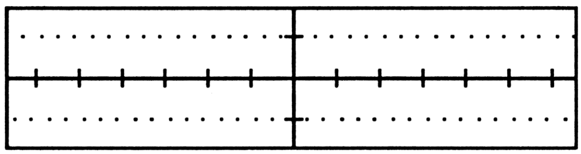
Input: coordinate pair, real, real, Effect: moves screen to a new center and zooms in or out horiz. or vert,

This program requires a three-part input: a coordinate pair on level 3, a real number on level 2 and a real number on level 1. The coordinate pair on level 3 is the point you want for the new center of the screen. It can be chosen from the graphing screen by moving the cursor to it and using $\boxed{\text{INS}}$, or it can be entered from the keyboard. The positive real number on level 2 is a "zoom" factor for the height of the screen. If this number is less than 1, you are zooming in; if it is greater than 1 you are zooming out by that factor. The positive real on level 1 works in a similar way on the width of the screen.

EXAMPLE 13.

Graph f(x) = sin(5πx).

'PPAR $\boxed{\text{PURGE}}$
' sin(5 * π * X) $\boxed{\text{ENTER}}$ $\boxed{\text{NEWG}}^{\text{M}}$

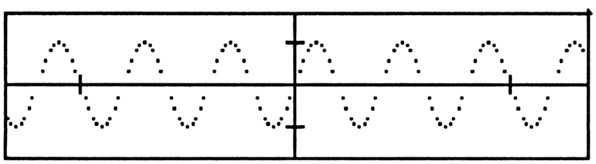


Why does this graph look like this? (Think of the points the 28S plots with the default **PPAR**.)

This function has period $\frac{2\pi}{5\pi} = \frac{2}{5}$ so obviously we would see more of its graph if we zoomed in by a factor of, say, .2. We shall keep the origin at the center, the height the same and multiply the width by a factor of .2. The simplest way to enter (0,0) on the stack is to press $\boxed{\text{INS}}$ with the graphing screen up, since the default position of the cursor is at the origin. So, do

$\boxed{\text{INS}}$ $\boxed{\text{ON}}$ **1** $\boxed{\text{ENTER}}$ **.2** $\boxed{\text{ENTER}}$ $\boxed{\text{ZOOM}}^{\text{M}}$

and the graph now looks like a sine curve.



The graphing programs given here are combinations and modifications of handout materials distributed by John Kenelly and Thomas Tucker at their presentations.

CHAPTER 1 EXERCISES
SET I

Enter the functions and display the graphs:

1. $x^3 - 2x^2 + 1$

2. $x^4 - x^2 - 1$

3. $\sin(\pi x)$

4. $\cos(\pi x/2)$

5. $\cos(10\pi x)$ Why does this graph look the way it does? Adjust the **PPAR** to make the graph look more like the usual cosine curve.

6. $\sin(x°)$ What change in **PPAR** would make this graph look more like sin x (x in radians)?

7. $\tan x$

8. $5 \sin(3x)$

9. $x^3 + 5x^2 - 2x - 5$

10. $x^4 - x^3 - 3x^2 + x - 2$

11. $x^3 - 1.3x^2 + .32x - .02$ Make the behavior of this function around and just right of $x = 0$ clearer by adjusting **PPAR**. This is probably simpler using the programs given in the last section although you can also correctly adjust without them.

12. $\dfrac{1}{x - 2}$

13. $\dfrac{x^3 - 1}{x - 1}$

14. $\dfrac{\sin x}{x}$

15. $\sqrt{1 - x^2}$

16. Graph the functions of Exercises 3, 4 and 8, all on one coordinate system:

(a) using the method of combining image strings,

(b) using the programs **NEWGRAPH** and **OVERDRAW**.

SET II

Graph on the HP-28S:

1. $\ln |x|$
2. 2^x
3. $3^{\cos x}$
4. $\ln(\sin x)$
5. $\ln |\sin x|$
6. $x e^x$
7. e^{-x^2}
8. $e^{-x/3}\sin(2x)$
9. $\cos x - \sin(2x)$
10. $\ln(e^x)$
11. $\ln(\ln x)$
12. $\sin(\sin^{-1} x)$
13. $\sin^{-1}(\sin x)$
14. $\tan(\tan^{-1} x)$
15. $\tan^{-1}(\tan x)$
16. $\tan^{-1}(1/x)$

CHAPTER 2
EVALUATING FUNCTIONS

Built-in functions (sin x, ln x, etc.) are evaluated in the usual way on the 28S, so this chapter will deal only with user-defined functions. Three techniques for evaluating a function will be discussed.

I. STORING AN X AND EVALUATING

Enter the function to be evaluated in symbolic mode (between ' ') and press [ENTER]. Key in the specific number at which you are evaluating the function and press [ENTER]. Key in **' X** [STO]. Press [EVAL]. The value of the function at the given x will replace the function on level 1.

EXAMPLE 1. For $f(x) = \sin^2 x + 3\cos(2x)$, find f(1.24561).

Enter **' SIN(X) ^ 2 + 3 * COS(2 * X)** [ENTER]

1.24561 'X [STO] [EVAL]

The value - 1.48965, which is f(1.24561), appears on level 1.

If you want the value of a function at several different values of x, this technique is the clumsiest of the three methods of evaluation presented here. However, it is close to the way we think of evaluating a function at a number: "just plug it in".

II. USING SOLVR :

If you want to compile a short table of values of a function, a convenient way to do this is to use **SOLVR**. This program is designed to find roots of equations, but the format of its menu makes it convenient for evaluating functions. With the function you want to evaluate on level 1, press [SOLVE] then [STEQ]^{M} then

$\boxed{\text{SOLVR}}^{M}$. The **SOLVR** menu has just two entries in this case, **X** and **EXPR=** . Keying in a number, then pressing $\boxed{\text{X}}^{M}$ (the menu key, not the alphabet key), and then pressing $\boxed{\text{EXPR=}}^{M}$, will give the value of the expression in **EQ** at the number entered.

EXAMPLE 2. Make a table of values for the function $f(x) = \dfrac{x+2}{2x+1}$ using x = 1, 10, 100 and 1000.

Key in **' (X + 2) / (2 * X + 1)** $\boxed{\text{ENTER}}$
$\boxed{\text{SOLV}}$ $\boxed{\text{STEQ}}^{M}$ $\boxed{\text{SOLVR}}^{M}$

Now, keying in **1** $\boxed{\text{X}}^{M}$ $\boxed{\text{EXPR=}}^{M}$ gives 1, which is f(1).

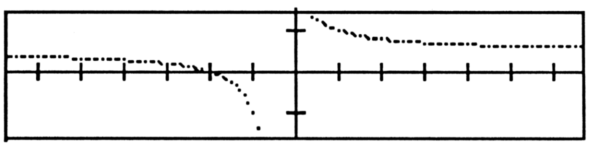
Now, keying in **10** $\boxed{\text{X}}^{M}$ $\boxed{\text{EXPR=}}^{M}$ gives .57143, which is f(10).

Now, keying in **100** $\boxed{\text{X}}^{M}$ $\boxed{\text{EXPR=}}^{M}$ gives .50746, which is f(100).

Now, keying in **1000** $\boxed{\text{X}}^{M}$ $\boxed{\text{EXPR=}}^{M}$ gives .50075, which is f(1000).

What would you say is the limit of f(x) as $x \to \infty$? The values above make .5 seem reasonable for this limit. Can you show that the limit actually is .5? Looking at the graph will give us another view of the situation:

' PPAR $\boxed{\text{PURGE}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{DRAW}}^{M}$



Although x only goes from -6.8 to 6.8 in this view, the idea that $f(x) \to \frac{1}{2}$ as x either increases or decreases without bound is suggested by the graph. This initial indication can be further confirmed by repeated expansions of the domain of x by use of the $\boxed{\text{* W}}^{M}$ key. That is,

$\boxed{\text{ON}}$	10	$\boxed{\text{*W}}^{\text{M}}$	$\boxed{\text{DRAW}}^{\text{M}}$	gives $-68 \le x \le 68$
$\boxed{\text{ON}}$	10	$\boxed{\text{*W}}^{\text{M}}$	$\boxed{\text{DRAW}}^{\text{M}}$	gives $-680 \le x \le 680$
$\boxed{\text{ON}}$	10	$\boxed{\text{*W}}^{\text{M}}$	$\boxed{\text{DRAW}}^{\text{M}}$	gives $-6800 \le x \le 6800$

When using **SOLVR**, if you enter an equation into **EQ** instead a function, the **SOLVR** menu has three entries, **X** , **LEFT=** and **RT=**. This gives a convenient way to make a table of values for two functions you want to compare.

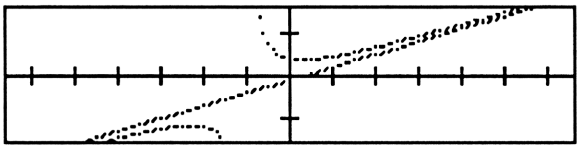
EXAMPLE 3. Make a table of values for the functions $f(x) = \dfrac{3x^2 + 2x + 4}{x + 1}$ and $g(x) = 3x - 1$ for $x = 1, 10, 100$ and 1000. Note that $f(x)$ is the left side of the equation, so pressing $\boxed{\text{LEFT=}}^{\text{M}}$ after entering a value for x will give $f(x)$ on level 1. Similarly, $g(x)$ is the right side of the equation, so pressing $\boxed{\text{RT=}}^{\text{M}}$ will give $g(x)$ on level 1.

Key in **' (3*X^2 +2*X+4)/(x+1)=3*X-1** $\boxed{\text{ENTER}}$ $\boxed{\text{SOLV}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{SOLVR}}^{\text{M}}$

1	$\boxed{\text{X}}^{\text{M}}$	$\boxed{\text{LEFT=}}^{\text{M}}$	gives 4.5 for f(1)
	and	$\boxed{\text{RT=}}^{\text{M}}$	gives 2 for g(1)
10	$\boxed{\text{X}}^{\text{M}}$	$\boxed{\text{LEFT=}}^{\text{M}}$	gives 29.455 for f(10)
	and	$\boxed{\text{RT=}}^{\text{M}}$	gives 29 for g(10)
100	$\boxed{\text{X}}^{\text{M}}$	$\boxed{\text{LEFT=}}^{\text{M}}$	gives 299.0495 for f(100)
	and	$\boxed{\text{RT=}}^{\text{M}}$	gives 299 for g(100)
1000	$\boxed{\text{X}}^{\text{M}}$	$\boxed{\text{LEFT=}}^{\text{M}}$	gives 2999.005 for f(1000)
	and	$\boxed{\text{RT=}}^{\text{M}}$	gives 2999 for g(1000)

If you perform the division indicated in f, you can see that $\frac{3x^2 + 2x + 4}{x + 1} = 3x - 1 + \frac{3}{x+1}$ and that the line y = 3x - 1 is an asymptote for the graph of y = f(x). The values above give evidence for this. Again the graph gives more evidence:

'PPAR $\boxed{\text{PURGE}}$ $\boxed{\text{SOLV}}$ $\boxed{\text{RCEQ}}^{\mathbf{M}}$ $\boxed{\text{USER}}$ $\boxed{\text{GSHO}}^{\mathbf{M}}$ $\boxed{\text{NEWG}}^{\mathbf{M}}$ doesn't give many points, so we will do $\boxed{\text{INS}}$ $\boxed{\text{ON}}$ **10** $\boxed{\text{ENTER}}$ **1** $\boxed{\text{ENTER}}$ $\boxed{\text{ZOOM}}^{\mathbf{M}}$ and get



We compressed the graph vertically by a factor of 10.

We can see on the graph what was mentioned above: f → 3x - 1 as x → ∞ and as x → -∞ .

III. USING A SIMPLE PROGRAM

This is the most useful way to store a function for later recall as an algebraic expression or for later evaluation at a specific value of x. We will have occasion to use this method many times. The procedure is simple:

For a function f, key in « → **X ' F(X)** $\boxed{\text{ENTER}}$ **' FEV** $\boxed{\text{STO}}$.

This program is now on the **USER** menu under the name you chose (my choice was **FEV**). If you key in **X** $\boxed{\text{FEV}}^{\mathbf{M}}$ the algebraic expression for the function will appear on level 1. If you key in a real number and then press $\boxed{\text{FEV}}^{\mathbf{M}}$, the value of the function at that real number will appear on level 1.

EXAMPLE 4. Evaluate $f(x) = \frac{x^3 - 1}{x - 1}$ at x = 2, 1.5, 1.1 and 1.01.

Key in **« → X ' $(X^3 - 1)/(X - 1)$** $\boxed{\text{ENTER}}$ **'FEV** $\boxed{\text{STO}}$

2 $\boxed{\text{USER}}$ $\boxed{\text{FEV}}^{\text{M}}$ gives 7 as f(2)

1.5 $\boxed{\text{FEV}}^{\text{M}}$ gives 4.75 as f(1.5)

1.1 $\boxed{\text{FEV}}^{\text{M}}$ gives 3.31 as f(1.1)

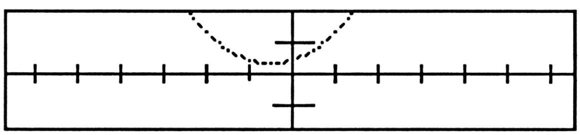
1.01 $\boxed{\text{FEV}}^{\text{M}}$ gives 3.0301 as f(1.01)

It appears that as $x \to 1$ the limit of f(x) is 3 and a little algebraic simplification of the expression for f(x) will allow you to confirm that the limit of f(x) is 3 as x approaches 1. Also note that X $\boxed{\text{FEV}}^{\text{M}}$ returns the expression for f(x) to level 1. If a numerical result is presented where you enter x $\boxed{\text{FEV}}^{\text{M}}$, then the variable X has a stored value. The stored value can be removed with x' $\boxed{\text{PURGE}}$.

It is interesting to graph f and see how the graph relates to what we found above. Since we are interested in values of f close to 3, we will do 3 $\boxed{\text{* H}}^{\text{M}}$ before graphing.

X $\boxed{\text{FEV}}^{\text{M}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{\text{M}}$

'PPAR $\boxed{\text{PURGE}}$ $\boxed{\text{NEXT}}$ 3 $\boxed{\text{* H}}^{\text{M}}$ $\boxed{\text{PREV}}$ $\boxed{\text{DRAW}}^{\text{M}}$



CHAPTER 2 EXERCISES

SET I

1. Make a table of values of the function $f(x) = \dfrac{x^7 - 128}{x - 2}$ for x = 3, 2.5, 2.1, 2.01 and 2.001.

2. Make a table of values of the function $f(x) = \dfrac{\sin x}{x}$ for x = 1, 0.5, 0.1, 0.01 and 0.001. What number does f(x) approach as x approaches 0?

3. Make a table of values of the function $f(x) = 36x^4 - 216x^3 - 18900x^2 + 195x + 250$ for x = -3, -2, -1, 0 and 1. Between which consecutive pairs of these integers does f have zeroes?

4. Make a table of values of the function $f(x) = x^4 + 2x^3 - 19x^2 - 29x + 37$ for x = -5, -4, -3, -2, -1, 0, 1, 2, 3 and 4. Between which consecutive pairs of these integers does f have zeroes?

5. Make a table of values of the function $f(x) = \dfrac{3x^2 + 4x}{x + 1}$ and the linear function $g(x) = 3x + 1$ for x = 10, 100, 1000 and 100000. The relation between f and g is further illustrated by graphing both on the same set of axes.

SET II

1. Make a table of values of the function $f(x) = \ln x$ for $x = 0.5, 0.2, 0.1, 0.01$ and 0.001.

2. Make a table of values of the function $f(x) = x \ln x$ for $x = 0.5, 0.2, 0.1, 0.001$ and 0.001.

3. Make a table of values of the function $f(x) = x\,e^{x}$, for $x = -1, -10, -100, -1000$ and -10000.

4. Make a table of values of the function $f(x) = (1 + \frac{1}{x})^{x}$ for $x = 10, 100, 1000$ and 100000. What number does f approach as x increases without bound? If you choose a very large number for x, say x = 1.0E20, the calculator gives the wrong answer for f(x). Can you explain this?

5. Make a table of values of the function $f(x) = \frac{\ln x}{x}$ for $x = 10, 100, 1000$ and 100000.

CHAPTER 3
FINDING DERIVATIVES

INTRODUCTION

The derivative of a function is a source of much useful information about the function itself. The value of the derivative at x , f'(x), is the slope of the tangent line to the graph of f at the point (x,f(x)). If the number f(x) represents some physical quantity, the number f'(x) represents the rate of change of f at x, with respect to x. If a function has a derivative everywhere, then its local maximum and local minimum points can occur only at those points where its derivative is 0, or at the endpoints of its domain.

In some applications, you are interested in knowing where the ***derivative*** of a function f has a maximum or minimum point. That is, you would like to know not only where f has an extreme value, but also where its ***rate of growth or decline*** has a maximum or minimum value. At the values of x where f'(x) has a local maximum or minimum value, the graph of f has an inflection point and f''(x) = 0 (if it exists). Geometrically, the points of f whose x coordinates are those where f' has an extreme value are points where the graph of f changes from bending left to bending right, or the reverse. Clearly the derivative of f, as well as the derivative of the derivative, are of interest in analyzing the behavior of f.

DIFFERENTIATION WITH THE HP-28S

For your own efficiency and protection, you should learn the mechanics of finding derivatives without the HP-28S. For the functions we will consider, the differentiation process is simple enough that you can usually take a derivative by hand in less time that it takes you to enter it into the calculator. There is also the possibility that you might be in a situation where you need derivatives and do not have access to the calculator. So, ***if all you want*** is the derivative of a function, do it by hand.

However, there will be many problems in which we shall make use of the derivative of a function within the calculator. For example, we will on many occasions graph a function on the 28S, find its zeroes on the calculator, and then find the second derivative and its zeroes. The graphing and equation-solving will be done on the 28S and so the differentiation might as well be done there also to keep the entire problem within the calculator.

There are two ways of taking derivatives on the HP-28S. One takes the derivative in a single stroke of the $\boxed{\frac{d}{dx}}$ key (with the proper input); the other takes the derivative one step at a time with each use of the **EVAL** key. The direct method is the one we will normally use to find derivatives. However, the"one-step-at-a-time" method can be useful to someone first learning to find derivatives because of the insight it gives into the differentiation process.

The procedure for direct differentiation with the $\boxed{\frac{d}{dx}}$ command requires a two-part input. Before executing this command, you should have the function to be differentiated in symbolic mode (between ' ') on level 2 and the variable of differentiation on level 1, also in symbolic mode:

4:	
3:	
2:	' f(x)'
1:	'x'

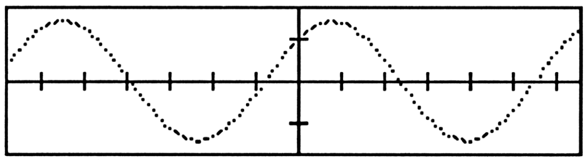
Then press $\boxed{\frac{d}{dx}}$. The derivative will appear on level 1.

EXAMPLE 1. We will find the derivative of f(x) = sin x + cos x on the 28-S and then graph both f and f' to see some of the relations between them mentioned above.

Key in **' SIN(X) + COS(X)** $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$

$\boxed{\text{USER}}$ $\boxed{\text{GSHO}}^{\text{M}}$ **'PPAR** $\boxed{\text{PURGE}}$ $\boxed{\text{NEWG}}^{\text{M}}$

This will give the graph of f:



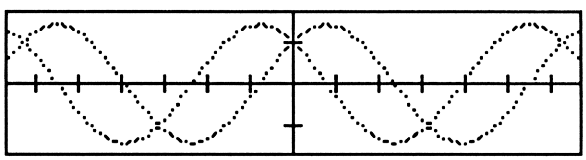
Now we shall find the derivative of f and graph it. I chose **NEWGRAPHS** to graph f so I could draw the graph of f' over the graph of f. This makes the two graphs more easily distinguishable than when they both appear simultaneously.

Key in (or recall from **EQ** or use the extra copy of the expression that was entered two times above) **' SIN(X) + COS(X)** $\boxed{\text{ENTER}}$

' X $\boxed{\text{ENTER}}$ $\boxed{\frac{\text{d}}{\text{dx}}}$

This should give you the derivative of f, cos x - sin x on level 1.

Now do $\boxed{\text{OVERD}}^{\text{M}}$ which draws f' over f:



Notice that where the graph of f' crosses the x axis, the graph of f has an extreme point; and where the graph of f' has an extreme point, the graph of f has an inflection point.

EXAMPLE 2. We will find the derivative of $f(x) = \frac{1}{2}\sin^3 x$ and then graph both f and f'.

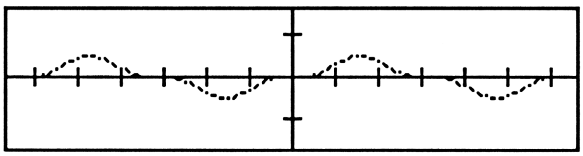
' .5 * SIN(X) ^ 3 [ENTER] [ENTER]

' X [ENTER] [$\frac{d}{dx}$] [SWAP]

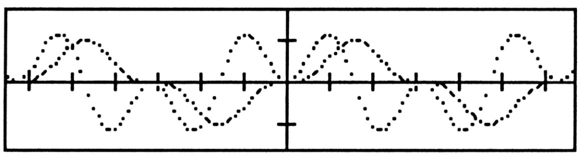
Notice that we duplicated f so that it would still be on the stack after differentiation and then swapped positions with it and f' so that we could graph f first. Your stack display should look like:

```
4:
3:
2:        ' .5 * (COS(X) * 3 * SIN(X ...
1:                    ' .5 * SIN(X) ^ 3 '
```

Now to graph: [USER] [NEWG]^{M} gives the graph of f:



To draw the graph of f' over f: [ON] [OVERD]^{M} :



Notice again that the graph of f has a maximum or minimum point where the graph of f' crosses the x axis. There are also points of the graph of f where it levels out and is tangent to the x axis but does not have an extreme value. At these points f' has value 0 but its graph does not cross the x axis.

When the default plotting parameters are used, i.e., **'PPAR** $\boxed{\textbf{PURGE}}$, the graphs in the example are somewhat compressed. Expand the vertical axis with a .5 $\boxed{\textbf{* H}}^{\textbf{M}}$ command followed with a $\boxed{\textbf{REDRA}}^{\textbf{M}}$ in the user menu and the graph will be more attractive.

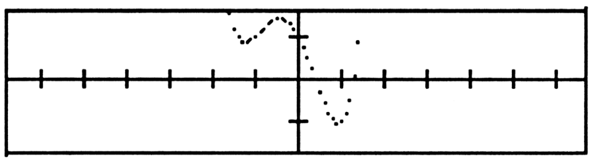
EXAMPLE 3. Find the derivative of $f(x) = x^4 + x^3 - 2x^2 - 2x + 1$ and sketch the graphs of both f and f'.

Key in **' X ^ 4 + X ^ 3 - 2 * X ^ 2 - 2 * X + 1** $\boxed{\textbf{ENTER}}$

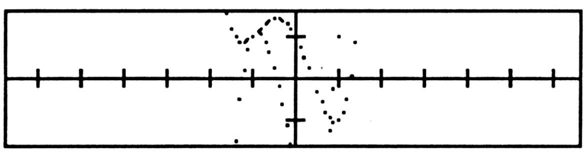
$\boxed{\textbf{ENTER}}$ **' X** $\boxed{\textbf{ENTER}}$ $\boxed{\frac{d}{dx}}$

which will give **' 4 * X ^ 3 + 3 * X ^ 2 - 4 * X - 2** on level 1 and f on level 2. To graph f first

do $\boxed{\textbf{SWAP}}$ **'PPAR** $\boxed{\textbf{PURGE}}$ $\boxed{\textbf{NEWG}}^{\textbf{M}}$ and get

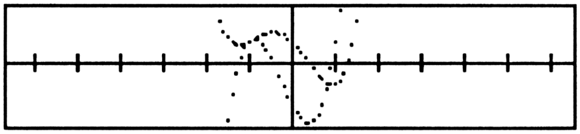


You should have f' now on level 1, so to draw its graph over the graph of f press $\boxed{\textbf{OVERD}}^{\textbf{M}}$:



Since we cannot see all of f' we will do

$\boxed{\text{INS}}$ $\boxed{\text{ON}}$ 2 $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$ $\boxed{\text{ZOOM}}^{M}$ and get



Now we can see the relations between f and f' noted before. At the values of x where f has an extremum, f' crosses the x axis. At the values of x where f' has an extremum, f has an inflection point.

THE "ONE STEP AT A TIME" METHOD

To take the derivative of f one step at a time key in ' $\boxed{\frac{d}{dx}}$ X (f(X)) $\boxed{\text{ENTER}}$. Then each use of the $\boxed{\text{EVAL}}$ key will perform one step of differentiation.

EXAMPLE 4. Take the derivative of $f(x) = x\,(x^2 + 4)^3 + x^4$ one step at a time.

Key in ' $\boxed{\frac{d}{dx}}$ X (X * (X ^ 2 + 4) ^ 3 + x^ 4 $\boxed{\text{ENTER}}$

This will appear on the screen as ' ∂X (X * (X ^ 2 + 4) ^ 3 + X ^ 4) '. For functions of one variable, the HP-28S uses ∂X( ) to mean the same thing as $\frac{d}{dx}$ ( ). Note also that the "X" after the ∂ symbol must be keyed in.

To proceed with the differentiation, press $\boxed{\text{EVAL}}$ and this will give ' ∂X(X * (X ^ 2 + 4) ^ 3) + ∂X (X ^ 4) '. It used the sum rule. Another $\boxed{\text{EVAL}}$ gives

' ∂X (X) * (X ^ 2 + 4) ^ 3 + X * ∂X ((X ^ 2 + 4) ^ 3) + ∂X (X) * 4 * X ^ (4 - 1) '

The product rule and the power rule.

Another $\boxed{\text{EVAL}}$ gives

' (X ^ 2 + 4) ^ 3 + X * (∂X (X ^ 2 + 4) * 3 * (X ^ 2 + 4) ^ (3 - 1)) + 4 * X ^ 3 '

It used the fact that $\frac{d}{dx}(x) = 1$ and the power rule for $[\,u(x)\,]^n$.

Another $\boxed{\text{EVAL}}$ gives

' (X ^ 2 + 4) ^ 3 + X * (∂X (X ^ 2) * 3 * (X ^ 2 + 4) ^ 2) + 4 * X ^ 3 '

It used the sum rule and 3 - 1 = 2.

Another $\boxed{\text{EVAL}}$ gives

' (X ^ 2 + 4) ^ 3 + X * (∂X (X)* 2 * X ^ (2 - 1)* 3 * (X ^ 2 + 4) ^ 2) + 4* X* 3 '

Finally, another $\boxed{\text{EVAL}}$ gives

' (X ^ 2 + 4) ^ 3 + X * (2 * X * 3 * (X ^ 2 + 4) ^ 2) + 4 * X ^ 3 ', an answer without any differentiation symbols. You could clean up this expression some by using the $\boxed{\text{COLCT}}^{\text{M}}$ command on the $\boxed{\text{ALGEBRA}}$ menu. This was a somewhat tedious way to take a derivative, but it does illustrate the steps in the differentiation process.

CHAPTER 3 EXERCISES

SET I

In problems 1 through 4, take the derivative by the direct method and then show the graphs of both f and f' on the HP-28S on the same set of axes. By noting where $f'(x) = 0$, estimate the values of x where f has a local maximum or minimum.

1. $f(x) = x^4 - 2x^3 + 3x - 2$
2. $f(x) = \dfrac{1}{1 + x^2}$
3. $f(x) = (x^2 - 1)^3$. Compressing the graph in height and expanding it in width are useful here in interpreting the combined graph.
4. $f(x) = \sqrt{x^2 + 1}$
5. $f(x) = \dfrac{1}{x - 1}$
6. $f(x) = | x^2 - x - 2 |$
7. $f(x) = \sin |x|$
8. $f(x) = \cos(2x) - \sin x$
9. $f(x) = \cos^2 x - \sin x$
10. $f(x) = \sin(\frac{x^2}{2})$

Use the "one-step-at-a-time" method to find the derivative of each function given below. Press the $\boxed{\text{EVAL}}$ key repeatedly until you get an expression with no differentiation symbol in it. Note the differentiation rule used at each step. You can "clean up" the final expression a little by using the **COLCT** command on the **ALGEBRA** menu.

11. $7x^4 - 5x^3 + 6x^2 -3x + 2$

12. $\dfrac{x^2}{x^2+4}$. Try this one by hand using the quotient rule.

13. $(x^2 + 3x + 5)^7$

14. $(x + \sqrt{x}\,)^3$

SET II

In problems 1 through 4, take the derivative by the direct method and then show the graphs of both f and f' on the HP-28S on the same set of axes. By noting where f'(x) = 0, estimate the values of x where f has a local maximum or minimum.

1. $f(x) = \ln |x|$ (= ln (ABS(X)))
2. $f(x) = |\ln x|$
3. $f(x) = \sin(e^{.3x})$
4. $f(x) = e^{.3 \sin x}$
5. $f(x) = \tan^{-1} x$
6. $f(x) = \sin^{-1}(x/2)$
7. $f(x) = \sin^{-1}(\sin x)$
8. $f(x) = e^{-x^2}$. Note the relation between the inflection points of f and the extreme points of f'.
9. $f(x) = e^{-x/3}\sin(2x)$

CHAPTER 4
ILLUSTRATING LINEAR APPROXIMATIONS AND THE MEAN-VALUE THEOREM

INTRODUCTION

In this chapter we will use the HP-28S to illustrate the underlying geometry for the mean-value theorem for derivatives and the idea of approximating a function with a linear polynomial. Understanding of both concepts is enhanced by keeping the graphical aspects in mind.

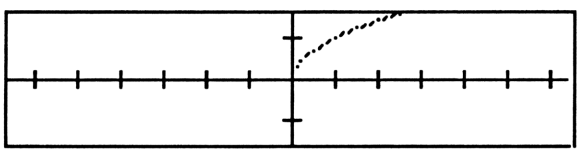
LINEAR APPROXIMATIONS

The linearization, L(x), for a function f at a is the linear function whose graph is the tangent line to the graph of f at the point (a,f(a)). So it is the linear function with slope f'(a) whose graph contains (a,f(a)): $L(x) = f(a) + f'(a)(x - a)$. Sometimes a problem that is unworkable or very difficult with f is much simpler if f(x) is replaced by L(x), and valid conclusions can be reached for values of x near a. The graph of L is the straight line that "best fits" the graph of f at the point (a,f(a)).

EXAMPLE 1. Find the linearization of $f(x) = \sqrt{x}$ at x = 1, draw a graph on the 28S which shows both graphs together on the same axes and make a table of values of both functions for x = .5, -.5, .9, -.9, .99, -.99 for comparison.

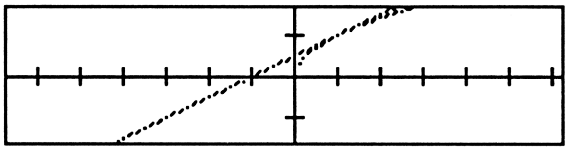
$$f'(x) = \frac{1}{2\sqrt{x}} \text{ so } f'(1) = \frac{1}{2} \text{ and } L(x) = 1 + \frac{1}{2}(x - 1) = .5x + .5$$

Do ' √X $\boxed{\text{ENTER}}$ $\boxed{\text{USER}}$ 'PPAR $\boxed{\text{PURGE}}$ $\boxed{\text{NEWG}}^{\text{M}}$ to get the graph of f:



Now draw the linearization over this with

' .5 * X + .5 [ENTER] [OVERD]^M to get:



For the table, one convenient technique is to key in

' √X = .5 * X + .5 [ENTER] [SOLV] [STEQ] [SOLVR]^M

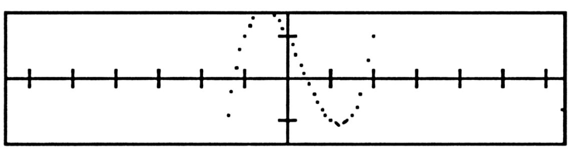
and proceed as outlined in Chapter 2 (I FIXed my screen display at 5 decimal places):

x	0.5	0.9	0.99	0.999
f(x)	0.70711	0.94868	0.99499	0.99950
L(x)	0.75000	0.95000	0.99500	0.99950

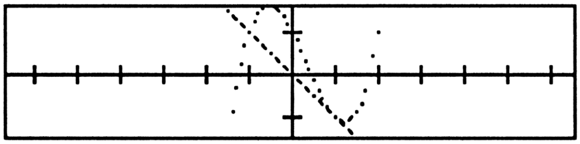
Notice the increasing agreement between the two as x gets closer to 1.

EXAMPLE 2. Find the linearization of $f(x) = x^3 - x^2 - 2x + 1$ and draw the graph on the calculator showing both f and L together on the same axes. Make a table which shows the values of both f and L for x = .5, -.5, .9, and .99.

$f'(x) = 3x^2 - 2x - 2$ and so $f'(1) = -1$ and $f(1) = -1$ so $L(x) = -1 -(x - 1) = -x$. To get the graph of f do ' X ^ 3 - X ^ 2 - 2 * X + 1 [ENTER] [NEWG]^M :



Now get both graphs with ' - X [ENTER] [OVERD]^M :



For the table we can use the same procedure as in Example 1 and get

x	0.5	-0.5	0.9	.99
f(x)	-0.12500	1.62500	-0.88100	-0.98980
L(x)	-0.50000	0.50000	-0.90000	-0.99000

THE MEAN-VALUE THEOREM FOR DERIVATIVES

The mean-value theorem is of the greatest importance in our effort to learn about the behavior of a function by investigating its derivative. Almost all of the important theorems we use depend on the ideas of the mean-value theorem for proof:

"if a function has zero derivative for all x on an interval, it has constant value on that interval"

"if the derivative of a function has positive value for all x on an interval, the function is increasing in value on that interval"; and so on.

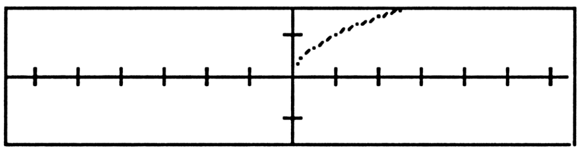
The mean-value theorem says:

"if a function is continuous on the interval [a,b] and differentiable on the interval (a,b), then there is a number c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$."

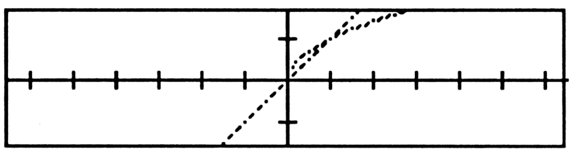
Graphically, the conclusion of the mean-value theorem says that there is a number c such that the tangent line to the graph of at (c,f(c)) is parallel to the line joining (a,f(a)) and (b,f(b)). It is this geometry that we will illustrate in the examples and exercises.

EXAMPLE 3. The function $f(x) = \sqrt{x}$ satisfies the hypothesis of the mean-value theorem on the interval [0,1]. It is continuous on [0,1] and differentiable on (0,1) ($f'(0)$ does not exist). $\frac{f(1) - f(0)}{1 - 0} = 1$, so the mean-value theorem says that there must be a number c between 0 and 1 such that $f'(c) = 1$. $f'(c) = \frac{1}{2\sqrt{c}}$ so $\frac{1}{2\sqrt{c}} = 1$ and $\sqrt{c} = \frac{1}{2}$ so $c = \frac{1}{4}$. Draw the graph of f on the HP-28S, then overdraw the graph of the line joining (0,0) and (1,1) and then on this second graph draw the graph of the tangent line at the point $(\frac{1}{4}, \frac{1}{2})$.

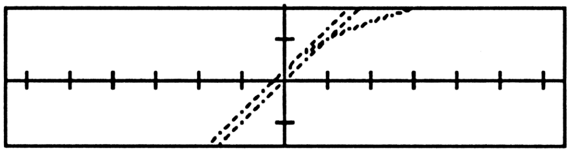
Do '$\sqrt{X}$ [ENTER] [NEWG]^{M} to get the graph of f:



The line joining (0,0) and (1,1) is the line y = x so do 'X [ENTER] [OVERD]^{M} to get this line on the graph:



The line through $(\frac{1}{4}, \frac{1}{2})$ with slope 1 is $y = x + \frac{1}{2}$ so do ' X + .5 $\boxed{\text{ENTER}}$ $\boxed{\text{OVERD}}^{\mathbf{M}}$ to get the tangent line on the graph:



CHAPTER 4 EXERCISES

Problems 1 through 5 of this assignment involve graphing both a function and its linear approximation (tangent line) at a point (a, f(a)), and also making a short table of values of both the function and its linear approximation for some values of x near a to see the closeness of the approximation. To graph: find the equation for the linearization L(x) as described in the text and **DRAW** the graph of the equation f(x) = L(x). To make up the table of values, use one of the methods described in an earlier chapter ro evaluate f(x) and L(x).

1. Graph, simultaneously, on the HP-28, the function $f(x) = x^4$ and the tangent line to this graph at the point (1,1). (You will get a better picture of this if you multiply,say 3, #H .) Make up a table which shows values of both f(x) and the tangent line function L(x) at x = 1.5, 1.1, 1.01 and 1.001.

2. Graph, simultaneously, on the HP-28, the function f(x) = sin(2x) and the tangent line to this graph at (0,0). Make up a table which shows values of both f(x) and L(x) at x = 0.5, 0.2, 0.1, 0.01 and 0.001.

3. Graph, simultaneously, on the HP-28, the function $f(x) = \sqrt{x^2 + 9}$ and the tangent line to this graph at the point (-4,5). (Again, multiply 4 or 5 #H.) Make up a table which shows values of both f(x) and L(x) at x = -4.5, -4.1, 4.01 and -4.001.

4. Graph, simultaneously, on the HP-28, the function $f(x) = x^3 - x - 1$ and the tangent line to this graph at the point (1,-1). Make up a table which shows values of both f(x) and L(x) for x = -1.5, -0.5, -1.1, -0.9, -1.01 and -0.99.

5. Graph, simultaneously, on the HP-28, the function sin x + cos x and the tangent line to this graph at the point (0,1). Make up a table which shows values of both f(x) and L(x) for x = -1, 1, -0.5, 0.5, -0.1 and 0.1.

In Problems 6 through 9, graph the given function f and graph the line through the points $(a,f(a))$ and $(b,f(b))$. Find the number c which satisfies the conclusion of the mean-value theorem and show the line tangent to the graph of f at $(c,f(c))$ on your graph.

6. $f(x) = x^3$, $a = 0$ and $b = 1$.

7. $f(x) = \frac{1}{x^2}$, $a = \frac{1}{2}$ and $b = 4$.

8. $f(x) = x^2 - x - 1$, $a = 0$ and $b = 2$.

9. $f(x) = x^3 - x - 1$, $a = 0$ and $b = 2$.

CHAPTER 5
SOLVING EQUATIONS

One of the ways in which the HP-28S calculator widens our horizons in a calculus course is by greatly enlarging the set of equations that we can solve. Some of the most important procedures of calculus involve finding zeroes of a function. In traditional calculus texts such problems are limited to polynomials which will factor easily and to very simple trigonometric, exponential and logarithmic functions. Using **SOLVR** on the HP-28S, we can solve a considerably expanded set of equations.

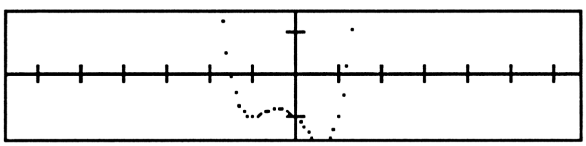
To find a root of a function on the HP-28S, the **SOLVR** operation requires an initial estimate of the root, supplied by you. Probably the best way to get these initial estimates is from the graph of the function. Get the graph of the function displayed so that you can see all the roots (x-intercepts) if possible (or at least all the roots in one period for a periodic function). This requires some thought and a little knowledge of algebra and calculus.

EXAMPLE 1. Graph $f(x) = x^4 + x^3 - x^2 - x - 1$.

Key in : **' X ^ 4 + X ^ 3 - X ^ 2 - X - 1** $\boxed{\text{ENTER}}$

$\boxed{\text{PLOT}}$ $\boxed{\text{NEXT}}^{\text{M}}$ **'** $\boxed{\text{PPAR}}^{\text{M}}$ $\boxed{\text{PURGE}}$

$\boxed{\text{PREV}}^{\text{M}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$

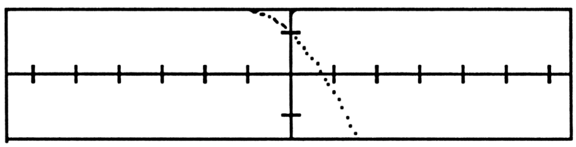


A fourth degree polynomial can have at most four roots. So, if you can see four x-intercepts for a fourth degree polynomial, you can see all the roots. The derivative of a fourth-degree polynomial is a third degree polynomial and can have at most three roots; so the graph of a fourth degree polynomial can have at most three points where it has a horizontal tangent. If you can't see four x-intercepts of the graph of a fourth degree polynomial but can see three points where it has a horizontal tangent, then you are seeing all the roots or probably can see them with an adjustment of the width of the plotting screen. The polynomial in Example 1 above has two roots; you will be asked to find them in an exercise.

We will now illustrate finding roots of functions with an example which would be very tedious to do without some sort of numerical root-finding program like **SOLVR**:

EXAMPLE 2. Find all the values of x for which cos x - x = 0.

To graph cos x - x, key in ' $\boxed{\text{PPAR}}$ $\boxed{\text{PURGE}}$
$\boxed{\text{TRIG}}$ ' $\boxed{\text{COS}}^{\text{M}}$ X) - X $\boxed{\text{ENTER}}$
$\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{\text{M}}$ $\boxed{\text{DRAW}}^{\text{M}}$



Not much of the graph shows but we know that -1 ≤ cos x ≤ 1 so the equation can not have a root outside of -1 ≤ x ≤ 1. Thus the root that shows on the graph is the only one. On the graphing screen, the cursor is moved with the four white menu keys that have arrowheads above them. Its default position is at the origin. Move the cursor to the right until it coincides with the root shown on the graph, then press

$\boxed{\text{INS}}$, the leftmost of the white menu keys, to record the coordinates of this point on the stack.

Return to the stack display with $\boxed{\text{ON}}$. The point that I chose is shown as (.7,0). You may have positioned the cursor a little differently and obtained a slightly different point - which is fine. The x coordinate of the point is just a starting point for **SOLVR** .

Now key in: $\boxed{\text{SOLV}}$ $\boxed{\text{SOLVR}}^{M}$

You should have two commands showing on the menu line: **X** and **EXPR=**.

Press the $\boxed{\text{X}}^{M}$ ***menu*** key which enters your point as a first guess for **SOLVR.**

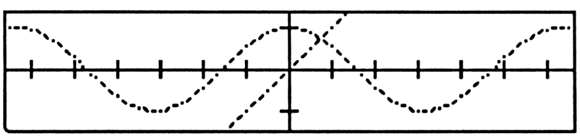
X: (.7, 0) will appear at the top of the screen.

Now press $\boxed{\text{RED}}$ $\boxed{\text{X}}^{M}$ to solve for x. **SOLVING FOR X** will appear at the top of the screen.

When the root is found, it will appear both at the top of the screen and on level 1. In this case the root is .739085133215.

EXAMPLE 3. The equation cos x - x = 0 in Example 2 is equivalent to the equation cos x = x. The solution of this equation is the point where the graph of y = cos x crosses the graph of y = x. First we will graph both of these functions on the same axes. The simplest way to do this is to graph the equation cos x = x:

'PPAR $\boxed{\text{PURGE}}$ **'COS(X) = X** $\boxed{\text{ENTER}}$
$\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{M}$ $\boxed{\text{DRAW}}^{M}$



Now move the cursor to the point of intersection of the two graphs. Record its coordinates on the stack with $\boxed{\text{INS}}$ and return to the stack display with $\boxed{\text{ON}}$.

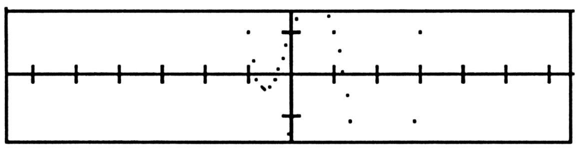
The point I got was (0.8, 0.7). Your point may be a little different, depending on where you stopped the cursor. Now do

SOLV SOLVR^M
X^M RED X^M

We get for the solution, of course, x = .739085133215.

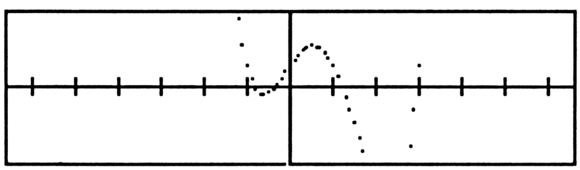
EXAMPLE 4. Find zeroes of the function $x^4 - 3x^3 - x^2 + 3x + 1$.

Key in: **'X ^ 4 - 3 * X ^ 3 - X ^ 2 + 3 * X + 1** ENTER
'PPAR PURGE PLOT STEQ^M DRAW^M



Not very much of the graph shows, so we compress the graph in height by a factor of 2:

Press ON NEXT **2** * H^M
PREV DRAW^M



That's better. The low point of the graph to the right is still off the screen, but we can see all four x intercepts and that is what we are interested in here.

Now with the graph still displayed on the screen, use the cursor keys to move the crosshairs near the rightmost x-intercept. When you have it on this intercept, press $\boxed{\text{INS}}$ to record the coordinates of this point on the stack. Now move the crosshairs to the next x-intercept to the left and press $\boxed{\text{INS}}$ to enter this point on the stack. Repeat this procedure with the intercept just to left of x = 0 and finally with the leftmost one near x = -1. Return to the stack display with $\boxed{\text{ON}}$.

You may have chosen slightly different points but my stack display looks like this:

4:	(3,0)
3:	(1.2,0)
2:	(-.3,0)
1:	(-.8,0)

Now we want to use these points as our initial guesses for the zeroes of our function.

Press $\boxed{\text{SOLV}}$ $\boxed{\text{SOLVR}}^{\text{M}}$ $\boxed{\text{X}}^{\text{M}}$

This enters your point on level 1 as a first estimate for x. This point is now displayed at the top of the screen as x. (**SOLVR** will use only the first coordinate of the point.)

To solve for x with this estimate,
press $\boxed{\text{RED}}$ $\boxed{\text{X}}^{\text{M}}$:

When the calculations are complete, the root -.827090915283 is displayed on level 1.

4:	(3,0)
3:	(1.2,0)
2:	(-.3,0)
1:	-.827090915283

Save it:

' R1 $\boxed{\text{STO}}$

You now need to repeat this procedure with the remaining three estimates to get the other roots. Since you need the point that is currently on level 2 to be on level 1 to proceed, press $\boxed{\text{SWAP}}$. Then to solve for x:

$\boxed{\text{X}}^{\text{M}}$ $\boxed{\text{RED}}$ $\boxed{\text{X}}^{\text{M}}$

This gives the root -.338261212718.

4:	(3,0)
3:	(1.2,0)
2:	-.827090915283
1:	-.338261212718

To get the point now on level 3 to level 1 to use as an estimate for the third root, do

3 $\boxed{\text{ROLL}}$ **, which will give**

4:	(3,0)
3:	-.827090915283
2:	-.338261212718
1:	(1.2,0)

Then do $\boxed{\text{X}}^{\text{M}}$ $\boxed{\text{RED}}$ $\boxed{\text{X}}^{\text{M}}$ which will give the third root on level 1:

4:	(3,0)
3:	-.827090915283

2:	-.338261212718
1:	1.20905692654

Doing 4 $\boxed{\text{ROLL}}$ to get the point on level 4 to level 1 and then repeating the above to solve for x gives all four roots:

4:	-.827090915283
3:	-.338261212718
2:	1.20905692654
1:	2.95629520147

Notice the basic procedure: with your initial estimate for x on level 1, press $\boxed{\text{X}}^{\text{M}}$ on the **SOLVR** menu to enter the estimate, then press $\boxed{\text{RED}}$ $\boxed{\text{X}}^{\text{M}}$ on the **SOLVR** menu. The root that **SOLVR** finds from the initial estimate will then appear on level 1. This procedure should work on almost all the problems we will encounter.

However, if you should have trouble getting **SOLVR** to home in on the root you want, the procedure can be refined. Instead of an initial estimate for x, you can give **SOLVR** two values of x, one on either side of the root. These should be entered, in the form of a **LIST** of the two values, as x on the **SOLVR** menu . Again, these can be points chosen from the graph by the crosshairs and **INS**, and **SOLVR** will use only the x coordinates of the points. A still further refinement is to present **SOLVR** with a three-value estimate in which the first value is near the root and the next two bracket it on either side; again this should be entered as x on the **SOLVR** menu in the form of a **LIST** of the three values. As before the three values can be points chosen graphically.

CHAPTER 5 EXERCISES

SET I

Find zeroes of the functions or solutions of the equations:

1. $x^3 + 5x^2 - 2x - 5$

2. $x^4 - x^3 - 3x^2 + x - 2$

3. $6x^4 - 29x^3 + 49x^2 - 34x + 8$

4. $\sin x = x^2$

5. $\sin x = \frac{x}{3}$

6. $\cos x = \tan x$. Find the solutions between 0 and 2π. Can you work this problem easily without using **SOLVR** ?

7. $x^3 = \sqrt{x^2 + 1}$

8. $\sin x = \cos(3x)$, $0 \leq x \leq 2\pi$.

9. $x = \tan x$, $0 \leq x \leq 2\pi$.

SET II

1. $\ln x + x = 0$
2. $e^{-x} = \ln x$
3. $e^{x} = \cos^{-1}$
4. $\ln x = \sin x$
5. $e^{x} = \cos x$. Find the solution for which $-2 \leq x \leq 0$. How many times do these graphs intersect?
6. $e^{x} = \tan x$
7. $e^{-x} = x$
8. $e^{-x^2} = x$
9. $e^{-x} = \tan^{-1} x$

CHAPTER 6
CURVE SKETCHING

INTRODUCTION

The "important" points of the graph of a function are the intercepts, and the local maximum, local minimum and inflection points. The local extreme points are the points where the graph "changes direction", that is, goes from increasing to decreasing or the opposite. The inflection points are the points where the graph changes from bending left to bending right, from being concave up to being concave down, or the opposite. (A fact of importance in many applications is that the x-coordinates of the inflection points of f are the x coordinates of points where f' has an extreme value.) If we can display on the calculator screen a graph that shows all the extreme points, inflection points and intercepts of a function f and we know the behavior of f as $x \to \infty$ and as $x \to -\infty$, we can feel confident that our graph gives us a good picture of the behavior of f, even though it displays f only for a limited range of values of x. For functions involving trigonometric functions, if the function is periodic, knowing the period will tell us when we have enough of the graph to get a full picture of the function's behavior.

Since, even though we can expand the range of x coordinates on the screen, we are always looking at the graph on a finite interval, the matter of knowing that we have ***all*** the intercepts, extreme points and inflection points becomes a matter of importance. A polynomial function of degree n can have at most n roots, so if we can see n x-intercepts on our graph, we have all of them. It's derivative is a polynomial of degree n - 1, so the graph of f can have at most n - 1 extreme points. Even when the function may have less than n intercepts, when we see n - 1 extreme points, we have essentially the full picture. For functions that have vertical or horizontal asymptotes, we need to know this. From such considerations , you can often tell that you have all the important points of a graph .

We can, of course, get an approximate idea of the coordinates of the extreme points and inflection points of the graph of a function by plotting its graph on the calculator. Usually, however, we want to know these coordinates with more accuracy than we can estimate from the screen display.

You will recall from your textbook that the local extreme points of f occur at those values of x where f' either does not exist, or is equal to 0 ***and*** where f' is of opposite sign left and right of the point. The inflection points occur at values of x where f" has similar properties. We will outline a procedure below for finding these values of x and thus the local extreme points and inflection points of the graph of a function f for a much larger class of functions than is usually treated in calculus textbooks. The behavior of f as $x \to \infty$ and as $x \to -\infty$ is sometimes clear by inspection and sometimes requires something like l'Hopital's rule.

The use of the HP-28S in a calculus course greatly enlarges the family of functions whose graphs can be sketched. Most of the functions given as curve-sketching exercises in calculus textbooks are carefully selected so that their derivatives and second derivatives will factor easily or at least are simple enough so that the critical values and the similar values for the second derivative can be found by the techniques of high-school algebra. With the 28S, using $\boxed{\text{SOLVR}}^{\text{M}}$, along with graphing to get initial estimates for the roots, critical values can be found for almost any function whose graph you would want to sketch. Many problems from engineering and science have coefficients that are not whole numbers but numbers given to several decimal places. Once such a problem is entered into the calculator, it is no harder to go through the procedures outlined below than it is for a problem with "nice" coefficients.

THE GENERAL TECHNIQUE

Given an equation of the form y = f(x) we would like to be able to sketch its graph, showing the exact (to the 12-figure accuracy of the HP-28) coordinates of any intercepts and any local minimum, local maximum or inflection points. A procedure for doing this is outlined below.

Enter f(x) and store it under a name of your choice. A very useful way to enter f is as a simple program, as discussed in Section III of Chapter 2; with this you can both recall f in algebraic form and evaluate it at specific values of x. Now graph f changing the plotting parameters as needed to get a graph that shows the important points. If you are unable to get a graph which shows all the important points, go on to the next step.

Recall f, take its derivative using the $\boxed{\frac{d}{dx}}$ command and store the derivative under a name of your choice.

Recall $\frac{df}{dx}$, take its derivative using the $\boxed{\frac{d}{dx}}$ command and store the second derivative under a name of your choice.

You are now ready to find critical numbers for f. You need to use $\boxed{\text{SOLVR}}^{\text{M}}$ to find roots of the equation f'(x) = 0. Use the procedure given in Chapter 5 to find roots of f'(x) = 0. For each critical value, evaluate f there. With the values of f at the critical numbers, you probably can now set the plotting parameters so that you can get a good representation of the graph of f on the screen if you could not do so originally. To find the coordinates of the inflection points, you need to repeat, for f'', the procedure used above to find roots of f'. To find the x intercepts, you now need to find the roots of f, using this same procedure.

On a test, you may sometimes be asked, when it is feasible, to sketch your graph of f on a coordinate system in which the unit distance is the same on both axes. The graph you sketch for yourself using the max/min and inflection points found will probably be a truer picture of f than the final graph shown on the HP-28, since your graph will not have the vertical compression that is usually necessary to fit a graph to the calculator screen.

EXAMPLE 1.

Sketch the graph of $G(x) = x^5 + 3x^4 - 2x^3 - 8x^2 + 2x + 6$ showing the coordinates of the important points correct to 3 decimal places.

Enter G(x) into the HP-28 in an evaluation program from which you can both recall G(x) and evaluate it for specific values of x:

$$\text{«} \rightarrow x \ 'x^5 + 3x^4 - 2x^3 - 8x^2 + 2x + 6' \text{»}$$

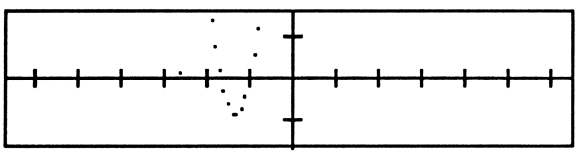
Save this under a name of your choice, for example: 'GEV $\boxed{\text{STO}}$

Recall G(x): x $\boxed{\text{GEV}}^{M}$

Graph G: $\boxed{\text{PLOT}}$ $\boxed{\text{STEQ}}^{M}$

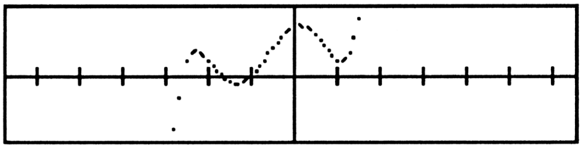
($\boxed{\text{NEXT}}$ $\boxed{\text{PPAR}}^{M}$ $\boxed{\text{PURGE}}$ $\boxed{\text{PREV}}$)

$\boxed{\text{DRAW}}$



Not much with the default **PPAR**. Try

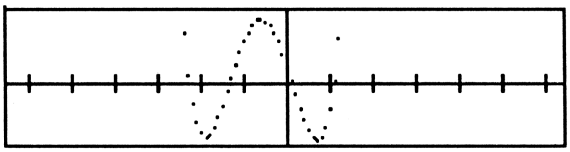
$\boxed{\text{NEXT}}$ 5 $\boxed{\text{* H}}$ $\boxed{\text{PREV}}$ $\boxed{\text{DRAW}}$



This is better. Note that although we see only three x intercepts of f, four extreme points are shown and this is all that the graph of f can have since f' is a fourth-degree polynomial. Thus we can see all the important points. Furthermore, since f is a fifth-degree polynomial with positive fifth-degree term, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$. To find the coordinates of the extreme points shown, get G(x) on stack level 1; take its derivative and store it:

'x, $\boxed{\text{ENTER}}$, $\boxed{\frac{d}{dx}}$. 'DG, $\boxed{\text{STO}}$.

Now recall the derivative and $\boxed{\text{DRAW}}^{M}$ its graph:



Notice that we can see all the x intercepts.

Move the crosshairs along the x axis with the cursor keys to a point close to each root and record the coordinates of each point with $\boxed{\text{INS}}$, then return to the stack display screen with $\boxed{\text{ON}}$. For my choice of points, I got:

4:	(1.1,0)
3:	(.1,0)
2:	(-1.3,0)
1:	(-2.3,0)

Solve for the roots of $\frac{dG}{dx}$:

$\boxed{\text{SOLV}}$ $\boxed{\text{SOLVR}}$ $\boxed{\text{x}}^{\text{M}}$ $\boxed{\text{RED}}$ $\boxed{\text{x}}$.

This will give you the root corresponding to your point estimate on level 1.

Store it: **'DGR1** $\boxed{\text{STO}}$.

Repeat this process to get the remaining three roots and store them. You can jot the roots of $\frac{dG}{dx}$ down on a piece of paper if you don't want to store them, However, it is convenient to keep everything in the calculator. Also, unless you jot down all 12 digits, you lose some accuracy. If you want to keep your **USER** menu neat, you can store the 4 roots in a **LIST**. You can now find the y coordinates of the extreme points by recalling each of the roots of $\frac{dG}{dx}$ found and finding the value of G there with your evaluation program for G. Store each of these

'GXTR1 $\boxed{\text{STO}}$ **'GXTR2** $\boxed{\text{STO}}$ etc.

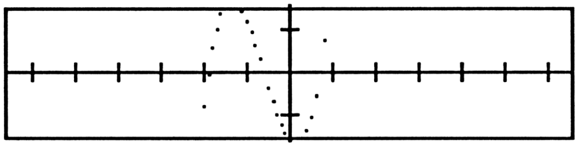
Here is an alternate method. With a root of G' on level 1, press $\boxed{\text{ENTER}}$. This duplicates the entry on level 1 and moves everything else up. Then do $\boxed{\text{USER}}$ $\boxed{\text{GEV}}^{\text{M}}$. This gives you the root on level 2 and the value of G at the root on level 1.

Now do $\boxed{\text{COMPLX}}$ then $\boxed{\text{R}\rightarrow\text{C}}^{\text{M}}$. This command takes real numbers on levels 2 and 1 and makes them the coordinates of a point (complex number). Thus you now have on level 1 an extreme point of G. Store it under a name: **'GXP1** $\boxed{\text{STO}}$. Repeat with the other roots of G'.

Now find $\frac{d^2G}{dx^2}$ and store it:

$\boxed{\text{DG}}^{\text{M}}$ **'x** $\boxed{\text{ENTER}}$ $\boxed{\frac{d}{dx}}$ **'D2G** $\boxed{\text{STO}}$.

Now recall $\boxed{\text{D2G}}^{\text{M}}$ and $\boxed{\text{DRAW}}^{\text{M}}$ its graph:



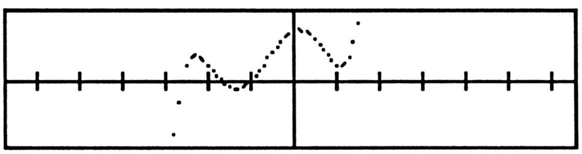
After doing 2 *H, we produced the graph shown above, which is fine since it shows what we want, the roots.

Get initial approximations for the roots by moving the crosshairs close to the x-intercept points and entering these points with $\boxed{\text{INS}}$. Now use **SOLVR** with these initial approximations to find each of the roots of $\frac{d^2G}{dx^2}$, storing them as you find them. These are the x-coordinates of the inflection points of G. Now find the values of G at each, using your evaluation program.

Finally, recall G(x), $\boxed{\text{DRAW}}^{\text{M}}$ its graph, and find its x-intercepts by entering points near them on the stack as initial estimates and using **SOLVR** to find the roots.

Usually you would now be asked to sketch your own graph of G, with equal unit distances on the two axes if possible, showing both coordinates of important points. This would give a graph of G without the vertical compression often needed to fit a graph to the screen.

Here, however, I will just show the screen graph of G again and list the points:



x-intercepts: -2.607, -1.668, -1.000

local maximum points: (-2.275,3.015), (0.121,6.122) local minimum points: (-1.335,-0.881),(1.089,1.858)

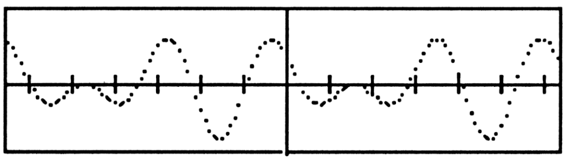
inflection points: (-1.894,1.333), (-0.605,2.627), (0.699,3.692)

A TRIGONOMETRIC EXAMPLE

Plot the graph of f(x) = cos(2x) - sin(3x) and find both coordinates of the important points. Again, enter this function as a simple program:

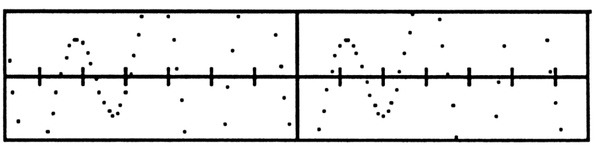
« → x ' cos(2*x) - sin(3*x)' » ENTER **' SCF** STO

Now do x SCF to recall f to level 1. Graphing with the default **PPAR** puts the lower extreme points off the screen so do **1.5** * H M before executing the DRAW M command again. This gives the following graph:



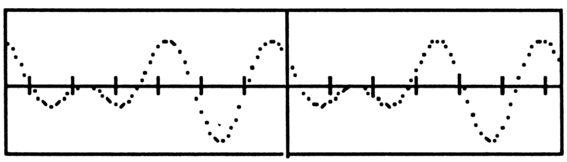
This function has period 2π. You can tell this by inspection of the graph or by noting that the period of sin(3x) is $\frac{2\pi}{3}$ and the period of cos(2x) is π. So we need to find the extreme points, inflection points and intercepts between 0 and 2π.

Recall the function to level 1 and take its derivative. DRAW M the graph of the derivative:



Although we can't see the extreme points, we can see the zeroes which is all we need here. Move the crosshairs to each of the points, starting at the right, where the graph of $\frac{df}{dx}$ crosses the x axis and record the coordinates on the stack with $\boxed{\text{INS}}^{\mathbf{M}}$. Now, get back to the stack display screen and use $\boxed{\text{SOLVR}}^{\mathbf{M}}$ to find the zeroes of $\frac{df}{dx}$ using these points as initial guesses. A hint on stack manipulation here: when you enter the point on level 1 as x and then solve for x, you will have this solution on level 1. To proceed with the second point you need it on level 1. Executing **2** $\boxed{\text{ROLL}}$ will bring the point on level 2 down to level 1 and move everything else up. After you have found this solution, executing **3** $\boxed{\text{ROLL}}$ will bring the point on level 3 down to level 1 and move everything else up. And so on. These roots of f'(x) = 0 are the x coordinates of the extreme points of the graph of f. You will need to store these roots and the corresponding values of f in some way.

Repeat the process above with f" to find the inflection points, and then find the roots of f(x) = 0 for the intercepts. The graph of f together with the important points:



x intercepts: 0,31416, 1.57090, 2.82743, 4.08407, 5.34071

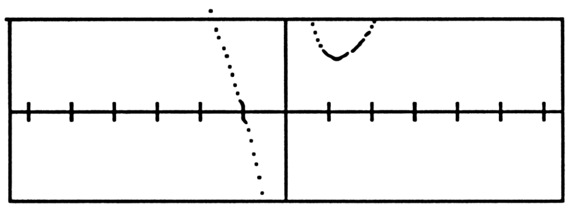
local maximum points: (1.57080,0), (3.51027,1.63418), (5.91451,1.63418)

local minimum points: (0.76663,-0.70825), (2.37496,-0.70825), (4.71239.-2)

inflection points: (0.14651,0.53188), (1.14662,-0.36735), (1.99497,-0.36735), (2.99509,0.53188), (4.12998,-0.21940), (5.24980,-0.21940)

AN EXAMPLE WITH AN ASYMPTOTE

We consider the function $f(x) = \dfrac{x^3 - x^2 + 2}{x}$. Since f(1) = 2, with the default **PPAR** this part of the graph would be off the screen so we will do **5** $\boxed{\text{* H}}$ and then $\boxed{\text{DRAW}}$.



This graph has the y axis as a vertical asymptote and just one local extreme point - a minimum to the right of the y axis - and just one inflection point. If we find and graph $\frac{df}{dx}$ and then use **SOLVR** to find its root we get x = 1.19743. Evaluating f here, we get (1.19743, 1.90665) as the minimum point. If we find and graph $\frac{d^2f}{dx^2}$ and then use **SOLVR** to find its root and then evaluate f there, we get (-1.25992, 1.25992) as the inflection point. The x intercept is -1.

CHAPTER 6 EXERCISES

Set I

For each of the functions given below, sketch its graph showing both coordinates of any local maximum, local minimum or inflection points.

1. $f(x) = x + 3 \sin x$, on the interval $[0,2\pi]$.

2. $f(x) = x^3 - x + 2$, on the interval $[-2,2]$.

3. $f(x) = \sin x + 2 \cos(3x)$

4. $f(x) = x^3 - 3x^2 - 5x + 2$

5. $f(x) = x^4 - x^3 - 3x^2 + x - 2$

6. $f(x) = x^3 -(1.3)x^2 + (.32)x - .02$

7. $f(x) = x^5 + 3x^4 - x^3 -3x^2 - x + 3$

8. $f(x) = x \sin x,\ 0 \le x \le 2\pi$

9. $f(x) = \sin(3x) - \cos(2x),\ 0 \le x \le 2\pi$

10. $f(x) = \sin(x^2/2),\ 0 \le x \le \pi$

Set II

For each of the functions given below, sketch its graph showing both coordinates of any local maximum, local minimum or inflection points.

1. $f(x) = x \ln x$

2. $f(x) = x\,e^{2x}$

3. $f(x) = e^{-x}\sin x$ (find the first high, low and inflection points right of $x = 0$).

4. $f(x) = e^{x} - x\,e^{-x}$

5. $f(x) = x^{x}$

6. $f(x) = x^{\sin x}$, $0 \le x \le 2\pi$

7. $f(x) = (\sin x)^{x}$, $0 \le x \le \pi$

CHAPTER 7

PARAMETRIC EQUATIONS

INTRODUCTION

The first type of plane curves treated in calculus are the graphs of equations like $y = f(x)$. However, there are many plane curves which are not the graphs of functions. For instance, the unit circle is not the graph of a function since pairs of its points have the same x coordinate and a function can have only one value for each x in its domain.

Though the unit circle is the graph of the relation $x^2 + y^2 = 1$, this relation will not accomplish very readily a task that is often required in applications: to describe a path in the plane which goes around the circle with increasing values of the independent variable. But we can describe such a path by introducing a parameter t and expressing the x and y coordinates of points as functions of t. If we let $x = \cos t$ and $y = \sin t$, then $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$ and so all the points in the plane whose coordinates satisfy this pair of parametric equations for some real number t lie on the unit circle. A little thought shows that if we take $0 \le t \le 2\pi$ then this will give one "orbit" around the unit circle, starting at (1,0) when $t = 0$, travelling along the circle counterclockwise and reaching (1,0) again when $t = 2\pi$. For $2\pi \le t \le 4\pi$, we will get another trip around the circle. We should also note that any plane curve which is the graph of an equation of the form $y = f(x)$ is the graph of the parametric equations $x = t$, $y = f(t)$. Thus the set of parametric curves in the plane is a large family of curves which includes as a subset the set of graphs of equations $y = f(x)$.

PROGRAMS FOR PLOTTING PARAMETRIC CURVES WITH THE HP-28S

You will need to enter some programs to plot parametric curves with the 28-S. Create a new directory named **PARM**. Do

[MEMORY] **'PARM**
[CRDIR]^{M} **[USER]** **[PARM]**

Now store the following in the new directory:

QUIT

« HOME » **[ENTER]** **'QUIT** **[STO]**

Input: none Effect: return to main directory

Establish memory locations for **SCR , N, B, A, Y** and **X.** Simply store 0 under each of these names.

RCLGR

« SCR →LCD DGTIZ » **[ENTER]** **'RCLGR** **[STO]**

Input: none Effect: recalls graph

(**[→LCD]**^{M} is on the **[STRING]** menu.)

CLDRA

« B A - N / → H	I sets $H = \frac{B - A}{N}$
CLLCD DRAX	I clear screen , draw axes
A 'T' STO	I store A in 'T'
1 N START	I start loop of N steps
X →NUM Y →NUM	I evaluate X and Y at T
R→C PIXEL	I form point (X,Y), plot
H 'T' STO+ NEXT	I add H to T for new T, proceed

'T' PURGE LCD→

'SCR' STO DGTIZ » »

Input: none — Effect: draws graph

($\boxed{\text{STO+}}^M$ is on the $\boxed{\text{STORE}}$ menu.)

ABSTO

« 'B' STO 'A' STO »

Input: A on level 2, B on level 1 — Effect: stores upper and lower limits on t

YSTO

« 'Y' STO »

Input: y(t) — Effect: stores y as a function of t

XSTO

« 'X' STO »

Input: x(t) — Effect: stores x as a function of t

NSTO

« 'N' STO »

Input: n, the number of points to be plotted — Effect: sets the number of points

(These programs are from Tom Tucker [2].)

SOME EXAMPLES OF PARAMETRIC CURVES

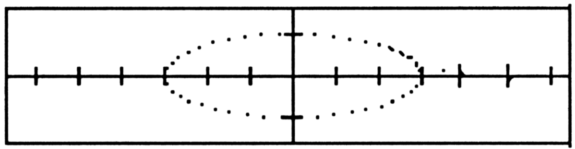
EXAMPLE 1. Plot the graph of $x = 3 \cos t$, $y = \sin t$, $0 \le t \le 6.29$.

Key in ' PPAR $\boxed{\text{PURGE}}$ $\boxed{\text{USER}}$ $\boxed{\text{PARM}}^M$ 50 $\boxed{\text{NSTO}}^M$ '

3*COS(T) $\boxed{\text{ENTER}}$ $\boxed{\text{XSTO}}$ 'SIN(T) $\boxed{\text{ENTER}}$ $\boxed{\text{YSTO}}^M$

0 [ENTER] 6.29 [ENTER] [ABSTO]^{M} [CLDRA].

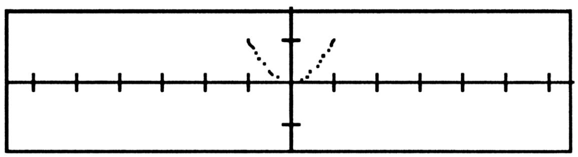
You should get the ellipse shown below:



The value of 50 set for n above seems to work well for many graphs. If a graph seems too "dotty" or disconnected to you, you can increase n.

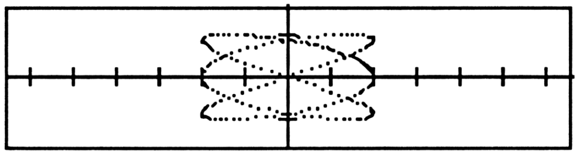
EXAMPLE 2. Plot the graph of x = cos t, y = $\cos^2$ t, $0 \le t \le 6.29$. Do [PLOT] [NEXT] ' [PPAR] [PURGE] 'COS(T [ENTER] [ENTER] [x^2] [YSTO] [XSTO] [CLDRA].

You should get the portion of a parabola shown below. If you watch closely while the curve is plotting, you can see that this curve is traced over twice by the plotting points.



Note that for this graph we left n, a and b at their former settings.

EXAMPLE 3. Plot the graph of x = 2 cos(3t), y = sin(2t), $0 \le t \le 6.29$. Taking n = 50 and n = 100 do not give very intelligible graphs, so we try n = 200. This graph shows up well with the default **PPAR**, so if you have something else, purge **PPAR**. In fact, the graph shows up even better if you take 0.7[* H]. Then do **200** [NSTO] **' 2*COS(3*T** [ENTER] **' SIN(2*T** [ENTER] [YSTO] [XSTO] [CLDRA]. You should get the curve:



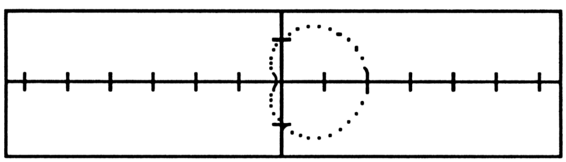
This type of curve, x = a cos(bt), y = c sin(bt) is a sometimes called a "Lissajou curve".

A PARAMETRIC CURVE USING CURVE-SKETCHING TECHNIQUES TO FIND HIGHEST, LOWEST, LEFTMOST AND RIGHTMOST POINTS

EXAMPLE 4. Sketch the parametric curve x = (1 + cos t) cos t, y = (1 + cos t) sint, 0 ≤ t ≤ 6.29, and find both coordinates of the highest, lowest, leftmost and rightmost points of the curve. Since we want to find values of x and y for specific values of t, it is convenient to enter both functions as simple evaluation programs.

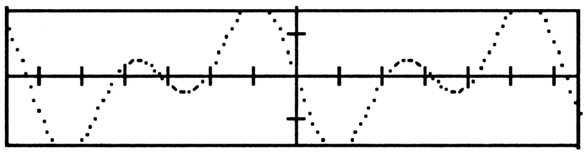
« → T ' (1 + COS(T))* COS(T) ' » ENTER 'XEVL STO, then
« → T ' (1 + COS(T))*SIN(T) ' » ENTER 'YEVL STO.

To plot the parametric curve, return to the default **PPAR**. Taking n = 50 works well here, so enter 50 for n. You can recall x and y to the screen by doing 'T ENTER XEVL then 'T ENTER YEVL **M**. Now store these with YSTO **M** then XSTO **M**. Plotting with CLDRA **M** gives the heart-shaped graph shown below.



We would now like to find coordinates for the highest and lowest and leftmost and rightmost points on this graph. The high and low points will occur at those values of t for which y(t) = (1 + cos t) sin t has a maximum or minimum. The rightmost and leftmost points will occur at those values of t for which x(t) = (1 +

cos t) cos t has a maximum or minimum. To find the leftmost and rightmost points, must recall x(t), find its derivative, plot the derivative and find its roots by the techniques of Chapter 5, and then find the x and y coordinates of the points corresponding to these values of t. From the parametric plot above, the rightmost point of the graph is obviously (0,1), which corresponds to both t = 0 and t = 2π so we can ignore these roots and direct our attention to the other values of t where x'(t) = 0. Do **T** $\boxed{\text{XEVL}}^{\text{M}}$ **'T** $\boxed{\text{ENTER}}$ $\boxed{\frac{d}{dx}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{DRAW}}^{\text{M}}$. This should give you the following graph:



Moving the cursor to each of the three points between 0 and 2π, recording their coordinates on the stack, then using **SOLVR** with these points as initial guesses gives for the roots the three numbers 2.09440, 3.14159 and 4.18879. The second number is clearly π and the point on the graph corresponding to t = π is (0,0) which, from the graph of the parametric curve is a *local* rightmost point. Evaluating x and y at the first value of t gives x(2.09440) = -0.2500 and y(2.09440) = 0.43301. Evaluating x and y at the third value gives x(4.18879) = -.25000 and y = -0.43301. So the two leftmost points of the graph are (-0.25, 0.43301) and (-0.25, -0.43301).

Now you need to get the expression for y(t) on level 1, take its derivative, plot it, get first estimates for the roots from the t intercepts on the graph and use **SOLVR** to find these roots. Note that the root at t = π gives y'(π) = 0, but this is neither a max nor a min for y since y' is negative on both sides of it. So we can ignore t = π. The other two roots of y' are 1.04720 and 5.23599. Finding the points of the parametric graph corresponding to these two values of t gives (0.75000, 1.29904) as the high point and (0.75000, -1.29904) as the low point.

CHAPTER 7 EXERCISES

In exercises 1 through 4, **DRAW** the graph of the given parametric curve:

1. $x = 4 \cos t$, $y = 3 \sin t$, $0 \le t \le 6.29$

2. $x = \sin^2 t$, $y = \sin t$, $0 \le t \le 6.29$

3. $x = t - \sin t$, $y = 1 - \cos t$, $-6.29 \le t \le 6.29$

4. $x = 2 \cos (3t)$, $y = 3 \sin(5t)$, $0 \le t \le 6.29$

5. $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le 6.29$

6. $x = \sec t$, $y = \tan t$, $0 \le t \le 6.29$

7. $x = \cos t + \sin t$, $y = \sin t - \cos t$, $0 \le t \le 6.29$

8. $x = \cos t - \cos(4t)$, $y = \sin t - \sin (4t)$, $0 \le t \le 6.29$

9. $x = \cos t + \cos(2t)$, $y = \sin t - \sin(2t)$, $0 \le t \le 6.29$

10. **DRAW** the graph of the parametric curve $x = t^2$, $y = \frac{1}{3} t^3$, $-5 \le t \le 5$, and find the x and y coordinates of the rightmost, leftmost, highest and lowest points of the graph.

11. **DRAW** the graph of the parametric curve $x = (1 + 2 \cos t) \cos t$, $y = (1 + 2 \cos t) \sin t$, $0 \le t \le 6.29$, and find the x and y coordinates of the leftmost, rightmost, highest and lowest points of the graph.

CHAPTER 8
INTEGRATION

INTRODUCTION

In this chapter, we shall exploit the calculating ability of the HP-28S in connection with integration. Some programs will be given which will allow the calculation of several kinds of Riemann sums for increasing values of n to help illustrate the limiting process that defines the definite integral. We will also introduce the idea of numerical integration with very simple methods and expand the programs to include calculations by the trapezoidal rule and Simpson's rule. The calculator's built-in programs for evaluating definite integrals and for finding antiderivatives will also be briefly treated.

The simplest way to evaluate a definite integral is to use the HP-28S' built-in routine. But the programs presented here may help give insight into the convergence of Riemann sums and serve as an introduction to numerical integration.

USING THE HP-28S AS AN AID TO UNDERSTANDING THE DEFINITE INTEGRAL

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of Riemann sums $\sum_{i=1}^{n} f(u_i)\Delta x_i$, where the numbers $x_1, x_2, \dots, x_n$ form a partition of the interval [a,b]:that is, $a=x_0 < x_1 < x_2 < \dots < x_n=b$. The "evaluation" points $u_1, u_2, \dots, u_n$, satisfy $x_{i-1} \le u_i \le x_i$ for each i from 1 to n inclusive, and $\Delta x_i = x_i - x_{i-1}$. If the integrand is never negative-valued on [a,b], this Riemann sum approximates the area under the graph of f for x on [a,b] as the sum of the areas of rectangles.

When the limit of such sums exists, as the width of the longest subinterval in the partitions goes to 0, the value is called the definite integral of f to a to b, i.e.,

$\int_a^b f(x)\,dx$. By installing a few simple programs into the HP-28S, we can evaluate several different kinds of Riemann sums for functions of our choice using various values of n and thereby gain some insight into the limiting process that defines the integral. (These programs all come from a minicourse on the HP-28S conducted by Tom Tucker of Colgate University at the Joint Mathematics Meetings at Phoenix, Arizona, in January, 1989.)

Create a directory named **INTG** by keying in **'INTG** $\boxed{\text{CRDIR}}^{\text{M}}$ (this key is on the first line of the $\boxed{\text{MEMORY}}$ menu). Now press $\boxed{\text{INTG}}^{\text{M}}$.

Key in the programs shown below. They will appear on the $\boxed{\text{INTG}}^{\text{M}}$ subdirectory under the names given to each program.

QUIT

« HOME » $\boxed{\text{ENTER}}$ **'QUIT** $\boxed{\text{STO}}$

Input: none Effect: returns to **HOME** directory

SUM

« 'X' $\boxed{\text{STO}}$ **0 1 N START EQ** $\boxed{\to\text{NUM}}$ **+ H 'X'** $\boxed{\text{STO+}}$ **NEXT H ***
'X' $\boxed{\text{PURGE}}$ **»** $\boxed{\text{ENTER}}$ **'SUM'** $\boxed{\text{STO}}$

Input: real number (given by another program)

Effect: this is a utility program which is used for computation by each of the Riemann sum programs. It takes the initial value of **'X'** from the other program, A for **LRECT**, **A + H** for **RRECT** and $\mathbf{A + \frac{H}{2}}$ for **MID**. It evaluates the integrand N times, starting at the initial value of X, increments X by H each time and adds to the running sum of integrand values kept on the stack. The final sum is multiplied by H.

RRECT

« A H + SUM » $\boxed{\text{ENTER}}$ 'RRECT $\boxed{\text{STO}}$

Input: none

Effect: for the integrand, A, B and N already stored, it uses SUM to compute the Riemann sum with the integrand evaluated at the right endpoint of each subinterval.

LRECT

« A SUM » $\boxed{\text{ENTER}}$ 'LRECT $\boxed{\text{STO}}$

Input: none

Effect: for the integrand, A, B and N already stored, it uses SUM to compute the Riemann sum with the integrand evaluated at the left endpoint of each subinterval.

MID

« 'A + H/2 ' EVAL SUM » $\boxed{\text{ENTER}}$ 'MID $\boxed{\text{STO}}$

Input: none

Effect: for the integrand, A, B and N already stored, it uses SUM to compute the Riemann sum with the integrand evaluated at the midpoint of each subinterval.

NSTO

« 'N' $\boxed{\text{STO}}$ B A - N / 'H' $\boxed{\text{STO}}$ » $\boxed{\text{ENTER}}$ 'NSTO $\boxed{\text{STO}}$

Input: positive integer N

Effect: it sets the number of subintervals at N and sets $H = \frac{B - A}{2}$ for use with the programs **LRECT, RRECT** and **MID**.

FABST

« 'B' $\boxed{\text{STO}}$ 'A' $\boxed{\text{STO}}$ STEQ » $\boxed{\text{ENTER}}$ 'FABST $\boxed{\text{STO}}$

Input: 'f(x)' on level 3, real number A on level 2, real number B on level 1

Effect: it stores f as the integrand and A and B as the left and right endpoints of the interval of integration.

The three programs **RRECT, LRECT** and **MID** evaluate three different Riemann sums for the function, interval and value of N you store in the calculator. In the **LRECT** program, the function is evaluated at the left endpoint of each subinterval. In the **RRECT** program, the evaluation points are taken to be the right endpoints of each subinterval and, in the **MID** program, they are the midpoints of each subinterval.

EXAMPLE 1. We shall use these programs to evaluate some Riemann sums for a specific integral, $\int_1^4 1/x \, dx$. If you are taking first-term calculus and have not had any calculus before, you may be unable to evaluate this integral using the Fundamental Theorem which says that $\int_a^b F'(x) \, dx = F(b) - F(a)$, since you may not know a function whose derivative is 1/x. Thus, we shall get some idea of the size of the number which is this integral by evaluating a few Riemann sums.

Press $\boxed{\text{USER}}$ then $\boxed{\text{INTG}}^{\text{M}}$.

Now key in ' 1/x $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$ 4 $\boxed{\text{FABST}}^{\text{M}}$.

This enters the function and the lower and upper limits of integration into the program. Now choose a value of n and enter it with **NSTO**. We start with n = 20:

20 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{LRECT}}^{\text{M}}$

gives the result 1.50576102986 on level 1. This is the Riemann sum for the function $f(x) = 1/x$ on the interval [1,4], when the interval is partitioned into twenty subintervals of equal length and the function is evaluated in each subinterval at the left endpoint. Since $1/x$ is a decreasing function for all positive x, this Riemann sum must be larger than the actual integral.

Pressing $\boxed{\text{RRECT}}^{\text{M}}$ gives the result 1.28076102986. This is like the **LRECT** estimate except that the evaluation point in each subinterval is chosen to be the right endpoint. Since the function is decreasing, this sum must be less than the actual integral.

Now keying in **40** $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{LRECT}}^{\text{M}}$ gives 1.41485855232

and $\boxed{\text{RRECT}}^{\text{M}}$ gives 1.35860655232.

Repeating with **100** $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{LRECT}}$ gives 1.39761466688

and $\boxed{\text{RRECT}}$ gives 1.37511466688.

So $1.37511466688 \le \int_1^4 \frac{1}{x}\,dx \le 1.39761466688$. Using larger values of n will, of course, narrow the gap still further.

For a decreasing function like $1/x$, evaluating the function at the left and right endpoints of each subinterval has the advantage of bracketing the answer. However, evaluating the function at the midpoint of each subinterval will usually give a better approximation for a given value of n.

Keying in 40 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{MID}}^{\text{M}}$ gives 1.3860748637 for the Riemann sum with the midpoint evaluation and 40 subintervals.

Doing 80 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{MID}}^{\text{M}}$ gives 1.38623944384 with 80 subintervals.

Doing 100 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{MID}}^{\text{M}}$ gives 1.38625921076 with 100 subintervals.

These midpoint estimates all agree to three decimal places, and the last two agree to four places. This is much better than we obtained with the left and right endpoint estimates.

USING THE HP-28S TO ILLUSTRATE NUMERICAL INTEGRATION

Most of the definite integrals that you evaluate in a calculus course can be obtained by finding an antiderivative of the integrand and using the fundamental theorem $\int_a^b F'(x)\,dx = F(b) - F(a)$. However, many applications involve evaluating integrals for which this is not possible. For example, finding the length of the graph of $f(x) = x^3$ between the points (1,1) and (4,64), involves evaluating the integral $\int_1^4 \sqrt{1 + 9x^4}\,dx$, and an antiderivative for $\sqrt{1 + 9x^4}$ can be found only as an integral itself, that is, $\int_1^x \sqrt{1 + 9t^4}\,dt$. However, this arclength and, in fact, any definite integral with constant limits, is just a number. What is needed is a technique for computing this number to the degree of accuracy needed in a given case. Riemann sums give one way of approximating a definite integral. But these sums often converge rather slowly and so more efficient algorithms for approximating integrals would be useful.

In Example 1 above, since the function is decreasing, we note that the **RRECT** estimate is below the value of the integral and the **LRECT** is above it. In fact, in this example the exact value is obtainable through the use of the fundamental theorem of calculus, (the function ln x is an antiderivative of $\frac{1}{x}$) and if we were to do the evaluations we would see that the errors in the two approximations are close to the same magnitude. In any case, the decreasing function observation insures that the errors are opposite in sign and thus tend to cancel out at least part of each other. Accordingly, this suggests that a better approximation may be derived through a simple average of the two approximations. This yields the trapezoidal rule. Enter the program below in the **INTG** subdirectory.

TRAP

« **LRECT RRECT + 2 /** » $\boxed{\text{ENTER}}$ **'TRAP** $\boxed{\text{STO}}$

Geometrically, you can think of the trapezoidal rule as approximating the function on each subinterval by the line segment joining the two endpoints of the graph on the subinterval, rather than by a horizontal line segment.

EXAMPLE 2. Approximate $\int_1^4 \frac{1}{x}\,dx$ using the trapezoidal rule with with n = 40, 80 and 100. The program **TRAP** gives the trapezoidal rule approximation for the given function, interval and n. We already have $\frac{1}{x}$ stored as f and 1 and 4 stored as a and b, so

40 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{TRAP}}^{\text{M}}$ gives 1.38673355232 as the trapezoidal rule approximation with n = 40,

80 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{TRAP}}^{\text{M}}$ gives 1.386404208 with n = 80 and

100 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{TRAP}}^{\text{M}}$ gives 1.38636466688 with n = 100

We can see with a little geometric analysis that for a concave up function like $\frac{1}{x}$, the trapezoidal line segments remain above the graph, except at the endpoints of the subintervals. The resulting approximation exceeds the true value of the definite integral and the error is consequently positive.

It can be noted with techniques that are not immediately apparent here that the error in the estimation by the midpoint rule is negative androughly half the size of the error generated by the trapezoidal approximation. This suggests that a weighted average which assigns twice as much weight to the midpoint approximation as to the trapezoidal approximation would take advantage of the errors "cancelling" with each other. Such a rule is the widely used formula known as Simpson's rule for approximating definite integrals. Enter this program in the INTG subdirectory:

SIMP

« MID 2 * TRAP + 3 / » $\boxed{\text{ENTER}}$ 'SIMP $\boxed{\text{STO}}$

EXAMPLE 3. Approximate $\int_1^4 \frac{1}{x}\,dx$ using Simpson's rule with n = 40, 80 and 100. We already have the function, a and b stored so

40 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{SIMP}}^{\text{M}}$ gives 1.386294426577 with n = 40;

80 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{SIMP}}^{\text{M}}$ gives 1.38629436528 with n = 80; and

100 $\boxed{\text{NSTO}}^{\text{M}}$ $\boxed{\text{SIMP}}^{\text{M}}$ gives 1.3862943628 with n = 100.

EXAMPLE 4. We will now use Simpson's rule to approximate the arclength integral $\int_1^4 \sqrt{1 + 9x^4}\,dx$ mentioned above.

Keying in '$\sqrt{1 + 9x^4}$ ' $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$ 4 $\boxed{\text{ENTER}}$ $\boxed{\text{FABST}}^M$ will set up the calculator to approximate $\int_1^4 \sqrt{1 + 9x^4}\, dx$.

For Simpson's rule with n = 50, key in 50 $\boxed{\text{NSTO}}^M$ $\boxed{\text{SIMP}}^M$ and get 63.124102271 for the approximation to the integral.

Keying in 100 $\boxed{\text{NSTO}}^M$ $\boxed{\text{SIMP}}^M$ gives 63.1241022593 for Simpson's rule with n = 100.

Keying in 200 $\boxed{\text{NSTO}}^M$ $\boxed{\text{SIMP}}^M$ gives 63.1241022587 for n = 200.

The Simpson's rule approximations seem to be in better agreement than the other approximations we calculated, and indeed Simpson's rule is the "best" of the techniques we have treated. A more detailed look at the situation can be obtained by considering the formulas in your text which estimate the error in the approximations by the trapezoidal rule and Simpson's rule.

The discussion and examples above should give you some insight into the limiting process that defines the definite integral and into some simple procedures for the numerical evaluation of definite integrals. These are both topics that you should learn for future use.

The simplest way to evaluate a definite integral on the HP-28S is to use the program built into the calculator for that purpose. The HP-28S' program for evaluating the definite integral $\int_a^b f(x)\, dx$ requires a three-item input. You should have on the stack :

3:	'f(x)'
2:	{ x a b }
1:	accuracy factor

The entry on stack level 2 is a list containing of the variable of integration, the lower limit of integration and the upper limit of integration. There should be spaces separating the elements of the list. The last item, the "accuracy factor", is a real number which sets an upper limit on the error in the calculation as a fraction of the computed value; for example, if you wanted the error in the answer to be no more than 0.01% of the the answer, the accuracy factor would be 0.0001.

With the required input displayed, executing $\boxed{\int}$ will produce a two-item output from the HP-28S:

2:	calculated integral
1:	maximum on error

EXAMPLE 5. We will use the built-in program to evaluate $\int_1^4 \frac{1}{x}$ dx, arbitrarily deciding that we want the error to be less than .000001 times the calculated integral:

Keying in '1/x $\boxed{\text{ENTER}}$ { x 1 4 } $\boxed{\text{ENTER}}$ .000001 $\boxed{\text{ENTER}}$

gives the display :

3:	'1/x'
2:	{ x 1 4 }
1:	.000001

Executing $\boxed{\int}$ gives

2:	1.38629436197
1:	1.38617494355E - 6

This output tells us that $\int_1^4 1/x \, dx$ is approximately 1.38629436197 with an error less than 1.38617494355 x 10^{-6}. The error term should be approximately the error factor you entered times the calculated integral. This is true here. (Wickes' states in his book [1] that if this is not true, the calculated approximation is suspect and, in particular, if the error term is -1, the approximations failed to converge.) The program in the calculator for approximating definite integrals uses the "Romberg technique", a more sophisticated procdure than those mentioned above in the section on numerical integration. A description of this procedure can be found in most books on numerical analysis.

FINDING ANTIDERIVATIVES WITH THE HP-28S

Finding an antiderivative for a given function is often a considerably harder problem than finding its derivative; in fact, finding an antiderivative as a simple function, not involving integral signs or power series, is not always possible. For example, an antiderivative of $\sin(x^2)$ is $F(x) = \int_0^x \sin(t^2)\, dt$, but expressing

$\int_1^3 \sin(x^2)\, dx$ as $F(3) - F(1) = \int_0^3 \sin(x^2)\, dx - \int_0^1 \sin(x^2)\, dx$ is not of much help in evaluating it as a number. (For those in beginning calculus, the subject "power series" will be investigated later.) When possible, finding antiderivatives involves recognizing patterns in integrals, such as $\int [u(x)]^r u'(x)\, dx$ or $\int \cos[u(x)]\, u'(x)\, dx$, for example. When one of the known patterns is not obviously present, there are trial-and-error methods such as integration by parts or substitution.

Since the HP-28S is not equipped to do this sort of pattern recognition, it can find antiderivatives for only the simplest kind of functions, polynomial functions. For functions other than polynomial functions, the calculator finds the Maclaurin polynomial for the function to the degree that you specify and then integrates this polynomial. The program for finding antiderivatives on the HP-28S, like the one for numerical integration and the Taylor series program we will discuss later, requires a three-item input :

3:	'integrand'
2:	'independent variable'
1:	degree

The items on Levels 3 and 2 are self-explanatory. The item on level 1 should be the degree of the polynomial for a polynomial function, or, for a non-polynomial function, the degree of the Maclaurin series you want the calculator to integrate. The larger the degree, the better the series will simulate the function, and the longer it will take to generate.

EXAMPLE 6. Entering

3:	'x^3-5*x^2+3*x+4
2:	'x'
1:	3

and then executing $\boxed{\int}$ produces

1:	'4*x+1.5*x^2- 1.66667*x^3+ .25*x^4

EXAMPLE 7. Entering

3:	'sin(x)'
2:	'x'
1:	5

and then executing $\boxed{\int}$ produces

```
1:            .5*x^2+4.16667E-2*x^4
                  +1.38889E-3*x^
                                6
```

We will seldom use the HP-28S simply to find an antiderivative. For polynomial functions you can find the antiderivative by hand quicker than you can enter the function into the calculator. But the program is worth keeping in mind. If you want to find definite integrals of a polynomial function over several sets of limits, it may be quicker to evaluate the antiderivative at these limits than to use the definite integral program several times. For a non-polynomial function, if all you want is a general idea of the shape of the graph of an antiderivative, you can get this by using the antiderivative program to get a Taylor polynomial approximation for the function and then plotting this.

CHAPTER 8 EXERCISES

1. Approximate $\int_0^3 \frac{1}{1+x^2}\,dx$

   (a) using **LRECT** and **RRECT** with n = 50, 100 and 200. What is the size relation between **LRECT, RRECT** and the actual value of the integral for a given n? It would probably be helpful to **DRAW** the graph of the integrand to see the size relations between the Riemann sums.

   (b) using **MID** with n = 50, 100 and 200,

   (c) using **TRAP** with n = 50, 100 and 200

   (d) using **SIMP** with n = 50, 100 and 200

   (e) find an antiderivative for $\frac{1}{1+x^2}$ and evaluate the integral using the fundamental theorem. Compare this answer with those above.

2. Repeat parts (a) through (d) of Exercise 1 for $\int_0^3 \frac{1}{1+x^3}\,dx$. Can you repeat part (e) for this integrand?

3. Approximate $\int_0^2 \sqrt{1+\sin^2 x}\,dx$

   (a) using Simpson's rule with n = 100

   (b) using the built-in numerical integration program with error factor .000001.

4. Approximate $\int_1^3 \sqrt{1 + 9x^4}\ dx$ using the bulit-in program with accuracy factor .0001.

5. Repeat Exercise 4 for $\int_0^{1.5} \sqrt{1 + \sec^4 x}\ dx$.

6. Repeat Exercise 4 for $\int_1^3 \sqrt{1 + \frac{1}{x^4}}\ dx$.

The integrals in Problems 3 through 7 represent arc lengths on the curves $y = \sin x$, $y = x^3$, $y = \tan x$ and $y = \frac{1}{x}$.

7. Find an antiderivative for $x^2 + 3x + 4$ using the built-in program.

8. Find an "antiderivative" for $\cos x$ with the built-in program, taking $n = 8$.

CHAPTER 9

TAYLOR POLYNOMIALS AND INFINITE SERIES

INTRODUCTION

When you enter a number x into your calculator and press the $\boxed{\text{SIN}}^{M}$ key, how does the machine determine sin x? There is no little man inside the calculator who draws triangles and measures sides when you press the SIN button. Most calculators, and computers as well, actually compute it; they do not have all the values of the common functions stored.

Among other things, this chapter will answer the question posed above. A method using Taylor polynomials will be presented which, when given a real number x, shows how to calculate sin x, cos x, e^x, etc. using only basic arithmetic operations and a little calculus. In fact, this is the method used to produce trigonometric tables, exponential tables, etc., for centuries. However, since the advent of the modern calculator in recent years, more efficient numerical methods are now generally programmed into them. See the *AMERICAN MATHEMATICAL MONTHLY*, volume 90, number 5, page 317, for a discussion.

Using a sequence of Taylor polynomials as a set of approximating functions for a given function leads to the notions of infinite series and power series. Since power series are a very basic way of defining functions in a variety of settings, infinite series and power series are presented in your text in some detail.

TAYLOR POLYNOMIALS

Polynomial functions are perhaps the simplest functions we meet in calculus. They are easy to understand because their values are readily calculated by finitely many additions and multiplications. In Chapter 4 we used a polynomial to obtain approximate values for a function f near x = a, the linearization $L(x) = f(a) + f'(a)(x - a)$. This is a polynomial in x - a which agrees with both f and f' at x = a. We will use the notation $P_1(x)$ for L(x) and call it the first degree *Taylor polynomial* for f about x = a.

We may reasonably ask for a second degree polynomial $P_2(x)$ in x - a to use in approximating values of f near x = a. Keying on the above, we would require that P_2 and its first two derivatives coincide with f and its first two derivatives at x = a. For $P_2(x) = a_0 + a_1(x - a) + a_2(x - a)^2$, we have $P_2(a) = a_0 = f(a)$; $P_2'(x) = a_1 + 2a_2(x - a)$, and so $P_2'(a) = a_1 = f'(a)$. Finally, $P_2''(x) = 2a_2$ and so $P_2''(a) = 2a_2 = f''(a)$, so $a_2 = \frac{f''(a)}{2}$. Thus the second-degree Taylor polynomial for f about x = a is $P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.

If this process is continued to get a third-degree polynomial $P_3(x)$ which agrees with f and its first three derivatives, then $P_3(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{3(2)}(x - a)^3$. If continued to the n-th degree, the same procedure gives $P_n(x) = f(a) + f'(a)(x - a) + ... + \frac{f^{(n)}(a)}{n!}(x - a)^n$, the nth degree *Taylor polynomial* for f about x = a. In the more concise summation notation, this is $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x - a)^k$. For these polynomial approximations to f to be useful, we must know the value of f and its first n derivatives at x = a.

You should know the definition of Taylor polynomials and be able to find them by hand for simple functions. However, the HP-28S will find Taylor polynomials about a = 0, and this is quite handy for large degree polynomials.

The command for finding a Taylor polynomial is $\boxed{\text{TAYLR}}^{\text{M}}$ located on the second line of the $\boxed{\text{ALGBRA}}$ menu. This command requires a three-item input before execution: the function f whose Taylor polynomial is desired, the independent variable and the degree of the desired Taylor polynomial.

EXAMPLE 1. Find the eleventh degree Taylor polynomial for f(x) = sin x about x = 0.

Key in ' sin(x $\boxed{\text{ENTER}}$ 'x $\boxed{\text{ENTER}}$ 11 $\boxed{\text{ENTER}}$.

Your stack display should look like :

1:	'sin(x)'
2:	'x'
3:	11

Now press $\boxed{\text{TAYLR}}^{\text{M}}$ to get:

```
1:          'x - 0.1667*x^3 + 0.0083
                *x^5 - 0.0002*x^7 +
                   2.7557E-6 * x^9 -
                    2.5052E-8*x^11
```

By hand $P_{11}(x)$ for f(x) = sin x, about a = 0, is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \frac{1}{11!}x^{11}$. This, of course, is what we have above with the coefficients in decimal form. Now find $P_7(x)$ in the same way. We will make a table of values for sin x, $P_{7(x)}$ and $P_{11}(x)$ for a few values of x near 0 to get an idea of how good the P_7 and P_{11} approximations are. We must store both P_9 and P_{11} in some way either as a procedure to use in compiling the values in the table, or under a name on the $\boxed{\text{USER}}$ menu to use with the $\boxed{\text{SOLVR}}$ routine to compile the table.

x	.1	.3	.6	1	1.5
sin x	.099833	.295520	.564642	.841471	.997495
P7(x)	.099833	.295520	.564642	.841469	.997391
P11(x)	.099833	.295520	.564642	.841469	.997391

Notice that for the values of x in the table the P_{11} approximation agrees with sin x to six decimal places and the P_7 approximation deviates from the true value only for x = 1 and x = 1.5. We could have predicted this by estimating the remainder terms $R_9(x)$ and $R_{11}(x)$ as explained in your textbook.

The $\boxed{\text{TAYLR}}^{M}$ command will only find Taylor polynomials centered about a = 0. You can find them about numbers other than 0 by entering the following program (see [1]).

```
« 31 SF 1 CF 3 PICK IFERR RCL THEN 1 SF END → x x0 d xv
« x SHOW 'XPRIM' DUP x0 + x STO d TAYLR xv x0 - 'XPRIM' STO
EVAL xv IF 1 FC?C THEN x STO ELSE PURGE END 'XPRIM' PURGE
» »
```

$\boxed{\text{ENTER}}$ 'TAYX0 $\boxed{\text{STO}}$

This program requires a four-item input: the function on level 4, the independent variable on level 3, the value x_0 about which you want the polynomial centered on level 2 and the degree of the desired Taylor polynomial on level 1.

EXAMPLE 2. Find the 7th degree Taylor polynomial for $f(x) = \sqrt{x}$ about a = 4. Your input stack display should look like this:

4:	'$\sqrt{x}$'
3:	'x'
2:	4
1:	7

Executing $\boxed{\text{TAYX0}}^{\text{M}}$ gives:

1: '2 + .25*(x-4) - .015625
*(x-4)^2 + .001953125
(x-4)^3 -
3.0517578125E - 4(x-4)

You will have to scroll down to see the rest of this polynomial.

INFINITE SEQUENCES AND INFINITE SERIES

An infinite sequence of real numbers is a real-valued function whose domain is the set of positive integers. That is, an infinite sequence has a first number f(1), a second number f(2), a third number f(3) and so on.

An infinite series $\sum_{i=1}^{\infty} a_i$ is a notation for the limit of $\sum_{i=1}^{n} a_i$ as $n \to \infty$. There are two infinite sequences associated with each infinite series, the sequence of numbers being added and the sequence of partial sums whose limit is defined to be the "infinite" sum. It is often a simple matter to determine the limit of an infinite sequence. For many of the sequences we consider, the limit can be found by inspection or by a simple application of l'Hopital's rule. On the other hand, it is seldom a simple matter to determine the limit of an infinite series, or to determine whether it converges or not. In fact, the infinite series itself is often both the definition of the number to which it converges and an algorithm for calculating this number. We include a program, below, which will find the limits of many common infinite series

- accurate to the twelve-digit display of the calculator - **PROVIDED YOU KNOW THAT THE SERIES CONVERGES.** To determine convergence, you will have to use one of the convergence tests presented in your text. The program shows the convergence of the series dynamically, by showing the partial sums as a single number with the last digits changing as more terms are added. Key in the following program (from W. Wickes' book [1]) :

« 0 SWAP DO DUP 4 PICK EVAL SWAP 1 + ROT ROT OVER + DUP 1 DISP DUP 4 ROLLD UNTIL == END ROT DROP2 » [ENTER] **' INFSM** [STO] .

This program returns a 12-digit number as the sum of the infinite series $\sum_{n=1}^{\infty} f(n)$.

Actually, this number represents the partial sum the calculations have reached at the time the display is accurate to 12 digits. The program requires a two-item input: the general term of the series written as a procedure, « → n ' f(n) ' » on level 2 and the value of n where you want to begin on level 1.

EXAMPLE 3. By the integral test, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$, converges (you should verify this). We can find its sum by using program **INFSM**:

Key in **« → n ' 1 / n ^ 4 »** [ENTER] **1** [ENTER]

The stack should look like this :

4:	
3:	
2:	« - > n ' 1/n^4'»
1:	1

Now execute **INFS** from the **USER** menu and a twelve-digit number will appear above the stack with the last digits changing. When all the digits of this number are fixed, the number is 1.08232323295. This is the sum of the series, correct to twelve digits.

CHAPTER 9 EXERCISES

1. Find the Taylor polynomials $P_6(x)$ and $P_{10}(x)$ for $f(x) = e^x$ about a = 0. Graph f and the two polynomials on the same axes. Make a table of values for all three functions using x = .1, .2, .6, 1 and 1.5.

2. Use the program given to find the Taylor polynomials $P_4(x)$ and $P_8(x)$ for $f(x) = \sqrt{x}$ about a = 1. Graph f and the two polynomials on the same set of axes.

3. Find the Taylor polynomials $P_4(x)$, $P_{10}(x)$ and $P_{16}(x)$ for f(x) = cos x about a = 0. Graph all three polynomials and f on the same set of axes.

4. Find the Taylor polynonials $P_6(x)$ and $P_{10}(x)$ for $f(x) = e^{-x^2}$ at a = 0. Graph these two polynomials and f on the same axes.

5. Show that $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges and find its sum using the **INFSM** program.

6. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and find its sum using **INFSM.** This series converges slowly.

7. Show that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges and use **INFSM** to find its sum. This series converges very fast.

8. What do you think will happen if you use INFSM on $\sum_{n=1}^{\infty} \frac{1}{n}$?

9. Show that $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges and use **INFSM** to find its sum.

10. Show that $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ converges and use **INFSM** to find its sum.

Bibliography

1. Wickes, William C. *HP-28 Insights*, 1988 Larken Publications (Corvallis, OR)

2. Tucker, Tom. Handout from MAA Mini-course #13 "Notes on the HP-28 for More Experienced Users. (Given out at Phoenix Math Meetings, Jan. 1989)

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