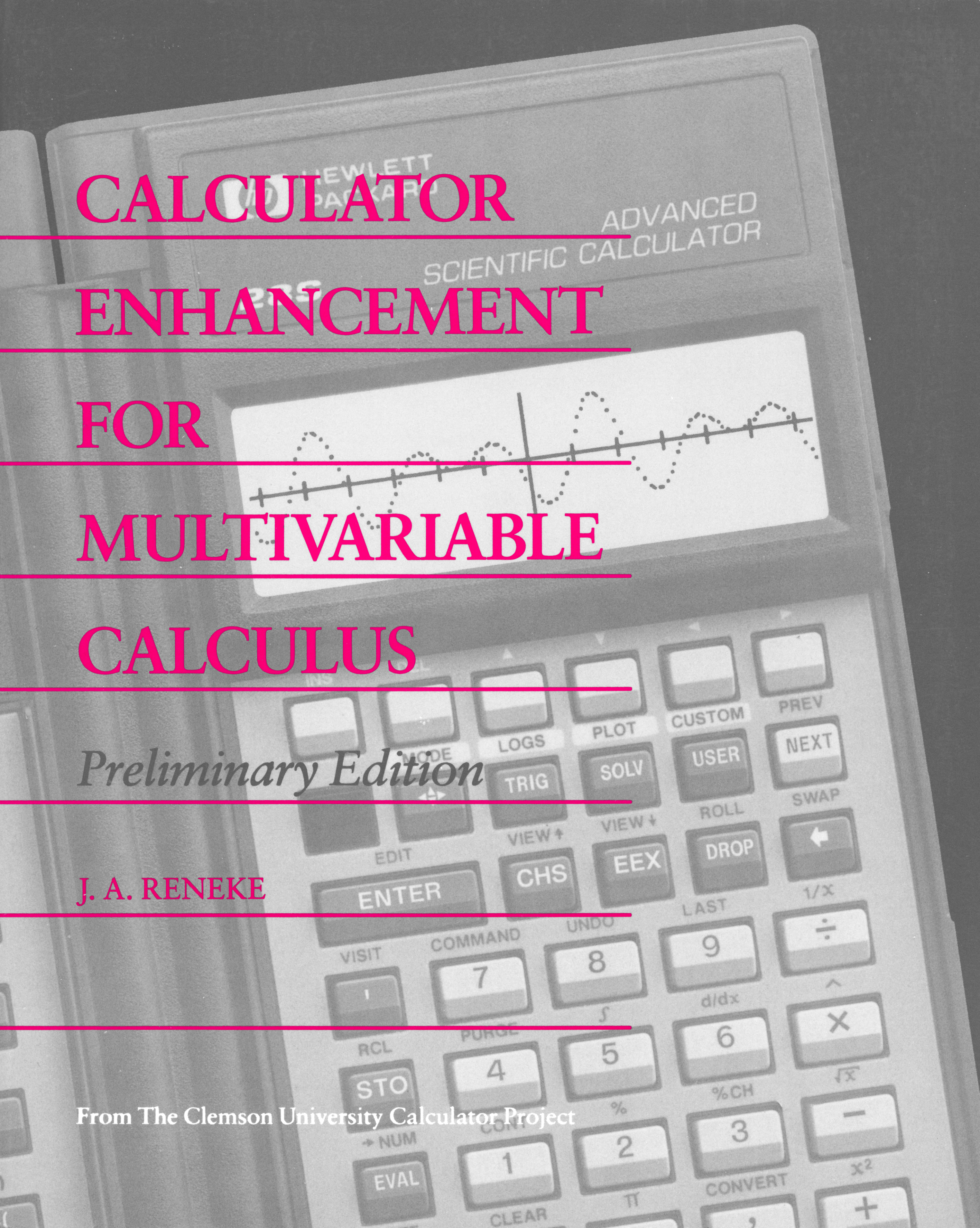
CALCULATOR ENHANCEMENT FOR MULTIVARIABLE CALCULUS

Preliminary Edition



J. A. RENEKE

From The Clemson University Calculator Project

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CALCULATOR ENHANCEMENT FOR MULTIVARIABLE CALCULUS

A Manual of Applications Using the HP-28S Calculator

Preliminary Edition

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PREFACE

We have sought to provide in this manual a new emphasis on geometrical thinking in multivariable calculus using the new technology of graphical calculators. We feel each topic in calculus should be introduced with an appeal to the student's geometric intuition stressing geometric definitions of concepts. We should motivate the calculus with geometric applications constructing geometric proofs wherever possible.

The HP-28S can be thought of as an integrated software package providing capabilities in graphics, symbolic manipulation, numerical linear algebra, integration, statistical analysis and equation solving. The calculator should be the student's handbook, scratch pad and number cruncher, i.e., the student's resource of unstructured computational power.

Chapter 1 contains an introduction to the philosophy of calculator enhanced calculus and suggestions for using the manual. Chapter 2 deals with graphing polar and parametric curves, conic sections and level curves of quadric surfaces. Chapter 3 is concerned with representing curves and surfaces with functions, classifying critical points for functions of two variables and the method of level curves. Chapter 4 presents material on describing regions of the plane bounded by curves and surfaces, single and double integrals. Chapter 5 finishes up with vector fields, line integrals and Green's Theorem.

The material in the manual was tested at Clemson University in a third semester multivariable calculus course which meets four hours a week for fifteen weeks. Students have been enthusiastic about the calculator and its integration into calculus. The sacrifice of class time required for the introduction of the calculator or programs was offset by the possibilities for a more in depth exploration of important concepts.

Table of Contents

CHAPTER 1
INTRODUCTION

1.1 Introduction to the Philosophy of Calculator Enhanced Calculus

The current state of calculus instruction. The reality of a crisis in calculus instruction is evidenced in recent years by the number of talks and panels at professional meetings, reports of various ad hoc groups, papers in the professional literature and the number of news stories in the popular press. My feeling is that the disquiet we all feel about the appropriateness of the content of calculus and the effectiveness of traditional methods of instruction stems not just from increased enrollments but from the broadening of the class of students who now populate a typical calculus course. Additionally over the last twenty years, while calculus is increasingly taught to students from a widening circle of curricula, there has been an accompaning ballooning of material included in the course (a response to appeals for relevance.)

Compounding the difficulties is the new technological enviornment tin which students and instructors find themselves. A bewildering array of new hardware and software offers powerful tools for graphical display, symbolic manipulation and computational ease. Is there a way out of our present situation?

Our response to the crisis in calculus instruction. We have sought a new emphasis on geometrical thinking in multidimensional calculus using the new technology of graphical calculators.

Objective of multidimensional calculus redesign: Reward geometric thinking.

Each topic should be introduced with an appeal to the students' geometric intuition. Build on the base of understanding that each student already has developed. Of course, the new topic represents an extension of that understanding.

Stress geometric definitions of concepts. Concepts which are arrived at geometrically will stick with the student. We aren't talking about a rigorous geometric definition here, but rather an intuitive, visual presentation of the new concepts.

Motivate the calculus with geometric applications. Calculus is overloaded with applications from science, engineering and business. In most university courses there is no common experience or interest among the students. The instructor is forced to pick and choose never satisfying everyone. A better approach is to concentrate on geometric applications which serve as an interface between calculus on one hand and applications outside of mathematics on the other. We are convinced that each application has to start with a geometric formulation before calculus can be brought to bear in any significant way. Let the instructors in the application areas show the students how this is achieved for their own area. Having done this, the properly trained student can then apply the methods of calculus on his own.

Construct geometric proofs wherever possible. Again the emphasis should not be on rigor but visual thinking. We are trying to encourage students to adopt a way of thinking about problems that will stay with them long after they have completed their calculus requirement. When instructors present proofs of various propositions they should at the same time demonstrate how a visual or geometric approach can be applied to the problem.

<u>Opportunities</u> deriving from the introduction of calculators:

1) We encounter the problem of representing abstract objects as functions or computer programs in a concrete setting. In some sense, this representation problem is central to applying mathematics and the interplay of the

calculator with abstract mathematics, i.e., calculus, encourages a step by step approach which makes the whole process very natural.

2) The problem of approximation also must be faced early on in the course. The false idea that mathematics is about the exact and the rest of science and engineering only the approximate never gets a hold.

Danger: Using the calculator in calculus may encourage dependence on the calculator.

1.2 Introduction to the Manual

The organization of the manual. The manual is organized in chapters roughly following the usual pattern adopted by calculus texts. Some of the material for a given topic will go beyond what is required at a first introduction. For instance, the material on polar curves might be visited at several points in the course: a first introduction, the area of regions bounded by single polar curve, regions whose boundary is made up of more than one polar curve, and line integrals along polar curves. We suggest that the material in the manual be taken up as required by the course text returning to the early chapters several times.

Outline of chapters

Chapter 1 Introduction to the manual and HP-28S
Introduction to the philosophy of calculator enhanced calculus and suggestions for using the manual.

Chapter 2 Graphs of curves and surfaces
Graphing polar and parametric curves, conic sections and level curves of quadric surfaces.

Chapter 3 Functions of two variables/Optimization
Representing curves and surfaces with functions, classifying critical points for functions of two variables and the method of level curves.

Chapter 4	Intersections of curves and surfaces/Integration Describing regions of the plane bounded by given curves and surfaces, single and double integrals.
Chapter 5	Vector fields and line integrals Vector fields, line integrals and Green's Theorem.

The role of the calculator. The HP-28S can be thought of as an integrated software package providing capabilities in graphics, symbolic manipulation, and numerical linear algebra, integration, statistical analysis and equation solving. The calculator is the student's handbook, scratchpad and number cruncher, i.e., the student's resource of unstructured computational power. The calculator exercises are designed to

1) Build a proficiency with the calculator which can be used in courses outside the mathematics department

2) Lead the student to explore examples beyond the usual pencil and paper textbook problems, i.e., problems which require for their solution some computational power

3) Lead to a deeper understanding by making connections with previous work, i.e., the calculator provides a level of technical skills (graphing, symbolic manipulation, numerical calculation) which in the past we have not assumed the student brought forward from previous experiences.

Background reading. The Owner's Manual is a great resource which can be used in conjunction with this manual. To start, read Chapter Two of the Owner's Manual. Each time a new capability of the calculator is applied to calculus for the first time, either read or review the appropriate section in the Owner's Manual. The Reference Manual is useful if you become confused about the use of specific commands. Wickes' book[1] provides a readable, but more advanced look at the HP-28S with numereous programs and applications.

1.3 How to Make Use of the Manual

Conventions for displaying commands and programs. Outside of programs, expressions and procedures, commands are executed immediately. On the otherhand, objects (numbers, arrays, programs and expressions) have to be put on the stack before they can be acted upon. When we write a sequence of objects and commands the assumption is that each object has been put on the stack, i.e., there is an implied ENTER following each object. For instance,

2 3 +

really stands for 2 ENTER 3 ENTER + which results in 5 on level 1 of the stack.

In expressions, some commands are represented differently from the corresponding key. For example 2 3 + inside an expression or program becomes '2 3 /' or << 2 3 />>. Hence we adopt the convention that in programs, procedures and expressions we display exactly what appears on the screen.

Programs in the manual are displayed in formatted form. Of course, when entering a program, the format (except for spaces) must be ignored. When the program is recalled it will appear formatted on the calculator screen.

---

[1]William C. Wickes, HP-28 Insights, Larken Publications, Corvallis, OR; 1988.

Outline of possible course with accompaning sections from the manual

Topic	Comments
Conic sections	The programs for graphing conic sections could be used here. I preferred using this period to introduce the calculator, postponing graphing conic sections with the calculator until quadric surfaces. Section 2.3.
Polar graphs	Discuss the differences between expressions and procedures. Introduce the conventions for indicating commands and displaying screens used in the manual. Introduce the tasks of entering, debugging and editing programs. Section 2.1.
Area of regions bounded by polar curves	Introduce IGL. The emphasis is on tracing out the region just once. I suggest using a naive approach leaving the graphing utilities until later. Section 4.2.

Parametric curves	At this point the students should be able to enter and debug the programs in the manual. I like to introduce some interesting parametric curves and graph some polar curves parametrically. Section 2.5.
Vectors and vector functions	I do not take up DOT and CROSS until later, but enterprising students tend to learn these on their own.
Tangent lines	The graphing utilities for parametric curves can be introduced here. Also machine differentiation starts to be useful in the context of tangents to polar graphs. Graph the curves parametrically. Section 2.6.
Arc length/surface area	The symbolic manipulation capabilities of the HP-28S start to payoff here. Also numerical integration frees us up from the usual restrictive textbook examples. We can actually compute the arc lengths and surface areas for figures generated with familiar functions. Section 4.2
Functions of two variables	Compute some partial derivatives.

Quadric surfaces	I use this opportunity to introduce the distinctions between explicit, implicit and parametric definitions of curves and surfaces. Introduce the utilities for graphing conic sections and use them to contruct level curve representations of the quadric surfaces. Sections 2.3, 2.4.
Limits	Explore limits along various paths and the relationship of these limits with the limit. Section 3.2.
Differentials	How nice can a function be and still not be differentiable? Section 3.3.
Tangent planes	Introduce Taylor polynomials. Section 3.4.
Classification of critical points	Use second degree Taylor polynomials to classify the critical points. Section 3.4
Lagrange multipliers	The method of level curves can be explored using the graphics capability of the HP-28S. Section 3.5.

Multiple integrals	Use the material on the intersections of curves and surfaces to set up iterated integrals. This is a good time to implement all of the graphing utilities. A lot of time can be made up here by concentrating on numerical solutions. Sections 4.1, 4.2.
Vector fields	Use gradient fields for explicitly defined quadric surfaces to introduce vector fields. Section 5.1.
Line Integrals	Again the symbolic capabilities come in very handy. Section 5.2.
Green's Theorem	The students now have the tools for attacking a host of nonstandard problems. Section 5.3.
The Divergence Theorem and Stokes' Theorem	

Suggestions for integrating the material into a standard course. The material in the manual was tested in a third semester multidimensional calculus course which meets four hours a week for fifteen weeks. Support from the Fund for the Improvement of Postsecondary Education (FIPSE) for this project is gratefully acknowledged. The students had no previous experience with the HP-28S. My goal was to devote no more than seven classes specifically to the calculator. One class was used to introduce the basic operations on the calculator; storing, recalling and editing expressions; and graphing simple functions using the default scaling. The next two classes devoted to the calculator were used for entering and using the programs for graphing polar curves and numerical integration of functions of one variable. The last four classes spent exclusively on applications of the calculator were used for level curves of quadric surfaces, one of Polya's problems, describing subsets of the plane and three space, and Green's theorem. The calculator was used at least once a week for smaller applications: an integration problem, recalling the shape of some parametric or polar curve, examples illustrating limits or differentials, line integrals, etc.. Since use of the calculator was allowed on tests and the final exam, the students were encouraged to work on their own to develop skills that could be applied to standard textbook problems.

CHAPTER 2
GRAPHS OF CURVES AND SURFACES

2.1 Curves in Polar Coordinates

In order to enter the program POLAR given below, keystroke in the program starting at << CLLCD DRAX and concluding at >> DGTIZ >> . Do <u>not</u> worry about line breaks. After the program is entered it will be formatted automatically. You must enter the spaces, they will be entered automatically for commands entered from the menus. Use the correct font, i.e., lower case and upper case. After the program has been keyed in, hit ENTER. If the program compiles it is entered on the stack. Otherwise you must find and correct the errors using the cursor control keys, INS and DEL. Assuming the program is compiled on the first stack level enter 'DRPOL' STO. The program can now be executed or recalled for editing from the USER menu.

Polar Draw

level 4	level 3	level 2	level 1	
procedure	θ_{min}	θ_{max}	$\delta\theta$	$\Rightarrow$

DRPOL:

```
<< CLLCD DRAX                 | Initialize the screen.
   4 ROLL → delta r           | Save the radius procedure
   <<                         | and increment.
     FOR theta
       theta r EVAL           | Compute r(θ).
       theta R→C              | Polar coordinates.
       P→R PIXEL              | Plot the point.
     delta STEP               | Increment θ.
   >> DGTIZ                   | Activate cursor.
>>
```

Recall the difference between an 'expression' and a 'procedure'. When the expression 'SIN(X)' is evaluated the sine function is evaluated at the current value store in the global variable 'X'. When the procedure << → T 'SIN(T)' >> is evaluated, a value is taken from the stack and stored in the local variable 'T'. Then the sine function is evaluated at the value stored in 'T'.

The program DROPL requires the function r = f(θ) to be entered as a procedure in level 4 of the stack. For instance, after storing the program in DRPOL, to draw r = sin(θ) enter the following:

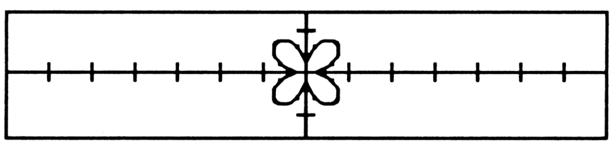
<< → T 'SIN(T)' >> -90 90 5 DRPOL

Notice the calculator must be in degree mode to produce the "right" picture with the data entered above. Since most of our work will be in the radian mode, try the following after putting the calculator in radian mode.

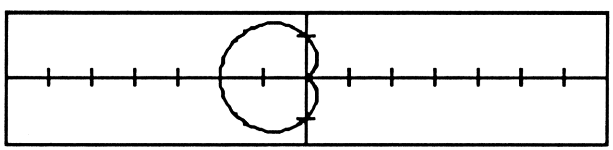
<< → T 'SIN(T)' >>
'π' CHS →NUM 2 + ENTER CHS ENTER 18 +
DRPOL

Examples. A couple of the standard figures are graphed below.

1) r = sin 2θ



2) r = 1 - cos θ



Exercises

1) Redo some of the other standard figures: $r = \sin(\theta)$, $r = \cos(\theta)$, $r = 1 - \sin(\theta)$, $r = 1 + 2\sin(\theta)$, $r = \cos(2\theta)$, and $r = \cos(3\theta)$.

2) Produce a graph containing both $r = \sin(\theta)$ and $r = \cos(\theta)$ on the same axis.

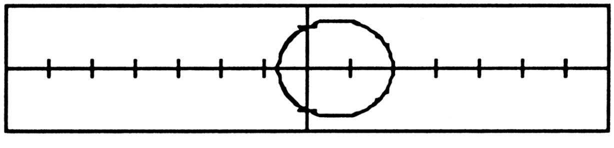
After producing a graph the cursor is active, i.e., the cursor can be moved to a new location and its position recorded by pushing INS. This useful feature can be exploited to capture new lower lefthand and upper righthand corners for a later graph.

Example. You might have produced the figure $r = \sin(\theta)$ using the default graphing parameter values in PPAR. The picuture can be improved by capturing a new lower lefthand and upper righthand corners using INS. Clear the screen with ON. The stack is then seen to contain two values, the points captured with INS. These can be entered into PPAR using PMAX and PMIN. Each of these keys, found in the PLOT menu, takes a value from the stack and stores it in the appropriate variable. Now, produce the figure $r = \sin(\theta)$ a second time. You should have a better view.

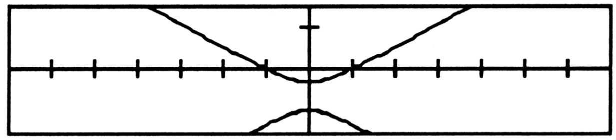
The conic sections also have polar coordinate representation. The general forms are $r = \pm ep/(1 \pm e \cos \theta)$ and $r = \pm ep/(1 \pm e \sin \theta)$. If $0 < e < 1$ the figure is an ellipse. If $e = 1$ the figure is a parabola. If $e > 1$ the figure is an hyperbola.

Examples

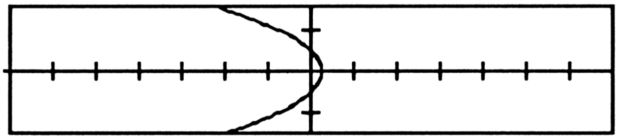
1) $r = 2/(2 - \cos \theta)$



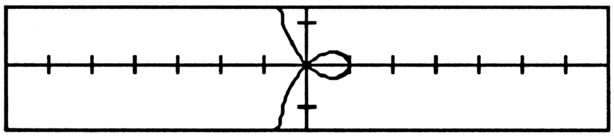
2) $r = 1/(1 - 2\sin\theta)$



3) $r = 2/(4 - 3\sin\theta)$



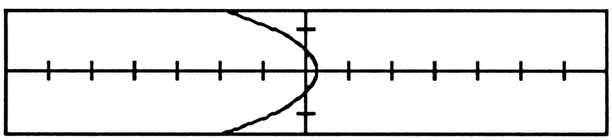
4) Other figures have a polar representation include Tschirnhausen's cubic, namely, $r = (\cos 2\theta)(\sec\theta)$.



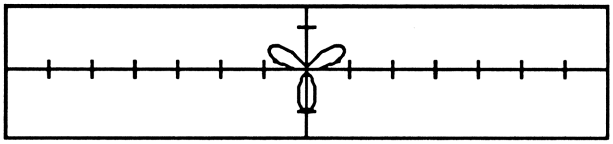
Exercises. Match the following curves given in polar coordinates with the figures given below:

1) $r = .5/(\cos\theta - 1)$ 2) $r = .25\sec(\theta/2)^2$ 3) $r = \sin 4\theta$

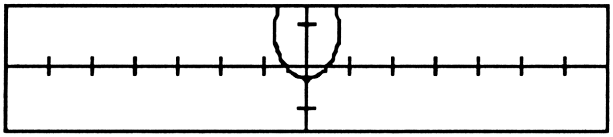
4) $r = 1 - 2\cos\theta$ 5) $r = \sin 3\theta$

A) 

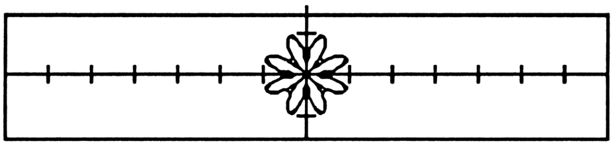
B)



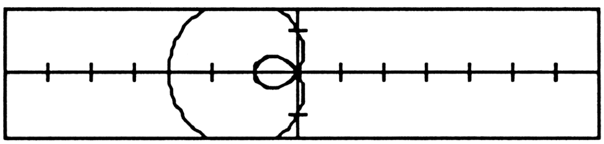
C)



D)



E)



2.2 Graphing Utilities for Polar Curves

Create a subdirectory called POLAR as follows:

'POLAR' CRDIR

QUIT:

<< HOME >> | Returns to home directory

We need a version of DRPOL which does not begin by clearing the screen. Since the new program appears in the subdirectory, we can use the same name (using lower case letters) and still avoid any confusion

level 4	level 3	level 2	level 1	
procedure	θ_{min}	θ_{max}	$\delta\theta$	⇒

Drpol:

```
<< DRAX                          | Initialize the screen.
   4 ROLL → delta r              | Save the radius procedure and
   <<                            | increment.
     FOR theta
       theta r EVAL              | Compute r(θ).
       theta R→C                 | Polar coordinates.
       P→R PIXEL                 | Plot the point.
     delta STEP                  | Increment θ.
   >>
>>
```

Drpol in this subdirectory (as did DRPOL in the home directory) requires the plotting parameters θ_{max}, θ_{min} and $d\theta$. Since the utilities are designed to produce and manipulate several polar graphs on the same axis, we will use PPAR to store the plotting parameters. There is no automatic feature for generating PPAR. Hence you must initialize PPAR, for instance, {(-6.8, -1.4) (6.8, 1.5) T 1 (0, 0)} PPAR STO.

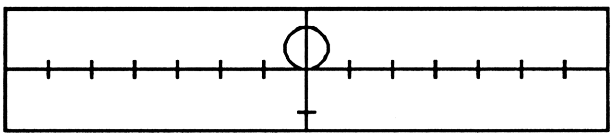
level 1

procedure ⇒

NEWGRAPHS:

```
<< CLLCD DUP
     1 →LIST 'PROCS' STO          | Creates a list containing the
                                  | procedure and names it PROCS
     PPAR 1 GET C→R DROP          | Gets the plotting parameters
     PPAR 2 GET C→R DROP          | required for DROPL from PPAR
     DUP2 SWAP - 136 /
     Drpol LCD→ 'SCR' STO         | Draws to screen, saves the
                                  | screen
     DGTIZ                        | and leaves the cursor active
>>
```

Example. << → T 'SIN(T)' >> NEWGRAPHS results in a graph something like the following which is saved in SCR.



GETG:

```
<< SCR →LCD DGTIZ >>
```

Example. GETG restores to the screen the graph stored in SCR.

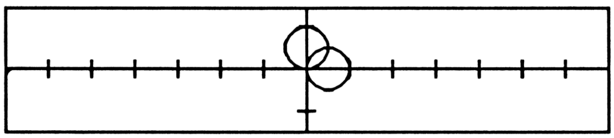
level 1

procedure ⇒

OVERDRAW:

```
<<  DUP 1 →LIST PROCS +          | Adds new procedure to
    'PROCS' STO                  | list PROCS
     PPAR 1 GET C→R DROP         | Gets the plotting PPAR
     PPAR 2 GET C→R DROP         | parameters required
     DUP2 SWAP - 136 /           | for Drpol from PPAR
     SCR →LCD                    | Reproduces old screen
     Drpol LCD→                  | and adds new graph.
     'SCR' STO DGTIZ             | Stores the result
    >>
```

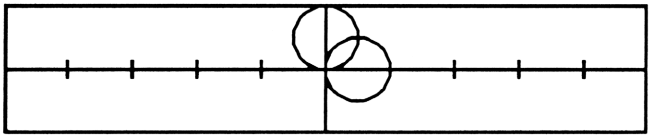
Example. << → T 'COS(T)'>> OVERDRAW produces the following which is a little small.



REDRAW:

```
<< CLLCD
   PPAR 1 GET C→R DROP           | Gets the plotting PPAR
   PPAR 2 GET C→R DROP           | parameters required
   DUP2 SWAP - 136 /             | for Drpol stored in PPAR
   PROCS SIZE 1
   SWAP FOR I 3 DUPN             | Reproduces plotting
                                 | parameters
   PROCS I GET 4 ROLLD           | Gets procedure stored
                                 | in PROCS
   Drpol NEXT 3 DROPN            | Redraws the graphs
                                 | Saves the screen
   LCD→ 'SCR' STO DGTIZ          | and leaves the cursor
                                 | active
>>
```

Example. The figures above can be regraphed with 2 3 + DUP *H *W REDRAW



This method of choosing the graphing window can be simplified with the following commands.

	level 2	level 1	
	coordinate pair	coordinate pair	⇒

BOX:

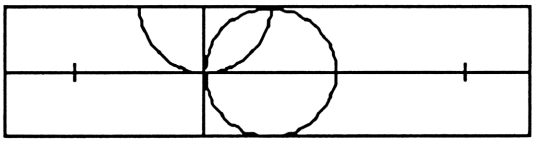
```
<< PMIN PMAX REDRAW
>>
```

level 3	level 2	level 1	
coordinate pair	real	real	⇒

ZOOM:

```
<< *W *H CENTR REDRAW
>>
```

Example. To return to the original parameters enter 'PPAR PURGE REDRAW Then to enlarge the region bounded by both curves try capturing the center of the region with INS and then enter .67 .67 ZOOM You should get something like the following:



2.3 Graphs of Conic Sections

We can create a facility for producing graphs of conics on the HP-28S by entering the following programs.

Test If 2nd Degree Polynomial Defines a Function

level 1		level 1
expression	⇒	flag

FNC?:

```
<< → rel                          | Input expression.
   << 'Y' PURGE rel
      'Y' ∂ 'Y' ∂ 0 ==            | Test if function.
                                  | Return flag.
   >>
>>
```

Solve Expression for Y and Save in EQ

level 1	
expression	⇒

LDR:

```
<< 'X' PURGE → rel                | Input expression.
   << rel FNC?                    | Test if function.
      << rel 'Y' 1 TAYLR 'Y' ISOL
      >>                          | Solve for Y.
      << rel 'Y' QUAD
      >> IFTE STEQ                | Store in EQ.
   'X' INDEP                      | Make X the plotting
                                  | variable
   >>
>>
```

Fix Up DRAW

Apparently DRAW is not well behaved when used in a program, i.e., after an error something unintended might be left on the stack. We can fix this unfortunate flaw by using the following program[1] in place of DRAW in a program.

DFix:

```
<<
   [ 1 ]                          | Marks the end of the
                                  | existing stack
      IFERR  DRAW
                                  | While executing DRAW,
                                  | look for an error
      THEN  DROP  ERRM 1
                                  | If there is drop [ 1 ]
                                  | and display the
   DISP                           | message
      ELSE                        | If there is no message
   [ 1 ]                          | return the stack to its
        WHILE SAME NOT            | original state
        REPEAT
   [ 1 ]
        END                       | End WHILE
      END                         | End IFERR
>>
```

[1]John and Annie Selden, Graphing with the HP-28S, preprint.

Draw Relation

DRAR:

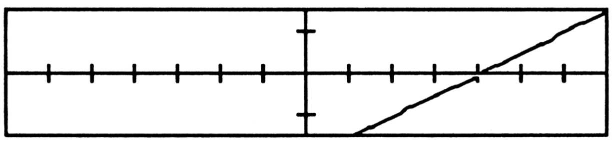
```
<< -1 1 2 → stp                          | Initialize loop parameters.
   << CLLCD
      FOR x x 's1' STO
        DFix stp                         | Draw branch of curve.
      STEP
   >> DGTIZ                              | Activate cursor.
>>
```

Example. Sketch $4y^2 - 8x - x^2 + 32y + 49 = 0$.

Enter

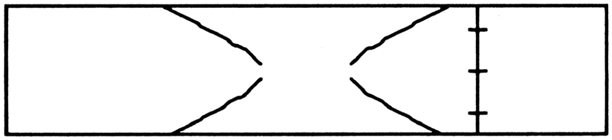
'PPAR' PURGE

'4*y^2 - 8*x - x^2 + 32*y + 49' LDR DRAR



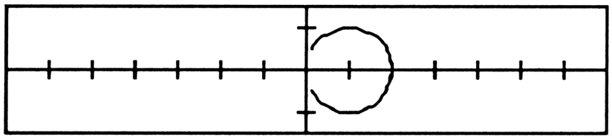
The resulting sketch isn't very satisfying. How can we improve it? Completing squares our equation becomes $(x + 4)^2 - 4(y + 4)^2 = 1$, i.e., the figure must be a hyperbola with center (-4, -4). Set the center in PPAR as follows:

4 CHS DUP R→C CENTER DRAR

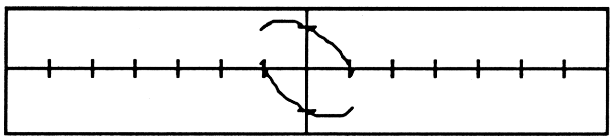


Examples. A couple of the standard figures are graphed below.

1) $x^2 = y^2 - 2x = 0$



2) $x^2 + xy + y^2 - 1 = 0$



Exercises

1) Redo some of the other standard figures: $x + y + 1 = 0$, $y^2 + x - 1 = 0$, $x^2 + y^2 - 1 = 0$ and $2x^2 - y^2 - 2 = 0$.

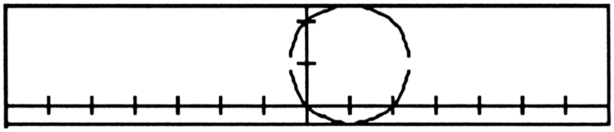
2) Produce a graph containing both $x + y - 1 = 0$ and $y = 2 - x^2$ on the same axis.

Example. You might have produced the figure $x^2 + y^2 - 1 = 0$ using the default graphing parameter values in PPAR. The picuture can be improved by capturing a new lower lefthand and upper righthand corners using INS. Clear the screen with ON. The stack is then seen to contain two values, the points captured with INS. These can be entered into PPAR using PMAX and PMIN. Each of these keys, found in the PLOT menu, takes a value from the stack and stores it in the appropriate variable. Now, produce the figure $x^2 + y^2 - 1 = 0$ a second time. You should have a better view.

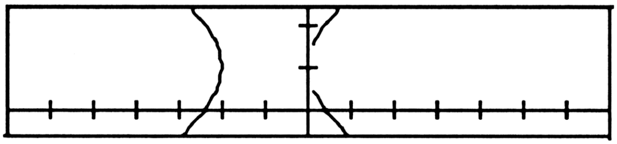
All second degree polynomials $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ (allowing for degenerate figures) define conic sections.

Examples

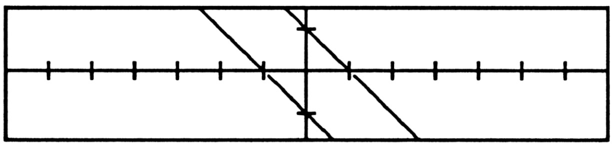
1) $x^2 + y^2 -2x -2y = 0$



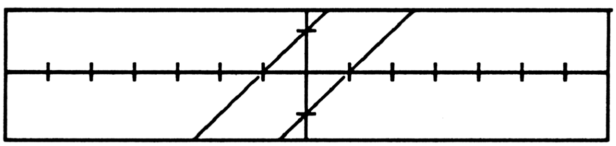
2) $x^2 - y^2 + 2x + 2y - 1 = 0$



3) $x^2 + 2xy + y^2 - 1 = 0$



4) $x^2 - 2xy + y^2 - 1 = 0$

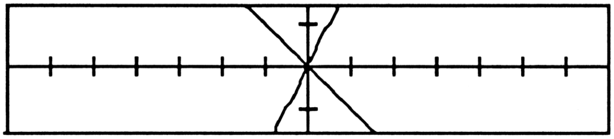


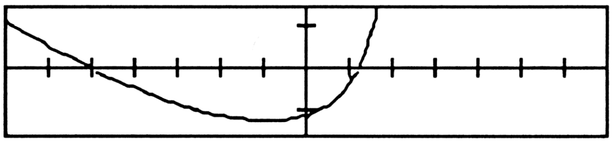
Exercises. Match the following curves with the figures given below:

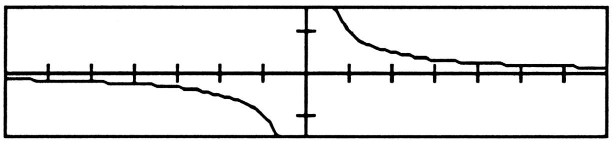
1) $2x^2 + xy - y^2 = 0$ 2) $y^2 - x = 0$ 3) $xy = 1$

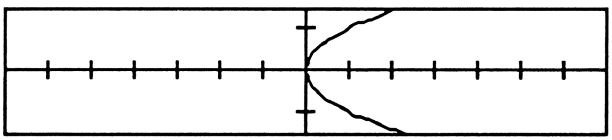
4) $16x^2 + 24xy +60x - 80y -100 = 0$

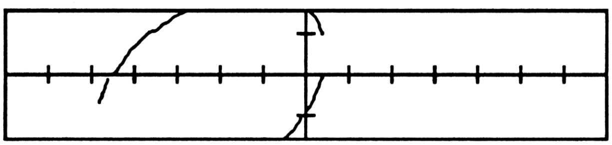
5) $29x^2 - 24xy + 36y^2 + 118x -24y - 55 = 0$

A)


B)


C)


D)


E)


2.4 Graphing Utilities for Conics

Create a subdirectory called Conics as follows:

```
'CONICS'   CRDIR
```

level 1	
expression	⇒

QUIT:

```
    << HOME >>                         | Returns to home directory
```

We need a version of DRAR which does not begin by clearing the screen. Since the new program appears in the subdirectory, we can use the same name (using lower case letters) and still avoid any confusion.

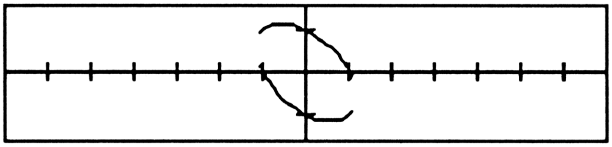
Drar:

```
<< -1 1 2 → stp                        | Initialize loop
                                       | parameters.
    <<  CLLCD
        FOR x x 's1' STO
            DFix stp                   | Draw branch of curve.
        STEP
    >>
>>
```

NEWGRAPHS:

```
<< CLLCD  LDR  RCEQ
    1 →LIST                            | Creates a list containing the
    'EQS'  STO                         | expression and names it EQS
    Drar  LCD→  'SCR'  STO             | Draws to screen, saves the screen
    DGTIZ                              | and leaves the cursor active
>>
```

Example. 'X^2 + X*Y + Y^2' NEWGRAPHS results in a graph something like the following which is saved in SCR.



GETG:

<< SCR →LCD DGTIZ >>

Note. GETG restores to the screen the graph stored in SCR.

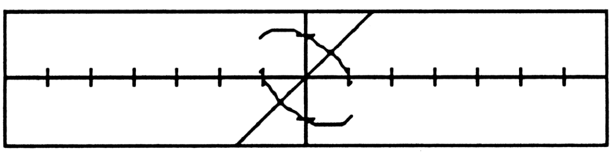
level 1

expression ⇒

OVERDRAW:

```
<< CLLCD LDR RCEQ                  | Adds new expression to list EQS
   1 →LIST EQS +
   'EQS' STO
   SCR →LCD Drar                   | Reproduces old screen and adds
   LCD→ 'SCR' STO DGTIZ            | new graph. Stores the result
>>
```

Example. 'Y - X' OVERDRAW produces the following divides the region bounded by the previous conic.

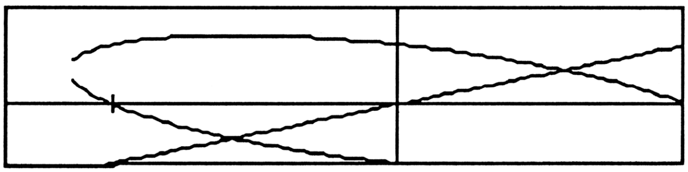


REDRAW:

```
<<  CLLCD  EQS  SIZE  1
    SWAP  FOR  I  EQS  I           | Redraws the graphs of
                                   | expressions
    GET STEQ  Drar  NEXT           | stored in EQS.  Saves the screen
    LCD→  'SCR'  STO  DGTIZ        | and leaves the cursor active
>>
```

Example. The upper region bounded by the curves can be regraphed with

(-1.4, -1) PMIN (1, 1.4) PMAX REDRAW



This method of choosing the graphing window can be simplified with the following commands.

	level 2	level 1	
	coordinate pair	coordinate pair	⇒

BOX:

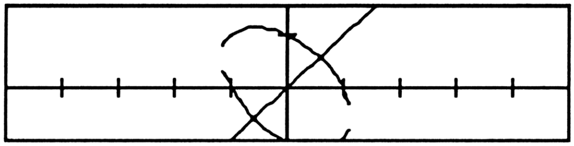
```
<<  PMIN  PMAX  REDRAW
>>
```

	level 3	level 2	level 1	
	real	real	oordinate pair	⇒

ZOOM:

```
<<  *W  *H  CENTR  REDRAW
>>
```

Example. To return to the original parameters enter 'PPAR PURGE REDRAW Then to enlarge the upper bounded region try capturing the center of the region with INS and then enter .75 .75 ZOOM You should get something like the following:



2.5 Parametric Curves

Parametric Draw

level 5	level 4	level 3	level 2	level 1	
procedure	procedure	T_{min}	T_{max}	δT	⇒

DRPAR:

```
<< CLLCD DRAX                          | Initialize the screen.
   5 ROLL  5 ROLL → delta x  y         | Save the x  and y
                                       | procedures and increment.
   <<
     FOR  theta
       theta  x  EVAL
       theta  y  EVAL                  | Compute x(t) and y(t).
       R→C  PIXEL                      | Plot the point.
       delta  STEP                     | Increment t.
>> DGTIZ                               | Activate cursor.
>>
```

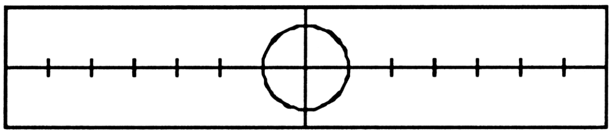
The program DRPAR requires the functions x = x(t) and y = y(t) to be entered as a procedures in levels 5 and 4 of the stack. For instance, after storing the program in DRPAR, to draw r = (cos(t), sin(t)) enter the following:

<< → T 'COS(T)' >> << → T 'SIN(T)' >> 0 'π' 2 * →NUM

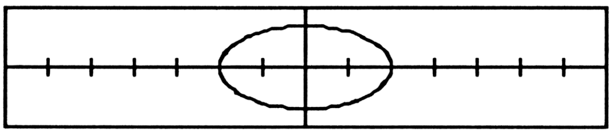
DUP 120 + DRPAR

Examples. A couple of the standard figures are graphed below.

1) r = (cos t, sin t)



2) r = (2 cos t, sin t)



Exercises

1) Redo some of the other standard figures: r = (t + 1, t - 1), r = (1 - t, t^2), r = (t^2, t^3) and r = (t + 1/t, t - 1/t)

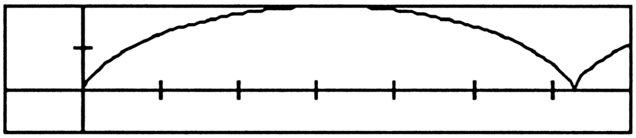
2) Produce a graph containing both r = (t, t^2) and r = (t^2, t) on the same axis.

Example. You might have produced the figure r = (cos t, sin t) using the default graphing parameter values in PPAR. The picuture can be improved by capturing a new lower lefthand and upper righthand corners using INS. Clear the screen with ON. The stack is then seen to contain two values, the points captured with INS. These can be entered into PPAR using PMAX and PMIN. Each of these keys, found in the PLOT menu, takes a value from the stack and stores it in the appropriate variable. Now, produce the figure r = (cos t, sin t) a second time. You should have a better view.

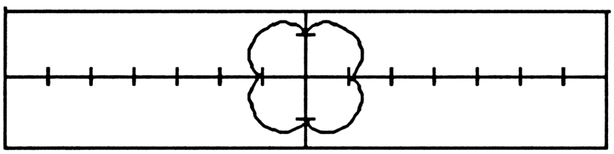
Several types of "circular" motion have nice parametric representations.

Examples

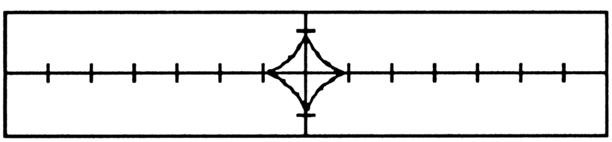
1) The cycloid is generated by tracing a fixed point on the circumference of a circle rolling along the x-axis, $r = (R(t - \sin t), R(1 - \cos t))$. Try $R = 1$.



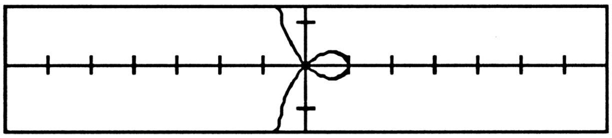
2) The epicycloid is generated by tracing a fixed point on the circumference of a circle rolling around outside the circumference of another circle, $x = (R_1 + R_2)\cos t - R_2\cos((R_1 + R_2)t/R_2)$, $y = (R_1 + R_2)\sin t - R_2\sin((R_1 + R_2)t/R_2)$. Try $R_1 = 1$ and $R_2 = 1/4$.



3) The hypocycloid is generated by tracing a fixed point on the circumference of a circle rolling around inside the circumference of another circle, $x = (R_1 - R_2)\cos t + R_2\cos((R_1 - R_2)t/R_2)$, $y = (R_1 - R_2)\sin t - R_2\sin((R_1 - R_2)t/R_2)$. Again, try $R_1 = 1$ and $R_2 = 1/4$.



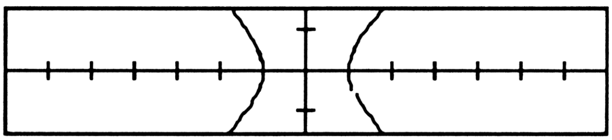
4) Other figures have a parametric representation include Tschirnhausen's cubic, namely, r = $(t^2 - 3, t^3/3 - t)$.

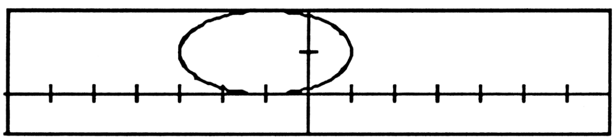


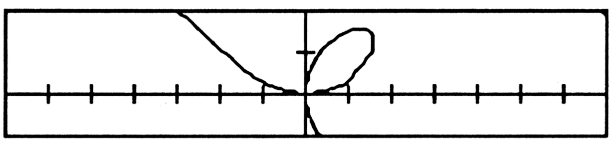
Exercises. Match the following curves given parametrically with the figures given below:

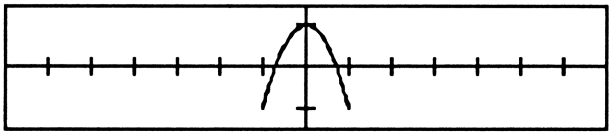
1) r = (sin t, cos 2t) 2) r = (sec t, tan t) 3) r = (2sin t -1, 1 + cos t)

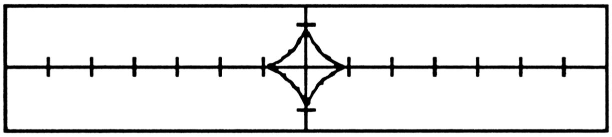
4) r = $(3t/(1 + t^3), 3t^2/(1 + t^3))$ 5) r = $((\cos t)^3, (\sin t)^3)$

A) 

B) 

C) 

D) 

E) 

2.6 Graphing Utilities for Parametric Curves

Create a subdirectory called PARAM as follows:

'PARAM' CRDIR

level 1

expression ⇒

QUIT:

<< HOME >> | Returns to home directory

We need a version of DRPAR which does not begin by clearing the screen. Since the new program appears in the subdirectory, we can use the same name (using lower case letters) and still avoid any confusion.

level 5	level 4	level 3	level 2	level 1	
procedure	procedure	T_{min}	T_{max}	δT	⇒

Drpar:

```
<< DRAX                                  | Initialize the screen.
   5 ROLL  5 ROLL → delta x  y           | Save the x and y
                                         | procedures and increment.
   <<
     FOR  theta
       theta  x  EVAL
       theta  y  EVAL                    | Compute x(t) and y(t).
     R→C  PIXEL                          | Plot the point.
     delta  STEP                         | Increment t.
   >>
>>
```

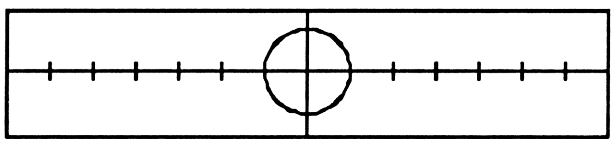
Drpar in this subdirectory (as did DRPAR in the home directory) requires the plotting parameters T_{max}, T_{min} and δT. Since the utilities are designed to produce and manipulate several parametric curves on the same axis, we will use PPAR to store the plotting parameters. There is no automatic feature for generating PPAR. Hence you must initialize PPAR, for instance, {(-6.8, -1.4) (6.8, 1.5) T 1 (0, 0)} 'PPAR' STO.

level 1	level 2	
procedure	procedure	⇒

NEWGRAPHS:

```
<<  CLLCD DUP2
    2 →LIST 'PROCS' STO       | Creates a list containing the
                              | procedure and names it PROCS
   PPAR 1 GET C→R DROP        | Gets the plotting parameters
   PPAR 2 GET C→R DROP        | required for Drpar from PPAR
   DUP2 SWAP - 136 /
   Drpol LCD→ 'SCR' STO       | Draws to screen, saves the screen
   DGTIZ                      | and leaves the cursor active
>>
```

Example. << → T 'SIN(T)' >> << → T 'SIN(T)' >> NEWGRAPHS results in a graph something like the following which is saved in SCR.



GETG:

```
<< SCR →LCD DGTIZ >>
```

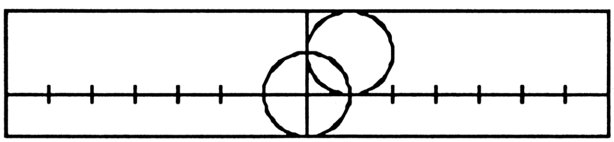
Example. GETG restores to the screen the graph stored in SCR.

level 1	
procedure	⇒

OVERDRAW:

```
<< DUP2 2 →LIST PROCS +        | Adds new procedure to
   PROCS' STO                  | list PROCS
   PPAR 1 GET C→R DROP         | Gets the plotting PPAR
   2 GET C→R DROP              | parameters required
   DUP2 SWAP - 136 /           | for Drpar from PPAR
   SCR →LCD                    | Reproduces old screen
   Drpar LCD→                  | and adds new graph.
   'SCR' STO DGTIZ             | Stores the result
>>
```

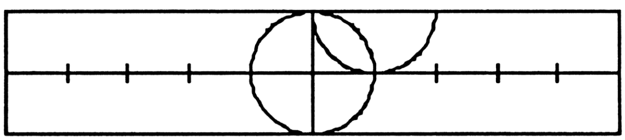
Example. << → T '1 + COS(T)' >> << → T '1 + SIN(T)' >> OVERDRAW produces the following which is a little small.



REDRAW:

```
<< CLLCD
   PPAR  1 GET C→R DROP             | Gets the plotting PPAR
   PPAR  2 GET C→R DROP             | parameters required
   DUP2 SWAP - 136                  | for Drpar stored in
                                    | PPAR
   PROCS SIZE 2 + 1
   SWAP FOR I 3 DUPN                | Reproduces plotting
                                    | parameters
   PROCS I 2 × 1 – GET 4 ROLLD      | Gets procedure
                                    | stored in PROCS
   Drpar NEXT 3 DROPN               | Redraws the graphs
                                    | Saves the screen
   LCD→ 'SCR' STO DGTIZ             | and leaves the cursor
                                    | active
>>
```

Example. The figures above can be regraphed with 2 3 + DUP *H *W REDRAW



This method of choosing the graphing window can be simplified with the following commands.

	level 2	level 1	
	coordinate pair	oordinate pair	⇒

BOX:

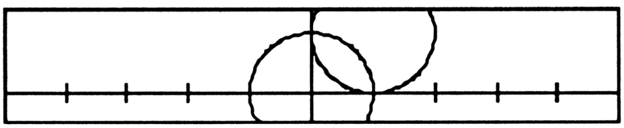
```
<< PMIN PMAX REDRAW
>>
```

	level 3	level 2	level 1	
	real	real	coordinate pair	⇒

ZOOM:

```
<< *W *H CENTR REDRAW
>>
```

Example. To return to the original parameters enter 'PPAR PURGE REDRAW Then to enlarge the region bounded by both curves try capturing the center of the region with INS and then enter .67 .67 ZOOM You should get something like the following:



CHAPTER 3
FUNCTIONS OF TWO VARIABLES/OPTIMIZATION

3.1 Curves, Surfaces and Functions

In one-dimensional calculus we applied our methods to problems associated with functions: max/min problems for functions, average value of a function, etc.. We usually did not distinguish a function and its graph nor make a big deal out of the application of calculus to implicitly defined curves, for instance the conic sections. That is, implicit differentiation was introduced as a natural extension of differentiation of functions. A consequence of this blurring of distinctions is a heightened emphasis on functions as opposed to geometric curves.

In multidimensional calculus the more fundamental objects of our study are the geometric curves and surfaces. Many of the concepts to be introduced are inherently geometric and functions enter as an aid for the study of these concepts. Furthermore, we can usually apply functions to the concepts in more than one way, though some particular way will likely be most useful.

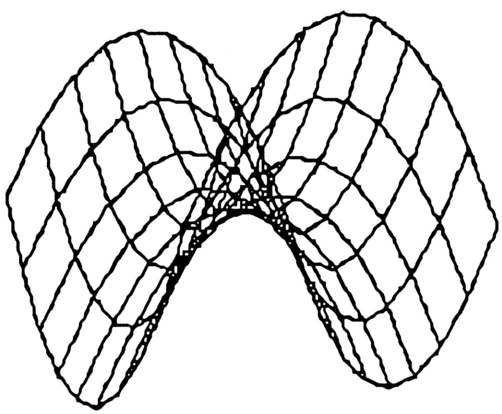
Thus the task now is to shift our attention from functions to curves and surfaces, solving some of the same problems introduced in one-dimensional calculus. However, we introduce others which have not been studied before ____ functions of several variables have more than one kind of derivative, for instance, directional derivatives; line and surface integrals are not just multidimensional versions of the Riemann integral. Obviously, in thinking about curves and surfaces we are not going to abandon what we know of differentiating and integrating functions. Therefore, how can we use functions to describe curves and surfaces in higher dimensional spaces? How is calculus applied to those functions to study the curves and surfaces they describe?

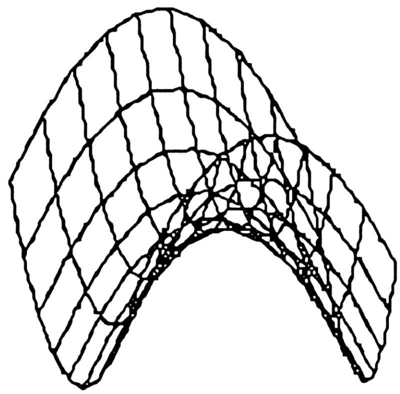
Basic definitions with examples

A function is best understood as a collection of ordered pairs no two of which have the same first term. In the one-dimensional calculus, functions are ordered pairs of real numbers which can be displayed graphically as subsets of the Cartesian plane. This graphical representation of the function is called the graph of the function. For real valued functions of more than one variable, the first members of the ordered pairs (elements of the domain of the function) will themselves be n-tuples. In the case of real valued functions of two variables the first members will be 2-tuples or ordered pairs. The graphical representation of these functions will be in 3-space. A surface S is said to be given explicitly by a function f, usually from R^n to R, provided S is the graph of f.

a) From the one-dimensional calculus consider the function $f(x) = x^2 + x + 1$. The curve $C = \{(x, y) \mid y = x^2 + x +1\}$, the graph of f, is said to be given explicitly by f.

b) The surface $S = \{(x, y, z) \mid z = x^2 - y^2\}$ is the graph of $z = x^2 - y^2$. Two views of S are shown below.





Exercises. If you have the use of a 3-D grapher on a personal computer try the following exercises.

1) Reproduce the pictures given above. Try changing the rotation and aiming point.

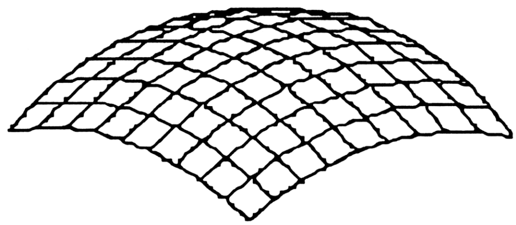
2) Produce the surface z = sin(x + y) using the viewing cylinder [0, 10] by [0, 10].

3) Try some of the other explicit quadric surfaces from your calculus text.

Among the conic sections the ellipses and hyperbolas are usually defined implicitly by equations. For example, the unit circle is given by the equation $x^2 + y^2 = 1$. Note that the graph of the circle cannot be the graph of a function, there are lots of verticle lines which intersect a circle more than one time. A surface S is said to be given implicitly by a function f, usually from R^n into R, provided S is a level set of f, i. e., there is a number k such that $S = \{ x \mid f(x) = k\}$. Note that S is a subset of the domain of f.

a) The unit circle C given by $x^2 + y^2 = 1$ is a level set of the function $f(x, y) = x^2 + y^2$, i.e., $C = \{(x, y) \mid f(x, y) = x^2 + y^2 = 1\}$.

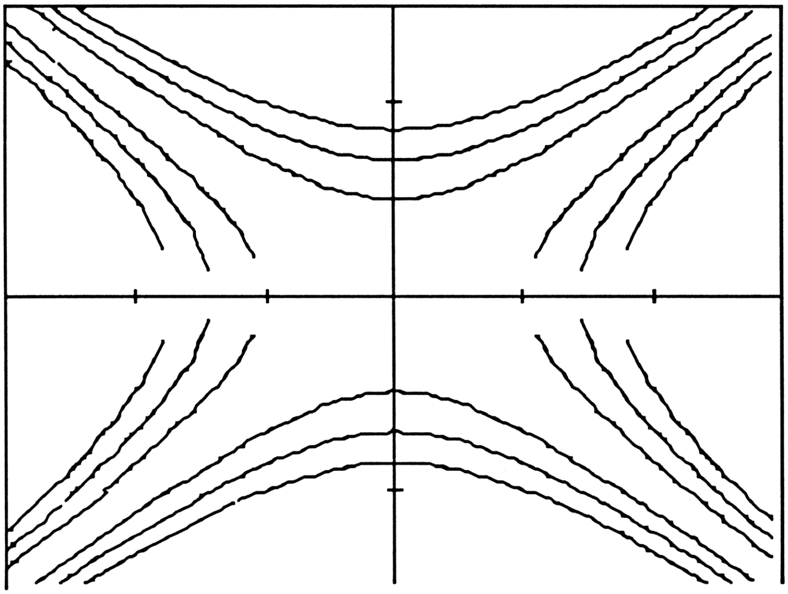
b) A portion of the implicitly defined surface $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is given below.



This picture was produced with a 3-D grapher by solving $x^2 + y^2 + z^2 = 1$ explicitly for z. Of course, each point (x, y) produces two values of z. We only used the nonnegative values of z. Thus we can use software for producing explicit surfaces to represent portions of implicitly defined surfaces.

Exercises. Using 3-D grapher on a personal computer, try to produce a portion of some of the implicitly defined quadric surfaces in your calculus text.

Implicitly defined curves can also be used to study explicitly defined surfaces. Given an explicit surface $z = f(x, y)$ we produce the implicitly defined level curves $k = f(x, y)$. The level curves of a surface, in this case, are a two dimensional representation of a three dimensional surface. Consider the level curve representation of the surface $z = x^2 - y^2$ given below.



Exercises. Using the conic graphing utilities reproduce the level curves for $z = x^2 - y^2$ shown above.

A curve C (sometimes a surface) is said to be given <u>parametrically</u> by a function **f**, usually from some subset U of R to R^n, provided C is the image of U under **f**, i.e., C = **f**(U). Note that C is a subset of the range of **f**.

a) The straight line through $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ is given

parametrically by the set of functions $\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \end{cases}$, a real number

or, more economically, in vector form by $\mathbf{x} = (x_0, y_0, z_0) + t(x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

b) The plane consisting of all linear combinations of two linearly independent vectors **u** and **v** is given parametrically by f(a, b) = a**u** + b**v**, (a, b) in $R \times R$.

Some standard representations

a) Lines (explicit: $y = mx + b$; implicit: $Ax + By + C = 0$; parametric: $\mathbf{r}(t) = \mathbf{u} + t\mathbf{v}$, t a real number)

b) Conic sections

Implicit:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Parametric:

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}, 0 \le \theta \le 2\pi \qquad \begin{cases} x = a\cosh t = \frac{a}{2}(e^t + e^{-t}) \\ y = b\sinh t = \frac{b}{2}(e^t - e^{-t}) \end{cases}, -\infty < t < \infty$$

Shifting the representation

a) A curve C given explicitly by $y = f(x)$, x in D, can be described implicitly by $C = \{ (x, y) \mid y - f(x) = 0\}$ or parametrically by

$$\begin{cases} x = x \\ y = f(x), \ x \text{ in } D \end{cases}$$

b) A curve C given implicitly by $f(x, y) = 0$ might not have have an explicit representation, for instance the circle $x^2 + y^2 = 1$. Even so, we can frequently give a "branch" of the curve an explicit representation, as the upper semicircle $y = \sqrt{1 - x^2}$, $-1 \le x \le 1$.

c) Sometimes a curve C originally given parametrically can also be represented implicitly by "eliminating the parameter". See the conic sections above.

Exercises.

1) Find both parametric and explicit representations of the line segment between P = (1, 1) and Q = (2, -3).

2) Find both implicit and parametric representations of the ellipse with vertices (-1, 0), (1, 0), (0, -2) and (0, 2).

3) Find both explicit and implicit representations of the plane which contains (1, 0, 0), (0, 1, 0) and (0, 0, 1).

3.2 Limits

If $\lim_{(x, y) \to (a, b)} f(x, y) = L$ then for any continuous function $y = g(x)$ such that $g(a) = b$ we have $\lim_{x \to a} f(x, g(x)) = L$. This fact is useful for showing that alimit does <u>not</u> exist. If you suspect that a limit doesn't exist then search for <u>two</u> continuous functions $y = g_1(x)$ and $y = g_2(x)$ with the property that $g_1(a) = g_2(a) = b$ and $\lim_{x \to a} f(x, g_1(x) \neq \lim_{x \to a} f(x, g_2(x))$.

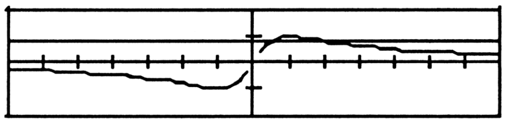
Example 1. Returning to that familiar example

$$f(x, y) = \begin{cases} 0 & x^2 + y^2 \neq 0 \\ \dfrac{2xy}{x^2 + y^2} & x^2 + y^2 \neq 0 \end{cases}$$

we know that the partial derivatives exist at (0, 0). In fact $f_x(0, 0) = f_y(0, 0) = 0$. However, z = f(x, y) does <u>not</u> have a limit at (0, 0). You can demonstrate this as follows:

```
<< → Y '2 * X * Y / (X ^ 2 + Y ^ 2)>>
    'Z' STO
    '2 * X' Z STEQ DRAW DEL ON
    'X ^ 2' Z STEQ DRAW DEL ON
    →LCD
```

The resulting screen should look something like this.



For $g_1(x) = 2x$ and $g_2(x) = x^2$ we have the limit of $f(x, g_1(x))$ at x = 0 is 4/5 while the limit of $f(x, g_2(x))$ at x = 0 is 0. Thus f(x, y) does not have a limit at (0, 0).

Example 2. The following example shows that a function f(x, y) might fail to have a limit at (a, b) while for every straight line y = g(x) through (a, b), the limit of f(x, g(x)) exists and , furthermore, the limits are the same no matter which straight line you choose.

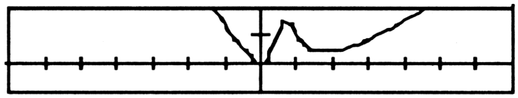
Consider $f(x, y) = \dfrac{(y^2 - x)^3 + x^2 y}{(y^2 - x^2)^2 + |y|^5}$. We proceed as follows:

```
<< → Y '((Y ^ 2 - X) ^ 3 + X ^ 2 * Y) / ((Y ^ 2 - X) ^ 2 +  ABS(Y) ^ 5)'
>>
    'Z'  STO
```

You should get something like this.

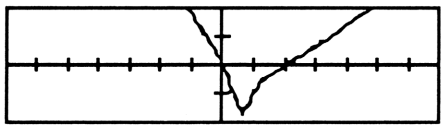
For

'X' Z STEQ DRAW

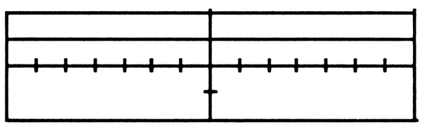


For

'- X' Z STEQ DRAW



Notice we are not saying that the graphs of f(x, g(x)) for various straight lines y = mx coincide but rather that they have the same limit at x = 0. On the other hand, for $g(x) = |x|^{1/2}$ the sketch of f(x, g(x)) looks something like the following:



i.e., f(x, g(x)) =1 and so has limit 1 not 0. This shows that f(x, y) does not have a limit at (0, 0).

Exercises. Show that the following do not have limits at (0, 0):

1) $\dfrac{x^2+y^2}{x^2+y^4}$ 2) $\dfrac{x^4-y^2}{x^2+y^2}$ 3) $\dfrac{xy}{y^2-x+x^2}$

4) $\dfrac{y}{x}$ 5) $\sqrt{y-x}$ 6) $\dfrac{\sqrt{xy}}{\sqrt{x^2+y^2}}$

3.3 Differentials

We say the surface given explicitly by $z = f(x, y)$ has an approximating plane at $(x_0, y_0, f(x_0, y_0))$ provided the partial derivatives f_x and f_y exist at (x_0, y_0) and

$$\lim_{(x, y) \to (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

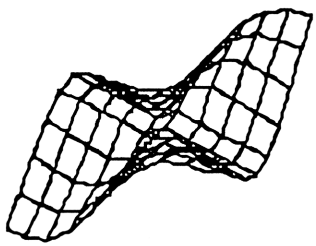
If the surface $z = f(x, y)$ has an approximating plane at $(x_0, y_0, f(x_0, y_0))$ then we say the function $z = f(x, y)$ is differentiable at (x_0, y_0). In this case, the function clearly must be continuous at (x_0, y_0).

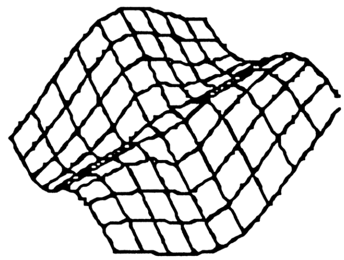
Furthermore, if $z = f(x, y)$ is differentiable at (x_0, y_0) and $y = g(x)$, a function differentiable at x_0 with $g(x_0) = y_0$, then $z = f(x, g(x))$ is differentiable at $x = x_0$ and $z'(x_0) = f_x(x_0, y_0) + f_y(x_0, y_0)g'(x_0)$. How "nice" can a function be and still not be differentiable, i.e., not have an approximating plane?

Consider the surface given by $z = f(x, y)$ where

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & x^2 + y^2 \neq 0 \\ 0 & x^2 + y^2 = 0 \end{cases}$$

Two views of the surface are given below.





The point (0, 0, 0) is the center of each picure. The function certainly appears to be continuous at (0, 0). We can see that this is the case by noting that

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \le |y|$$

for all (x, y) π (0, 0).

If we section the surface with a plane $y = mx$ for some constant m the resulting curve $z = f(x, mx) = (m/(1 + m^2))x$ is a straight line, i.e., each section is a differentiable curve with slope $m/(1 + m^2)$ at (0, 0). This would seem to say that $z = f(x, y)$ is well behaved at (0, 0). However, recall that if $z = f(x, y)$ is differentiable at (0, 0) and $z = f(x, mx)$ then $z'(0) = f_x(0, 0) + mf_y(0, 0) = 0$. This means $z = f(x, y)$ is not differentiable at (0, 0). What went wrong?

To see what the problem is look at the quotient $Q(x, y) =$

$$\frac{f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} = \frac{x^2 y}{(x^2 + y^2)^{3/2}}$$

For z = f(x, y) to be differentiable at (0, 0), the quotient Q(x, y) must have limit 0 at (0, 0). In fact, the quotient Q(x, y) has no limit at all which can be verified by letting, in turn, y = x and y = 2x. The limit as $x \to 0$ of the quotient along y = x is $2^{-3/2}$. The limit along y = 2x is $2 \cdot 5^{-3/2}$. Since these two limits are not the same the function Q(x, y) does not have limit 0 at (0, 0). Thus z = f(x, y) is not differentiable at (0, 0).

The HP-28S can be used in the above analysis as follows:

```
'X ^ 2 * Y / ( X ^ 2 + Y ^ 2 )'  'Z'  STO
```

You compute z_x by

```
Z  'X'  PURGE  'X'  d/dx
```

and z_y by

```
Z  'Y'  PURGE  'Y'  d/dx
```

To compute z = f(x, 2x) try

```
'M * X'  'Y'  STO
2  'M'  STO  Z
EVAL  EVAL
```

At this point, you should have something like 'X^2*(2*X)/(X^2+(2*X)^2)' on the first level of the stack. Using COLCT and EXPAN from the ALGEBRA menu, you can transform the expression to '.4*X'. The problem is that common factor from the denominator and numerator have to be cancelled. Now, for (d/dx)f(x, 2x)

```
X'  PURGE  'X'  d/dx
```

returns .4 , i.e., for $z = f(x, 2x)$ we get $z' = .4$. Of interest here is the symbolic manipulation capability of the HP-28S. We stored $f(x, y)$ in Z, mx in Y, 1 in M and then computed $f(x, 2y)$ by executing Z followed by two EVALS.

Exercises. Compute dz/dx given:

1) $z = u/v$; $u = x$, $v = 1 + x^2$

2) $z = \arctan(u/v)$; $u = xe^x$, $v = 1 + x^2$

3) $z = 1/(u^2 + v^2)$; $u = \sin x$, $v = \ln x$

4) Repeat the above analysis for $f(x, y) = \dfrac{x^2 \sin y}{x^2 + y^2}$

i.,e., show that $z = f(x, y)$ is continuous at (0, 0), $z = f(x, mx)$ is differentiable at $x = 0$ for all m, and yet $z = f(x, y)$ is not differentiable at (0, 0). It's hard to use the calculator for the first of these tasks, but for the second task the symbolic manipulation capability of the HP-28S can be put to work.

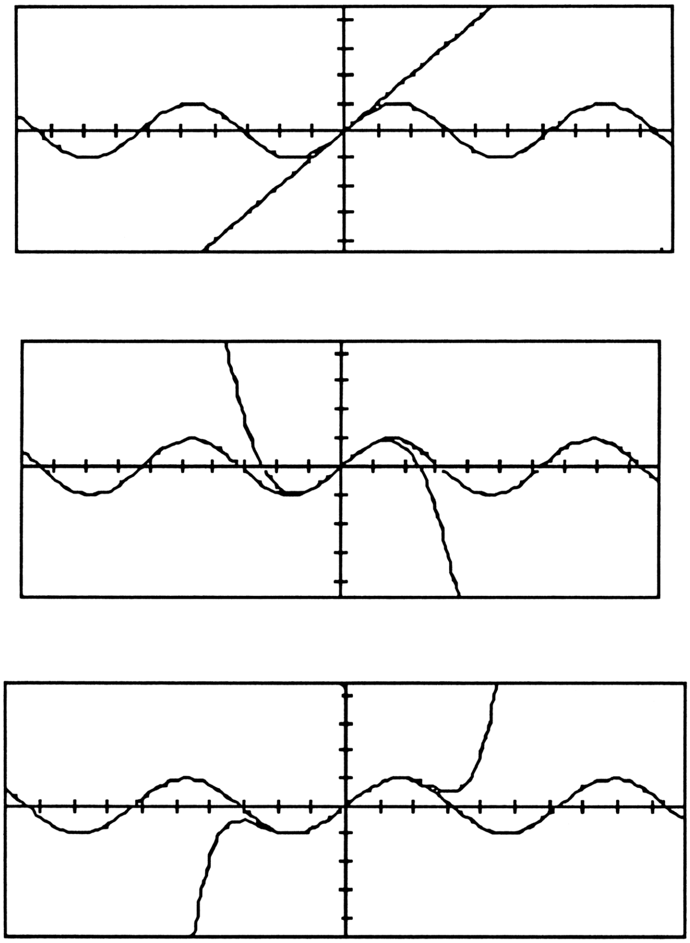
3.4 Classification of Critical Points for Functions of Two Variables

Taylor series for functions of one variable. The Taylor series expansion of f(x) about x_0 is given by

$$f(x) = \sum_{p=0}^{\infty} \frac{f^{(p)}(x_0)}{p!}(x - x_0) .$$

The sum of the first $n + 1$ terms, i.e., for $p = 0, 1, \ldots, n$, is called the nth degree Taylor polynomial $P_n(x)$. Of course, $P_1(x)$ is the line tangent to the graph of f(x) at $(x_0, f(x_0))$. Similarly, each of the polynomials approximates the function. We can get a better idea of the nature of this approximation by considering the graphs of f(x) and $P_n(x)$ plotted on the same axis.

Example 1. Sketch the graphs of y = sin(x) and the first three nonzero Taylor polynomials at $x_0 = 0$.



The same pictures can be reproduced on your HP-28S by entering the following series of commands.

'PPAR' PURGE
'SIN(X)' STEQ DRAW DEL ON ENTER
'SIN(X)' 'X' 1 TAYLR STEQ DRAW DEL ON OR →LCD ON ENTER
'SIN(X)' 'X' 3 TAYLR STEQ DRAW DEL ON OR →LCD ON ENTER
'SIN(X)' 'X' 5 TAYLR STEQ DRAW DEL ON OR →LCD

Exercise. Repeat the process for $1/(1 + x^2)$ using n = 1, 3, 5.

We look for critical points of $y = f(x)$ by finding values for which the graph of $P_1(x)$ is a horizontal line. A critical point x_0 of $y = f(x)$ is classified as a local minimum if the second degree Taylor polynomial $P_2(x)$ for $f(x)$ is a parabola which opens up. Notice that $P_2(x) = f(x_0) + f''(x_0)(x - x_0)^2$, which opens up provided $f''(x_0) > 0$. Similarly, x_0 is classified as a local maximum if $P_2(x)$ opens down.

Example 2. TAYLR always computes the polynomial expansions at zero, i.e., the MacLaurin expansions. In order to find an expansion about some other point we must introduce a change of variables. Suppose we want the expansion of $1/x$ about 1. Notice there is <u>no</u> expansion about 0.

Enter the following commands:

'X' 1/x 'Y + 1' 'X' STO EVAL
'Y' 3 TAYLR
'X' PURGE 'X - 1' 'Y' STO EVAL

Exercise. Try expanding sec x about π.

Classifying critical points. For $z = f(x, y)$ we proceed as follows: let $w(t, x, y) = f(x_0 + t(x - x_0), y_0 + t(y - y_0))$, a section of $z = f(x, y)$ in the direction of (x, y). The second degree Taylor polynomial at 0 for $w(t, x, y)$ (holding (x, y) fixed) is

$$P_2(t, x, y) = f(x_0, y_0) + [f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)]t + (1/2)[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2]t^2$$

which reduces to

$$P_2(t, x, y) = f(x_0, y_0) + (1/2)[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2]t^2$$

when (x_0, y_0) is a critical point.

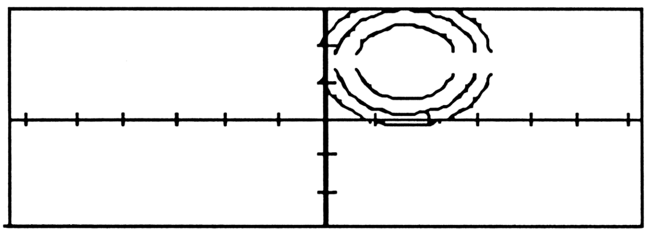
Example. Find the second degree Taylor polynomial for $w(t, x, y)$ given $z = (\sin x)(\sin y)$ and $(\pi/2, \pi/2)$ is a critical point . Enter the following:

```
'SIN(U)*SIN(V)'  'Z'  STO
'π/2'  →NUM  DUP  CHS  'X'  +  'T'  ×  +  'U'  STO
U  EDIT  (change X to Y)  ENTER  'V'  STO
Z  EVAL  'T'  2  TAYLR
```

In order to classify $(\pi/2, \pi/2)$ as either a local maximum. a local minimum or a saddlepoint, we must determine if $P_2(t, x, y)$ is a parabola opening down (or up) for all (x, y) or opening down for some (x, y) and up for others. We can decide by sketching the level curves of $P_2(1, x, y)$. Proceed as follows:

```
1 'T' STO EVAL
'K' PURGE 'K' + LDR
0 'K' STO DRAR DEL ON
1 CHS 'K' STO DRAR DEL ON OR
2 CHS 'K' STO DRAR DEL ON OR →LCD
```

Clearly, we have produced the level curves of an elliptic paraboloid, i.e., the coefficient of t^2 in $P_2(t)$ must always be either positive or negative. Therefore $(\pi/2, \pi/2)$ must be a local extremum. Furthermore, since the paraboloid looks down $(\pi/2, \pi/2)$ must be a local maximum.



The example illustrates this general result: if $f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2$ is an ellipse then (x_0, y_0) is an extremum. In which case, (x_0, y_0) is a maximum if $f_{xx}(x_0, y_0)$ is ngative and a minimum if $f_{xx}(x_0, y_0)$ is positive. If $f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2$ is a hyperbola then (x_0, y_0) is a saddlepoint. This can be developed as an efficiently applied critera as follow: Let $\Delta = (f_{xy}(x_0, y_0))^2 - f_{xx}(x_0, y_0)f_{yy}(x_0, y_0)$.

1) If $\Delta > 0$, then (x_0, y_0) is an extremum. A maximum if $f_{xx}(x_0, y_0) < 0$ and a minimum if $f_{xx}(x_0, y_0) > 0$.

2) If $\Delta > 0$, then (x_0, y_0) is a saddlepoint.

3) If $\Delta = 0$ then the test fails.

Exercise. Use the method outlined above to classify the critical points (-1, 11/6) and (1, 1/2) of $z = x^3 + y^2 + 2xy - 4x - 3y + 5$ as either local extrema or saddlepoints.

3.5 Polya's Problems[1]

Example. If the sum of two numbers is 6, what is the maximum of their product?

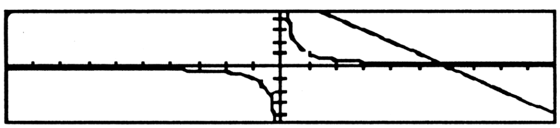
Let the numbers be x and y. We are given that x + y = 6 and we wish to maximize the function f(x, y) = xy subject to that constraint. From the handout, we seek a point (x_0, y_0) on x + y = 6 where the level curve $f(x, y) = x_0y_0$ is tangent. We proceed as follows:

Create a program that combines two screen images and leaves the cursor active.

```
<< OR →LCD DGTIZ >> 'TNGT' STO
```

We then produce a graph containing both x + y = 6 and xy = 1, i.e., the constraint and an arbitrary level curve of f(x, y).

```
'PPAR' PURGE
'6 - X' STEQ DRAW DEL ON ENTER
'K' PURGE 'X*Y - K' LDR 1 'K' STO DRAR DEL ON
TNGT
```



Of course, xy = 1 is not tangent to the constraint x + y = 6 at any point. We can choose a more appropriate level curve of f(x, y) = xy by moving the cursor to a point

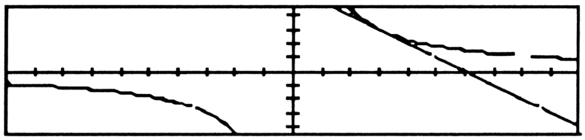
---

[1] George Polya, Mathematics and Plausible Reasoning, Vol. 1, Princeton University Press, Princeton, NJ, 1954.

on the constraint $x + y = 6$ where we think some level curve is tangent. Capture that point with INS and then continue. (Our guess resulted in (3.1, 2.9).)

ON C→R × 'K' STO ENTER DRAR DEL ON TNGT

This will produce something like the following:



The picture is pretty convincing that either (3.1, 2.9) is a point of tangency or near such a point. To check our answer, we can zoom in by setting the center at (3.1, 2.9) and repeating the process from the top.

Exercise. Find the minimum of $x^2 + y^2$ on the curve $x = y^2 + 1$.

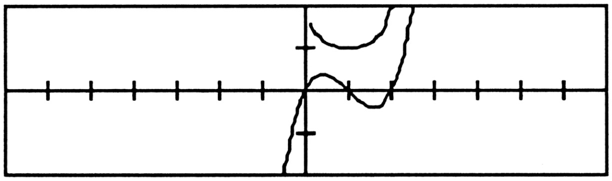
Example. Find the distance of the point (1, 2) from the curve $y = x^3 - 3x^2 + 2x$. If we find the point which minimizes the sqare of the distance that point will also minimize the distance. This latter problem can be stated as:

Minimize: $f(x, y) = (x - 1)^2 + (y - 2)^2$
Subject to: $g(x, y) = y - x^3 + 3x^2 - 2x$

The method of Lagrange multipliers leads to a system of polynomial equations with no rational solution. Let us try the graphical method outlined above.

```
'PPAR' PURGE (-6.8, -1) PMIN (6.8, 2.1) PMAX
'X*(X - 1)*(X - 2)' 'Z' STO
Z STEQ DRAW DEL ON 'CONST' STO
'K' PURGE '(X - 1)' x² '(Y - 2)' x² + 'K' - LDR
DRAR DEL ON TNGT
```

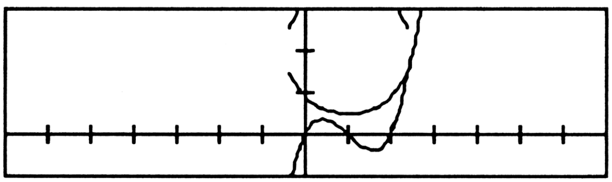
This produces something like the following:



Use the cursor to estimate the point on the curve nearest (1, 2), for us (2.5, 1.92). We chose PMIN to be (-6.8, -1) and PMAX to be (6.8, 3). We update the graph as follows:

C→R 2 →ARRY 1 2 2 →ARRY - DUP DOT 'K' STO
RCEQ Z STEQ DRAW DEL ON'CONST' STO
STEQ DRAR DEL ON CONST TNGT

This produces the following:



The current estimate of the minimum seems pretty good, but we could improve our estimate by repeating the process with a better guess. Our current estimate of the distance is the square root of K, i.e., 1.50.

Exercise. Find the distance of the point (1, 2) from the curve y = ln x.

Example. We want to find a graphical solution to a more difficult version of the milkmaid problem. Suppose the house is located at P(0, 1), the barn at Q(-2, 1) and the river bank is given by y = sin x. If each morning the milkmaid walks in a straight line from the house P to a point R on the riverbank to fill her pail and then in a straight line to the barn Q, the total distance she must travel is d(P, R) + d(R, Q). The problem is to choose the point R on the riverbank to minimize the total distance she must travel.

The problem can be stated as follows:

Minimize: $f(x, y) = ((x + 1)^2 + y^2)^{1/2} + (x^2 + (y - 1)^2)^{1/2}$

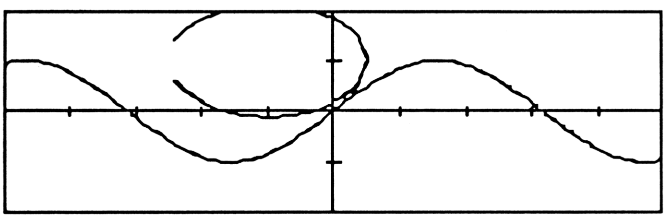
Subject to: $g(x, y) = y - \sin x = 0$

The method of Lagrange Multipliers leads to some difficult equations. However, the graphical method outlined above still works reasonably well.

The level curves of the distance the milkmaid walks are ellipses. Recall from the derivation of the general form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ of an ellipse with major axis horizontal that for this problem (h, k) = (-1, 1), 2a is the distance the milkmaid walks, 2c = 2 is the distance between the house and barn (the foci), and $b^2 = 1 - a^2$. The calculator analysis proceeds as follows:

'SIN(X)' STEQ 'PPAR' PURGE DRAW DEL ON 'CONST' STO
'K' PURGE 'X' 1 + x^2 'K' x^2 + 'Y' .5 - x^2 'K' x^2
1 - + + 1 - 'Z' STO Z 'Y' 2 TAYLR 'Y' QUAD STEQ
1.5 'K' STO DRAR DEL ON
CONST TNGT

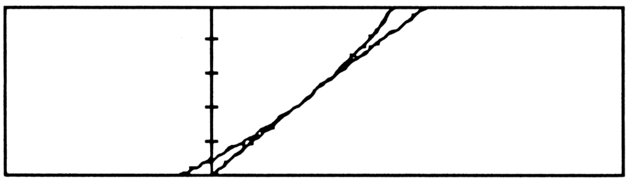
The graph looks something like.



Use the cursor to capture a guess for the point which minimizes the distance and to bracket the region of interest. Our guess is (.3, .3) , PMIN is (-.5, .1) and PMAX is (1, .6). Duplicate (3., .3) and store one copy in CNTR. To compute the updated K proceed as follows:

C→R 2 →ARRY DUP 0 1 2 →ARRY - DUP DOT $\sqrt{x}$
SWAP 2 CHS 1 2 →ARRY - DUP DOT $\sqrt{x}$ + 2 + 'K' STO
RCEQ 'SIN(X)' STEQ DRAW DEL ON 'CONST' STO
STEQ DRAR DEL ON TNGT

The graph looks something like this.



Of course, the estimate can be improved by repeating the above process.

Exercise. Find a graphical solution to the milkmaid problem given the house is at (0,1), the barn is at (0, 2) and the river bank is given by y = ln x.

CHAPTER 4
INTERSECTIONS OF CURVES AND SURFACES/INTEGRATION

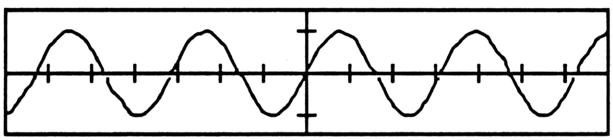
4.1 Intersections of Curves and Surfaces

Solving systems of equations forms the core of mathematics. Restricting ourselves to pencil and paper methods means that most equations, even equations with a single unknown, cannot be solved. Even simple equations such as $e^x = 10x$, which are easy to picture using the HP-28S, are best avoided without some kind of computational capability. The HP-28S gives us a powerful tool for solving equations and largely frees us from the narrow world of textbook problems.

Solution of one variable equations using SOLVE. Starting with something simple, suppose we want the zeros of $f(x) = \sin(2x)$. Of course there are infinitely many zeros and we know some of them, namely, $x = n\pi$ for any integer n. Are there any more? Obtain a graph of f(x) as follows:

'PPAR' PURGE

'X' 2 × SIN STEQ DRAW



Since the function is periodic, estimate the first three zeros starting at (0, 0). Capture each estimate using INS. You should have something like (0, 0), (1.5, 0) and (3.2, 0). Enter SOLVR from the SOLV menu. The first entry on the new menu is X and the second EXPR=. Enter the estimate of the right most zero, namely (3.2, 0), by pressing X. The top line of the screen display becomes X: (3.2, 0). Now enter

ALERNATE (the red key) X. The top line becomes SOLVING FOR X followed quickly by X: 3.1415926539 and the second line becomes Sign Reversal.

Quoting from the Reference Manual, "The Solver found two points where the procedure values have opposite signs, but it can't find an intermediate point where the procedure value is zero because (a) the two points are neighbors or (b) the procedure is not real valued between the points. The Solver returns the point where the procedure value is closer to zero. If the procedure is a continuous real function, this point is the calculator's best approximation to the actual root."

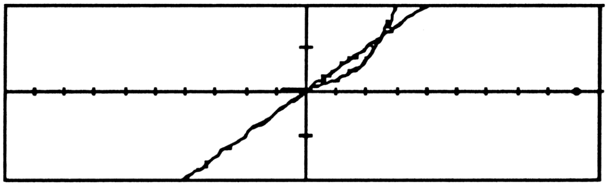
We know the true answer to be π. If you check the calculator value of π you see it agrees exactly with the root returned by the solver.

DROP the answer and repeat the process with (1.5, 0) and again with (0, 0). For (0, 0) something new emerges. This time the second line reads Zero. This indicates the Solver has found a point where the procedure is zero, i.e., an "exact" answer. Drawing on our knowledge of $f(x) = \sin(2x)$ we now can conclude the zeros are $n\pi/2$ for n an integer.

Exercise. Repeat the process for $f(x) = x^3 - 3x^2 + 2x$.

The process breaks down when we can't obtain a reasonable sketch of the function for forming an initial estimate. Try the repeating the process for $f(x) = e^x - 10x$. We're in trouble. the graph seems to consist of only a couple of scattered points. Try to graph just $y = 10x$ without changing the plotting parameters. Again, nothing seems to happen.

In order to produce graphs of two or more functions, such as the one given below of $y = e^x$ and $y = 10x$ properly scaled, we introduce a set of graphing utilities to make the process more efficient.



Graphing Utilities for Functions[1]. Create a subdirectory called GFNCS as follows:

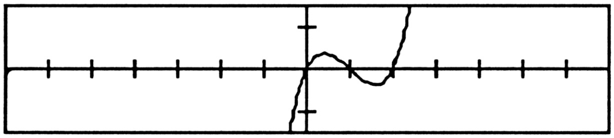
```
                          level 1
                          expression  ⇒
QUIT:
     << HOME >>                                | Returns to home
                                               | directory

                          level 1
                          expression  ⇒
NEWGRAPHS:
     <<  CLLCD  DUP  1  →LIST                   | Creates a list containing
         'EQS'  STO                            | the expression and names
                                               | it EQS
         STEQ  DRAW  LCD→  'SCR'  STO           | Draws to screen, saves
                                               | the screen
         DGTIZ                                 | and leaves the cursor
                                               | active
    >>
```

---

[1] John Kenelly and Thomas Tucker, private communication.

Example. 'X*(X - 1)*(X - 2)' NEWGRAPHS ON EQS results in a graph something like the following which is saved in SCR.



GETG:

```
<< SCR →LCD DGTIZ >>
```

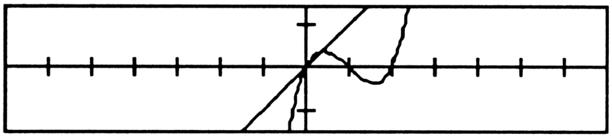
Example. GETG restores to the screen the graph stored in SCR.

level 1	
expression	⇒

OVERDRAW:

```
<<  DUP 1 →LIST EQS +            | Adds new expression to
                                 | list EQS
    'EQS' STO STEQ SCR
    →LCD DRAW LCD→               | Reproduces old screen
                                 | and adds
    'SCR' STO DGTIZ              | new graph.  Stores the
                                 | result
>>
```

Example. 'X' OVERDRAW produces the following which clips one of the regions bounded by the two curves.

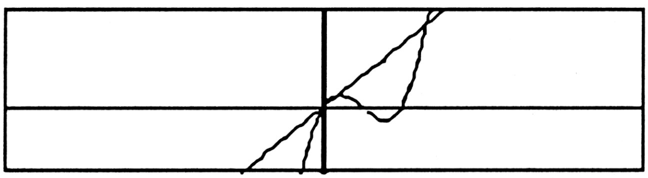


REDRAW:

```
<<  CLLCD  EQS  SIZE  1
    SWAP  FOR  I  EQS  I              | Redraws the graphs of
                                      | expressions
    GET STEQ  DRAW  NEXT              |  stored in EQS.  Saves
                                      | the screen
    LCD→  'SCR'  STO  DGTIZ           | and leaves the cursor
                                      | active
>>
```

Example. The regions bounded by the curves can be regraphed with

(-7, -1) PMIN (7, 3) PMAX REDRAW



This method of choosing the graphing window can be simplified with the following command.

level 2	level 1	
coordinate pair	coordinate pair	⇒

BOX:

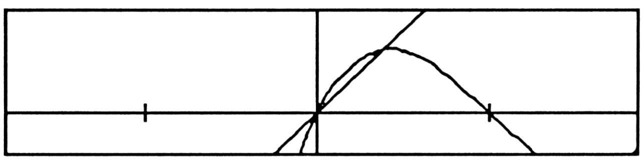
```
<< PMIN PMAX REDRAW
>>
```

level 3	level 2	level 1	
real	real	coordinate pair	⇒

ZOOM:

```
<< *W *H CENTR REDRAW
>>
```

Example. To return to the original parameters enter 'PPAR PURGE REDRAW Then to enlarge the smallest of the bounded regions try capturing the center of the region with INS and then enter .25 .25 ZOOM You should get something like the following:



Describing Regions of the Plane. Suppose we want to describe in set-builder notation the region R pictured immediately above. Our estimate of the top intersection captured with INS is (.375, .375). To use SOLVR proceed as follows:

EQS →LIST DROP - STEQ X (ALTERNATE)X

The Solver returns .381966. Thus we have R = $\{(x, y) \mid 0 \leq x \leq 0.381966 \text{ and } x \leq y \leq x^3 - 3x^2 + 2x\}$.

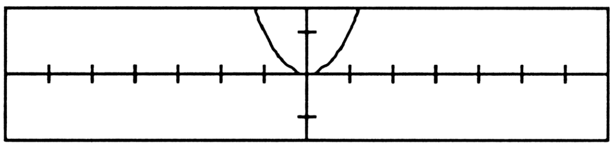
Exercises. In each of the following describe in set-builder notation using rectangular coordinates the given region R^2.

1) The region bounded by $f(x) = x^3 - 4x$ and $g(x) = 1 - x^2$.

2) The region bounded by $f(x) = x$ and $g(x) = (0.1)e^x$.

Example. Describe the region R bounded by $y = x^2$ and $x = y^2$. The new difficulty presented here is dealing with a relation, namely $x = y^2$, as opposed to a function. We can proceed as follows:

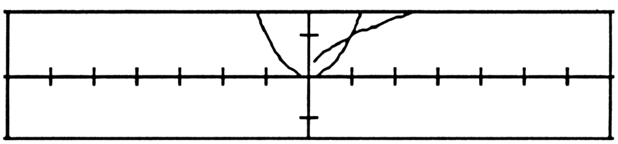
'PPAR PURGE 'X ^ 2' NEWGRAPHS



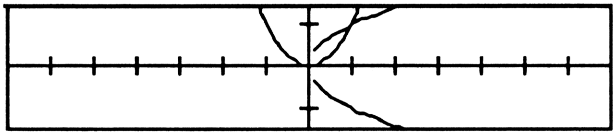
ON 'X - Y ^ 2' 'X' PURGE 'Y' QUAD

The last command found in SOLV returns 's1*√(4*X)/(-2)' , i.e., the symbolic solution for y (using the quadratic formula) of $x - y^2 = 0$. The parameter s1 is to have values ±1.

1 's1' STO OVERDRAW



1 CHS 's1' STO OVERDRAW



To describe the region, we need the upper intersection. (Of course, we know the answer, namely (1, 1).) Here's how to proceed with the calculator. First get an estimate with the cursor. We got (.9, 1).

ON EQS LIST→ DROP SWAP DROP

We need to use the upper branch of $x = y^2$.

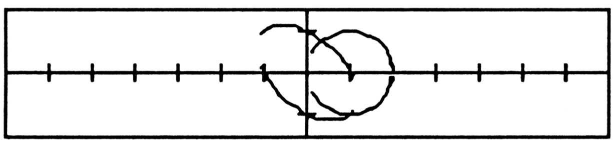
- STEQ SOLVR X (ALTERNATE)X

The answer returns 1. Thus $\{(x, y) \mid 0 \le x \le 1 \text{ and } x^{1/2} \le y \le x^2\}$.

Exercise. Describe the region bounded by $y = x^2 + x + 1$ and $x + 3/2 = y^2$.

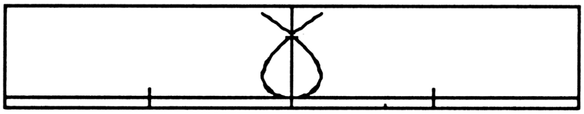
These problems are really about regions bounded by conic sections. Using the graphing utilities for conic sections makes the whole job easier.

Exercise. Describe the region R bounded by $x^2 + xy + y^2 - 1 = 0$ and $x^2 + y^2 - 2x = 0$. The graphs look like

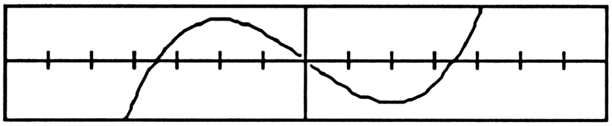


Turning to regions bounded by parametric curves and curves in polar coordinates, the problem of interest becomes that of determining a minimum interval of parameters for the curves.

Example. Consider the curve C: $r(t) = (t^3/8 - 3t/2, 3t^2/8)$, $-4 \le t \le 4$.



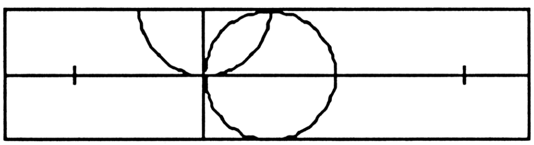
Clearly, we only need the three zeros of x(t).



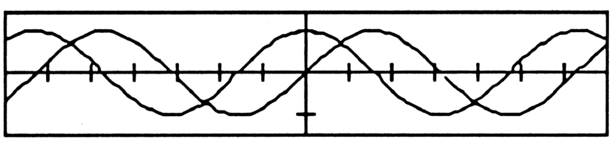
Since $x(t) = 0$ factors, we don't need to use SOLVR to obtain the roots -3.4641, 0, and 3.4641. Thus the region R bounded by C has boundary given parametrically by $r(t) = (t^3/8 - 3t/2, 3t^2/8)$, $-3.4641 \le t \le 3.4641$.

Exercise. Find the boundary of the region bounded by C: $r(t) = (t^2 - 3, t^3/3 - t), -2 \le t \le 2)$.

Example. Find the boundary C of the intersection of the regions bounded by the polar curves $r_1 = \cos t$ and $r_2 = \sin t$.



The two intersections are (0, 0) and (.7071, 7071). Graphing the two curves in rectangular coordinates we realize there is no value of t such that $r_1(t) = r_2(t) = 0$.



However, we can write C as a sum $C_1 + C_2$ of curves where C_1 is given by $r_1 = \cos t$, $.7071 \le t \le 1.5708$ and C_2 is given by $r_2 = \sin t$, $0 \le t \le .7071$.

Intersections of surfaces in 3-space. The intersection of two surfaces in 3-space will usually be a curve. Dealing with the most general surfaces is too hard for us at this time. We can look at the problem of describing regions bounded by a quadric surface and a plane.

Example. Describe the region D of R^3 between $z^2 = 2x^2 + 3y^2 + 1$ and $x + y + z = 2$.

The two surfaces intersect in a 3-dimensional curve. We begin by looking at the cylinder set parallel to the z-axis which contains this intersection. The equation for the cylinder set is found by eliminating z from the equations, i.e., solve the equation for the plane for z and substitute the result into the equation for the elliptic paraboloid of two sheets. We the want to look at the xy-section of the cylinder set in order to check that we have a bounded region.

```
'X + Y + Z - 2'  'Z'  ISOL  'Z'  STO
'2*X^2 + 3*y^2 - Z^2 + 1'  EVAL  LDR  DRAR
```

The resulting picture is difficult to analyze. Step back by reducing the resolution, i.e.,

```
5 *W 5 *H DRAR
```

The region does appear to be an ellipse. We can confirm this by shifting the axis. Try capturing the upper right hand corner of a new window by using INS. My attempt yields (3, 3) which I save using PMAX. You have to estimate something for the lower left hand corner. I tried

```
-20 -10 R→C PMIN
```

The resulting picture obtained using DRAR is convincing, i.e., the cylinder set intersects the xy-plane in a bounded elliptic region D_2.

The surface $z^2 = 2x^2 + 3y^2 + 1$ is an elliptic paraboloid. For $x = 0$, $y = 0$ we have $z^2 = 1$. The corresponding point on the plane $x + y + z = 2$ is $(0, 0, 2)$, i.e., the plane intersects the upper sheet of the elliptic paraboloid. Hence for (x, y) in the bounded region D_1, the ellipse obtained using the calculator, we must have

$$(2x^2 + 3y^2 + 1)^{1/2} \leq 2 - x - y$$

In order to complete the description of D_1, we must in essence obtain a description of the region D_1 in set builder notation. Look at what is stored in EQ. Thus for (x, y) in D_1 we have

$$2 - x - \sqrt{10 - 12x - x^2} \leq y \leq 2 - x + \sqrt{10 - 12x - x^2}$$

Call this interval of y values I_x. To find the interval of x values proceed as follows:

'EQ' RCL

The x-coordinates of the vertices of the ellipse will occur when the radical is zero. Why? Use the editor to obtain just the radical.

EDIT '√((2*(2 - X)^2 - 3*(2*X^2 - (2 - X)^2 + 1))'
x² 'X' QUAD

We find the x-interval with

ENTER -1 's1' STO EVAL
SWAP 1 's1' STO EVAL

The stack is 0.7828 ... , -12.7828 ... . We obtain the right order with SWAP.

Thus $D_1 = \{(x, y) \mid -12.7828 \leq x \leq 0.7828 \text{ and } y \text{ in } I_x\}$. Finally, $D = \{(x, y) \mid -12.7828 \leq x \leq 0.7828, y \text{ in } I_x \text{ and } (2x^2 + 3y^2 + 1)^{1/2} \leq z \leq 2 - x - y\}$.

4.2 Integration on the HP-28S

We will use the integration program IGL written by Wickes instead of the keyboard command '∫ ' for numerical integration. Before using IGL or the program IGL2 for evaluating iterated integrals you must first store a positive number repesenting an acceptable error in ACC. For instance, this can be accomplished by entering

0.001 'ACC' STO

Integral

level 4	level 3	level 2	level 1		level 1
a	b	f(x)	'x'	⇒	integral

IGL:

```
<<  4 ROLL →NUM 4 ROLL →NUM
    3 →LIST  ACC                    | Set up arguments
    ∫                               | Compute the
                                    | integral
    IF 0 <                          | Check the error
    THEN  "Bad Integral" 1  DISP    | If it's negative,
                                    | display
    END                             | a message
>>
```

Examples

1) Use IGL to evaluate $\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos}\ \ dt$.

2 √2 0 2 'π' →NUM × '1 - COS(T)' √x 'T' IGL ×

2) Compute the circumference of the ellipse given parametrically by $x = 2 \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

0 2 'π' →NUM × '4*SQ(SIN(T)) + SQ(COS(T))' $\sqrt{x}$ 'T' IGL

Exercises

1) Compute the circumference of the unit circle using the standard parameterization.

2) Compute the length of the parabola $y = x^2$ from ((1, 1) to (3, 9).

3) Compute the surface area of the figure generated by revolving about the x-axis the curve $y = \ln x$, $1 \le x \le 2$.

4) Compute the surface area of the figure generated by revolving about the x-axis the curve parameterized by $x = t^2$, $y = t^3$, $0 \le t \le 1$.

As you become more skilled with the HP-28S you will want to minimize the pencil and paper preprocessing that the previous examples employ. Consider the following example which uses the differentiation capabilities of the calculator.

Example 3. Compute the length of the curve $y = \dfrac{1}{1 + x^2}$ from (0, 1) to (3, 0.1). Enter the following:

0 3 '1 + SQ(X)' 1/x 'X' PURGE 'X' d/dx x^2 1 + $\sqrt{x}$ 'X' IGL

Exercises

5) Compute the surface area of the figure generated by revolving about the x-axis the curve discussed in the example above.

6) Redo the previous set of exercises.

7) Compute the circumference of the circle given in polar coordinates by $r = \sin\theta$.

Wickes' program IGL2 for evaluating iterated integrals works just like IGL, the program for definite integrals of functions of a single variable. Notice also that IGL2 uses IGL. Remember to store a positive real number in ACC before executing IGL2.

Iterated Integral

Level 7	6	5	4	3	2	1		level 1
a	b	'x'	c	d	f(x, y)	'y'	⇒	integral

IGL2:

```
<<  → c d f y                              | Arguments of
                                           | the "inner"
                                           | integral
    <<
       << c d f y IGL                      | Compute
       >>                                  | inner integral
       SWAP IGL                            | Compute outer
                                           | integral
    >>
>>
```

Example 4. Evaluate $\int_0^1 \int_x^{\sqrt{x}} \sqrt{x+y}\ dydx$.

0 1 'X' 'X' 'X' $\sqrt{x}$ 'X + Y' $\sqrt{x}$ 'Y' IGL2

Notice that the calculator takes a while to return the answer of 0.1522, about 32 seconds. Iterated integrals are harder to obtain numerically than definite integrals for functions of a single variable. Also, since the stack is complicated you should save the stack before executing IGL2, i.e., the commands should be

0 1 'X' 'X' 'X' $\sqrt{x}$ 'X + Y' $\sqrt{x}$ 'Y'
7 →LIST ENTER LIST→ DROP
IGL2

If everything looks right then SWAP DROP and you are left with only the answer.

Exercises

8) Find the volume of the figure bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$ and $z = 0$.

9) Find the volume of the figure bounded by $z = 0$, $x^2 + y^2 = 1$ and $z = x^2 + y^2$.

10) Compute the mass of a flat plate, the quarter disk $x^2 + y^2 \leq 1$, $x \geq 0$ and $y \geq 0$, with density $\rho(x, y) = xy$.

CHAPTER 5
VECTOR FIELDS AND LINE INTEGRALS

5.1 Vector Fields

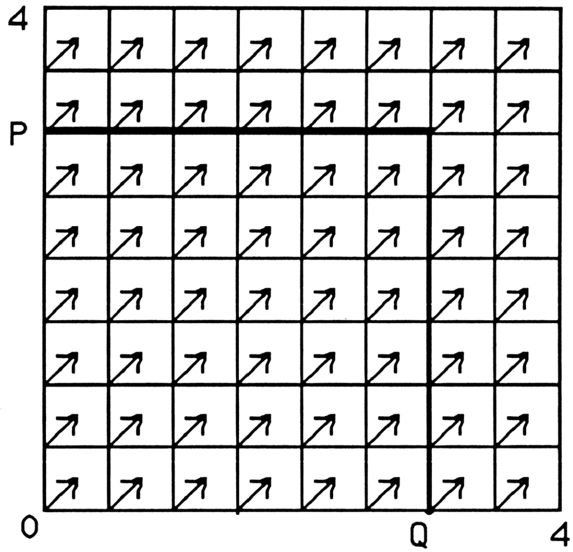
A vector valued function defined on a subset of R^n, n > 1, is called a vector field. Similarly, a scalar valued function is called a scalar field. We will tend to use a standard notation, for instance, **f**(x, y) = P(x, y)**i** + Q(x, y)**j**. Of course, **f** is a vector field with component functions P and Q which are scalar fields.

If a constant force c (constant in both direction and magnitude) is applied is applied in moving a particle along a straight line (the x-axis) from a to b (a < b) then the work W done is c(b - a). Notice that if c is positive then W is positive and the physical interpretation is that we have done work on the system. If c is negative then the system does work on us.

Any problem where motion is in a straight line and the force acts in a direction parallel to the direction of motion could be recoordinatized to fit our standard formulation.

Suppose the particle is to be moved from **P** = (a, b) to **Q** = (c, d) along the straight line connecting **P** and **Q**, but the force **f** no longer is assumed to act in a direction parallel to **u** = $(c - a, b - d)^T$ The component of the force in the direction **u** is given by $(\mathbf{f}\cdot\mathbf{u})/\|\mathbf{u}\|$. The distance traveled in moving from **P** to **Q** is $\|\mathbf{u}\|$. Hence the work done is $((\mathbf{f}\cdot\mathbf{u})/\|\mathbf{u}\|)\|\mathbf{u}\| = \mathbf{f}\cdot\mathbf{u}$.

Example 1. Compute the work required to move a particle in the force field illustrated below from the point P to the point Q.

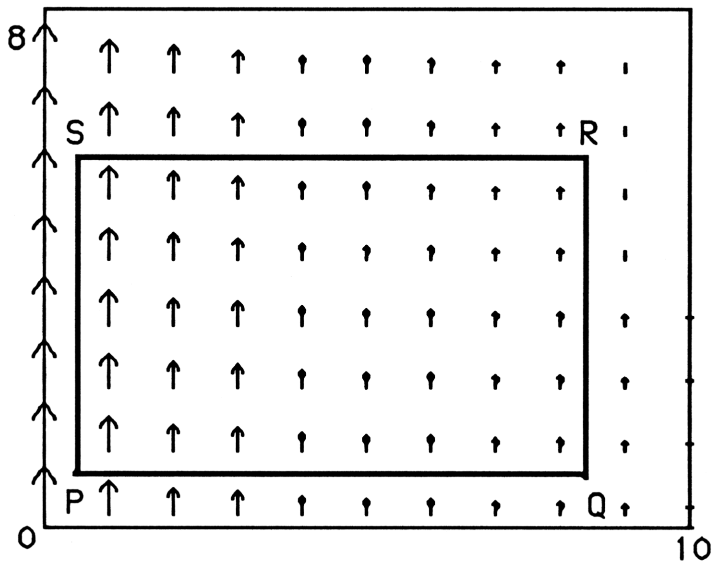


The force field is constant, say $\mathbf{f} = (1/4)\mathbf{i} + (1/4)\mathbf{j}$. The path the particle must travel can be split up into two parts, from P = (0, 3) to (3, 3) and from (3, 3) to Q = (3, 0). The total work W is the sum of the work on each part. Let $\mathbf{u} = 3\mathbf{i}$ and $\mathbf{v} = -3\mathbf{j}$. The force to be exerted in moving the particle must balance the force exerted by the field, i.e., the force exerted in moving the particle must be $-\mathbf{f}$. Thus $W = -\mathbf{f}\cdot\mathbf{u} - \mathbf{f}\cdot\mathbf{v} = -(3/4) + (3/4) = 0$.

Example 2. The picture below represents the force field $\mathbf{f} = (1/(1 + x))\mathbf{j}$. Compute the work done in moving a particle around the path PQRSP.

Of course, in moving from P to Q and from R to S, no work is done. Again, a force which balances the field must be exerted on the particle, but work is the component of the force exerted in the direction of motion. On those two segments of the path the force is orthogonal to the direction of motion so no work is done.

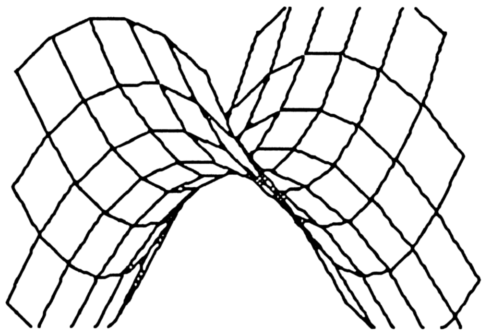
Assume that P = (1/2, 1), Q = (17/2, 1), R = (17/2, 6) and S = (1/2, 6). Let **u** = 5j and **v** = -**u** = -5j. Since the field on the path from Q to R is **f**(17/2, y) = (2/19)j and from S to P is **f**(1/2, y) = (2/3)j, we have W = -**f**(17/2, y)·**u** - **f**(1/2, y)·**v** = -(10/19) + (10/3) = 160/57. Note that in moving from Q to R the system does work on us. From S to P we do work on the system. Since the total work is positive, we do work in traversing the path PQRSP. If we move in the opposite direction PSRQP then the system does net work on us.



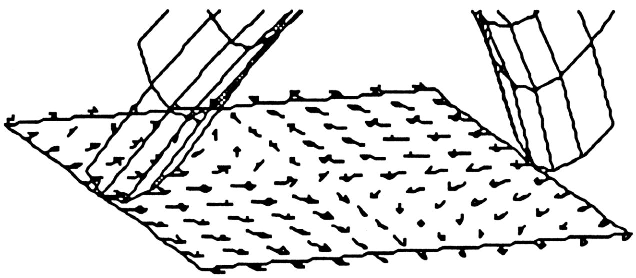
Exercise 1. Show that W = 0 if the particle is moved in a straight line from P to Q.

Exercise 2. Which path PQR or PSR requires the most work to move a particle from P to R.

Of course, in order to construct a vector field **f**(x, y) = P(x, y)**i** + Q(x, y)**j** one just has to specify P and Q. An interesting way to do this is to start with a function z = g(x, y) and let **f** = ∇g = $(g_x, g_y)^T$ = g_x**i** + g_y**j**. A vector field defined this way is called a <u>gradient field</u>. Consider the surface below given explicitly by $z = x^2 - y^2$.

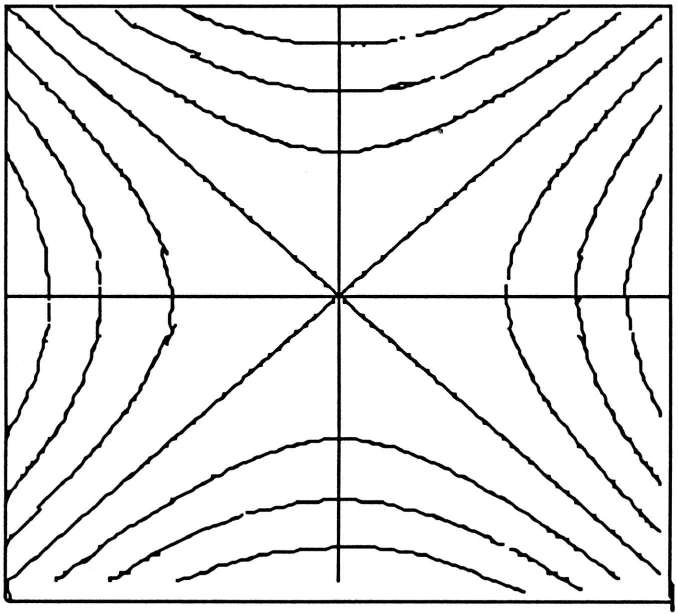


The with attached gradient field appears below. The vectors have all been normalized to simplify the picture.



Exercise 3. Use the fact that a gradient vector at a point is normal to the level curve through the point to add normalized vectors to the level curves of $z = x^2 - y^2$

given below and recapture the previous picture. Remember which direction is up hill.



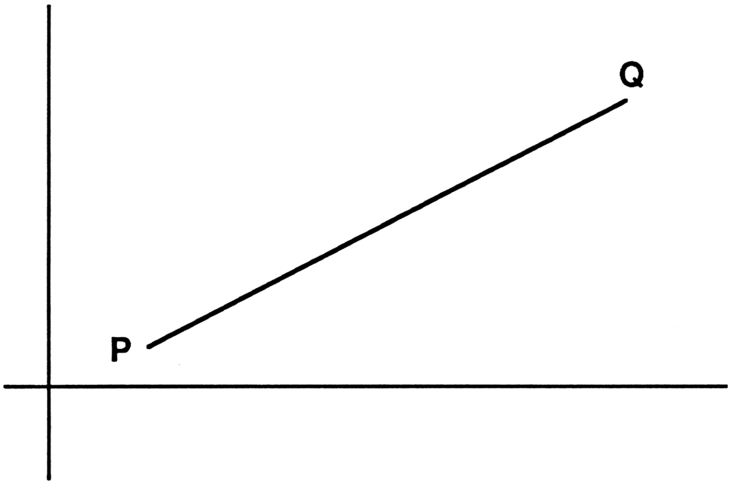
Exercise 4. Produce the gradient fields for $z = y^2$ and $z = x^2 + y^2$.

5.2 Line Integrals

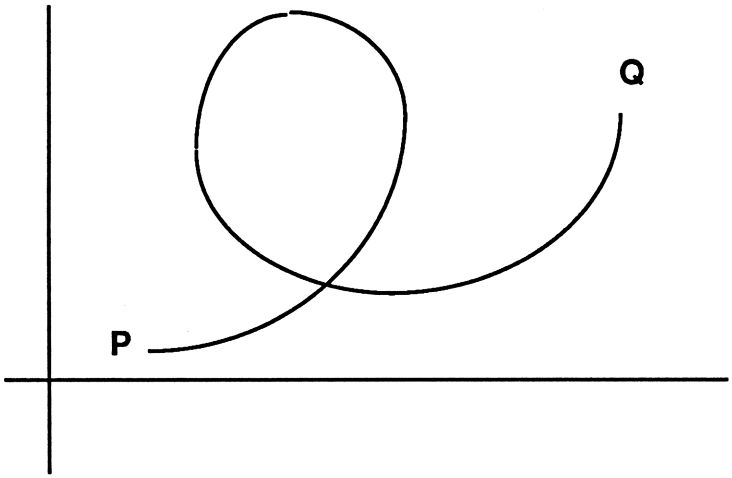
If a constant force c (constant in both direction and magnitude) is applied is applied in moving a particle along a straight line (the x-axis) from a to b ($a < b$) then the <u>work</u> W done is $c(b - a)$. Notice that if c is positive then W is positive and the physical interpretation is that we have done work on the system. If c is negative then the system does work on us.

If the force $f(x)$ is <u>not</u> constant but acts in only a direction parallel to the direction of motion then $W = \int_a^b f(x)\,dx$ and again we interpret positive $f(x)$ as a force in the direction of motion. If W is positive then we are doing a net positive work on the system. Notice that any problem where motion is in a straight line and the force acts in a direction parallel to the direction of motion could be recoordinatized to fit our standard formulation.

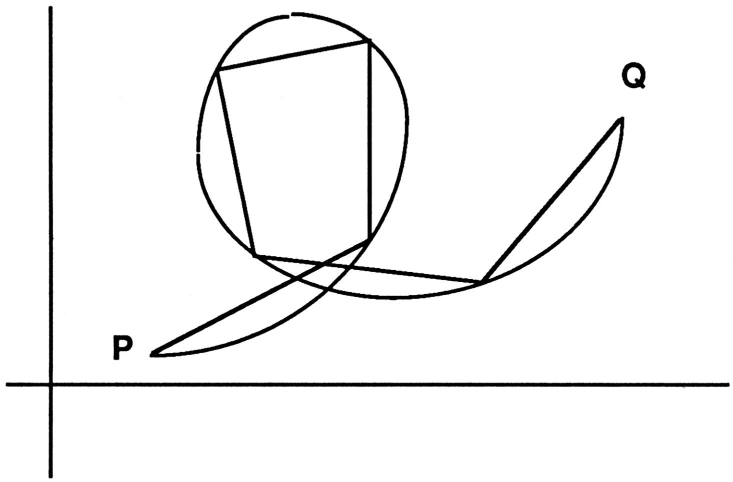
Suppose the particle is to be moved from P = (a, b) to Q = (c, d) along the straight line connecting P and Q, but the force f(x, y) no longer is assumed to act in a direction parallel to u = (c - a, b - d).



The component of the force in the direction u is given by $(f(x, y)\cdot u)/\,||\,u\,||$. If the straight line segment from P to Q is parameterized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, then the work done in moving the particle from P to Q is approximated, for a partition $a = t_0 < t_1 < \ldots < t_n = b$ by $\sum_{i=1}^{n} (f(x_{i-1}, y_{i-1})\ u)(t_i - t_{i-1}) = \ (f(x(t_{i-1}), y(t_{i-1}))$ $u)\,||(x(t_i) - x(t_{i-1}), y(t_i) - y(t_{i-1}))\,||/\,||\,u\,||$. Thus $W = \int_a^b f(x(t), y(t))\ u\ \ dt$. Moving on to the general case where the curve from P to Q is <u>not</u> a straight line, our picture becomes as follows.



Again we assume that the curve from P to Q has been parameterized by $x = x(t)$, $y = y(t)$, $a \le t \le b$, and consider the polygonal approximation to the curve based on the partition $a = t_0 < t_1 < \ldots < t_n = b$.



The approximation to the work done in moving the particle from P to Q becomes $\sum_{i=1}^{n} f(x_{i-1}, y_{i-1})\,(x_i - x_{i-1}, y_i - y_{i-1}) = \sum_{i=1}^{n} f(x_{i-1}, y_{i-1})\left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}}, \frac{y_i - y_{i-1}}{t_i - t_{i-1}}\right)(t_i - t_{i-1})$ which approaches in the limit $\int_a^b f(x(t), y(t))\,(x'(t), y'(t))\,dt = \int_C f\,dr$, where C is the curve from P to Q parameterized by r.

We use the numerical integration program IGL to compute the various ine integrals in the next three examples. Notice that the calculator performs the task of constructing the integrand symbolically from the pieces, i.e., P(x, y), Q(x, y), x = x(t) and y = y(t). In fact, the keystrokes for computing much more complex line integrals differs very little from these simple examples.

Example 1. $\int_C x^2y\,ds$; C: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi/2$.

```
'X^2*Y' 'P' STO
'COS(T)' 'X' STO
'SIN(T)' 'Y' STO
'T' PURGE
'X' 'T' d/dx 'XP' STO
'Y' 'T' d/dx 'YP' STO
0 'π' →NUM 2 /
'P*((XP)^2 + (YP)^2)^.5'
'T'
IGL
```

Example 2. $\int_C (x^2y\,dx + xy\,dy)$; C: $x^2 + y^2 = 1$ form (1 ,0) to (0, 1).

C: $y = \sqrt{1 - x^2}$, $0 \leq x \leq 1$

```
'X^2*Y' 'P' STO
'X*Y' 'Q' STO
'(1 - x^2)^.5' 'Y' STO
'X' PURGE
'X' 'X' d/dx 'XP' STO
'Y' ''X' d/dx 'YP' STO
1 0
'P*XP + Q*YP'
'X'
IGL
```

C: $x = \cos t$, $y = \sin t$, $0 \leq t \leq \pi/2$

```
'X^2*Y' 'P' STO
'X*Y' 'Q' STO
'COS(T)' 'X' STO
'SIN(T)' 'Y' STO
'T' PURGE
'X' 'T' d/dx 'XP' STO
'Y' ''T' d/dx 'YP' STO
0 'π' →NUM 2 /
'P*XP + Q*YP'
'T'
IGL
```

Example 3. $\int_C F \cdot dr$; F(x, y) = x + 2y)I + (2x + y)j, C: r(t) = ti + t^2j, $0 \le t \le 1$.

```
'X + 2*Y' 'P' STO
'2*X + Y' 'Q' STO
'T' 'X' STO
'T^2' 'Y' STO
'T' PURGE
'X' 'T' ∂ 'XP' STO
'Y' "T' ∂ 'YP' STO
0 1
'P*XP + Q*YP'
'T'
IGL
```

Exercises

1) Compute $\int_C F \cdot dr$ given F(x, y) = (x + 2y)i + (2x + y)j, C: $x = \sqrt{2}\,t$, $y = \sqrt{2}$ sin t, $0 \le t \le \pi/4$. Compare with Example 3.

2) Compute $\int_C F \cdot dr$ given F(x, y) = x^2yi + xyi, C = $C_1 + C_2$, C_1: $x = 1 + (\sqrt{2} - 2)\,t/2$, $y = \sqrt{2}t\,/2$, and C_2: $x = \sqrt{2}\,(1 - t)\,/2$, $y = \sqrt{2}\,(1 - t)\,/2 + t$.

Compare with Example 2. What happens as C is approximated with shorter straight line segments?

Example 4. Compute $\int_C F \cdot dr$, where F(x, y) = xyi + (x - y)j, and C is given in polar coordinates by r = cos θ.

Of course, our first task is to obtain a parametric representation of C in rectangular coordinates. This is easily done since $x(\theta) = r(\theta)\cos\theta = \cos^2(\theta)$ and $y = r(\theta)\sin\theta = \cos\theta \sin\theta$.

```
'X*Y'  'P'  STO
'X - Y'  'Q'  STO
'SQ(COS(T))'  'X'  STO
'COS(T)*SIN(T)'  'Y'  STO
'T'  PURGE
'X'  'T'  d/dx  'XP'  STO
'Y'  'T'  d/dx  'YP'  STO
0  'π'
'P*XP + Q*YP'
'T'
IGL
```

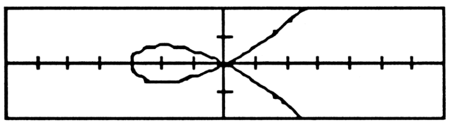
5.3 Green's Theorem

Green's Theorem may be stated as follows: Suppose **F**(x, y) is a vector field, i.e., **F**(x, y) = P(x, y)**i** + Q(x, y)**j** where P(x, y) and Q(x, y) are scalar functions (fields). Assume that $P_y(x, y)$ and $Q_x(x, y)$ are continuous in a bounded region R with a piecewise smooth boundary C which is oriented positively. (C is given parametrically by **r**(t) = x(t)**i** + yt)**j**, a ≤ t ≤ b, and x(t) and y(t) are piecewise smooth. Furthermore, as t varies from a to b, **r**(t) traces out C keeping R on the left.) Then $\int_C \mathbf{F}\bullet d\mathbf{r} = \int_a^b P(x, y)\,dx + Q(x, y)\,dy = \int\int_R Q_x(x, y) - P_y(x, y)\,dxdy$.

Example 1. Use Green's Theorem to compute the area of the unit disc R. Since the area is given by $\int\int_R dxdy$, we can apply Green's Theorem provided P(x, y) and Q(x, y) can be found so that $Q_x(x, y) - P_y(x, y) = 1$. Of course, this can be done in many ways. Why not P(x, y) = 0 and Q(x,y) = x. Thje boundary of the unit disc R is the unit circle C which could be parameterized with $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$. Hence the area is

$$\int_C x\, dy = \int_0^{2\pi} (\cos t)^2\, dt = \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = \pi$$

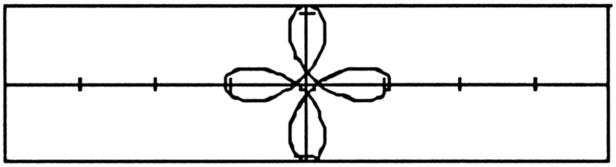
Example 2. Find the area inside the loop of Tschirnhausen's cubic C parameterized by $\mathbf{r}(t) = (t^2 - 3)\mathbf{i} + (t^3/3 - t)\mathbf{j}$, $-3 \le t \le 3$. The curve C looks something like



We need to restrict the range of t to the values which give the boundary of just the loop, call it C_1. This can be done by solving x(t) = y(t) = 0. Clearly, $t = \pm 3^{1/2}$, i.e., we restrict the parameter to the interval $-3^{1/2} \le t \le 3^{1/2}$. Check to see that the loop (the boundary of the region inside) has a positive orientation. The area is then given by

$$\int_{C_1} x\, dy = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2 - 3)(t^2 - 1)\, dt = \left(\frac{t^5}{5} - \frac{4t^3}{3} + 4t\right)\Big|_{-\sqrt{3}}^{\sqrt{3}} = 6.235$$

Example 3. Find the area of the four loops in the hypotrochoid C given parametrically by $\mathbf{r}(t) = (6\cos t + 5\cos 3t)\mathbf{i} + (6\sin t - 5\sin 3t)\mathbf{j}$, $0 \le t \le 2\pi$. The figure looks something like



The loops are all equal. The boundary of each loop is negatively oriented. Only the boundary of the little region in the center of the figure is positively oriented. Let's work with the top loop. The first step is to restrict the range of the parameter and call the resulting boundary C_1. We need to solve the equation x(t) = 6cos t + 5cos 3t = 0. Using the calculator proceed as follows:

'6 * COS(T) + 5 * COS(3 * T)' STEQ
'PPAR' PURGE DRAW

The resulting picture isn't much help, but you can improve it by changing the plotting parameters. Use the cursor and INS to capture the bottom of the y-axis. It should be approximately, (0, -1.5). Store the result in PPAR by hitting PMIN. Redraw the graph. Again you probably want to change PPAR. This time move the cursor to the top of a vertical line just to the right of the first two zeros, say (2, 1.6). Capture this point with INS and store in PPAR with PMAX. Now we can estimate the first two zeros of x(t), i.e., the beginning and end of the top vertical loop of the hypotrochoid.

We use ROOT to estimate the two zeros more accurately. Try this: use the cursor and INS to estimate the leftmost zero. <u>Then</u> bracket that guess in the same way. Having done that enter the following:

3 →LIST EQ SWAP 'T' SWAP ROOT

My answer comes back as 0.83548. Repeating this process for the zero on the right (remember to capture your best guess of the zero first) I got 1.5708. Of course, the true answer is $\pi/2$. The area is

$$\int_{C_1} x\,dy = \int_{.83548}^{1.5708} (6\cos t + 5\cos 3t)\,(6\cos t - 15\cos 3t)\ dt$$

which can be evaluated as follows: (assuming the two zeros we found are still on the stack)

```
'6 * COS(T) + 5 * COS(3 * T)'  '6 * SIN(T) - 5 * SIN(3 * T)'
'T' PURGE 'T' d/dx × 'T' IGL
```

The answer returned is -15.831. We're trying to find an area and we've ended up with a negative number. What's wrong? As noted earlier C_1 has a negative orientation, so the area is 15.831. The area of the region bounded by the four loops is 63.324.

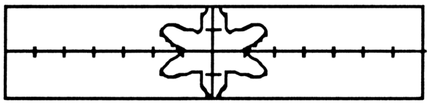
Exercises.

1) Compute the area of the region R bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Our standard parameterization of the boundary C of R is given by $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

2) Find the area bounded by one arch of the cycloid generated with a circle of radius one and the x-axis. The portion of the cycloid of interest, call it C_1, is parameterized by $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$, $0 \le t \le 2\pi$. For the relevant portion of the x-axis C_2 use $\mathbf{R}(t) = t\mathbf{i}$, $0 \le t \le 2\pi$. The boundary of the region C with positive orientation then becomes $C = C_2 - C_1$.

Sometimes one side of the equation in Green's Theorem is easier to evaluate than the other. This usually comes about because the integral on one side or the other is easier to set up.

Example 4. Evaluate $\int\int_R y - x \, dxdy$, where R is the region bounded by the curve C given in polar coordinates by $r(\theta) = 2 - \cos^2(3\theta)$. The region looks something like



Clearly, the 'snowflake' region R would be difficult to describe in rectangular coordinates. We proceed as follows:

```
'X * Y' ENTER 'Q' STO 'P' STO
'2 - COS(3 * T) ^ 2'
ENTER 'COS(T)' × 'X' STO
'SIN(T)' × 'Y' STO
X 'T' PURGE 'T' d/dx 'XP' STO
Y 'T' d/dx 'YP' STO
0 6.3 'P * XP + Q * YP' 'T' IGL
```

After a wait of several minutes the answer 0.000 returns.

Exercises.

3) Integrate y - x over the region bounded by the loop of Tschirnhausen's cubic parameterized by $\mathbf{r}(t) = (t^2 - 3)\mathbf{i} + (t^3/3 - t)\mathbf{j}$, $-3 \le t \le 3$. (Use the boundary C_1 developed in Example 2 above.)

4) $\int_C x \, dy$, where C is the polygonal path from (0, 0) to (1, 0) to (1, 1) to (0, 1) to (0, 0).

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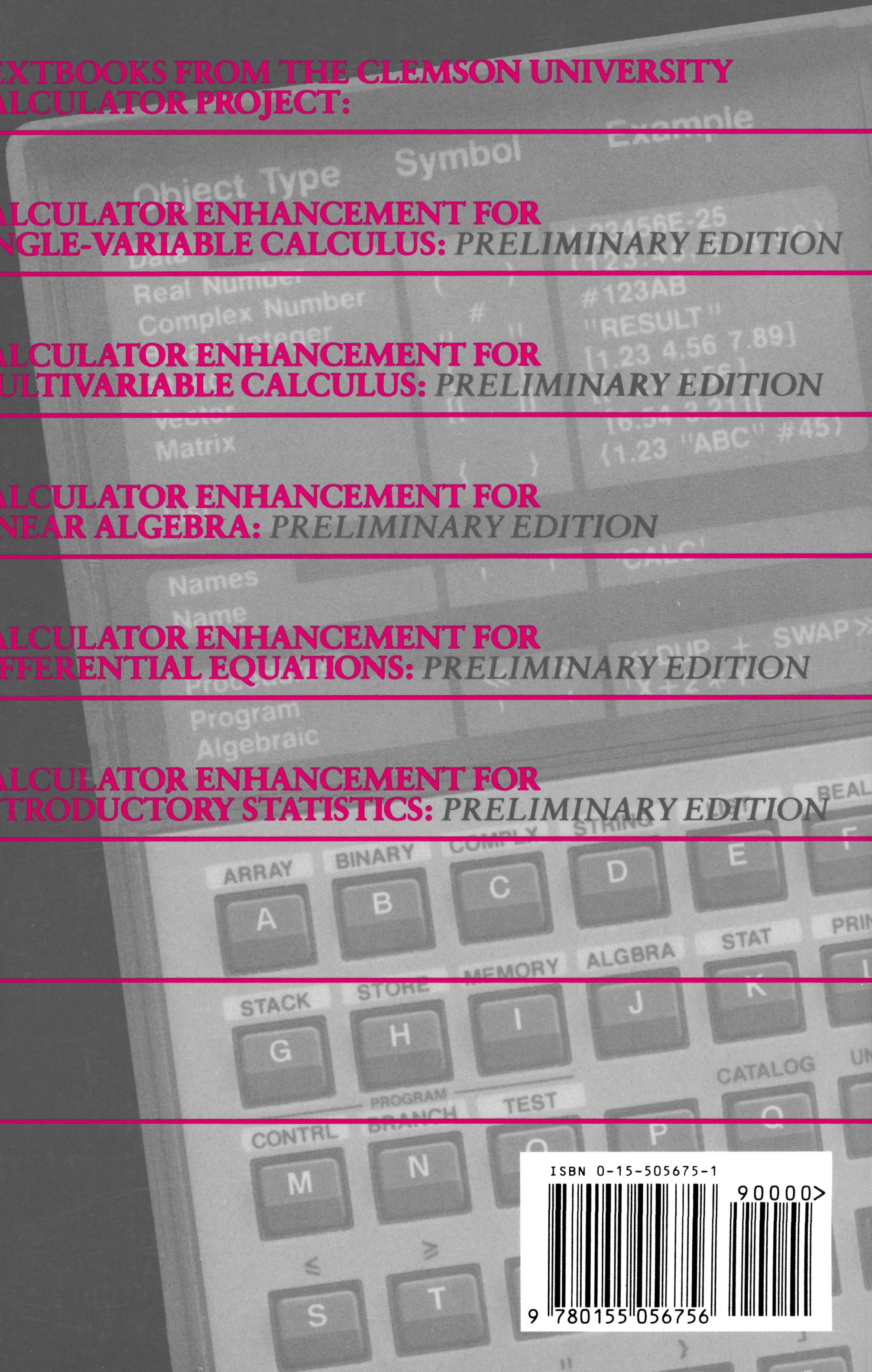

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