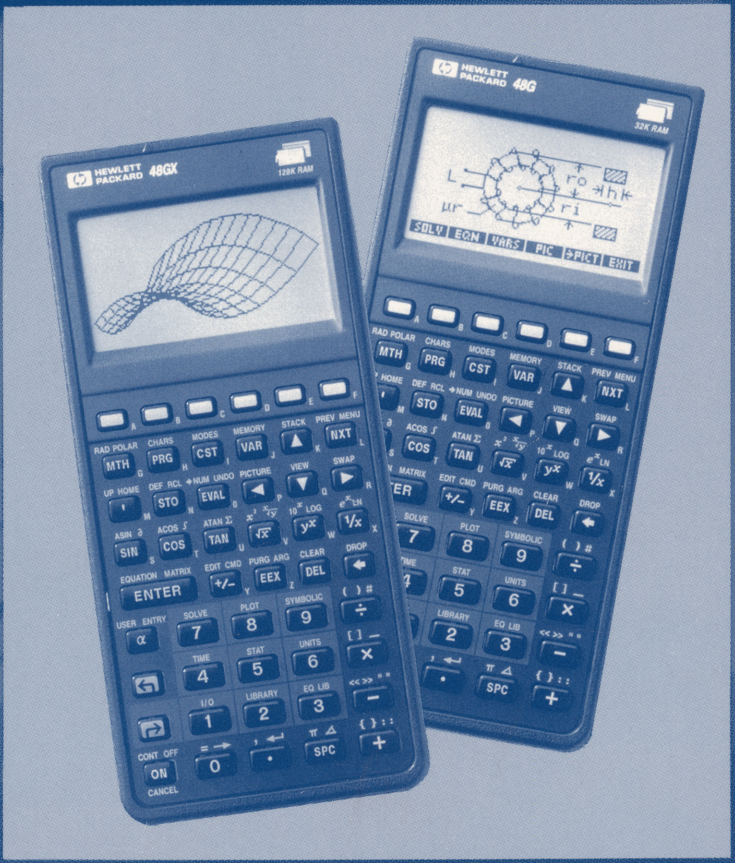
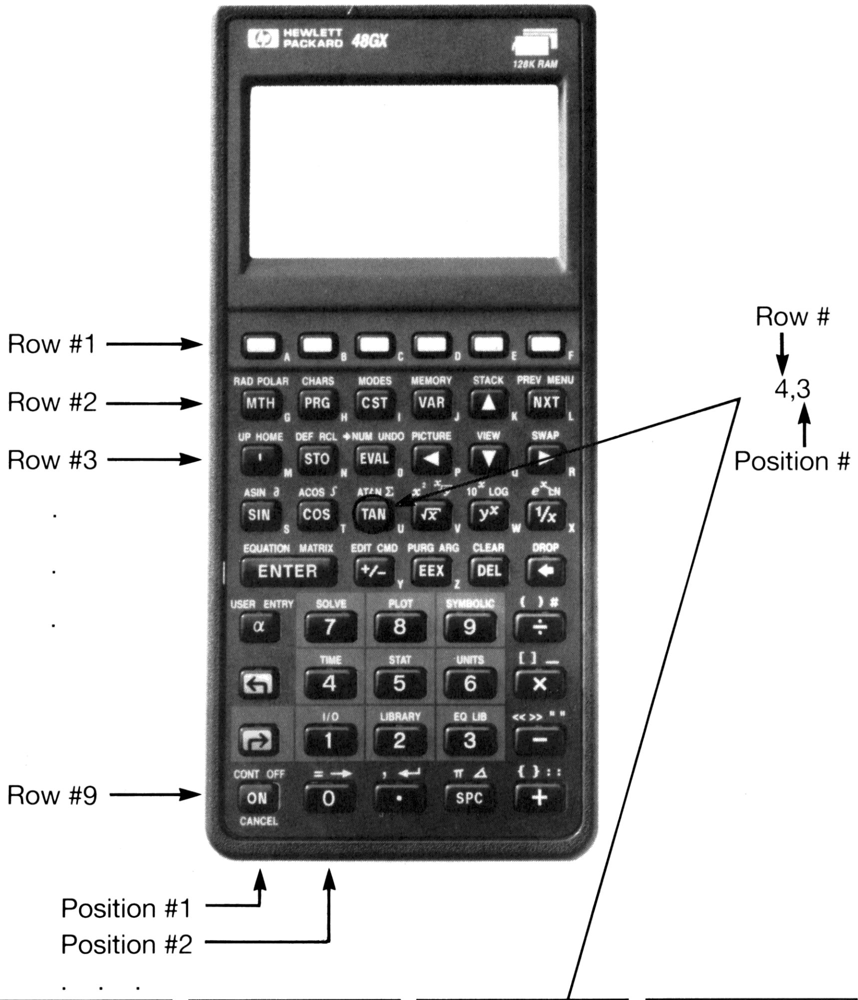


Jerold Mathews
Jack Eidswick

An HP 48G Calculus Companion



ACOS	4,2
ARG	5,3
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ATAN	4,3
CANCEL	9,1
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SIN	4,1
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STAT	7,3
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TAN	4,3
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$1/x$	4,6
10^x	4,5
+/−	5,2
+	9,5
−	8,5
X	7,5
÷	6,5
=	9,2
α	6,1

π	9,4
Σ	4,3
∂	4,1
∫	4,2
∡	9,4
◀	5,5
→	9,2
↵	9,3
'	3,1
•	9,3
,	9,3
#	6,5
_	7,5
( )	6,5
[ ]	7,5
<< >>	8,5
" "	8,5
{ }	9,5
: :	9,5
▲	2,5
◀	3,4
▼	3,5
▶	3,6
↰	7,1
↱	8,1

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Preface

Introduction

The words *calculator*, *calculate*, and *calculus* come from the Latin word "calculus," which refers to small stones used in reckoning. The *Calculus Companion* features the HP 48G or HP 48GX calculator, provides a rich selection of problems to support our assumption that calculus is best learned through problem solving and calculation, and is organized to facilitate the learning of calculus. The *Calculus Companion* is intended to supplement an "average" calculus text and to be a "companion" to a student studying calculus.

In the last decade of the twentieth century we are at the beginning of a revolution in the use of computing technology in the physical, engineering, and mathematical sciences. Peter Lax, a leading U.S. applied mathematician, said that this revolution is altering the face of applied mathematics. Programmable handheld calculators with numerical, graphical, and symbolical capabilities are part of this revolution.

Numerical, Graphical, and Symbolical Exposition, Problem Solving, and Learning

Placed in the middle of the the HP 48 keyboard, on the 7, 8, and 9 keys, are SOLVE, PLOT, and SYMBOLIC. These keys give quick access to the numerical, graphical, and symbolical power of the HP 48. Numerical, graphical, and symbolical approaches to exposition, problem solving, and learning are used throughout the *Calculus Companion*. We have tried to express the viewpoint, mostly by example, that picking up a graphics calculator to explore an idea or problem numerically or graphically is a normal and valuable part of learning or using mathematics.

Problems for Examples, Problems for Exercises, and Student Projects

Our examples, exercises, and projects were chosen to more nearly resemble real problems and to make use of the power of the HP 48. We have not thought it useful to simply give lists of traditional problems and ask that they be solved with the help of a calculator.

- We have given problems that help students learn, understand, and apply calculus better, through encouraging students to approach problem solving using numerical, graphical, and symbolical techniques
- We have divided the exercises into A, B, and C sets, in an attempt to provide for a variety of students. Some students will be interested in programming their calculators, some will be interested in more challenging problems, and some will be interested in a deeper analysis of a topic. The A exercises are more or less at the same level and cover the same ideas as the examples. The B exercises ask the student to go beyond the content and level of the examples and may include some programming. The C exercises are more ambitious yet or explore topics such as error analysis.
- We give several "Projects," which are problems suitable for a student writing project or group work.

Order of Chapters and Content

The order of the chapters is close to that of an average calculus text. However, topics or ideas on which the impact of the HP 48 was seen (by us) as not important were omitted. We have omitted entirely or given brief treatments of other topics for reasons of space. The main chapter on graphing, Chapter 3, follows Chapter 1 (Functions and Limits) and Chapter 2 (Derivatives) because the latter two chapters are at the beginning of most calculus courses. We want to keep the student focused on calculus, not on the bells and whistles of the HP 48. Chapter 3 comes at a point where students have reasons for wanting to graph a function. On the other hand, graphing is a strong feature of the HP 48 and is in fact useful in Chapters 1 and 2. We have therefore included pieces of "how to graph" in Chapters 0 and 1. Chapter 0 (HP 48 Nuts and Bolts) is a good introduction to the HP 48 and may be studied at different levels of intensity, depending on the student. Chapter 6 (Solving Systems of Linear Equations with the HP 48) was written for use in §7.5, Partial Fraction Calculations. For this purpose we give programs for exact arithmetic. These programs may be adapted for use in a simplex algorithm for solving a linear program.

Other Distinguishing Features

The *Calculus Companion* has several useful features.

- Front and back end papers give tips on trouble-prevention and trouble-shooting, and include a Program Index.
- In *To the Student*, which precedes Chapter 0, we urge students to take advantage of five tools for learning calculus: themselves, their instructor, their calculus textbook, the *Calculus Companion*, and their HP 48. We also summarize the key strokes needed for numerical, graphical, and symbolical calculations, and for differentiating or integrating a function.
- Each chapter and section begins with a preview of the main ideas to be discussed.
- The main ideas in a typical section are presented through a series of examples, written in a problem/solution format.
- An input/output format is used in the examples to make it easier to learn to use the HP 48. The key strokes used to input numbers, functions, and commands are shown on the left and the resulting HP 48 screen is shown on the right.
- We explore more of the numerical side of calculus than has been included in traditional courses, not simply because the capabilities of the HP 48 make this possible but because many students will use calculus in this way.
- Most chapters include programs. Students can use these programs in learning/using calculus, whether or not they are interested in learning to program. Students who have programmed in BASIC, PASCAL, or FORTRAN quickly learn to program the HP 48. Students and instructors differ widely on the value of programming. Some wish to use the rich array of HP 48 built-ins. Their interest is more nearly in finding, say, the numerical value of an integral than in understanding numerical integration. By offering programs for various algorithms we have given students, instructors, and other readers a choice. Most programs are listed in a Program Index.
- Answers to most exercises are included.
- Although we have written with the HP 48G in mind, we use the broader term *HP 48* when we need to refer to a calculator. Most of the graphical, numerical, and symbolic calculations we discuss can be done on either the HP 48G or the HP 48S. Many of the calculations and programs can be done on the HP 28S.

Acknowledgments

The *Calculus Companion* has grown out of our combined experience in teaching calculus with the HP 48 or in giving workshops featuring this calculator. Our students or audiences have ranged from junior high students, to mainline calculus students, to faculty members of teachers' colleges in Zimbabwe, and to participants of the Nebraska Mathematics Scholars Program and the Western Mathematics Scholars Program.

We have many people and organizations to thank. We particularly want to thank university colleagues Tom Dick, Lynn Garner, Frank Gilfeather, Leon Hall, John Kenelley, Don LaTorre, Gary Meisters, Gil Proctor, Mel Thornton, and Tom Tucker; and Hewlett-Packard colleagues Diana Byrne, Ron Brooks, and Clain Anderson. We have also received help or support from Hewlett-Packard Company and United States Information Agency, and from Iowa State University, University of Nebraska-Lincoln, California State University-San Bernardino, and The University of Montana. For help with AMS-TeX we thank Clifford Bergman of Iowa State University and Lori Pickert of Archetype Publishing, Inc.

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We would like to thank, particularly, George Duda, our friend and editor at HarperCollins.

Finally, we are deeply indebted to our families and, in particular, to our wives, Eleanor Mathews and Lily Eidswick, who have given us unceasing support, time, and encouragement over the years. Special thanks to Lily, who proofread and TeXed large portions of the manuscript and made numerous suggestions, leading to a more readable book. We thank Dick and Patti Eidswick for generously allowing us to convert their Colorado condominium into a summer workshop for our project.

Reader/User Comments

We welcome comments from students or faculty members on their experiences with the *Calculus Companion* in learning or teaching calculus. We may be reached by writing to the Department of Mathematics at either of the universities listed on the title page.

To the Student

Calculus and your two HPs

There now exist machines that can work most routine calculus problems quickly and accurately. The HP 48 calculator is such a machine and with calculus is the focus of this book. Sound easy? You might even wonder why bother studying calculus if a machine can do it all. Of course, the answer is that you need to know the *ideas* of calculus before you can effectively use the HP 48. At the same time, these calculators provide an interesting context in which you can learn the ideas of calculus. You will soon see that the HP 48 is a very powerful machine, one that can be teamed with an even more powerful machine: your head! By combining your two HPs—your HP 48 and your Head Power—with what you read in this book, you will find that learning and using calculus is both rewarding and enjoyable.

What this book is and what it is not

First, this book does *not* systematically cover the usual calculus syllabus. For that you need a regular calculus book. You also need a regular instructor. Friendly advice: go to class regularly, listen carefully to what your instructor is trying to teach you, *then* use your calculus book, this book, and the HP 48 to *supplement not replace* your instructor's efforts.

This book is *not* a calculator manual. On the other hand, we have tried to anticipate what you must know about the HP 48 and discuss it as needed. No previous calculator/computer experience is required. The object of the book is to help you understand the concepts and the value of calculus through the use of the HP 48. The HP 48 is a very sophisticated machine, having many other uses besides in calculus. We do *not* attempt to teach you everything there is to know about this machine; nor do we even attempt to teach you everything about it that relates to calculus. We concentrate only on what is needed, introducing it gradually so that you will be able to develop confidence in the machine. For our purposes, the most basic aspects of the HP 48—the "nuts and bolts"—are outlined in Chapter 0.

Beginners are advised to read Chapter 0 carefully with an HP 48 in hand. The ideas and notation introduced there will be used throughout the book. Even if you are an experienced HP 48 user, you should at least scan Chapter 0. If you get impatient and want to learn more or learn faster, read the HP 48 Owner's Manual. This manual is written for a broad readership and contains more information and terminology than we need. Those who want additional breadth and depth concerning the HP 48 may want to consult William C. Wickes' book *HP-48 Insights*. [Larken Publications, Corvallis, Oregon]. For those who want additional challenges, there are a few unusual calculator applications in the (C) exercises of this book.

A basic feature of this book is a collection of problems on both the concepts of calculus and its applications. The problems often require a fair amount of calculation. Without a machine like the HP 48 to do the "dirty work" of number-crunching and symbolic manipulation, taking on such problems would not be practical. Another feature of the book is a collection of short programs which may be used both to doublecheck answers to textbook problems and to shorten certain algorithms. Except for Chapter 0, the book parallels most calculus textbooks in context and order of topics.

HP 48G or HP 48GX?

If you have not yet purchased an HP 48, you may be wondering which machine to get, the HP 48G or the HP 48GX? The X in HP 48GX stands for expandability, which means that the HP 48GX can accommodate certain expansion cards. That, cost, and added memory are the only differences; everything else, including I/O capability, is the same. Expansion cards include memory cards, application cards, and a card for connecting the HP 48 to an overhead projector—none of which is needed to learn calculus. The practical choice for most students will be the HP 48G; for a few—like teachers who want to use it in their instruction and students with special needs—the added expense for the HP 48GX might be worth it.

Friendly advice

Scattered throughout this book you will find occasional boxed-in remarks like the ones shown below. Pay special heed to these; they contain advice that can save you a lot of trouble down the road.

It is important that you use the graphing capabilities of the calculator to supplement rather than bypass what your instructor is trying to teach you.

Leave your calculator in radians mode throughout your study of calculus.

Always look for ways to check the reasonableness of your answer.

KEY TO RPN: Think action key last!

To get out of a jam: PRESS $\boxed{\text{ON}}$

Don't forget to use your other HP!

Timesaving Keystrokes

To do this	Key in this	See page
$2+3$	2 $\boxed{\text{SPC}}$ 3 + or '2 + 3' $\boxed{\text{EVAL}}$	3–5
$\sqrt{25}$	25 $\boxed{\sqrt{x}}$ or '$\boxed{\sqrt{x}}$ 25' $\boxed{\text{EVAL}}$	3–5
2^3	2 3 $\boxed{y^x}$ or '2 $\boxed{y^x}$ 3' $\boxed{\text{EVAL}}$	3–5
graph $y = f(x)$	f(X) $\boxed{\text{↰}}$ $\boxed{\text{EQ}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$	26–27
evaluate $f(x)$ at $x = a$	f(X) a $\boxed{\text{FEVAL}}$	19
write a program	use ≪ ≫	14
calculate $\frac{d}{dx}f(x)$	f(X) X ∂	85
calculate $\int_a^b f(x)dx$	a b f(X) X $\int$ $\boxed{\text{→NUM}}$	200–202

Contents

Chapter 0

HP 48 Nuts & Bolts

0.0 PREVIEW

In this chapter we summarize those capabilities and features of the HP 48 to be used and built upon in this book. We also introduce a small number of short programs (§ 4–8), some of which you may wish to transfer over to your own calculator. You should read this chapter with an HP 48 in hand, but wait until Example 1 before trying it out.

0.1 CALCULATION METHODS

Imagine carrying out the following calculation. How would you do it?

$$\sin(0.68 + 1.2\sqrt{|6 - 4!|})$$

Notice that you would necessarily carry out the operations in the following order:

$4! = 24$
$6 - 24 = -18$
$|-18| = 18$
$\sqrt{18} \approx 4.24264068712$
$1.2 \times 4.24264068712 \approx 5.09116882454$
$0.68 + 5.09116882454 \approx 5.77116882454$
$\sin(5.77116882454) \approx -0.489936127325$

Concerning the calculation, note the following:

(i) The order of calculation is *opposite* from the way the expression is written. The first thing calculated, the factorial, is the last thing to appear in the expression; the last thing calculated is the first thing to appear in the expression.

(ii) The expression involves the following seven functions:

$$\begin{array}{cccc l} \sin & \sqrt{\ } & |\cdot| & ! & \text{(functions of one variable)} \\ + & - & \times & & \text{(functions of two variables)} \end{array}$$

(iii) Commonly used functional notation is not uniform. For example, sin is written *in front of* (i.e., to the left of) its argument, ! is written *behind* its argument, $|\cdot|$ is written *around* its argument, $\sqrt{\ }$ is written *in front of and above* its argument, + and − are written *between* their arguments, and, often, × (for multiplication) isn't even written!

Polish Notation and Reverse Polish Notation

A general purpose notational system, credited to Polish logician, Jan Łukasiweicz, is based on the idea of always writing function names *in front of* their arguments. Thus, $2 + 5$ in this so-called "Polish system" would be written $+(2, 5)$, and the expression above would be written as follows:

$$\sin(+(0.68, \times(1.2, \mathrm{sqrt}(\mathrm{abs}(-(6, \mathrm{fact}(4)))))))$$

Unfortunately, this notation appears strange to people accustomed to conventional notation, and, like the metric system in this country, acceptance does not come easily. Nevertheless, it is a logical, algorithmic way of evaluating expressions. The Polish notation enables us to drop parentheses altogether:

$$\sin\ +\ 0.68\ \times\ 1.2\ \mathrm{sqrt}\ \ \mathrm{abs}\ \ -\ \ 6\ \ \mathrm{fact}\ \ 4$$

The evaluation scheme would be to move from right-to-left, evaluating functions as we come to them. *Reverse Polish notation*, RPN for short, is the left-to-right analogue of the above system. Thus, in RPN function names are always written *after* their arguments. For example, $2 + 5$ in RPN would be $(2, 5)+$, $\sin(\pi/6)$ would be $((\pi, 6)/)\sin$, $\sqrt{7}$ would be $(7)\sqrt{\ }$, etc.. The above expression would then be written as

$$((0.68, (1.2, (((6, (4)\mathrm{fact})-)\mathrm{abs})\mathrm{sqrt})\times)+)\sin$$

As with the Polish system, parentheses can be eliminated:

$$0.68\quad 1.2\quad 6\quad 4\quad \mathrm{fact}\quad -\quad \mathrm{abs}\quad \mathrm{sqrt}\quad \times\quad +\quad \sin \tag{1}$$

To evaluate an expression, read from left-to-right evaluating functions as you come to them, remembering that functions act on numbers to their left. In the above illustration, the first function is *fact*, a function of one variable, so it operates on 4 (the number to the left of it); next comes −, a function of two variables, which operates on the two numbers to its left, 6 and 24 (= 4 *fact*); and so on.

The Three HP 48 Choices: RPN, "Tick" & the EquationWriter

The HP 48 offers a choice of notational systems for calculations. First, there is RPN (discussed above and below). Second, there is the "usual system" where $*$ denotes multiplication, ^ exponentiation, etc. This system is implemented on the HP 48 by putting single quotes ' ' around expressions to be evaluated. We refer to single quotes as "ticks" and to the corresponding evaluation system as the "tick" system.

Finally, there is the EquationWriter system which allows entry of mathematical expressions in close-to-textbook form. This allows you to neglect some multiplication signs (e.g., the EquationWriter understands what you mean when you type $2x - 3y$), enter fractions in the customary horizontal-bar form, make radicals over long complicated expressions, type elevated exponents, enter summation and integration signs with upper and lower limits, etc. The EquationWriter system works well if you don't make mistakes; however, if you *do* make mistakes, correcting them can be annoying and time-consuming.

We suggest that you try out all three notational systems and decide which one suits you best. It is likely that you will find that one system works best in certain situations and another one in other situations. Whatever you decide, we strongly recommend that you take time to practice RPN. It will be time well spent, and the payoff will be less typing, fewer mistakes (caused by misplaced parentheses), and a better understanding and appreciation of HP 48 programming.

Below we illustrate RPN and "tick". You will find additional examples in your Owner's Manual. We leave it to you to learn about the EquationWriter system from your Owner's Manual. In this book, we will make little use of that system.

The key to success in understanding, using, and appreciating RPN is to always think:

$$\boxed{\textit{ACTION KEY LAST!}}$$

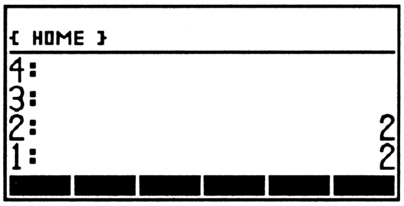
To do $2 + 2$, $+$ is the action, so $+$ comes last (i.e., after 2 2); to do $\sqrt{49}$, $\sqrt{\ }$ is the action, so $\sqrt{\ }$ comes last (i.e., after 49); to do $\sin x$, sin is the action, so sin comes last (i.e., after x); etc..

In the following examples, we assume that you are familiar with the layout of the HP 48 keyboard. The primary keys are shown on the front cover of this book and that's all you need to worry about for now. You will find more detailed diagrams in your Owner's Manual. The table shown on the front cover gives names and locations of 100 special keys. Here, the notation m, n following a key name means that the key is located in the mth row from the top, nth key to the right. For instance, ATAN, designated by 4, 3, is located in the 4th row, 3rd key to the right (in orange letters, next to TAN).

EXAMPLE 1. Use your HP 48 to calculate $2 + 2$ using: (a) RPN; (b) tick notation.

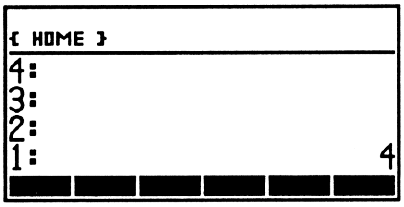
SOLUTION.

(a) $\boxed{2}$ $\boxed{\text{ENTER}}$
$\boxed{2}$ $\boxed{\text{ENTER}}$



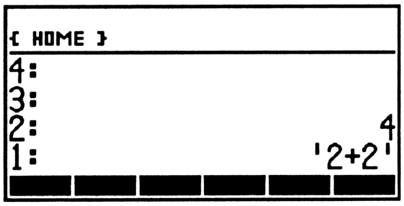
```
{ HOME }
4:
3:
2:                    2
1:                    2
```

+



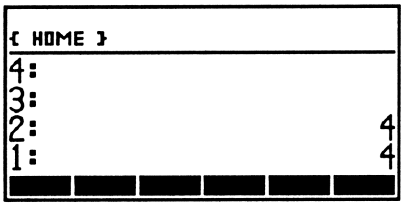
```
{ HOME }
4:
3:
2:
1:                 4
```

(b) ' 2 + 2
ENTER



```
{ HOME }
4:
3:
2:                 4
1:             '2+2'
```

EVAL



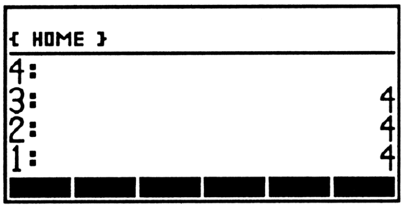
```
{ HOME }
4:
3:
2:                 4
1:                 4
```

✓Points to note

1. Boxed-in words like ENTER and EVAL represent single keystrokes. For key locations, see the front cover of this book.
2. Observe how the right-hand tick mark moves to the right as you enter new data.

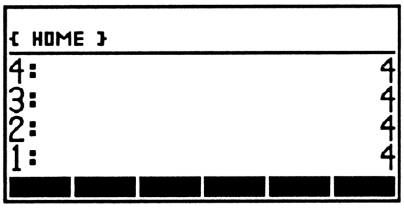
Thanks to user-friendly features of the HP 48, the above solutions can be shortened as follows:

(a') 2 SPC 2
+



```
{ HOME }
4:
3:                 4
2:                 4
1:                 4
```

(b') ' 2 + 2
EVAL



```
{ HOME }
4:                 4
3:                 4
2:                 4
1:                 4
```

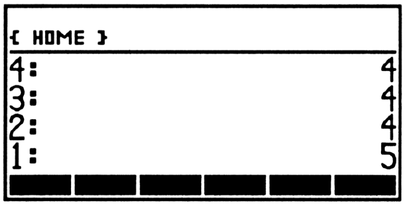
✓Points to note

1. You don't have to bother pressing ENTER or SPC before an action key.
2. When you enter non-action data (like numbers and letters), you can enter them together with spaces in between instead of one-by-one.

EXAMPLE 2. Calculate $\sqrt{16+9}$ using: (a) RPN; (b) tick.

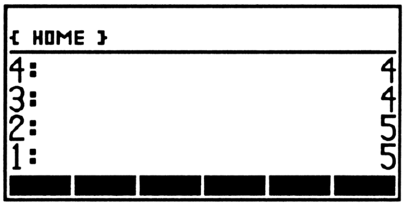
SOLUTION.

(a) $\boxed{1}$ $\boxed{6}$ $\boxed{\text{SPC}}$
$\boxed{9}$ $\boxed{+}$ $\boxed{\sqrt{x}}$



```
{ HOME }
4:                    4
3:                    4
2:                    4
1:                    5
```

(b) $\boxed{'}$ $\boxed{\sqrt{x}}$ $\boxed{↰}$ $\boxed{\div}$
$\boxed{1}$ $\boxed{6}$ $\boxed{+}$ $\boxed{9}$
$\boxed{\text{EVAL}}$



```
{ HOME }
4:                    4
3:                    4
2:                    5
1:                    5
```

✓Points to note

1. The purple and green shift keys, $\boxed{↰}$ and $\boxed{↱}$, allow you to access the expressions printed in purple and green just above the primary keys; e.g., $\boxed{↰}$ $\boxed{\div}$ produces the parentheses ( ). Notice also how the right-hand parenthesis, like the right-hand tick, stays to the right as you enter more data.
2. Observe how the outputs stack up as they are generated. This "stacking" feature of the HP 48 is the topic of the next section.

You can clear the stack by pressing $\boxed{\text{DEL}}$ or by pressing $\boxed{\blacktriangleleft}$ repeatedly. Do this now. In what follows, we assume that your VAR menu is empty (i.e., if you press the VAR key, you should see six blank rectangles at the bottom of the display.)

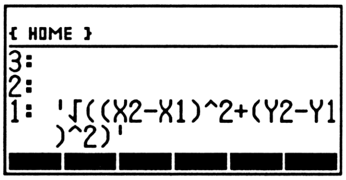
EXAMPLE 3. Use your HP 48 to form the expression

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

using: (a) RPN; (b) tick notation.

SOLUTION.

(a) $\boxed{\alpha}$ $\boxed{1/x}$ $\boxed{2}$
$\boxed{\text{ENTER}}$ $\boxed{\alpha}$ $\boxed{1/x}$
$\boxed{1}$ $\boxed{-}$ $\boxed{2}$ $\boxed{y^x}$
$\boxed{\alpha}$ $\boxed{\pm}$ $\boxed{2}$ $\boxed{\text{ENTER}}$
$\boxed{\alpha}$ $\boxed{\pm}$ $\boxed{1}$ $\boxed{-}$
$\boxed{2}$ $\boxed{y^x}$ $\boxed{+}$ $\boxed{\sqrt{x}}$



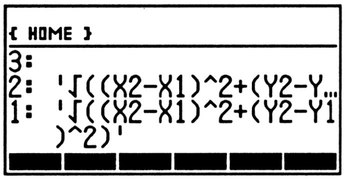
```
{ HOME }
3:
2:
1: '√((X2-X1)^2+(Y2-Y1
   )^2)'
```

✓Points to note

1. The alpha key $\boxed{\alpha}$ (position 6,1) allows you to access the alphabet, e.g., $\boxed{\alpha}$ $\boxed{1/x}$ produces the letter X located just to the right of the $\boxed{1/x}$ key. (Later we will adopt a simpler notational scheme.)
2. If you make a mistake while entering the above input, you can correct it by using the cursor keys $\boxed{\blacktriangleleft}$ $\boxed{\blacktriangleright}$, and the left-delete key $\boxed{\leftarrow}$ (or simply press $\boxed{\text{ON}}$ and start over). For additional details, see the Owner's Manual.
3. If you detect a mistake *after* entering the above input, you can correct it by first pressing $\boxed{\text{EDIT}}$ ($\boxed{\Lsh}$ $\boxed{\pm}$), then correct it using $\boxed{\blacktriangleleft}$, $\boxed{\blacktriangleright}$, and $\boxed{\leftarrow}$. For additional details, see the Owner's Manual.

(b)

$\boxed{'}$	$\boxed{\sqrt{x}}$	$\boxed{\Lsh}$	$\boxed{\div}$
$\boxed{\Lsh}$	$\boxed{\div}$	$\boxed{\alpha}$	$\boxed{1/x}$
$\boxed{2}$	$\boxed{-}$	$\boxed{\alpha}$	$\boxed{1/x}$
$\boxed{1}$	$\boxed{\blacktriangleright}$	$\boxed{y^x}$	$\boxed{2}$
$\boxed{+}$	$\boxed{\Lsh}$	$\boxed{\div}$	$\boxed{\alpha}$
$\boxed{\pm}$	$\boxed{2}$	$\boxed{-}$	$\boxed{\alpha}$
$\boxed{\pm}$	$\boxed{1}$	$\boxed{\blacktriangleright}$	$\boxed{y^x}$
$\boxed{2}$	$\boxed{\text{ENTER}}$		



```
{ HOME }
3:
2: '√((X2-X1)^2+(Y2-Y…
1: '√((X2-X1)^2+(Y2-Y1
    )^2)'
```

Before starting the next example, clear the stack $\boxed{\text{DEL}}$ and set your calculator in *radians mode*. If you see RAD in the upper left-hand corner of your display as shown in the following examples, you're in radians mode; if you don't, you're not. To get in or out of radians mode, press $\boxed{\Lsh}$ $\boxed{\text{MTH}}$.

> **ADVICE:** *leave your calculator in radians mode throughout your study of calculus.*

If you must work with degrees, be sure to return to radians mode when you are done.

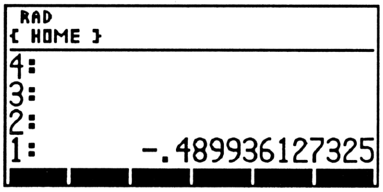
EXAMPLE 4. Use your HP 48 to evaluate the expression

$$\sin(0.68 + 1.2\sqrt{|6 - 4!|})$$

using: (a) RPN; (b) tick notation.

SOLUTION.

(a) $\boxed{\cdot}$ $\boxed{6}$ $\boxed{8}$ $\boxed{\text{ENTER}}$
$\boxed{1}$ $\boxed{\cdot}$ $\boxed{2}$ $\boxed{\text{ENTER}}$
$\boxed{6}$ $\boxed{\text{ENTER}}$ $\boxed{4}$
$\boxed{\text{MTH}}$ $\boxed{\text{NXT}}$ $\boxed{\text{PROB}}$
$\boxed{!}$ $\boxed{-}$ $\boxed{\text{MTH}}$
$\boxed{\text{VECTR}}$ $\boxed{\text{ABS}}$ $\boxed{\sqrt{x}}$
$\boxed{\times}$ $\boxed{+}$ $\boxed{\text{SIN}}$

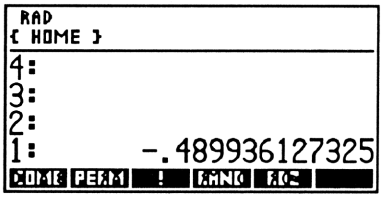


```
RAD
{ HOME }
4:
3:
2:
1:       -.489936127325
```

✓**Point to note**

We have treated the menu names $\boxed{\text{PROB}}$, $\boxed{!}$, $\boxed{\text{VECTR}}$, and $\boxed{\text{ABS}}$ as if they were keys you could press. *What you actually press are the white keys in row 1 directly below those names.*

(b) $\boxed{'}$ $\boxed{\text{SIN}}$ $\boxed{\cdot}$ $\boxed{6}$
$\boxed{8}$ $\boxed{+}$ $\boxed{1}$ $\boxed{\cdot}$ $\boxed{2}$
$\boxed{\times}$ $\boxed{\sqrt{x}}$ $\boxed{\text{MTH}}$
$\boxed{\text{VECTR}}$ $\boxed{\text{ABS}}$ $\boxed{6}$
$\boxed{-}$ $\boxed{4}$ $\boxed{\text{MTH}}$
$\boxed{\text{NXT}}$ $\boxed{\text{PROB}}$ $\boxed{!}$
$\boxed{\text{EVAL}}$



```
RAD
{ HOME }
4:
3:
2:
1:       -.489936127325
```

Exercises 0.1

In Exercises 1–5, write out the exact HP 48 keystrokes needed to carry out each calculation using (a) RPN and (b) tick notation. Shorten the total number of keystrokes whenever possible.

A1. 58×37

A2. $\sqrt{2} + \sqrt{3}$

A3. $2^3 - 3^2$

A4. 2^{3^4}

A5. $\sin(\pi/12) - \cos(\pi/12)$

Use an HP 48 to carry out the indicated calculations as written using (a) RPN and (b) tick notation. In each case, use as few keystrokes as possible.

A6. $\dfrac{1.23 - 4.5}{6.78}$

A7. $\dfrac{1}{1/2 + 1/3}$

A8. $5\sqrt{17}+2^{3.1}$

A9. $(\log 12-\log 10)^{-4}$

A10. $\left(\dfrac{1+\sqrt{5}}{2}\right)^{2}$

A11. $\sin(\sin 1)-\sin^{2} 1$

A12. $\sqrt{2+\sqrt{2+\sqrt{2}}}$

A13. $\sin^{2}(X-\pi)+\cos 2Y+1$

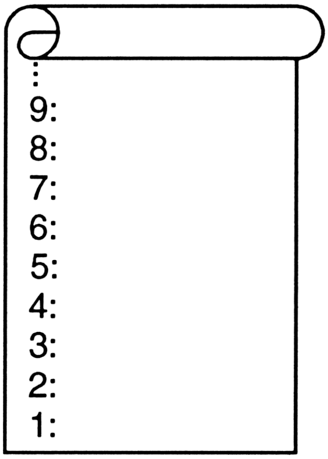
A14. $[A(3-4B)+7((C-4)^{2}+1)]^{3/2}-D^{3/2}$

A15. $3X^{4}-5X^{3}+7X^{2}-2X+4$

A16. $\dfrac{1}{\dfrac{1}{4}+1}\div\left(\dfrac{1}{\dfrac{1}{2}+1}+\dfrac{1}{\dfrac{1}{3}+1}\right)$

0.2 THE STACK

The stack of the HP 48 is like a long scroll where you can see only the bottom 4 lines



For example:

29 $\boxed{\text{ENTER}}$ 8 $\boxed{\text{ENTER}}$
41.7 $\boxed{\text{ENTER}}$ 33 $\boxed{\text{ENTER}}$
85 $\boxed{\text{ENTER}}$

```
4:            8
3:         41.7
2:           33
1:           85
```

You can view any upper portion of the stack by using the $\boxed{▲}$ key.

▲ ▲ ▲ ▲ ▲

```
5:           29
4:            8
3:         41.7
2:           33
```

To move back down, use ▼. To return to normal stack activities, press $\boxed{\text{ON}}$. For all practical purposes, the stack contains an unlimited number of levels starting with level 1 at the bottom and going on up. Except for certain special stack commands (see Owner's Manual), *the stack can be manipulated only from the bottom.* Notice that when a new line is added, it is added to the bottom and everything gets pushed up one level. Similarly, if the bottom line is dropped, everything drops one level. Normally, your concern will be with what's on levels 1 and 2 because, typically, you will operate on these levels with functions of one or two variables, as the following example illustrates.

Start with the above stack configuration and press the "change sign" key $\boxed{+/-}$, a function of one variable. Observe the effect: 85 (the bottom of the stack) is instantly replaced by −85.

```
4:            8
3:         41.7
2:           33
1:           85
```

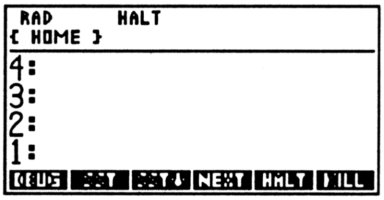
$\xrightarrow{\boxed{+/-}}$

```
4:            8
3:         41.7
2:           33
1:          -85
```

The following example illustrates how RPN and the stack work together. It also illustrates the SST (single-step) feature of the calculator, which proves to be a valuable programming aid.

EXAMPLE. Carefully key in the following expression exactly as you see it. If you have trouble doing this, see ✓Points to note below.

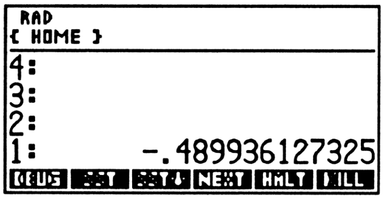
≪ HALT .68 1.2 6 4
! − ABS √ *
\+ SIN $\boxed{\text{ENTER}}$
$\boxed{\text{EVAL}}$ $\boxed{\text{PRG}}$ $\boxed{\text{NXT}}$ $\boxed{\text{RUN}}$



```
RAD      HALT
{ HOME }
4:
3:
2:
1:
```

$\underbrace{\boxed{\text{SST}}\cdots\boxed{\text{SST}}}_{\text{13 times}}$

You will see the calculation being carried out step-by-step.



```
RAD
{ HOME }
4:
3:
2:
1:      -.489936127325
```

✓Points to note

1. The French quotes ≪ ≫, called “program delimiters”, are located next to $\boxed{-}$ (in purple). You can access them by pressing $\boxed{↰}$ then $\boxed{-}$. As with ' ' and ( ) you get both of the delimiters ≪ ≫ at the same time and the right-hand one ≫ will move to the right as you key in the rest of the expression.
2. Spacing is important. Use the $\boxed{\text{SPC}}$ key.
3. You may key in the words like HALT and ABS (even SIN) either letter-by-letter using the α key or all at once by accessing the appropriate menu. For a complete set of menu names, see the back of your Owner’s Manual. Once you learn your way around the menus, you will find that taking words off of a menu is a little easier than keying them in letter-by-letter. Also, you will probably find that the easiest way to key in text is to *press and hold down the* α *key* as you do it. For example, here are two ways of entering the word HALT:

   INPUT: $\boxed{\text{PRG}}$ $\boxed{\text{NXT}}$ $\boxed{\text{RUN}}$ $\boxed{\text{HALT}}$
   OUTPUT: HALT

   INPUT: $\boxed{\alpha}$ (hold down) $\boxed{\text{H}}$ $\boxed{\text{A}}$ $\boxed{\text{L}}$ $\boxed{\text{T}}$ (release α key)
   OUTPUT: HALT

4. Key in ! by pressing $\boxed{\text{MTH}}$ $\boxed{\text{NXT}}$ $\boxed{\text{PROB}}$ $\boxed{!}$, ABS by pressing $\boxed{\text{MTH}}$ $\boxed{\text{VECTR}}$ $\boxed{\text{ABS}}$, $\sqrt{\ }$ by pressing $\boxed{\sqrt{x}}$, and ∗ by pressing $\boxed{\times}$.

Exercises 0.2

In Exercises 1–4, predict the outcome of applying the indicated operations to the given stack configuration. Use your calculator to check your conclusions.

A1.

```
4:
3:                    66
2:                    73
1:                   −12
```

(a) $\boxed{+}$ $\boxed{-}$
(b) $\boxed{-}$ $\boxed{+}$
(c) $\boxed{+}$ $\boxed{+}$ $\boxed{+}$
(d) $\boxed{+/-}$ $\boxed{+/-}$
(e) $\boxed{+/-}$ $\boxed{-}$ $\boxed{-}$

A2.

```
4:                 −97.6
3:                    33
2:                    18
1:                    15
```

(a) $\boxed{+}$ $\boxed{-}$ $\boxed{\times}$
(b) $\boxed{+}$ $\boxed{-}$ $\boxed{\div}$
(c) $\boxed{+}$ $\boxed{-}$ $\boxed{\text{SWAP}}$ $\boxed{\div}$
(d) $\boxed{+}$ $\boxed{-}$ $\boxed{y^x}$
[Note that $\boxed{\text{SWAP}}$ = $\boxed{▶}$]

A3.

```
4:
3:                     0
2:                  1.79
1:                  1.79
```

(a) $\boxed{\text{SIN}}$ $\boxed{x^2}$ $\boxed{\text{SWAP}}$ $\boxed{\text{COS}}$ $\boxed{x^2}$ $\boxed{+}$ $\boxed{\text{SWAP}}$ $\boxed{\text{COS}}$ $\boxed{-}$
(b) $\boxed{\text{TAN}}$ $\boxed{x^2}$ $\boxed{\text{SWAP}}$ $\boxed{\text{COS}}$ $\boxed{x^2}$ $\boxed{1/x}$ $\boxed{-}$ $\boxed{\text{SWAP}}$ $\boxed{\text{COS}}$ $\boxed{+}$
[Note that $\boxed{x^2}$ = $\boxed{↰}$ $\boxed{\sqrt{x}}$]

A4.

4: 2
3: 3
2: 4
1: 5

(a) $\boxed{\times}$ $\boxed{\times}$ $\boxed{\times}$ $\boxed{4}$ $\boxed{\ !\ }$ $\boxed{\div}$
(b) $\boxed{\times}$ $\boxed{\times}$ $\boxed{\times}$ $\boxed{5}$ $\boxed{\ !\ }$ $\boxed{-}$
(c) $\boxed{\ !\ }$ $\boxed{\ !\ }$

A5.

4:
3: 3
2: 2
1: 1

(a) $\boxed{y^x}$ $\boxed{y^x}$
(b) $\boxed{\text{SWAP}}$ $\boxed{y^x}$ $\boxed{y^x}$
(c) $\boxed{y^x}$ $\boxed{\text{SWAP}}$ $\boxed{y^x}$
(d) $\boxed{\text{SWAP}}$ $\boxed{y^x}$ $\boxed{\text{SWAP}}$ $\boxed{y^x}$
(e) $\boxed{y^x}$ $\boxed{y^x}$ $\boxed{\text{SWAP}}$

A6. Suppose that -1 is on line 1 of the stack. Predict the effect of applying the following sequences of keystrokes:

(a) $\boxed{x^2}$ $\boxed{\sqrt{x}}$
(b) $\boxed{\sqrt{x}}$ $\boxed{x^2}$
(c) $\boxed{1/x}$ $\boxed{1/x}$
(d) $\boxed{x^2}$ $\boxed{1/x}$
(e) $\boxed{\text{SIN}}$ $\boxed{\text{ASIN}}$
(f) $\boxed{\text{ASIN}}$ $\boxed{\text{SIN}}$

A7. Repeat Exercise A.6 with -2 on line 1 of the stack. Explain the difference.

A8. Suppose that for each $i = 1, \cdots, 10$, the number i is on the ith level of the stack. What would happen if you pressed

(a) $\boxed{\times}$ 9 times?
(b) $\boxed{+}$ 9 times?

B1. Predict the outcome of applying the following sequences of keystrokes. What can you deduce in general?

(a) $\boxed{2}$ $\boxed{\text{SPC}}$ $\boxed{2}$ $\boxed{\text{SPC}}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{y^x}$
(b) $\boxed{2}$ $\boxed{\text{SPC}}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{2}$ $\boxed{y^x}$
(c) $\boxed{3}$ $\boxed{\text{SPC}}$ $\boxed{3}$ $\boxed{\text{SPC}}$ $\boxed{3}$ $\boxed{y^x}$ $\boxed{y^x}$
(d) $\boxed{3}$ $\boxed{\text{SPC}}$ $\boxed{3}$ $\boxed{y^x}$ $\boxed{3}$ $\boxed{y^x}$

B2. (Compare with A.8.) Suppose that for each $i = 1, \cdots, n$, the number i is on the ith level of the stack. What would happen if you pressed

(a) $\boxed{\times}$ $n-1$ times?
(b) $\boxed{+}$ $n-1$ times?

0.3 THE KEYS; NOTATION TO BE USED IN THIS BOOK

Several-Keys-in-One Principle

Each of the 49 keys of the HP 48 acts as several keys in one. Principal uses are shown at the beginning of this book (see inside front cover).

To get the idea of the several-keys-in-one principle, consider the $\boxed{\text{SIN}}$ key of the HP 48. This key serves six different purposes, which are distinguished through the use of the alpha key $\boxed{\alpha}$, the left-shift key $\boxed{↰}$, and the right-shift key $\boxed{↱}$. Table 1 shows how it works.

Table 1

key name	keystrokes
SIN	$\boxed{\text{SIN}}$
ASIN (purple)	$\boxed{\text{↰}}$ $\boxed{\text{SIN}}$
∂ (green)	$\boxed{\text{↱}}$ $\boxed{\text{SIN}}$
S	$\boxed{\alpha}$ $\boxed{\text{SIN}}$
s	$\boxed{\alpha}$ $\boxed{\text{↰}}$ $\boxed{\text{SIN}}$
σ	$\boxed{\alpha}$ $\boxed{\text{↱}}$ $\boxed{\text{SIN}}$

Notice that ASIN and ∂ are color-coded and printed on the appropriate sides of the SIN key so it's easy to keep things straight.

Other keys have similar multiple uses. You will find a complete list of keys and their uses at the end of your HP 48 Manual and you may find it useful to scan through that list. As we mentioned earlier, there is far more here than what is needed for calculus and you shouldn't try to learn everything.

Henceforth, we will refer to multiple use keys as if they were single keys independent of the others. For example, log(arctan(x)) in RPN will take the form:

$$\boxed{\text{X}} \quad \boxed{\text{ATAN}} \quad \boxed{\text{LOG}}$$

instead of

$$\boxed{\alpha} \quad \boxed{1/x} \quad \boxed{\text{↰}} \quad \boxed{\text{TAN}} \quad \boxed{\text{↱}} \quad \boxed{y^x}$$

Classification of Keys

For our purposes the HP 48 keys can be divided into two categories:

I. those that play a direct role in calculations and programs, and
II. those that don't.

Category I keys can be further divided into *action keys* and *non-action keys.*

Non-action keys include:

1. Number keys: $\boxed{0}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ $\boxed{7}$ $\boxed{8}$ $\boxed{9}$ $\boxed{\pi}$ $\boxed{\text{e}}$ $\boxed{\text{i}}$
2. Letter keys: $\boxed{\text{A}}$ $\boxed{\text{B}}$ $\cdots$ $\boxed{\text{Z}}$ $\boxed{\text{a}}$ $\boxed{\text{b}}$ $\cdots$ $\boxed{\text{z}}$ $\boxed{\alpha}$ $\boxed{\beta}$ $\boxed{\gamma}$ $\cdots$
3. Punctuation keys: $\boxed{(}$ $\boxed{)}$ $\boxed{[}$ $\boxed{]}$ $\boxed{\{}$ $\boxed{\}}$ $\boxed{\ll}$ $\boxed{\gg}$ $\boxed{\rightarrow}$ $\boxed{\text{"}}$ $\boxed{\text{'}}$ $\boxed{,}$
4. A few others: $\boxed{\#}$ $\boxed{\%}$ $\boxed{\measuredangle}$ $\boxed{\bar{x}}$ $\cdots$

Action keys include:

1. Operation keys: $\boxed{+}$ $\boxed{\times}$ $\boxed{-}$ $\boxed{\div}$ $\boxed{y^x}$ $\boxed{x^2}$ $\boxed{1/x}$ $\boxed{\sqrt{x}}$ $\boxed{\int}$ $\boxed{\partial}$ $\boxed{\Sigma}$ $\boxed{\pm}$
2. Relation keys: $\boxed{=}$ $\boxed{\neq}$ $\boxed{<}$ $\boxed{>}$ $\boxed{\leq}$ $\boxed{\geq}$
3. Special function keys: $\boxed{\text{SIN}}$ $\boxed{\text{COS}}$ $\boxed{\text{TAN}}$ $\boxed{\text{LN}}$ $\boxed{e^x}$ $\boxed{\text{ATAN}}$ $\boxed{\text{LOG}}$ $\boxed{10^x}$
4. Special command keys: $\boxed{\text{STO}}$ $\boxed{\text{EVAL}}$ $\boxed{\text{RCL}}$ $\boxed{\text{DROP}}$ $\boxed{\text{SWAP}}$ $\boxed{\rightarrow\text{NUM}}$ $\boxed{\text{PURG}}$

5. Menu keys (functions and commands): $\boxed{\text{ABS}}$ $\boxed{!}$ $\boxed{\text{MIN}}$ $\boxed{\text{CROSS}}$ $\boxed{\text{DOT}}$ $\boxed{\text{DET}}$ $\boxed{\text{BEEP}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$ $\boxed{\text{CLLCD}}$ $\boxed{\text{R}\rightarrow\text{C}}$ $\boxed{\rightarrow\text{LCD}}$ $\boxed{\text{PIXON}}$ $\boxed{\text{FREEZ}}$ $\boxed{\text{ROT}}$ $\boxed{\text{DUP}}$ $\boxed{\text{TAYLR}}$ $\boxed{\text{AND}}$ $\boxed{\text{OR}}$
6. Combinations: $\boxed{\text{IF}}\boxed{\text{THEN}}\boxed{\text{ELSE}}$ $\boxed{\text{FOR}}\boxed{\text{NEXT}}$ $\boxed{\text{FOR}}\boxed{\text{STEP}}$ $\boxed{\text{DO}}\boxed{\text{UNTIL}}\boxed{\text{END}}$

Category II keys are also of two types:

1. Menu and directory names: $\boxed{\text{MTH}}$ $\boxed{\text{PRG}}$ $\boxed{\text{MEMORY}}$ $\boxed{\text{PLOT}}$ $\boxed{\text{MODES}}$ $\boxed{\text{SYMBOLIC}}$ $\boxed{\text{CHARS}}$ $\boxed{\text{STACK}}$ $\boxed{\text{TEST}}$ $\boxed{\text{GROB}}$ $\boxed{\text{RUN}}$ $\boxed{\text{MATR}}$ $\boxed{\text{PROB}}$ ···
2. Calculator operation keys: $\boxed{\text{ENTER}}$ $\boxed{\text{ON}}$ $\boxed{\text{EDIT}}$ $\boxed{\text{OFF}}$ $\boxed{\text{DEL}}$ $\boxed{\text{NXT}}$ $\boxed{\text{PREV}}$ $\boxed{\text{CLEAR}}$ $\boxed{\leftarrow}$ $\boxed{\text{SPC}}$ $\boxed{\blacktriangle}$ $\boxed{\blacktriangledown}$ $\boxed{\blacktriangleleft}$ $\boxed{\blacktriangleright}$ ···

For a complete list of operations, see the back of your HP 48 Manual.

Notation Used in This Book

When describing calculations or programs, we will normally *not* write Category II keys, often eliminate boxes around names and symbols, and tend to use the program notation of Table 2.

Table 2

Preferred notation	instead of
*	×
/	÷
^	y^x
$\sqrt{\ }$	$\sqrt{x}$
INV	$1/x$
SQ	x^2
EXP	e^x

For example, we would write:

$$\pi \quad R \quad SQ \quad *$$

instead of

$$\boxed{\Lsh} \ \boxed{\text{SPC}} \ R \ \boxed{\Lsh} \ \boxed{\sqrt{x}} \ \boxed{\times},$$

and the calculation

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

would be written

$$\text{X2 X1} - \text{SQ Y2 Y1} - \text{SQ} + \sqrt{\ } \qquad (1)$$

in RPN and as

$$'\sqrt{\ }(\text{SQ(X2}-\text{X1)}+\text{SQ(Y2}-\text{Y1))}'$$

in tick notation.

Boxed-in words and symbols will always represent single keystrokes; e.g., ENTER refers to the ENTER key, *not* E N T E R . And we will generally *not* break numerals or words into single keystrokes; e.g.,we would write

123 ELM STREET

instead of

1 2 3 SPC E L M SPC S T R E E T

0.4 SIMPLE PROGRAMMING

A program on the HP 48 is any expression of the form ≪ ··· ≫. Here's a simple and useful example. Carefully key in the following:

```
≪ CLLCD "···your name···" 1 DISP "···your address···"
  2 DISP "···rest of address···" 3 DISP
  "···your phone number···" 4 DISP 7 FREEZE ≫
```

Note that quotation marks *are* to be keyed in and that ···*your name*···, for instance, is to be replaced by your actual name—limited to 22 characters including spaces.

Next press ENTER .

If you didn't make a mistake keying in the program, the HP 48 will put the program on line 1 of the stack (but you won't see it all); if you *did* make a mistake, you can easily correct it using the keys marked:

◀ DEL ▲ ▼ ◀ ▶

You can probably guess how to use these keys, but if you need details see the Owner's Manual under the topic of *editing*. Be warned that one mistake often leads to others. If you get hopelessly tangled in mistakes, just press ON (which serves as an escape key) and CLEAR (DEL) to start over.

To get out of a jam:
PRESS ON

✓Points to note

1. CLLCD, DISP, and FREEZE are commands (hence, action keys) that stand for "clear liquid crystal display", "display", and "freeze", respectively. You can easily access all three of these commands by pressing PRG NXT OUT . The command CLLCD tells the calculator to get rid of all displayed information like stack numbers, messages, the horizontal line near the top, and any other information in the display area. (However, it will *not* get rid of the menu.) DISP *follows* its arguments. Remember, this is the nature of RPN. Thus, "···*your name*···" 1 DISP means: display ···*your name*··· on line 1 (the top line in this context); "···*your address*···" 2 DISP means: display ···*your address*··· on line 2; etc..
2. When you key in a program like the one above you can either do it as a single line (which may scroll off the display) or in multiple lines using ↩ (↱ ·).
3. You are limited to eight display lines (so 9 DISP, e.g., has no meaning) and, as mentioned above, 22 characters per line.

Next step: *Store* your program in memory under the name ID by keying in:

ID STO

When you now press VAR, you'll see ID in the left-most rectangle of that menu and when you press the white key under ID you'll see:

```
YOUR NAME
YOUR ADDRESS
REST OF ADDRESS
YOUR PHONE NO.
```

✓Points to note

1. Notice that the quotes you put around ···*your name* ···, etc. don't appear on the display.
2. If you don't like the name ID, call it something else. The HP 48 allows for a great variety of menu names (which Hewlett-Packard calls *global variables*). As examples, you could name the above program any of the following: $\mu\beta3\delta$ or oWnEr or ME.MYSELF.AND.I.
3. Instead of ID STO, you could have pressed 'ID' STO; in fact, that is what is suggested by the HP 48 manual. However, as a safeguard against accidentally replacing the contents of a name already in use, we recommend that you enter new names *without tick marks.* An exception to this rule is when you *want* to replace the contents of a menu name with new contents. (This is often the situation when names are introduced *inside of programs.*)

About Programming Formulas

Among the simplest programs are those for putting *formulas* on your calculator. In a typical situation, the formula will consist of an expression E with variables v_1, v_2,..., v_n. To program it, carry out the following steps:

Step 1. Introduce variable symbols, say V1, V2, ··· , VN, corresponding to v_1, v_2, ..., v_n that make sense to the calculator (e.g., that don't involve subscripts).

Step 2. Use RPN or tick notation to formulate the calculation of E in terms of the variable symbols introduced.

Step 3. Key in the following program:

```
« → V1 V2 ··· VN ************ »,
```

where ************ is to be replaced by « E » or 'E', depending on whether E is in RPN or tick notation. Variable symbols like V1, V2, ···, VN, introduced by → , are called *local variables.*

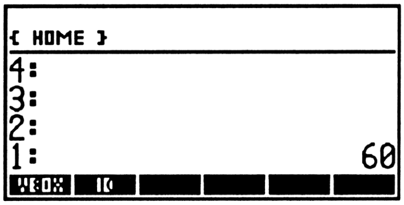
To enter the program, press ENTER; to save the program, type the name of your choice then STO; to run the program, enter the values of V1, V2, ···, VN on the stack or on the command line in that order. The simplest way is to separate them by spaces on the command line.

EXAMPLE 1. (a) Write a program for finding the volume of a box of length l, width w, and height h; (b) use the program to find the volume of a $3 \times 4 \times 5$ box.

SOLUTION.

(a) Enter ≪ → L W H ≪ L W H ∗ ∗ ≫ ≫
or ≪ → L W H 'L∗W∗H' ≫
and name it VBOX: VBOX $\boxed{\text{STO}}$

(b) 3 4 5 $\boxed{\text{VBOX}}$



```
{ HOME }
4:
3:
2:
1:                  60
```

✓Points to note

1. The HP 48 automatically puts spaces around action keys like → and ∗.
2. You may want to use lowercase letters for local variables as is suggested by the Owner's Manual, but uppercase letters require less typing.
3. The above syntax is quite rigid. Introduced local variables must always be preceded by → (with spaces around it) and either ≪ or ' must always follow the last local variable.

A more useful example is the *distance formula*. Using (1) of 0.3, we obtain the following program:

≪ → X1 Y1 X2 Y2 ≪ X2 X1 − SQ

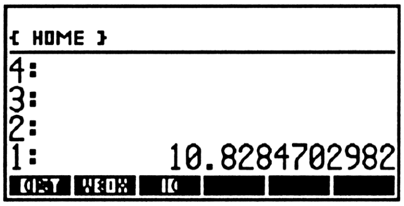
Y2 Y1 − SQ + √ ≫ ≫

Store this under the name DIST: DIST $\boxed{\text{STO}}$.

EXAMPLE 2. Use DIST to find the distance between $(2.8, 4.56)$ and $(-7.1, 0.173)$.

SOLUTION.

2.8 4.56 −7.1 .173
$\boxed{\text{DIST}}$



```
{ HOME }
4:
3:
2:
1:       10.8284702982
```

✓Point to note

To enter negative numbers, use $\boxed{+/-}$. For example, to enter −7.1, first enter 7.1 then $\boxed{+/-}$.

About Avoiding Errors

When you use a program like DIST, make sure your other HP (Head Power) is fully engaged at the time. It's like applying the distance formula itself; you've got to know what the symbols mean. For instance, look what would happen if you got mixed up and entered the x-coordinates first, then the y-coordinates:

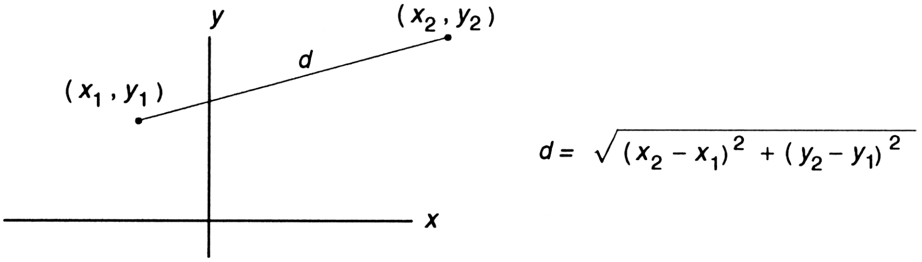
2.8 − 7.1 4.56 .173 DIST

This would give you a wrong answer: 7.48292249058.

In this book we will help you guard against making such mistakes by presenting programs in a two-column format, making it clear just what goes in and what goes out — plus providing a brief description of what the program steps mean. We will sometimes also include "Checksum" and "Bytes" as a check against typing errors. You can check these by putting the name of the program on line 1 of the stack and pressing BYTES in the ↰ MEMORY menu (yes, we do mean ↰). See the program box labeled DIST. In addition, it will help if you can formulate a clear geometric picture in your mind and on paper as to exactly what the program/formula is supposed to accomplish. For example, see Fig. 2.

DIST

Inputs: x_1, y_1, x_2, y_2	*Output: Distance between* (x_1,y_1) *and* (x_2,y_2)
« → X1 Y1 X2 Y2 « X2 X1 − SQ Y2 Y1 − SQ + √ » »	Introduce local variables Calculate the distance using reverse Polish notation Checksum: #19764d Bytes: 85



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Figure 2

Here is another tip for avoiding errors:

> *Always look for ways to check the reasonableness of your answer.*

The above DIST error could have been caught by noting that the distance sought is approximately equal to the distance between $(3,4)$ and $(-7,0)$ which by hand calculation is $\sqrt{116}$—obviously bigger than 10.

Formulas containing formulas are also simple to program. Mainly you must be careful to follow the syntax rules for introducing local variables as described in point (3), Example 1. As an example, consider Heron's formula for the area of a triangle with sides a, b, c.

$$AREA = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semiperimeter

$$s = \frac{1}{2}(a+b+c).$$

Here, the inputs are a, b, and c, so we use local variables A, B, and C. The only question is what to do about s? One way would be to eliminate s altogether, replacing it by $\frac{1}{2}(a+b+c)$. A better way is to introduce a fourth local variable S as is illustrated in the program in the box labeled HERON. Note that to store this program under the name HERON, enter HERON then press $\boxed{\text{STO}}$.

HERON

Inputs: *a* , *b* , *c*	Output: *Area of triangle with sides a , b , c*

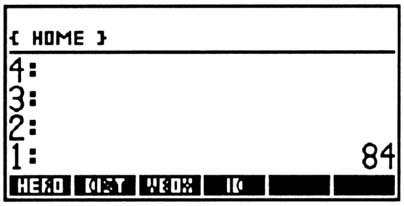
```
« → A B C «                    Introduce local variables A,B,C
    A B C + + 2 /              Calculate semiperimeter
    → S «                      Introduce local variable S
    S S A − S B −              Calculate Heron's formula
    S C − * * * √                using reverse Polish notation
    » » »
```

Checksum: #43697d Bytes: 125

EXAMPLE 3. Find the area of a triangle with side lengths 13, 14, 15.

SOLUTION.

13 14 15 $\boxed{\text{HERON}}$



```
{ HOME }
4:
3:
2:
1:                    84
```

About Function Evaluation

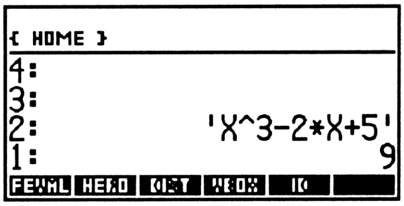
How can we get the machine to evaluate $f(x) = x^3 - 2x + 5$ at $x = 2$? We discuss three methods of doing this and compare their merits.

FEVAL

Inputs: $f(X)$, a	*Output:* $f(X)$, $f(a)$
`« 'X' STO`	Stores a under the name X
`DUP`	Makes a second copy of $f(X)$
`EVAL`	Evaluates f at a
`'X' PURGE »`	Purges X from the VAR menu
	Checksum: #47353d Bytes: 48.5

1. *The FEVAL method.* First enter and store the program FEVAL (see program box). Then all you have to do is enter the function and the number you want to evaluate it at.

X 3 ^ 2 X
* − 5 + 2
$\boxed{\text{FEVAL}}$



```
{ HOME }
4:
3:
2:                'X^3-2*X+5'
1:                          9
```

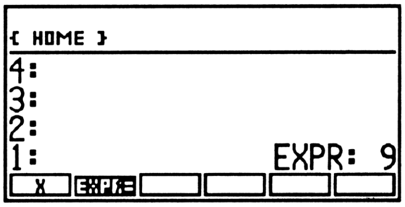
This method is easy to apply and does not clutter the VAR menu with unwanted symbols. To reapply the method to the same function and a different value of a, you need only DROP the old value off the stack before entering the new value. A slight disadvantage of this method is that you have to use X as the independent variable (however, see Exercise B.4).

2. *The SOLVE method.* First key in the function X 3 ^ 2 X * − 5 + Then press $\boxed{↰}$ $\boxed{\text{SOLVE}}$ ($\boxed{↰}$ 7) $\boxed{\text{ROOT}}$ $\boxed{↰}$ $\boxed{\text{EQ}}$ $\boxed{\text{SOLVR}}$ to store the function under EQ and obtain the following menu:

```
EQ: 'X ^ 3 - 2 * X + 5'
4:
3:
2:
1:
   X |EXPR=|    |    |    |
```

Now use the white keys under X and EXPR= to evaluate $f(x)$ at any number.

2 $\boxed{\text{X}}$ $\boxed{\text{EXPR=}}$

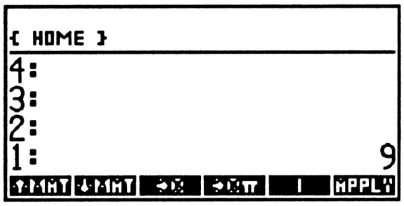


```
{ HOME }
4:
3:
2:
1:                   EXPR: 9
```

One advantage of this method is that it easily extends to functions of several variables. One drawback is that it leaves variables on the VAR menu (and that can sometimes be a big nuisance). To delete X from the VAR menu, first press $\boxed{\text{VAR}}$, then enter 'X' $\boxed{\text{PURG}}$ ($\boxed{↰}$ $\boxed{\text{EEX}}$).

3. *The WHERE method.* First key in the function X 3 ^ 2 X $*$ − 5 +. (If you went through the SOLVE example above, pressing $\boxed{\text{EQ}}$ will do this for you.) Then enter in braces { } ($\boxed{↰}$ +) the variable X and the number at which you want to evaluate the function. Then press the "where" key $\boxed{|}$ which you will find in the $\boxed{\text{SYMBOLIC}}$ directory ($\boxed{↰}$ 9 $\boxed{\text{NXT}}$).

{X 2} $\boxed{|}$
$\boxed{\text{ENTER}}$



```
{ HOME }
4:
3:
2:
1:                    9
```

Dialog Boxes

The HP 48 uses dialog boxes or "input forms" to make the calculator more friendly for inexperienced users. For example, when you press $\boxed{↱}$ $\boxed{\text{SOLVE}}$, you get a dialog box asking what you want to solve. When you then select "solve equation", you get another dialog box asking you to fill in blanks for EQ and X. Finally, pressing $\boxed{\text{EXPR=}}$ puts the desired result on the stack. While dialog boxes can be quite helpful—especially for beginners, we will make little use of them in this book because they require extra keystrokes. If you've made it this far, you are no longer a "beginner" and you will appreciate knowing how to take short-cuts.

For keys with green names over them like $\boxed{\text{SOLVE}}$ (green 7), you can bypass dialog boxes by using $\boxed{↰}$ (purple) instead of the $\boxed{↱}$ (green). In this way, you may access command menus directly and thereby, in general, cut down on keystrokes. For example, in method 2 above, we used $\boxed{↰}$ $\boxed{\text{SOLVE}}$ instead of $\boxed{↱}$ $\boxed{\text{SOLVE}}$ (even though $\boxed{\text{SOLVE}}$ is colored green). Similarly, in method 3, we used $\boxed{↰}$ $\boxed{\text{SYMBOLIC}}$ (the 9 key) instead of $\boxed{↱}$ $\boxed{\text{SYMBOLIC}}$ (a green key).

SHORT-CUT: *for keys with green names, use* $\boxed{↰}$ *(purple) instead of* $\boxed{↱}$ *(green).*

We will use such short-cuts regularly throughout this book.

For further information about dialog boxes, see your Owner's Manual under "input forms".

Exercises 0.4

For Exercises 1–15, write an HP 48 program to produce the indicated output from the given input.

A1. Input: x_1, y_1, x_2, y_2
Output: midpoint of line segment joining (x_1, y_1) and (x_2, y_2).
[*Hint*: a b R→C yields (a, b).]

A2. Input: $(x_1, y_1), (x_2, y_2)$
Output: midpoint of line segment joining (x_1, y_1) and (x_2, y_2).
[*Hint*: arithmetic operations affect pairs just as you might guess; e.g., 5 (1, 2) * yields (5, 10) and (1, 2) (3, 4) + yields (4, 6).]

A3. Input: x_1, y_1, x_2, y_2
Output: slope of the line segment joining (x_1, y_1) and (x_2, y_2).

A4. Input: (x_1, y_1), (x_2, y_2) where $x_1 \neq x_2$
Output: slope of the line segment joining (x_1, y_1) and (x_2, y_2).
[*Hint*: (a, b) C→R yields a b]

A5. Input: m, b
Output: equation of line with slope m and y-intercept b.
[*Hint*: S T = yields S = T]

A6. Input: m, a
Output: equation of line with slope m and x-intercept a.

A7. Input: a, b, m
Output: equation of line through (a, b) with slope m.
[*Suggestion*: store this program under the name PTSL (for point-slope).]

A8. Input: x_1, y_1, x_2, y_2 with $x_1 \neq x_2$
Output: equation of the line through (x_1, y_1) and (x_2, y_2)
[*Suggestion*: store this program under the name SLIN (for slanted line).]

A9. Input: a
Output: equation of the vertical line through $(a, 0)$.
[*Suggestion*: store this program under the name VLIN (for vertical line).]

A10. Input: x_1, y_1, x_2, y_2
Output: equation of the line through (x_1, y_1) and (x_2, y_2)
[*Hint*: use IF THEN ELSE END together with SLIN and VLIN from A.8 and A.9.]

A11. Input: x_1, y_1, x_2, y_2 with $y_1 \neq y_2$
Output: equation of perpendicular bisector of the line segment joining (x_1, y_1) and (x_2, y_2).

[*Hint*: use PTSL from A.7.]

A12. Input: a, b, $b \neq 0$
Output: equation of line through the origin parallel to the line $ax + by = 1$.

A13. Input: l, w, h
Output: surface area of the box with dimensions $l \times w \times h$.

A14. Input: r
Output: area and circumference of circle with radius r.

A15. Input: b, h
Output: area of parallelogram with base b and height h.

A16. Use FEVAL to calculate the function $\tan^{-1} x^2$ at the points 0, 1, 100, 10000 and 1000000.

A17. Repeat Exercise A.16 using a user-defined function F. When you are finished, purge F: 'F' $\boxed{\text{PURG}}$.

A18. Repeat Exercise A.16 using SOLVR. When you are finished, purge X: 'X' $\boxed{\text{PURG}}$.

A19. Use FEVAL to evaluate the function

$$f(x) = \frac{x + \log x}{\sqrt{1 + x^2}}$$

at $x = 1$, 10, 1000, 000000.

A20. Repeat Exercise A.19 using WHERE.

A21. Repeat Exercise A.19 using SOLVR. When you are finished, purge X: 'X' $\boxed{\text{PURG}}$.

For Exercises 1–2, write an HP 48 program to produce the indicated output from the given input.

B1. Input: a, b
Output: equation of the line through (a, b) which is tangent to the circle $x^2 + y^2 = a^2 + b^2$.

B2. Input: a, b, c, $a \neq 0$
Output: solutions to $ax^2 + bx + c = 0$

B3. Write an HP 48 program that will produce the vertex of the parabola $y = \mathrm{A}x^2 + \mathrm{B}x + \mathrm{C}$ where $\mathrm{A} \neq 0$.

B4. Write an HP 48 program that will take as input an arbitrary function f (var), an arbitrary variable symbol `'var'`, and an arbitrary number in the domain of f to produce as output the value $f(a)$.

0.5 ORGANIZING AND CHANGING THINGS

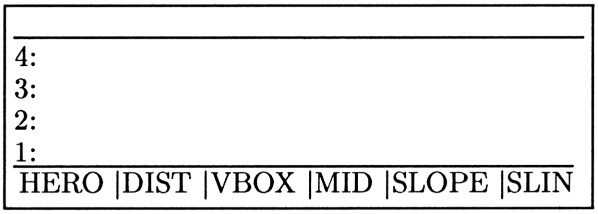
If you worked through the last section with an HP in hand, your VAR menu now contains names like these:

FEVAL	X	EQ	F	SLIN
HERO	DIST	VBOX	ID	

If we were to go on, adding more and more names, it wouldn't be long before confusion would set in. One way around this problem is to use *subdirectories*. We will restrict discussion to a two-level structure: (1) the HOME directory—the one that presently contains ID, DIST, FEVAL, etc.; and (2) a set of subdirectories immediately below the HOME directory. The reader interested in knowing about a general multi-level structure is referred to the HP 48 Owner's Manual.

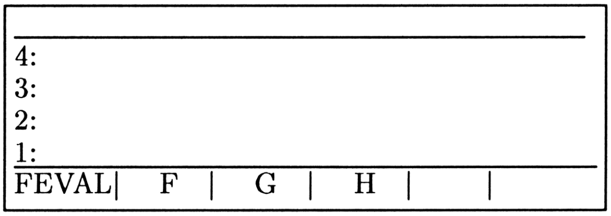
The idea of directories is simple: you put like things together and unlike things apart. For example, you might want to put together all of your formula programs in a directory named FORMS, all function-related stuff in a directory named FUN, all personal stuff in PERS, addresses of friends in ADD, etc. As you will see, it's easy to create directories, and, after learning a few tricks, you'll be able change things around at will.

Your subdirectories might then look like this:



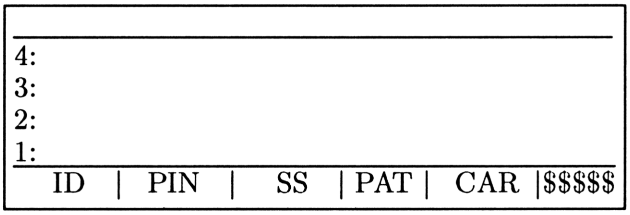
```
4:
3:
2:
1:
HERO |DIST |VBOX |MID |SLOPE |SLIN
```

FORMS



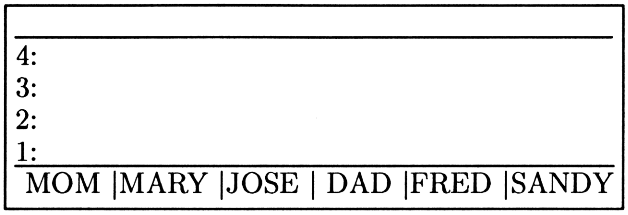
```
4:
3:
2:
1:
FEVAL|  F  |  G  |  H  |     |
```

FUN



```
4:
3:
2:
1:
  ID  | PIN  |  SS  | PAT | CAR |$$$$$
```

PERS



```
4:
3:
2:
1:
 MOM |MARY |JOSE | DAD |FRED |SANDY
```

ADD

To create a FORMS directory, type FORMS, then press [CRDIR] from the [MEMORY] ([↰] [MEMORY] [DIR]) menu. When you now press [VAR], you'll see FORM in the leftmost rectangle. Press [FORM] to get a brand new directory for your formulas. You may be wondering: must I retype programs like DIST which are already stored in the HOME directory? Also, how can I get back home? The answer to the first question is no; there's a way to transfer menu names from one directory to another and we'll get to that shortly. To get back HOME, press the [HOME] key ([↱] [']).

In a similar manner, you can create any directory you want.

To manipulate menu names and their contents, you need to know about STO, RCL, EDIT, and PURG. The term *object* as used by the Owner's Manual is quite broad and includes numbers, algebraic expressions, and programs. Here are four useful tricks:

(1) To *store* an object under the name NAME, type NAME [STO] or
['] [NAME] [STO]. (Cf. point (3) following ID STO, sec. 0.4.)
(2) To *recall* the contents of NAME, type ['] [NAME] [RCL].
(3) To *edit* the contents of NAME, type ['] [NAME] [ENTER] [EDIT], then
use the cursor keys to make the changes you want, then press [ENTER].
(Note: if after making some changes you change your mind, you can press [ON]
to get back where you started).
(4) To *delete* a name NAME from a menu, type ['] [NAME] [PURG].

The best way to move names and their contents from one directory to another is by using the MEMORY dialog box. Here's how to move HERO, DIST, and VBOX from the HOME directory to the FORMS directory:

(1) Open the MEMORY dialog box by pressing [MEMORY] ([↱] [VAR]).

(2) Use the cursor keys to highlight HERON, then press [CHK]; similarly, highlight and check DIST and VBOX.

(3) Press [MOVE] to open the MOVE VARIABLE(S) dialog box.

(4) Press [CHOOS] to access the list of DIRECTORIES.

(5) Use the cursor keys to highlight FORMS, then press [OK] [OK] [ON].

The following example shows how to use an existing program to create a new program. We will use FEVAL to create and store a new program called COMP for forming the composition of two given functions. [Recall that the *composition* of functions $f(x)$ and $g(x)$ is the function $f(g(x))$.]

(i) Recall FEVAL and use EDIT ([↰] [+/−]) to omit the word DUP inside of it:

['] [FEVAL] [RCL] [EDIT]

▶ ··· *(10 times)* ··· ▶

$\boxed{\text{DEL}\rightarrow}$ $\boxed{\text{ENTER}}$;

(ii) Store the edited program under the name COMP: COMP $\boxed{\text{STO}}$

COMP

Inputs: $f(x)$, $g(x)$	*Output: $f(g(x))$*
« 'X' STO	Stores $g(x)$ under the name X
EVAL	Evaluates f at $g(x)$
'X' PURGE »	Purges X from the VAR menu
	Checksum: #35841d Bytes: 45

Exercises 0.5

A1. (a) Create a subdirectory called TEST of your HOME directory.

(b) Enter the following program: « HOME » in the TEST directory and try to name it HOME. What happens? Why? Try to name it hOME. What happens?

(c) Try to purge the TEST directory. What happens? Purge hOME from the TEST directory, then purge TEST.

A2. Create a subdirectory called FORMS of your HOME directory and move HERON into the FORMS directory.

A3. Create a subdirectory called FUN of your HOME directory and move FEVAL into the FUN directory.

0.6 GRAPHING

The subject of calculus is very much concerned with *graphs*. In fact, it wouldn't be far off to say that calculus *is* the study of graphs. Of particular interest are high points, low points, steepness, concavity, and points at which concavity changes. The HP 48 is a powerful grapher and, as such, it can help you significantly in your study of calculus. But be careful not to bypass the main ideas!

> **REMINDER:** *It is important that you use the graphing capabilities of the calculator to supplement rather than bypass what your instructor is trying to teach you.*

In this section we will discuss the nuts and bolts of HP 48 graphing. For this, we will assume that you already know how to graph by hand. It's important that you know that when the technology seems to fail (as it will from time to time), you could always take pencil in hand and make a careful point-by-point plot in an xy-coordinate system. It's also important that you understand how the HP 48 makes graphs.

We will discuss and illustrate the following:

I How to make a standard plot using $\boxed{\text{DRAW}}$.
II How to change scales using $\boxed{\text{ZOOM}}$.
III How to change the center of a plot using $\boxed{\text{CENT}}$.
IV How to "zoom-in" to another viewing window using $\boxed{\text{BOXZ}}$.
V How to store a plot using $\boxed{\text{STO}}$.
VI How to plot two or more curves at once.
VII How the HP 48 makes graphs.

Most of the information here can be found in the Owner's Manual which is fairly complete. Our purpose is to focus on what is simplest and what is most important for calculus. In later chapters we will build on these ideas (see § 1.2 and Chapter 3).

We suggest that you begin by deactivating the "connect" feature of your calculator so that your graphs will be more accurate and will compare with those illustrated in this book.

To deactivate the connect feature, just change $\boxed{\text{CNC}\blacksquare}$ to $\boxed{\text{CNCT}}$ in the $\boxed{\text{FLAG}}$ subdirectory of the $\boxed{\text{PLOT}}$ directory (press $\boxed{\Lsh}$ 8 $\boxed{\text{NXT}}$ $\boxed{\text{FLAG}}$ $\boxed{\text{CNC}\blacksquare}$).

I. *Standard plots.* The standard viewing window for the HP 48 is the rectangle $-6.5 \le x \le 6.5$, $-3.1 \le y \le 3.2$ in which you can easily plot any function as follows:

(1) Key in the function of your choice in either RPN or tick notation;
(2) Store the function under the name EQ;
(3) Press $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$.

Conveniently, the commands EQ, DRAX, and DRAW are grouped together in the first page of the PLOT menu ($\boxed{\Lsh}$ 8).

Before starting Example 1—for that matter almost any new plot—it's a good idea to make sure that the standard plot parameters are set. You can do this by pressing $\boxed{\text{RESET}}$ in the PPAR subdirectory of PLOT directory. A simpler way is to use the following program, also named RESET, which does everything the built-in RESET does plus a little more. Note that you can access the character Σ through the CHARS directory.

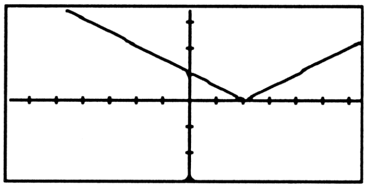
```
≪ ERASE {PPAR ΣPAR ZPAR VPAR ΣDAT X Y Z S T A N EQ
  PRTPAR IOPAR IERR} PURGE #131 #64 PDIM {RESET}
  ORDER ≫ RESET STO
```

The program erases the previous graph, sets the standard plot parameters (PPAR), sets the standard picture dimensions (PDIM), and acts as a general "trash remover". It also moves itself to the front of the VAR directory. You may find it useful to put this extended RESET program on all of your working directories. You may also want to add other "trash" items to it, like frequently used user-defined function symbols. (See "About function evaluation" § 0.4). Just be careful not to throw out treasures with the trash!

EXAMPLE 1. Make a standard plot of the function $f(x) = |0.5x - 1|$.

SOLUTION.

RESET .5 X *
1 − ABS ↰ PLOT
↰ EQ DRAX DRAW



✓Points to note

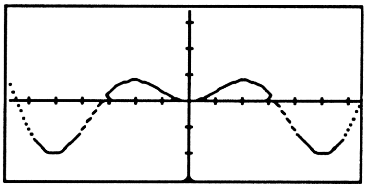
1. When you use the above procedure, be sure to enter the function not the equation, i.e., don't enter $f(x)$ and $=$.

2. For standard plots, tickmarks on the coordinate axes represent single units of length.

EXAMPLE 2. Make a standard plot of the function $f(x) = .4x \sin x$

SOLUTION.

ON .4 X * X SIN
* ↰ EQ ERASE
DRAX DRAW



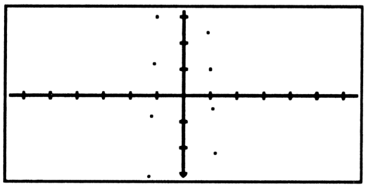
Generally, to obtain a meaningful plot—for that matter any plot at all—you'll have to change the scale of one or both of the coordinate axes. In addition, you may want the center of the viewing window to be someplace else other than at the origin. Features II–IV are for these purposes.

II. *Changing scales.* When you change scales on the HP 48, the axes and tickmarks on the display stay the same; what changes are the numbers the tickmarks represent. The ZOOM menu makes the process easy.

EXAMPLE 3. Graph the curve $f(x) = x^3 - 8x^2 - 2x + 10$.

SOLUTION.

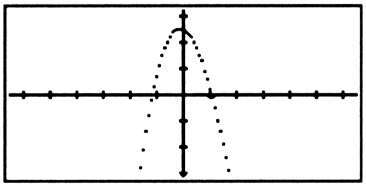
ON X 3 ^ 8 X
SQ * − 2 X *
− 10 + ↰ PLOT
↰ EQ
ERASE DRAX DRAW



All we get is a few dots. Obviously, this is less than satisfactory (but at least we got something!).

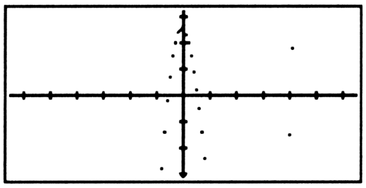
We will now use ZOOM in the graphics menu to magnify the values on the y-axis by a factor of 4. Put another way, we will rescale the y-axis so that we will see 4 times as much of it. (You could replace 4 by any positive number you'd like but 4 is easiest because that's the "default" value.)

ZOOM NXT VZOUT



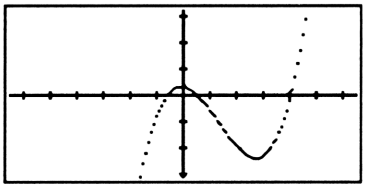
A great improvement! If you're impulsive you might even think that we're done (if you're *really* impulsive, you'll think the graph is a parabola!). However, if you remember Descartes' rule of signs or, better still, know about the asymptotic behavior of polynomials, you know that there's more to this graph than meets the eye. In particular, you know that it must cross the x-axis one more time to the right. Let's take a look at twice as much of the x-axis.

ZOOM ZFACT 2 OK
OK NXT HZOUT



Progress! Let's now try to see if we can find the turn-around point between the two positive zeros. Try another VZOUT , this time using a factor of 8.

ZOOM ZFACT ▼ 8
OK OK NXT
VZOUT



Now that's more like it! Indeed, this captures the essential features of the graph and that's about all we can do for now.

Later, after you've learned some calculus, you'll want to analyze various aspects of graphs very accurately. For now, we can at least make some fairly good approximations. For example, where does the graph cross the x-axis on the right? It appears to be at exactly the fourth tickmark, which would be $x = 4$ in the standard viewing window. Of course, it isn't standard because we've magnified both axes. Since the only adjustment to the x-axis was by a factor of 2, the fourth tickmark is really $x = 2 \cdot 4 = 8$. Is this the *exact* place where the graph crosses? You can easily check by calculating $f(8)$ by hand (is it *exactly* 0?). Better still, let's use FEVAL. To do this, press ON (to get back to the stack), VAR , EQ (that's where $f(x)$ is stored), 8 , then FEVAL (HOME FUN FEVAL). The answer: $f(8) = -6$. Definitely not 0! Does that mean that there is something wrong with the calculator? Not at all; it only means that the scale on the y-axis is so magnified that we can no longer distinguish by eye the difference between -6 and zero. What *is* the magnification of the y-axis? We did magnifications of factors 4 and 8. Therefore, the magnification is

$4 \cdot 8 = 32$. Do we always have to keep track of all of the scaling factors in order to figure out the scales? Fortunately not. Read on.

Hewlett-Packard has provided a little + cursor for finding coordinates of points in graphics displays, e.g., points corresponding to tickmarks. To see how it works, return to the graph of $f(x)$ by pressing $\boxed{◀}$. You can then activate the + cursor (which is hiding at the origin) by using the cursor keys ▲ ▼ ◀ ▶. To determine the coordinates of any point, move the + cursor to that point and then press the white key under $\boxed{\text{(X,Y)}}$. The coordinates will then be exhibited on the lower line of the display. To get the graphics menu back, press any white key. For the graph under consideration, you will find that the first tickmarks on the positive x- and y-axes have coordinates (2,0) and (0,32), respectively.

✓Points to note

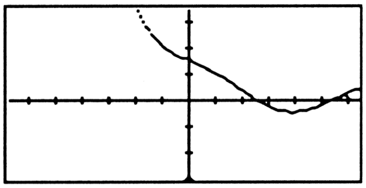
1. The above trial-and-error/refinement method for scaling works quite well for most functions you will encounter in calculus. A more systematic approach would be to start with a sample of functional values over a specified x-interval or intervals (either by hand or by machine). In this way, you could make "an educated guess" for a suitable scaling factor for the y-axis. This is exactly what ZAUTO does. You will find ZAUTO in the ZOOM directory, and you may find this feature useful in some situations. If you use this feature, be warned that there is a danger that you may overlook essential aspects of the graph.
2. You may find it useful to put copies of FEVAL in all directories in which you will be graphing. The above example is typical. Most graphing problems require a certain amount of evaluation of functions.

III. *Changing the center.* For most calculus textbook problems, you'll be satisfied with a viewing window centered around the origin. Often, however, the interesting part of the graph will occur far away from the origin and some other viewing window will be more appropriate. The CENTR command in PLOT PPAR allows you to specify whatever viewing window you want; just enter the coordinates (a, b) of your choice of center in the form (a, b), then press $\boxed{\text{CENT}}$. Note that even if you choose a viewing window that does not contain the coordinate axes, you can always determine coordinates of points by using the + cursor as explained in the previous example. You can also specify a new center by positioning the + cursor at the desired point, and then press the white key under $\boxed{\text{CNTR}}$ in the ZOOM directory.

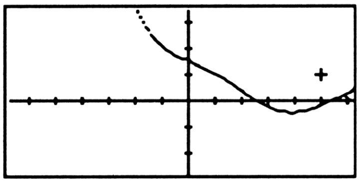
EXAMPLE 4. Graph the function $f(x) = 2^{-x} + 0.5\sin(x + 1)$.

SOLUTION. Start with a standard plot. Press $\boxed{\text{RESET}}$. Then

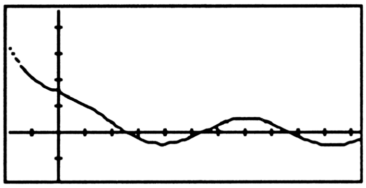
2 X $\boxed{+/-}$ ^ .5 X
1 + SIN * +
$\boxed{↰}$ $\boxed{\text{PLOT}}$ $\boxed{↰}$ $\boxed{\text{EQ}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



Since it appears that the more interesting part of the graph is to the right, we walk the cursor over to the position shown. This will be the new center.



CNTR



IV. *Zooming-in.* Often in the process of graphing with the HP 48, you'll want to take a closer look at a particular portion of the graph. A nice way to do this is to use BOXZ in the ZOOM directory to "zoom-in". Just do the following:

(1) Imagine a rectangle around the portion of the graph you want to zoom in on.
(2) Move the + cursor to any corner of your imaginary rectangle, and then press BOXZ .
(3) Use the cursor keys to draw your imaginary rectangle, and then press the white key under ZOOM .

For a "zoom-in" illustration, see Example 2, § 3.1. For further details see the Owner's Manual.

V. *Storing graphs.* Sometimes it's handy to store graphs so that they can be easily recalled later or even combined with other graphs. The simplest way of doing this is to press STO when the graph is being displayed. When you return to the stack you'll see a representative of a coded version of the graph in the form of "Graphic $m \times n$", where m and n are the pixel dimensions of the graph. You can store this code like you would any other object: NAME STO . To retrieve the graph, press NAME then key in: PICT STO ◄ .

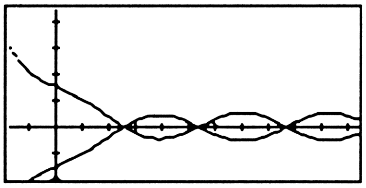
We will pursue this topic further in subsequent chapters.

VI. *Plotting two or more curves.* The HP 48 has a built-in procedure for plotting two functions $f_1(x)$ and $f_2(x)$ at the same time. You simply enter the expression $f_1(x) = f_2(x)$ in RPN or tick, then press ↰ EQ DRAX DRAW .

EXAMPLE 5. Graph the function $f(x)$ of Example 4 and its negative $-f(x)$ together.

SOLUTION. We use the same PPAR (so don't press RESET).

EQ ENTER +/−
= ↰ EQ ERASE
DRAX DRAW

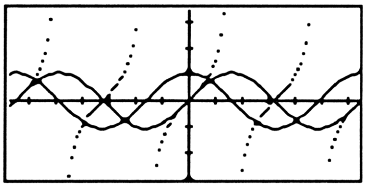


To plot three or more functions on the same display, put them in a list, then press $\boxed{\Lsh}$ $\boxed{\text{EQ}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$.

EXAMPLE 6. Graph $y = \sin x$, $y = \cos x$, and $y = \tan x$ on the same display.

SOLUTION. First press $\boxed{\text{RESET}}$. Then

X $\boxed{\text{SIN}}$ X $\boxed{\text{COS}}$ X
$\boxed{\text{TAN}}$ 3 $\boxed{\to\text{LIST}}$
$\boxed{\Lsh}$ $\boxed{\text{EQ}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



✓Points to note

1. If you enter →LIST character by character, be sure to delete any spaces between → and L. An easier way is to take →LIST off the LIST subdirectory of the PRG directory.
2. Another way to plot three or more functions on the display is to do them one at a time making sure not to use ERASE between plots.

VII. *How the HP 48 makes graphs.* The HP 48 display is really just a grid that contains a large number of tiny squares or *pixels.* Each pixel can be either shaded (on) or unshaded (off). Pixel size is roughly 0.47 mm × 0.47 mm. In terms of pixels, the display dimensions are 131 × 64. Put another way, the distance between adjacent tickmarks is equal to ten pixels. Given good lighting, the right angle, and good eyesight, these pixels can be seen with the naked eye. One way to see them clearly is to look closely at the screen image projected by a classroom display unit.

When it comes to graphing functions, the HP 48 cares only about pixels. This results in two imperfections. First, only 131 x-values are considered: $x = 0, \pm 0.1, \pm 0.2, \cdots, \pm 6.5$. Second, the corresponding y-values are rounded off to the nearest one-tenth. Thus, for example, when you tell the HP 48 to draw $\sin x$ it will systematically make dots at $(-6.5, -0.2)$, $(-6.4, -0.1)$, $(-6.3, 0)$, $(-6.2, 0.1)$, $(-6.1, 0.2)$, $(-6.0, 0.3)$, $(-5.9, 0.4)$, $(-5.8, 0.5)$, $(-5.7, 0.6)$, $(-5.6, 0.6)$, $(-5.5, 0.7)$, $(-5.4, 0.8)$, $(-5.3, 0.8)$, and so on.

To have real control over the graphics environment, you need to know how to *make* pixels. One way to do this is by using $\boxed{\text{DOT+}}$ in the graphics environment. Just press $\boxed{\blacktriangleleft}$ $\boxed{\text{EDIT}}$, then position the graphics + cursor wherever you like and press $\boxed{\text{DOT+}}$ to make a dot at that point. By using DOT+ and DOT− (which erases dots) you can make pictures like the one shown in Fig. 3.

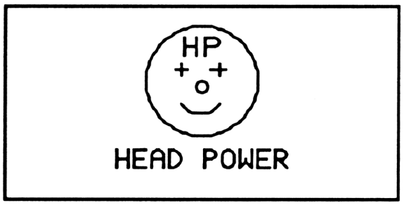


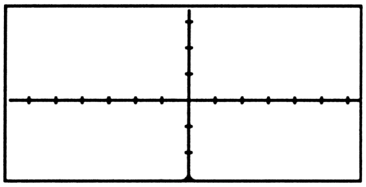
Figure 3

Another way to shade pixels is to use the PIXON command. You just specify the coordinates in the form (a, b) and press PIXON .

EXAMPLE 7. Shade the pixel at $(3, 1)$.

SOLUTION.

RESET DRAX (3,1)
PIXON ◄



Exercises 0.6

A1. Graph each of the following functions using standard plot parameters. In each case, try to predict the outcome before keying in the function.

(a) $\sin x$
(b) $-\sin x$
(c) $|\sin x|$
(d) $-|\sin x|$
(e) $-\frac{1}{4}x + \frac{1}{2}$
(f) $0.2x \cos x$
(g) $0.2x \cos^2 x$
(h) $\sin x \cos x$
(i) $\sin 2x$
(j) $\sin^2 x + \cos^2 x$
(k) 1
(l) 3.2
(m) -3.1
(n) 0
(o) $\cos^2 x - \sin^2 x$
(p) $\cos 2x$
(q) $\sin^2 x$
(r) $\sin^3 x$
(s) $\sin^4 x$
(t) $\sin^5 x$

A2. Graph each of the following functions using standard plot parameters. In each case, try to predict the outcome before keying in the function.

(a) $\sqrt{x}$
(b) $-\sqrt{x}$
(c) $\sqrt{-x}$
(d) $-\sqrt{-x}$
(e) x^2
(f) $0.1x^2$
(g) $-x^2$
(h) $-0.1x^2$
(i) $1 - x^2$
(j) $1.5 - x^2$
(k) $2 - x^2$
(l) x^3
(m) $0.1x^3$
(n) $0.01x^3$
(o) $-0.01x^3$
(p) $|x| - x$
(q) $|x| + x$
(r) $x^{1/3}$
(s) $-(-x)^{1/3}$
(t) $x^{1/3} - (-x)^{1/3}$

A3. Graph the upper half of the circle $x^2 + y^2 = 1$.

A4. Graph the upper half of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{2.25} = 1$.

A5. Graph the lower half of the parabola $x = y^2$.

A6. Graph the upper half of the hyperbola $\dfrac{x^2}{4} - y^2 = 1$.

A7. For each of the following, plot the given pair of functions using standard plot parameters. In each case, try to predict the outcome before keying in the functions.

(a) $\sin x$, $\sin 2x$

(b) $\sin x$, $1.5 \sin x$

(c) $\sin x$, $-\sin x$

(d) $\pm|.5x + 1|$

(e) $|.5x + 1|$, $|.5x - 1|$

(f) $\pm 0.2x \cos x$

A8. Graph the circle $x^2 + y^2 = 6.25$.

A9. Graph the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{6.25} = 1$.

A10. Graph the hyperbola $\dfrac{x^2}{4} - y^2 = 1$.

A11. Graph the circle $(x - 2)^2 + (y + 0.5)^2 = 4$.

A12. Graph the ellipse $(x + 3.5)^2 + (3y - 1.5)^2 = 9$.

A13. For each of the following, obtain a graph of the given function that contains all of its interesting features. In each case, specify the scales used.

(a) $-x^2 + 6x - 4$

(b) $x^3 - 8x^2 + 7x + 17$

(c) $x^4 - 4x^2$

(d) $x^3 - 100x + 1$

(e) $2 \sin x - \cos x$

(f) $\sin 15x$ [*Hint*: use BOXZ]

B1. Let D, M, and Y denote, respectively, the day, month, and year (last two digits) you were born. For example, if your birth date was April 14, 1973, then $D = 14$, $M = 4$, and $Y = 73$. Make a careful sketch of the quadratic $y = Dx^2 - Yx + M$.

B2. (a) Suppose you enter '$X = 1$' $\boxed{\Lsh}$ EQ ERASE DRAX DRAW. What output do you predict? Check your answer.

(b) Use PIXON to graph the line segment from $(1, 0)$ to $(1, 1)$.

0.7 LOOPS & BRANCHES

In some of the program structures in this book we use loops and branches. Generally these will be of the following three types:

I FOR-NEXT loops;
II FOR-STEP loops; and
III IF-THEN-ELSE branches

This section consists of a brief introduction to I–III. The reader interested in a more extensive treatment of these topics is referred to the Owner's Manual.

I. *FOR-NEXT loops.* This is a technique for carrying out repetitive calculations. Suppose, for example, that you wanted to perform five similar calculations C_1, C_2, C_3, C_4, C_5. Of course, you could always get the calculator to do each calculation separately, but that could amount to quite a bit of work. A more efficient way would be to express the ith calculation in terms of i then tell the calculator to do C_i for $i = 1, 2, \ldots, 5$. As a particular example, suppose you wanted to get the calculator to calculate the cubes 1^3, 2^3, 3^3, 4^3, 5^3. One way to do this would be to key in: 1 3 ^ ; then 2 3 ^ ; then 3 3 ^ ; then 4 3 ^ ; and, finally, 5 3 ^ . A simpler way would be to tell the machine: "do i 3 ^ , for $i = 1, \cdots, 5$". Of course you have to tell it in the language the calculator understands, namely, reverse Polish. The repetitive procedure "do i 3 ^ , for $i = 1, \cdots, 5$" in reverse Polish becomes: 1 5 FOR I I 3 ^ NEXT. In general, a FOR-NEXT procedure on the HP always has the following format:

$$\boxed{\text{A B FOR I ************ NEXT}}$$

where ************ represents any calculation or instructions (usually) involving I. Translated into English, it says:

$$\boxed{\text{"do ************ for each I = A , A+1 , . . . , B"}}$$

The above syntax is strict except that there is nothing special about the index symbol I. Any other letter would do just as well. Note that the index letter *must* follow FOR and is *not* to follow NEXT. Note also that NEXT is not the same as NXT. All loop and branch commands (FOR, NEXT, STEP, IF, THEN, ···) are located in the BRCH subdirectory of the PRG directory.

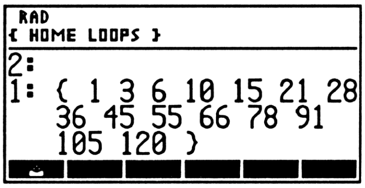
In this book we will use FOR-NEXT loops only in situations where A and B are integers with A < B.

EXAMPLE 1. (a) Write an HP 48 program to generate the first N triangular numbers $1, 3, 6, \cdots, \frac{N(N+1)}{2}$; (b) illustrate the program with $N = 15$.

SOLUTION.

(a) ≪ → N ≪ 1 N FOR I I I 1 + * 2 / NEXT N →LIST ≫ ≫ Δ $\boxed{\text{STO}}$
[To enter Δ press α ↱ C.]

(b) 15 $\boxed{\Delta}$



```
RAD
{ HOME LOOPS }
2:
1: { 1 3 6 10 15 21 28
   36 45 55 66 78 91
   105 120 }
```

About Polynomials

Recall that a *polynomial* is any function of the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

where a_0, a_1, $\cdots$, a_n are numbers and n is a nonnegative integer (= the *degree* of the polynomial). Examples:

$$A(x) = 1 - 2x + 5x^2$$
$$B(x) = 4 + 7x^2 + 2x^4 - 53x^6$$
$$C(x) = x^3 - 1$$

are polynomials of degrees 2, 6, and 3, respectively.

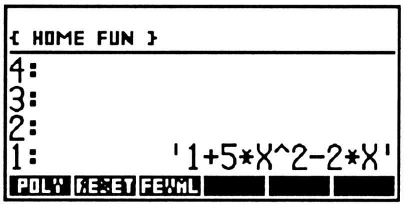
Polynomials are basic, polynomials are conceptually simple, polynomials are the building blocks for many other functions, polynomials are important in calculus. Throughout this book we will be working with polynomials. After you key in a few polynomials, you'll realize that the process is quite repetitious, so guess what? That's right, it's time for another program.

POLY	
Inputs: a_0, a_1, $\cdots$, a_n	*Output:* $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
`« DEPTH → M «`	Assigns local variable M= $n + 1$
`0 'P' STO`	Initializes P= 0
`1 M FOR I`	Sets up a loop from I = 1 to M
`'X' M I - ^`	Calculates x^j where j = M − I
`*`	Forms a_jx^j
`P + 'P' STO`	Increments P by a_jx^j
`NEXT`	Ends loop; go on to the next I
`P 'P' PURGE`	Puts polynomial on the stack & purges P from the VAR menu
`COLCT » »`	Cleans up parentheses
	Checksum: #59317d Bytes: 128

EXAMPLE 2. Use POLY to enter the polynomials $A(x)$, $B(x)$, $C(x)$ above.

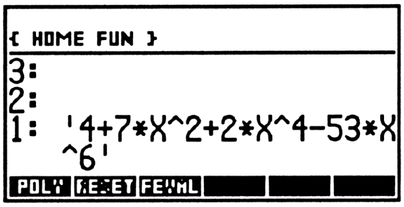
SOLUTION.

(a) 1 −2 5 POLY



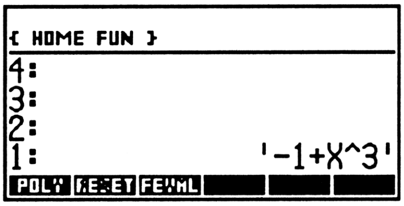
```
{ HOME FUN }
4:
3:
2:
1:          '1+5*X^2-2*X'
```

(b) $\boxed{\blacktriangleleft}$ 4 0 7 0 2
0 −53 $\boxed{\text{POLY}}$



```
3:
2:
1: '4+7*X^2+2*X^4-53*X
   ^6'
```

(c) $\boxed{\blacktriangleleft}$ −1 0 0 1
$\boxed{\text{POLY}}$



```
4:
3:
2:
1:                '-1+X^3'
```

✓Points to note

1. For a polynomial of degree n, the input will always consist of exactly $n + 1$ numbers.
2. When you apply POLY it is important that you start with an empty stack.

II. *FOR-STEP loops.* These are similar to FOR-NEXT loops, the essential difference being that you get from A to B through noninteger steps. FOR-STEP loops have the following structure:

$\boxed{\text{A B FOR I ************ S STEP}}$

where ************ represents any calculation or instructions (usually) involving I and S is the step size. Translated into English it says "do ************ for each I=A, A+S, A+2S, etc. until you get to B (more precisely, until you get to last number of the form A+kS which is less than or equal to B). As with FOR-NEXT structures, the syntax is strict except for the use of the particular index symbol I. We can use a FOR-STEP loop together with the PIXON command (§ 0.6) to make a vertical line segment as is illustrated by VSEG (see program box).

VSEG

Inputs: a , c , d	*Output: line seg. from (a,c) to (a,d)*

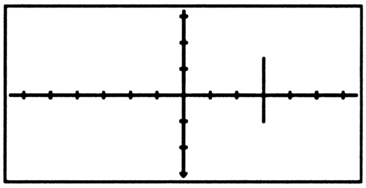
```
≪ DRAX                          Clears the LCD & draws axes
   → A C D ≪                    Defines local vars. corr. to a, c, d
   C D FOR Y                    Starts loop from C to D
   A Y R→C                      Forms the ordered pair (A,Y)
   PIXON                        Puts a dot at the point (A,Y)
   .1 STEP PICTURE ≫ ≫          Ends loop; increments Y by 0.1

                                   Checksum: #24217d  Bytes: 87.5
```

EXAMPLE 4. Draw the line segment from $(3, -1)$ to $(3, 1.4)$.

SOLUTION.

$\boxed{\text{RESET}}$ 3 −1 1.4

$\boxed{\text{VSEG}}$



III. *IF-THEN-ELSE branches*. We illustrate this idea with the following function which is defined one way for $x > 1$ and a different way for $x \leq 1$:

$$f(x) = \begin{cases} 2 - 0.6x & \text{if } x > 1 \\ 0.5(x+1) & \text{if } x \leq 1 \end{cases}$$

Put another way, the formula says: IF "$x > 1$" THEN "do $2 - 0.6x$" ELSE "do $0.5(x+1)$". A complete translation into HP 48 language is `'IFTE (X > 1 , 2 − .6 * X , .5 * (X+1))'`. In general, `'IFTE (A,B,C)'` translated into broken English means: IF *A is true* THEN *do B* ELSE *do C*. Functions defined using IFTE can be graphed in the usual way (↰ EQ DRAX DRAW). A standard plot of the above function is shown in Fig. 4.

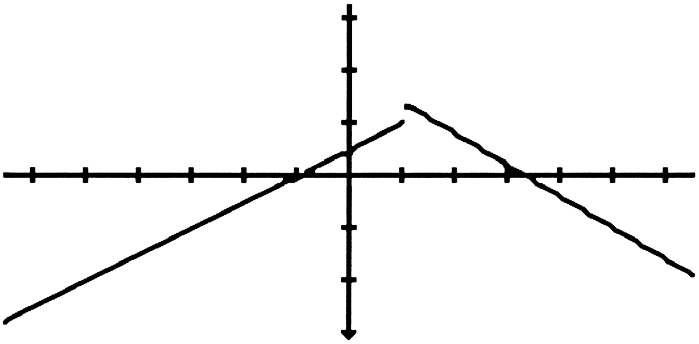


Figure 4

Exercises 0.7

For Exercises 1–7, write HP 48 programs to perform the indicated chores.

A1. Put the first 100 positive integers on the stack in order (so that 1 is on the top and 100 is at the bottom).

A2. Put the first 100 positive integers on the stack in reverse order (so that i is on level i for $i = 1, 2, \cdots, 100$).

A3. Input: N
Output: $\{1^2, 2^2, \cdots, N^2\}$

A4. Input: N
Output: $\{1, 2^2, 3^3, \cdots, N^N\}$

A5. Input: N
Output: $\{1, 2, 6, \cdots, N!\}$

A6. Draw the line segment from $(-1,2)$ to $(5,2)$.

A7. Draw the line segment from $(0,0)$ to $(3,3)$.

A8. Predict the outcomes of the following programs. Check your answers by running the programs.

(a)
```
« DRAX 3 6 FOR X X 1 R→C PIXON .1 STEP
  PICTURE »
```

(b)
```
« DRAX 1 4 FOR X X −1 R→C PIXON X 1 R→C
  PIXON .1 STEP −1 1 FOR Y 1 Y R→C PIXON 4
  Y R→C PIXON .1 STEP PICTURE »
```

A9. [*Trick question*] What is the degree of the polynomial $p(x) = 3x^{-1} + 1 + 2x^3$?

A10. [*Bad joke*] What would you call the following program? `« 'POLY' PURGE »`

A11. [*Your personal polynomial*] Let A, B, C, D, E, F, G, H, I be the digits of your social security number, except replace 0s by 10s. Form your *personal polynomial* as follows: $P(x) = A + Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 - Ix^8$. [For example, if your SSN is 190-34-7053, then $P(x) = 1 + 9x + 10x^2 - 3x^3 + 4x^4 - 7x^5 + 10x^6 - 5x^7 - 3x^8$.] Use POLY to enter $P(x)$ on the stack and store it under the name of your choice (MYPOLY ?) for later use.

A12. Use the HP 48 to graph the function

$$f(x) = \begin{cases} -2 & \text{for } x < 0 \\ 3 & \text{for } x \geq 0. \end{cases}$$

For Exercises 1–6, write HP 48 programs to perform the indicated chores.

B1. Input: N
Output: $\{1,\ 1 + \tfrac{1}{2},\ 1 + \tfrac{1}{2} + \tfrac{1}{4},\ \cdots,\ 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{N-1}}\}$

B2. Input: N
Output: $\{1,\ 1 + \tfrac{1}{2},\ 1 + \tfrac{1}{2} + \tfrac{1}{3},\ \cdots,\ 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{N}\}$

B3. Input: a, b, c
Output: The line segment from (a,c) to (b,c).

B4. Draw the line segment from $(-3,0)$ to $(5,3)$.

B5. Draw the arc of $\sin x$ from $x = 0$ to $x = \pi$.

B6. Input: N
Output: $\{\sqrt{N},\ \sqrt{N-1},\ \cdots,\ \sqrt{2},\ \sqrt{1}\}$

Chapter 1

Functions and Limits

1.0 PREVIEW

The main focus of calculus is the study of functions and the geometric or physical quantities they describe. Functions are central to the major ideas of calculus and to their applications in engineering and physics. Calculus is of importance in the study of dynamics, which is concerned with the motion of material objects subject to forces acting upon them. Functions are used to describe the position, velocity, and acceleration of such objects. For example, the function

$$s = s(t) = 10.3e^{-1.25t}\sin 0.194t, \quad 0 \le t \le t_1$$

gives the displacement s of a railroad "snubber" (a shock absorber at the end of a track) when hit by a freight car with mass 10^4 kilograms and speed 2 meters per second. This equation holds for $0 \le t \le t_1$, where t_1 is the time upon rebound when the car loses contact with the snubber. A graph of s and a schematic of the snubber spring and box car (seen shortly after contact) are shown in Fig. 1. In designing a snubber it is important to know the maximum displacement of the contact point for a typical impact. An approximate value for the maximum displacement can be read from the graph.

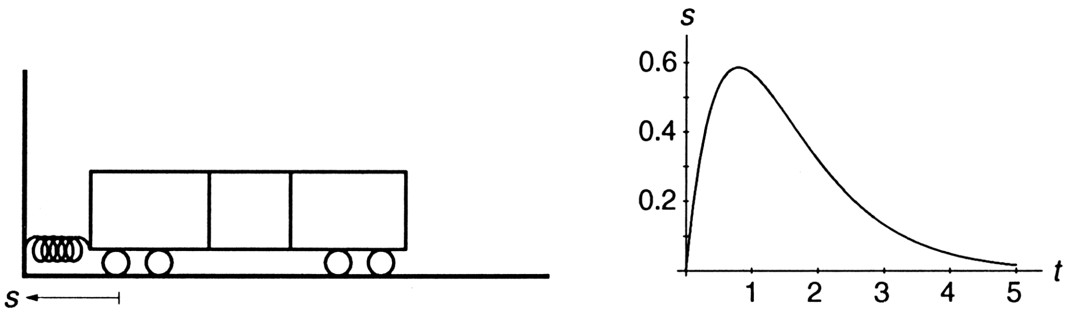


Figure 1

A more powerful method for finding the maximum value of s depends upon finding the time t for which the velocity $v(t)$ of the contact point is zero. For this we must differentiate s to form v, calculate the solution $t_{\max}$ of the equation $v(t) = 0$, and, finally, calculate $s(t_{\max})$. In solving $v(t) = 0$ it is helpful to graph the function v. Learning to use an HP 48 facilitates your ability to graph v, find $t_{\max}$, and evaluate $s(t_{\max})$.

Our main purpose in this chapter is to help you learn to use your calculator in evaluating functions, graphing functions, and in finding the zeros of functions. We use it also in discussing the idea of the limit of a function, one of the main ideas of calculus.

1.1 DEFINING AND EVALUATING FUNCTIONS

One of the most basic concepts in calculus is that of *function*. All of the functions used in elementary calculus are either "built-in" to the HP 48 or can be formed by combining several built-ins. Such functions are easily evaluated and graphed on the HP 48. Typical built-ins include the trigonometric functions, the inverse trigonometric functions, the squaring function, the square root function, the exponential function and its inverse (called the natural logarithm function), and the hyperbolic functions. Functions which are combinations of built-ins, such as the snubber response function, can be defined so that they become virtually indistinguishable from built-ins. We illustrate these methods in either examples or problems. In one further example we illustrate the composition of functions.

We begin by defining and evaluating a function by means of the SOLVR, which is on the ↰ SOLVE ROOT menu. First, go to this menu by pressing [↰], [SOLVE], and [ROOT]. Secondly, key in an equation and store it as EQ by pressing [↰] EQ (press ↰ and then the white key beneath EQ). Now press [SOLVR].

EXAMPLE 1. The polynomial

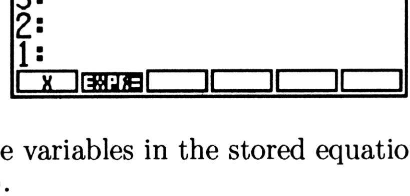
$$P(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

is sometimes used as an approximation to $\sin x$, particularly for values of x not too far from 0. Compare the values of $P(x)$ and $\sin x$ by calculating their values for $x = 0.0, 0.1, \ldots, 1.0$.

SOLUTION. The comparison of $P(x)$ and $\sin x$ is shown in Table 1. Verify the second column entries using the built-in SIN function. Use radian mode (↰MODES ANGL RAD) and set the calculator to show 6 decimals (↰MODES FMT 6 FIX).

The third column was computed with the SOLVR.

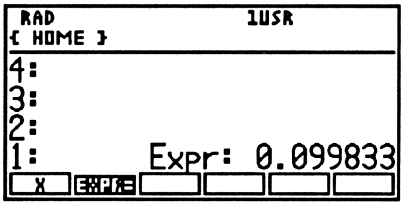
[↰] [SOLVE] [ROOT]
'X−X^3/6+X^5/120'
[↰] EQ
[SOLVR]



RAD 1USR
{ HOME }
4:
3:
2:
1:

The menu contains the names of the variables in the stored equation and EXPR=. Next we calculate $P(0.1)$.

.1
X (on SOLVR menu)
$\boxed{\text{EXPR=}}$



```
RAD                1USR
{ HOME }
4:
3:
2:
1:        Expr: 0.099833
```

Please verify the remaining third column entries in Table 1.

These calculations show that the polynomial $P(x)$ closely approximates $\sin x$ for $x \in [0, 1]$. We compare these two functions graphically in Example 2. Leave EQ on the VAR menu for use in Example 2.

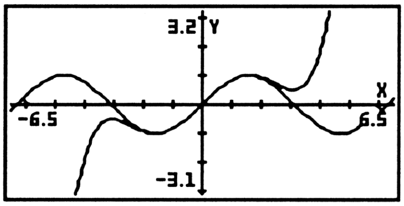
Table 1

x	$\sin x$	$P(x)$
0.0	0.000000	0.000000
0.1	0.099833	0.099833
0.2	0.198669	0.198669
0.3	0.295520	0.295520
0.4	0.389418	0.389419
0.5	0.479426	0.479427
0.6	0.564642	0.564648
0.7	0.644218	0.644234
0.8	0.717356	0.717397
0.9	0.783327	0.783421
1.0	0.841471	0.841667

EXAMPLE 2. Compare the polynomial $P(x)$ given in Example 1 and $\sin x$ by plotting them together.

SOLUTION. If necessary, key in $P(x)$ from Example 1 and store as EQ. To get short plot labels, use standard mode (↰MODES FMT STD).

'SIN(X)' $\boxed{\text{ENTER}}$
'EQ1' $\boxed{\text{STO}}$ $\boxed{\text{↱PLOT}}$
$\boxed{\text{NXT}}$ $\boxed{\text{RESET}}$
Reset plot $\boxed{\text{OK}}$ $\boxed{\text{NXT}}$
$\boxed{\text{CHOOS}}$
Check (✓) EQ and EQ1
$\boxed{\text{OK}}$ $\boxed{\text{DRAW}}$



To produce the labels in the figure, press $\boxed{\text{EDIT}}$ $\boxed{\text{NXT}}$ $\boxed{\text{LABEL}}$ just after DRAW is complete. To view the entire screen press $\boxed{-}$.

EXAMPLE 3. In Examples 1 and 2 we used algebraic style in defining and evaluating functions. We now discuss *program style*. In algebraic style "tick" marks are used as delimiters. We continue with the function $P(x)$ given in the first two examples.

SOLUTION. A program defining $P(x)$ is given by

```
≪  →  X  'X–X^3/6+X^5/120'  ≫
```

The program delimiters ≪ and ≫ must enclose any program. When the program is run, the command → X results in one number being removed from the stack and stored as X. This variable—called a "local variable"—is accessible only within the program and does not appear on the VAR menu. The last part of this program is an expression defining the function. The program calculates the value of $P(x)$ and puts it on the stack.

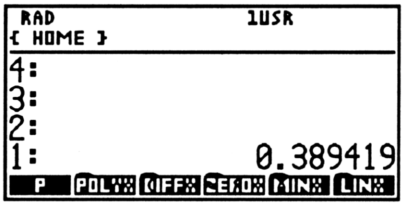
Please either key in directly the above program and store it under the name P or use the HP 48 built-in DEFINE. For this, enter

```
'P(X)= X–X^3/6+X^5/120'
```

and then press DEF. This puts the above program on the VAR menu under the name P.

The value $P(0.4)$ may be calculated in several ways. The easiest way is to put the number 0.4 on the stack and press the key beneath P on the VAR menu. The value 0.389419 is returned to the stack.

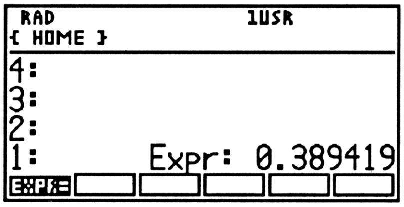
.4
P (on the VAR menu)



```
RAD                 1USR
{ HOME }
4:
3:
2:
1:              0.389419
```

If we place the expression 'P(0.4)' on the stack and press EVAL we obtain the same value as before. Finally, we may use the SOLVR, though the procedure is slightly different in program style. For this we recall a copy of the program stored as P to the stack.

'P' RCL
↰SOLVE ROOT
↰ EQ SOLVR
.4 EXPR=



```
RAD                 1USR
{ HOME }
4:
3:
2:
1:        Expr: 0.389419
```

The value of $P(0.4)$ is on the screen. On the SOLVR, evaluation of functions defined in program style takes one key stroke less than when algebraic style is used.

We often prefer to define functions in program style. If a function f is stored as F, then to x and $f(x)$ correspond the acts of keying in x and pressing the white key beneath F on the VAR menu. This is mathematically natural and leaves no trace of x on the VAR menu. On the other hand, program style takes extra key strokes at the time the function is defined. A function defined in program style and stored as, say, F, may be used for plotting by putting 'F(X)' on the stack and using it as an algebraic.

EXAMPLE 4. Calculate the amount P_k in a bank account after k quarterly compoundings at 7.5% interest per year of an initial amount P_0.

SOLUTION. If an amount P_0 dollars is deposited in a bank at 7.5% per year and this amount is compounded 4 times per year, then after 1/4 years the account will have increased to P_1 dollars, after 2/4 years it will have increased to P_2 dollars, ..., where

$$\begin{aligned}
P_1 &= P_0 + P_0 \cdot 0.075 \cdot (1/4) = P_0(1+0.075/4)^1 \\
P_2 &= P_1 + P_1 \cdot 0.075 \cdot (1/4) = P_1(1+0.075/4) = P_0(1+0.075/4)^2 \\
P_3 &= P_2 + P_2 \cdot 0.075 \cdot (1/4) = P_2(1+0.075/4) = P_0(1+0.075/4)^3
\end{aligned}$$

In these calculations we have added to the principal P at the beginning of an interest period the interest earned during the period, where the interest is found from the formula $I = Prt$. After k such (1/4)-year compoundings we have

$$\begin{aligned}
P_k &= P_{k-1} + P_{k-1} \cdot 0.075 \cdot (1/4) = P_{k-1}(1+0.075/4) \\
&= P_0(1+0.075/4)^k
\end{aligned} \tag{1}$$

We may use the SOLVR to evaluate the function P_k or we may define a function P using a program, as in Example 3. Letting $P_0 = 100$, a program for the function described in (1) is

```
≪ → K '100*1.01875^K' ≫        (2)
```

We used 1.01875 in the program instead of 1+0.075/4 to avoid dividing and adding each time the program is executed. Please enter this program and store it as P. To calculate the amount P_4, key in 4 and press P on the VAR menu. The result is $107.71 (2 FIX). Experiment to find how long it takes for money to double at 7.5%? (The answer is given at the end of Example 6.)

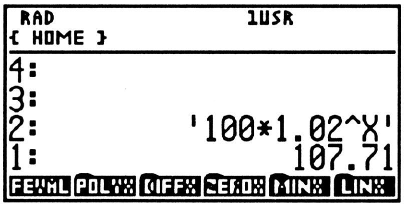
EXAMPLE 5. Use the program FEVAL in calculating the amounts P_4, P_8, and P_{12}, where P_k is given in (1). Use 2 FIX (*Lsh*MODES FMT 2 FIX).

SOLUTION. We repeat/review FEVAL here. The program is

```
≪ 'X' STO DUP EVAL 'X' PURGE ≫
```

If you do not have this program on some menu, please enter and store it as FEVAL. One feature of FEVAL is that the function variable must be x, not k or some other variable. We calculate P_4 using FEVAL as follows.

'100*1.01875^X'
4 [FEVAL]



```
RAD                 1USR
{ HOME }
4:
3:
2:          '100*1.02^X'
1:                107.71
```

Note that with 2 FIX in effect, the number 1.01875 in the expression on level 2 shows up as 1.02. To calculate P_8, drop P_4, key in 8, and press $\boxed{\text{FEVAL}}$. The program returns 116.02. Drop this number, key in 12, and press $\boxed{\text{FEVAL}}$. The program returns 124.97.

FEVAL stores as X the number x in level 1, duplicates the expression for $f(x)$, evaluates one of these expressions using the stored value of x, and then purges X. FEVAL uses the same algebraic form of $f(x)$ as used by SOLVR and PLOT, which is often an advantage.

EXAMPLE 6. Form the composition of the functions $F(x) = \sin x$ and $G(w) = w^3 + 1$ with your HP 48. Do it in both orders.

SOLUTION. The compositions $P = F \circ G$ and $Q = G \circ F$ are

$$P(w) = F(G(w)) = \sin(w^3+1) \qquad \text{and} \qquad Q(x) = G(F(x)) = \sin^3 x + 1$$

We use sine and not F for the sine function. We may form these compositions of sine and G with the built-in SIN function and the function

```
« →W 'W^3+1' »
```

Storing the latter as G, we first put `'SIN(G(W))'` on the COMMAND LINE and then press $\boxed{\text{EVAL}}$. We obtain the expression `'SIN(W^3+1)'`. Next we put `'G(SIN(X))'` on the COMMAND LINE and press $\boxed{\text{EVAL}}$. We obtain `'SIN(X)^3+1'`. (From Example 4: To double your money at 7.5% requires 38 quarters or 9.5 years.)

Exercises 1.1

A.1 Use SOLVR in computing the entries of a table in which the values of $\cos x$ and the polynomial $Q(x) = 1 - (1/2!)x^2 + (1/4!)x^4$ are compared for $x = 0.0, 0.1, \ldots, 1.0$. Graph these two functions together, as in Example 2.

A.2 An approximation to $\sin x$ (called a Padé approximation) is the rational function

$$\frac{2520x - 360x^3 + 11x^5}{2520 + 60x^2}$$

Use SOLVR in computing the entries of a table in which $\sin x$ and the given Padé approximation are compared for $x = 0.0, 0.1, \ldots, 1.0$. Graph these two functions together, as in Example 2.

A.3 Using the rational function—call it $r(x)$—given in problem A.2, compare the following three methods of defining and evaluating a function: (i) using program style, discussed in Example 3; (ii) using the program FEVAL discussed in Example 5; and (iii) using the program FOFX discussed below. Base your comparison on ease of use, number of key strokes needed to compute $r(.5)$, the relative speeds of these methods (estimate as well as you can the time each takes to compute $r(.5)$, starting with the final key stroke), and the number of things left on the VAR menu. You may also wish to consider how these methods compare when both evaluation and plotting of the function are needed. The program FOFX is

```
«  'X'  STO  EQ  →NUM  'X'  PURGE  »
```

To use FOFX, the function must be stored as EQ on the VAR menu. To evaluate the function put a number on the stack and press $\boxed{\text{FOFX}}$.

A.4 In Example 4 an expression P_k for the amount of money in the bank after k quarterly compoundings was given, namely

$$P_k = P_0(1 + 0.075/4)^k$$

Letting $P_0 = 100$, compounding n times per year instead of 4, and letting $A(t)$ denote the amount of money in the bank at time t (in years), show that

$$A(t) = 100(1 + 0.075/n)^{nt} = 100((1 + 0.075/n)^n)^t$$

A.5 (*continuation*) Write a program for calculating $A(t)$ and use it to prepare a table comparing the amounts obtained after 5 years with compounding once a year, semi-annually, quarterly, daily (assume 365 days), and hourly. What does this mean for the small investor?

A.6 A commercial for a bank claims that "A penny saved is \$1,000 trillion earned. And that's only 4% interest for 1, 000 years." Is this honesty in advertising?

A.7 Let the functions f, g, and h be defined by

$$f(x) = \sqrt{x-2},\ x \geq 2; \quad g(y) = y^2 + 2,\ -\infty < y < \infty; \quad h(w) = \frac{1-w}{1+w},\ w \neq -1$$

Using the methods of Example 6, form and simplify the nine possible composition pairs. The HP 48 will need your help in simplifying some of the results. Include in your answer the domain of each composite function.

A.8 Repeat Example 6 but use the program FEVAL instead of the program method.

B.1 Graph the "snubber function" given in §1.0. Using this graph, estimate the maximum displacement of the snubber and the time t_{max} this occurs. Calculate t_1.

Table 2

n	L_n	$n!$	U_n	$U_n/n!$
20	2.42278684677E18	2.43290200818E18	2.45307168235E18	1.00829037672
50	3.03634459417E64			1.00332641743
100		9.33262154439E157		1.00166493399
200	7.88329328671E374			1.00083289978

B.2 Verify empirically Stirling's approximation L_n of $n!$, where

$$L_n = \sqrt{2\pi n}(n/e)^n \;<\; n! \;<\; \sqrt{2\pi n}(n/e)^n(1+1/(4n)) = U_n$$

Recompute Table 2 and fill in the missing entries. Use STD mode. Notes: 20! can be calculated on the MATH NXT PROB menu. The number e is a built-in constant. It is on the MTH NXT CONS menu.

B.3 Closely related to Stirling's inequality is John Wallis' (1616–1703) inequality

$$\pi n \le \left(\frac{2^{2n}(n!)^2}{(2n)!}\right)^2 \le \frac{\pi(2n+1)}{2}$$

Obtain estimates for π by taking $n = 50$ and $n = 100$ in this inequality.

B.4 A complex number $x + iy$ is represented in the HP 48 by the pair (x,y). Once entered, complex numbers such as $2 + 3i = (2,3)$ and $-7 + 4i = (-7,4)$ can be subtracted with the $\boxed{-}$ key. Simply enter $(2,3)$ and $(-7,4)$ on the stack and subtract to get $(9,-1)$. The HP 48 has a built-in function ABS for the *length* or *modulus* of a complex number. The modulus of (x,y) is $\sqrt{x^2+y^2}$. To calculate the modulus of $(-7,4)$, put $(-7,4)$ on the COMMAND LINE and press $\boxed{\text{ABS}}$, which may be found on the MTH REAL NXT or MTH NXT CMPL menus. We obtain 8.06⋯. We may use these features of the HP 48 to shorten the calculation of the distance between points (x_1,y_1) and (x_2,y_2) of the (x,y)-plane. Write a program to calculate the distance between points (x_1,y_1) and (x_2,y_2).

B.5 If from an initial point (x_1,y_1) in the (x,y)-plane we walk a distance r on a "heading" θ to a destination point (x_2,y_2), find the coordinates x_2 and y_2 in terms of x_1, y_1, r, and θ. It is understood that $r \ge 0$ and $0 \le \theta < 360°$. Write a program which has stack inputs (x_1,y_1), r, and θ and returns (x_2,y_2). Use this program to find the final destination if the starting point is $(3.4,-5.9)$ and we are given successive distance/heading pairs (2,45°), (5.2,144.7°), (3.5,269.4°), and (22.8,4.8°). Include DEG and RAD as the first and last commands in your program.

B.6 Write a program to help you estimate the coordinates of the point on the parabola with equation $y = x^2$ closest to the point (2,0).

C.1 Write a program that calculates the angle $\alpha_2 - \alpha_1$ (in degrees or radians, as you think best) from line L_1 to line L_2, given the slopes m_1 and m_2 of these lines. The program should take m_1 and m_2 from the stack, where m_1 is on level 2 and m_2 on level 1, and return $\alpha_2 - \alpha_2$. Recall that most analytic geometry or calculus texts define the inclination α of a line with slope m to satisfy $\tan\alpha = m$ and $0 \le \alpha < \pi$. Test your program on (at least) the following data: (1) $m_1 = \tan 20°$, $m_2 = \tan 45°$; program should return $0.436\cdots$ or 25°. (2) $m_1 = \tan 70°$, $m_2 = \tan 120°$; program should return $0.873\cdots$ or 50°. (3) $m_1 = \tan 170°$, $m_2 = \tan 120°$; program should return $2.267\cdots$ or 130°. Return your calculator to radian mode after completing this problem.

C.2 Write a program for the amount of money in the bank at time t if the interest rate, the initial amount, and the number of compoundings per year are stored on the VAR menu. The program should take t from the stack and return $A(t)$.

1.2 GRAPHING FUNCTIONS

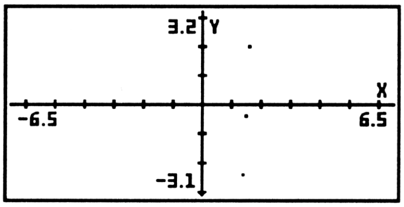
The PLOT menu was discussed in Chapter 0, including the meaning of the plot parameters PPAR and the use of RESET. In this section we discuss the scaling keys SCALE, *W, and *H, the XRNG and YRNG keys, and the zoom key BOXZ. We restrict ourselves to plot type FUNCTION. The purpose of graphing a function f is to reveal its "interesting" features. These include x-intercepts of the graph of f (that is, the zeros of f), vertical and horizontal asymptotes, intervals in which f is decreasing or increasing, and intervals in which the graph of f lies below or above its tangent line. For most of the functions found useful in the applications of calculus, the interesting features are relatively close to the origin. At points far from the origin most functions have settled into a stable pattern.

The scaling keys control the horizontal (width) and vertical (height) scales. In the default PPAR the scale in both vertical and horizontal directions is 1 unit per tick mark. Putting 2 on the COMMAND LINE and pressing *H multiplies the distance between the tick marks on the vertical axis by 2. The plot parameters in PPAR are modified accordingly. Horizontal units (independent variable) can be modified by *W. The default screen, as specified by the default PPAR, goes from -6.5 to 6.5 in the horizontal direction and from -3.1 to 3.2 in the vertical. After we have multiplied the height by 2, the parameters governing the horizontal remain the same and the vertical parameters become -6.2 to 6.4. To increase the horizontal scale of an existing viewing rectangle by a factor of 1.7 and the vertical scale by 2.3 we may either enter 1.7 followed by *W and then 2.3 followed by *H or simply put 1.7 and 2.3 on the stack, in this order, and press SCALE.

EXAMPLE 1. Graph the function $f(x) = 5x^3 - 7x^2 + 9x - 15$, starting with the default PPAR.

SOLUTION. In entering $f(x)$ in the input below, note the X on the PLOT menu after you type '. Using it saves a key stroke for each X.

↱PLOT NXT RESET
Reset plot OK NXT
'5*X^3-7*X^2+9*X-15'
OK DRAW
EDIT NXT LABEL

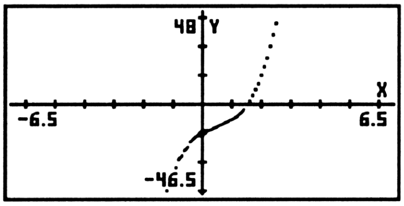


A disappointing graph. If CONNECT is checked, the graph is a near-vertical jagged line. If CONNECT is not checked, which we often prefer, the graph is two or three points (three until the PICTURE menu covers up the lowest point; to see the entire screen press −). CONNECT is on the ↱PLOT OPTS menu.

To explain what has happened, press TRACE and (X,Y) on the PICTURE menu. Move the cursor by pressing ▶ to find that the pairs $(1.3, -4.1)$, $(1.4, -2.4)$, $(1.5, -0.4)$, $(1.6, 2.0)$, and $(1.7, 4.6)$ are on the graph of f. (We have rounded the y-values to 1 decimal.) Since the default PPAR gives a viewing window with y-values between -3.1 and 3.2, we see why only two or three of these 5 points were plotted. Since f changes sign between 1.5 and 1.6, the graph has an x-intercept between $x = 1.5$ and $x = 1.6$. Since $f(1.6) = 2.0$ and $f(1.7) = 4.6$, we guess that f increases rapidly for $x \geq 1.5$. For as x increases by just 0.1 from 1.6

f increases rapidly for $x \geq 1.5$. For as x increases by just 0.1 from 1.6 to 1.7, f increases by $4.6 - 2.0 = 2.6$. Finally, we find that $f(2) = 15$, $f(3) = 84$, and $f(-1) = -36$ by further cursor movement. We guess that we should multiply the vertical scale by 15. After two CANCELs,

↰PLOT PPAR NXT
15 *H NXT PLOT
ERASE DRAX DRAW
(DRAX draws the axes)

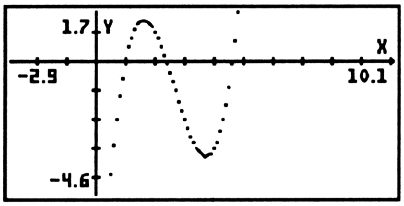


Before restoring the normal screen, move the cursor to what appears to be the single real zero of f, then press FCN and ROOT. After a few seconds the screen will display ROOT: 1.51703347723.

EXAMPLE 2. By graphing the function $f(x) = x^3 - 8x^2 + 18x - 11$, locate approximately its local minimum.

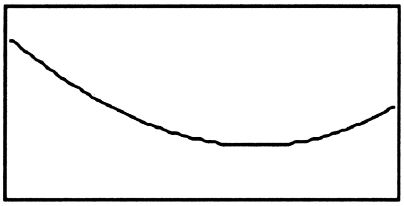
SOLUTION. Graph f using the default PPAR. We use CNTR to view the local minimum of the graph. CNTR shifts the viewing rectangle, it does not change scale.

ZOOM NXT +
Move cursor to (3.6, −1.5)
+ CNTR



Press TRACE to turn TRACE on and then use ◀ or ▶ to locate approximately the local minimum. Press (X,Y) to obtain coordinates (3.7, −3.3). Results may vary. Press any key in the top row and then turn TRACE off. With the cursor located at the local minimum, more or less, we use ZOOM to locate the minimum more accurately.

ZOOM
Move cursor 3 clicks left and
3 clicks up
BOXZ
Move cursor 6 clicks right and
6 clicks down
ZOOM



Turning TRACE on again we find the local minimum at (3.72, −3.27), approximately. Results may vary a little.

EXAMPLE 3. In graphing a function it often saves time to make some preliminary rough estimates. We may use the SOLVR to do this or we may try PLOT with the default PPAR. We may wish to use AUTO, which attempts to choose a reasonable vertical scale. Each of these may help us to display the main features of the graph. We illustrate with the function $f(x) = x^4 - 6x^3 + 8x^2 + 4$.

SOLUTION. Key in, store as EQ, and plot f using the default PPAR. We may significantly reduce the time needed to plot by storing $f(x)$ in the factored form

'X∗X∗(X∗(X−6)+8)+4'

The displayed graph has 10 points. The $\boxed{-}$ key may be used to view the full screen. After $\boxed{\text{CANCEL}}$, go to the ↱PLOT menu, check AUTOSCALE, and then press $\boxed{\text{ERASE}}$, and $\boxed{\text{DRAW}}$. AUTOSCALE gives a parabolic shape, with the vertical range from −573 to 3775. The vertical range can be verified by pressing $\boxed{\text{CANCEL}}$ and then highlighting and editing V-VIEW. In a moment we will see that this scale washes out most of the interesting features of the graph.

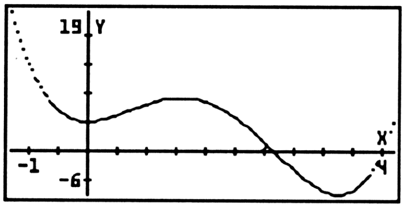
What to do next? We may guess at a scale change, perhaps 5∗H (applied to the default PPAR). The resulting figure is quite good. Or we may use the SOLVR to obtain information to reset the PPAR. The function $f(x)$ is stored under EQ and is immediately available to the SOLVR. Use ↰SOLVE ROOT SOLVR. After evaluating $f(x)$ for a few values of x we decided it would be sufficient to evaluate $f(x)$ for $x = -2.0, -1.5, \ldots, 4.5, 5.0$. The results are listed in Table 3. We used 1 FIX.

Table 3

x	$f(x)$	x	$f(x)$
−2.0	100.0	2.0	4.0
−1.5	47.3	2.5	−0.7
−1.0	19.0	3.0	−5.0
−0.5	6.8	3.5	−5.2
0.0	4.0	4.0	4.0
0.5	5.3	4.5	29.3
1.0	7.0	5.0	79.0
1.5	6.8		

We change the viewing rectangle using the above calculations. The useful range of x appears to lie between −1.0 and 4.0 and that of y between −6 and 19.

$\boxed{\text{↰PLOT}}$ PPAR
−1 SPC 4 $\boxed{\text{XRNG}}$
−6 SPC 19 $\boxed{\text{YRNG}}$
$\boxed{\text{NXT}}$ $\boxed{\text{NXT}}$ $\boxed{\text{PLOT}}$
$\boxed{\text{ERASE}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



This graph shows the main features of f. We may locate the three "relative extrema" with the cursor, either reading them directly from the screen (press $\boxed{+}$) or putting them on the stack with ENTER. We find the left-most relative minimum is at $(0.0, 3.9)$, the relative maximum at $(1.2, 7.1)$, and the other minimum at $(3.3, -6.0)$. Answers may vary. If $\boxed{+}$ is used to obtain coordinates of several points, it saves time in moving the cursor to a new location if you press $\boxed{+}$ again. This key is a toggle for displaying cursor coordinates.

In this brief section we discussed CNTR, which shifts the viewing window, the use of the scaling keys SCALE, ∗H, ∗W and BOXZ, and the more direct scaling keys XRNG and YRNG.

Exercises 1.2

A.1 Graph $y = \cos x$ using the default PPAR.

A.2 Graph $y = \cos x$ and $y = x$ on the same screen using the default PPAR. Through moving the cursor and pressing $\boxed{+}$ to display coordinates, find the point where these two graphs intersect. Using the built-in COS function, find by systematic trial and error (or any method you know) a value of x for which $\cos x = x$, accurate to two decimal places.

A.3 Graph $y = \tan x$ and $y = -x$ on the same screen using the default PPAR. Display the point with the least positive value of x where these two graphs intersect. Using the built-in TAN function find a value of x for which $\tan x = -x$, accurate to two decimal places. In how many points do the full graphs intersect?

A.4 Graph $y = \cot x$ using the default PPAR.

A.5 Graph $y = \csc x$ using the default PPAR.

A.6 Graph $y = x^2 + 3.9x + 3.1$ using the default PPAR. Display the x-intercepts and compare with the roots of the quadratic equation $x^2 + 3.9x + 3.1 = 0$.

A.7 Graph $y = x^2 - 3x - 4$. Display the coordinates of the vertex.

A.8 Graph the parabola having the equation $y - 0.1 = (x + 0.3)^2$.

A.9 Graph the function $f(x) = x^4 - 3x^2 + 15$. Use the SOLVR in finding a viewing rectangle. Note that since the graph is symmetric about the y-axis we may restrict the viewing rectangle to the right half-plane. Find approximate coordinates of the minimum with $x > 0$.

A.10 Graph the equation $y = (x^3 - 10x^2 + x + 50)/(x - 2)$. Find approximate coordinates of the three x-intercepts and the local minimum.

A.11 Graph the function $f(x) = x^4 + x^2 - 4x + 4$ and find the approximate coordinates of the local minimum.

A.12 Graph the function $f(x) = (3x^4 - 20x^3 + 24x^2 + 128)/((x^2 - 6x + 10)(x^2 + 5))$. Find the approximate coordinates of the three extrema.

B.1 Graph $y = x^3 - 7.5x^2 + 17.5x - 11$. Find estimates of the coordinates of the local minimum and local maximum. Find coordinates of the single x-intercept.

B.2 Graph $y = \sin x/(1 - 0.7\cos(x - \pi/20))$. Find estimates of the coordinates of the (repeated) local maximum and local minimum.

B.3 Graph $y = x/2 - \sin x$. Graph $y = x/2$ and $y = x/2 - \sin x$ on the same screen.

Describe the graph of $y = x/2 - \sin x$ in relation to the line $y = x/2$.

B.4 Using the default PPAR and DRAW, graph the function

$$f(x) = \frac{x^5 - x^4 + x - 1}{x^2 - x - 12}$$

Since the denominator of $f(x)$ has zeros at -3 and 4, we may suspect that we have not yet seen the main features of the graph. Hint: Use the SOLVR to evaluate f for $x = -6, -5, \ldots, 5, 6$.

B.5 Graph $f(x) = x \sin(1/x)$ with the default PPAR. To get a reasonably good picture of the oscillatory behavior near the origin you will need to experiment with both $*$H and $*$W.

B.6 Graph the function $f(x) = 1 + (5x - |x|)/2 - x^2 - (x-1)|x-1|$. Recall the built-in function ABS. Explain the three parts of the graph.

B.7 Graph the function $f(x) = x^{3/2}/\sqrt{2-x}$. Use the default PPAR. Note that PLOT, when it evaluates the expression stored in EQ at the 137 values -6.8, -6.7,..., 6.7, 6.8, ignores complex numbers and zeros in denominators. This has the effect of implicitly recognizing the "natural" domain of f. What is the natural domain of f?

B.8 Graph the function $f(x) = -0.25/(|x - 0.75| - 0.75|x + 0.25|)$, first using the default PPAR. Use XRNG and YRNG to enlarge the most interesting portions of the graph. Find approximate coordinates of the local minimum.

B.9 The function

$$f(x) = \frac{q_2}{|x - q_1|} - \frac{q_1}{|x - q_2|} - \frac{1}{2}x^2,$$

where $q_1 = M_2/(M_1 + M_2)$ and $q_2 = -M_1/(M_1 + M_2)$, occurs in the study of the motion of a satellite in the gravitational field of two bodies of masses M_1 and M_2. It is understood that the mass of each of the two bodies is much larger than that of the satellite. For a satellite in the earth/moon system, $M_1 = 1$ and $M_2 = 81.3015$, where we have taken the moon as a unit mass. These data result in a graph that is difficult to scale. Instead, graph f for $M_1 = 0.2$ and $M_2 = 0.6$, working to obtain a well presented figure.

B.10 Describe the interesting features of the graph of $f(x) = (3x^4 - 20x^3 + 24x^2 + 128)/(64(x^4 + 1))$.

C.1 Graph the function $f(x) = (1 - \cos x^6)/x^{12}$ using the default PPAR and DRAW. Although the resulting graph is useful for studying f near the origin, its scale hides other features of the function. First, find a viewing rectangle showing the oscillatory behavior of this function away from the origin. Next, zoom-in towards $(0, 0.5)$ and note the chaotic features of the graph of f. To say "the" graph of f may be misleading; say, rather, "a" graph, based upon calculator approximations of $f(x)$ for values of x near 0. Explain what is happening. It may be useful to use the fact that

$$0 \le \frac{1}{2} - \frac{1 - \cos x^6}{x^{12}} \le \frac{x^{12}}{24}, \qquad 0 < x \le 0.22$$

Finally, we note that f may be tamed by multiplying numerator and denominator by $1 + \cos x^6$ and simplifying.

C.2 Graph the function $f(x) = \sin(36x)$. Start with the default PPAR. Try scaling with 20 ∗W. Although the graph is attractive and each pixel is correct, it gives little idea of the graph of f. To set the scaling correctly consider the period of this trigonometric function.

C.3 Locate the local maximum of the function

$$\frac{1}{x^2\sqrt{\dfrac{1}{x^4}+\dfrac{1}{(x-2\pi)^4}+\dfrac{1}{(x-4\pi)^4}}}$$

between 2π and 4π. What is the horizontal asymptote of the graph of f?

C.4 Graph the "difficult" case given in problem B.9.

1.3 LIMITS OF FUNCTIONS

We remarked in the Preview of this chapter that the main focus of calculus is the study of functions and the geometric or physical quantities they describe. In §1.1 and §1.2 we discussed how to define, evaluate, and graph functions with the help of a calculator. In this section we discuss the idea of the limit of a function. Our purpose will not be, however, to explain the idea of limit. This is done in all calculus books. Our purpose, rather, is to give you some examples of how a calculator may be used in making reasonable guesses about the values of several kinds of limits. We hope that from these examples and from solving some related problems your understanding of limit will be deepened.

Most calculus texts introduce the idea of the limit of a function f at a point a, written as $\lim_{x\to a} f(x)$, with an informal definition. For example, $\lim_{x\to a} f(x) = L$ means that we can be sure that $f(x)$ is close to L provided that x is close to a. The function f need not be defined at a.

An equivalent (also informal) definition of the statement $\lim_{x\to a} f(x) = L$ is that the numbers $f(x_1), f(x_2), \ldots$ must approach L whenever the numbers $x_1, x_2, \ldots$ approach a. This definition suggests what is often done in practice. Choose a convenient sequence $x_1, x_2, \ldots$ of numbers approaching a and check if the corresponding sequence $f(x_1)$, $f(x_2), \ldots$ approaches a number L. If $a = 0$, for example, we may take $x_1 = 0.1$, $x_2 = 0.01$, $x_3 = 0.001, \ldots$, and calculate $f(0.1)$, $f(0.01)$, $f(0.001), \ldots$. Although such a numerical calculation is not a proof that $\lim_{x\to a} f(x) = L$, the "trend" of the sequence of function values often gives persuasive evidence about the behavior of $f(x)$ near a.

EXAMPLE 1. For our first example we consider the function $f(x) = \sin x/x$, defined for all $x \neq 0$. We wish to compute the limit of this function at $a = 0$, that is,

$$\lim_{x\to a} f(x) = \lim_{x\to 0} \frac{\sin x}{x}$$

SOLUTION. Note that the number 0 is not in the domain of f. This is echoed by the HP 48 in that if you store 'SIN(X)/X' as EQ on the VAR menu, go to the ⮢SOLVE ROOT SOLVR menu and attempt to evaluate this function at $x = 0$, the HP 48 will complain about an "Undefined Result." Please verify the entries of Table 4. Use the built-in SIN function for the second column, the SOLVR for the third, and STD mode.

Table 4

x	$\sin x$	$\sin x/x$
0.1	$9.98334\,166468 \times 10^{-2}$	0.99833 4166468
0.01	$9.99983\,333417 \times 10^{-3}$	0.99998 3333417
0.001	$9.99999\,833333 \times 10^{-4}$	0.99999 9833333
0.0001	$9.99999\,998333 \times 10^{-5}$	0.99999 9998333
0.00001	$9.99999\,999983 \times 10^{-6}$	0.99999 9999983
0.000001	0.000001	1
0.0000001	0.0000001	1

Table 4 gives persuasive evidence that $\lim_{x \to 0} \sin x/x = 1$, although the abrupt transition between the fifth and sixth rows raises questions about the calculations. In the bottom two rows of the table, the entries under x and $\sin x$ are the same. Without attempting to explain how the HP 48 calculates $\sin x$, it appears to be true that when x is less than 0.00001, the HP 48 does not distinguish between x and $\sin x$. Indeed, since to 25 decimals the true value of sin(0.000001) is

$$0.00000\,09999\,99999\,99983\,33333,$$

the HP 48, which is limited to 12 significant digits, can not distinguish between this function value and 0.000001. The HP 48 has given the best possible answers within its constraints.

As we consider further examples, we will occasionally notice consequences of the limited accuracy of the HP 48. The same is true for all calculators or computers. There are practical limits to the number of significant digits a computing device is able to calculate, store, or display. For our purposes, this limited accuracy will not matter; usually we will have long since gotten sufficient numerical evidence to guess a limiting value.

EXAMPLE 2. The bacterium *Escherichia coli*, usually called *E. coli*, is found in the human gut. Under ideal conditions each E. coli cell divides into two cells 1/3 hour after its own "birth." The mass of one E. coli cell is approximately $5 \cdot 10^{-13}$ grams. If at $t = 0$ we have one cell, the mass m of cells present at any time t (assuming no deaths, adequate food supply, ideal habitat, etc.) is

$$m = f(t) = 5 \cdot 10^{-13} 2^{3t} \tag{1}$$

We show a graph of this function in Fig. 2. The axes are scaled so that the graph has convenient proportions.

Equation (1) must eventually fail to describe the growing E. coli population since, according to (1), its mass would eventually be larger than that of the earth. (In problem A.2 we ask you to calculate Doomsday.) To monitor the growth, estimate the "birth rate" at any time t.

SOLUTION. We may take the birth rate as the increase in the total mass of E. coli per hour. The points A and B in the figure give the

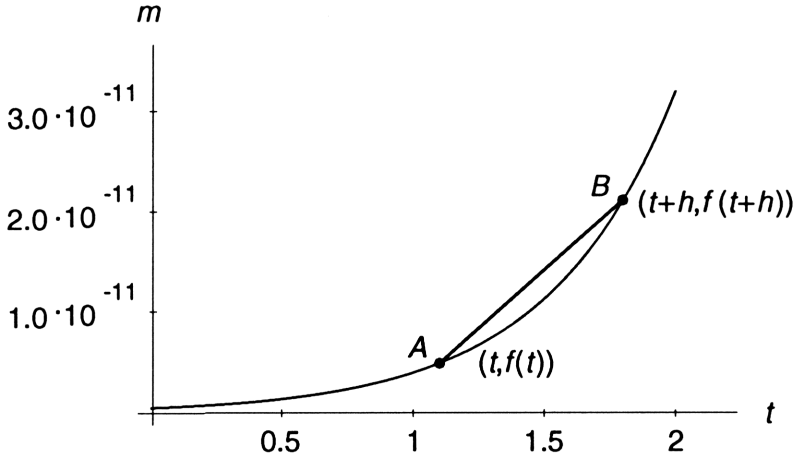


Figure 2

masses of the population at a fixed time t and a slightly later time $t+h$. The increase in mass in this time interval is $f(t+h)-f(t)$. If we divide this difference by the elapsed time we obtain the "average birth rate" (grams per hour) in the time interval $[t,t+h]$. Note that this is the same as the slope $S(h)$ of the line joining A and B, namely

$$\begin{aligned} S(h) &= \frac{f(t+h)-f(t)}{(t+h)-t} = \frac{5\cdot 10^{-13}\,2^{3(t+h)} - 5\cdot 10^{-13}\,2^{3t}}{h} \\ &= (5\cdot 10^{-13}\,2^{3t})\frac{2^{3h}-1}{h} \end{aligned}$$

The rearrangement in the last step (based on the fact that $a^{b+c}=a^b a^c$) shows that the average birth rate, or the slope, in the interval $[t,t+h]$ may be written as a product of two factors, one of which depends upon h alone. The birth rate $R(t)$ at time t would be the limiting value of the average birth rate as $h\to 0$, that is,

$$R(t) = \lim_{h\to 0} S(h) = 5\cdot 10^{-13}\,2^{3t}\lim_{h\to 0}\frac{2^{3h}-1}{h}$$

We may gain an idea of $R(t)$ by evaluating the expression $(2^{3h}-1)/h$ for small values of h. For this please verify the entries of Table 5.

Table 5

h	2^{3h}	$(2^{3h}-1)/h$
0.1	1.23114 441334	2.31144 41334
0.01	1.02101 212571	2.10121 2571
0.001	1.00208 160508	2.08160 508
0.0001	1.00020 796578	2.07965 78
0.00001	1.00002 079463	2.07946 3
0.000001	1.00000 207944	2.07944

We observe in the third column a steady decrease in the number of significant digits. The reason for this is that we are subtracting numbers which are very nearly equal. For example, in the fourth row we find $2^{3\cdot 0.0001}$ in the second column, with 12 significant digits; in the third column we subtract 1 from this and obtain $1.00020\,796578 - 1.00000\,000000 = 0.00020\,796578$, which has 8 significant digits. Since 2^{3h} approaches 1 as h approaches 0 (see column 2), the losses will increase in subsequent rows. Despite the loss in accuracy, we have no reason to doubt that

$$R(t) \approx 2.0794 \cdot 5 \cdot 10^{-13}\, 2^{3t} \tag{2}$$

Please use (2) to show that the "birth rate" of our population of E. coli at $t = 14$ hours is approximately 4.6 grams per hour. Perhaps you know that the limit of the third column entries is $3 \ln 2 = 2.07944154168 \cdots$.

EXAMPLE 3. The amount $A(t)$ of money in the bank after t years, assuming \$100 was invested at 7.5% and the amount is compounded n times per year, is given by

$$A(t) = 100(1 + 0.075/n)^{nt} = 100((1 + 0.075/n)^n)^t$$

This function was discussed in Example 4 and problems A.4–A.5 of §1.1. In problem A.5 we asked about the size of $A(5)$ for $n = 1, 2, 4, 365$, and $24 \cdot 365 = 8760$, which correspond to compounding once a year, semiannually, quarterly, daily, and hourly. We continue this numerical experiment here, first rearranging the expression for $A(t)$. Letting $m = n/0.075$ we may write

$$A(t) = 100\left(\left(1 + \frac{1}{m}\right)^m\right)^{0.075t} \tag{3}$$

Since we are interested in how $A(t)$ varies with n, it is enough to examine the subexpression $(1 + 1/m)^m$ of (3). Please verify the values of this subexpression, given in Table 6, as m, and so n, becomes large.

Table 6

n	m	$(1+1/m)^m$
1	13.33333 33333	2.62288 665433
2	26.66666 66667	2.66900 555024
4	53.33333 33333	2.69322 824158
365	4866.66666 667	2.71800 257835
$365 \cdot 24$	116800	2.71826 897430
	200,000	2.71827 503279
	400,000	2.71827 843061
	800,000	2.71828 012953

SOLUTION. Evaluate the expression '(1+1/M)^M' with SOLVR. Use the values of m given in Table 6. These calculations offer convincing evidence—but not proof—that for "continuous compounding"

$$A(t) = 100\left(\lim_{m\to\infty}(1+1/m)^m\right)^{0.075t} \approx 100 \cdot 2.71828^{0.075t} \tag{4}$$

The limiting number 2.71828··· is denoted by e, after Leonhard Euler, a Swiss mathematician. Using this notation we may rewrite (4) as

$$A(t) = 100\, e^{0.075t} \tag{5}$$

The *exponential function* $\exp(x) = e^x$ is built-in to the HP 48. To calculate e^2, for example, we may put 2 on level 1 and then press the e^x key. The value of $A(5)$ for "continuous compounding" is $100e^{0.075 \cdot 5}$, which, with the help of the EXP key, is equal to \$145.50. Please verify this. Finally, we note that the limiting process $\lim_{m\to\infty} f(m)$ considered in this example is different from the limits in Examples 1 and 2. It is different in that the variable t is not approaching a finite value, but is required to become larger than any given number, that is, to approach (positive) infinity.

EXAMPLE 4. Estimate the value of π by estimating the area of a circle of radius 1. Start with a circle of radius 1 and inscribe and circumscribe regular polygons with $n = 4, 8, 16, \ldots, 512$ sides. Calculate the average A_n of the areas of the nth inscribed and circumscribed polygons. The number A_n is an estimate of π.

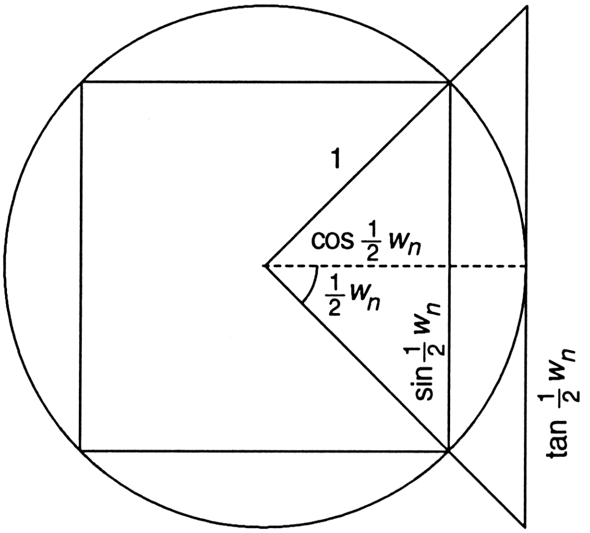


Figure 3

SOLUTION. If the vertices of the nth polygon are joined to the center of the circle, we obtain n isoceles triangles, all of which have an angle $w_n = 2\pi/n$ at the center. One of these triangles is shown in Fig. 3.

We have bisected the "central" angle w_n for the calculation of the two areas. Let L_n and U_n denote the areas of the inscribed and circumscribed polygons. Please verify that

$$L_n = \frac{n \sin w_n}{2} \quad \text{and} \quad U_n = \frac{n \sin w_n}{1 + \cos w_n} = \frac{2L_n}{1 + \cos w_n}$$

These formulas together with the half-angle formulas (6) show how we may estimate π. For polygons with 4, 8, 16, 32, ... sides, the number of sides is doubled at each step and the central angles are halved. We start with $n = 4$. Since $w_4 = 2\pi/4 = \pi/2$, we know the values of $\sin w_4$ and $\cos w_4$ without knowing the value of π. We have, then,

$$L_4 = \frac{4 \cdot 1}{2} = 2 \qquad U_4 = \frac{4 \cdot 1}{1 + 0} = 4$$

If we know $\cos(2\pi/4)$ and $\sin(2\pi/4)$, we may compute $\cos(2\pi/8)$ and $\sin(2\pi/8)$ using the half-angle formulas

$$\sin \tfrac{1}{2}\theta = \sqrt{\frac{1 - \cos\theta}{2}} \quad \text{and} \quad \cos \tfrac{1}{2}\theta = \sqrt{\frac{1 + \cos\theta}{2}} \tag{6}$$

Letting $\theta = w_4$ we have

$$\sin w_8 = \sqrt{\frac{1 - \cos w_4}{2}} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \cos w_8 = \sqrt{\frac{1 + \cos w_4}{2}} = \frac{1}{\sqrt{2}}$$

From these results we may compute the areas L_8 and U_8 of the inscribed and circumscribed polygons. We have

$$L_8 = \frac{8 \sin w_8}{2} = \frac{4}{\sqrt{2}} \quad \text{and} \quad U_8 = \frac{2L_8}{1 + \cos w_8} = \frac{8}{\sqrt{2} + 1}$$

In what follows it is convenient to denote $\sin w_n$ by S_n and $\cos w_n$ by C_n. Using this notation we may describe the steps from w_n to w_{2n}. Assuming we know n and C_n, we may compute S_{2n}, C_{2n}, L_{2n}, and U_{2n} using the results

$$S_{2n} = \sqrt{\frac{1 - C_n}{2}}\,, \; C_{2n} = \sqrt{\frac{1 + C_n}{2}}\,, \; L_{2n} = nS_{2n}\,, \; U_{2n} = \frac{2L_{2n}}{1 + C_{2n}} \tag{7}$$

We have used these formulas in obtaining Table 7. The last column contains the average $A_n = (L_n + U_n)/2$ of the inscribed and circumscribed areas.

Table 7

n	L_n	U_n	A_n
4	2	4	3
8	2.82842 712475	3.31370 849898	3.07106 781186
16	3.06146 745891	3.18259 787807	3.12203 266849
32	3.12144 515224	3.15172 490742	3.13658 502983
64	3.13654 849056	3.14411 838526	3.14033 343791
128	3.14033 115734	3.14222 363034	3.14127 739384
256	3.14127 725116	3.14175 036939	3.14151 381028
512	3.14151 380744	3.14163 2087	3.14157 294722

We stopped with $n = 512$ for two reasons. First, since our intent here is to give an example of a limit process associated with area and not to obtain an extremely accurate approximation to π, eight values of n are enough to give a strong sense that $\lim_{n\to\infty} A_n = 3.1415\cdots$. The second reason is the sudden change in the number of significant digits in the U_n column. One of the steps in the calculation can lead to a loss of accuracy (this will be explored in a problem) and we thought U_{512} was the first visible sign of this.

It is not difficult to verify the entries of Table 7. We give a program which more or less duplicates the steps suggested by (7). Given n and C_n as inputs, we want the program to output $2n$ and C_{2n} (for use in the next step) as well as L_{2n}, U_{2n}, and A_{2n}. For convenience we write the program so that it begins with the assumption that the first five levels of the stack are the previous output. Since only levels 5 and 4 are needed for the next step, the first two steps in the program CIRC are to drop the (contents of the) first three levels of the stack and to store (the contents of) levels 4 and 5. To duplicate the table the initial stack must be 4, 0, *, *, *, where * stands for any expression or number you care to put on levels 1, 2, and 3. Executing CIRC once gives the second line of the table. If CIRC is executed a second time, using for input the output of the first run, the third line of table is computed. Please enter CIRC and verify the table.

CIRC

Inputs: n C_n * * *	*Outputs:* $2n$ C_{2n} L_{2n} U_{2n} A_{2n}
`« 3 DROPN → N C «`	Drop numbers in levels 1,2,3; store numbers in levels 4 and 5 as local variables N and C
`1 C - 2 / √`	Compute S_{2n}
`1 C + 2 / √`	Compute C_{2n}
`→ S2 C2 «`	Store S_{2n} and C_{2n} as local variables S2 and C2
`2 N * C2 N S2 *`	$2n$, C_{2n}, and L_{2n} to stack
`DUP 2 * 1 C2 + /`	U_{2n} to stack
`DUP2 + .5 * » » »`	Duplicate U_{2n}; A_{2n} to stack
	Checksum: #2075d Bytes: 158.5

Exercises 1.3

A.1 Use your calculator to estimate each of the following limits.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}$

(b) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{3x^2+2x}$

(c) $\displaystyle\lim_{x\to 0}\frac{\sin(5x^2)}{3x^2+2x}$

(d) $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$

(e) $\lim_{\theta\to 0} \dfrac{\tan\theta - \sin\theta}{\sin^3\theta}$ (f) $\lim_{x\to 2} \dfrac{\sqrt[3]{x} - \sqrt[3]{2}}{x-2}$

A.2 Referring to Example 2, how long will it take for the mass of E. coli to exceed that of the earth (whose mass is $5.979{\cdot}10^{24}$ kilograms)?

A.3 Compute the slope of the line segment AB in Fig. 2 of Example 2, when $t = 4$ and $h = 0.05$. Compare this with the birth rate $R(4)$. Explain the result from a geometric point of view.

A.4 The mass of E. coli present at $t = 10$ hours is, from (1) of Example 2, $5 \cdot 10^{-13} 2^{30}$ grams. Find the time t at which the mass is three times that present at $t = 10$. You may either solve for t using logarithms and then use your HP 48 to obtain a numerical result or you may estimate the value of t using the SOLVR to repeatedly refine estimates of t.

A.5 To how many compoundings per hour does the last line of the table in Example 3 correspond?

A.6 Assuming that \$20 is a "significant" amount of money, for what amount P_0 of initial capital invested for 10 years at 7.5% would the difference between daily and continuous compounding become significant?

A.7 Verify the formulas for L_n and U_n given in Example 4.

A.8 In Example 4, compute the areas L_{16} and U_{16} in terms of L_8 and U_8.

B.1 Use your calculator to estimate each of the following limits.

(a) $\lim_{x\to 0} \dfrac{\arcsin x - \arctan x}{x^3}$ (b) $\lim_{\alpha\to\pi} \dfrac{2}{\pi}\left[\dfrac{\pi}{2} - \arctan\left(\dfrac{1+\cos\alpha}{\sin\alpha}\right)\right]$

(c) $\lim_{x\to 0} \sqrt{1-x^2}\left(\dfrac{\arcsin x}{x} + \sqrt{1-x^2}\right)$

(d) $\lim_{v\to 1-} \sqrt{\dfrac{1+v}{(1-v)^3}}\left(\sqrt{1-v^2} - v\sqrt{1-v^4} + \dfrac{\arcsin v - \arcsin v^2}{v}\right)$

B.2 For each number x assume the following limit exists and denote it by $g(x)$, that is

$$g(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

We wish to graph the function g. For this, define the function

$$G(x) = \frac{\sin(x+0.01) - \sin x}{0.01}$$

We expect that $g(x) \approx G(x)$ and, moreover, the graphs of g and G to be indistinguishable for the default PPAR. We may study g through G, which may be entered as a user-defined function. Note that $G(0)$ is given in Example 1. Graph $G(x)$. What function do you think g is?

B.3 What would $R(t)$ in (2) of Example 2 become if E. coli were to divide every 1/4 hour?

B.4 Graph $w(x) = (f(x+0.01) - f(x))/0.01$ to find the approximate maximum displacement of the railroad snubber function in §1.1.

C.1 Identify the probable cause of the loss of signicant digits in computing the table just following (7) in Example 4. Recalling that CIRC's outputs include cos w_n, estimate the probable number of significant digits for each of the values $n = 4, 8, \ldots, 512$. Assume (the true fact) that the computed values of cos w_n are within 1.0×10^{-10} of their true values.

C.2 The accuracy of Table 7 may be improved by modifying the formula for S_{2n} in (7). Show that

$$S_{2n} = \sqrt{\frac{1 - C_n}{2} \cdot \frac{1 + C_n}{1 + C_n}} = \frac{S_n}{\sqrt{2}\sqrt{1 + C_n}}$$

This modification may be used within CIRC to improve the accuracy of its output.

C.3 For a limiting process in which a number A is known to be larger than each entry of an ascending sequence L_n and smaller than each entry of a descending sequence U_n, explain why $|A - A_n| < |U_n - L_n|/2$, where $A_n = (L_n + U_n)/2$. (In Example 4, L_n and U_n are the areas of certain inscibed and circumscribed polygons and it is clear that the area A of the circle satisfies the inequalities $L_n < A < U_n$, $n = 1, 2, 3, \ldots$.) Note that the inequality $|A - A_n| < |U_n - L_n|/2$ makes it possible to state that the error made in approximating the unknown A with the calculated value A_n is less than a certain calculated value. Apply this result and the corrected table values computed in problem C.2 to estimate $|A_n - \pi|$ for $n = 8$, 16, …, 512.

C.4 The loss of accuracy in the third column of Table 5 in Example 3 can be avoided by using the EXPM key on the MATH HYP menu. If x is input, EXPM returns a highly accurate value of $e^x - 1$. Recalculate the last entry of column three of the table, showing that for $h = 0.000001$, $(2^{3h} - 1)/h \approx 2.07944370372$. You will need to use the fact that $2^x = e^{x \ln 2}$ for all x.

PROJECT

ARCHIMEDES' ALGORITHM

Find π to within 0.00001 by calculating inscribed and circumscribed approximations to the circumference of a unit circle, following the method used by the Greek mathematician and physicist Archimedes (287–212 BC). Include an error estimate and an analysis of possible sources of error.

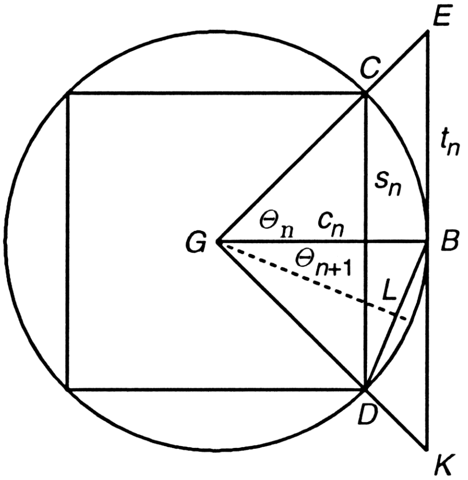


Figure 4

Start by dividing the central angle of a unit circle into 8 parts, so that $\theta_1 = \pi/4$. The angle $2\theta_1$ has chord CD and tangent EK. The lengths $4 \cdot CD$ and $4 \cdot EK$ are lower and upper approximations to the circumference. Bisect θ_1 to get θ_2. To θ_n correspond lengths c_n, s_n, and t_n. From s_n and t_n the circumference can be approximated. To θ_{n+1} correspond c_{n+1}, s_{n+1}, and t_{n+1}. In the figure these are GL, BL, and (not shown). Show that

$$c_{n+1}^2 = \frac{1 + c_n}{2}, \qquad \frac{2s_{n+1}}{s_n} = \frac{1}{c_{n+1}}, \qquad t_n = \frac{s_n}{c_n}$$

Finally, letting $S_n = 2^{n+2}s_n$ and $T_n = 2^{n+2}t_n$, which are lower and upper approximations to the circumference, show that

$$c_{n+1} = \sqrt{\frac{1 + c_n}{2}}, \qquad S_{n+1} = \frac{S_n}{c_{n+1}}, \qquad T_{n+1} = \frac{S_{n+1}}{c_{n+1}}$$

These *recursion relations* hold for $n = 1, 2, 3, \ldots$. Initial values of c_1 and S_1 are easily found. Values of c_2, S_2, and T_2 may be found from the recursion relations. Next, c_3, S_3, and T_3, and so on.

1.4 EVALUATION AND ZEROS OF POLYNOMIALS

In this section we discuss methods for evaluating and finding the zeros of polynomial functions. We use a combination of a synthetic division program and the powerful built-ins PLOT and SOLVR. In this we have tried to build on prior knowledge and to use the calculator as an accurate and rapid plotting or arithmetic device, not as a replacement for understanding. We minimize dependence on high-level built-ins until the end of Example 3, where we discuss PEVAL and PROOT. These built-ins provide fast and accurate methods for evaluating and finding the zeros of polynomials.

Polynomials have been part of mathematics and its applications for over 3500 years. Both linear and quadratic polynomials were solved in Mesopotamia prior to 1500 B.C.. A thousand years later Mesopotamian (same region but several political systems later) astronomers used piece-wise linear functions in their tables of motions of the sun, moon, and planets. Two thousand years after that, near 1500 AD, Italian mathematicians learned how to solve polynomial equations of the third and fourth degrees. In 1846 the French mathematician Évariste Galois showed that for polynomials of fifth and higher degrees there is no hope of finding formulas comparable to those found for the linear, quadratic, cubic, and quartic cases. It's like trisecting an angle with straightedge and compass, it just can't be done!

Polynomials are essential to the evaluation of the trigonometric, exponential, and logarithmic functions. For example, in plotting 'SIN(X)' your HP 48 repeatedly evaluates a polynomial approximation to $\sin x$, once for each of the 131 pixels across the screen. To speed up such calculations, efficient methods of polynomial evaluation are required.

We start with the problem of how to evaluate polynomials efficiently and relate this to synthetic division. We give a synthetic division program. We then turn to approximating the zeros of polynomials. We use PLOT or synthetic division to gain a rough idea of the location of the zeros and the SOLVR to find highly accurate approximations to them. We discuss deflation. We include a project in which we give formulas and programs for solving cubic and quartic polynomials. We end the section with examples showing the use of PEVAL and PROOT.

We show in Table 8 the number of multiplications (in columns headed with "×") and additions ("+") required to evaluate polynomials of degrees 1 through 4. On the left the evaluation is done with the polynomials in traditional form. On the right the evaluation is done with the polynomials in *nested form.* We used nested form in Example 3 in §1.2.

Table 8

×	+	Traditional Form	Nested Form	×	+
1	1	$a_1x + a_0$	$xa_1 + a_0$	1	1
3	2	$a_2x^2 + a_1x + a_0$	$x(a_2x + a_1) + a_0$	2	2
5	3	$a_3x^3 + a_2x^2 + a_1x + a_0$	$x(x(a_3x + a_2) + a_1) + a_0$	3	3
7	4	$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$	$x(x(x(a_4x + a_3) + a_2) + a_1) + a_0$	4	4

It is apparent from the table that evaluations using nested form are considerably faster than evaluations using the traditional form. Although the parentheses used in the nested form may give an impression of greater complexity, they in fact only specify the order in which the multiplications and additions are done.

The nested form is closely connected to *synthetic division*, which you may recall from your study of polynomial equations. We recall that a (real or complex) number c is a zero of a polynomial $p(x)$ if and only if $(x - c)$ is a factor of $p(x)$, that is, $p(x) = (x - c)q(x)$, where $q(x)$ is a polynomial. This result shows that we may test whether a number c is a zero of $p(x)$ by dividing $p(x)$ by the polynomial $x - c$ and noting whether the remainder is zero. We have

$$\frac{p(x)}{x-c} = q(x) + \frac{r}{x-c}$$

where $q(x)$ and r are the quotient and remainder of the division. Multiplying both sides of this expression by $(x - c)$ gives

$$p(x) = q(x)(x - c) + r \tag{1}$$

We use (1) in three ways:

I If $r = 0$, then c is a zero of $p(x)$.
II If $r = 0$, then the remaining zeros of $p(x)$ are those of $q(x)$.
III For all numbers c, $r = p(c)$.

Synthetic division is often presented as simply a condensed version of the division of $p(x)$ by $x - c$. We recall synthetic division by dividing $x^2 - 5x + 6$ by $x - 3$, thereby testing whether $x - 3$ is a factor and 3 a zero. The coefficients 1, -5, and 6 are put on the top row, with $c = 3$ displaced a little to one side. After the initialization step in which a 0 is placed in the first column of the second row, just under the leading coefficient, the arithmetic steps are (i) add the two numbers in the first and second row of the "active" column (initially, the active column is column 1, the left-most) and (ii) multiply the sum by c and record the product in the second row of the next column. Repeat this pattern until the last column of the third row has been calculated.

$$\begin{array}{rrr|r} 1 & -5 & 6 & 3 \\ 0 & 3 & -6 & \\ \hline 1 & -2 & 0 & \end{array}$$

The remainder r is the last entry of the third row, 0 in this case, which shows that $c = 3$ is a zero. The coefficients of the quotient polynomial $q(x)$ precede r in the third row. We have $x^2 - 5x + 6 = (x - 3)q(x) = (x - 3)(1 \cdot x - 2)$. By II, the remaining zero of $p(x)$ is the zero of $x - 2$, namely 2.

The arithmetic of this synthetic division is precisely that found when the nested form of $x^2 - 5x + 6$ is evaluated at $x = 3$. Please compare the arithmetic operations in the above synthetic division to those occurring when x is replaced by 3 in the nested form of $x^2 - 5x + 6$:

$$x^2 - 5x + 6 = x(x - 5) + 6$$

Before giving examples of finding the real and complex zeros of polynomials, we give the synthetic division program SYND. This program helps with the calculations needed to locate and find the zeros of polynomials. For SYND we use [a b c] to represent the polynomial $ax^2 + bx + c$, and similarly for polynomials of higher (or lower) degree. If a coefficient is 0, the brackets [⋯] must contain a corresponding 0. To represent $x^2 + 1$, for example, we use [1 0 1]. The program assumes that the initial stack contains a polynomial P, in the form [⋯], and a number c. These inputs correspond to the first row of the pencil and paper synthetic division algorithm discussed above. The output of SYND is two polynomials: P on level 2 and Q, which corresponds to the quotient polynomial and the remainder, on level 1. The polynomial P is returned to the stack for possible further use. Please enter SYND and store it under this name. The program accepts a variable such

SYND	
Inputs: P, c	*Outputs: P, Q or P, R*
≪ → P C	P, c → local variables P, C
≪ P SIZE OBJ→	Store size of P as
DROP → M	$\mathrm{M} = n + 1$
≪ 0 P 1	0 initializes the second row;
	P & 1 initialize third row
1 M 1 − START	Start an (m−1)-fold loop
GETI 4 ROLL + DUP	Add rows 1 and 2
4 ROLLD C ∗ 3 ROLLD	Multiply by c
NEXT GET +	End loop & do last addition
M C TYPE 6 If ≠	Is c number or variable?
THEN ROW→ P SWAP	Number: Outputs P & Q
ELSE ROLLD M 1 −	Variable: Outputs P & R
DROPN P SWAP END	
≫ ≫ ≫	
	Checksum: #2579d Bytes: 190.5

as 'X' in place of a number c. In this case SYND outputs P and R, where R corresponds to the remainder. We discuss this after Example 3.

In using SYND it is usually best to go into STD mode. If we wish to divide $x^2 - 5x + 6$ by $x - 3$, put the polynomial [1 −5 6] on the stack, then 3. Press the key beneath SYND on the VAR menu. You will see [1 −5 6] on level 2 and [1 −2 0] on level 1. This shows that 3 is a zero of the polynomial $x^2 - 5x + 6$.

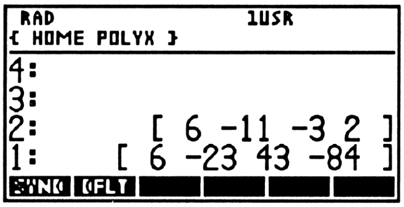
EXAMPLE 1. Use SYND in finding the zeros of $p(x) = 6x^3 - 11x^2 - 3x + 2$.

SOLUTION. How do we locate the real zeros, if any, of $p(x)$? We may calculate $p(c)$ for several values of c, attempting to locate the zeros by finding sign changes, or we may sketch the graph of $p(x)$. In this example we do the first of these, using SYND to evaluate $p(x)$. We first look for rational zeros. Recall that for a polynomial with integer coefficients the only rational numbers u/v which can be zeros are those for which u is a divisor of the constant term (2 here) and v is a divisor of the coefficient of the term of highest degree (6 here). The divisors of 2 are 1 and 2. The divisors of 6 are 1, 2, 3, and 6. The possiblilites for u/v are (with duplicates)

$$\pm\{1/1, 1/2, 1/3, 1/6, 2/1, 2/2, 2/3, 2/6\}$$

We try the smallest of the possible rational roots first.

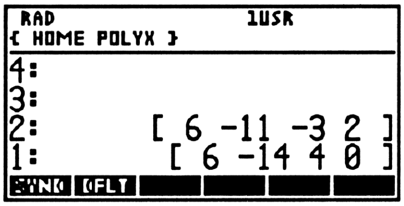
[6 −11 −3 2]
−2
SYND



```
RAD                    1USR
{ HOME POLYX }
4:
3:
2:      [ 6 -11 -3 2 ]
1:    [ 6 -23 43 -84 ]
```

Since $r = -84 \neq 0$, -2 is not a zero. In the same way we try -1, $-2/3$, and, finally, $-1/2$. We find that $c = -1/2$ is a zero.

[6 −11 −3 2]
−.5
SYND



```
RAD                 1USR
{ HOME POLYX }
4:
3:
2:      [ 6 -11 -3 2 ]
1:       [ 6 -14 4 0 ]
```

The quotient polynomial, whose zeros are the remaining zeros of $p(x)$, is $q(x) = 6x^2 - 14x + 4$. We could solve this quadratic and be done. Or we may continue with the list of possible rational roots. We try the latter and quickly find the remaining zeros 1/3 and 2.

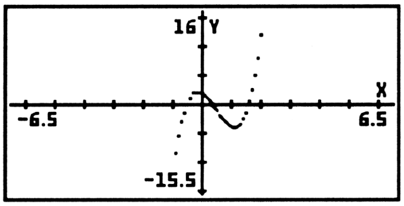
EXAMPLE 2. Find the zeros of the polynomial $p(x) = 7x^3 - 10x^2 - 3x + 2$.

SOLUTION. This polynomial has no rational zeros, as may be verified by trying the possiblilities $\pm\{1/1, 1/7, 2/1, 2/7\}$. We use PLOT to help locate any real zeros. For PLOT we must enter the polynomial as an algebraic expression. We choose the nested form. We may generate the nested form for use in PLOT by putting [7 − 10 − 3 2] and 'X' on the stack and running SYDN. It returns

'(−3+(−10+7∗X)∗X)∗X+2'

We explain this use of SYND later.

↰PLOT ↰ EQ
PPAR RESET NXT
5 ∗H NXT PLOT
DRAX DRAW

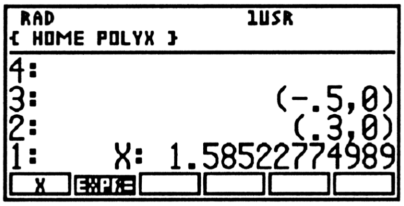


Move the cursor to each of the three zeros (please do this from left to right) and put it on the stack by pressing ENTER . After a CANCEL or two we find

$$(-.5, 0),\ (.3, 0),\ \text{and}\ (1.6, 0)$$

Answers may vary. We use these estimates in the SOLVR. The first X below takes the first number of the pair in level 1 as an estimate of a zero. The second X, which is preceded by ↰, starts the solution algorithm.

↰SOLV ROOT SOLVR
X (on SOLVR menu)
↰ X



```
RAD                 1USR
{ HOME POLYX }
4:
3:              (-.5,0)
2:               (.3,0)
1:    X: 1.58522774989
```

The SOLVR displays the message "SOLVING FOR X" during the calculation, followed by "Sign Reversal" and the zero 1.58522774989. Please find the other two zeros in this way. You may want to use PICK on the INTERACTIVE STACK (press ▲) to pick the next pair. Otherwise, you may record the zero and then remove it from the stack. You should find .353378662978 and −.510034984295. Answers may vary slightly, depending upon the initial estimate. You may have gotten the message "Zero" instead of "Sign Reversal." "Zero" means that the HP 48 has found a number c for which $p(c)$, as evaluated by the calculator, is 0. "Sign Reversal" means that no number has been found for which $p(c)$ is exactly 0, but two values were found, differing only in the least significant digit, for which the values of the polynomial have different signs.

There is an automated version of what we have just done. For this, return to the graph by pressing ◄. The PICTURE menu shown contains a FCN menu. Press FCN. Now move the cursor to a zero and press ROOT. The HP 48 calculates one of the zeros found above. Press any key in the top row to return to the FCN menu. You may obtain all zeros visible on the screen in this way. To return to the VAR menu, press CANCEL and VAR. Purge 'X' from VAR menu when done.

EXAMPLE 3. Find the zeros of the polynomial $L_4^0(x) = (x^4 - 16x^3 + 72x^2 - 96x + 24)/24$, which is one of the Laguerre polynomials used in quantum mechanics. It is known that the zeros of these polynomials are real, positive, and distinct.

SOLUTION. A rough idea of the size of the largest zero may be obtained by rearranging $L_4^0(x)$ slightly. We have

$$24L_4^0(x) = x^3(x - 16) + x(72x - 96) + 24$$

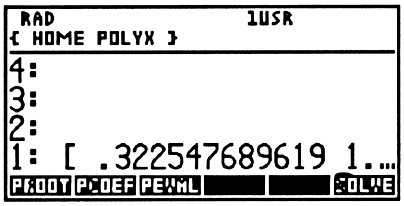
Since all terms are positive for $x > 16$ we may restrict our search to the interval $[0, 16]$. In finding the zeros of L_4^0 we may ignore the factor of $1/24$. Key in [1 − 16 72 − 96 24] and use SYND to search for points at which $p(x) = 24L_4^0(x)$ changes sign as x varies from 0 to 16. You should find changes of sign between 0 and 1, 1 and 2, 4 and 5, and 9 and 10. We use the SOLVR to find these zeros. For this we may use SYND to generate an algebraic expression for $p(x)$. Assuming that [1 − 16 72 − 96 24] is on level 2, DROP level 1, and key in 'X'. Running SYND returns

'(−96+(72+(−16+X)∗X)∗X)∗X+24'

Go to the SOLVR by ⬑SOLVE ROOT, store the polynomial on the stack as EQ, and press SOLVR. In Example 2 the initial estimate was a pair of numbers (of which the SOLVR uses only the first). We have a choice of a single number, a pair (a, b), or a list containing one, two, or three numbers. In this example we use a list containing two numbers, which together define an interval containing a zero. This is generally a better choice than a single number. For the zero between 0 and 1 we enter the list { 0 1 }. Put this list on the stack and store it by pressing X on the SOLVR menu. Then press ⬑X. You should obtain the zero .322547689619. Please find the other three zeros in this way. You should obtain 1.74576110115, 4.53662029697, and 9.3950709123.

In Examples 1, 2, and 3 we used synthetic division, plotting, or polynomial evaluation in either finding zeros directly or estimates for the SOLVR. The powerful built-in PROOT makes these procedures unnecessary.

[1 −16 72 −96 24]
↰SOLV POLY PROOT
To view, press ▼
Use cursor keys to view zeros



```
RAD                        1USR
{ HOME POLYX }
4:
3:
2:
1: [ .322547689619 1.…
```

The built-in PROOT calculates all zeros of polynomials with real or complex coefficients. It uses a highly sophisticated algorithm for its calculations. On the same menu as PROOT is PEVAL, which has the same general purpose as SYND. To explain PEVAL recall equation (1).

$$p(x) = q(x)(x - c) + r \tag{1}$$

Given $p(x)$, the synthetic division algorithm gives both $q(x)$ and r. The output Q of SYND gives both $q(x)$ and r.

The main difference between SYND and PEVAL is that SYND will accept two forms of input. For SYND, the bracketed form P of the polynomial $p(x)$ must be on level 2 and either a number c or a variable `'X'` must be on level 1. When the input is a number, SYND returns P and Q, where Q is the quotient polynomial and remainder combined in a bracketed form. When the input is a variable, SYND returns P and an algebraic form of the remainder, which, as noted, is the nested form of $p(x)$. PEVAL requires input P and c and returns the remainder.

We give a second program, whose output is the bracketed form of $q(x)$. It is used when c is a zero of $p(x)$. In this case, the quotient polynomial $q(x)$, whose zeros are the remaining zeros of $p(x)$, is called a *deflated* polynomial. The program is called DFLT. The input to DFLT should be P and c, where $p(c) = 0$.

```
≪ SYND OBJ→ OBJ→ DROP 1 −

SWAP DROP ROW→ SWAP DROP ≫
```

Store this program as DFLT. Try DFLT on the polynomial $p(x) = x^2 - 5x + 6$, whose zeros are 2 and 3. If you deflate $p(x)$ with the zero 2, you should obtain the linear polynomial $x - 3$ whose zero is the remaining zero of $p(x)$.

EXAMPLE 4. Use the SOLVR, SYND, DFLT, and QUAD in finding the zeros of the polynomial $p(x) = 3x^4 + 4x^3 - 2x^2 + 2x + 3$.

SOLUTION. Begin by keying in $P = [\ 3\ \ 4\ \ -2\ \ 2\ \ 3\ ]$. We use PLOT to locate the real zeros of p, if any. For this enter `'X'` and run SYND. Store and plot by ↰PLOT, ↰EQ, PPAR, RESET, NXT, 10, ∗H, NXT, PLOT, DRAX, and DRAW. It appears that p has two complex zeros and two real zeros. Put the real zeros on the stack by locating them with the cursor (left-most zero first, please) and then ENTER. This gives $(-1.8, 0)$ and $(-.7, 0)$. Answers may vary. Although we could give the SOLVR initial estimates in the form of lists with two elements, we choose convenience over caution and use the pairs we got from PLOT. Since the equation is already stored, we may go to the SOLVR, enter the first initial guess, and solve. We obtain $-.695164924913$ as one zero. Use SWAP to put the estimate for the second real zero in level 1 and use the SOLVR again, obtaining -1.7459736875 as the second real zero of p. Answers may vary slightly. Now go to the VAR menu and purge `'X'`. The stack should contain the two real zeros. We are ready to deflate twice, ending with a quadratic, which we may solve for the complex zeros of p.

We use the interactive stack to prepare for DFLT. Press ▲, move the cursor to level 3, and then ROLLD. Leave the interactive stack by CANCEL. Run DFLT, then SWAP, and DFLT a second time. The quadratic

$$[3 \quad -3.32341583724 \quad 2.47169972328\ ]$$

should be on the stack. We may solve this with QUAD. Start by converting the bracketed form to nested form. For this enter `'X'` and then $\boxed{\text{SYND}}$. The quadratic equation solver QUAD takes two stack inputs, an algebraic expression on level 2 and the variable `'X'` on level 1. Enter `'X'` once more. Before running QUAD go to ↱MODES FLAG to check on system flag 01. We want it "unchecked," that is, to read "General solutions." After OK and CANCEL, run QUAD, which is on the *Lsh*SYMBOLIC menu. If in the expression QUAD leaves on the stack we give to s1 the values 1 and -1 we obtain both zeros of the quadratic. For this we may use the SOLVR. Store the expression as `'EQ'`. In the SOLVR, key in 1 and then press the keys beneath s1 and EXPR=. Repeat this procedure using s1=-1. The complex zeros of p are found to be

$$(.55390263954, \pm .719090935607) = .55390263954 \pm .719090935607i$$

Answers may vary a little.

We may reduce this lengthy calculation to a few key strokes with PROOT. Key in $P = [\,3 \quad 4 \quad -2 \quad 2 \quad 3\,]$ again and run PROOT, which is found on the ↰SOLVE menu under POLY. The result may be inspected by pressing $\boxed{\text{EDIT}}$.

In this section we have discussed a few of the many known techniques for finding zeros of polynomials. We recalled that a real or complex number c is a zero of a polynomial if and only if $x - c$ is a factor, noted the connection between synthetic division of $p(x)$ by $x - c$ and the nested form of $p(x)$, and promoted the advantages for efficient calculation the nested form has over evaluation of $p(x)$ in its traditional form. We recalled the "rational roots" theorem. We discussed the use of the SOLVR and the two supplementary programs SYND and DFLT. We gave a brief description of PROOT and PEVAL.

Exercises 1.4

Problems A.1–A.8 use the rational zeros result discussed in Example 1. Only SYND and QUAD need be used. Optionally, DFLT may be used to reduce the number of rational numbers tested.

A.1 Use SYND in finding the rational zeros of $x^3 - 6x^2 - x + 30$.

A.2 Use SYND in finding the rational zeros of $15x^4 + 17x^4 - 249x^2 + 199x - 30$.

A.3 Use SYND in finding the rational zeros of $175x^4 - 45x^3 - 633x^2 + 37x + 18$. You may wish to use DFLT to reduce the number of possible rational zeros.

A.4 Use SYND and QUAD in finding the zeros of the polynomial $x^6 + 3x^5 - 36x^4 - 45x^3 + 93x^2 + 132x + 140$, given the real zeros are in the interval $[-8, 6]$.

A.5 Find any rational zeros of $x^3 - 2x^2 - 25x + 50$. Why may the search be restricted to integers?

A.6 Find the zeros of $6x^4 - 7x^3 + 8x^2 - 7x + 2$. By noting the alternating signs of the terms give an argument why this polynomial can have no negative real zeros.

A.7 Find the zeros of $2x^3 + 12x^2 + 13x + 15$. Give an argument why this polynomial can have no positive real zeros.

A.8 Find a rational root of the polynomial in Example 5.

Problems A.9–A.14 may be solved using the techniques discussed in Example 2. Use PLOT and the SOLVR. If the polynomial has only one real zero, use DFLT and QUAD to find all zeros.

A.9 Find the zeros of $x^3 - 3x + 1$.

A.10 Find the zeros of $x^3 + 3x^2 - 3$.

A.11 Find the zeros of $x^3 + 6x^2 + 8x - 1$.

A.12 Find the zeros of $x^3 + 4x^2 - 10$.

A.13 Locate a plane parallel to the base of a hemisperical solid that divides it into two parts of equal volume. The following formula may be used. If a sphere of radius a is cut by a horizontal plane h units above the center of the sphere, the volume of the spherical cap above the plane is $\pi(2a^3 - 3a^2h + h^3)/3$.

A.14 A spherical plastic float of radius 10 centimeters is made from material whose density is 1/4 that of water. To what depth will the float sink in water? Assume Archimedes' principle that the float will displace a volume of water equal in weight to that of the float.

Problems A.15–A.21 may be solved using the techniques discussed in Example 3, using SYND to detect changes in sign of the given polynomial and the SOLVR, with initial estimate a list with two numbers, to find the zeros. We give other Laguerre polynomials as well as Jacobi polynomials, Hermite polynomials, Gegengauer polynomials, and Legendre polynomials, all of which are important in mathematical physics.

A.15 Find the zeros of $L_4(x) = (-x^3 + 9x^2 - 18x + 6)/6$.

A.16 Find the zeros of $L_5(x) = (-x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120)/120$.

A.17 Find the zeros of the Jacobi polynomial $(21/2)(x-1)^3 + 28(x-1)^2 + 21(x-1) + 4$.

A.18 Find the zeros of the Hermite polynomial $16x^4 - 48x^2 + 12$.

A.19 Find the zeros of the Gegenbauer polynomial $192x^5 - 160x^3 + 24x$.

A.20 Find the zeros of the Legendre polynomial $(35x^4 - 30x^2 + 3)/8$.

Problems A.21-A.23 may be solved using the techniques discussed in Example 4, which include PLOT, SYND, DFLT, and the SOLVR, as appropriate.

A.21 Find the zeros of $36x^4 - 157x^3 + 490x^2 - 49x - 390$.

A.22 Find the zeros of $14x^5 - 64x^4 + 747x^3 - 3097x^2 + 2989x + 1911$, given that $7i$ is one zero.

A.23 Find the zeros of $20x^5 + 72x^4 + x^3 - 31x^2 - 41x - 21$.

A.24 Find where the curves $y = 5(3x^4 - 6x^2 + 1)$ and $y = 2(3x^5 - 5x^3)$ intersect.

A.25 A hollow steel ball sinks in water to the depth of its outer radius. If the thickness of the metal is 1 cm and the density of steel is 7.5 times that of water, find the outer radius of the ball. Recall problem A.14.

The remaining A problems may have multiple zeros. Use any technique we have discussed.

A.26 Find the zeros of $343x^5 - 392x^4 + 2002x^3 - 4745x^2 + 2416x + 448$.

A.27 Find the zeros of $7x^5 - x^4 - 42x^3 + 6x^2 + 63x - 9$. There are two multiple zeros.

A.28 Find the zeros of $729x^5 - 810x^4 + 342x^3 - 72x^2 + 8x - 32/81$.

B.1 Explain the programs SYND and DFLT.

B.2 For polynomials with real coefficients complex zeros occur in conjugate pairs, that is, whenever $a + bi$ is a zero, so is the conjugate $a - bi$. The HP 48 representation of $a + bi$ is (a, b). All of the programs we have used in this section accept complex numbers as input. Find the zeros of $x^4 - x^3 - 37x + 91$, given that $-2 + 3i$ is a zero. Use DFLT twice and then solve the resulting simple quadratic by hand.

B.3 When using deflation on several zeros it is preferable to find and remove zeros in the order of their absolute values, smallest first, to minimize error. Check this using the polynomial

$$x^4 - 1111\pi x^3 + 112110\pi^2 x^2 - 1111000\pi^3 x + 1000000\pi^4$$

The zeros of this polynomial are known to be $x_j = 10^j\pi$, j=0,1,2,3. Determine these zeros using SYND and DFLT alternately. Use 3, 30, 300, and 3000 as inital estimates for the SOLVR. Do the entire procedure twice, once in increasing order, once in decreasing order. Which zeros are more accurate?

B.4 Find the zeros of the Chebyshev polynomial

$$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1,$$

finding and deflating as in problem B.3. You may check your own answers by using the fact that the zeros $x_1, x_2, \ldots, x_8$ are given by

$$z_k = \cos\frac{(2k+1)\pi}{16} \qquad k = 1, 2, \ldots, 8\,.$$

B.5 Find all zeros of the polynomial

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 3-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

C.1 Let $C(p) = \mu f_1 + \alpha f_2$ be the "arithmetic cost" of the polynomial $p(x)$, where μ and α are the smallest number of multiplications and additions (or subtractions) required to evaluate $p(x)$ in its given form. The factors f_1 and f_2 are the costs of doing one multiplication and addition, respectively. We will use the term "monic" for a polynomial with leading coefficient 1.

(a) The polynomial $p_1(x) = x^3 + 7x^2 + 2x - 5$ can be written in either of the forms $p_2(x) = x(x(x+7)+2)-5$ or $p_3(x) = (x^2+1)(x+7)+(x-12)$. Show that the arithmetic costs of $p_1(x)$, $p_2(x)$, and $p_3(x)$ are $4f_1+3f_2$, $2f_1+3f_2$, and $2f_1 + 4f_2$. The form $p_3(x)$ of $p_1(x)$ is called the *preconditioned* form of $p_1(x)$.

(b) For a third degree monic $x^3 + ax^2 + bx + c$, find formulas for coefficients r, s, and t such that

$$x^3 + ax^2 + bx + c = (x^2 + r)(x + s) + (x + t)$$

Show that the arithmetic costs of evaluating this polynomial once, using traditional, nested, and preconditioned forms (including the cost of preconditioning) are, respectively, $4f_1 + 3f_2$, $2f_1 + 3f_2$, and $(2f_1 + 3f_2) + (f_1 + 2f_2)$.

(c) If it required to evaluate the monic given in part (b) for all values of x between 0 and 10 at intervals of 0.001, determine which form has least cost.

C.2 Extend the procedure discussed in problem C.1 to monics of degree $n = 2^k - 1$, where $k \geq 2$ is a positive integer. Specifically, work out the case $k = 3$, first expressing $x^7 + a_6x^6 + \cdots + a_1x + a_0$ in the form

$$(x^4 + a)(x^3 + q_2x^2 + q_1x + q_0) + (x^3 + r_2x^2 + r_1x + r_0)$$

Apply the procedure of problem C.1 to the two cubic polynomials. Find, separately, the arithmetic cost of the preconditioning and the cost of one evaluation of the preconditioned form.

C.3 If it is required to evaluate the monic in C.2 for all values of x between 0 and 10 at intervals of 0.001, what is the cheapest method of the three?

C.4 In this problem and the next two problems we ask you to establish a simple method for finding an upper limit for the real zeros of a polynomial. Problem C.4 may be viewed as an elaboration of the data occurring in synthetic division. We begin by recalling that the entries on the third line of the synthetic division of $a_nx^n + \cdots + a_1x + a_0$ by $x - c$ are a_n, $ca_n + a_{n-1}$, ... For what follows it is useful to replace c by x and to define polynomials $f_k(x)$, $k = 0, 1, \ldots, n$. On each line of the definition (after the first) we give the relation of the function to the preceding function. These relations are easy to verify.

$$\left[\begin{array}{ll} f_0(x) = a_n & \\ f_1(x) = xa_n + a_{n-1} & = xf_0(x) + a_{n-1} \\ f_2(x) = x^2a_n + xa_{n-1} + a_{n-2} & = xf_1(x) + a_{n-2} \\ \vdots & \\ f_n(x) = x^na_n + \cdots + xa_1 + a_0 & = xf_{n-1}(x) + a_0 \end{array}\right. \tag{2}$$

Using these polynomials, show that

$$f_j(x) = (x - c)\left[f_0(c)x^{j-1} + f_1(c)x^{j-2} + \cdots + f_{j-1}(c)\right] + f_j(c) \tag{3}$$

for $j = 0, \ldots, n$.

C.5 Use (3) in problem C.4 to justify the following procedure for finding an upper limit for the positive zeros of a polynomial $p(x) = a_nx^n + \cdots + a_1x + a_0$, where

$a_n > 0$. Choose a positive number c and try synthetic division. If the entries of the third row are all nonnegative and the last is positive, then c is an upper limit for the positive zeros of $p(x)$. (Try this on the polynomial of Example 5. You will find that $c = 0.8$ is an upper bound.)

C.6 How may the result of C.5 be used in case $a_n < 0$? How may the result of C.5 be used to find a lower bound for negative zeros?

PROJECT

CARDANO'S AND FERRARI'S METHODS FOR SOLVING CUBICS AND QUARTICS

Gerolamo Cardano (1501–1576), an Italian physician, mathematician, and astrologer, was the first to give an algebraic solution of polynomial equations of degree three, usually called cubics. Lodovico Ferrari (1522–1565), a student of Cardano, was the first to give an algebraic solution of polynomial equations of degree four, usually called quartics. Both solutions were given in Cardano's book *Ars magna* ("The Great Skill"), published in 1545 and important in the history of algebra.

Cardano's Method

Fill in the details of the following outline of Cardano's algebraic solution of the cubic equation

$$x^3 + ax^2 + bx + c = 0, \qquad \text{where } a,\ b, \text{ and } c \text{ are real.} \tag{1}$$

Step 1 Show that if x is replaced by $y - a/3$ in (1) we obtain

$$y^3 + py + q = 0, \qquad \text{where } p = b - \frac{a^2}{3} \text{ and } q = c - \frac{ab}{3} + \frac{2a^3}{27}. \tag{2}$$

Step 2 Show that if y is replaced by $u + v$ in (2) we may rearrange the result to obtain

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0 \tag{3}$$

This arrangement suggests the system of equations

$$\begin{cases} 3uv + p = 0 \\ u^3 + v^3 + q = 0 \end{cases} \tag{4}$$

Step 3 To express the solutions of (4) in a convenient form we consider the polynomial $x^3 - s$, where s is a given real or complex number. The *Fundamental Theorem of Algebra* states that a polynomial of degree n has at least one solution. By deflation and successive applications of this theorem, it follows that a polynomial equation of degree n has exactly n zeros, some of which may be multiple. Thus the equation $x^3 - s$ has 3 zeros. Show that these zeros are

$$\sqrt[3]{s}, \qquad \omega\sqrt[3]{s}, \qquad \omega^2\sqrt[3]{s}$$

where $\sqrt[3]{s}$ denotes one of the cube roots of s and $\omega = -1/2 + i\sqrt{3}/2$ is a "cube root of unity."

Show that the system (4) can be solved by using the first equation to eliminate v from the second equation. The result is a polynomial equation of degree 6, which may be considered as a quadratic in the variable u^3. Show that the roots of the quadratic equation are

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \quad \text{and} \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Show that for any solution (u, v) of (4) either

$$u^3 = A,\ v^3 = B \qquad \text{or} \qquad u^3 = B,\ v^3 = A$$

Show that the solutions of the system (4) are

$$(u_1, v_1), \qquad (u_2, v_2) = (\omega u_1, \omega^2 v_1), \qquad (u_3, v_3) = (\omega^2 u_1, \omega v_1)$$

where

$$u_1 = \sqrt[3]{A} \qquad \text{and } v_1 \text{ satisfies} \quad 3u_1 v_1 + p = 0$$

Show that the solutions of (1) are

$$x_1 = u_1 + v_1 - a/3, \qquad x_2 = u_2 + v_2 - a/3, \qquad x_3 = u_3 + v_3 - a/3$$

Step 4 Write two programs which, together, find the zeros of (1). First, write a program called RCUB (reduced cubic), with input [1 0 p q] and output y_1, y_2, and y_3 on levels 1, 2, and 3. A second program, CARD, with input [1 a b c] and output x_1, x_2, x_3 on levels 1–3, can calculate the input of RCUB and call RCUB. Try RCUB on $f(x) = x^3 + 9x - 2$. The zeros are

$$.22102253739$$
$$-.110511268697 + i\,3.00610016825,$$
$$-.110511268699 - i\,3.00610016824, \quad \text{and}$$

Try both programs on $f(x)$. Try CARD on $g(x) = x^3 + x^2 - 2$, whose zeros are 1 and $-1 \pm i$. Verify with RCUB that the zeros of $x^3 - 3x + 1$ are $2\cos 40°$, $-2\cos 20°$, $2\cos 80°$.

Ferrari's Method

Using Cardano's work on cubics, Ferrari gave an algebraic solution of the general quartic

$$x^4 + ax^3 + bx^2 + cx + d, \qquad \text{where } a, b, c, \text{ and } d \text{ are real.} \tag{5}$$

We give Birkhoff and MacLane's arrangement of Ferrari's method. Fill in the details of the following outline.

Step 1 Show that the substitution $y = x + a/4$ in (5) gives an equation of the form

$$y^4 + py^2 + qy + r \tag{6}$$

Step 2 Show that for any u whatever, (6) can be rewritten as

$$(y^2 + u/2)^2 - [(u-p)y^2 - qy + (u^2/4 - r)] \tag{7}$$

Step 3 Recalling that a quadratic polynomial $Ax^2 + Bx + C$ is a perfect square provided that $B^2 - 4AC = 0$, show that the second term of (7) is a perfect square provided that u is chosen so that

$$4(u-p)(u^2/4 - r) - q^2 = 0 \tag{8}$$

Step 4 Letting the polynomial in (8) be denoted by $f(u)$, show that $f(u)$ changes sign in the interval (p, ∞). It follows that $f(u)$ has a real zero $u_1 > p$. What happens if $q = 0$?

Step 5 With $u = u_1$, (7) has the form $S^2 - T^2$, which factors into two quadratic factors, the zeros of each of which are easily found. From this result the zeros of (5) can be found.

Step 6 Write a program FERR for finding the zeros of (5). Test your program on the polynomials $2x^4 + 5x^3 - 8x^2 - 17x - 6$, $x^4 + 2x^2 + x + 2$, $x^4 - 3x^2 + 6x - 2$, and $x^4 - x^2 - 2x - 1$. The zeros are -3, -1, $-1/2$, 2; $(-1 \pm i\sqrt{3})/2$, $(1 \pm i\sqrt{7})/2$; $-1 \pm \sqrt{2}$, $1 \pm i$; $(1 \pm \sqrt{5})/2$, $(-1 \pm i\sqrt{3})/2$.

Chapter **2**

Derivatives

2.0 PREVIEW

A function f is differentiable at x if under increasing magnification its graph looks more and more like a straight line through $(x, f(x))$. Geometrically, the derivative $f'(x)$ of f at x is the slope of the line towards which the magnifications tend. Symbolically, we say that $f'(x)$ is the limit of a quotient function Q defined in terms of x. In this chapter we discuss the difference quotient, the symmetric difference, the angle of inclination of a tangent line to the graph of f, the angle between intersecting graphs, the tangent line approximation, the reflection properties of conics, and symbolic differentiation.

2.1 DIFFERENTIABLE FUNCTIONS AND THE DIFFERENCE QUOTIENT

Through the zoom features of the HP 48 it is easy to make immediate, gut-level sense of the statement that a function f is differentiable at a point x if it is increasingly line-like under magnification. We give an example using ZIN on the ZOOM menu.

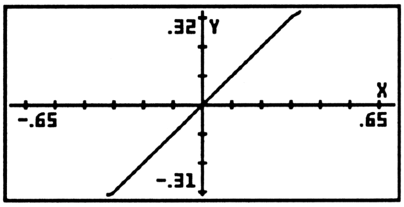
EXAMPLE 1. Give graphical evidence that the function $f(x) = \sin x$ is differentiable at $x = 0$.

SOLUTION. Graph $f(x) = \sin x$ in radian mode and with the default PPAR. With the graph on the screen, we use ZFACT on the ZOOM menu to set the horizontal (H-FACTOR) and vertical (V-FACTOR) scale factors to 0.1. We zoom in with ZIN.

$\boxed{\text{ZOOM}}$ $\boxed{\text{ZFACT}}$

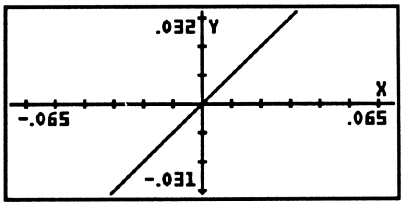
.1 $\boxed{\text{OK}}$ .1 $\boxed{\text{OK}}$

$\boxed{\text{OK}}$ $\boxed{\text{ZIN}}$ $\boxed{-}$



Since we started with RESET, after ZIN the tick marks on the axes are separated by 0.1 units. Move the cursor to the first tick mark to the right of the origin and press $\boxed{+}$. The coordinates of this point should be (0.1,0). It appears, then, that for $-0.3 \leq x \leq 0.3$ the graph of $\sin x$ is not distinguishable from a straight line through the origin with slope 1. Press $\boxed{+}$ again and repeat ZIN.

$\boxed{\text{ZOOM}}$ $\boxed{\text{ZIN}}$



The coordinates of the first tick mark to the right of the origin are (0.01,0). It thus appears that for $-0.032 \leq x \leq 0.032$ the graph of $\sin x$ is not distinguisable from that of $y = x$. Further magnification would show the same thing.

The derivative of a function f at a point x is defined as the limit of the *difference quotient*

$$Q(h) = \frac{f(x+h) - f(x)}{h}, \qquad h \neq 0 \tag{1}$$

as $h \to 0$. If this limit exists we denote its value by $f'(x)$ and say that f is differentiable at x.

If f is given no physical interpretation, the quotient $Q(h)$ is usually interpreted geometrically, as the slope of the "secant" line joining the points $(x, f(x))$ and $(x+h, f(x+h))$ on the graph of f. This familiar interpretation is shown in Fig. 1.

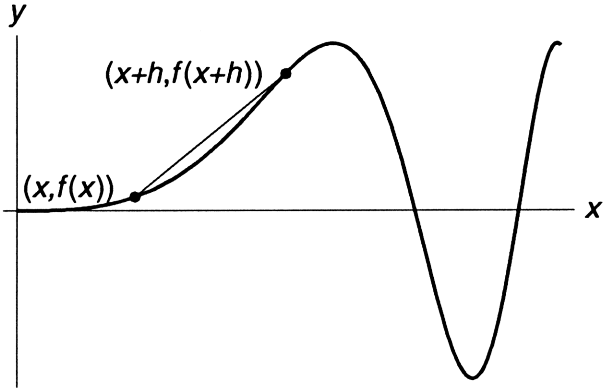


Figure 1

If $f(x)$ is the position of a particle at time x, then $Q(h)$ is the average velocity of the particle on the time interval $[x, x+h]$. In either of these interpretations, x is thought of as fixed and h as a variable. Geometrically, as $h \to 0$ the secant line turns about $(x, f(x))$, tending towards coincidence with the tangent line to the graph at this point. For the particle, the average velocity approaches the "instantaneous velocity" at time x.

To calculate the derivative $f'(x)$ of f at x we must know or be able to calculate the value of f at x and at all points $x+h$ near x. In this case, $Q(h)$ can be calculated for all h near but not equal to 0. There are, however, applications in which a function is neither known nor can be calculated at all points $x+h$ near x. For example, the position of a particle may be known only at times $x = 0.0, 0.1, 0.2, \ldots, 10.0$ seconds. We may, nonetheless, calculate its average velocity for each of the intervals $[0.0, 0.1]$, $[0.1, 0.2]$, …, $[9.9, 10.0]$. We use the formula

$$\frac{f(x+0.1)-f(x)}{0.1}, \qquad x = 0.0, 0.1, 0.2, \ldots, 9.9$$

Plotting the average velocity data on these short intervals would give some idea of how the velocity of the particle is changing with time. Why? Because on small time intervals we expect the average velocity to be very nearly equal to the velocity.

In our discussion of the difference quotient we use the common "square bracket" notation. We write

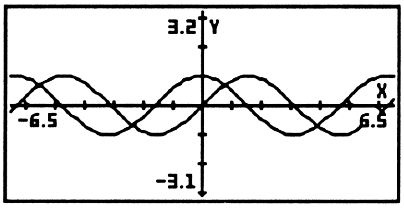
$$f[x, x+h] = \frac{f(x+h)-f(x)}{h} \tag{2}$$

In what follows we think of (2) with h fixed and x variable. For each x, we freeze the limit as $h \to 0$ at a small, fixed value of h, say, $h = 0.1$. In terms of slope, this amounts to studying the slope of the graph of f at points $(x, f(x))$ by approximating them with the slopes $f[x, x+h]$ of the secant lines. In terms of the velocity of a particle, we approximate the velocity at times x by calculating the average velocities $f[x, x+h]$. We give two examples.

EXAMPLE 2. Compare the graphs of $\sin x$ and its difference quotient $\sin[x, x+0.1]$ by plotting them simultaneously.

SOLUTION. Store 'SIN(X)' as EQ1 and '(SIN(X+.1)−SIN(X))/.1' as EQ2. To plot simultaneously, the first thing we do in ↱PLOT is to check (✓) the SIMULT option on the ↱PLOT OPTS screen.

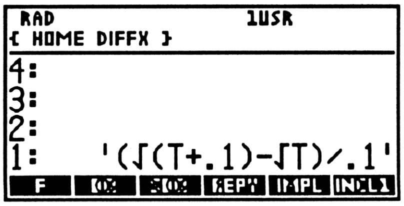
↱PLOT OPTS
Check the SIMULT option
OK NXT RESET
NXT CHOOS
Check EQ1 and EQ2
OK DRAW EDIT
NXT LABEL −



In the resulting figure, the heights of the two graphs at x are $\sin x$ and the approximate slope of the sine function at the point $(x, \sin x)$. Starting at $x = 0$, where the slope of the sine curve appears to be near 1 (the angle of inclination is near 45°), the slope decreases as x moves towards $\pi/2$, where it appears that the tangent line is horizontal. The graph of $\sin[x, x+0.1]$ is a record of the slopes of the secant lines to the graph of $\sin x$. As the two graphs are being plotted, try to see them as curve and "slope curve." As the plotting starts at the left, the sine curve starts out on the bottom. Perhaps you noticed that the slope curve is barely distinguishable from the cosine curve. This demonstration is strong evidence that $d(\sin x)/dx = \cos x$.

Keying in $f[x, x+h]$ can become tedious, even for relatively simple functions. We discuss a way of generating $f[x, x+h]$ for functions defined in program style. We demonstrate with the function $f(t) = \sqrt{t}$. In what follows, the letter F refers to the F variable put on the VAR menu by DEF.

'F(T)= √T' $\boxed{\text{DEF}}$
'T+.1' $\boxed{\text{ENTER}}$ F
'T' $\boxed{\text{ENTER}}$ F
$\boxed{-}$ .1 $\boxed{\div}$



```
RAD                    1USR
{ HOME DIFFX }
4:
3:
2:
1:    '(√(T+.1)-√T)/.1'
```

This calculation is illustrated in the next example.

EXAMPLE 3. Plot the position $y = f(t)$ and the average velocity $f[t, t+0.05]$ of a mass hanging from a spring, where

$$y = f(t) = 0.7\sin(2\pi t/3)$$

Estimate from the graph the position at $t = 4.2$ and the average velocity of the mass on the interval $[4.2, 4.25]$.

SOLUTION. The velocity of the mass at any time t may be approximated by calculating the average velocity on the interval $[t, t+0.05]$. The calculation is

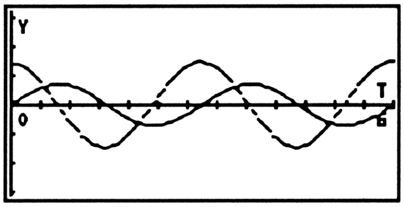
$$f[t, t+0.05] = \frac{0.7\sin(2\pi(t+0.05)/3) - 0.7\sin(2\pi t/3)}{0.05}$$

The period of oscillation is $2\pi/(2\pi/3) = 3$ seconds. We plot two periods to get a sense of the continuing motion. Begin by using DEF on

'F(T)=.7*SIN(2*π*T/3)'

This puts F on the VAR menu, replacing any earlier function stored as F. We plot $f(t)$ and $f[t, t+0.05]$ simultaneously.

'T+.05' $\boxed{\text{ENTER}}$ F 'T'
$\boxed{\text{ENTER}}$ F $\boxed{-}$ .05
$\boxed{\div}$ 'EQ1' $\boxed{\text{STO}}$ 'T'
$\boxed{\text{ENTER}}$ F 'EQ2' $\boxed{\text{STO}}$
$\boxed{\text{↱PLOT}}$ $\boxed{\text{NXT}}$ $\boxed{\text{RESET}}$
$\boxed{\text{NXT}}$ $\boxed{\text{CHOOS}}$
Check EQ1 and EQ2;
then set INDEP to T and
H-VIEW from 0 to 6
$\boxed{\text{DRAW}}$



The curve with the larger amplitude is the average velocity. The other curve is the coordinate position of the mass. To find the position and average velocity at $t = 4.2$ turn on TRACE. Use ▲ or ▼ to put the cursor on the position curve, press $\boxed{+}$ to display coordinates, and then trace until $t = 4.2$. Do the same for the velocity curve. You should find $f(4.2) \approx 0.41$ and the average velocity on the interval $[4.2, 4.25]$ to be -1.2, approximately.

In the above examples, the difference quotient $f[x, x+h]$ was a good approximation to the function $f'(x)$, for the reason that h was sufficiently small relative to the change in f'. By contrast, the sketch in Fig. 2 shows a case in which the difference quotient (slope of the line CB) is at best a fair approximation to $f'(x)$ (slope of the line DCE). The stepsize h is not small enough or, better, the stepsize is not small enough relative to how rapidly the slope of the curve is changing. From the figure we see that the slope of line AB is much closer to the slope of the tangent line DCE than the slope of CB.

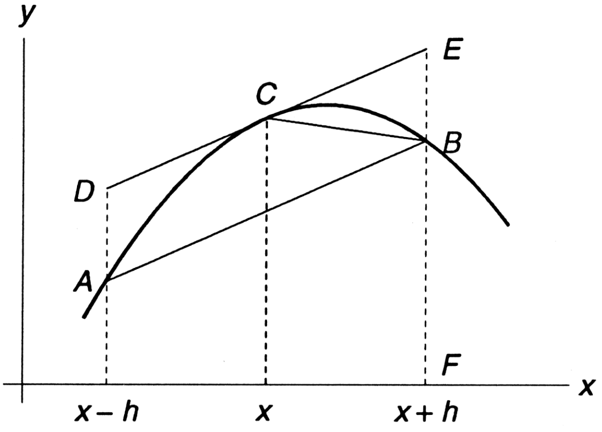


Figure 2

The slope of AB is called the *symmetric difference* of f. From the figure the symmetric difference is

$$\frac{f(x+h) - f(x-h)}{2h}$$

If f is differentiable at x, the limit of the symmetric difference as $h \to 0$ is $f'(x)$.

We give one example of how information about f at one point can be used to approximate f at nearby points. Using Fig. 2, the question is how can we approximate $BF = f(x+h)$, given the values h, $f(x)$, and $f'(x)$? The standard answer, referred as the *tangent line approximation*, is that $EF \approx BF$. The distance EF is the height of the tangent line DCE above F. Since the equation of this line is

$$Y - f(x) = f'(x)(X - x),$$

where the running variables of the line equation are X and Y, we see that

$$f(x+h) = BF \approx EF = Y = f(x) + f'(x)((x+h) - x) = f(x) + hf'(x),$$

so that

$$f(x+h) \approx f(x) + hf'(x) \tag{3}$$

EXAMPLE 4. A car is moving on a straight road. The car is observed passing the 119 mile marker with velocity 63.2 miles per hour (the car is moving towards the 120 mile marker). Where will the car be after 5 seconds?

SOLUTION. We may answer the question without explicitly using the tangent line approximation. In the absence of other information, we assume the velocity is constant and use the formula $D = RT$. The result would be that the car is at the $119 + 63.2(5/3600) \approx 119.087$ mile point. The assumption of constant velocity is equivalent to estimating the car's position at $t = t_0 + 5/3600$ seconds knowing its position and velocity at $t = t_0$ seconds and using the tangent line approximation. We are neither given t_0 nor does it matter. We may imagine the position $f(t)$ of the car at any time t is a specific but unknown function. What we do know is that $f(t_0) = 119$ and $v = f'(t_0) = 63.2$. Using the tangent line approximation (3) we have

$$f(t_0 + 5/3600) \approx f(t_0) + hf'(t_0) = 119 + (5/3600)63.2 \approx 119.087$$

In this section we have studied the idea of derivative from two viewpoints. First we used a graphical approach, magnifying the graph of f at a point $(x, f(x))$. If f is differentiable there, its graph is increasingly line-like under magnification. Secondly, we used the difference quotient and symmetric difference to numerically approximate the derivative. In the next section we study the derivative from a symbolic viewpoint.

Exercises 2.1

A.1 Repeat all of Example 1 with $f(x) = \cos x$. What is your conclusion?

A.2 Repeat all of Example 1 with $f(x) = \tan x$. Compare $f[x, x + .1]$ with $\sec^2 x$ by graphing them simultaneously. What is your conclusion?

A.3 Find an approximate value of the maximum velocity of the mass discussed in Example 3. If you know how to differentiate this function, check your result.

A.4 You may know that the derivative of $f(x) = \sqrt{x}$ is $1/(2\sqrt{x})$ for $x > 0$. Plot $f[x, x + 0.1]$ and $f'(x)$ together for $x \geq 0.1$. How do they compare? Where on $[0.1, \infty)$ do they differ most? Explain.

B.1 If the position $s(t)$ (in meters) of a particle moving on a line is known only at times $t = t_0, t_1, t_2, \ldots, t_n$ (in seconds), the velocity can not be found by differentiating $s(t)$. However, we may approximate the velocity by calculating $s[t_i, t_{i+1}]$ for $i = 0, 1, 2, \ldots, n-1$. The calculation is easily done by hand when the number of data points is small. When the data set is larger, the calculation is better left to a calculator or computer.

(a) The pairs

$$(0.0, 0.0),\ (0.2, 0.06),\ (0.4, 0.18),\ (0.6, 0.34),\ (0.8, 0.54),\ (1.0, 0.78),$$
$$(1.2, 1.04),\ (1.4, 1.32),\ (1.6, 1.62),\ (1.8, 1.92),\ (2.0, 2.22)$$

give the positions $(t_i, s(t_i))$ of a particle at times $t_0, t_1, \ldots, t_{10}$. Calculate and sketch the velocity $v(t_i)$ using $v(t_i) \approx s[t_i, t_i + (t_{i+1} - t_i)]$ for $i = 0, \ldots, 10$.

(b) We give a program to calculate velocities as in part (a). The program assumes that pairs $(t_i, s(t_i))$ are on the stack, with $(t_0, s(t_0))$ at the top. The output is a graph of the velocities. Try the program on the data given in part (a). In the explanations column of the program we use the brief notation p_k to denote the data point (t_k, s_k). We use the arrow to denote stack changes. For example, $p_{k-1}, p_k \rightarrow p_{k-1}, p_k, p_{k-1}$ suggests the way the command OVER changes the stack.

VEL

Inputs: $(t_0, s(t_0)), \dots, (t_n, s(t_n))$	*Outputs: Velocity Graph*
`« CLΣ`	Clear statistical matrix
`DEPTH`	Find number $(n+1)$ of data points
`1 - 1 SWAP`	Put 1 and n on the stack
`START OVER DUP`	Set $k = n$; $p_{n-1}, p_n \rightarrow$ $p_{n-1}, p_n, p_{n-1}, p_{n-1}$
`RE 3 ROLLD`	$p_{n-1}, p_n, p_{n-1}, p_{n-1} \rightarrow$ $p_{n-1}, t_{n-1}, p_n, p_{n-1}$
`- C→R`	$p_{n-1}, t_{n-1}, p_n, p_{n-1} \rightarrow$ $p_{n-1}, t_{n-1}, t_n - t_{n-1}, s_n - s_{n-1}$
`SWAP /`	$p_{n-1}, t_{n-1}, t_n - t_{n-1}, s_n - s_{n-1} \rightarrow$ $p_{n-1}, t_{n-1}, v_{n-1}$
`2 →ARRY Σ+ NEXT` `DROP`	$p_{n-1}, t_{n-1}, v_{n-1} \rightarrow$ $p_{n-1}, [t_{n-1}\ v_{n-1}]$; store $[t_{n-1}\ v_{n-1}]$ in statistical matrix
`SCATRPLOT DRAX PICTURE »`	Draw graph and axes
	Checksum: #38097d Bytes: 77.5

(c) Enter the position data

$$(0.0, 0.0),\ (0.2, 0.2),\ (0.4, 0.6),\ (0.6, 1.1),\ (0.8, 1.7),\ (1.0, 2.2),\ (1.2, 2.2),$$
$$(1.4, 1.9),\ (1.6, 1.5),\ (1.8, 0.9),\ (2.0, 0.1),\ (2.2, -0.8),\ (2.4, -1.4),\ (2.6, -1.6),$$
$$(2.8, -1.3),\ (3.0, -0.9),\ (3.2, -0.4),\ (3.4, 0.2),\ (3.6, 0.8),\ (3.8, 1.3),\ (4.0, 1.7)$$

Before using VEL to plot the velocity data, key in 21 (the number of data items) and use →LIST on the OBJ menu on the PRG TYPE menu to save a copy of this (somewhat tediously entered) data. Store it as DAT. After recalling it to the stack for use in VEL, use OBJ→ to spread it up the stack for VEL. You will need to DROP one number (the number of entries in the list) from the stack. Now press VEL . These programs use the STAT menu to store and plot. SCATRPLOT autoscales the plot. Use the plot to estimate the maximum and minimum velocities and the times they occur.

(d) If the data in part (c) were velocity data instead of position data, how could VEL be used to compute approximate values of the acceleration at times $t = 0,\ 0.2, \dots,\ 2.8$?

B.2 Experiment with FEVAL to find a method of calculating $f[x, x+h]$, using w instead of x to avoid a problem with FEVAL.

B.3 The program FEVAL assumes that the stack contains an expression for f in level 2 and a number x in level 1. FEVAL returns the function and the value $f(x)$. Write a program DQEV that assumes the initial stack contains an expression for f in level 3, a number x in level 2, and a number h in level 1. The program should return the function and the value $f[x, x+h]$. The program may itself call FEVAL. Test your program on a simple function, such as $f(x) = \sqrt{x}$.

B.4 Write a program DQ which assumes a function f has been stored as F by DEF on the VAR menu. The inputs to DQ should be x and h on the stack. The output should be $f[x, x+h]$. The program DQ should have the form ≪ → X H '*an expression*' ≫. Test your program on a simple function such as $f(x) = \ln x$, using 'X' and 'H' as inputs.

B.5 The difference quotient $f[x, x+h]$ is closely connected to the definition of $f'(x)$. For functions whose tangent lines are not turning too fast relative to the step-size h, we have observed that $f'(x) \approx f[x, x+h]$. The "cost" of the difference quotient is two function evaluations, a subtraction, and a division. At almost the same cost a considerably better approximation to $f'(x)$ can be calculated. It is called the symmetric difference. See Fig. 2. Write a program SDQ which calculates the symmetric difference. Follow the suggestions in problem B.4. Use the function $f(x) = \sqrt{x}$ to compare the difference quotient and symmetric difference approximations. Let $h = 0.1$. Plot both approximations simultaneously, first setting H-VIEW: 0.1 4 and V-VIEW: 0 1.6. To look more closely at the steepest part of the graph, set H-VIEW: 0 2. Using the same viewing rectanagle, graph simultaneously the difference quotient, the symmetric difference, and the derivative $1/(2\sqrt{x})$.

B.6 Use the Mean Value Theorem in showing that the symmetric difference is the average of two values of the derivative. This suggests why the symmetric difference often better approximates the derivative than the difference quotient.

C.1 The sketch in Fig. 2 of the symmetric difference and the difference quotient suggests the former is at least in some cases a better approximation to the derivative than the latter. Problem B.5 gives evidence for this in a specific case. In problems C.1 and C.2 we study the two approximations more closely, with x fixed and h variable. In particular we take $f(x) = \sqrt{x}$, $x = 1$, and let h take on values 0.1, ,0.01, 0.001, ... We define the functions Δ_1 and Δ_2 by

$$\Delta_1(h) = \frac{\sqrt{1+h}-1}{h} - \frac{1}{2} \quad \text{and} \quad \Delta_2(h) = \frac{\sqrt{1+h}-\sqrt{1-h}}{2h} - \frac{1}{2}$$

These functions measure how well the difference quotient and symmetric difference approximate the derivative $f'(x) = 1/(2\sqrt{x})$ at $x = 1$. Verify the entries of Table 1 (you may wish to use the short program given below) and attempt to reconstruct the reasoning that led to deciding the third and fifth columns would be of interest. How significant is the improvement?

This program uses DQ from problem B.4, has input h, and outputs the second and third columns. It may be modified to output all columns.

```
≪ → H ≪ 1 H DQ .5 - DUP H / ≫ ≫
```

The ✱ symbols denote a loss of accuracy due to the subtraction of two numbers of nearly equal value. Most of the missing values may be calculated by "rationalizing the numerators" of the difference quotient and symmetric difference.

Table 1

h	$\Delta_1(h)$	$\Delta_1(h)/h$	$\Delta_2(h)$	$\Delta_2(h)/h^2$
.1	−.0119	−.119	.000628	.0628
.01	−.00124	−.124	.00000625	.0625
.001	−.000125	−.125	.0000000615	.0615
.0001	−.0000125	−.125	*	*
.00001	−.000001	−.1	*	*
.000001	*	*	*	*

C.2 Use cases $n = 1$ and $n = 2$ of Taylor's Formula to explain the numerical study of the functions Δ_1 and Δ_2 in problem C.1. Taylor's Formula, which is included in most calculus texts, may be stated as follows: If f and its first n derivatives are defined and continuous on an interval $[a, b]$ and its $(n + 1)$st derivative exists on (a,b), then for any two numbers $x, x + h \in [a, b]$, there is a number c between x and $x + h$ for which

$$\begin{aligned} f(x+h) = f(x) + f^{(1)}(x)h + \frac{1}{2!}f^{(2)}(x)h^2 + \cdots + \frac{1}{n!}f^n(x)h^n \\ + \frac{1}{(n+1)!}f^{(n+1)}(c)h^{n+1} \end{aligned} \tag{4}$$

The case $n = 0$ is the Mean Value Theorem, which you have seen. (The most common form of the Mean Value Theorm follows from Taylor's Formula by letting $x + h = b$ and $x = a$.) The expression for $\Delta_1(h)$ is easily formed. For $\Delta_2(h)$ apply Taylor's Formula twice, once to get an expression for $f(x + h)$ and again for $f(x - h)$. There is no reason why the two values of c you will get are equal.

C.3 Knowing something about how the error in an approximation process behaves can lead to a significant improvement in estimation. For the function $f(x) = \sqrt{x}$, problems C.1 and C.2 suggest that for a given value of x, the error $\Delta_2(h) \approx kh^2$, where k is a constant. Assuming this is the case we may write

$$f'(x) - SD(x, x+h) \approx kh^2 \qquad \text{and} \qquad f'(x) - SD(x, x+h/2) \approx kh^2/4 \tag{5}$$

where $SD(x, x + h)$ is the symmetric difference of f at x with stepsize h. If the second of the relations in (5) is multiplied by −4 and the result is added to the first relation, show that

$$f'(x) \approx \frac{4}{3}SD(x, x+h/2) - \frac{1}{3}SD(x, x+h) \tag{6}$$

This is called *Richardson's extrapolation.* We now have three approximations to $f'(x)$: the difference quotient $f[x, x + h]$, the symmetric difference $SD(x, h)$, and the result in (6). Show that for $f(x) = \sqrt{x}$ these approximations to $f'(1) = 0.5$, with $h = 0.1$, are 0.4880884817, 0.500627750595, and 0.499999311388.

2.2 SYMBOLIC DERIVATIVES

The procedure or algorithm for calculating the derivative $f'(x)$ of a differentiable function f at a point x can be stated in the form of a number of rules. For example, if f is a sum of several differentiable functions u, v, and w, $f'(x)$ may be found by calculating and adding the derivatives of u, v, and w. If f is a product of two functions u and v, then we use the "product rule," according to which $f'(x)$ is a certain combination of u, v, u', and v'. If $f(x) = \sin(u(x))$ is a composite function, where u is a differentiable function, then we apply the "chain rule" to obtain $f'(x) = \cos(u(x))u'(x)$. There are altogether between 11 and 33 differentiation rules used in elementary calculus. (The count depends upon several things. Does one count both addition and subtraction? Does one count both the natural logarithm function and logarithms to other bases? What about the inverse cosecant function? Etc.)

Your HP 48 "knows" all the differentiation rules of elementary calculus. It does not always simplify the derivative it obtains in the ways most of us have learned, but the results are algebraically the same as those obtained by hand. The HP 48 can be used to calculate the numerical value of $f'(x)$ for specific values of x as well as a symbolic expressions for an unspecified value of x. It can differentiate equations implicitly and be used to illustrate the chain rule. We illustrate these features of HP 48 differentiation in a series of examples.

To differentiate symbolically your HP 48 must be in the symbolic mode. Go to the ↰MODES menu and press the key beneath SYM. The SYM mode should be ON, that is, there should be a small square next to SYM.

EXAMPLE 1. Differentiate the function $f(x) = x^2$ symbolically.

SOLUTION. Put the expression `'X^2'` on the stack, followed by `'X'`, and then press $\boxed{\partial}$. The result `'2*X'` is returned to the stack. If a number was returned to the stack instead of this expression, you must have a variable X stored somewhere on the path between your current directory and HOME. If this is the case, the result returned to the stack was the derivative $2x$ evaluated at the stored value of X.

EXAMPLE 2. Calculate the slope of the graph of the function $f(x) = x \sin x$ at $x = 1.0$.

SOLUTION. First, store the value 1.0 under the name X. Next, put `'X*SIN(X)'` on level 2 and `'X'` on level 1. Now press $\boxed{\partial}$. The number $1.38177\cdots = \sin 1 + \cos 1$ is returned to the stack. This is the slope of the graph at $(1, f(1))$. Purge X from the VAR menu.

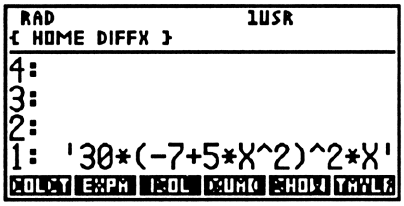
EXAMPLE 3. The method used in Examples 1 and 2 is called "complete differentiation" to distinguish it from "stepwise differentiation." The latter may be used to illustrate the chain rule. Differentiate $f(x) = (5x^2 - 7)^3$ using stepwise differentiation.

SOLUTION. To motivate the HP 48 notation used for stepwise differentiation, recall that there are several notations used for the derivative of f at a point x.

$$\frac{d}{dx}f(x) \qquad \text{or} \qquad f'(x) \qquad \text{or} \qquad D_x f(x)$$

The HP 48 notation imitates the third of these. To differentiate $f(x) = (5x^2 - 7)^3$ stepwise, first purge X (if necessary) and then

'∂X((5*X^2−7)^3)'
$\boxed{\text{ENTER}}$ $\boxed{\text{EVAL}}$ $\boxed{\text{EVAL}}$
$\boxed{\text{EVAL}}$ $\boxed{\text{EVAL}}$ $\boxed{\text{EVAL}}$
$\boxed{\text{↰SYMBOLIC}}$ $\boxed{\text{COLCT}}$



```
RAD                    1USR
{ HOME DIFFX }
4:
3:
2:
1:  '30*(-7+5*X^2)^2*X'
```

The first EVAL differentiates only the outermost function, in this case $g(w) = w^3$. One of the factors in the derivative is $g'(5x^2-7) = 3(5x^2-7)^2$. The derivative of $5x^2-7$ is indicated but not evaluated. The second EVAL differentiates the sum $5x^2 - 7$, the third the product $5x^2$, the fourth differentiates x^2, and the fifth EVAL differentiates x. COLCT simplifies the result. If before starting a step-by-step differentiation a number is stored as X, the final result is a number, the derivative evaluated at the stored value. If we store the number 4 as X and differentiate as before, we find $f'(4) = 639480$ after 5 EVALs.

EXAMPLE 4. Differentiate the function $f(x) = x^3$, assuming it is defined in program style and stored as F, that is

$$\ll \;\; \rightarrow \;\; \mathrm{X} \;\; \texttt{'X\^{}3'} \;\; \gg$$

SOLUTION. A function defined using "program style" can be differentiated using either the stepwise or complete method. To differentiate f using the complete method put 'F(X)' and 'X' on the stack and then press $\boxed{\partial}$. To differentiate f using the stepwise method, we EVAL the expression '∂X(F(X))'. In either case, if a number x is stored as X the final result will be the number $f'(x)$.

EXAMPLE 5. Differentiate the composite function $\cot(x^3)$, first by direct entry and then by defining each of the functions $f(u) = \cot u$ and $g(x) = x^3$ separately in program style.

SOLUTION. To differentiate $\cot(x^3)$ by direct entry, key in

'INV(TAN(X^3))'

followed by 'X' and $\boxed{\partial}$. The result may be rearranged by hand to give the more traditional form $-3x^2\csc^2(x^3)$.

To do the same calculation using functions defined in program style, press $\boxed{\text{DEF}}$ after entering each of

'COT(U)=INV(TAN(U))' and 'G(X)=X^3'

To differentiate the composite function $\cot(x^3)$, put 'COT(G(X))' and 'X' on the stack and press $\boxed{\partial}$.

EXAMPLE 6. It is sometimes useful to think of a composite function as the result of combining several equations by substitution. For example, the composite function $\sin\sqrt{x}$ may be thought of as the "chained" equations

$$y = \sin w \qquad \text{and} \qquad w = \sqrt{x}$$

where w is the "chaining" variable. In this context the chain rule is often written as

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = \cos w \frac{1}{2\sqrt{x}} = \cos\sqrt{x}\,\frac{1}{2\sqrt{x}}$$

Model this approach to composite functions and the chain rule on your HP 48.

SOLUTION. Store the expression 'SIN(W)' as Y and store '√X' as W. To calculate dy/dx put 'Y' and 'X' on the stack and press $\boxed{\partial}$. We find 'COS(√X)*(1/(2*√X)).

Implicit Differentiation on the HP 48

The functions

$$f(x) = \sqrt{1-x^2} \qquad \text{and} \qquad g(x) = -\sqrt{1-x^2}$$

"satisfy" the equation $x^2+y^2-1=0$ for $-1 \le x \le 1$. This means that if in the equation y is replaced by either of the expressions $\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$, the left side of the equation is identically equal to 0 for all $x \in [-1,1]$. This is summarized by saying that the equation defines the functions *implicitly*. The equation $x^2+y^2-1=0$ is sufficiently simple that f and g can be found *explicitly* by "solving" the equation for y in terms of x. Many equations cannot be solved in this way but nonetheless define one or more functions implicitly. For example, the equation $\sin y + (x^2+2)y + x = 0$ defines y as a function of x, but cannot be solved (in finite terms) for y. Whether an equation can be solved or not, we may wish to calculate the derivative of any differentiable function f defined by the equation. To prepare for the discussion that follows, we recall the technique of "implicit differentiation."

We show the main steps in implicit differentiation for each of the two equations we have discussed.

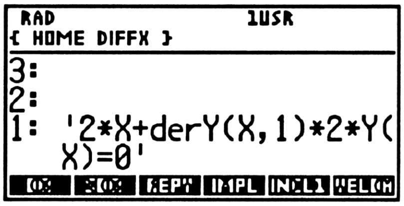
$$\begin{aligned} x^2+y^2-1&=0 \\ 2x+2yy'&=0 \\ y'&=-\frac{x}{y} \end{aligned} \qquad\qquad \begin{aligned} \sin y+(x^2+2)y+x&=0 \\ y'\cos y+2xy+(x^2+2)y'+1&=0 \\ y'&=\frac{-2xy-1}{\cos y+x^2+2} \end{aligned}$$

When we differentiate implicity we make a mental note that, say, y is a function of x, but rarely write anything down. If we are to use the HP 48 to differentiate implicitly, we must write something down for it. The way we do this is to replace y by $y(x)$ in equations like $x^2+y^2-1=0$. We discuss this in the next example.

EXAMPLE 7. Differentiate the equation $x^2 + y^2 - 1 = 0$, which implicitly defines y as a function $y(x)$ of x.

SOLUTION. Purge any variables X or Y.

```
'X^2+Y(X)^2 -1=0
ENTER
'X'  ENTER
∂
```



```
RAD                    1USR
{ HOME DIFFX }
3:
2:
1: '2*X+derY(X,1)*2*Y(
   X)=0'
```

This result corresponds to $2x + 2yy' = 0$. Leave the final result on the stack for use in the next example.

We give a short program to rewrite expressions like that left on the screen in Example 7. This program, which depends upon the ↑MATCH command on the ↰SYMBOLIC menu, replaces all subexpressions Y(X) in an expression by Y and all subexpressions derY(X,1) by DY.

```
« { 'Y(X)' Y } ↑MATCH DROP { 'derY(X,1)' DY } ↑MATCH DROP »
```

Please store this program in a DIFFX directory, naming it, say, REPY.

EXAMPLE 8. Differentiate the functions defined implicitly by

$$x^2 + y^2 - 1 = 0 \qquad \text{and} \qquad \sin y + (x^2 + 2)y + x = 0$$

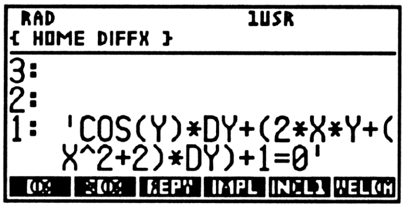
with the help of the program REPY.

SOLUTION. For the first equation, given that the final result of Example 7 is on the stack, we simply press $\boxed{\text{REPY}}$. The result is

'2∗X+DY∗2∗Y=0'

The second equation leads to a more complex result.

'SIN(Y(X))+
  (X^2+2)∗Y(X)+X=0'
$\boxed{\text{ENTER}}$ 'X' $\boxed{\text{ENTER}}$
$\boxed{\partial}$ $\boxed{\text{REPY}}$



```
RAD                    1USR
{ HOME DIFFX }
3:
2:
1: 'COS(Y)*DY+(2*X*Y+(
   X^2+2)*DY)+1=0'
```

Leave this result on the stack for use in Example 9.

Using the methods of Example 8, the result of differentiating an equation implicitly is an equation in terms of the variables x, y, and y' (X, Y, and DY). Usually we want to solve for y'. If the equation has been copied from the screen (after pressing $\boxed{\blacktriangledown}$ to produce a more readable form), it is easy to solve by hand for y' since the equation is "linear" in y'. If we do not wish to copy from the screen and do further calculations by hand, there are several things we may do.

If we wish to calculate y' for specific values of x and y, then we may use the SOLVR and the ALGEBRA menu. This is discussed in Example 9. If we want to write y' as a function of x and y and wish this done by the HP 48, the situation is not quite so easily handled. We give in Exercise C.1 a program that returns y' as a function of x and y.

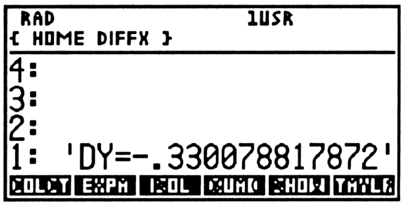
EXAMPLE 9. The equation

$$\sin y + (x^2 + 2)y + x = 0 \tag{1}$$

defines y as a function of x. Find the derivative of this implicitly defined function at the point (0.1, –0.0332246220278), which satisfies (1).

SOLUTION. Given that the final result from Example 8 is on the stack,

$\boxed{\text{↰SOLVE}}$ $\boxed{\text{ROOT}}$
↰ EQ $\boxed{\text{SOLVR}}$
.1 X (on the SOLVR menu)
.0332246220278 $\boxed{+/-}$ Y
$\boxed{\text{EXPR=}}$ $\boxed{\text{DROP}}$
$\boxed{\text{↰SYMBOLIC}}$ $\boxed{\text{COLCT}}$
'DY' $\boxed{\text{ISOL}}$


```
RAD                  1USR
{ HOME DIFFX }
4:
3:
2:
1:  'DY=-.330078817872'
```

So, $y' = -0.330078817872$ at (0.1, –0.0332246220278).

In this section we used the symbolic differentiation features of the HP 48 to illustrate the chain rule and implicit differentiation. When y is defined implicitly, we used the REPY program and the SOLVR to calculate y' at a given point (x, y).

Exercises 2.2

A.1 Differentiate each of the following functions symbolically. Purge X from all directories on the current path before starting.

(a) $-3x^2 - 5x + 2$
(b) $\sqrt{3x - 5}$
(c) $\dfrac{2x^3 - 5}{4x^2 + 7}$
(d) $\sin\sqrt{x}$
(e) $[11\sqrt{2x^2 + 1}/(7 + 11x)]^{3/2}$
(f) $\sqrt{x + \sqrt{x + \sqrt{x}}}$

A.2 Repeat problem A.1 but instead of obtaining a symbolic result find the values of the derivatives at $x = 2$.

A.3 Repeat problem A.1 but use the SOLVR to evaluate the symbolic expressions at $x = 2$.

A.4 Differentiate each of the following functions using stepwise differentiation.

(a)	$(x^2+1)^3$	(b)	$\sqrt{2x-5}$
(c)	$\left(\dfrac{2x+3}{x^3-1}\right)^5$	(d)	$\sin\sqrt{3x+5}$
(e)	$\tan(\sin(\sqrt{x}))$	(f)	$\ln(\arctan x)$

A.5 Differentiate the function $f(x) = \sqrt{x^2+7x-2}$ using the method illustrated in Example 4. Compute the value of $f'(9)$ using two different methods.

A.6 Differentiate the composite function $f(x) = \sin(\cos x)$ by (i) direct entry and complete differentiation, (ii) direct entry and stepwise, (iii) storing

≪ → X 'SIN(X)' ≫ as, say, S and ≪ → X 'COS(X)' ≫ as C

and using complete differentiation, and (iv) storing as in (iii) and using stepwise differentiation.

A.7 Use the idea of "chaining variables" as in Example 6 to differentiate the function defined by

$$y = \frac{3w-7}{5w^2+1} \qquad \text{and} \qquad w = \sqrt{x}$$

Check the result by hand to see if the HP 48 got the correct answer.

A.8 The length L of a beam in inches is given by $L = 0.073T + 34.7$, where T is the ambient temperature in Fahreheit degrees. The temperature T is given by $T = 51.0 + 12.3\sin(\pi t/12)$, where t is measured in hours from 6:00 AM. Compute the rate of change of the length of the beam at 2:00 PM.

A.9 In this problem we assume you have stored the program REPY discussed just before Example 8. For each of the following equations, use REPY in finding the derivative y' of the implicitly defined functions(s). After REPY, solve for y' by hand. You may enter the equations as given, or rewrite them in the form $f(x,y) = 0$.

(a) $y^3 + 2x^2y + 1 = 0$
(b) $x^3 + y^3 - 1 = 0$
(c) $y^2(x+y) = -x + y$
(d) $y = \sin(x+y)$
(e) $\sqrt{y-3x} + y^2 = x + 2,\ -2 \le x \le 0$
(f) $x^2y^3 + xy^2 = 1,\ x > 0$
(g) $\sqrt{1 - x\sqrt{1+y}} = x^2 + y^2,\ 0 \le x \le 1$
(h) $\sqrt{x} - x\sqrt{y} - xy\sqrt{x} = y - 1,\ x \ge 0$

A.10 The items (a)–(h) in this problem correspond to the equations given in problem A.9. In each case, a point (x_0, y_0) satisfying the corresponding equation is given. Give the equation of the tangent line to the implicitly defined function at the given point.

(a)	$(1, -0.453397651518)$	(b)	$(0.5, 0.95646559158)$
(c)	$(-1, 1.83928675521)$	(d)	$(0.1, 0.753750156832)$
(e)	$(-0.5, 0.366026175736)$	(f)	$(1, 0.754877666371)$
(g)	$(0.5, 0.597246449686)$	(h)	$(1, 0.609611796798)$

A.11 Most calculus texts discuss the differential *df* of a function f. The value of the differential *df* depends upon two numbers, the value of x and an "increment," usually denoted by Δx or h. Using the latter notation we have $df = f'(x)h$. The differential is used in what is often called the "tangent line approximation" to $f(x+h)$. The approximation is written as

$$f(x+h) \approx f(x) + df = f(x) + f'(x)h$$

Use this result to compute approximations for each of the following cases. In each case compare the tangent line approximation to $f(x+h)$ with the "HP 48 approximation" obtained by a function evaluation at $x+h$. (The fact that with a key press or two you are able to obtain better approximations than that given by the tangent line approximation shows that this kind of problem is becoming artificial. The idea of the tangent line approximation remains, however, very important.)

(a) $f(x) = \sqrt[3]{x^2+4}, \qquad x = 2, \qquad h = 0.14$
(b) $f(x) = \sqrt{(7x-3)/(x+5)}, \qquad x = 4, \qquad h = -0.2$
(c) $f(x) = \sin\sqrt{x}, \qquad x = \pi^2/36, \qquad h = 0.18$
(d) $f(x) = e^{-x^2}, \qquad x = -1, \qquad h = -0.2$

B.1 By looking through the *Owner's Manual*, determine that there are 27 functions or operations with which there may be associated a "differentiation rule." Notes: Count such functions as ABS and + but do not count such functions as SIGN, which, when differentiated returns `'derSIGN(X,1)'`.

B.2 At the beginning of this section we said "there are between 11 and 33 differentiation rules ...". Look though your calculus book with the idea of checking this assertion.

B.3 Demonstrate that the HP 48 knows the differentiation rules by placing each of `'U(X) + V(X)'`, `'U(X)*V(X)'`, and `'U(X)/V(X)'` on the stack and differentiating with respect to X. Also try `'SIN(U(X))'` and `'EXP(SIN(U(X)))'`.

B.4 Write a short program for computing the value of the tangent line approximation to $f(x+h)$ discussed in problem A.11.

B.5 Suppose the equation $f(x,y) = 0$ defines y as a function $y(x)$ of x. Using the tangent line approximation (see problem A.11), estimate $y(x+h)$ for one or more of the equations defined in problem A.9. Use the specific points given in problem A.10 and take $h = 0.1$ in all cases.

B.6 Explain how the program REPY works.

C.1 The "natural" method of calculating the derivative y' of a function $y(x)$ defined implicitly by an equation $f(x,y) = 0$, is based on the Implicit Function Theorem. You may find this result in your calculus text, probably in the chapter in which *partial derivatives* are discussed. It is stated in terms of the partial derivatives $D_x f(x,y)$ and $D_y f(x,y)$ of f. It is easy to calculate partial derivatives. The *partial derivative of f with respect to x*, written $D_x f(x,y)$, is calculated by regarding y as a constant and differentiating with respect to x as usual. The *partial derivative of f with respect to y*, written $D_y f(x,y)$, is similar. If $f(x,y) = x^2y^3 + 5x + 13y + 9$, for example, then $D_x f(x,y) = 2xy^3 + 5$ and $D_y f(x,y) = 3x^2y^2 + 13$. The Implicit Function Theorem states that if at a point

(x_1, y_1) the partial of f with respect to y is non-zero, so that, $D_y f(x_1, y_1) \neq 0$, then

$$y'(x_1) = \frac{-D_x f(x_1, y_1)}{D_y f(x_1, y_1)}$$

For the function $f(x, y) = x^2 + y^2 - 1$, for which we know that $y' = -x/y$, we have $D_x f(x, y) = 2x$ and $D_y f(x, y) = 2y$. At the point $(\sqrt{3}/2, 1/2)$ we have $y' = -D_x f(\sqrt{3}/2, 1/2)/D_y f(\sqrt{3}/2, 1/2) = -\sqrt{3}$. A program implementing this result is

```
≪ → F ≪ F X ∂ NEG F Y ∂ / ≫ ≫
```

With input 'X^2+Y^2 −1', this program returns '−(2*X/(2*Y))'. Try some of problems in A.9.

C.2 The program given in problem C.1 has input $f(x, y)$ and output y'. Write a program with inputs

$$f(x, y),\ x_1,\ \text{and}\ y_1, \quad \text{and output} \quad y' = \frac{-D_x f(x_1, y_1)}{D_y f(x_1, y_1)}$$

2.3 INCLINATION AND INTERSECTION; REFLECTIONS FROM CONICS

Apollonius (born 262 B.C.), a near contemporary of Euclid (fl. 295 B.C.), wrote a work called *Conics*, the first systematic book on the conic sections. Apollonius discussed an application of conics to optics in *On the Burning Mirror*, where he studied the focal properties of spherical and parabolic mirrors. The use of conic sections in optics continues today in NASA's Hubble Space Telescope, put into earth orbit in 1990.

In this section we use the idea of derivative to describe and calculate several geometric properties of the graph of a function. We touch on the angle of inclination of the tangent line to a point on a graph and the angle between two graphs at a common point. We discuss the reflection properties of conics and an application of these to telescopes. We give programs as appropriate. We assume the idea of implicit differentiation is familiar.

Most calculus texts define the angle of inclination at a point $(x, f(x))$ of the graph of a differentiable function f as the angle α such that $0 \leq \alpha < \pi$ and, if $\alpha \neq \pi/2$, $\tan\alpha = f'(x)$. If $f'(x) \neq 0$, the tangent line to the graph at $(x, f(x)$ intersects the x-axis at a point P. The angle α is measured from the part of the x-axis to the right of P, counterclockwise around to the tangent line. We show in Fig. 3 a graph with points at which $0 < \alpha < \pi/2$ and $\pi/2 < \alpha < \pi$.

EXAMPLE 1. Find the angles of inclination α_1 and α_2 at the points

$$(x_1, y_1) = (0.85, 0.62973\cdots) \quad \text{and} \quad (x_2, y_2) = (1.25, -0.79790\cdots)$$

of the graph of $f(x) = \sin((x + 0.5)^3)$, shown in Fig. 3.

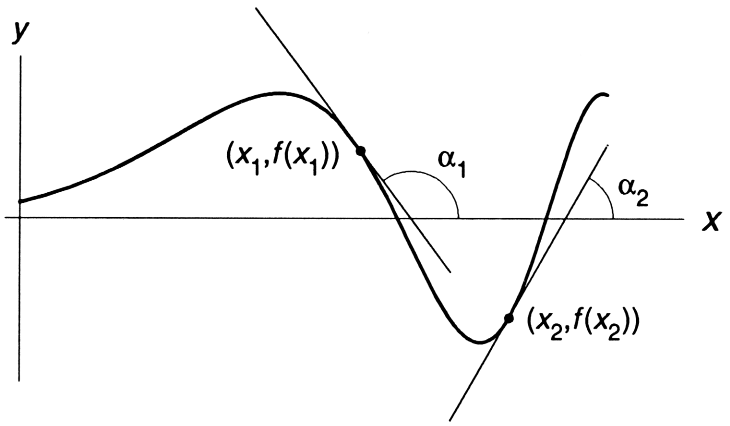


Figure 3

SOLUTION. To find the angles of inclination at these points we must find the derivatives $f'(x_1)$ and $f'(x_2)$. These values are $\tan \alpha_1 = f'(x_1)$ and $\tan \alpha_2 = f'(x_2)$. We use the inverse tangent function to find α_1 and α_2.

The values of $\tan \alpha_1$ and $\tan \alpha_2$ may be found by hand or through the symbolic differentiation key. In any case,

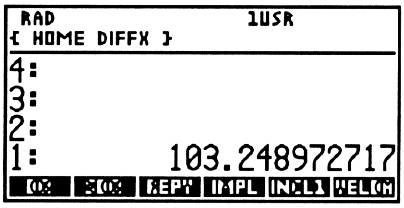
$$\tan \alpha_1 = f'(0.85) = -4.24\cdots \quad \text{and} \quad \tan \alpha_2 = f'(1.25) = 5.53\cdots$$

A glance at Fig. 3 and the definition of the angle of inclination (the definition included the condition that $0 \leq \alpha < \pi$) shows that α_1 is a quadrant II angle and α_2 is a quadrant I angle. With the values of $f'(0.85)$ and $f'(1.25)$ on the stack, two presses of the ATAN key, with an intervening SWAP, results in $\alpha_2 = 1.392\cdots$ and $\alpha_1 = -1.339\cdots$ on the stack. These are radian measures of the two angles. In degrees, $\alpha_1 \approx -76.75°$ and $\alpha_2 \approx 79.76°$. The first of these is not a quadrant II angle, contrary to expectation. We must add π or 180 to convert the angle measures of the inverse tangent function to the usual definition of the angle of inclination. This is clear when you recall that the domain of the inverse tangent function is $R = (-\infty, \infty)$, that is, $\arctan(x)$ or $\tan^{-1} x$ is defined for all values of x, and its range is the interval $(-\pi/2, \pi/2)$. We find, then, $\alpha_1 \approx 103.25°$ and $\alpha_2 \approx 79.76°$.

If several angles of inclination must be calculated, it is worth writing a program to reduce the work. We give a program with function f and the number x as inputs and, in imitation of FEVAL, outputs f and α. The program is named INCL1, for version 1 of a program to compute the angle of inclination.

We may test this program by recalculating α_1 in Example 1.

'SIN((X+.5)^3)' [ENTER]
.85 INCL1
[π] [÷] 180 [×]



```
RAD                    1USR
{ HOME DIFFX }
4:
3:
2:
1:          103.248972717
```

We leave as an exercise a version of INCL1 in which the function is defined in program

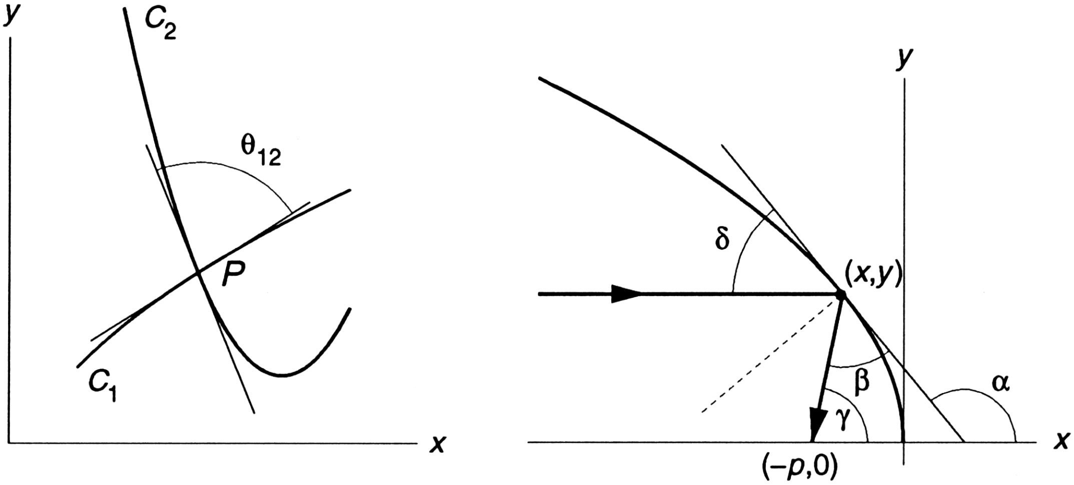
INCL1

Inputs: $f(x)$, x	*Output: α*

```
«  'X'  STO                 Store x
'X'  ∂                      Compute f'(x)
ATAN  DUP                   Compute arctan(f'(x)) and DUP
IF  0  <  THEN  π  +        Add π if arctan(f'(x)) < 0
END  'X'  PURGE             Purge X
→NUM  »                     Convert symbolic result

                              Checksum: #25857d   Bytes: 85.5
```

style. Also, versions of both of these programs in which the output is in degrees (while leaving, however, the calculator in radian mode, as always).

EXAMPLE 2. In Fig. 4 we show two curves, a point P at which they intersect, and the angle from C_1 to C_2. The coordinates of point P are $(1.38, 1.26)$, the slope of C_1 is $m_1 = 0.638$, and the slope of C_2 is $m_2 = -2.47$. Find the angle θ_{12} from C_1 to C_2.



Figures 4 and 5

SOLUTION. Most calculus texts give a formula for calculating the angle between two curves at a point where they intersect. Suppose f_1 and f_2 are functions and x a point for which $f_1(x) = f_2(x)$. We assume both functions are differentiable at x, denote their slopes at x by m_1 and m_2, and refer to the graphs of these two functions as curves C_1 and C_2. The *angle θ_{12} from C_1 to C_2* is given by

$$\theta_{12} = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right) \tag{6}$$

provided that $1 + m_1 m_2 \neq 0$. (If this condition holds, the curves are *orthogonal* and $\theta_{12} = \pi/2$.) Applying (6) we have

$$\theta_{12} = \arctan\left(\frac{-2.47 - (0.638)}{1 + (-2.47)(0.638)}\right) \approx \arctan 5.40 \approx 1.39 \; (\approx 79.5^\circ)$$

A program for calculating the angle between two curves is given in problem A.8.

The Hubble Space Telescope

In the remainder of this section we discuss reflection properties of the three conic sections and the application of these to the Hubble Space Telescope.

The upper part of the parabola with equation $y^2 = -4px$ is shown in Fig. 5. The focus of this parabola is the point $(-p, 0)$. Parabolas have the reflection property shown in Fig. 5. We assume the parabola is acting as a mirror, so that the angles of incidence and reflection are equal. A horizontal ray of light comes in from the left, is reflected from the parabola at the point (x, y), and, given that the angles δ and β are equal, subsequently passes through through the focus $(-p, 0)$ of the parabola. The angles of incidence and reflection are usually measured from the "normal" to the mirror (the normal is shown as a dotted line in the figure), so that δ and β are complements of these angles.

To verify the reflection property of a parabola, it is sufficient to show that

$$\tan\delta = \tan\beta$$

or, since $\pi - \alpha = \delta$ and $\tan(\pi - \alpha) = -\tan\alpha$, that

$$-\tan\alpha = \tan\beta \tag{7}$$

We begin by calculating the slope at the point (x, y). Differentiating the equation $y^2 = -4px$ implicitly, we find that

$$\tan\alpha = y' = -4p/(2y), \quad \text{so that} \quad -\tan\alpha = 2p/y$$

We show that $\tan\beta = 2p/y$, which will complete the verification. Starting with the observation from Fig. 5 that $\gamma + \beta + (\pi - \alpha) = \pi$, we have

$$\begin{aligned}
\tan\beta = \tan(\alpha - \gamma) &= \frac{\tan\alpha - \tan\gamma}{1 + \tan\alpha\tan\gamma} \\
&= \frac{-\dfrac{2p}{y} - \dfrac{y}{x+p}}{1 + \left(-\dfrac{2p}{y}\right)\left(\dfrac{y}{x+p}\right)} = -\frac{2px + 2p^2 + y^2}{xy + yp - 2py} \\
&= -\frac{-2px + 2p^2}{xy - py} = -\frac{2p(p-x)}{y(x-p)} = \frac{2p}{y}
\end{aligned} \tag{8}$$

The hyperbola also has a reflection property. Referring to Fig. 6, where a (silvered) hyperbola with foci F and F' is shown, if a ray of starlight comes in on ray AB, which,

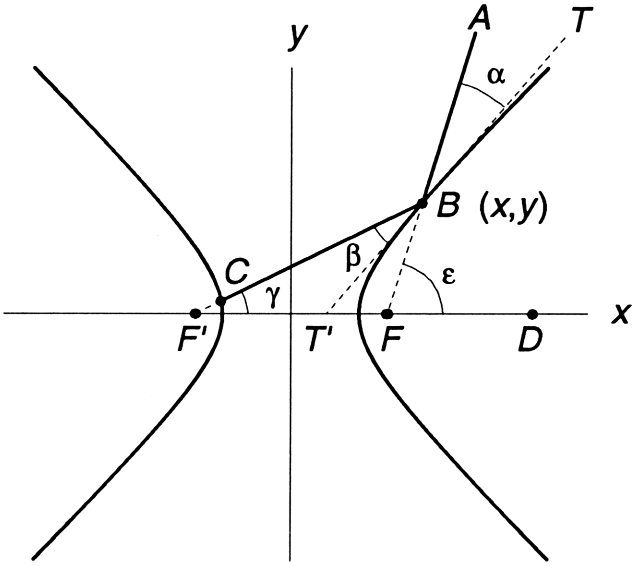


Figure 6

if extended, would pass through the focus F, then it will reflect from the hyperbola along line BC, which, if extended, would pass through the other focus F'.

The reflection property will follow if we can show that $\tan\alpha = \tan\beta$. The argument is similar to the one given above for the parabola. Let the angles $CF'T'$, $BT'F$, and BFD be denoted by γ, δ, and ε, respectively. Note that $\beta = \delta - \gamma$ and $\alpha = \varepsilon - \delta$. The hyperbola shown in Fig. 6 has equation $x^2/a^2 - y^2/b^2 = 1$. The foci F and F' are at $(\pm c, 0)$, where $c^2 = a^2 + b^2$.

The slope of the hyperbola at B, with coordinates (x, y), may be found by differentiating implicitly. From

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{we find} \quad y' = \frac{xb^2}{ya^2} \tag{9}$$

Using this result we calculate $\tan\alpha$ and $\tan\beta$ separately.

$$\begin{aligned}
\tan\beta &= \tan(\delta - \gamma) \\
&= \frac{\tan\delta - \tan\gamma}{1 + \tan\delta\tan\gamma} \\
&= \frac{\dfrac{xb^2}{ya^2} - \dfrac{y}{x+c}}{1 + \dfrac{xb^2}{ya^2}\dfrac{y}{x+c}} \\
&= \frac{x^2b^2 + xb^2c - y^2a^2}{ya^2(x+c) + xyb^2} \\
&= \frac{b^2(a^2 + cx)}{c^2xy + ya^2c} \\
&= \frac{b^2}{cy}
\end{aligned}
\qquad
\begin{aligned}
\tan\alpha &= \tan(\varepsilon - \delta) \\
&= \frac{\tan\varepsilon - \tan\delta}{1 + \tan\varepsilon\tan\delta} \\
&= \frac{\dfrac{y}{x-c} - \dfrac{xb^2}{ya^2}}{1 + \dfrac{y}{x-c}\dfrac{xb^2}{ya^2}} \\
&= \frac{y^2a^2 - x^2b^2 + xcb^2}{xya^2 - ca^2y + xyb^2} \\
&= \frac{-a^2b^2 + xcb^2}{xya^2 - ca^2y + xyb^2} \\
&= \frac{b^2(cx - a^2)}{c^2xy - ca^2y} = \frac{b^2}{cy}
\end{aligned}$$

The Hubble Space Telescope uses a *Cassegrain* optical system, which consists of two mirrors. The larger mirror is parabolic in cross-section and the smaller secondary mirror is hyperbolic in cross-section. We show in Fig. 7 the arrangement of the two mirrors.

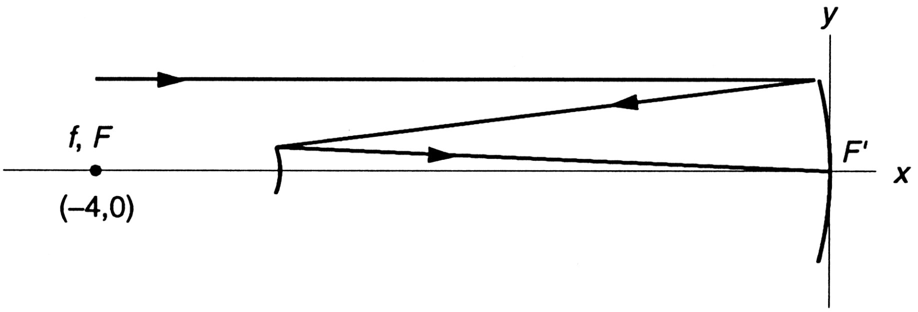


Figure 7

The primary parabolic mirror is at the origin and has its focus at f. The secondary hyperbolic mirror has foci at F and F'. Only the left branch of the hyperbola is used. The key fact in the arrangement is that the foci f and F are coincident. This couples the reflective properties of the two mirrors together. A ray of starlight parallel to the common axis is shown coming towards the primary mirror. It reflects from the primary to its focus f, or, rather, the light would pass through f were it not for the hyperbolic mirror. Since the light is directed towards the focus F of the hyperbolic mirror, it is reflected so that it passes through the second focus F'. The primary mirror often has a circular hole in its center so that the light coming into the focus F' can be gathered and directed to film or a measuring instrument. We give equations for the cross-sections of the primary and secondary mirrors in the next example and illustrate two of the many calculations required in designing a Cassegrain telescope.

EXAMPLE 3. The equations for the primary and secondary mirrors are

$$y^2 = -16x \qquad \text{and} \qquad \frac{(x+2)^2}{1} - \frac{y^2}{3} = 1 \tag{10}$$

If the diameter of the primary mirror is to be 1 meter, what is the minimum required diameter D of the secondary mirror and what is the diameter of the "dark spot" in the secondary mirror?

SOLUTION. To answer these questions, let $Q = (-q^2/16, q)$ be an arbitrary point on the upper part of the primary mirror. We find the coordinates of the point on the secondary mirror though which the line from Q to the focus $(-4, 0)$ passes. Since the slope of this line is $16q/(64-q^2)$, we may write the equation of the line and solve it simultaneously with the equation of the hyperbola. We have

$$\begin{cases} \dfrac{(x+2)^2}{1} - \dfrac{y^2}{3} = 1 \\ y - 0 = \dfrac{16q}{64-q^2}(x+4) \end{cases} \tag{11}$$

Solving the second equation for x and substituting into the first equation we have

$$\left(\frac{64-q^2}{16q}y - 2\right)^2 - \frac{y^2}{3} = 1 \tag{12}$$

which is quadratic in y. We may now answer the two questions raised above. For the first we take $q = 0.5$ and solve (12) for y. This gives the minimum required diameter of the secondary mirror. For the second we take q to be the y just found and solve (12) for y. This will give the diameter of the "dark spot" in the secondary mirror.

We may use QUAD to do most of the work. After purging any variables Y and Q from the VAR menu, enter (12), put 'Y' on the stack, go to the ↰SYMBOLIC menu, and press $\boxed{\text{QUAD}}$. The result, which takes about 20 seconds for the HP 48 to calculate, will be a complex expression with variables s1 and Q. We use the SOLVR to evaluate this expression. We expect (12) to have two solutions for a given value of q, these corresponding to the two intersections of the line and hyperbola whose equations are given in (11). Store the result from QUAD as EQ and go to the SOLVR. Let $q = 0.5$ and take s1 as -1 (why?). Pressing $\boxed{\text{EXPR=}}$ gives $y \approx 0.1252$. The minimum required diameter of the secondary mirror is $2 \cdot 0.1252 = 0.2503$ meters. With 0.1252 still on the stack, use it as the next value of q (why?). We find the diameter of the dark spot on the secondary to be 0.0626 meters.

In this section we have given examples illustrating possible uses of the HP 48 in using the derivative to study the graphs of one or more functions. Examples of calculating the inclination of a graph at a given point and the angle at which two curves intersect were given. Conics were studied in the context of their reflection properties.

Exercises 2.3

A.1 Sketch the graph of $f(x) = x^2 - 5x + 2$ and find the angles of inclination of the tangent lines at the points $(1, -2)$ and $(3, -4)$. Do this with and without the program INCL1. Convert the angles to degree measure.

A.2 Find the largest angle of inclination for the function $f(x) = x^2 - 7x + 10, \quad 2 \leq x \leq 5.5$.

A.3 The graphs of the equations $y = \sin x$ and $y = (2\sqrt{2}/\pi)x$ intersect at the point $(\pi/4, \sqrt{2}/2)$. Sketch these graphs and calculate the two angles of inclination. Subtract these to get the angle between the two curves.

A.4 Find the angles of inclination at the x-intercepts of the graph with equation $y = x^3 - 5x^2 - x + 5$.

A.5 Find the angles of inclination at the points on the curve with equation $y = x^2 + x + 1$ at which a line through the origin is tangent.

A.6 A shell from a mortar located at the origin of an (x, y)-coordinate system is fired upwards and to the right, with a muzzle velocity of v_0 feet per second and at an angle of θ, measured in degrees from the positive x-axis. The path of the shell is the graph of the equation

$$y = \frac{-g}{2v_0^2 \cos^2 \theta} x^2 + \tan \theta \, x$$

where g is the acceleration due to gravity. Find the angles of inclination of the shell's path as it leaves the mortar and at the point it hits the earth (the x-axis). Infer these angles without calculation. What are the angles of inclination of the shell at the 1/4, 1/2, and 3/4 range points?

A.7 Referring to Fig. 5, find the angle of incidence of a ray at height $y = 0.4$ coming into the parabola $y^2 = -16x$. What is the slope of the line along which the reflected ray travels?

A.8 Find the angles (in degrees) from the parabola with equation $y = x^2 + x + 1$ to the line with equation $y = x + 2$ at each of their points of intersection. Do the calculation by hand first. Check your results by using the program ABCS (angle between curves), which follows. The program takes the slopes m_1 and m_2 of the two curves C_1 and C_2 as input and returns the degree measure of the angle from C_1 to C_2.

```
   ≪  →  M1  M2  ≪  M2  M1  DUP2  -
      3  ROLLD  *  1  +  /  ATAN  DUP
IF  0  <  THEN  π  +  END  π  /  180  *  ≫  ≫
```

Problems B.2, B.7, and C.1 are related to this program or improvements upon it.

A.9 A parabola having horizontal axis, vertex at (h, k), and focus at $(h - p, k)$ has equation $(y - k)^2 = -4p(x - h)$. If the parameter p is assumed to be positive, this parabola "opens" to the left.

(a) Locate the vertex and focus of the parabola with equation $y^2+4y-3x+7 = 0$. Sketch the graph.

(b) Write an equation of a parabola with vertex at $(3, -2)$ and focus at $(-8, -2)$.

(c) Show that for the "generic" parabola $(y - k)^2 = -4p(x - h)$, each point (x, y) on the parabola is equally distant from the focus and the line with equation $x = h + p$.

(d) What is the "depth" of a parabolic mirror with a 5 meter focal distance and diameter 1 meter?

A.10 A hyperbola having its "center" at (h, k), vertices at $(h-a, k)$ and $(h+a, k)$, and foci at $(h-c, k)$ and $(h+c, k)$ has equation $(x-h)^2/a^2 - (y-k)^2/b^2 = 1$, where $c^2 = a^2 + b^2$.

(a) Locate the center, vertices, and foci of the hyperbola with equation $(x - 1)^2 - y^2 = 1$. Sketch the graph.

(b) Locate the center, vertices, and foci of the hyperbola with equation $3x^2 - 2y^2 + 6x + 8y - 6 = 0$. Sketch the graph.

(c) Find the standard equation $(x-h)^2/a^2-(y-k)^2/b^2 = 1$ for the hyperbola with foci at $(1, -3)$ and $(6, -3)$ and vertices at $(2, -3)$ and $(5, -3)$.

A.11 The parabolic mirror of a Cassegrain telescope has equation $y^2 = -16x$. The prime focus of the telescope is to be at $(0.2, 0)$ and the vertex of the hyperbolic mirror is to be at $(-3, 0)$. Find the equation of the hyperbolic mirror.

B.1 Modify the program INCL1 so that it returns angles in degrees.

B.2 Explain how the program ABCS given in problem A.8 works.

B.3 Discuss how a horizontal ray coming in from the right is reflected from the parabola $(y - k)^2 = -4p(x - h)$, where $p > 0$.

B.4 Find the minimum required diameter of the hyperbolic mirror of the Cassegrain telescope described in problem A.11, assuming that the diameter of the primary parabolic mirror is 1 meter. Also find the diameter of the dark spot on the secondary mirror and the diameter of the smallest hole in the primary through which all possible light may reach the focus.

B.5 The equations

$$y^2 = -22.08x \qquad \text{and} \qquad \frac{(x+2.472)^2}{2.459^2} - \frac{y^2}{1.801^2} = 1$$

describe the Hubble Space Telescope, where all dimensions are in meters. The diameter of the primary mirror is 2.4 meters. By tracing appropriate rays, find the minimum required diameter of the hyperbolic mirror, the diameter of the dark spot on the secondary, and the diameter of the smallest hole in the primary through which all possible light may reach the focus.

B.6 Radio telescopes designed to detect X-rays often use hyperbolic and parabolic reflectors. The reflectors are arranged as in Fig. 8, although the mirrors are configured so that the "grazing angles" are much smaller than shown. The coordinate axes are often chosen so that the y-axis passes through the intersection of the parabola and the right branch of the hyperbola. The equations of the parabola and hyperbola are

$$y^2 = 4p(x-h) \qquad \text{and} \qquad \frac{(x-H)^2}{a^2} - \frac{y^2}{b^2} = 1$$

Letting $p = 1$, $a = 4$, and $b = 3$, and noting that $h = H - c - p$, find H so that the y-axis passes through the intersection, as shown in Fig. 8.

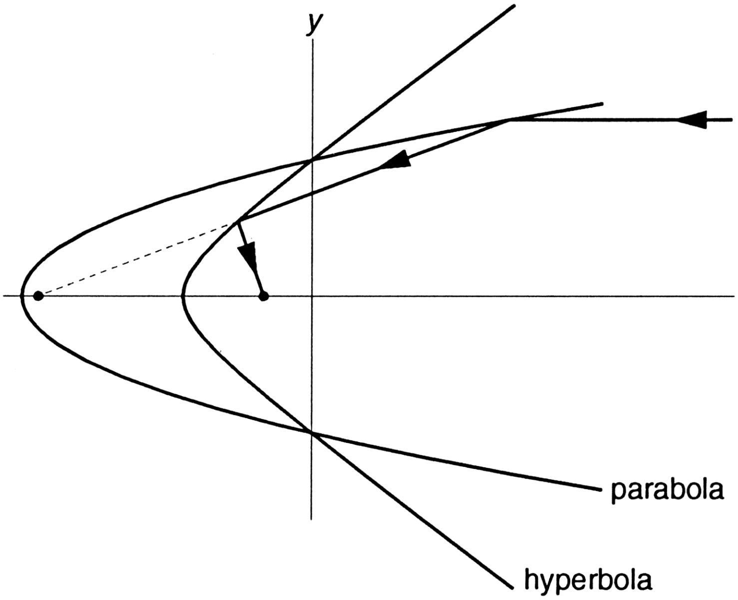


Figure 8

B.7 Modify the program ABCS in problem A.8 so that it detects the "perpendicular case" and gives an appropriate response.

B.8 Write an alternate version of ABCS using INCL1 or INCL2. The inputs should be $f_1(x)$, $f_2(x)$, and x.

C.1 Write a program INCL2 that does the same calculation as INCL1 but assumes the input function is written in program style. The stack inputs should be a program defining $f(x)$ and x. The output should be the single number α (the angle of inclination of the tangent line to the graph of f at the point $(x, f(x))$, in radians or degrees, as preferred; we suggest, however, that the calculator remain in radian mode) or α and the program for f.

C.2 The graph of the equation $x^2/a^2 + y^2/b^2 = 1$, where $a > b$, is an ellipse with horizontal axis, center at the origin, and foci at $(-c, 0)$ and $(c, 0)$, where $c^2 = a^2 - b^2$. Show that a ray passing through one focus and reflecting from the ellipse so that the angles of incidence and reflection are the same passes through the other focus.

C.3 A variation of the Cassesgrain telescope is the Gregorian, which uses parabolas and ellipses. Sketch the Gregorian telescope with parabola $y^2 = -16x$, ellipse $(x + 3/2)^2/16 + y^2/(39/4) = 1$, and diameter of the primary mirror equal to 2 feet. Find the minimum required diameter of the elliptical mirror and the size of the hole that must be cut from the primary mirror.

Chapter 3

Graphing

3.0 PREVIEW

In § 0.6 we discussed the nuts and bolts of HP 48 graphing from a "how-to" point of view. In § 1.2 we continued the discussion of graphing functions. In this chapter, we assume that you know how to graph using an HP 48 and that you know the geometric interpretations of the first and second derivatives. We also discuss parametric and polar plotting which may not appear until somewhat later in your textbook.

Throughout your calculus book you will find problems of the following type:

- a specific function or equation is given; you are asked to
- locate such "interesting features" as zeros, critical points, points of inflection, extrema, intervals of monotonicity, intervals of concavity, $\cdots$; often you are asked to
- make a careful sketch of the graph.

The graph often comes last since until recently that has been the hardest part; now, thanks to graphing calculators, things have changed, and in a way, graphing is now the easiest part. But one must be careful. You can't just sit back passively and watch the machine do the graphing and expect all to turn out right. For one thing, what is *right* depends on why you are working the problem. If the problem is modeled after some real world situation, then that must be taken into account; if you are working the problem for its own sake, then you need to make sure the machine includes all "interesting features" of the graph. Moreover, you always have to be on guard for tricks the machine might play on you.

In this chapter, we will take the following approach to graphing:

- start with a function or an equation;
- use the HP 48 to make a careful sketch;
- ask questions about the interesting features of the graph; then
- answer whatever questions we can.

The only thing "wrong" with this approach is that sometimes we'll run into questions we're not quite ready for. But that's not really bad because such questions motivate work that lies ahead.

Conceptually, this chapter is broken up into three parts corresponding to the following three kinds of curves:

1. curves defined by a function;
2. curves defined parametrically;
3. curves defined by an equation.

These are not mutually distinct classes; indeed, most familiar curves are definable in all three ways. All curves to be considered are two-dimensional—typically cast in an (x, y)-coordinate system. Be alert to the fact that a good understanding of two-dimensional curves is a prerequisite for studying curves and surfaces in three-dimensional space.

3.1 CURVES DEFINED BY A FUNCTION

This section is about those curves that pass the so-called "vertical line test": any vertical line will intersect the curve in either one or zero points. Given such a curve C, there will always correspond a function f whose graph coincides with the curve; that is,

$$C = \{ (x, f(x)) : x \in D \},$$

where D is the domain of the function.

Curves not of this type often can be divided into pieces that are. For example, the unit circle $x^2 + y^2 = 1$ can be divided into two half-circles defined by $f_1(x) = \sqrt{1 - x^2}$ and $f_2(x) = -\sqrt{1 - x^2}$.

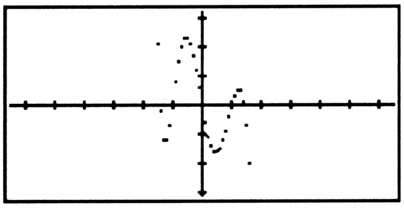
To capture all of the interesting features of a graph generally requires making more than one drawing—usually a large-scale drawing to see what is going on globally and one or more close-ups to see what's going on locally. In this section, we will concentrate on polynomials. If you haven't already keyed in the programs RESET and POLY from Chapter 0, now might be a good time to do so.

EXAMPLE 1A. Sketch the following sixth degree polynomial:

$$p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$$

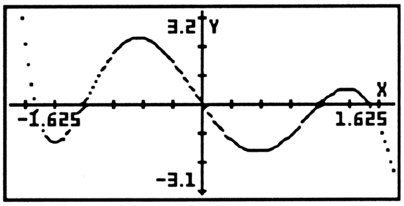
SOLUTION. The tacit instructions here are to include all interesting features. Begin with a standard plot:

$\boxed{\text{RESET}}$ 0 −6 2 9
−3 −3 1 $\boxed{\text{POLY}}$
$\boxed{↰}$ $\boxed{\text{EQ}}$ $\boxed{\text{DRAX}}$
$\boxed{\text{DRAW}}$



Next, adjust the scale on the x-axis (to stretch it out horizontally).

$\boxed{\text{ZOOM}}$ $\boxed{\text{HZIN}}$



Apparently this sketch captures all of the graph's interesting features. Does it really and how can we tell?

Generally speaking, this is not an easy question to answer; however, with polynomials and a few other familiar functions, there are *clues.* One such clue is provided by *the Fundamental Theorem of Algebra* which tells us that *a polynomial of degree n has exactly n complex zeros (counting multiplicities).* In particular, this means that a polynomial of degree 6 will have *at most 6 real zeros.* In other words, the graph will cross or touch the x-axis at most 6 times. Notice that the above figure shows 5 crossing points so one more crossing is possible.

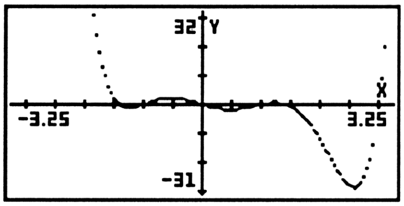
Another clue is that a polynomial behaves asymptotically (as $x \to \infty$ or $x \to -\infty$) like its highest degree term. Thus, the given polynomial will behave asymptotically like x^6. Therefore, $p(x) \to \infty$ as $x \to \pm\infty$. It follows that the graph has to eventually turn around and head upward somewhere to the right. Two more clues (both easy to take for granted) are the *Intermediate Value Theorem* and the fact that polynomials are continuous. (Do you see how these facts fit in?)

All things considered, we see that $p(x)$ must have exactly *one additional zero* and it must be *somewhere to the right.* The following scaling locates the errant zero.

$\boxed{\text{ZOOM}}$ $\boxed{\text{ZFACT}}$ 2

$\boxed{\text{OK}}$ 10 $\boxed{\text{OK}}$ $\boxed{\text{OK}}$

$\boxed{\text{ZOUT}}$



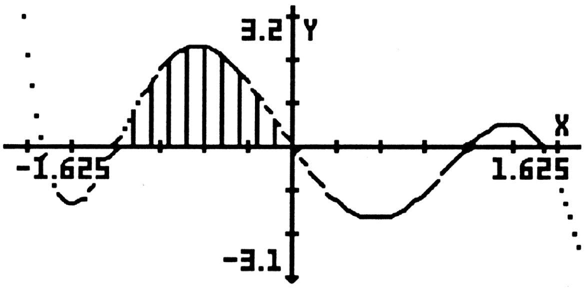


Figure 1

EXAMPLE 1B. What is interesting and what are the questions?

What is interesting about a graph is, of course, relative. Not only does it depend on the "eye of the beholder", but it also depends on the coordinate system. A better question to ask might be "what is *uninteresting*?" Most people with the background to study calculus would agree that items of interest include variations in the graph and how the graph interacts with the coordinate axes. On the other hand, most people wouldn't find much of interest about a portion of the graph that gets steeper and steeper indefinitely.

> TO THE READER: *Before reading on, what five questions would you ask about the graph in Fig. 1?*

SOLUTION. Reasonable questions to ask include the following:

1. What are the zeros (i.e., the x-intercepts)?
2. What are the extrema (i.e., the high and low spots)?
3. What are the inflection points (i.e., the places where the concavity changes)?
4. What is the equation of the tangent line to the curve at the smallest positive zero? At the largest positive zero?

Other reasonable questions (see Fig. 1):

5. What is the area of the shaded region?
6. What is the perimeter of the shaded region?
7. What is the "center" of the shaded region?

We will answer Questions 1–4 below, then return to Questions 5–7 in Chapter VI.

More About Zeros

One way to find zeros is by factoring, but most polynomials of degree n ≥ 5 won't factor in a nice way. For a historical account of this phenomenon and other related facts, see § 1.4. As it turns out, the above polynomial *does* factor nicely, and you might want to stop here and see if you can do it (the factorization is given below). A more powerful way of finding zeros is by using the HP 48. However, *proceed with caution!* In the first place, it is important to realize that:

Calculators don't give correct answers

Most of the time calculators give only *good approximations*; seldom do they give *exact answers.* For instance, when the HP tells you that it has found a zero, the real truth is that it found a zero correct to twelve digits. (When it says "sign reversal," that means it can guarantee *only* eleven digits.) Of course, in most practical situations, eleven- or twelve-digit accuracy would be more than enough. Nevertheless, it's important that you realize that there is a big difference between an exact answer and an approximation (no matter how good the appproximation)—this is especially true in the subject of calculus where very large numbers, very small numbers, and numbers very close to other numbers occur in major ideas. Another reason for being cautious about approximations (one that may be more easily appreciated by calculus beginners) is that *small errors can accumulate and become big errors.*

In the process of finding zeros, the HP 48 may do some surprising things; e.g., it may find the "wrong" zero. Worse yet, it may even find an extreme point! See, for example, Exercise B.2; also see § 1.4. While this sort of thing won't happen often, it surely *can* happen and when it does, *you* have to be the one to come to the rescue. To be a good rescuer, you need to constantly relate (1) what you see with (2) what you are trying to do with (3) what the calculator is telling you. You also need to have an understanding of how the calculator finds zeros. In particular, you need to understand the connection between zero-finding and the *Intermediate Value Theorem* and *Newton's Method.* Chapter 4 gives you some insight in these matters. For now, if you have any doubts as to whether a zero is *really* a zero, you can always check it out by evaluating the function at the alleged zero.

As you already know from § 1.4, here's the procedure for finding zeros with the HP 48:

Step 1 Graph the function to "see" the zero(s).

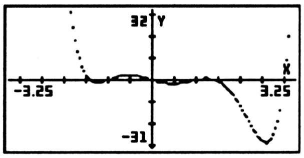
Step 2 Move the + cursor to the approximate location(s) of the zero(s) (the closer the better) and press $\boxed{\text{ROOT}}$ in the FCN submenu of the graphics menu.

When you return to the stack (press ON twice), you'll find the zero(s) waiting for you.

EXAMPLE 1C. What are the zeros of $p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$?

SOLUTION.

INPUT:



◀ ··· ◀ $\boxed{\text{ROOT}}$ ▶ ··· ▶ $\boxed{\text{ROOT}}$ ▶ ··· etc. $\boxed{\text{ON}}$

OUTPUT: Root: −1.41421356238; Root: −1; Root: 0; Root: 1; Root: 1.41421356238; Root: 3.

This suggests the factorization

$$p(x) = x(x-1)(x+1)(x^2-2)(x-3)$$

which is correct as may be checked.

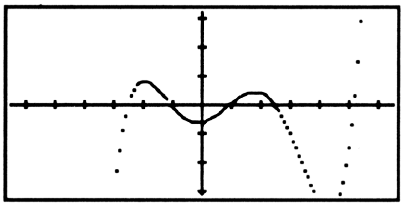
REMINDER: *$f'(a)$ = the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.* (1)

EXAMPLE 1D. What are the extreme points of $p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$?

SOLUTION. The above display (Example 1C) reveals two high spots (*relative maxima*) and three low spots (*relative minima*). At such points the tangent lines will surely be horizontal; thus, the derivative (which is the slope of the tangent line) will be equal to zero. Put another way, *the extrema can be found by finding the zeros of $p'(x)$.* How can we find the zeros of $p'(x)$? Answer: apply the above 2-step process to $p'(x)$.

Before graphing $p'(x)$, store $p(x)$ under the name P and clear the stack. Then graph $p'(x)$ using the previous scaling factors:

$\boxed{\text{EQ}}$ P $\boxed{\text{STO}}$ $\boxed{\text{DEL}}$ $\boxed{\text{P}}$
X $\boxed{\partial}$ $\boxed{↰}$ $\boxed{\text{EQ}}$
$\boxed{\text{ERASE}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



Notice that $p'(x)$ is of degree 5 and that we can "see" all 5 of its zeros. We may now use ROOT as above to obtain the zeros of $p'(x)$: -1.24561109254, -0.537807896525, 0.46975521485, 1.22987914899, 2.58378462523.

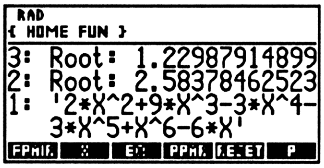
From the graph of $p(x)$, it is clear which of these are relative minima and which are relative maxima. Also, recalling that $p'(x) > 0$ corresponds to $p(x)$ increasing and $p'(x) < 0$ corresponds to $p(x)$ decreasing, we could deduce the same information from the above graph of $p'(x)$. For example, at the left-most zero notice how $p'(x)$ goes from being negative to being positive. Therefore, $p(x)$ goes from decreasing to increasing. Therefore, $p(x)$ has a relative minimum at that point.

The corresponding y-values can be calculated using one of the function evaluation methods discussed in Chapter 0 (just make sure to use $p(x)$ not $p'(x)$). A simpler way is to use the program FPAIRS. See program box.

FPAIRS

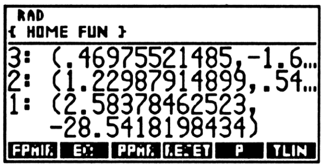
Inputs: $x_1, \cdots, x_n, f(x)$	*Output:* $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$
« → F «	Introduces local variable F
DEPTH → N «	Introduces local variable N
1 N FOR I	Sets up FOR-NEXT loop
RE DUP 'X' STO	Makes copy of x_{N-I+1} & stores it under the name X
F EVAL	Calculates f at x_{N-I+1}
R→C	Forms pair $(x_{N-I+1}, f(x_{N-I+1}))$
N ROLLD	Puts $(x_{N-I+1}, f(x_{N-I+1}))$ on top of
NEXT	stack; ends FOR-NEXT loop
'X' PURGE » » »	Purges X; ends program
	Checksum: #31114d Bytes: 109

With the zeros of $p'(x)$ on the stack, we may press P to adjoin $p(x)$ then FPAIRS to obtain the following:



```
RAD
{ HOME FUN }
3: Root: 1.22987914899
2: Root: 2.58378462523
1: '2*X^2+9*X^3-3*X^4-
   3*X^5+X^6-6*X'
```

$\boxed{\text{FPAIRS}}$
$\longrightarrow$



```
RAD
{ HOME FUN }
3: (.46975521485,-1.6…
2: (1.22987914899,.54…
1: (2.58378462523,
   -28.5418198434)
```

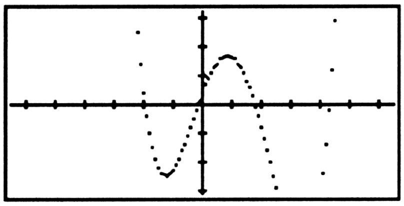
✓Point to note

When you apply FPAIRS make sure there is nothing on the stack other than the x_i's.

EXAMPLE 1E. What are the inflection points of $p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$?

SOLUTION. Upward concavity corresponds to $f''(x) > 0$; downward concavity to $f''(x) < 0$. See your book for a discussion of concavity. *Points of inflection* are points at which concavity changes. Generally, the real line breaks up into intervals of upward concavity and downward concavity with inflection points in between. Inflection points occur among the points where $f''(x) = 0$. In other words, *inflection points can be found by finding the zeros of* $f''(x)$. Now how do you think we'll do that? Same song, different dance.

DEL EQ X ∂
↰ EQ ERASE
DRAX DRAW



Then use ROOT and FPAIRS to get the inflection points:

$$(-0.97594868615, 0.19317293182)$$
$$(-0.0711162399389, 0.433504283806)$$
$$(0.891011984907, -0.46710353142)$$
$$(2.15605294118, -17.5835736841)$$

Table 1 summarizes our findings.

Table 1

FUNCTION: $p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$		
ZEROS	EXTREMA	INFLECTION POINTS
$-\sqrt{2}$	$(-1.24561109254, -1.30804335209)$ (relative minimum)	$(-0.97594868615, 0.19317293182)$
-1	$(-0.537807896525, 2.31353350649)$ (relative maximum)	$(-0.0711162399389, 0.433504283806)$
0	$(0.46975521485, -1.648208216)$ (relative minimum)	$(0.891011984907, -0.46710353142)$
1	$(1.22987914899, 0.54391290609)$ (relative maximum)	$(2.15605294118, -17.5835736841)$
$\sqrt{2}$	$(2.58378462523, -28.5418198434)$ (absolute minimum)	————————
3	————————	————————

EXAMPLE 1F. Find the equation of the tangent line to the curve $y = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$ at (a) the smallest positive zero; (b) the largest positive zero.

SOLUTION. The smallest positive zero of $p(x)$ is $x = 1$. Thus we want to find the tangent line to $y = p(x)$ at the point $(1,0)$. By (1), the slope of this line is $p'(1)$. It is easy to obtain $p'(1) = 4$ [do this two ways]. Therefore, the equation of the tangent line is $y - 0 = 4(x - 1)$ or $y = 4x - 4$. [Is this a reasonable answer? Look at the first display in Example 1A, in which the axes have the same scale, and you'll see that the answer is yes.]

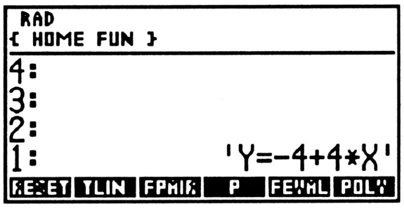
The largest positive zero is $x = 3$ and the slope is $p'(3) = 168$. Therefore, the equation of the tangent line is $y - 0 = 168(x - 3)$ or $y = 168x - 504$ [a *very steep* line!]

About Tangent Lines

Notice how algorithmic our solution to Example 1F is; in fact, you could write out a general formula for the tangent line in terms of a given function f and a given x-value a (your book may do this). What does that suggest? Memorizing the formula? Surely not. Writing a tangent line program? Certainly. But let's proceed with caution. Even though the program below will trivialize the process of finding tangent lines and give you quick solutions to many textbook problems, you can still benefit by working a few such problems by hand. (Note that "by hand" does *not* mean using the formula!) The pay-off will be a better understanding of the geometric interpretation of the derivative as well as the idea of obtaining the equation of a line from its slope and a point on it. Good advice for most students: *Use the program TLIN (see box) as a double-check on hand calculations, later as a tool.*

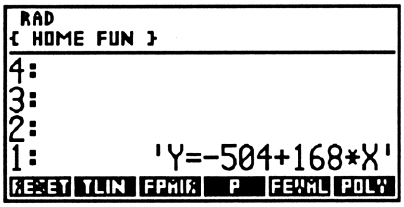
Let's double-check the above results.

P 1 TLIN



```
RAD
{ HOME FUN }
4:
3:
2:
1:                'Y=-4+4*X'
```

P 3 TLIN



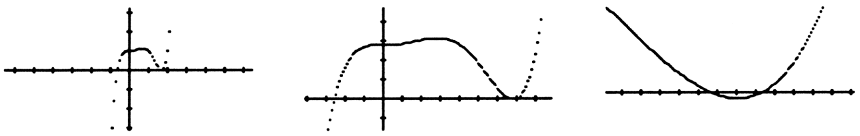
```
RAD
{ HOME FUN }
4:
3:
2:
1:            'Y=-504+168*X'
```

Example 1 illustrates a polynomial of degree 6 that has 6 real zeros, 5 extrema, and 4 inflection points. Such regularity is *not* typical. In other words, it is *not* typical that a polynomial of degree n will have n real zeros, $n - 1$ extrema, and $n - 2$ inflection points. The following example illustrates some of the things that can go "wrong".

TLIN	
Inputs: $f(x)$, a	*Output: Equation of tangent line to* $y = f(x)$ *at* $x = a$
« 'X' STO → F «	Stores a in X; defines local variable F
Y F EVAL F 'X' ∂ EVAL 'X' X − * + =	Calculates the formula $y = f(a) + f'(a)(x - a)$ in RPN
EXPAN COLCT	Simplifies formula
'X' PURGE » »	Purges X, ends program
	Checksum: #46668d Bytes: 114

EXAMPLE 2. Sketch the polynomial $p(x) = x^5 - 3x^4 + 2x^3 + 1$ and find its zeros, extrema, and inflection points.

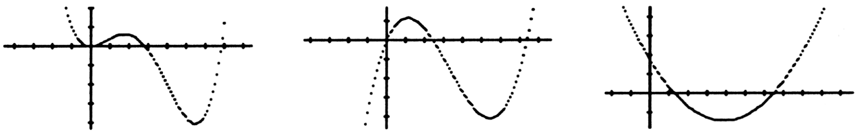
SOLUTION. The standard plot in Fig. 2(a) suggests that there are exactly two real zeros, at least one relative maximum, and at least one relative minimum. The zoom-in on these features shown in Fig. 2(b) doesn't help much but suggests that we ought to take a closer look at the zero to the right because it seems to stay zero throughout a short interval. Another zoom-in confirms our doubts and we see that there are really two zeros where it first looked like there was only one. See Fig. 2(c). (For a discussion of the zooming-in procedure, see IV of § 0.6.)



(a) Standard plot (b) Zoom-in #1 (c) Zoom-in #2

Figure 2

We can get more information by looking at the derivatives of $p(x)$. First store $p(x)$ under the name P: EQ 'P' STO .



(a) $y = p'(x)$ (b) $y = p''(x)$ (c) $y = p'''(x)$

Figure 3

Observe that $p'''(x)$ and $p''(x)$ display the same regular behavior we observed earlier. In particular, $p''(x)$ is of degree 3 and obviously has exactly 3 real zeros, 2 extrema, and 1 inflection point. (See Figs. 3(b)–(c)). On the other hand, since

$$p'(x) = 5x^4 - 12x^3 + 6x^2 = x^2(5x^2 - 12x + 6),$$

it is clear from Fig. 3(a) that $p'(x)$ has exactly 3 real zeros, 3 extrema, and 2 inflection points. It follows that Fig. 2 tells the full story: $p(x)$ has precisely 3 real zeros, 2 extrema, 3 inflection points, and behaves asymptotically like x^5. (Note that if $p(x)$ had more than 3 real zeros, then there would be at least 4 extrema, which would mean that $p'(x)$ would have at least 4 real zeros.) Using ROOT and FPAIRS, we obtain the summary given in Table 2.

Table 2

FUNCTION: $p(x) = x^5 - 3x^4 + 2x^3 + 1$		
ZEROS	EXTREMA	INFLECTION POINTS
−0.61803398875	(0.710102051445, 1.13389451796) (relative maximum)	(0, 1)
1.61803398876	(1.68989794855, −0.032454518) (relative minimum)	(0.441742430505, 1.07498617736)
1.75487766627	———————	(1.35825756949, 0.42389382264)

Exercises 3.1

For Exercises 1–7, (a) make one or more plots of each function to capture its interesting features; (b) determine all real zeros, extrema, and inflection points.

A.1 $x^4 - 2x^3 - 9x^2 + 2x + 7$

A.2 $15x^4 - 32x^3 + 24x^2 - 8x - 5$

A.3 $16x^5 - 8x^4 - 7x^3 - x^2$

A.4 $x^4 - 9x^3 + x^2 - 11x + 1$

A.5 $x^6 - 24x^5 + 150x^4 - 3$

A.6 $480x^4 - 1440x^3 + 240x^2 + 2520x - 1890$

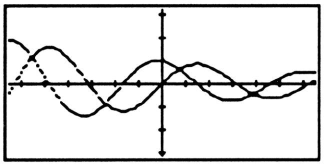
A.7 $P^{(4)}(x)$ where $P(x)$ is your personal polynomial as defined in Exercise A.11, § 0.7.

For Exercises 8–10, (a) find the equation of the tangent line to the given curve corresponding to the given x-value; (b) make a sketch of the curve and tangent line.

A.8 $y = (x + \frac{1}{x})^3$; 2

A.9 $y = x \sin 4x$; $\pi/2$

A.10 $y = x - 2\sin x$; the positive zero



A.11 The display to the right represents a function f and its derivative f'. Which is which?

A.12 Find the equation of the tangent line to your personal polynomial at the y-intercept (see A.11, § 0.7) and make a sketch of both.

B.1 Let $q(x) = p(x) + 0.0324545$, where $p(x)$ is as in Example 2. Use the Intermediate Value Theorem to explain why $q(x)$ has two real zeros near $x = 1.7$. Make an HP 48 plot that clearly shows the existence of these two zeros.

B.2 Let $q(x) = p(x) + 0.0325$ where $p(x)$ is as in Example 2. Graph $q(x)$, move the + cursor near $x = 1.7$, and press $\boxed{\text{ROOT}}$. What does the calculator tell you? Do you believe it? Make an HP 48 plot that clearly shows the nonexistence of zeros near $x = 1.7$.

B.3 Make a careful sketch of $P''(x)$ where $P(x)$ is your personal polynomial (see A.11, § 0.7) and determine all real zeros, extrema, and inflection points for $y = P''(x)$.

C.1 Make a careful sketch of your personal polynomial (see A.11, § 0.7) and determine all real zeros, extrema, and inflection points.

3.2 CURVES DEFINED PARAMETRICALLY

If $x(t)$ and $y(t)$ are continuous functions on an interval $[a, b]$, then the set of points of the form $(x(t), y(t))$, where t ranges over $[a, b]$, generates a curve C. In this situation, we say that the curve C is *defined parametrically*; t is called the *parameter*, and $[a, b]$ is called the *parametrization interval.* Many motion problems fit this model and even when there is no underlying motion, it is convenient to think of a curve as the path of a moving particle, its position at time t being $(x(t), y(t))$.

All familiar curves can be represented parametrically. If a curve is defined by $y = f(x)$, where $a \leq x \leq b$, then a parametrization is given by:

$$\begin{cases} x = t \\ y = f(t) \end{cases} \quad a \leq t \leq b$$

If a curve is defined by $x = g(y)$, $c \leq y \leq d$, then a parametrization is given by:

$$\begin{cases} x = g(t) \\ y = t \end{cases} \quad c \leq t \leq d$$

If a curve is defined in polar coordinates by $r = r(\theta)$, $\alpha \leq \theta \leq \beta$, then a parametrization is given by:

$$\begin{cases} x = r(t)\cos(t) \\ y = r(t)\sin(t) \end{cases} \quad \alpha \leq t \leq \beta$$

The circle with center (h, k) and radius r can be parametrized by:

$$\begin{cases} x = h + r\cos(t) \\ y = k + r\sin(t) \end{cases} \quad 0 \leq t \leq 2\pi$$

and the line segment from (x_1, y_1) to (x_2, y_2) can be parametrized by:

$$\begin{cases} x = (1-t)x_1 + tx_2 \\ y = (1-t)y_1 + ty_2 \end{cases} \quad 0 \leq t \leq 1$$

Conceptually, graphing parametric curves is simple. Just plot points of the form $(x(t), y(t))$ and join them with a smooth curve. Generally this requires a fairly large number of calculations and a good deal of care in plotting points. Since the HP 48 is good at both of these things—calculating and plotting—it makes sense that we put it to work.

The program PARA given below simplifies the procedure for making a parametric plot described in the Owner's Manual. Since we will use PARA in the remainder of this chapter we ask that you enter it into your calculator.

In addition to PARA, you will find the following program useful.

```
« ERASE IN OBJ→ DROP » INPUTS [STO]
```

PARA

Inputs: $x(T)$, $y(T)$, a, b, s	Output: Graph of $x = x(t)$, $y = y(t)$, $a \leq t \leq b$

```
« → X Y A B S «                  Introduces local vars. X, Y, A, B, S
    X Y A B 4 →LIST              Forms list {X,Y,A,B}, stores it under
    'IN' STO PARAMETRIC          name IN, begins parametric plot
    X Y i * +                    Stores x(T) and y(T) in complex form
    'EQ' STO                     in EQ
    T A B 3 →LIST INDEP          Stores T and endpoints in INDEP
    S RES                        Sets step size = s
    DRAX DRAW                    Plots and displays graph
    PICTURE » »
                                     Checksum: #14830d   Bytes: 152
```

✓Points to note

1. When you use PARA, be sure to use T instead of t;
2. The input s represents the step size; for example, if $s = 0.1$ and the parametrization

interval is $[0, 2]$, then the program will plot the following twenty-one points: $(x(0), y(0))$, $(x(0.1), y(0.1))$, $(x(0.2), y(0.2))$, $\cdots$, $(x(1.9), y(1.9))$, $(x(2), y(2))$. The value of s is up to you; your choice should depend on scaling, the length of $[a, b]$, and the length of the curve. When in doubt, try $s = 0.1$. If that gives you too few or too many dots, then try $s = 0.05$ or 0.2, etc.

3. The purpose of storing {X, Y, A, B} (in IN) is so that you won't have to re-enter those quantities if you want to adjust the value of s or make other small changes. Just press $\boxed{\text{INPUT}}$ to get $x(T)$, $y(T)$, a, and b back on the stack, enter your new choice for s, then press $\boxed{\text{PARA}}$.
4. To enter i, press $\boxed{\alpha}$ $\boxed{\text{↰}}$ $\boxed{\text{CST}}$.

Peace of Pi?

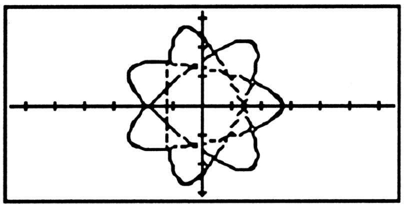
If you haven't already done so, you'll probably want to *set flag* −2. This will make it possible to deal with numbers like π more easily. Just key in −2 SF ENTER. The alternative is to press →NUM whenever you want to do a numerical calculation involving π. Put another way, if you want to work with 3.14159265359 instead of the symbol π (as you almost always will), then key in −2 SF $\boxed{\text{ENTER}}$.

EXAMPLE 1. Sketch the parametric curve

$$\begin{cases} x = 2\cos t + 0.75\cos \frac{4}{3}t \\ y = 2\sin t - 0.75\sin \frac{4}{3}t \end{cases} \qquad 0 \le t \le 6\pi$$

SOLUTION.

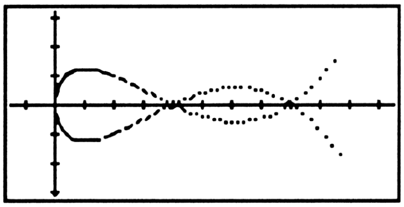
$\boxed{\text{RESET}}$ 2 T COS
* .75 4 T * 3
/ COS * + 2
T SIN * .75 4
T * 3 /
SIN * − 0 6
π * .05 $\boxed{\text{PARA}}$



EXAMPLE 2A. Sketch the parametric curve $x = t^2$, $y = t^5 - 6t^3 + 8t$, $-2.2 \le t \le 2.2$.

SOLUTION.

$\boxed{\text{ON}}$ $\boxed{\text{ERASE}}$ 3 $\boxed{\text{*H}}$ .5
$\boxed{\text{*W}}$ (2.5 0) $\boxed{\text{CENT}}$ T
SQ T 5 ^ 6 T 3 ^
* − 8 T * + 2.2
$\boxed{\text{+/−}}$ 2.2 .03 $\boxed{\text{PARA}}$



✓Point to note

$\boxed{*\text{H}}$, $\boxed{*\text{W}}$, and $\boxed{\text{CENT}}$ are in PLOT PPAR.

EXAMPLE 2B. What are the questions?

> **TO THE READER:** *Before reading on, what five questions would you ask?*

SOLUTION. If you gave the same questions as in Example 1B, § 3.1, that's fine, but let's be more imaginative. Do you see the region shaped like a top? (It's the one enclosed by the left-hand loop.) How about the football-shaped region? (That's the one enclosed by the right-hand loop.) Here are some questions. See how close you can come to guessing the correct answers.

1. Is the graph symmetric with respect to the x-axis?
2. Is the horizontal measurement of the top-shaped region equal to the horizontal measurement of the football-shaped region?
3. Is the vertical measurement of the top-shaped region equal to twice the vertical measurement of the football-shaped region?
4. Is the area of the top-shaped region equal to twice the area of the football-shaped region?
5. How does the perimeter of the top-shaped region compare with the perimeter of the football-shaped region?
6. Do the two inflection points coincide with the first positive x-intercept?
7. What are the equations of the two tangent lines to the curve at the first positive x-intercept? The second positive x-intercept?
8. If you rotate the curve about the x-axis, you'll get a 3-dimensional "top" and a 3-dimensional "football". How do their volumes compare? Their surface areas?

We will answer Questions 1, 2, 3, 6, and 7 here; Questions 4, 5, and 8 will be answered in Chapter 6.

EXAMPLE 2C. Determine whether the graph is symmetric with respect to the x-axis.

SOLUTION. The symmetry question amounts to this: if (x, y) is on the graph, must $(x, -y)$ also be on the graph? Suppose (x, y) is on the graph. Then we must have $(x, y) = (t^2, t^5 - 6t^3 + 8t)$ for some t between -2.2 and 2.2. But if t is between -2.2 and 2.2, then so is $-t$, and, therefore,

$$((-t)^2, ((-t)^5 - 6(-t)^3 + 8(-t)) = (t^2, -(t^5 - 6t^3 + 8t)) = (x, -y)$$

is also on the graph. Hence the graph *is* symmetric.

EXAMPLE 2D. Determine the horizontal measurement of (a) the top-shaped region; (b) the football-shaped region.

SOLUTION. This is really a question about the spacing of the x-intercepts. How can we find the x-intercepts? They are characterized by the zeros of $y(t)$ in the sense that $x(t)$ is an x-intercept if and only if $y(t) = 0$. What are the zeros of $y(t)$? Although we could haul out the HP 48 zero-finder, it isn't necessary here because

$$y = t^5 - 6t^3 + 8t = t(t^4 - 6t^2 + 8) = t(t^2 - 2)(t^2 - 4)$$

and we see that the zeros of $y(t)$ are precisely 0, $\pm\sqrt{2}$, and ± 2. It's important to note that these are t-values, not x-values. The corresponding x-values (x-intercepts) are: 0, 2, and 4. So both horizontal distances are 2 (and therefore they *are* equal).

About Derivatives

If we think of y as a function of x, we may apply the chain rule to get dy/dx and d^2y/dx^2 in terms of dx/dt, $dy/dt, d^2x/dt^2$, and d^2y/dt^2 as follows:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \tag{2}$$

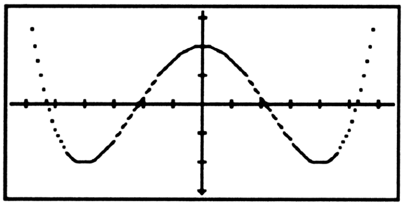
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d/dt\,(y'(t)/x'(t))}{dx/dt} = \frac{y''(t)x'(t) - y'(t)x''(t)}{x'(t)^3} \tag{3}$$

We will use equations (2) and (3) to obtain answers to Questions 3, 6, and 7.

EXAMPLE 2E. Determine the vertical measurement of (a) the top-shaped region; (b) the football-shaped region.

SOLUTION. Because of symmetry, the problem reduces to finding the maximum values of the curve for the two regions. By (2), we see that the t-values for these maxima will occur among the zeros of $y'(t)$. How can we find the zeros of $y'(t)$? One way is by hand (see Exercise B.8); another way is by using the HP 48 zero-finder.

$\boxed{\text{RESET}}$ $\boxed{\text{INPUTS}}$ $\boxed{\text{DROP}}$
$\boxed{\text{DROP}}$ $\boxed{\text{SWAP}}$ $\boxed{\text{DROP}}$
T ∂ $\boxed{↰}$ $\boxed{\text{EQ}}$ T
$\boxed{\text{INDEP}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$
$\boxed{\text{ZOOM}}$ $\boxed{\text{VZOUT}}$
$\boxed{\text{ZOOM}}$ $\boxed{\text{ZFACT}}$ 3
$\boxed{\text{OK}}$ $\boxed{\text{OK}}$ $\boxed{\text{HZIN}}$



Next use ROOT to obtain the four zeros: ± 0.720676871083, ±1.7551708884. By looking at the sign of $y'(t)$ near these zeros, we can tell that −1.7551708884 and 0.720676871084 correspond to the two maxima. Finally, if we evaluate $y(t)$ at these two values we get:

$$y(-1.7551708884) = 1.7437616351$$
$$y(0.720676871084) = 3.71400799133$$

Hence, the vertical measurements of the two regions are 7.42801598266 and 3.4875232702.

It is often desirable to go from t-values to the corresponding x- and y-values. The program TPAIRS, analogous to FPAIRS, gives the pairs $(x(t), y(t))$ corresponding to an arbitrary set of t-values. Notice that the program takes $x(t)$ and $y(t)$ from INPUTS so you don't have to re-enter those functions.

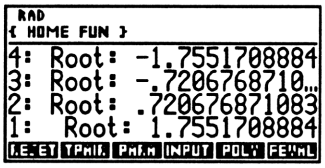
TPAIRS	
Inputs: $t_1, \cdots, t_n$	*Output:* $(x(t_1), y(t_1)), \cdots, (x(t_n), y(t_n))$
`« DEPTH INPUTS DROP`	Puts $n, x(T), y(T)$ on stack
`DROP → N X Y «`	Introduces local vars. N, X, Y
`1 N FOR I`	Starts FOR-NEXT loop
`RE 'T' STO`	Stores t_{N-I+1} under name T
`X EVAL Y EVAL`	Calculates $x(t_{N-I+1}), y(t_{N-I+1})$
`R→C`	Forms $x(t_{N-I+1}), y(t_{N-I+1})$
`N ROLLD`	Puts $(x(t_{N-I+1}), y(t_{N-I+1}))$ on top
`NEXT`	of stack, ends FOR-NEXT loop
`'T' PURGE » »`	Purges T, ends program
	Checksum: #24875d Bytes: 125

✓Point to note

When you apply TPAIRS make sure there is nothing on the stack other than the t_i's.

We illustrate TPAIRS by applying it to the roots of $y'(t)$.

INPUT:



```
RAD
{ HOME FUN }
4: Root: -1.7551708884
3: Root: -.7206768710…
2: Root: .720676871083
1:  Root: 1.7551708884
```

TPAIRS

OUTPUT:

(3.08062484749, 1.7437616351)	relative maximum
(.519375152513, -3.71400799134)	relative minimum
(.519375152513, 3.71400799134)	relative maximum
(3.08062484749, -1.7437616351)	relative minimum

EXAMPLE 2F. Find the two inflection points (of the curve shown in Example 2A).

SOLUTION. By (3), we see that the t-values for inflection points will be among the zeros of $y''(t)x'(t)-y'(t)x''(t)$. So we have yet another zero problem! To solve it, start by entering the function.

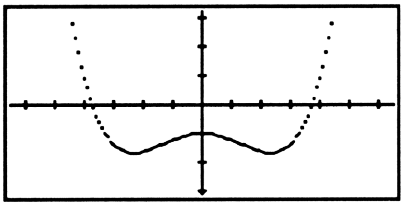
$\boxed{\text{INPUTS}}$ $\boxed{\text{DROP}}$ $\boxed{\text{DROP}}$ $\boxed{\text{SWAP}}$ $\boxed{\text{DROP}}$ T ∂

$\boxed{\text{ENTER}}$ T ∂ 2 T * * $\boxed{\text{SWAP}}$ 2 * −

[*Suggestion.* Double-check this result by hand. Also note that if you had to work many problems of this type—as you'll have the opportunity to do in the exercises—you might want to write a program to calculate the general expression $y''(t)x'(t) - y'(t)x''(t)$. See Exercise B.1.]

Now graph the function and adjust the PPAR.

$\boxed{↰}$ $\boxed{\text{EQ}}$ $\boxed{\text{ERASE}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$
$\boxed{\text{ZOOM}}$ $\boxed{\text{VZOUT}}$



The two zeros that you see correspond to the inflection points. Using ROOTS and TPAIRS, we easily obtain the values: (1.54516312524, ±1.38792331957). Since the first x-intercept occurs at $(2,0)$ the answer to Question 6 is no (as you probably guessed).

EXAMPLE 2G. Find the equations of the two tangent lines at (a) the first positive x-intercept; (b) the second positive x-intercept.

SOLUTION. The first positive x-intercept is

$$(2,0) = \left(x(\pm\sqrt{2}), y(\pm\sqrt{2})\right).$$

By (2), the slopes are

$$\begin{aligned}\frac{y'(t)}{x'(t)} &= \frac{(t^5-6t^3+8t)'}{(t^2)'} = \frac{5t^4-18t^2+8}{2t}\\ &= \pm 2\sqrt{2} \approx \pm 2.82842712475.\end{aligned}$$

Therefore, the equations of the two tangent lines are

$$y = \pm 2\sqrt{2}(x-2) \approx \pm 2.82842712475(x-2)$$

Similarly, the tangent lines at (4,0) are $y = \pm 4(x-4)$.

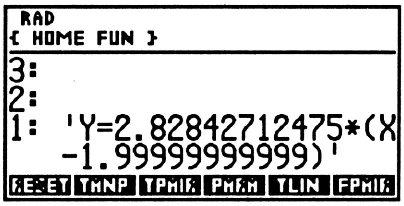
About Tangent Lines

Notice that formula (2) does not apply if $x'(t) = 0$. If $x'(t) = 0$, special care is required. If $x'(c) \neq 0$, the program TANPAR (see box) will find the equation of the tangent line to the curve at the point $(x(c), y(c))$.

TANPAR	
Inputs: $x(t)$, $y(t)$, c such that $x'(c) \neq 0$.	*Output: Equation of tangent line to curve $x = x(t)$, $y = y(t)$ at $(x(c), y(c))$.*
`« 'T' STO → X Y «`	Stores c in T; introd. loc. vars. X, Y
`Y 'T' ∂ EVAL`	Calculates $y'(c)$
`X 'T' ∂ EVAL`	Calculates $x'(c)$
`/`	Forms $m = y'(c)/x'(c)$
`'X' X EVAL - * Y`	Forms $m(\mathrm{X} - x(c)) + y(c)$
`EVAL +`	
`'Y' SWAP =`	Forms $\mathrm{Y} = m(\mathrm{X} - x(c)) + y(c)$
`'T' PURGE » »`	Purges T, ends program
	Checksum: #14777d Bytes: 147

For example, we can easily obtain the above tangent lines as follows.

$\boxed{\text{INPUTS}}$ $\boxed{\text{DROP}}$ $\boxed{\text{DROP}}$
$-\sqrt{2}$ $\boxed{\text{TANPAR}}$

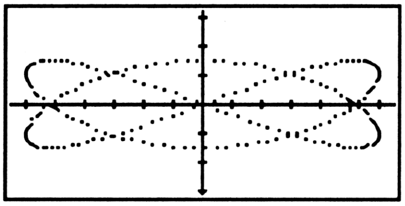


```
 RAD
{ HOME FUN }
3:
2:
1:  'Y=2.82842712475*(X
     -1.99999999999)'
```

EXAMPLE 3. Sketch the parametric curve $x = 6\sin 2t$, $y = 1.5\cos 3t$, $0 \leq t \leq 2\pi$.

SOLUTION.

$\boxed{\text{RESET}}$ 6 2 T *
SIN * 1.5 3 T
* COS * 0 2
π * .03 $\boxed{\text{PARA}}$



What are your questions? Do you see the two "hearts" ? Do you see the two "boomerangs" ? Do you see the big "football" ? You'll have the opportunity to explore a variety of interesting questions about this curve in the exercises. (See especially § 6.3.) Many such questions depend on being able to find the intersection points.

About Self-intersection Points

Notice that a parametric curve like the one in Example 3 intersects itself when two or more values of t correspond to the same point, i.e., when there exist values t_1 and t_2 with $t_1 \neq t_2$ such that $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$. Generally, to find such pairs (t_1, t_2) is a difficult job, but in special cases like this one, things are manageable.

For the curve above, the problem of finding pairs (t_1, t_2) is equivalent to solving the following system:

$$\begin{cases} \sin 2t_1 = \sin 2t_2 \\ \cos 3t_1 = \cos 3t_2 \end{cases} \quad 0 \le t_1 < t_2 \le 2\pi \tag{4}$$

If you look at how horizontal lines intersect the curves $y = \sin x$ and $y = \cos x$, you'll see that $\cos\alpha = \cos\beta$ if and only if $\beta = \pm\alpha + 2n\pi$ for some integer n, and $\sin\alpha = \sin\beta$ iff $\beta = \alpha + 2n\pi$ for some integer n or $\beta = -\alpha + (2n+1)\pi$ for some integer n. See Exercise B.6. Applying these conditions to (4) yields the following seven solutions for (t_1, t_2): $(\pi/12, 17\pi/12)$, $(\pi/6, 7\pi/6)$, $(5\pi/12, 13\pi/12)$, $(\pi/2, 3\pi/2)$, $(7\pi/12, \pi/12)$, $(5\pi/6, 11\pi/6)$, $(11\pi/12, 19\pi/12)$. See Exercise B.7. TPAIRS can then be used to obtain the corresponding points $(x(t), y(t))$ which we list in Table 3.

Table 3

t-values	$(x(t), y(t))$
$\pi/12$, $17\pi/12$	$(3, 1.06066017178)$
$\pi/6$, $7\pi/6$	$(5.19615242272, 0)$
$5\pi/12$, $13\pi/12$	$(3, 1.06066017177)$
$\pi/2$, $3\pi/2$	$(0, 0)$
$7\pi/12$, $23\pi/12$	$(3, 1.06066017177)$
$5\pi/6$, $11\pi/6$	$(-5.19615242266, 0)$
$11\pi/12$, $19\pi/12$	$(-3, -1.06066017178)$

If we interpret this information in terms of a moving particle, the particle would start at the point (0, 1.5) at time 0, reach the first intersection point (3, 1.06066017178) at time $t = \pi/12$, the second intersection point (5.19615242272, 0) at time $t = \pi/6$, the third intersection point (3, -1.06066017177) at time $t = 5\pi/12$, and so on, finally ending up back where it started at time 2π.

Generally, the problem of finding self-intersection points can be solved by using an approximation method similar to Newton's method.

About Polar Curves

As mentioned at the beginning of this section, any polar curve $r = r(\theta)$, $\alpha \le \theta \le \beta$, can be represented in parametric form by setting $x(t) = r(t)\cos t$, $y(t) = r(t)\sin t$, $\alpha \le t \le \beta$. Thus, any question about polar curves can be thought of as being a question about parametric curves. In particular, we can easily obtain a polar graphing program POLR from PARA. See program box.

The HP 48 also has a built-in POLAR program that works well.

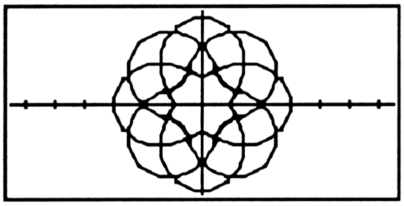
EXAMPLE 4. Graph the four polar curves $r = 2 \pm \sin 2\theta$, $r = 2 \pm \cos 2\theta$, $0 \le \theta \le 2\pi$, in the same coordinate system.

POLR

Inputs: $r(T)$, α, β, s	Output: Graph of $r = r(\theta)$, $\alpha \leq \theta \leq \beta$
« → R A B S «	Introduces local vars. R, A, B, S
R T COS * R T SIN *	Forms RcosT, RsinT
A B S PARA	Hooks onto program PARA
» »	Ends program
	Checksum: #5467d Bytes: 93

SOLUTION.

$\boxed{\text{RESET}}$ $\boxed{\text{POLAR}}$
2 2 θ *
SIN + 2 2 θ
* SIN − 2 2
θ * COS + 2 2
θ * COS − 4
$\boxed{\text{→LIST}}$ $\boxed{↰}$ $\boxed{\text{EQ}}$
θ $\boxed{\text{INDEP}}$ .03 $\boxed{\text{RES}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



✓Points to note

1. $\boxed{\text{POLAR}}$ is in PLOT PTYPE.
2. The θ key is $\boxed{\alpha}$ $\boxed{↱}$ F.

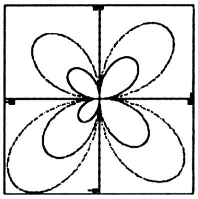
EXAMPLE 5. Graph the curve $r = \theta \sin 2\theta$, $\pi \leq \theta \leq 5.5\pi$.

SOLUTION. Begin by $\boxed{\text{RESET}}$ting, then set the PICTure DIMensions as follows:

(−13 −13) (13 13) $\boxed{\text{PDIM}}$

Then enter the function and use POLR:

T 2 T * SIN
* π $\boxed{\text{ENTER}}$
5.5 * .01 $\boxed{\text{POLR}}$



✓Points to note

1. $\boxed{\text{PDIM}}$ is in PRG PICT.
2. On the display of your calculator you will only be able to see one-

eighth of the entire OUTPUT at any one time. You can scroll through the other seven-eighths with the cursor keys.

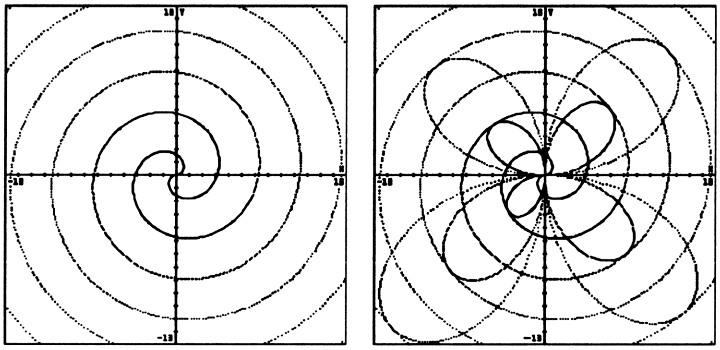


Figure 4

It is informative to compare this graph with the double spiral $r = \pm\theta$, $0 \le \theta \le 6\pi$. See Fig. 4. What are the coordinates of the intersection points? How far away from the origin are the extreme points of the lobes? Do these points coincide with the intersection points? See Exercises C.4.

Exercises 3.2

For Exercises 1–14, (a) make one or more plot of the given parametric curves to capture all interesting features; (b) find all x-intercepts, y-intercepts, extrema, and inflection points. For inflection points, you may first want to do Exercise B.1.

A.1 $x = 2\sin 2t$, $y = 2\cos 3t$, $0 \le t \le 2\pi$.

A.2 $x = 2\sin 3t$, $y = 2\cos 2t$, $0 \le t \le 2\pi$.

A.3 $x = 2\sin t - 4\cos 3t$, $y = \sin 2t + 3\cos 2t$, $0 \le t \le 2\pi$.

A.4 $x = \sin 2t + \cos 3t$, $y = \sin 3t + \cos 2t$, $0 \le t \le 2\pi$.

A.5 $x = 4\sin 2t$, $y = 1.5\sin t + 1.5\cos 2t + 1$, $0 \le t \le 2\pi$.

A.6 $x = t - \sin t$, $y = 1 - \cos t$, $-4\pi \le t \le 4\pi$.

A.7 $x = t - 0.4\sin t$, $y = 1 - 0.4\cos t$, $-4\pi \le t \le 4\pi$.

A.8 $x = t - 2\sin t$, $y = 1 - 2\cos t$, $-4\pi \le t \le 4\pi$.

A.9 $x = \sin t$, $y = \sin t + \cos 2t$, $0 \le t \le 2\pi$.

A.10 $x = 6\sin t$, $y = 1.5\sin 2t + 1.5\cos t$, $0 \le t \le 2\pi$.

A.11 $x = 5\sin t$, $y = 1.5\sin 3t + 1.5\cos 2t$, $0 \le t \le 2\pi$.

A.12 $x = 5\sin t$, $y = 1.5\sin 2t + 1.5\cos 3t$, $0 \le t \le 2\pi$.

A.13 $x = 5\sin t$, $y = 1.5\sin 2t + 1.5\sin 3t$, $0 \le t \le 2\pi$.

A.14 $x = \sin 2t + \cos t$, $y = \sin 2t$, $0 \le t \le 2\pi$.

A.15 Given the curve $x = t^2 - 3t - 2$, $y = t^4 - t^2 + t - 2$, $-2 \le t \le 2$, (a) make a sketch showing all interesting features; (b) find the absolute minimum point; (c) find the equation of the tangent line at the positive x-intercept.

A.16 Given the curve $x = t^4 - 4t^2 + 1$, $y = 0.5t^5 - 4t^3 + 7.5t$, $-2.5 \le t \le 2.5$, (a) find all intercepts; (b) find the dimension of the smallest rectangle that contains the curve.

A.17 Given the curve $x = \cos t - \sin 2t$, $y = \sin 3t$, $0 \le t \le 2\pi$, (a) make a careful sketch of this bunny-shaped curve; and (b) find the coordinates of the only self-intersection point and the four t-values corresponding to it.

A.18 Given the curve $x = 4\sin t - 2\cos 3t$, $y = 3\cos 2t$, $0 \le t \le 2\pi$, (a) sketch this curve that resembles two kissing ducks; and (b) find all self-intersection points.

A.19 Sketch the limaçon $r = 1 + 2\cos\theta$ and find the equations of the two tangent lines at the origin.

A.20 Sketch the limaçon $r = 3 + 2\cos\theta$ and find the points at which the tangent line is vertical.

A.21 Sketch the polar curve $r = 3 + \sin\theta$. What do you think it is? Find the horizontal and vertical distances across the curve. What do you conclude?

A.22 Sketch the four polar curves $r = 2.2 \pm \sin\theta$, $r = 2.2 \pm \cos\theta$, in the same coordinate system and determine the twelve intersection points.

A.23 (a) Sketch the double spiral shown in Fig. 6(b).
(b) Sketch the double spiral $r = \theta$, $-4\pi \le \theta \le 4\pi$.
(c) Sketch the double spiral $r = |\theta|$, $-4\pi \le \theta \le 4\pi$.

B.1 Write a program to calculate the general expression $y''(t)x'(t) - y'(t)x''(t)$. Assume that INPUTS contains $x(t)$ and $y(t)$.

B.2 Make a careful plot of the curve $x = t^2 - 3t - 4$, $y = 2t^2 - t - 12$, $-\infty < t < \infty$, showing its interesting features. What do you think it is? Determine the coordinates of the left-most point of this curve. Also find its "vertex".

B.3 Sketch the curve $x = \sin t$, $y = \sin t + \cos t$, $0 \le t \le 2\pi$. What do you think it is? Verify your conjecture.

B.4 Sketch the curve $x = \sin t - \cos t$, $y = \sin t + \cos t$, $0 \le t \le 2\pi$. What do you think it is? Verify your conjecture.

B.5 The curve $x = 5t^2 + 5t - 4$, $y = 2t^3 + 4.5t^2 + 3t - 1$, $-2 \le t \le 1$, contains a cusp P_0

(a) Find the exact coordinates of P_0;

(b) Even though there is no tangent line at P_0, the nearby tangent lines $T(P)$ approach a limiting line T as $P \to P_0$. Find the equation of T and make a sketch showing both the cusp and the line T;

(c) Does T pass through the origin? Explain.

B.6 Verify (a) $\cos\alpha = \cos\beta$ if and only if $\beta = \pm\alpha + 2n\pi$ for some integer n;

(b) $\sin\alpha = \sin\beta$ if and only if $\beta = \alpha + 2n\pi$ for some integer n or $\beta = -\alpha + (2n+1)\pi$ for some integer n.

B.7 Apply the results of Exercise B.6 to find the seven solution pairs (t_1, t_2) of the system (4).

B.8 Find the exact solutions of $5t^4 - 18t^2 + 8 = 0$.

B.9 Sketch the lemniscate $r^2 = 36\cos 2\theta$, $-\pi/4 \le \theta \le \pi/4$ and find the equations of the two tangent lines at the origin.

C.1 Prove that the curve in B.2 represents a parabola.

C.2 Prove that a curve of the form $x = a\sin t + b\cos t$, $y = c\sin t + d\cos t$, $0 \le t \le 2\pi$, represents an ellipse if and only if $ad - bc \ne 0$.

C.3 Prove that a curve of the form $x = a\sin(t+\alpha)$, $y = b\sin(t+\beta)$, $0 \le t \le 2\pi$, represents an ellipse if and only if $\beta - \alpha$ is not a multiple of π.

C.4 (a) Calculate the coordinates of the tips of the nine lobes shown in Fig. 6(a).

(b) Find the ten intersection points of the curves shown in Fig. 6(c).

(c) Compare (a) and (b) and deduce what happens as $\theta \to +\infty$.

3.3 CURVES DEFINED BY AN EQUATION

Many curves occur in the form $f(x, y) = 0$. Examples of what we mean are easy to give, though often not easy to graph:

$$\begin{aligned}
&x^2 + y^2 = 2.25\\
&5y^2 + 10^y = 8x + 19\\
&x^2 + 4y^2 + 4x - 8y - 8 = 0\\
&x^2 - 4xy + 4y^2 - x - 5y - 3 = 0\\
&\sin x^2 y + x^3 + 3^y = 17\\
&7y^2 = x^2(1 - x^2)
\end{aligned}$$

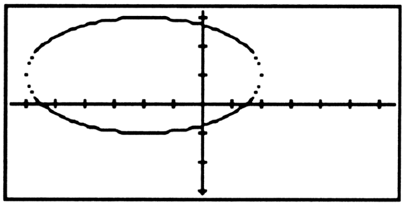
How can we graph such curves? The answer boils down to either (a) solve the equation for y in terms of x and use the methods of § 3.1 or (b) parametrize the curve and use the methods of § 3.2.

Generally speaking, neither (a) nor (b) is easy and quite often (b) is preferable to (a). In this section, we limit discussion to *second degree equations.* These equations describe the *general conic.* The discussion relates to material in your calculus text under the headings "conic sections" and "rotation of axes". In § 4.3, we discuss more general situations along with *inverse functions* and *implicitly-defined functions.* In § 9.2, we discuss the general case in conjunction with *level curves* for three-dimensional surfaces.

EXAMPLE 1. Graph the equation $x^2 + 4y^2 + 4x - 8y - 8 = 0$.

SOLUTION. Note that since the equation is quadratic in y, we may solve for y in terms of x to get $y = 1 \pm \sqrt{3 - x - x^2/4}$. Clearly, this equation is equivalent to the given one, and all we have to do is graph the functions determined by the two signs.

$\boxed{\text{RESET}}$ 1 3 X −
X SQ 4 / − $\sqrt{\ }$
+ $\boxed{\text{ARG}}$ − =
$\boxed{\text{↰}}$ $\boxed{\text{EQ}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



✓Point to note

ARG (GREEN EEX) puts the last two arguments on the stack (and nothing else). Here, it takes only two keystrokes to enter $1 - \sqrt{3 - x - x^2/4}$ (ARG and −).

Another way to work this problem is by approach (b). To see this, first use algebra to put the equation in the form:

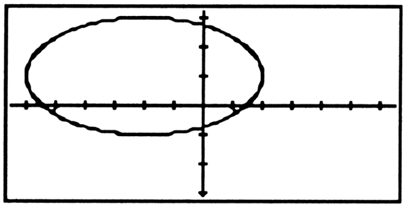
$$\frac{(x+2)^2}{4^2} + \frac{(y-1)^2}{2^2} = 1$$

Then think about the trig identity $\sin^2\alpha + \cos^2\alpha = 1$. This will get you to the following parametrization:

$$\begin{cases} x = -2 + 4\sin t \\ y = 1 + 2\cos t \end{cases} \quad 0 \le t \le 2\pi$$

The rest is easy; just use PARA.

$\boxed{\text{RESET}}$ −2 4 T
SIN * + 1 2
T COS * + 0
2 π * .02
$\boxed{\text{PARA}}$



Observe the difference between the two outputs. Typically, parametric representations will give sharper outputs.

A third way to plot the above equation is to use the HP built-in plotter. This works for any second degree equation and is based on the "solve for y" idea. By a second degree equation we mean one of the following form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

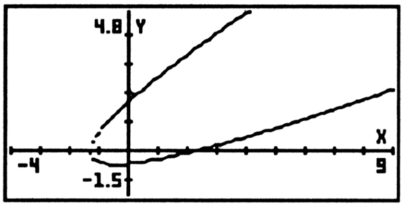
To apply the built-in plotter, enter the entire equation ($\boxed{\Lsh}$ $\boxed{\text{EQ}}$), press CONIC (in PTYPE), then press $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$.

EXAMPLE 2. Graph the following second degree equations:

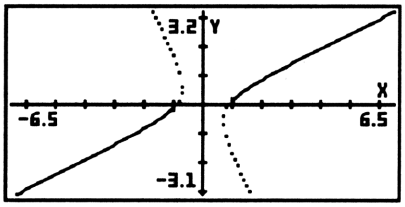
(a) $x^2 - 4xy + 4y^2 - x - 5y - 3 = 0$
(b) $4x^2 - 6xy - 4y^2 = 3$

SOLUTION.

(a) $\boxed{\text{RESET}}$ X SQ
4 X * Y *
− 4 Y SQ *
+ X − 5 Y
* − 3 − 0 =
$\boxed{\Lsh}$ $\boxed{\text{EQ}}$ $\boxed{\text{CONIC}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



(b) $\boxed{\text{RESET}}$ 4 X SQ
* 6 X * Y
* − 4 Y SQ
* − 3 −
$\boxed{\Lsh}$ $\boxed{\text{EQ}}$ $\boxed{\text{CONIC}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



What About Interesting Features?

Even though the above graphing process is quite satisfying (to *really* appreciate it, try graphing a few second degree equations by hand!), one gets an empty feeling when it comes to the question of "interesting features". What about "vertices", for example? How can we find them? For conic sections, interesting features include vertices, foci, directrices, centers, asymptotes, eccentricities, axes, semi-axes, and latus recta. If $B = 0$, things are easy. In this case, the equation can readily be put into one of the "standard forms" from which everything can be easily determined. If $B \neq 0$, that's a different story. Geometrically, $B \neq 0$ means that the curve is "tilted" with respect to the (x, y)-coordinate system. To effectively analyze tilted curves, one must have a good understanding of *rotations*.

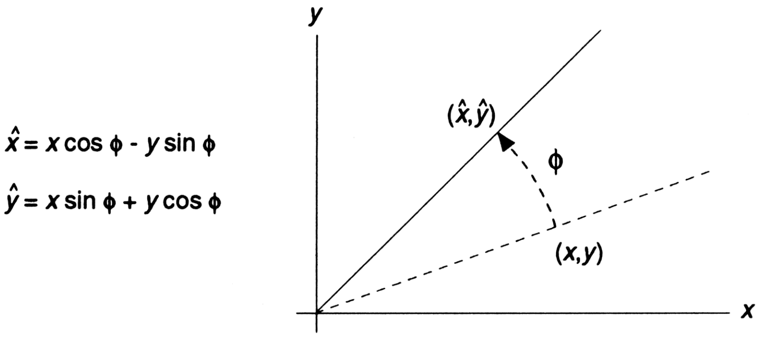


Figure 5

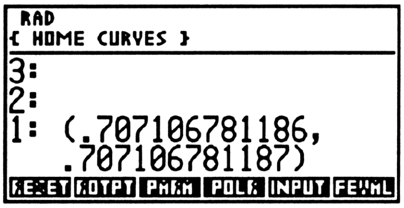
Rotation of Points

If a point with coordinates (x, y) is rotated counterclockwise about the origin through an angle ϕ, positive or negative, then the new coordinates $(\widehat{x}, \widehat{y})$ are related to the old ones as shown in Fig. 5.

ROTPT	
Inputs: (x, y), ϕ	*Output:* $(\hat{x}, \hat{y})$ = *counterclockwise rotation of* (x, y) *through* $\angle\ \phi$.
`« i * EXP * »`	Multiplies the complex numbers (a, b) and $(\cos\phi, \sin\phi)$ Checksum: #6592d Bytes: 29.5

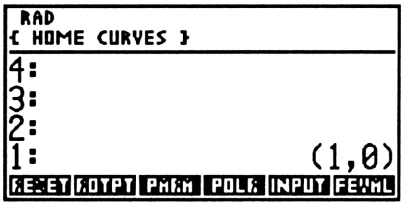
The program ROTPT (see box) will rotate any point about the origin through any angle. Examples:

(1 0) π 4 /
$\boxed{\text{ROTPT}}$



```
RAD
{ HOME CURVES }
3:
2:
1: (.707106781186,
   .707106781187)
```

(0 1) π 2 / $\boxed{+/-}$
$\boxed{\text{ROTPT}}$ 10 $\boxed{\text{RND}}$



```
RAD
{ HOME CURVES }
4:
3:
2:
1:                (1,0)
```

$$\Gamma : \begin{cases} x = x(t) \\ y = x(t) \end{cases} \quad a \le t \le b$$

$$\hat{\Gamma} : \begin{cases} \hat{x} = x(t) \cos \phi - y(t) \sin \phi \\ \hat{y} = x(t) \sin \phi + y(t) \cos \phi \end{cases}$$

$$a \le t \le b$$

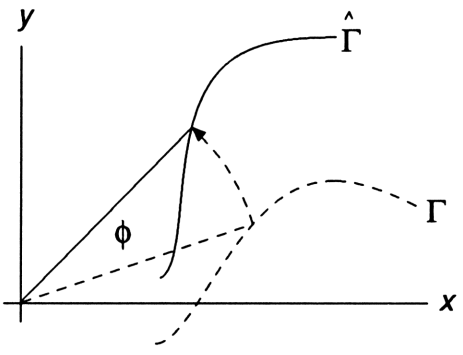


Figure 6

About Rotation of Curves

If a curve Γ with parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, is rotated counterclockwise about the origin through an angle ϕ, then the new curve $\widehat{\Gamma}$ can be parametrized in terms of the old parametrization as indicated in Fig. 6.

The program ROTCV will rotate any parametric curve about the origin through any angle.

ROTCV	
Inputs: $x(T)$, $y(T)$, a, b, s, ϕ	*Output: Counterclockwise rotation of curve through $\angle\ \phi$.*
« → X Y A B S P «	Introduces loc. vars. X, Y, A, B, S, P
X P COS * Y P SIN * −	Calculates x-value for rotation
X P SIN * Y P COS * +	Calculates y-value for rotation
A B S PARA » »	Graphs rotated curve
	Checksum: #51020d Bytes: 136

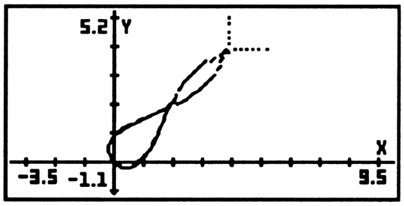
EXAMPLE 3. Rotate the curve

$$\begin{cases} x = \frac{4}{3}t^2 \\ y = \frac{1}{6}t^5 - t^3 + \frac{4}{3}t \end{cases} \quad -2.2 \le t \le 2.2$$

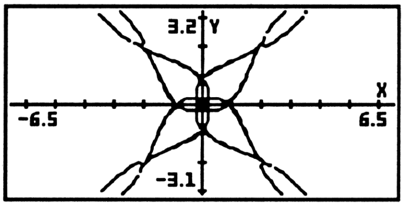
through angles 45°, 135°, 225°, and 315° (cf. Example 2A, §3.1).

SOLUTION. Start with a 45° rotation.

$\boxed{\text{RESET}}$ 4 3 / T
SQ * T 5 ^ 6
/ T 3 ^ − 4 3
/ T * +
−2.2 2.2 .03 π 4
/ $\boxed{\text{ROTCV}}$



Repeat the process
three times using
INPUTS from
the previous plot and
$\phi = \pi/2$
(see Exercise B6).



Rotations of Equations

If a curve Γ, defined by $f(x, y) = 0$, is rotated counterclockwise about the origin through an angle ϕ, then the new curve $\widehat{\Gamma}$ is given by the equation $f(x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi) = 0$. This is because a point (x, y) is on the curve $\widehat{\Gamma}$ if and only if its rotation through $-\phi$ is on the curve Γ. In particular, if the given curve is the general conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

then counterclockwise rotation of this curve through angle ϕ gives the equation

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0,$$

where

$$\begin{aligned}
A' &= A\cos^2\phi - B\cos\phi\sin\phi + C\sin^2\phi \\
B' &= B(\cos^2\phi - \sin^2\phi) + 2(A - C)\cos\phi\sin\phi \\
C' &= A\sin^2\phi + B\cos\phi\sin\phi + C\cos^2\phi \\
D' &= D\cos\phi - E\sin\phi \\
E' &= D\sin\phi + E\cos\phi \\
F' &= F.
\end{aligned}$$

You will probably find similar formulas in your textbook. If some of the signs differ from those in your book, it's because a counterclockwise rotation of the coordinate axes through angle ϕ is equivalent to a *clockwise* rotation of everything else through angle ϕ. (Most textbooks take the former point of view.)

As you can imagine, to find the equation of a rotated curve can involve a lot of tedious calculation. The HP 48 specializes in tedious calculations, so let's put it to work.

The program ROTEQ calculates the coefficients A', B', C', D', E', F', stores them under the name L, and forms the equation of the rotated curve.

EXAMPLE 4. Obtain parametric and rectangular representations and plots for the counterclockwise rotation of the curve $y = x^2$ through 45°.

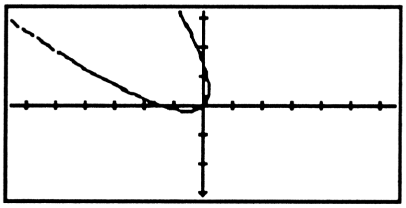
SOLUTION. First, use ROTCV to make a parametric plot.

ROTEQ

Inputs: Coefficients A, B, C, D, E, F of general conic, and ∠ φ	*Output: General conic rotated counterclockwise through ∠ φ*

```
≪ → A B C D E               Introduces local variables
  F P ≪
P COS P SIN → K             Introd. loc. vars. for cos φ, sin φ
S ≪ A K SQ * B
K S * * - C S SQ
* + 8 RND B K SQ
S SQ - * 2
A C - * K S                 Calculates A′, B′, C′, D′, E′, F′
* * +
RND A S SQ * B              Rounds to 8 places
K S *
* + C K SQ * +
8 RND D K * E S *
- 8 RND D S * E K
* + 8 RND F
6 →LIST 'L' STO             Forms {A′, B′, C′, D′, E′, F′},
                            stores as L
L 1 GET X SQ *
L 2 GET X Y * *
L 3 GET Y SQ *              Forms desired equation
L 4 GET X * L 5
GET Y * + F + +
+ + 0 = COLCT ≫ ≫ ≫         Simplifies equation

                              Checksum: #64582d   Bytes: 465
```

RESET T T SQ
−3 3 .05 π 4 /
ROTCV



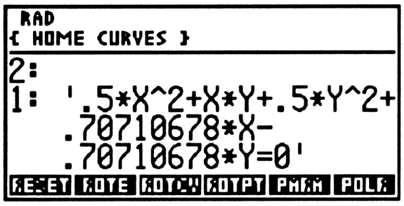
Since ROTCV uses PARA, we may use INPUTS to get a parametric representation.

INPUT: ON INPUTS

OUTPUT:
$$\begin{cases} x = .707106781186(t - t^2) \\ y = .707106781187(t + t^2) \end{cases} \quad -3 \leq t \leq 3$$

To get a rectangular representation rewrite the equation of the curve as $x^2 - y = 0$ and apply ROTEQ.

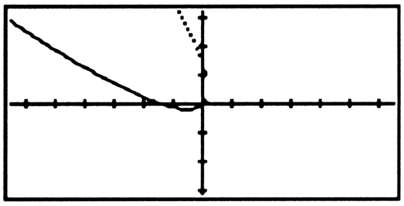
$\boxed{\text{RESET}}$ 1 0 0
0 −1 0 π
4 / $\boxed{\text{ROTEQ}}$



```
RAD
{ HOME CURVES }
2:
1: '.5*X^2+X*Y+.5*Y^2+
   .70710678*X-
   .70710678*Y=0'
```

Finally, to get a rectangular plot, use CONIC.

$\boxed{\text{RESET}}$ $\boxed{\text{CONIC}}$
$\boxed{↰}$ $\boxed{\text{EQ}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



Notice again that the rectangular plot is not quite as sharp as the parametric plot. (Can you figure out why this is so?)

A Strategy

As mentioned above, the case $B = 0$ is easy. If $B \neq 0$, the trick is to find an angle ϕ so that $B' = 0$ (so that the rotated curve will be easy to deal with). The following value will always work:

$$\phi = \tan^{-1}\left(\frac{A-C}{B} + \sqrt{\left(\frac{A-C}{B}\right)^2 + 1}\right)$$

Note that $\tan^{-1}$ denotes the inverse tangent function, not $1/\tan$; on the HP 48, it's ATAN. Your textbook probably gives a similar formula for rotation of axes. The calculation of ϕ (between $-\pi/2$ and $+\pi/2$) can be made easier with the help of the following program:

```
« → A B C « A C − B / DUP SQ 1 + √ +
ATAN DUP 'PHI' STO » » ANG [STO]
```

To find the "interesting features" of a second degree curve (vertices, foci, etc.), we may proceed as follows:

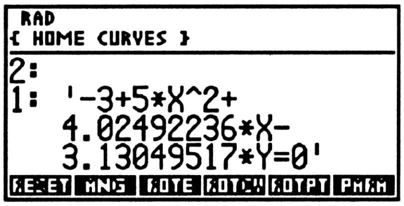
Step 1. Calculate the above angle ϕ and rotate the given curve through that angle.
Step 2. Use standard methods to find the "interesting features" of the rotated curve.
Step 3. Rotate the data from Step 2 through the angle $-\phi$.

EXAMPLE 5. Find vertex, focus, and directrix of the parabola $x^2 - 4xy + 4y^2 - x - 5y - 3 = 0$ (cf. Example 2(a)).

SOLUTION.

Step 1. Enter the coefficients of the given equation, calculate ϕ (you should get $1.10714871779 \approx 63.4°$), then apply ROTEQ:

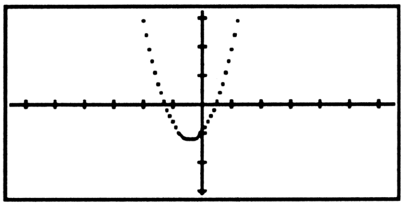
$\boxed{\text{RESET}}$ 1 −4 4
−1 −5 −3 1 −4
4 $\boxed{\text{ANG}}$ $\boxed{\text{ROTEQ}}$



```
RAD
{ HOME CURVES }
2:
1: '-3+5*X^2+
   4.02492236*X-
   3.13049517*Y=0'
```

This is the equation of the rotated curve. Notice that the xy-term is missing, as it should be. You can easily graph it using CONIC.

$\boxed{\text{↰}}$ $\boxed{\text{EQ}}$ $\boxed{\text{CONIC}}$
$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



Step 2. It is easy to obtain the following information (rounded to 8 places) for the rotated curve. Consult your book for details (also see Exercise B.1).

vertex $= (-0.40249224, -1.21705986)$
focus $= (-0.40249224, -1.06053510)$
directrix: $y = -1.37358461$

Step 3. Finally, rotate this information back by using ROTPT and ROTEQ:

INPUT: $(-.40249224, -1.21705986)$ $\boxed{\text{PHI}}$
$\boxed{+/-}$ $\boxed{\text{ROTPT}}$

OUTPUT: $(-1.26857143, -0.18428571)$ (= vertex)

INPUT: $(-.40249224, -1.06053510)$ $\boxed{\text{PHI}}$
$\boxed{+/-}$ $\boxed{\text{ROTPT}}$

OUTPUT: $(-1.12857143, -0.11428571)$ (= focus)

INPUT: 0 0 0 0 1 1.37358461 $\boxed{\text{PHI}}$
$\boxed{+/-}$ $\boxed{\text{ROTEQ}}$

OUTPUT: $1.37358461 + 0.89442719x + 0.4472136y = 0$
(= directrix)

You can put the last expression in more familiar form by using ISOL (in the SYMBOLIC menu).

INPUT: Y $\boxed{\text{ISOL}}$ $\boxed{\text{EXPA}}$ $\boxed{\text{EXPA}}$ $\boxed{\text{COLCT}}$

OUTPUT: $y = -3.07142853 - 2x$

Figure 7 shows the given curve together with vertex, focus, and directrix.

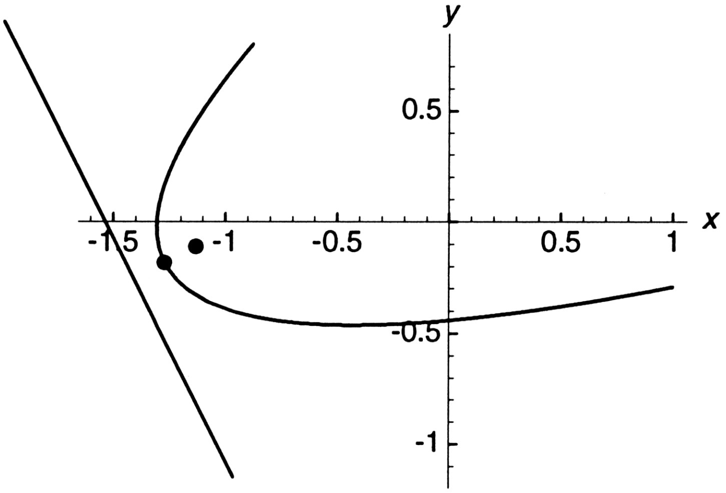


Figure 7

Exercises 3.3

For Exercises 1–6, (a) graph the equation and identify the conic section; (b) rotate the conic section through a suitable angle ϕ so that the rotated conic is in standard form; (c) as appropriate, identify vertices, foci, directrix, center, and asymptotes for the rotated conic; (d) as appropriate, identify vertices, foci, directrix, center, and asymptotes for the given curve.

A.1 $5x^2 - 20xy + 20y^2 - 6x + 8y - 7 = 0$

A.2 $8x^2 + 7xy + 8y^2 = 100$

A.3 $9x^2 + 12xy + 4y^2 - 8x + 7y = 0$

A.4 $4x^2 - 6xy - 4y^2 = 3$

A.5 $34x^2 - 24xy + 41y^2 = 25$

A.6 $x^2 - 24xy - 6y^2 - 26x + 12y = 17$

A.7 Find equations of two different parabolas through the points $(-1, 1)$, $(0, -1)$, $(2, 0)$.

B.1 Write an HP 48 program that will take as input three points (x_1, y_1), (x_2, y_2), (x_3, y_3), where x_1, x_2, x_3 are distinct, and produce as output the equation $y = ax^2 + bx + c$ of the parabola through them. Test your program on Exercise A.7.

B.2 Find equations of two different parabolas through the points $(-2, 0)$, $(0, -1)$, $(2, 0)$.

B.3 Write a program that will take as input an equation of the form $y = ax^2+bx+c$, $a \neq 0$, and produce as output vertex, focus, and directrix of the parabola. [*Hint*: use OBJ→ and differentiation to isolate the coefficients a, b, c.] Test your program on step 2 of Example 5.

B.4 Use the program in B.1 to help solve Exercise A.1 and A.3 above. [*Hint*: for A.3 rotate an additional 90°.]

B.5 Explain why rectangular plots are generally not as sharp as parametric plots.

B.6 Complete Example 3. [*Hint*: first delete ERASE from INPUTS.]

C.1 Graph several parabolas that pass through the points $(-2,0)$, $(0,-1)$, $(2,0)$. What do you conjecture about the union of all such parabolas?

C.2 Let P_1, P_2, P_3, P_4 be distinct points, no three of which are collinear. *Prove* or *disprove*: (a) there exists *at least one* parabola that passes through all of the P_i; (b) there exists *at most one* parabola that passes through all of the P_i.

Chapter **4**

Zeros of Functions and Applications

4.0 PREVIEW

A number c is a zero of a function f if $f(c) = 0$. Zeros of functions and roots of equations are closely related ideas. To find the roots of the equation $e^x = 2 - x$, for example, we may define the function $f(x) = e^x - 2 + x$. A number c is a zero of the function f if and only if c is a root of the equation $e^x = 2 - x$. Problems in which it is necessary to find the zeros of a function occur throughout calculus and its applications.

In this chapter we discuss several algorithms for finding real zeros of functions, some inefficient but requiring only that the function be continuous (the bisection algorithm), others efficient but requiring that the function be differentiable (Newton's algorithm). For most functions the algorithms SOLVR and ROOT on the SOLVE menu may be used in finding zeros efficiently and accurately.

The choice between SOLVR or ROOT and the bisection or Newton's algorithms is complex and will differ among users. If a primary goal is to survey several machine-independent methods for finding zeros, then bisection and Newton's methods should be explored.

In §4.2–4.3 we describe the bisection algorithm and Newton's algorithm and give programs for their implementation. We discuss the SOLVE menu in §4.4. We show in §4.5 how these algorithms may be used in the study of implicitly defined functions and inverse functions.

4.1 INTERMEDIATE VALUE THEOREM

The first steps in finding the real zeros of a function f are to determine if in fact f has one or more zeros and, if so, to locate each of its zeros in an interval. Often the existence of a zero and its location are obvious, either from the graph or a physical interpretation of the function. For example, if $y = f(t) = -gt^2/2 + v_0t + y_0$ is the position of a ball thrown upwards from height y_0 above the ground and with initial velocity v_0, it is clear that f has at least one zero since the ball eventually hits the ground (assuming, of course, that v_0

is less than escape velocity). If, for example, $f(t) = -16t^2 + 10t + 30$ for all $t \geq 0$, then the ball must hit the ground between $t = 0$ and $t = 2$ since $f(0) = y_0 = 30$ and $f(2)$ is obviously negative.

These steps—existence and approximate location of a zero—can be based on the intermediate value theorem. This theorem is almost certainly mentioned in your calculus book.

Intermediate Value Theorem (IVT). *If f is a continuous function defined on an interval I, then f takes on each value between any two of its values.*

Note that the domain of f is assumed to be an interval, which is a subset of the x-axis containing no gaps, that is, no numbers are skipped. Given this assumption, and that f is continuous, the IVT states that f skips no values. By interval we mean a gap-free piece of the real line, which may include end points or not. Rays such as $[-3, \infty)$ or $(-\infty, 5)$ are intervals. For finding zeros of functions, the IVT can be stated more briefly:

If a continuous function changes sign on an interval I, then it has a zero in I.

We illustrate this theorem with the function $f(x) = 2x^3 - 7x + 6$, whose graph is shown in Fig. 1. Since f is negative at $x = -3$ and positive at $x = -2$, it must take on —that is, cannot skip—the value 0, which lies between $f(-3) = -27$ and $f(-2) = 4$.

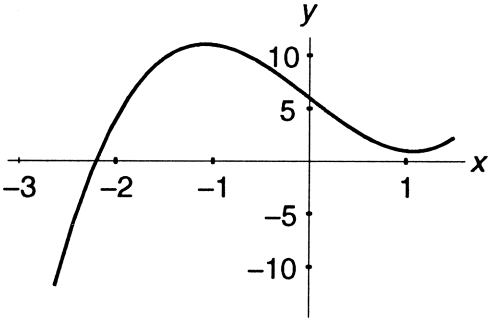


Figure 1

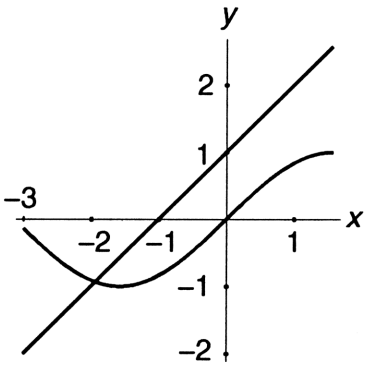


Figure 2

EXAMPLE 1. Use the intermediate value theorem to show that the graphs of $g(x) = \sin x$ and $h(x) = x + 1$ intersect.

SOLUTION. Fig. 2 makes it *geometrically* clear that the graphs of g and h intersect. We use the intermediate value theorem as a *numerical* test of the intersection. Since g and h are continous on any interval, the function $F(x) = g(x) - h(x) = \sin x - x - 1$ is continuous. Since $F(-\pi) = \pi - 1 > 0$ and $F(0) = -1 < 0$, F has a zero between $-\pi$ and 0. So there is a number c for which $F(c) = g(c) - f(c) = 0$. It follows that $(c, g(c))$ is a point common to the two graphs. It is easy in this case to locate c more closely. Using the graph as a guide and the HP 48 to evaluate the function, please check that $F(-2.0) \approx 0.09 > 0$ and $F(-1.9) \approx -0.05 < 0$. The x-coordinate of the point of intersection lies between -2.0 and -1.9.

Exercises 4.1

Using the IVT, locate at least one real zero or root of the following functions or equations. Locate the zeros or roots between successive "tenths," for example, between 3.7 and 3.8 or between −0.3 and −0.2.

A.1 $f(x) = x^3 - 2x - 5$

A.2 $f(x) = 2^x - 4x$

A.3 $x = 1 - x^3/10$

A.4 $x + \sin x = 1$

A.5 $f(x) = x \tan x - 1, \quad 0 < x < 1.5$

A.6 $f(x) = x^3 - 4x^2 - x + 3$

A.7 $f(x) = \ln x - x + 2, \quad x > 0$

A.8 $\tan^3 \theta - 8 \tan^2 \theta + 5 \tan \theta - 4 = 0, \quad 0 \leq x < 1.5$

B.1 Locate between successive tenths the zeros of the function

$$f(x) = (x-2)^{1/3} + 2x^2 - 15$$

B.2 Locate between successive thousandths the three smallest zeros of the equation

$$\cos(68.617\sqrt{x}) \cosh(68.617\sqrt{x}) = -1$$

B.3 Use the IVT to show that if f and g are continuous functions defined on an interval I and $[f(a) - g(a)][f(b) - g(b)] < 0$ for points a and b of I, then there is a point w between a and b for which $f(w) = g(w)$.

4.2 BISECTION ALGORITHM

The bisection algorithm is a relatively slow, reliable method for finding approximations to the real zeros of a function f. It assumes that f is continuous on $[a, b]$ and $f(a)f(b) \leq 0$. It follows from the Intermedidate Value Theorem that f has a zero in $[a, b]$. We use the function $f(x) = 2x^3 - 7x + 6$, with $a = -3$ and $b = -2$ as an example. The graph of this function is shown in Fig. 1. Note that $f(-3)f(-2) = (-27)(4) \leq 0$.

The idea of the bisection algorithm is easy. If f has a zero in an interval $I = [a, b]$ with midpoint m, then f must have a zero in either the left or right half of I. To determine which half, that is, which of $[a, m]$ or $[m, b]$ contains a zero, we calculate $f(a)$ and $f(m)$ and classify the nine possible outcomes. These outcomes are shown in Table 1. The column headed by "$f(a)f(m) > 0$?" gives a two-way classification of the product $f(a)f(m)$. The "New Interval" listed in the last column is a half-interval containing a zero.

The length of the half-interval is $h/2$, where $h = (b - a)/2$. We refer to h as the half-length of the interval $[a, b]$. We continue the bisection until we have found a sufficiently small interval containing a zero of f.

At each step we begin with an "old" interval $[a, b]$, old midpoint m, and old half-length h. Using Table 1 we choose a "new" interval, new midpoint m', and new half-length $h' = h/2$. If we wish to approximate a zero c of f (c is in the new interval) to within an *error tolerance* E, we continue the bisection algorithm until $h' < E$. For

$$|c - m'| < h/2 = h' < E \tag{1}$$

Table 1

Case	$f(a)$	$f(m)$	$f(a)f(m) > 0$?	New Interval
1	$-$	$-$	Yes	Right
2	$-$	0	No	Left
3	$-$	$+$	No	Left
4	0	$-$	No	Left
5	0	0	No	Left
6	0	$+$	No	Left
7	$+$	$-$	No	Left
8	$+$	0	No	Left
9	$+$	$+$	Yes	Right

We summarize one step of the bisection algorithm in a flowchart. Note that $a = m - h$.

$$m,\ h \quad \longrightarrow \quad \boxed{\begin{array}{l} \text{If } f(m-h)f(m) > 0, \text{ then} \\ \text{replace } [m-h, m+h] \quad \text{by} \quad [m, m+h] \\ \text{else replace } [m-h, m+h] \text{ by } \ [m-h, m] \end{array}} \quad \longrightarrow \quad m',\ h' \qquad (2)$$

EXAMPLE 1. Find the one zero of the function $f(x) = 2x^3 - 7x + 6$.

SOLUTION. A glance at Fig. 1 shows that f has a zero in the interval $[-3, -2]$. The first midpoint and half-length are $m = -2.5$ and $h = 0.5$. Referring to the flowchart, since $f(m-h)f(m) = (-27)(-7.75) > 0$ we can be certain that f has a zero in $[-2.5, -2.0]$. The values of m' and h' are -2.25 and 0.25. We drop the primes on m and h. Since $f(m-h)f(m) > 0$, the next values of midpoint and half-length are -2.125 and 0.125. At this point we may wish to use -2.125 as a sufficiently accurate estimate of c. We know that $|c-(-2.125)| \leq 0.125$. If this approximation is not sufficiently accurate we continue. Successive values of m are -2.5, -2.25, -2.125, -2.1875, -2.21875, -2.203125, and -2.2109375. The half-length for the last midpoint value is 0.0078125. If we use the (not usually known) information that the zero towards which this sequence of midpoints is converging is $c \approx -2.20470\ 54233\cdots$, we may check (1) by noting that

$$\begin{aligned} |c - (-2.2109375)| &\approx |-2.2047054233 - (-2.2109375)| \\ &\approx 0.0062320767 < 0.0078125 = h \end{aligned}$$

We give the program BSCT below. The program imitates the flowchart but includes a refinement that cuts the number of function evaluations in half. The refinement is based on the observation that in the > 0 case, the next left end point is the old m. Since we have calculated $f(m)$, we save it on the stack. In the ≤ 0 case, the next left end point is the old $a = m - h$. So, we save $f(m-h)$, which we have calculated. BSCT requires that f be written in program style and stored on the VAR menu as F. BSCT requires the initial stack to be $f(m-h)$, m, and h, does the calculations for one step, and returns $f(m'-h')$, m', and h' to the stack for use in the next step.

BSCT

Inputs: $f(m-h)$,m,h	*Outputs:* $f(m'-h')$,m',h'
≪ 2 /	Compute h'
→ FA M HP	Store $f(m-h)$, m, and h' as local variables FA, M, and HP
≪ FA M F DUP2 *	Calculate $f(m-h)f(m)$ & save the two factors for the next iteration
IF 0 >	If $f(m-h)f(m) > 0$
THEN SWAP DROP	then put $f(m)$ and
M HP +	$m' = m + h'$ on the stack
ELSE DROP	else put $f(m-h)$ and
M HP − END	$m' = m - h'$ on the stack
HP ≫ ≫	Last of output stack: h'
	Checksum: #32567d Bytes: 129

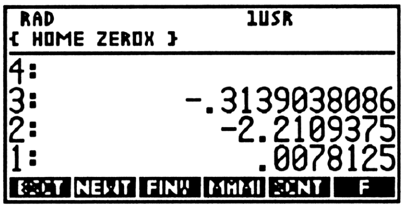
Once started, BSCT should be run repeatedly, until $h' < E$.

You may wish to create a new directory, perhaps called ZEROX, in which to store BSCT and the user-defined function F. We repeat part of Example 1 to show the operation of BSCT.

EXAMPLE 2. Find the zero $c \in [-3, 2]$ of the function $f(x) = 2x^3 - 7x + 6$ to within an error tolerance of 0.01.

SOLUTION. Use DEF to store f on the VAR menu in the ZEROX directory, where you have BSCT stored. Given that f has a zero in $[-3, -2]$, we take $a = -3$, $m = -2.5$, and $h = 0.5$. Put -3 on the stack and press F to put $f(m-h) = f(a)$ on the stack.

−2.5 ENTER
.5 BSCT
Now press BSCT
5 more times



```
RAD                 1USR
{ HOME ZEROX }
4:
3:         -.3139038086
2:           -2.2109375
1:             .0078125
```

We pressed BSCT until h' in level 1 is less than 0.01. We may say that $m' = -2.2109375$ is within 0.01 of c.

Exercises 4.2

Use the bisection algorithm in solving the following problems. In evaluating polynomials it is more efficient to write them in nested form. This is discussed in §1.4. Round your last calculator result to the nearest $10E$. For example, if $E = 0.01$, then round to the nearest 0.1. Answers will be reported this way.

A.1 Find the zero of the function $f(x) = 5x^3 - 7x^2 + 9x - 41$ in the interval $[0, 4]$. Let the error tolerance be $E = 0.0001$.

A.2 Find the zero of the function $f(x) = -6x^5 - 11x^4 + 2x + 2$ in the interval $[-1, 0]$. Let the error tolerance be $E = 0.0001$.

A.3 Find the two zeros of $f(x) = (x-2)^{1/3} + 2x^2 - 15$ in $[-5, 5]$. Use $E = 0.001$. See problem B.1 in §4.1 and its answer.

A.4 The volume V (in cubic meters) of 1 mole of a gas is related to its temperature T (degrees Kelvin) and pressure P (in atmospheres) by the ideal gas law $PV = RT$. A more accurate equation is van der Waals' equation

$$\left(P + \frac{a}{V^2}\right)\left(V - b\right) = RT$$

The constant R is 0.08207. For carbon dioxide, $a = 3.592$ and $b = 0.04267$. Find the volume V of 1 mole of carbon dioxide if $P = 2.2$ atmospheres and $T = 320°\text{K}$. Use $E = 0.01$.

A.5 Find the zeros of the function $f(x) = 2^x - 4x$. Use $E = 0.001$. See problem A.2 in Exercises 4.1.

A.6 Find the three smallest zeros of the equation $\cos(68.617\sqrt{x})\cosh(68.617\sqrt{x}) = -1$. Use $E = 0.0001$. See problem B.2 in Exercises 4.1.

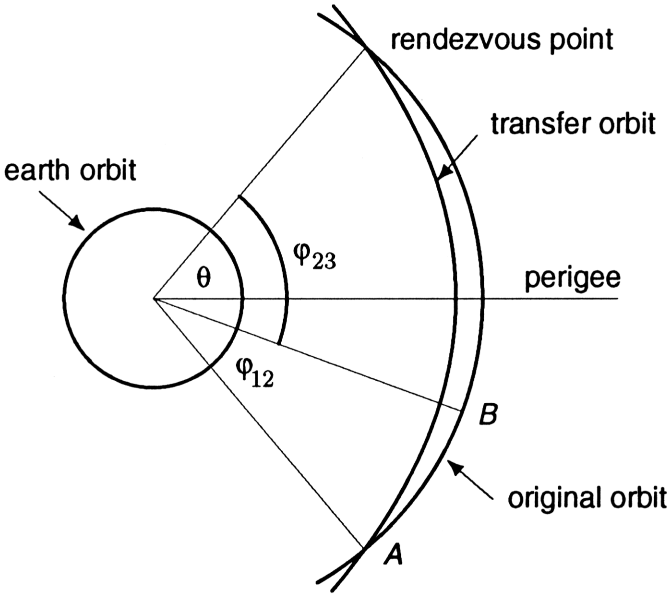


Figure 3

B.1 If two space vehicles A and B (see Fig. 3) are in the same circular orbit and it is required that they rendezvous, it is necessary for A to go into what is called a transfer orbit. To calculate the amount of thrust required to accomplish this maneuver, given the radius of the original orbit and the values of two angles, the eccentricity of the transfer orbit must be known. Using the equations of motion of a body in a central force field and the geometry of the conic sections, the equation satisfied by the unknown eccentricity ε is

$$\frac{\pi}{360}\varphi_{23} = \sqrt{\left[\frac{1+\varepsilon\cos\theta}{\varepsilon^2-1}\right]^3}\left[\frac{\varepsilon\sqrt{\varepsilon^2-1}}{1+\varepsilon\cos\theta}\sin\theta - \ln\left(\frac{\sqrt{\varepsilon+1}+\sqrt{\varepsilon-1}\,\tan(\theta/2)}{\sqrt{\varepsilon+1}-\sqrt{\varepsilon-1}\,\tan(\theta/2)}\right)\right]$$

In this equation it is assumed that $\varepsilon > 1$, which gives hyperbolic orbits. The variables φ_{23} and θ are described by

φ_{23} = angle between B and the point at which A and B rendezvous, measured at the time the manuever begins, and

$\theta = (\varphi_{12} + \varphi_{23})/2$, where

φ_{12} = the angle between A and B at the time the manuever begins

Use the bisection algorithm with error tolerance 0.001 to obtain values of the eccentricity ε for $\varphi_{12} = 30°$ and each of the values $\varphi_{23} = 20°$, 30°, 40°, and 50°.

Rough Outline of Solution

1. Put your calculator in MODE DEG.
2. Use the values $\varphi_{12} = 30°$ and $\varphi_{23} = 20°$ in first defining the function f. The entire set of data can be covered by modifying f later. (There are several ways of accomodating the changing values of φ_{23}, of which the one suggested here is not the most elegant.) It is probably easiest to enter the function in pieces. For example, define functions F1, F2, and F3, using the program method and the following expressions:

```
'((1+E*COS(25))/(E*E−1))^1.5'
'E*√(E*E −1)*SIN(25)/(1+E*COS(25))'
'LN((√(E+1)+√(E−1)*TAN(12.5))/(√(E+1)−√(E−1)*TAN(12.5)))'
```

3. The stack input function f can be defined in terms of F1, F2, and F3. Store $\pi \cdot 20/360 = \pi/18 = .174532925199$ as a constant.
4. Check your entry of f from $f(2) \approx 0.064311593091$.
5. Before using BSCT you will need to locate the zero in an interval $[a, b]$ such that $f(a) \le 0 \le f(b)$. Try $a = 4$ and $b = 5$..
6. With error tolerance 0.001 you should obtain $\varepsilon = 4.48$.
7. Return your calculator to radian mode.

B.2 The frequency equation of a vibrating beam is

$$\cos(k\lambda)\cosh(k\lambda) = -1$$

where $k^2 = p/a$, $a^2 = (EIg)/(A\sigma)$, p = the natural frequency of the beam in radians/second, $\lambda = 120$ in (length of the beam), $I = 170.6$ in^4 (elastic moment of inertia of the beam), $E = 3 \cdot 10^6$ lb/in^2 (elastic modulus of the beam material), $\sigma = 0.066$lb/in^3 (density of the beam material), $A = 32$ in^2 (cross-sectional area of the beam), and $g = 386$ in/sec^2 (acceleration of gravity). Find the three smallest natural frequencies of the beam.

C.1 In structural beams subject to eccentric loading ("eccentric" means that the direction of the applied force does not pass through the center line of the beam) the average unit load L is related to the slenderness ratio S (which describes the geometry of the beam) through the "secant formula"

$$L = \frac{\sigma_{max}}{1 + \frac{ec}{r^2} \sec\left(0.5S\sqrt{L/E}\right)}$$

where σ_{max} is the maximum stress, ec/r^2 is the eccentricity ratio, and E is the modulus of elasticity. This formula is derived in the book *Mechanics of Materials*, Third Edition, by Higdon, Ohlsen, Weese, and Riley. Fig. 4 was adapted from an illustration in this book. The figure was accompanied by the statement "Digital computers can also be programmed to solve the formula directly using iterative techniques." Using the bisection algorithm, compute data sufficient to plot the curve corresponding to the eccentricity ratio 0.4. Some of the data you need are found on the figure. The horizontal axis corresponds to the slenderness ratio S and the vertical axis to the average unit load L.

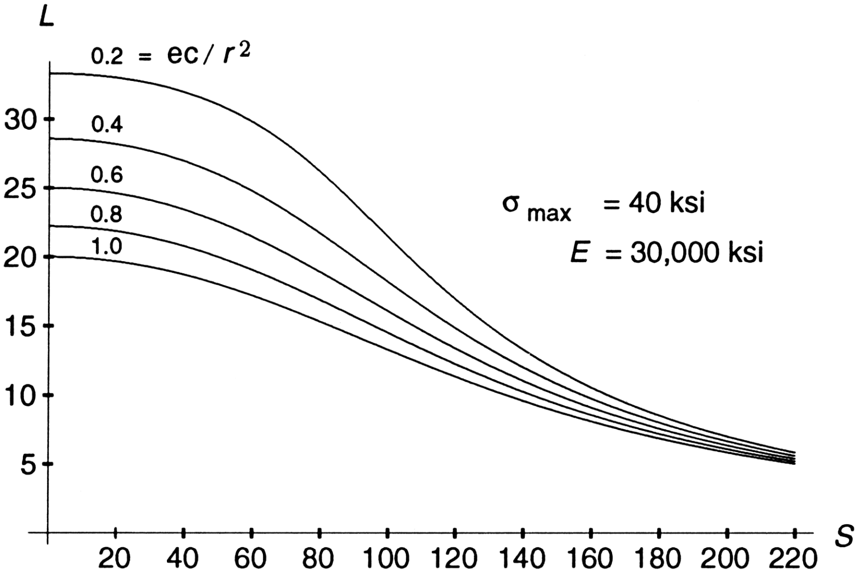


Figure 4

4.3 NEWTON'S METHOD

The bisection algorithm is slow, reliable, and requires only that f be continuous. The algorithm called Newton's method is usually much faster. For Newton's method we must assume that f is differentiable. This assumption is not, however, sufficient to guarantee success. The successive approximations may wander away from a nearby zero.

We show in Fig. 5 the graph of a function f near a point c where it crosses the x-axis. Let x_1 be an initial approximation to c. From x_1 we go up to the point $(x_1, f(x_1))$ on the graph of f. From there we slide on the tangent line down to its x-intercept x_2, then rise from x_2 to the point $(x_2, f(x_2))$, slide on the tangent to its x-intercept x_3, and so on. For functions similar to that in Fig. 5, where the concavity of the graph is the same on both sides of c, the points x_1, x_2, x_3, ... converge rapidly to the desired zero c of f.

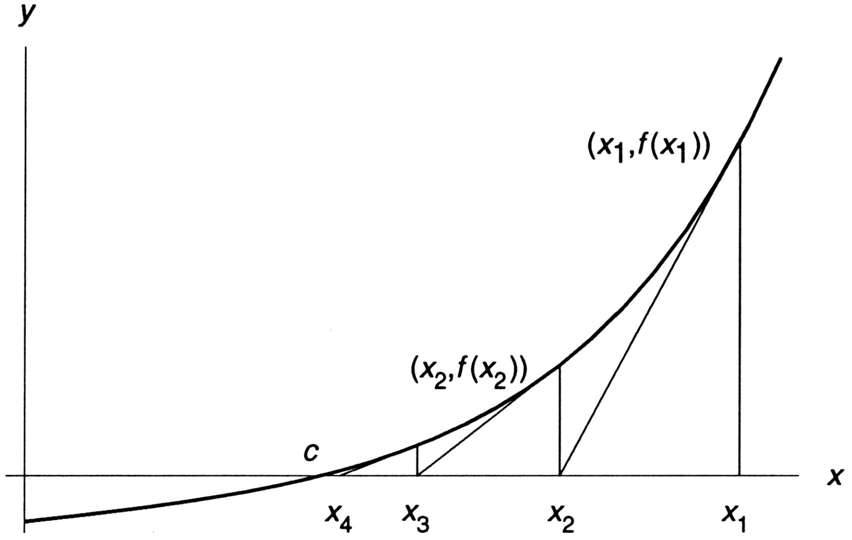


Figure 5

The algorithm for Newton's method is easily written out from the above description and Fig. 5. We assume now and in what follows that f has a non-zero derivative throughout an interval containing c. Let x_n be any one of the iterates. The next iterate, x_{n+1}, is the x-intercept of the tangent line at the point $(x_n, f(x_n))$. The equation of the tangent line to the graph of f at $(x_n, f(x_n))$ is

$$y - f(x_n) = f'(x_n)(x - x_n)$$

Setting $y = 0$ to get the x-intercept we have

$$x - x_n = -f(x_n)/f'(x_n)$$

Solving for x and denoting this value by x_{n+1} we have Newton's algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 1, 2, \ldots \tag{1}$$

We show in Fig. 6 a function f for which Newton's algorithm does not give an improving sequence of approximations to a zero c of f. The most prominent feature of this graph is that it changes concavity at $c = 0$.

Newton's algorithm requires more calculation per step than the bisection algorithm since the iteration step (1) includes two function evaluations, $f(x_n)$ and $f'(x_n)$. The extra calculation is often justified by the speed with which the iterates converge to a zero of f. The number of correct digits usually doubles at each step. Knowing when to stop is a more difficult question than for the bisection algorithm. We discuss this after we give an example and a program.

EXAMPLE 1. Use Newton's method to find the positive zero of the function $f(x) = x^2 - 2$.

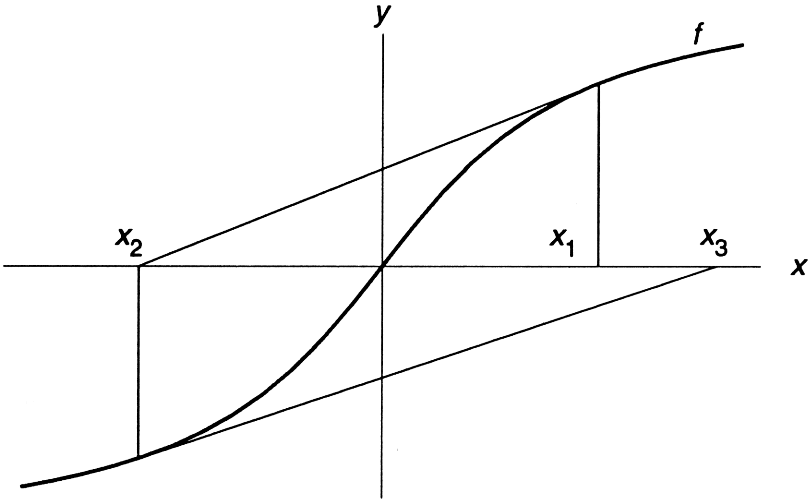


Figure 6

SOLUTION. The positive zero is $\sqrt{2} = 1.41421\ 35623\cdots$.We take the first guess x_1 to be 2. The first few steps of Newton's algorithm are

$$\begin{aligned}
x_2 &= x_1 - \frac{x_1^2 - 2}{2x_1} = 2 - \frac{2}{4} &&= 1.5 \\
x_3 &= x_2 - \frac{x_2^2 - 2}{2x_2} = 1.5 - \frac{0.25}{3.0} &&= 1.4166666666 \\
x_4 &= x_3 - \frac{x_3^2 - 2}{2x_3} = 1.41666666667 - \frac{0.00694444445}{2.83333333334} &&= 1.4142156862
\end{aligned}$$

Newton's method can be programmed in many different ways. Leaving aside the question of deciding when an iterate x_n is sufficiently close to a zero, any program requires at the very least access to f and x_1. The program NEWT given here is user-friendly but inefficient. To use NEWT, f must be stored on the ZEROX menu in program style. NEWT is inefficient in that it recalculates f' at each step. It is user-friendly in that besides f, only one input is required, namely, x_n. NEWT outputs x_n and x_{n+1} so that successive iterates can be compared. When x_n and x_{n+1} agree to, say, five decimals, it is often assumed that x_{n+1} approximates a zero of f to four or five decimals.

NEWT

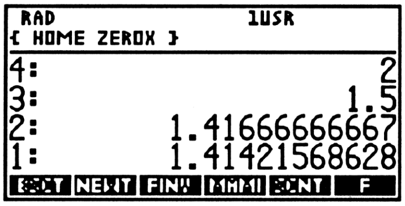
Inputs: x_n	*Outputs:* x_n, x_{n+1}
≪ → X	Store x_n as a local variable X
≪ X DUP F	Duplicate x_n for output stack and calculate $f(x_n)$
'F(X)' 'X' ∂	Calculate $f'(x_n)$
/ NEG X +	Calculate x_{n+1}
≫ ≫	
	Checksum: #24330d Bytes: 90

Suggestions for other Newton's method programs are given in problems C.1–C.3.

EXAMPLE 2. Use NEWT in repeating Example 1.

SOLUTION. Use STD mode.

```
'F(X)=X^2-2'
DEF
2  NEWT
NEWT
NEWT
```



```
RAD                1USR
{ HOME ZEROX }
4:                    2
3:                  1.5
2:       1.41666666667
1:       1.41421568628
```

The final stack shows x_1, x_2, x_3, and x_4. If we press NEWT twice more we get 1.41321356237 in levels 1 and 2. These agree with $\sqrt{2}$ as calculated on the HP 48. Note that the number of correct decimals is more than doubled at each iteration.

To illustrate one method of checking the accuracy of an approximation to a zero c of a function f, suppose in the example above we stop with $x_4 = 1.41421568628$, hoping that it will give us an approximation x within 0.0001 of $\sqrt{2}$. Let $x = 1.4142$.

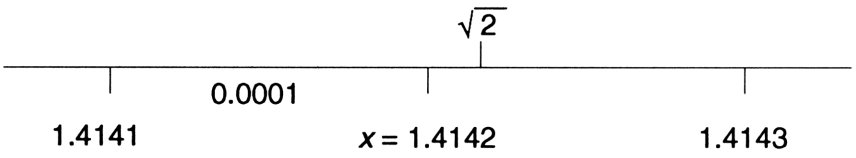


Figure 7

To show that $|x-\sqrt{2}| < 0.0001$ it is enough to show that $f(x-0.0001)f(x+0.0001) < 0$. For, referring to Fig. 7, this condition means that f crosses the axis between the end points. The crossing point ($\sqrt{2}$) must be within 0.0001 of the midpoint. We note that

$$f(x - 0.0001)f(x + 0.0001) = f(1.4141)f(1.4143) = (-.00032119)(.00024449) < 0$$

We leave as problem C.3 an adaptation of NEWT which uses this criterion as a stopping rule.

Exercises 4.3

In the A problems use the method associated with Fig. 7 to establish the accuracy of an approximation. In each problem we give an error tolerance E. It follows from $f(x - E)f(x + E) < 0$ that x is within E of a zero of f. For problems in which x_1 is not specified you may wish to plot the function to obtain a reasonable value of x_1.

A.1 Find the real zero of the function $f(x) = x^3 - 5$. Let $x_1 = 2.0$ and $E = 0.0001$.

A.2 Find the real zero of the function $f(x) = -x + \cos x$. Let $x_1 = 1.0$ and $E = 0.0001$.

A.3 Find the real zero of the function $f(x) = \tan x - x$, where $3.3 \le x \le 4.7$. Let $x_1 = 4.6$ and $E = 0.0001$.

A.4 In finding acceleration poles in a planetary gear train the equation

$$\tan^3\theta - 4\tan^2\theta + \tan\theta - 4 = 0$$

must be solved. Find the real root. Give your answer in degrees.

A.5 *(similar to problem 4)* For a second gear train the equation

$$\tan^3\theta - 8\tan^2\theta + 17\tan\theta - 8 = 0$$

must be solved. Find the three real roots. Give your answers in degrees.

A.6 Spherical four-bar linkage mechanisms can be designed to mechanically approximate given mathematical functions. To generate the function $\log_{10} x$, for $1 \le x \le 10$, the quintic equation

$$t^5 + 1.21355t^4 + 2.44461t^3 + 2.42633t^2 + 1.47224t + 1.18747$$

must be solved (as one step in calculating the lengths of the four bars). Find the real root. Use $E = 0.01$.

A.7 Determine the first four positive zeros of the function

$$f(x) = \cos x \cosh x + 1$$

to within 0.001 of their true values. This function becomes very large for relatively small values of x. For example, $f(6) \approx 194.68$. This feature makes it difficult to find approximate locations of its zeros using a graph. To locate approximations to the zeros you may wish to evaluate $f(x)$ for several different values of x, looking for sign changes. Try successive integers. Once you have found successive integers for which f changes sign, use their average as a starting value. For evaluation you may store f as a user-defined function, say as F, and simply press the key beneath the VAR menu variable F after entering a number. Or you may use the SOLVE menu.

A.8 Find all the zeros of the Chebyshev polynomial

$$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$

to within 0.00001 of their values. You may wish to plot this function to find initial guesses. You can cut your work in half by an observation. You may check your results by using the fact that the zeros x_k are given by

$$x_k = \cos[(2k+1)\pi/16], \qquad k = 0, 1, \ldots, 7$$

A.9 The equation $x^3 - 2x - 5 = 0$ was used by Wallis in 1685 to illustrate Newton's method. It has been included in most subsequent works dealing with the numerical solution of equations. Find the real root of this equation.

B.1 Pulleys of radii R and r are connected by a taut belt of total length L. Letting $R = 200$ cm, $r = 100$ cm, and denoting by x the distance between pulley centers, express L in terms of R, r, and x. Find x to within one decimal place for each of the values $L = 2000, 2100, \ldots, 2800$ cm.

B.2 If in the Newton's method equation (1) $f'(x_n)$ is replaced by the difference quotient

$$\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}},$$

the resulting formula for x_{n+1} is called the secant method. If x_n and x_{n-1} are close together, $f'(x_n)$ and the above difference quotient are very nearly equal. We may expect the secant method to converge at a rate between those of the bisection and Newton's methods. The secant method is useful if f either fails to have a derivative or has a derivative which is difficult to calculate. In any case, if it happens that $f(x_n) = f(x_{n-1})$, the secant method will fail, just as Newton's method fails if $f'(x_n) = 0$. The secant algorithm is given by

$$x_{n+1} = x_n - f(x_n)\,\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1}} \qquad n = 1, 2, \ldots$$

Use this algorithm in calculating the zero of the function $f(x) = \cos x - x$. Note that you will need two starting values, x_1 and x_2.

C.1 Modify the program NEWT so that it (1) assumes the user has stored f and f' in program style on the VAR menu as F and DF and (2) expects as input just x_1, does one step of Newton's method, and leaves on the stack the input for the next step as well as the preceding iterate, for comparison.

C.2 Modify the program NEWT or the program written in problem C.1 so that the functions are algebraic expressions, not functions written in program style.

C.3 Write a program with input x_1 and E and output the first iterate x_n for which $f(x_n - E)f(x_n + E) < 0$, so that $|x_n - c| < E$, where c is a zero between $x_n - E$ and $x_n + E$.

C.4 Newton's method may be used to find complex zeros. If you are interested in this topic you may wish to look up Bairstow's method in Peter Henrici's *Essentials of Numerical Analysis with Pocket Calculator Demonstrations*, John Wiley & Sons, New York, 1982. We use the program NEWT given in this section. One of the difficulties in finding complex zeros is in locating initial approximations.

We illustrate the procedure with the function $f(x) = x^3 - 1$, which we may factor as $f(x) = (x-1)(x^2+x+1)$. The zeros of f are 1 and $-1/2 \pm i\sqrt{3}/2$. We use NEWT in finding one of the complex zeros. Recall that the complex number $a + ib$ is entered as (a, b). Use DEF to store f on the ZEROX menu and let $x_1 = (-.4, .8)$. After NEWT has run, level 1 will be $(-.516666666667, .866666666667)$. Running NEWT three more times gives $(-.5, .866025403785)$ in level 1. It is already clear that the iterates are converging to $-1/2 + i\sqrt{3}/2 = .5 + i.866025403785$.

Use NEWT in finding all of the zeros of the polynomial

$$x^4 - 5x^3 + 21x^2 + 13x + 49$$

There are zeros near $3 + 4i$ and $-.5 + 1.3i$.

4.4 SOLVR AND ROOT

In this section we discuss how to use the HP 48 algorithms SOLVR and ROOT for finding zeros of functions. For most functions, SOLVR and ROOT find zeros efficiently and accurately. The HP 48 manuals contain well written descriptions of SOLVR and ROOT. Accordingly our discussion is brief.

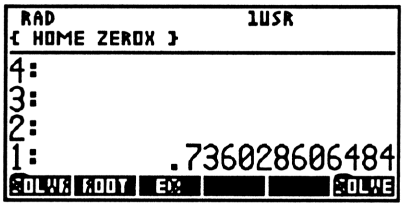
We start with ROOT, which is the heart of the SOLVR. The differences between the two are (1) ROOT does not include the user-friendly environment of SOLVR and (2) ROOT can be used in a program. In what follows we give an example of using ROOT, several examples of using the SOLVR, and a final example of how ROOT may be used in a program.

EXAMPLE 1. Use ROOT to find the real zero of the function $f(x) = 3x^3 - 5x^2 + 17x - 11$.

SOLUTION. The built-in ROOT algorithm is on the ↰SOLVE ROOT menu. ROOT will find a root of an equation $f(x) = 0$ or a zero of a function f. It requires three stack inputs: First-in is an algebraic expression for f. If f entered in program style is stored as F, then `'F(X)'` may be used. Second-in is the name of the variable. Third-in is a single number or a list of one to three numbers. If a single number or a list with one number is the input chosen, the number should be a reasonably good estimate of a zero of f. If a list $\{a, b\}$ of two numbers is input, the interval $[a, b]$ should narrowly contain a zero. And if a list $\{c, a, b\}$ is input, c should be your best guess for the zero and, as before, the interval $[a, b]$ should narrowly contain a zero.

Suppose we wish to use a list of two numbers containing a zero as input to ROOT. To locate bounding numbers of a zero, we evaluated $f(x)$ for several values of x using the SOLVR. We found $f(0.7) \approx -0.521$ and $f(0.8) = 0.936$. It follows from the Intermediate Value Theorem that f has a zero in the interval $[0.7, 0, 8]$. The input and output to ROOT are as follows.

```
↰SOLVE  ROOT
'3*X^3-5*X^2+
  17*X-11'
ENTER  'X'  ENTER
{.7 .8}  ENTER  ROOT
```



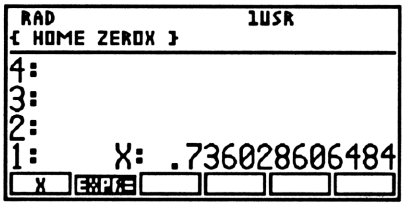
```
RAD                    1USR
{ HOME ZEROX }
4:
3:
2:
1:           .736028606484
```

The result, .736028606484, closely approximates one zero of f.

EXAMPLE 2. Repeat Example 1 but use the SOLVR instead of ROOT.

SOLUTION. The SOLVR is on the same menu as ROOT (ROOT is also the name of a directory). Perhaps using the nested form of f shown below, put f on the stack.

```
↰SOLVE  ROOT
'X*(X*(3*X-5)+17)-11'
↰ EQ  SOLVR
{.7 .8}  ENTER  X
[↰]  X
```



```
RAD                    1USR
{ HOME ZEROX }
4:
3:
2:
1:      X: .736028606484
 X  EXPR=
```

Pressing $\boxed{\text{SOLVR}}$ displays a menu showing the variable(s) occuring in the expression for f. It also shows $\boxed{\text{EXPR=}}$. We give the SOLVR our estimate {.7 .8} by keying this in and then pressing $\boxed{\text{X}}$ on the SOLVR menu. Pressing the ↰ and X keys does two things. It tells SOLVR that you wish to find a zero of the function f stored as EQ and that f is regarded as a function of the variable x. The result of ↰X is the number .736028606482, as in Example 1, and the word "Zero." This means that the displayed number is an approximation to a zero of f in $[0.7, 0.8]$ and, within the limitations of the calculator, the number is a zero of f. You may verify this by pressing the $\boxed{\text{X}}$ key in the SOLVR menu and then $\boxed{\text{EXPR=}}$.

EXAMPLE 3. Calculate $\sqrt[3]{2}$ by finding the real zero of the function $f(x) = x^3 - 2$.

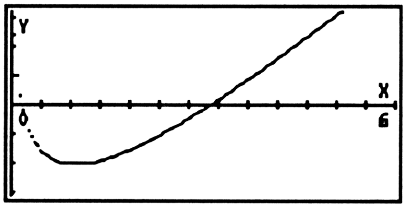
SOLUTION. Put an algebraic for the function $f(x) = x^3 - 2$ on the stack, store it as EQ, and use SOLVR in locating between successive "tenths" the zero of f. You should find $f(1.2) = -0.272$ and $f(1.3) = 0.197$. Using {1.2 1.3} as the guess, store this by pressing X. Pressing ↰X on the SOLVR menu gives 1.25992104989 ($\approx \sqrt[3]{2}$) and the message "Sign Reversal." This means that the HP 48 was not able to find a HP 48-representable number c such that $f(c) = 0$. Instead the SOLVR gave one of a pair of "adjacent" numbers p and q for which $f(p)$ and $f(q)$ have different signs. In this case the adjacent numbers are 1.25992104989 and 1.25992104990. You may check with the $\boxed{\text{EXPR=}}$ key that the corresponding function values are -0.00000000002 and 0.00000000002.

In the next example we show how ROOT can be accessed through the GRAPHICS FCN menu.

EXAMPLE 4. Find the zeros of $f(x) = x - \ln x - 2$, $x > 0$.

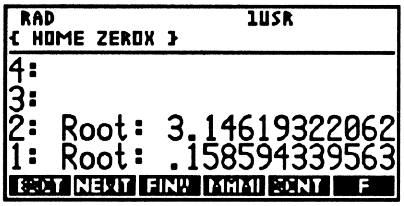
SOLUTION. We use PLOT to obtain estimates of the zeros of f.

```
'X-LN(X)-2'
↰PLOT  ↰ EQ
 PPAR  RESET
0  SPC  6  XRNG
-1.5  SPC  1.5  YRNG
PREV  PLOT
DRAX  DRAW
```



Next we use FCN on the PICTURE menu beneath the above display.

FCN
  Move cursor to larger zero
ROOT $\boxed{+}$
  Move cursor to smaller zero
ROOT $\boxed{+}$ CANCEL



```
RAD                    1USR
{ HOME ZEROX }
4:
3:
2: Root: 3.14619322062
1: Root: .158594339563
```

The zeros of f are on the stack.

EXAMPLE 5. In problem C.1 of Exercises 4.2 the "secant formula" was given as

$$L = \frac{\sigma_{max}}{1 + \frac{ec}{r^2}\sec\left(0.5S\sqrt{L/E}\right)}$$

In the formula, $\sigma_{max} = 40$ and $E = 30,000$. We assume here that $ec/r^2 = 0.2$. The variable L is the average unit load and is the dependent variable. The independent variable is the slenderness ratio S. We wish to plot L against S. Since L is implicitly defined by the secant formula, we must let S take on values in an interval of interest—taken as [0,220] in problem C.1—and solve the secant formula for L for selected values of S. This will result in a set of pairs (S, L) from which we sketch a graph, as in Fig. 4. Write a program whose output is a data set from which the the "0.2 curve" in Fig. 4 may be plotted. Use ROOT to generate the pairs and commands in the STAT menu to store and plot these data.

SOLUTION. We may plot a reasonably good graph if we let S increase from 0 to 220 in steps of 5 units. For each specific value of S we define the function

$$H(L) = L - \frac{\sigma_{max}}{1 + \frac{ec}{r^2}\sec\left(0.5S\sqrt{L/E}\right)}$$

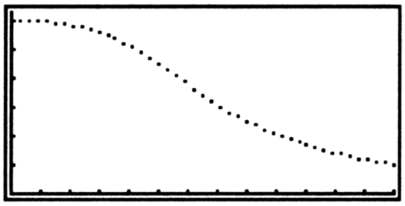
To use ROOT we need an algebraic expression for H on the stack, the variable L, and at least one estimate for the zero. Since L varies with S, we expect the estimate to vary as well. Since it easy to estimate the zero of H when $S = 0$, namely, $\sigma_{max}/(1 + ec/r^2)$, we have a starting place for the estimates. We may use the calculated zero for $S = 0$ as an estimate for the zero corresponding to $S = 5$, the calculated zero for $S = 5$ as an estimate for the zero corresponding to $S = 10$, and so on. The program SCNT prepares the stack for ROOT, calls ROOT, and stores the resulting pair (S, L). After SCNT generates the required data set, we plot using SCTR on the STAT menu. First, the program.

The program SCNT takes a minute or so to generate the 45 pairs of data from which the graph is plotted. The data are stored under the name ΣDAT, which appears on the VAR menu after SCNT is complete. To plot ΣDAT we go to the STAT menu. We use VZOUT with V-FACTOR 1.1 to make the horizontal axis visible.

SCNT

Inputs: None	*Outputs: ΣDAT*
« CLΣ −19 CF	Clears the statistical matrix ΣDAT & sets data flag
33.33 → L1	Put estimate L_1 for L (corresponding to $S = 0$) on stack for ROOT
« 0 220 FOR S	Set FOR/STEP loop
'L−40/(1+.2* INV(COS(.5*S*√(L/30000))))'	Define algebraic for H
'L' L1 ROOT	Complete stack for ROOT
DUP S SWAP →V2 Σ+	DUP L, build vector $[S, L]$, and store in ΣDAT
'L1' STO	Save L to use as new L_1
5 STEP	Increase S by 5
» 'L' PURGE −19 SF»	Clean up & restore data flag
	Checksum: #30209d Bytes: 240.5

↰STAT PLOT SCATR
ZOOM ZFACT
Set V-FACTOR equal to 1.1
OK OK NXT VZOUT
$\boxed{-}$



The resulting graph may be compared with the "0.2 curve" in Fig. 4 of problem C.1 in §4.2.

Exercises 4.4

A.1 Use ROOT in finding the zero of the equation $x\cosh(50/x) = x - 10, \quad x > 0$.

A.2 Use the SOLVR in finding all of the positive zeros of the Laguerre polynomial $x^3 - 9x^2 + 18x - 6$.

A.3 The polar curve with equation $r = \ln\theta + \theta$, $0 < \theta < 2\pi$, has a loop. To find the area of this loop we must find the coordinates of the point where the curve crosses itself. If you have not yet learned how to find the area of polar curves, find where the curve crosses itself. If you have learned how to find such areas, determine the limits of integration and calculate the area of the loop.

A.4 Find the zero of the derivative of the function

$$f(x) = 1 + 5.2x - \sec\sqrt{0.73x}, \quad 0 \le x \le 3.2$$

B.1 Repeat the work in Example 5 with $ec/r^2 = 0.4$. You will need to modify the program SCNT in only minor ways.

B.2 If we were required to find the zeros of each of the family of polynomials

$$x^5 - \alpha x^4 + 200x^3 - 600x^2 + 600x - 120, \quad \alpha = 24.8, 24.85, \ldots, 25.2,$$

we might decide to look for a way to speed the necessary calculations. Since in finding zeros most of the calculation time is spent in function evaluation, it is important that the polynomial be evaluated efficiently. We discuss one method of doing this in what follows. The method depends upon using the programs NSTD and DFLT discussed in Chapter I. For the discussion we take $\alpha = 24.8$.

We begin by finding the zero of smallest absolute value, call it x_1. To do this efficiently, we enter the list $L_5 = \{1 \; -24.8 \; 200 \; -600 \; 600 \; -120\}$. By using NSTD we easily generate the polynomial in factored form for efficient use by the SOLVR or PLOT. Using the SOLVR or PLOT we find the smallest zero is near 0.3 and, with this as the required estimate, find $x_1 = 0.263557337625$. Next, with L_4 and x_1 on the stack, use DFLT to obtain

$$\{1 \; -24.5364426624 \; 193.533240497 \; -548.992894392 \; 455.308894377\}$$

Using NSTD on this list we find the factored form of the deflated polynomial. Again using the SOLVR or PLOT we find the smallest zero is near 1.4. The SOLVR then gives $x_2 = 1.41846415306$. We deflate again and continue in this way until all five zeros (the given polynomial has five positive zeros) are determined. It is known that if the zeros are determined in order of size, from smallest in absolute value to largest in absolute value, the use of synthetic division to deflate the polynomial yields accurate results. We find $x_3 = 3.46437108483$ and, using QUAD, $x_4 = 7.8488644354$, and $x_5 = 11.8047429891$. Verify these computations using the SOLVR, NSTD, DFLT, and QUAD. The arrangement of these programs and what they leave on the stack works out very well for an easy algorithm, interupted only to record the zeros as they are calculated. Repeat for $\alpha = 24.85$.

C.1 (continuation of problem B.2) Write a program whose input is a value of α and whose output is the smallest zero of the polynomial in problem B.2. Use the program to find the smallest zero for $\alpha = 24.8, 24.85, \ldots, 25.2$.

C.2 (continuation of problem B.2) Find to within 0.1 the greatest value of α less than 24.8 at which some of the zeros become complex.

4.5 IMPLICITLY DEFINED AND INVERSE FUNCTIONS

We discussed implicitly defined functions in Chapter 2 and gave a program for finding their derivatives. In this section we use the methods of section 4.4 to graph implicitly defined functions. Inverse functions are a special case of implicitly defined functions. We give a program for graphing an invertible function and its inverse on the same graph.

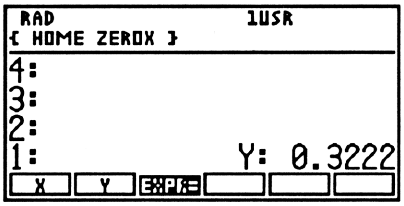
EXAMPLE 1. The equation

$$x^3y^3 + 3x^2y + x - 2 = 0, \qquad x \geq 1 \tag{1}$$

defines y as a function of x. (See problem C.1.) We denote this function by f. To study f we wish to prepare a rough graph. Use SOLVR to generate a small table of points $(x, y) = (x, f(x))$ and use these in sketching a graph of f.

SOLUTION. We let x take on the values 1.00, 1.25, 1.50, 1.75, 2.00, 2.50, 3.00, 4.00, 5.00, 7.00, 9.00, 11.00, 13.00, 15.00, and 19.00 and determine the corresponding values of $y = f(x)$. The spacing of the x values was determined by how much the computed y values change with a given change in x. When y is not changing much with x, the interval between successive x values may be increased. For $x = 1.00$ we need an initial guess y_1 for SOLVR. For subsequent values of x we can use the most recent y as a guess for the next. For example, we may take for y_1 corresponding to $x = 1.25$ the value of y found by SOLVR for $x = 1.00$. The starting value for $x = 1.00$ may be found graphically or by use of the SOLVR. We take $y_1 = 0.3$ for $x = 1.00$. Set 4 FIX.

```
↰SOLVE  ROOT
'X^3*Y^3+3*X^2*Y
  +X-2'
↰ EQ  SOLVR
1  X  .3  Y
[↰]  Y
```



```
RAD                     1USR
{ HOME ZEROX }
4:
3:
2:
1:                 Y: 0.3222
```

This gives the first line of Table 2. Leaving this value of y on the stack, enter 1.25 and press X, press Y (to give SOLVR an updated initial guess), and then ↰ and Y. This gives the second line of Table 2. Continuing in this way we easily generate the entire table.

These data are plotted in Fig. 8. We leave as problem A.5 the location of the minimum of this function. We leave as problem C.1 the explicit solution of (1) for y in terms of x using hyperbolic functions.

Table 2

x	y	x	y
1.00	0.3222	5.00	−0.0397
1.25	0.1583	7.00	−0.0339
1.50	0.0739	9.00	−0.0287
1.75	0.0272	11.00	−0.0247
2.00	0.0000	13.00	−0.0217
2.50	−0.0267	15.00	−0.0192
3.00	−0.0370	17.00	−0.0173
4.00	−0.0416	19.00	−0.0157

EXAMPLE 2. For $0 \le x \le 3$ and $y \ge 0$ the equation

$$e^{0.01y}(0.1x + y) = e^{-x}(1 + 2x - 0.2y) \qquad (3)$$

defines y as a function of x. Unlike (1), however, this equation cannot be solved explicitly for y in terms of x, which means that y can not be written as a combination of a finite number of "elementary functions" of x. Estimate the maximum of the function determined by (3).

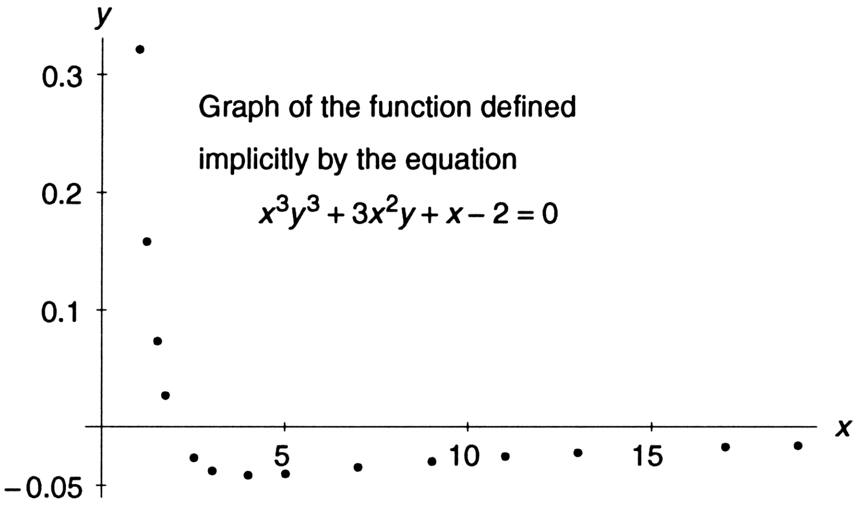


Figure 8

SOLUTION. Let g denote the function of x determined by (3). We may use the SOLVR as in Example 1 to calculate Table 3. We show in Fig. 9 a plot of these data.

Table 3

x	$g(x)$	x	$g(x)$
0.0	0.82	0.9	0.96
0.1	0.90	1.0	0.93
0.2	0.96	1.2	0.84
0.3	1.00	1.4	0.75
0.4	1.02	1.6	0.66
0.5	1.03	1.8	0.56
0.6	1.02	2.0	0.46
0.7	1.01	2.5	0.24
0.8	0.99	3.0	0.05

The maximum of g is approximately 0.95 and occurs at $x \approx 0.5$. For some purposes these values may be good enough. There are several ways to find x_{max} and $g(x_{max})$ more accurately. In Chapter 5 we return to this problem when we discuss the golden section algorithm. For the moment we may use the SOLVR to gain confidence in the approximate values of x_{max} and $g(_{max})$, even to refine them if we wish. For example, we easily find the following (x, y)-pairs satisfying (3):

$$(0.49, 1.0271), \quad (0.50, 1.0273), \quad (0.51, 1.0275), \quad (0.52, 1.0275)$$

We leave as a problem which of 0.51 or 0.52 better approximates x_{max}.

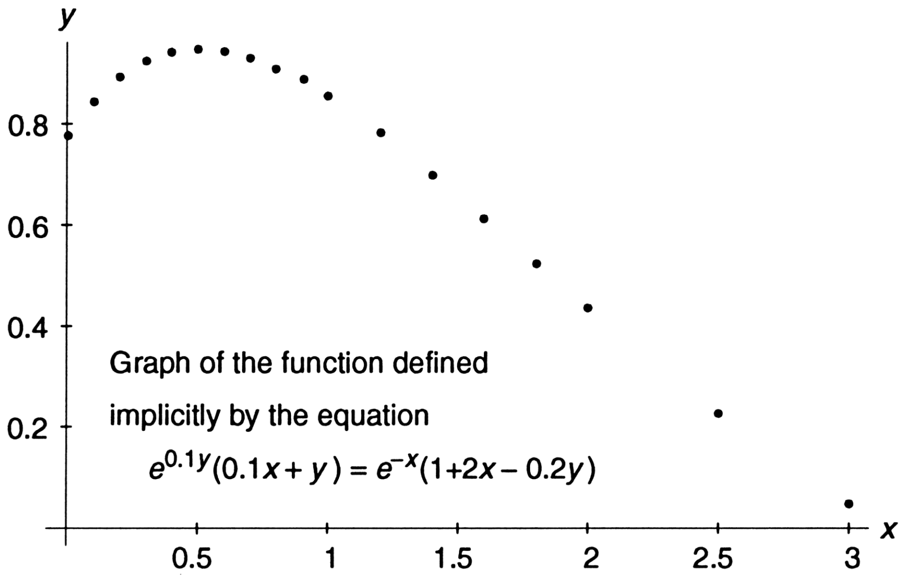


Figure 9

Inverse Functions

The problem of finding the inverse of a function is a special case of finding implicitly defined functions. If a function f is known to have an inverse f^{-1}, we may attempt to find an explicit formula for f^{-1} by writing an equation $y = f(x)$, solving this equation for x in terms of y, and, finally, interchanging x and y.

EXAMPLE 3. Find the inverse of the function f defined by

$$y = f(x) = \frac{2x-3}{x+1}, \qquad x \neq -1 \tag{4}$$

SOLUTION. Since

$$f'(x) = \frac{5}{(x+1)^2} > 0, \qquad x \neq -1$$

the function f is increasing and so invertible. Solving (4) for x in terms of y we find

$$x = -\frac{y+3}{y-2}, \qquad y \neq 2 \tag{5}$$

Interchanging x and y in (5) so that we may express f^{-1} in terms of the traditional variable x we find

$$f^{-1}(x) = -\frac{x+3}{x-2}, \qquad x \neq 2$$

The essential step in Example 3 came in solving (4) for x in terms of y. For the function f defined in (4) this was easy. For other functions it may difficult or impossible to find an explicit formula for f^{-1}. An example of an invertible function whose inverse can not be found explicitly is

$$f(x) = x - \ln x - 2, \qquad x > 1 \tag{7}$$

Since $f'(x) = (x-1)/x > 0$ for $x > 1$, f is increasing and hence invertible. To find f^{-1} we write the equation

$$y = x - \ln x - 2$$

Since f is invertible, this equation defines x as a function of y. If y is any fixed value in the range of f, we may use SOLVR (or Newton's method or the bisection algorithm) to find the zero of the function

$$g_y(x) = x - \ln x - 2 - y$$

(We have used the notation g_y since as y changes, the form of the function changes as well.) If we repeat this for several values of y, the result is a set S of ordered pairs of the form (y, x), where x is the zero of g_y. Plotting S gives an idea of the graph of f^{-1}.

Graphing f and f^{-1}

We may graph f^{-1} even though we have no explicit formula for it. Following the convention that the domain of a function should be on the horizontal axis, it is enough to reflect the graph of f across the 45°-line. If it is acceptable to use the vertical axis for the domain of f^{-1}, the graph of f^{-1} is identical to the graph of f! Another approach is described at the end of Example 4.

To graph f^{-1} and f on the same axes, we must arrange for the horizontal axis to include both the domain of f and that of f^{-1}. The first program, MAMI, does the calculations for this. Before listing MAMI, we give the background to understand this program.

If f is an invertible continuous function defined on the interval $[a, b]$, then either f is increasing or decreasing. The range of f is $[c, d]$, where $c = \min\{f(a), f(b)\}$ and $d = \max\{f(a), f(b)\}$. The graph of f is contained in the rectangle bounded by the lines $x = a$, $x = b$, $y = c$, and $y = d$. The graph of f^{-1} is contained in the rectangle bounded by the lines $x = c$, $x = d$, $y = a$, and $y = b$. In Fig. 10 we show a function f for which $a = -2$, $b = 4$, $c = -3$, and $d = 5$. To graph both f and f^{-1} on the same axes, we must be ready to graph anywhere in the rectangle bounded by $x = u$, $x = v$, $y = r$, and $y = s$, where

$$u = r = \min\{a, c\} \qquad \text{and} \qquad v = s = \max\{b, d\}$$

For the function f in the figure, $u = r = -3$ and $v = s = 5$.

MAMI

Inputs: a, b	Outputs: $\min\{a,c\}$,$\max\{b,d\}$

```
«  →  A  B                   Store a and b
«  A  F  B  F  MIN           Calculate c =min{f(a), f(b)}
LASTARG  MAX                 Calculate d =max{f(a), f(b)}
B  MAX                       Calculate v = s =max{b, d}
SWAP  A  MIN  SWAP           Calculate u = r =min{a, c}
»  »
```

Checksum: #25709d Bytes: 79.5

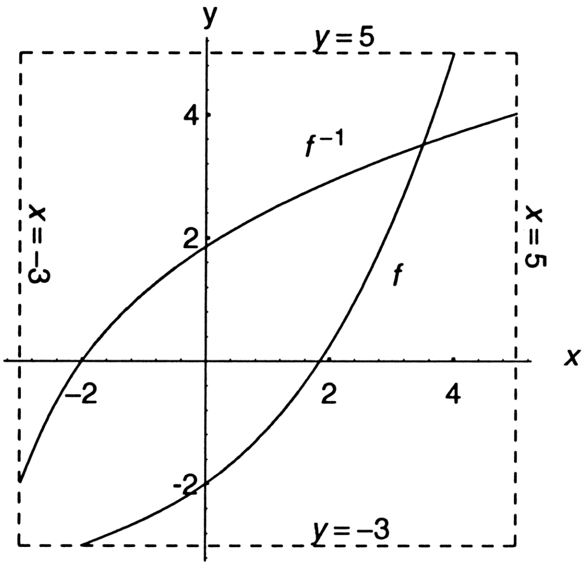


Figure 10

FINV	
Inputs: a, b	*Outputs: Graphs of f and f^{-1}*
« DUP2 'X' 3 ROLLD 3 →LIST INDEP	Duplicate a and b; prepare list for INDEP, to make parameter domain independent of XRNG & YRNG
{ 'X+i*F(X)' 'F(X)+i*X' }	Using HP 48 PARAMETRIC form, prepare equations for both graphs
'EQ' STO	Store equations
MAMI DUP2 XRNG YRNG	Calculate and store XRNG & YRNG
ERASE DRAX DRAW PICTURE »	Graph
	Checksum: #24605d Bytes: 158

MAMI takes advantage of the built-in functions MIN and MAX, found on the MTH REAL menu. MAMI assumes f is on the VAR menu and a and b are on the stack. As above, $c = \min\{f(a), f(b)\}$ and $d = \max\{f(a), f(b)\}$.

Please store MAMI on the ZEROX menu.

The graphing is done by the program FINV, which assumes that the PARAMETRIC PLOT TYPE has been set and that f is a user-defined function stored on the VAR menu. The inputs to FINV are a and b. It outputs a plot containing the graphs of f and f^{-1} on their domains. The idea of FINV is that the graphs of f and f^{-1} can be done parametrically.

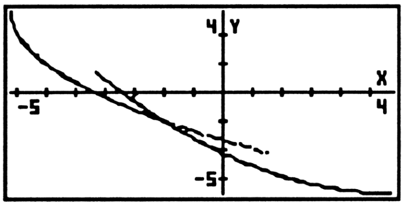
$$\begin{cases} x = x \\ y = f(x) \end{cases} \quad a \le x \le b \qquad\qquad \begin{cases} x = f(x) \\ y = x \end{cases} \quad a \le x \le b$$

Please store FINV on the ZEROX menu (store FINV on the same menu as MAMI).

EXAMPLE 4. Show that $f(x) = -5 + (6/49)(x-4)^2$, $-3 \leq x \leq 4$, is invertible and graph f and f^{-1} together.

SOLUTION. Since $f'(x) = (12/49)(x-4) < 0$ for $-3 \leq x < 4$, f is decreasing on its domain and is therefore invertible. We use FINV to graph f and its inverse. We assume that we start in the ZEROX menu and STD mode.

```
'F(X)=-5+6/49*(X-4)^2'
DEF
↰PLOT  PTYPE  PARA
VAR
-3  SPC  4  FINV
EDIT  NXT  LABEL
```



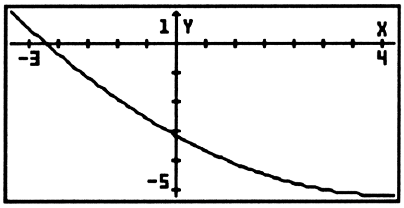
The window is bounded by $x = -5$, $x = 4$, $y = -5$, and $y = 4$. The calculations done by MAMI were

$$c = \min\{f(-3), f(4)\} = -5 \quad d = \max\{f(-3), f(4)\} = 1$$

$$u = r = \min\{a, c\} = -5 \qquad \text{and} \qquad v = s = \max\{b, d\} = 4$$

An alternative approach to graphing f^{-1} is to graph f and then turn your HP 48 counter-clockwise 90°. After turning, the domain of f^{-1} is on the horizontal axis but reversed, with the positive end to the left. To properly implement this approach to graphing f^{-1}, use RESET and set both XRNG and YRNG. For XRNG use -3 and 4. For YRNG use $f(4) = -5$ and $f(-3) = 1$. We show this in the screen dump below, which may be compared with the screen dump generated with FINV.

```
↰PLOT  PTYPE  FUNC
'-5+6/49*(X-4)^2'
↰ EQ
PPAR  RESET
-3  SPC  4  XRNG
-5  SPC  1  YRNG  PREV
PLOT  DRAX  DRAW
EDIT  NXT  LABEL  [−]
Turn 90° counter-clockwise
```



For implicitly defined functions and inverse functions for which there are no known finite solution procedures, the numerical schemes we have described may leave one with a sense that a job has not been completed. This is true, of course, but the observation needs to be balanced by reflecting upon what is known of most of the elementary functions. Functions such as sine, cosine, natural logarithm, exponential, and, even, square root are "known" through tables of values (or, their modern replacement, calculators or computers), through their graphs (visual tables), and through their properties and inter-relationships. We do not know these functions in the sense that we can apply a finite number of arithmetic operations and obtain their exact values. Polynomial functions are among the relatively few functions we can truly "know" in this way.

In this section we have shown how the SOLVR and PLOT applications can be used to obtain numerical or graphical information about functions defined implicitly.

Exercises 4.5

In these problems use your judgement as to the precision of your calculations. Use Newton's method for at least one problem. Use the bisection algorithm for at least one problem. When we say "graph f" we mean find sufficiently many points of the graph so that its main features are clear.

A.1 Show that the equation

$$x \tan y + y^3 - 4 = 0, \qquad x \geq 0, \quad 0 < y < \pi/2$$

determines y as a function f of x. Graph f for $0 \leq x \leq 10$. Calculate approximate values of $f(100)$ and $f(1000)$. Does f have a limit as $x \to \infty$? If so, give an heuristic argument based on the defining equation for the value of this limit.

A.2 Show that the equation

$$-x^3 + 6xy + 3y^2 = 1 + 8y\sqrt{xy}, \qquad x \geq 0$$

determines y as a function of x. For this you may need to recall that for positive numbers x and y, $(x+y)/2 \geq \sqrt{xy}$. Graph f for $0 \leq x \leq 3.0$.

A.3 The equation

$$x + 11y - 10\sin(x - y/9) = 0$$

determines y as a function of x. Graph f for $0 \leq x \leq 2.0$. Estimate the maximum value of f from your graph and determine where it occurs. Verify x_{max} by differentiating the given equation implicitly and solving the equation $y' = 0$. For the latter you will need a second equation since there are two variables.

A.4 Referring to Example 1 and using the data there, write an equation of the tangent line to the graph of f at the point $(1.25, f(1.25))$.

A.5 Find x_{min} and $f(x_{min})$ for the function f in Example 1.

A.6 Show that the function

$$f(x) = \frac{xe^{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}, \qquad -1 < x < 1$$

is invertible. Calculate $f^{-1}(0.9)$ to within 0.001 and compare it to a value obtained by interpolation, using the values of $f(0.7) = 0.8341$ and $f(0.8) = 0.9111$.

A.7 Use FINV in graphing $f(x) = \ln x$, $0.1 \leq x \leq 5$ and its inverse. What is f^{-1}?

A.8 Use FINV in graphing $f(x) = x^3$, $-2 \leq x \leq 2$. What is f^{-1}?

B.1 Show that (3) in Example 2 defines y as a function of x.

B.2 In many calculus books the natural logarithm function ln is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

These books first show that ln is invertible and then define the exponential function as the inverse of ln. Using the ideas of this section (and using the exponential key on your calculator only for checking), prepare a small exponential table in which e^x is tabulated against x, for $x = 0.0, 0.1, \ldots, 2.0$. You may wish to use problem C.3 in solving this problem.

B.3 Use the function h given in problem C.2 to verify the graph sketched in Fig. 8.

C.1 Fill in the details of the following argument establishing that the equation (1) in Example 1 defines y as a function of x. Let $x \geq 1$ be given, fixed but unspecified. Consider the function

$$g(y) = x^3y^3 + 3x^2y + x - 2 \tag{2}$$

The question is whether g has one and only one zero. If y is such a zero, then the values x and y satisfy (1) and we would have shown that (1) implicitly defines y as a function of x. With x fixed, $g(y)$ becomes negative for "sufficiently negative" values of y and becomes positive for sufficiently large values of y. By the intermediate value theorem, g has at least one zero. We wish to show it has exactly one zero. We do this by showing that g is always increasing. By differentiating with respect to y we have

$$g'(y) = 3x^2(xy^2 + 1) > 0$$

This makes it clear that for $x \geq 1$ the equation (1) defines y as a function of x.

C.2 Show that the function implicitly defined by (1) in Example 1 can be found explicitly and, in particular, is given by

$$h(x) = \frac{2}{\sqrt{x}} \sinh\left(\frac{1}{3}\sinh^{-1}\left(\frac{2-x}{2x\sqrt{x}}\right)\right)$$

Here is an outline of an argument. For cubics of the form

$$y^3 + py = q$$

change the variable by setting $y = hz$ and multiply the resulting equation by k, where $h = \sqrt{4|p|/3}$ and $k = 3/(h|p|)$. This will result in one of the two forms

$$4z^3 + 3z = C \qquad \text{or} \qquad 4z^3 = 3z + C$$

For the equation in (1) the first form results. To solve this form use the identity

$$\sinh 3\theta = 4\sinh^3\theta + 3\sinh\theta$$

C.3 Newton's method can often be used to give a fast algorithm for finding $f^{-1}(x)$ for given values of x. Show that the iteration scheme

$$y_{n+1} = y_n + \frac{x - f(y_n)}{f'(y_n)}, \qquad n = 1, 2, \ldots$$

where y_1 is a reasonably good approximation to $f^{-1}(x)$, would give approximations to $f^{-1}(x)$ in most cases. Do problem B.2 using Newton's method as outlined here.

Chapter 5

Minimization/Maximization of Functions

5.0 PREVIEW

Methods for minimizing or maximizing a function are important in many applications of calculus. For example, if the net profit $P(x)$ of manufacturing an item depends upon a labor costs variable x, we may wish to choose x so that $P(x)$ is as large as possible. Or, if the fraction $A(w)$ of carbon monoxide in the exhaust of an engine depends upon a variable w related to the air/fuel ratio w, we may be required to choose w so that $A(w)$ is minimized. Usually such functions as P or A depend upon several variables. In this chapter we minimize or maximize functions of just one variable. In Chapter 9 we discuss methods for minimizing or maximizing functions of several variables.

The usual method for minimizing or maximizing functions depends upon the following result.

Candidate Theorem. *Suppose f is defined on an interval $I \supset (a, b)$. If f is differentiable on (a, b) and has local minimum at a point $c \in (a, b)$, then $f'(c) = 0$.*

The Candidate Theorem is one of the reasons why finding zeros of functions is important. We use this result in an example and several problems. We include a minimization method not dependent upon having an explicit formula for the function.

We have divided this chapter into three short sections. In the first we review the traditional method "solve $f'(x) = 0$ and don't forget to check the end points", review the use of PLOT and Chapter IV in this regard, and discuss the use of the SOLVR in minimization.

In §5.2 we discuss what Arthur Engel (in *Elementary Mathematics from an Algorithmic Viewpoint*) calls the bisection method for minimization, a numerical method for finding the minimum of *unimodal functions.* The bisection method is relatively inefficient but is easily explained and understood. In §5.3 we describe the more efficient golden section search algorithm.

We discuss most of the algorithms in terms of minimization. This includes maximization since the local maxima of a function f are the local minima of the function $-f$.

5.1 THE TRADITIONAL METHOD AND USING THE SOLVR FOR MINIMIZATION

The traditional method for minimizing continuous functions is based on two theorems. The first does not assume the function is differentiable.

Low Point Theorem. *Every function f defined and continuous on a closed and bounded interval I has a minimum.*

Recall that a function f has a *local minimum* at a point c of its domain I if there is a positive number p such that if x is a point of I and $|x - c| \leq p$, then $f(c) \leq f(x)$. A function f has a *minimum* (sometimes called an *absolute minimum*) at a point c of its domain I if $f(c) \leq f(x)$ for all x in I. The number $f(c)$ is often called the minimum value of f. Informally, we may refer to c as a minimum point, sometimes denoting it by x_{min}. If f has a minimum at c, then f has a local minimum there as well.

The second result is the Candidate Theorem, stated above. We assume you have used these theorems to locate a minimum point for a function f defined and continuous on an interval $[a, b]$ and differentiable on (a, b). The Low Point Theorem states that f has a minimum at a point c *somewhere* in $[a, b]$. If c happens to be in (a, b), then $f'(c) = 0$ by the Candidate Theorem. So, the traditional method is to find the zeros of f' in (a, b). These zeros are *candidates* for the minimum point. The only other candidates are the end points a and b. To minimize f on $[a, b]$ we compare the values of f at each of the candidate points.

We give two examples. In the first we calculate the derivative symbolically and then use ROOT on the PICTURE FCN menu to find a zero of the derivative. In the second example we use the SOLVR directly in finding the minimum point.

EXAMPLE 1. Using the traditional method, find the (shortest) distance between the point $(0, 1)$ and the graph of $y = \sin x$.

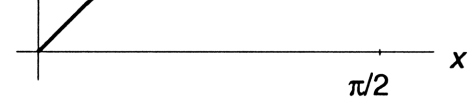


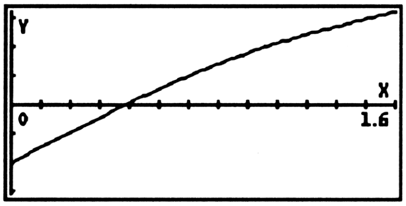
Figure 1

SOLUTION. From Fig. 1 the shortest distance is $\sqrt{f(x_{min})}$, where x_{min} is a minimum point for

$$f(x) = (x - 0)^2 + (\sin x - 1)^2, \qquad \text{where } 0 \leq x \leq \pi/2$$

We calculate $f'(x)$ and find its zero. For this, purge 'X' from your current directory and all directories above it.

'X^2+(SIN(X)−1)^2'
ENTER ENTER 'X'
ENTER ∂ ↰PLOT ↰EQ
PPAR RESET
0 SPC 1.6 XRNG PREV
PLOT DRAX DRAW
Move cursor to zero of f'
FCN ROOT CANCEL
SWAP ↰SOLVE
ROOT ↰EQ
SOLVR X (on SOLVR menu)
EXPR=



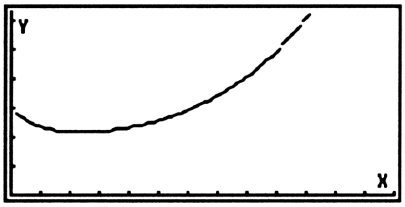
From the graph of f' we see that f' has only one zero c on $(0, \pi/2)$, near $x = 0.5$. ROOT displays $c \approx 0.47872$. We back out of PLOT with two presses of CANCEL and prepare to evaluate f at 0, $\pi/2$, and c using the SOLVR. We find that the minimum value of f is $f(c) \approx 0.52008$. The distance between the point $(0, 1)$ and the graph of $y = \sin x$ is $\sqrt{f(c)} \approx 0.72116$.

EXAMPLE 2. Use the SOLVR in finding a minimum point of the function

$$f(x) = \begin{cases} 1, & \text{for } x = 0 \\ x^x, & \text{for } 0 < x \leq 2 \end{cases}$$

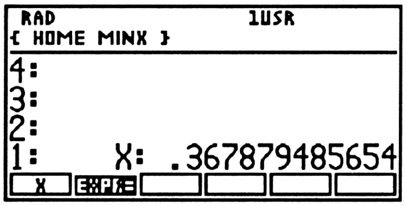
SOLUTION. The function f is continuous at $x = 0$ since $\lim_{x \to 0+} x^x = 1$. It is differentiable on $(0, 2)$. The function f has a minimum on $[0, 2]$ by the Low Point Theorem. We may locate it with the help of PLOT. To represent f we use the convenient IF-THEN-ELSE command IFTE, found on the PRG BRCH NXT menu. Key in 'F(X)=IFTE(X==0,1,X^X)' and then press DEF.

'F(X)' ENTER ↰PLOT
↰EQ PPAR
RESET 0 SPC 2 XRNG
0 SPC 2 YRNG PREV
PLOT DRAX DRAW



The low point on the graph is near (0.4,0.7). Next we use the SOLVR.

↰SOLVE ROOT SOLVR
.4 X
↰ X


```
RAD                 1USR
{ HOME MINX }
4:
3:
2:
1:     X: .367879485654
```

The HP 48 returned $0.3678\cdots$ and the message "Extremum." We find $f(0.3678\cdots) \approx 0.6922$ by using the SOLVR. An automated version of this is EXTR on the FCN menu. With the plot on the screen, press FCN, position the cursor, and then EXTR. All of this may be checked by direct calculation. The derivative of f is $f'(x) = x^x(\ln x + 1)$, for $0 < x \leq 2$. We see that $f'(x) = 0$ for $x = 1/e = 0.3678\cdots$. Since $f(0) = 1$ and $f(2) = 4$, the minimum of f on $[0,2]$ is $1/\sqrt[e]{e} = 0.6922\cdots$.

Exercises 5.1

A.1 Use the model of Example 1 in finding the (shortest) distance from $(1,2)$ to the graph of $y = \cos x$, where $0 \leq x \leq \pi/2$.

A.2 Solve problem A.1 using the SOLVR and its "extremum" feature.

A.3 Find the minimum of the function $f(x) = e^{-1.2\sqrt{x}} \sin(0.53x)$, $\quad 0 \leq x \leq 12$.

A.4 Find the maximum and minimum of the function $f(x) = \sqrt{x} \cos x$, $\quad 0 \leq x \leq 2\pi$.

A.5 Find the maximum and minimum of the function $f(x) = (x^2 + 1)/(x^3 + 1)$, $\quad 0 \leq x \leq 3$.

A.6 Locate all points where the function $f(x) = -2x^2 + \tan x$, $0 \leq x \leq 1.4$, has a local maximum or minimum.

A.7 Find the maximum of the function $f(x) = 2xe^{-x}$, $\quad 0 \leq x \leq 2$.

5.2 BISECTION METHOD FOR MINIMIZATION

Some algorithms for finding a minimum point for a function f defined on an interval I depend upon finding a subinterval $[a,b]$ of I in which f has exactly one local minimum. In this section we assume we have done the work necessary to select such a subinterval $[a,b]$. A function f defined on an interval $[a,b]$ is said to be *unimodal* if it is continuous and has exactly one local minimum.

We prove one preliminary result (called a *lemma*, which is a kind of "helping theorem"), state another result about unimodal functions but defer the proof to a problem, and then outline the bisection method for minimization.

Lemma. *Let f be a unimodal function on $[a,b]$. If $a \leq u < v \leq b$ and $f(u) = f(v)$, then $u < x_{min} < v$.*

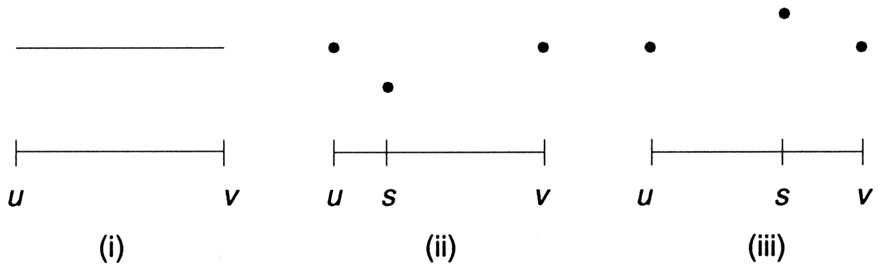


Figure 2

Proof. The argument is divided into the three cases shown in Fig. 2. First, suppose f is constant in $[u,v]$. If this is so, then each point in (u,v) would be a local minimum. This is too many local minima for a unimodal function. If f is not constant, then there is a point s for which either $f(s) < f(u)$ or $f(s) > f(u)$. These cases are shown in (ii) and (iii).

If (ii) holds, then since f is continuous in $[u, v]$, it has a minimum at, say, w. Since $f(s) < f(u) = f(v)$, it follows that $w \in (u, v)$, which is the conclusion we want. The last case is impossible for a unimodal function since f decreases on either side of s and so must have a minimum in each of $[a, s)$ and $(s, b]$.

The main result states an obvious property of unimodal functions.

Unimodal Theorem. *A unimodal function f on $[a, b]$ is strictly decreasing on $[a, x_{\text{min}}]$ and strictly increasing on $[x_{\text{min}}, b]$.*

The Unimodal Theorem guarantees the correctness of the bisection method for minimization (BMM), a procedure for locating x_{min}. The BMM algorithm starts with a unimodal function f on an interval $[m - h, m + h]$ and an error tolerance E. An outline of an argument showing the correctness of the BMM algorithm is given in problem C.2.

BMM Algorithm

Step 1 Divide the current interval $[m - h, m + h]$ into quarters. Denote the interval $[m - h, m]$ by I_1, $[m - h/2, m + h/2]$ by I_2, and $[m, m + h]$ by I_3. Go to Step 2.

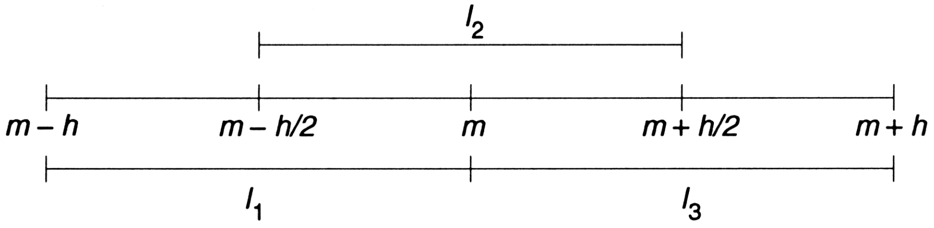


Figure 3

Step 2 Calculate $f(m - h/2)$ and $f(m)$. If $f(m - h/2) < f(m)$, then $x_{min} \in I_1$. In this case, replace m by $m - h/2$ and go to Step 5. Otherwise, go to Step 3.

Step 3 Calculate $f(m + h/2)$. If $f(m + h/2) < f(m)$, then $x_{min} \in I_3$. In this case, replace m by $m + h/2$ and go to Step 5. Otherwise, go to Step 4.

Step 4 In this case, $x_{min} \in I_2$. Go to Step 5.

Step 5 Replace h by $h/2$. The point x_{min} is in the new current interval $[m - h, m + h]$ and $|x_{min} - m| \leq h$. If $h < E$, stop. Otherwise, go to Step 1.

We may observe that to cut the "interval of uncertainty" in half, we need two function evaluations half the time and three function evaluations half the time. Thus on average we need 2.5 function evaluations to cut the interval containing x_{min} in half.

A program for the BMM algorithm follows. It uses a program SD to convert from the natural input E, a, and b to E, m, and h. The program SD (for sum and difference) takes two numbers a and b from the stack and returns $m = (a+b)/2$ and $h = (b-a)/2$. The program SD is

```
« OVER - 2 / + LASTARG SWAP DROP »
```

Please store SD and BMM in a directory MINX (for optimization). The program BMM assumes the function f is stored in program style as F on the MINX menu.

BMM

Inputs: E, a, b	*Outputs:* x_{min}
`« SD → E M H`	Calculate & store local variables for E, m, & h
`« WHILE E H ≤`	If $h < E$, end WHILE REPEAT
`REPEAT`	If $h \geq E$, select new interval
`IF M F M H 2 / - DUP2`	Calculate $f(m)$ & $m - h/2$ & DUP2
`F >`	If $f(m) > f(m - h/2)$, then
`THEN SWAP DROP`	$f(m)$ is not needed &
`'M' STO`	$m - h/2$ replaces m
`ELSE IF H + DUP 3 ROLLD`	If $f(m) > f(m + h/2)$, then
`F > THEN 'M' STO`	$m + h/2$ replaces m
`ELSE DROP`	m remains the same
`END`	End IF THEN command
`END H 2 / 'H' STO`	End IF THEN ELSE command & $h/2$ replaces h
`END M`	End WHILE REPEAT command & put x_{min} on stack
`» »`	Close program delimiters
	Checksum: #43752d Bytes: 217

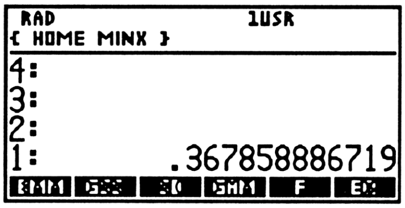
EXAMPLE 1. Find the minimum of f on $[0,1]$, where

$$f(x) = \begin{cases} 1, & \text{for } x = 0 \\ x^x, & \text{for } 0 < x \leq 2 \end{cases}$$

This is the same function used in Example 2 of §5.1.

SOLUTION. Enter f in program style, storing it as F on the MINX menu. Put $E = 0.0001$, $a = 0$, and $b = 1$ on the stack and run BMM.

```
'F(X)=IFTE(X==0,1,X^X)'
DEF
.0001 SPC 0 SPC 2
BMM
```



```
RAD                1USR
{ HOME MINX }
4:
3:
2:
1:      .367858886719
```

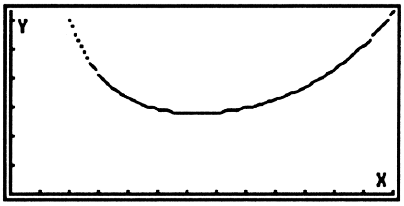
BMM returns 0.367858886719 within 2 seconds. This number is within 0.0001 of x_{min}. Recall that $x_{min} = 1/e = 0.36787\cdots$. We easily find $f(0.367858886719) = 0.692200627953$.

The BMM algorithm does not require that f be given explicitly, only that we can calculate its values. We illustrate this with the gamma function, which is a generalization of the factorial function. The relation between the gamma function and the factorial function is $\Gamma(x) = (x-1)!$, so that, for example, $\Gamma(1) = 0! = 1$, $\Gamma(2) = 1! = 1$, $\Gamma(3) = 2! = 2$, $\Gamma(4) = 3! = 6, \ldots$. The gamma function is defined for all real x except $0, -1, -2, \ldots$. Through the gamma function we can assign a value to such things as 0.5!, which otherwise would not appear to make sense. Specifially, we define $0.5! = \Gamma(1.5) = 0.886\cdots$. The gamma function may be found on the MTH PROB menu.

EXAMPLE 2. Use BMM in locating to within 0.001 the minimum of the gamma function on $(0, 3)$.

SOLUTION. For the BMM algorithm we need to store the gamma function as F. Enter 'F(X)=FACT(X−1)' and then press DEF. Enter several values of x and calculate $\Gamma(x)$. For example, $\Gamma(4) = 3! = 6$. Before running BMM we plot the gamma function for $x \in (0, 3]$.

```
↰PLOT  'F(X)'
↰ EQ  PPAR
0  SPC  3  XRNG
0  SPC  2  YRNG
PREV  PLOT
ERASE  DRAX  DRAW
```



The minimum is near $x = 1.5$. Recalling that the gamma function is not defined at $x = 0$, we may put 0.001, 1, and 2 on the stack as input for BMM. The algorithm returns $1.4619\cdots$ in a few moments.

Exercises 5.2

Use the BMM algorithm/program in solving the following problems.

A.1 Find the shortest distance from the origin to the graph of $y = e^x$. Take $E = 0.01$ and use a rough sketch by hand to estimate a and b.

A.2 Find x_{max} to within 0.01 for the function

$$f(x) = \begin{cases} 0, & \text{for } x = 0 \\ 1 - x^x, & \text{for } 0 < x \le 2 \end{cases}$$

A.3 Graph and find the minimum of the gamma function on $(-2, -1)$. Use $E = 0.001$.

A.4 Find the maximum of the function $f(x) = 1 + 5.2x - \sec\sqrt{0.73x}, \quad 0 \le x \le 3.2$.

B.1 Think through the program SD used in the BMM algorithm so that you can explain it to your friends. Writing out the stack history may help. Or use the built-in debugger. Put 'A', 'B', and 'SD' on the stack, press PRG, NXT, RUN, and DBUG. Now single step (SST) through the algorithm, watching the commands of SD appear at the top of the display and the results at the bottom.

B.2 Solve problem A.4 with exact methods. You should obtain $x_{min} = \sqrt[6]{2}$ and $f(x_{min}) = 1 - (2/3)2^{2/3}$.

B.3 Explain the program BMM with the aid of stack diagrams.

B.4 Give an argument for the assertion in Example 2 that x_{min} is in the interval $[3.4, 7.3]$.

C.1 Prove the Unimodal Theorem.

C.2 Verify the BMM algorithm. Use the Unimodal Theorem in verifying Steps 2 and 3. Use the lemma for Step 4. Show that the algorithm eventually stops and, when it does, $|x_{min} - m| < E$.

C.3 (Depends upon problems A.4 and B.2.) Fill in the details of the following discussion, whose purpose is to show that at least for some functions, we cannot improve the accuracy of our approximation to x_{min} by taking E very small. We use the function f in problem A.4 in the analysis but note that the phenomenon is common. First, run BMM twice more on this function. Use $a = 1$ and $b = 1.5$. First take $E = 0.0000001$ and then $E = 0.00000001$. The program BMM will return 1.12246131897 each time. Note that from problem B.2, this estimate of x_{min} is incorrect in the sixth decimal place. It appears that we can obtain no better estimate. (You may wish to try smaller values of E if you are in doubt on this point.) Why? We outline an answer with the help of Taylor's formula. Letting $m = x_{min}$, for each value of $h \neq 0$ there is a number v between m and $m + h$ for which

$$f(m+h) = f(m) + f'(m)h + f''(v)h^2/2!$$

Since $f'(m) = 0$ and $|f''(x)| < 10$ for $x \in [1, 1.5]$ we have

$$|f(m+h) - f(m)| < 5h^2, \qquad \text{provided that} \quad m, m+h \in [1, 1.5] \tag{1}$$

It follows from (1) that if in the BMM algorithm all of the points at which f is being evaluated are within 10^{-7} of m, then the function values are not distinguishable by the HP 48. (Try this on the SOLVR.) Consequently, the algorithm will always bop down to Step 4 and the estimate for x_{min} will no longer change. Note, however, that $f(x_{min})$ may be calculated with great accuracy.

5.3 GOLDEN SECTION SEARCH

The golden section search (GSS) algorithm is a second algorithm for approximating x_{min} for a unimodal function. It is somewhat more efficient than the BMM algorithm in that it requires a little less than 1.5 function evaluations to halve the interval of uncertainty while the BMM algorithm requires 2.5 function evaluations. The idea of a "golden section" is older than Euclid's *Elements*. It is thought by some that many Roman and Greek buildings have proportions related to the golden section. The profile of one side of the

Parthenon, for example, is a rectangle whose height to breadth ratio is not far from the value $(-1+\sqrt{5})/2 \approx 0.62$. Such rectangles are called golden rectangles.

The GSS algorithm arises from an attempt to minimize the number of function evaluations needed to locate the minimum. In searching for x_{min} for a unimodal function f on $[a,b]$, we must evaluate f at least twice. If we have no prior information, we would choose points u and v symmetrically placed about the midpoint, as in Fig. 4. The question is, how should we space these points?

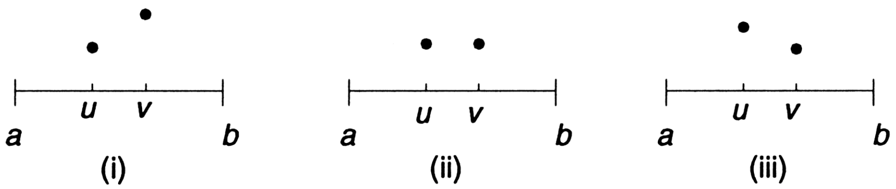


Figure 4

Before answering this question we note that no matter how we choose u and v, we may compare $f(u)$ and $f(v)$ and, from this, shorten the interval of uncertainty, which, at the beginning, is $[a,b]$. The three possible outcomes of the comparison are shown in Fig. 4. In (i) we show the case in which $f(u) < f(v)$. It follows from the Unimodal Theorem in §5.2 that $x_{min} \in [a,v]$. The second case is shown in (ii). It follows from the lemma in §5.2 that $x_{min} \in [u,v]$. In the third case we have $x_{min} \in [u,b]$. Ignoring the unlikely second case, we have reduced the interval of uncertainty from $[a,b]$ to either $[a,v]$ or $[u,b]$. It is reasonable to try to shorten these intervals of uncertainty as much as possible by choosing u and v close to the midpoint.

In the next iteration of the algorithm we would choose new values u' and v' of u and v, again choosing them near the (new) midpoint. We see, then, that for two function evaluations we can almost cut the interval of uncertainty in half. This is better than the BMM algorithm, where 2.5 evaluations are needed to cut the interval in half.

The GSS algorithm chooses u and v so that only one new function evaluation is needed at each step (after the first). We show this arrangement in Fig. 5. We have assumed in drawing the figure that in comparing the function values $f(u)$ and $f(v)$ we found $f(u) < f(v)$.

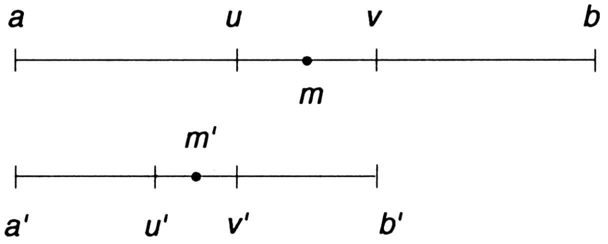


Figure 5

The points u and v are chosen in $[a,b]$ so that when u' and v' are similarly chosen in $[a',b']$, either $v'=u$ (shown in Fig. 5) or $v=u'$. The effect of such choices is that only one function evaluation is needed at each step of the algorithm, since one of the two required function evaluations will have been done in the preceding step. The condition $v'=u$ (or $v=u'$) is enough to give us the recipe for choosing u and v, as we show next.

In the interval $[a, b]$, let $m = (a+b)/2$ be the midpoint and $h = (b-a)/2$ the half-length. We may write

$$u = m - \gamma h \qquad \text{and} \qquad v = m + \gamma h,$$

where $0 < \gamma < 1$. To find γ we may assume that $a = 0$ and $b = 1$. In this case, $m = h = 1/2$. We write the new midpoint and half-length as m' and h'. Calculating v' and u separately we find

$$v' = m' + \gamma h' = \frac{v}{2} + \gamma\frac{v}{2} = \frac{v}{2}(1+\gamma) = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}\gamma)(1+\gamma) = \frac{1}{4}(1+\gamma)^2$$
$$u = m - \gamma h = \frac{1}{2} - \gamma\frac{1}{2} = \frac{1}{2}(1-\gamma)$$

Imposing the condition $v' = u$ gives

$$\frac{1}{4}(1+\gamma)^2 = \frac{1}{2}(1-\gamma)$$
$$\gamma^2 + 4\gamma - 1 = 0$$
$$\gamma = -2 + \sqrt{5} \approx 0.24$$

We used only the positive root since $\gamma > 0$.

After the first step of the GSS algorithm, we reduce the length of the interval by a factor of $(-1+\sqrt{5})/2 \approx 0.62$ (since, from above, the new interval has length $v = m + \gamma h \approx 0.5 + 0.24/2 = 0.62$) for each function evaluation. This is an improvement over the BMM algorithm factor of 0.5 for 2.5 evaluations. We refine this comment in a problem. Also in a problem is a discussion of how γ is related to Euclid's golden section.

It is a straightforward job to write out the GSS algorithm and an HP 48 program. We start with a unimodal function f on an interval $[m-h, m+h]$ and an error tolerance E.

GSS Algorithm

Step 1 Let $\gamma = -2 + \sqrt{5}$. Calculate $(1-\gamma)/2$, $(1+\gamma)/2$, $u = m - \gamma h$, and $v = m + \gamma h$. Calculate $f(u)$ and $f(v)$. Go to Step 2.

Step 2 If $f(u) < f(v)$, then $x_{min} \in [m-h, v]$. Calculate

$$m' = ((m-h)+v)/2 = m - h(1-\gamma)/2, \quad h' = (v - (m-h))/2 = h(1+\gamma)/2,$$
$$u' = m' - \gamma h', \quad \textit{and} \quad f(u')$$

Set $v' = u$ and $f(v') = f(u)$. Go to Step 5.

Step 3 If $f(u) > f(v)$, then $x_{min} \in [u, m+h]$. Calculate

$$m' = (u + (m+h))/2 = m + h(1-\gamma)/2, \quad h' = ((m+h) - u)/2 = h(1+\gamma)/2,$$
$$v' = m' + \gamma h', \quad \text{and} \quad f(v')$$

Set $u' = v$ and $f(u') = f(v)$. Go to Step 5.

Step 4 If $f(u) = f(v)$, then $x_{min} \in [u, v]$. In this case, which occurs only rarely, we note that x_{min} is in both intervals $[m-h, v]$, and $[u, m+h]$. We may therefore replace in Step 2 the inequality $f(u) < f(v)$ by $f(u) \leq f(v)$, thus including Step 4 in Step 2.

Step 5 Rename m' and h' as m and h. We have shown that $x_{min} \in [m-h, m+h]$. Hence $|x_{min} - m| \leq h$. If $h < E$, we may exit the algorithm. Otherwise, go to Step 2.

A program for the GSS algorithm is given below. Please store GSS on the MINX menu. We assume that the constant $-2+\sqrt{5}$ has been calculated and stored as GAM on the MINX menu. GSS uses the SD program discussed with BMM and assumes that f is written in program style and stored as F on the MINX menu.

GSS	
Inputs: E, a, b	*Outputs:* x_{min}
`≪ SD DUP2 GAM DUP`	Calculate m & h. Recall γ & DUP
`1 SD 8 ROLLD 8 ROLLD`	Calculate $(1 \pm \gamma)/2$ & stack down 2
`* SWAP SD 2 *`	Calculate u and v
`F SWAP 2 * F`	and then $f(u)$ and $f(v)$
`→ GP GM E M H FU FV`	Store local variables
	$(1 \pm \gamma)/2$, E, m, h, $f(u)$, and $f(v)$
`≪ WHILE E H ≤`	If $h < e$, end WHILE REPEAT
`REPEAT`	If $h \geq e$, select new interval
`IF FU FV DUP2 ≤`	If $f(u) \leq f(v)$, then
`THEN DROP 'FV' STO`	$f(u)$ not needed; store $f(v)$
`M GM H * - GP H *`	Calculate m' and h'
`DUP2 GAM * - F`	Calculate u' and $f(u')$
`'FU' STO`	Store $f(u')$
`ELSE 'FU' STO DROP`	$f(u') = f(v)$
`M GM H * + GP H *`	Calculate m' and h'
`DUP2 GAM * + F`	Calculate v' and $f(v')$
`'FV' STO`	Store $f(v')$
`END 'H' STO 'M' STO`	Store h' and m'
`END M`	Put x_{min} on stack
`≫ ≫`	Close program delimiters
	Checksum: #9913d Bytes: 374

EXAMPLE 1. Find $x_{\min}$ for the function f given in Example 2 of §5.1, namely,

$$f(x) = \begin{cases} 1, & \text{for } x = 0 \\ x^x, & \text{for } 0 < x \leq 2 \end{cases}$$

SOLUTION. Running GSS with $E = 0.0001$, $a = 0$, and $b = 1$, gives 0.3679 within 2 seconds. For these data the BMM algorithm is slightly faster than GSS. For more complex functions or smaller values of E, the GSS algorithm is faster. Apparently the more complex initialization and generally heavier "overhead" in the GSS program masks the savings from the smaller number of function evaluations in some cases.

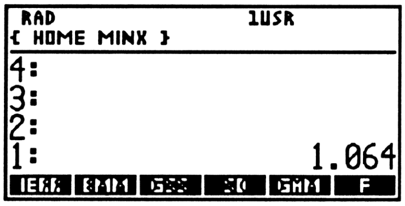
EXAMPLE 2. This example includes definite integrals and the HP 48 built-in numerical integration function. You may wish to defer it until Chapter VI. However, the main idea is minimization—using the GSS algorithm—of a function whose values must be calculated numerically. Find the minimum point for the function

$$f(x) = \frac{1}{\sqrt{x}+1} \int_0^{\pi/2} \sqrt{1 + (x^2+1)\sin^2 t}\, dt, \qquad 0 \le x \le 2$$

It is given that f is unimodal on $[0, 2]$.

SOLUTION. The GSS program will do the heavy labor, provided that we can define f. Note that f is entered as a program.

```
≪  →  X
  ≪  '∫(0,π/2,
  √(1+(X^2+1)*
  SQ(SIN(T))),T)'  →NUM
   1  1  X  √  +  /  *
≫  ≫
ENTER  'F'  STO
↰MODES  FMT  3  FIX
VAR
.01  SPC  0  SPC  2  GSS
```



```
RAD                    1USR
{ HOME MINX }
4:
3:
2:
1:                    1.064
```

We set the accuracy factor for the numerical integration by 3 FIX. This gives estimates for this function f within 0.003. Running GSS with $E = 0.01$, $a = 0$, and $b = 2$ gives $x_{min} \approx 1.06$ after approximately 20 seconds. The reason for this relatively long execution time is that each function evaluation requires a numerical integration. The variable IERR left on the MINX menu is an HP 48 estimate of the error in the numerical integration.

Exercises 5.3

Use the GSS algorithm/program in solving the following problems. Let $E = 0.001$.

A.1 Find the minimum of the unimodal function $f(x) = 2^x - x - 1$, $x \in [-1, 2]$.

A.2 Find the minimum of the unimodal function $f(x) = (x - \ln x)/x$, $x \in [1, 5]$.

A.3 Find all local maxima and minima as well as the maximum and minimum of the function $f(x) = x^4 - x^2 - 0.4x$, $x \in [-1.5, 1.5]$. Note that this function is not unimodal in its domain.

A.4 Choose θ so that the trapezoid in Fig. 6 has maximum area.

A.5 Find the minimum of the unimodal function

$$f(x) = \sqrt{x^2 + 100^2} + \sqrt{(200 - x)^2 + 50^2}, \qquad x \in [0, 200]$$

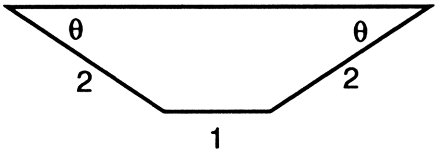


Figure 6

B.1 The BMM algorithm takes 2.5 function evaluations (on average) to reduce the length of the interval by a factor of 0.5. The GSS algorithm takes 1 function evaluation to reduce the interval by a factor of $(-1 + \sqrt{5})/2 \approx 0.62$. Show that to reduce an interval of length 1 to an interval of length 0.00001 requires the BMM algorithm approximately 41 evaluations and the GSS algorithm approximately 24 evaluations.

B.2 An electrician's cart of width 2 feet is to be moved around a square corner, from an 8 foot hallway into a 5 foot hallway. Find the longest possible length of the cart, given that it must be rolled around the corner and not tipped in any way.

B.3 A ray of light is traveling from a source at $(0, 100)$ to a 200 centimeter mirror on the x-axis, where it reflects to a receiver at $(200, 50)$. This is shown in Fig. 7.

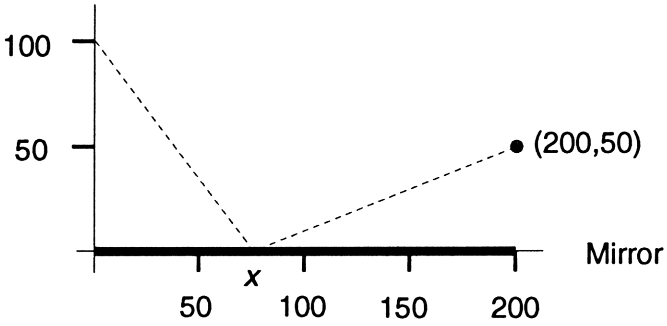


Figure 7

We do not assume that the angle of incidence of the ray at the mirror is the same as its angle of reflection. Rather, we assume the ray reaching the receiver is the one taking the least time. This is Fermat's principle. Find an expression for the length $L(x)$ of the path followed by the ray shown in the figure Use GSS to find the minimum of this unimodal function defined on $[0, 200]$. From your result show that the angles of incidence and reflection of the ray on the x-axis are equal.

B.4 Use GSS in finding the maximum of the function in Example 3 of §4.5.

C.1 The idea of a "golden section" was known by the time of the Pythagoreans, several centuries before Euclid. It occurs in the pentagon, where the diagonals intersect at the golden section point. The rectangle with sides PR and RQ is a "golden rectangle." The ratio PR/RQ is $(-1 + \sqrt{5})/2 \approx 0.62$. Euclid's *Elements* contains several results related to the pentagon and its diagonals as well as a simple construction of the golden section. We show in Fig. 8 a pentagon and the drawing usually given with Euclid's result. The construction is given in Book II, Proposition 11 of the *Elements*.

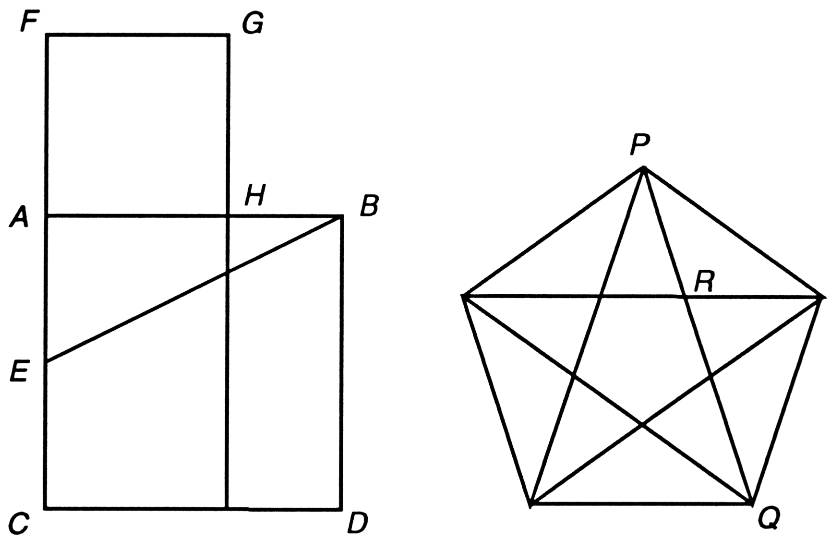


Figure 8

Proposition 11. *To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.*

Partial proof. Let AB be the given line. The problem is to construct point H so that $AB \cdot BH = AH^2$. The method given by Euclid is to construct square $ABDC$, bisect AC at E, construct $EF = BE$, and, finally, construct square $AFGH$. This gives a point H. Prove that H is the desired point.

Some mathematical historians argue that II.11 was seen by the Greeks as a constructive means to find the positive root of the quadratic $x^2 + ax = a^2$. Letting AH be x and $AB = BD = a$, II.11 may be restated as $a \cdot (a - x) = x^2$ or $x^2 + ax = a^2$. Solving the equation $x^2 + ax - a^2$ we find one positive root,

$$x = a\frac{-1+\sqrt{5}}{2} \approx a \cdot 0.618\cdots$$

Note that if $AB = 1$, then $AH = (-1 + \sqrt{5})/2 \approx 0.62$.

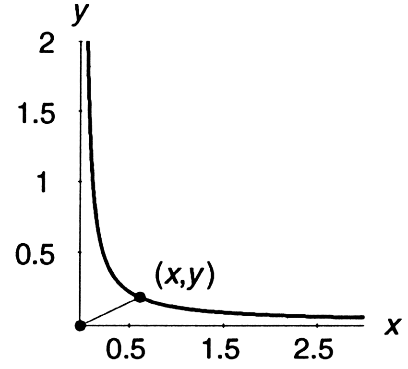


Figure 9

C.2 A particle is moving on the graph defined by the equation

$$4x^2y^3 - 2x^3y^2 + 9xy - 1 = 0, \qquad \text{for} \quad 0 < x < 3 \tag{1}$$

The graph is shown in Fig. 9. Find the point P_{min} of closest approach to the origin. One way of solving this problem is to use a modification of the GSS program. First, we imagine that the equation (1) has been solved for y in terms of x, giving $y = g(x)$. We must minimize the function $f(x) = x^2 + y^2 = x^2 + (g(x))^2$. From Fig. 9 it is clear that f is unimodal on $(0, 3)$. The GSS algorithm requires only that we be able to calculate f at certain points. Suppose, for example, that the GSS algorithm requires the value $f(0.5)$. We set $x = 0.5$ in (1) and solve the resulting equation for y, using the SOLVR, Newton's method, or the bisection method (for finding zeros), whichever method is preferred. Having found y, we can calculate $f(x)$ and the GSS algorithm/program can continue. It should be apparent that the GSS algorithm will be more efficient here than the BMM algorithm.

PROJECT

OPTIMAL SPRAYER PROBLEM

DISCUSSION

A field irrigation system moves on water-driven wheels w1, w2, ... and sprays water through evenly-spaced emitters e1, e2, ..., as shown in Fig. 10. The entire apparatus moves through the field at a uniform speed. We assume that each emitter distributes water uniformly in a circular pattern. Ignoring the special cases of the first and last emitters, determine the spacing of the emitters so that the water is distributed as uniformly as possible and no point in the field receives water from more than two emitters.

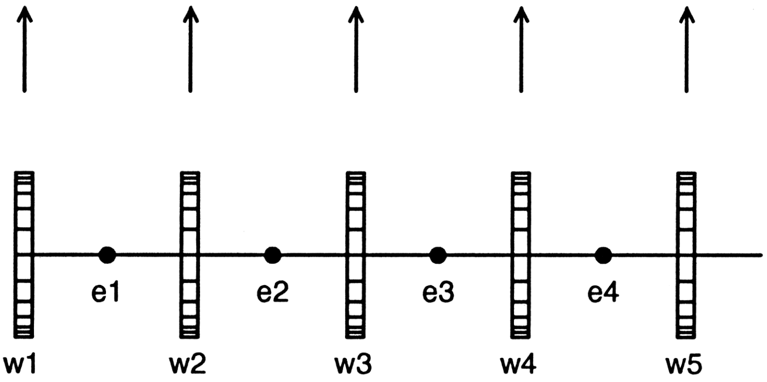


Figure 10

OUTLINE OF SOLUTION

(1) Let the emitter spray radius be 1 unit. Since no point of the field receives water from more than two emitters, the problem reduces to determining the center $(c, 0)$ of the second circle. See Fig. 11. Show that $1 \leq c \leq 2$.

(2) Explain why it is useful to find an expression $w(x)$ whose value is proportional to the water received by points along the vertical line through $(x, 0)$, where $0 \leq x \leq c/2$. Show that $w(x)$ is given by

$$w(x) = \begin{cases} \sqrt{1-x^2}, & 0 \leq x \leq c-1 \\ \sqrt{1-x^2} + \sqrt{1-(x-c)^2}, & c-1 \leq x \leq c/2 \end{cases}$$

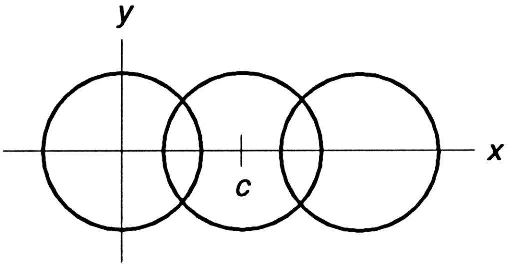


Figure 11

(3) The total amount of water received on $[0, c/2]$ is

$$\int_0^{c/2} w(x)\,dx$$

Assuming here and elsewhere that the proportionality constant mentioned above is equal to 1, show that the total amount of water received on $[0, c/2]$ is $\pi/4$. You may infer this result from the figure, without calculation.

(4) Show that the average amount of water received on $[0, c/2]$, that is, the average value of w on $[0, c/2]$, is $\pi/(2c)$. (Recall that the average value of a function f on $[a, b]$ is $(\int_a^b f(x)\,dx)/(b-a)$.

(5) Give an informal argument as to why we may optimize the uniformity of coverage by minimizing the "average variation" function

$$g(c) = (2/c)\int_0^{c/2} (w(x) - \pi/(2c))^2\,dx \qquad 1 \le c \le 2$$

(6) Show that $g(c)$ may be reduced to a sum of the form

$$\frac{4}{c}\int_{c-1}^{c/2} \sqrt{1-x^2}\sqrt{1-(x-c)^2}\,dx + \frac{k_1}{c} + \frac{k_2}{c^2}$$

where k_1 and k_2 are specific, numerical constants. You will need to determine the values of k_1 and k_2.

(7) Use the GSS algorithm in showing that $c_{min} \approx 1.7$. The function g is not unimodal throughout $[1, 2]$. Use $E = 0.05$. Set 2 FIX.

Chapter 6

Solving Systems of Linear Equations with the HP 48

6.0 PREVIEW

Systems of equations like

$$\begin{aligned} x + y + 2z &= 6 \\ 2x - y + z &= 6 \\ x - y - z &= -1 \end{aligned} \tag{1}$$

are important in both single- and multi-variable calculus. Both integration by "partial fractions" and using linear functions to approximate functions of several variables require an understanding of linear systems and facility with an algorithm for solving them.

Using matrices and vectors we may rewrite (1) to resemble a linear equation of the form $ax = b$, where a and b are constants and x is the unknown. For this we identify in (1) the 3×3 coefficient matrix A, the 3×1 matrix B of "right-hand-sides," and the 3×1 matrix X of "unknowns."

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 6 \\ -1 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

The 3×1 matrices B and X are often called *column vectors.*

Using the definition of matrix multiplication we may rewrite (1) as the matrix equation

$$AX = B \tag{3}$$

Some of you may know that the system (3)—and, hence, (1) as well—may be solved by calculating the *inverse* A^{-1} of A and then multiplying both sides of (3) by A^{-1}. This is analogous to solving a system like $ax = b$ by multiplying both sides by the reciprocal of a, that is,

$$\begin{aligned} ax &= b \\ a^{-1}(ax) &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= b/a \end{aligned}$$

For matrices

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

We return to this topic in problem B.1 in §6.4.

Our goals in this short chapter are modest. We begin by describing how matrices and vectors are represented and entered on the HP 48. In §6.2 we discuss the Gauss-Jordan algorithm for solving a linear system. It is likely that you are familiar with the basic ideas of this algorithm. Our goals are to stay close to your experience and to provide procedures and programs to reduce the amount of by-hand arithmetic you must do. We include in this section programs providing for exact arithmetic when working with rational numbers. In solving systems of equations, these programs eliminate roundoff error and duplicate the rational arithmetic occurring in solving equations with rational coefficients by hand. In §6.3 we describe the powerful system solver resident on the HP 48 divide key. We limit our comments to how-to-do-it since the algorithm used by the HP 48 is beyond our chosen scope.

6.1 ARRAYS ON THE HP 48

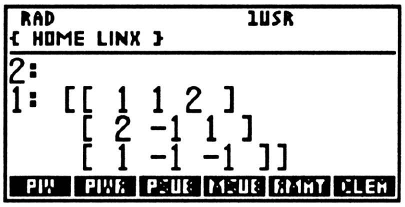
Arrays on the HP 48 can represent either matrices or vectors. An array is enclosed by square bracket [ ] delimiters. Inside these delimiters individual rows are also enclosed by [ ]. The matrices A and B in (2) would appear on the stack as

$$\begin{matrix} [[& 1 & 1 & 2 &] \\ [& 2 & -1 & 1 &] \\ [& 1 & -1 & -1 &]] \end{matrix} \qquad \begin{matrix} [[& 6 &] \\ [& 6 &] \\ [& -1 &]] \end{matrix} \quad \text{or} \quad [\ 6\ \ 6\ \ -1\ ] \tag{4}$$

Either of the two ways shown in (4) of representing B—as an array or as a vector—allows B to combine with other matrices as a 3 × 1 column vector.

The matrix A can be entered using the MatrixWriter Application. The Owner's Manual explains this clearly. We prefer the faster command line entry. We may enter the matrix A as follows. Use STD mode, to which, by the way, you may change by simply typing STD and then press ENTER.

$\boxed{\Lsh}$ $\boxed{[\ ]}$ $\boxed{\Lsh}$ $\boxed{[\ ]}$
1 SPC 1 SPC 2 ▶
2 SPC −1 SPC 1 SPC
1 SPC −1 SPC −1
ENTER



```
RAD                  1USR
{ HOME LINX }
2:
1: [[ 1 1 2 ]
     [ 2 -1 1 ]
     [ 1 -1 -1 ]]
```

Store this matrix as A. For B, press $\boxed{\Lsh}$ $\boxed{[\ ]}$, followed by the entries of B separated with spaces, and then ENTER. Store as B.

Another fast method of entering matrices uses →ARR, which is on the PRG TYPE menu. To enter the matrix A, for example, key in

1 SPC 1 SPC 2 SPC 2 SPC −1 SPC 1 SPC 1 SPC −1 SPC −1

and press ENTER. Now enter the size of the matrix, in this case { 3 3 }, and press $\boxed{\to\text{ARR}}$.

Matrix arithmetic is easy on the HP 48. We give one illustration. The solution to (3) is the column vector $X = [\ 1\ -1\ 3\ ]$. It is easy to check this by forming the product AX. If the product is the same as B, then X is a solution. Bring A back to the stack, enter X, and press $\boxed{\times}$. We observe that the product AX is B. Also try forming the product of A and X in reverse order. You will get the "Invalid Dimension" error message. Matrix multiplication is not commutative or, as here, always defined.

If you already know about the inverse of a matrix you may wish to form A^{-1} by putting A on the stack and pressing the $\boxed{1/x}$ key. In STD mode the result does not fit on the screen. To view A^{-1} element-by-element, press $\boxed{\blacktriangledown}$. The high-lighted element (actually, the darkened area) is displayed on the command line. You may inspect all elements by using the cursor keys. Press CANCEL to get back to the normal display. Now recall a copy of B and form the product $A^{-1}B$. You should obtain the solution $X = [\ 1\ -1\ 3\ ]$.

Exercises 6.1

A.1 Show that

$$\begin{bmatrix} 2 & -1 \\ 3 & 10 \end{bmatrix}\begin{bmatrix} -2 & 4 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 8 & -5 \\ 4 & 12 & 4 \end{bmatrix}$$

A.2 Show that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 3 & 8 \end{bmatrix}$$

A.3 Show that

$$\begin{bmatrix} 1 & 5 & 4 \\ -1 & 1 & 0 \\ 0 & 7 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ -10 \\ 1 \end{bmatrix} = \begin{bmatrix} -45 \\ -11 \\ -64 \end{bmatrix}$$

A.4 Show that

$$[3\quad 1\quad 2]\begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} = [-3]$$

Enter these matrices as [[3 1 2]] and [[1] [4] [−5]].

A.5 Show that

$$\begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}[3\quad 1\quad 2] = \begin{bmatrix} 3 & 1 & 2 \\ 12 & 4 & 8 \\ -15 & -5 & -10 \end{bmatrix}$$

Enter these matrices as [[1] [4] [−5]] and [[3 1 2]].

A.6 Calculate A^3, where

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

B.1 Using any legitimate method, solve the system of equations $AX = B$ by hand, where A, X, and B are

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} -5 \\ 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1/3 \\ -1 & 2/3 \end{bmatrix}$$

Next, solve $AX = B$ by multiplying both sides of this equation by C, forming $C(AX) = CB$. Since $C(AX) = (CA)X$, we may form CA first. By hand, show that the product CA times X is just X. Thus the solution of $AX = B$ is CB. The matrix C is called the inverse of A. It is analogous to the reciprocal of a non-zero number. Enter A and then press $\boxed{1/x}$. Is the resulting matrix the same as C? Comment.

6.2 THE GAUSS-JORDAN ALGORITHM

Systems such as (1) in §6.1, which is repeated in Example 1 below, may be solved in many different ways. One way is "by hand." We give two alternatives, methods which save labor and reduce error. In this section we solve systems $AX = B$ through row operations, using the Gauss-Jordan algorithm. We give a program PIV that implements the Gauss-Jordan algorithm. This algorithm and program are based on familiar ideas. In §6.4 we use the divide key $\boxed{\div}$ to "divide" two matrices. This is faster than PIV but depends upon ideas not familiar to many calculus students.

We explain the Gauss-Jordan algorithm through examples. We assume that in solving a system of equations you have had some experience with dropping the names of the unknowns and working with the coefficient matrix or the *augmented matrix.* The augmented matrix of a system of equations is the coefficient matrix augmented by the column vector of "right-hand-sides." We form an augmented matrix in Example 1.

EXAMPLE 1. Solve the system of equations given on the left in (1) using row operations. The augmented matrix for this system is shown on the right.

$$\begin{aligned} x + y + 2z &= 6 \\ 2x - y + z &= 6 \\ x - y - z &= -1 \end{aligned} \qquad \longrightarrow \qquad \begin{bmatrix} 1 & 1 & 2 & 6 \\ 2 & -1 & 1 & 6 \\ 1 & -1 & -1 & -1 \end{bmatrix} \tag{1}$$

SOLUTION. We use the number in the $\{1\ 1\}$ position (that is, in the first row, first column position) to eliminate x from the remaining two equations. Said differently, we "pivot" on the matrix entry in the $\{1\ 1\}$ position. In terms of the augmented matrix, we use row operations so that in all rows other than the first the first column entry becomes zero. We may accomplish this by (i) dividing the entire first row by the coefficient of x, (ii) adding to the second row -2 times the (now modified) first row, and (iii) adding to the third row -1 times the (now modified) first row. After all this, the pivot algorithm continues by "pivoting" on the $\{2\ 2\}$ entry, that is, on the -3 in the second row and second column of the second matrix in (2). Finally, we pivot on the -1 in the $\{3\ 3\}$ position in the third matrix in (1). We have omitted the usual large parentheses or brackets.

$$\begin{array}{cccc} 1 & 1 & 2 & 6 \\ 2 & -1 & 1 & 6 \\ 1 & -1 & -1 & -1 \end{array} \quad \longrightarrow \quad \begin{array}{cccc} 1 & 1 & 2 & 6 \\ 0 & -3 & -3 & -6 \\ 0 & -2 & -3 & -7 \end{array} \quad \longrightarrow$$

$$\begin{array}{cccc} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & -3 \end{array} \quad \longrightarrow \quad \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array} \tag{2}$$

From the last augmented matrix we may read the solution of the system of equations given in (1). It is $x = 1$, $y = -1$, and $z = 3$.

Much of the arithmetic required for each step in (2) can be condensed into one easily remembered pattern. We show this in (3). Suppose a pivot element p has been chosen in the "old matrix." (We do not choose pivots from the last column, which is not part of the coefficient matrix.) The "new matrix" resulting from this pivot is formed from the old matrix by following certain "rules." We explain (3) in what follows.

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & p & \cdots & b & \cdots \\ & \vdots & & \vdots & \\ \cdots & c & \cdots & a & \cdots \\ & \vdots & & \vdots & \end{array} \quad \rightarrow \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & \dfrac{b}{p} & \cdots \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & \dfrac{ap - bc}{p} & \cdots \\ & \vdots & & \vdots & \end{array} \tag{3}$$

Gauss-Jordan Pivot Algorithm

Pivot Rule 1 For each choice of pivot, all entries of the new matrix other than those in the row or column of the pivot are calculated using the formula $(ap - bc)/p$. This formula is easily remembered by thinking of it as the "rectangle rule." Referring to (3), suppose we wish to calculate the new matrix entry corresponding to the old matrix entry a. The entries a and p lie at the ends of a diagonal of a rectangle. The entries b and c lie at the ends of the other diagonal. The new matrix entry corresponding to the old matrix entry a is the difference of the products on the two diagonals, which is then divided by the pivot. It makes no difference whether the position of a is above or below p, or to the right or left of p.

Pivot Rule 2. The entries in the pivot row of the new matrix are formed from the pivot row entries in the old matrix by dividing them by the pivot p. Thus the entry in the same position as b is b/p.

Pivot Rule 3. Each entry in the pivot column of the new matrix is 0, except for the entry in the pivot position, which is 1 (Pivot Rule 2).

Try these rules on the calculations in (2), perhaps circling the successive pivots to anchor the rectangles for the new elements.

We give an HP 48 program for pivoting. The program PIV assumes the stack contains a matrix A on level 2 and a list $\{r\ s\}$ on level 1. The list specifies the entry of A on which

PIV

Inputs: A_{old}, $\{r\ s\}$

Outputs: A_{new}

```
« DUP2 GET ROT DUP SIZE        A, {r s} → {r s}, p, A, {m n}
4 ROLL + OBJ→ DROP             {r s}, p, A, {m n} →
6 ROLL                           A, m, n, r, s, p
→ M N R S P «                  Store m, n, r, s, p as local variables
1 → I « DO IF I R ≠            Set up outer and inner loops, where
THEN 1 → J «                     I and J are row & column indices;
DO IF J S ≠ THEN DUP             skip row & column of pivot
DUP2 {I J} GET 3 ROLLD         A, A, A, A → A, a_ij, A, A
{I S} GET SWAP { R J} GET      A, a_ij, A, A → A, a_ij, a_is, a_rj
* P / - {I J} SWAP PUT         a_ij − a_rj a_is/a_rs replaces a_ij in A
END 1 J + 'J' STO UNTIL        Advance index j of
J N > END »                      inner loop until j > n
END 1 I + 'I' STO UNTIL        Advance index i of
I M > END »                      outer loop until i > m
1 N FOR J {R J} DUP2 GET       A → A, {r s}, a_rj
P / PUT NEXT                   Finish dividing pivot row by p
1 M FOR I {I S} 0 PUT NEXT     Put zeros in entire pivot column
{R S} 1 PUT » »                Restore 1 to pivot position

                                 Checksum: # 49953d   Bytes: 428
```

to pivot. PIV returns the new matrix to level 1, ready for further use. The old matrix is not saved.

For the purpose of understanding the program PIV we restate the pivot rules in terms of subscripted matrix entries. We denote the augmented matrix by A and agree that its dimensions are m rows and n columns. In most applications, $n = m + 1$. The element in the i th row and j th column of the old matrix is a_{ij}. The new matrix is formed by replacing the elements of A by the new entries. Suppose we have chosen $a_{rs} \neq 0$ as the pivot, where $s \neq n$.

Pivot Rule 1. Replace each element a_{ij}, where $i \neq r$ and $j \neq s$, by

$$\frac{a_{rs}a_{ij} - a_{rj}a_{is}}{a_{rs}} = a_{ij} - \frac{a_{rj}a_{is}}{a_{rs}}$$

Pivot Rule 2. Replace each element a_{rj} of the r th row by a_{rj}/a_{rs}.

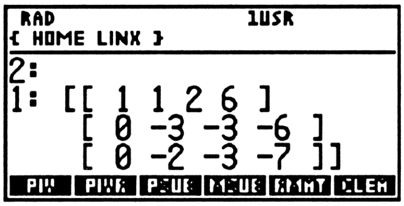
Pivot Rule 3. Replace each element a_{is}, $i \neq r$, of the s th column by 0.

It is possible to give pivot programs half the size of PIV, programs relying upon matrix operations available on the HP 48. We have chosen to give a longer version of PIV for two reasons. First, we think the longer version is easier to understand, not being based upon more advanced ideas. Secondly, we may convert this program to an exact arithmetic program with only six changes (and several auxiliary programs). We do this after an example.

EXAMPLE 2. Use PIV to solve the system given in Example 1.

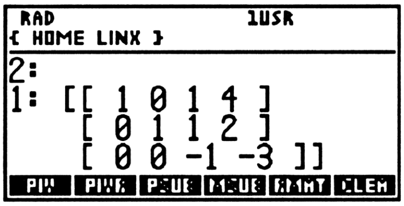
SOLUTION. In what follows we have not reminded you to press the SPC key between successive matrix elements. We do this to save space.

1 1 2 6 2 −1
1 6 1 −1 −1 −1
ENTER
{3 4} ENTER →ARR
{1 1} ENTER PIV



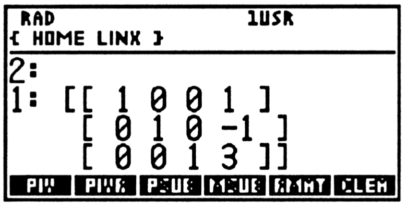
```
RAD                 1USR
{ HOME LINX }
2:
1: [[ 1 1 2 6 ]
    [ 0 -3 -3 -6 ]
    [ 0 -2 -3 -7 ]]
```

{2 2} ENTER PIV



```
RAD                 1USR
{ HOME LINX }
2:
1: [[ 1 0 1 4 ]
    [ 0 1 1 2 ]
    [ 0 0 -1 -3 ]]
```

{3 3} ENTER PIV



```
RAD                 1USR
{ HOME LINX }
2:
1: [[ 1 0 0 1 ]
    [ 0 1 0 -1 ]
    [ 0 0 1 3 ]]
```

The above results duplicate (2).

Modifying PIV to Do Exact Arithmetic

To obtain exact solutions to systems of equations with augmented matrices having rational entries, we may either solve the systems by hand or attempt to use the program PIV. If we use PIV and want exact solutions, we must be prepared to examine the output and hope to recognize decimals such as .142857142857 and .666666666667 as 1/7 and 2/3, thereby recovering the rational solution. In what follows we give programs supporting a minor revision of PIV called PIVR. The purpose of PIVR is to return exact results. We note, however, that for input or calculations including rational numbers with very large numerators or denominators, these programs may fail, for without further programming the HP 48 cannot store integers having more than 12 digits.

We use the HP 48 representation of complex numbers as a means of storing rational numbers compactly and within existing data structures. If a and b are integers and $b \neq 0$, we represent the rational number a/b by (a, b). The HP 48 has the convention that all entries of a matrix must be of the same type, either real or complex. This would mean that if a 2×2 matrix had entries 0, 2/7, 13/(−7), and −3, we would have to enter $(0, 1)$, $(2, 7)$, $(13, -7)$, and $(-3, 1)$, which is clumsy. The purpose of the first program, RMAT, is to allow mixed entries on the stack. Upon running RMAT, the entries are put into a standard format. To enter the given 2×2 matrix, for example, we may put 0, $(2, 7)$, $(13, -7)$, and −3 on the stack. Next we tell RMAT the size of the matrix by entering { 2 2 }. Upon pressing RMAT, 0 becomes $(0, 1)$, $(2, 7)$ becomes $(2, 7)$, $(13, -7)$ becomes $(-13, 7)$, and −3 becomes $(-3, 1)$. Entries of the form $(3, 0)$ produce a beep, give an error message, and KILL the program. The program combines these results into a matrix and returns the matrix to the stack.

In using RMAT (and other programs in this package), we use STD mode to save space in the display. Also, to simplify entering complex/rational numbers, we suggest going to the ↱MODES FLAG menu and checking 19. With this flag set, if we display the MTH VECTR menu we may enter (2, 3) by pressing 2 SPC 3 and then →V2.

Since RMAT is somewhat longer than the other programs in the package, we display RMAT and list the others. As we noted earlier, the inputs to RMAT are the entries of an $m \times n$ matrix A, spread on the stack in mixed format, followed by the size { m n } of A. The output of RMAT is a matrix of complex/rational numbers.

RMAT

Inputs:Mixed format a_{ij}; { m n } | *Output: A, formatted*

```
«  DUP  →  S                      Duplicate { m n } and store as S
«  OBJ→  DROP  *  →  K            Calculate number of entries (mn)
                                     to be processed; store as K
«  1  K  START                     Set up main loop
IF  DUP  TYPE  0  ==               If stack entry is a real number x,
THEN  1  R→ C                        then convert it to (x, 1)
ELSE  IF  DUP  IM  0  ==           If stack entry is (x, 0),
THEN  1024  .5  BEEP  CLLCD          then beep and prepare message;
"(X,0) not allowed!"  4  DISP        display message in level 4
2  WAIT  KILL                      Wait 2 seconds & kill program
ELSE  IF  DUP  IM  0  <            If stack entry is (x, y), where
THEN  NEG                            y < 0, then convert it to (−x, −y)
END  END  END                      End three IF statements
K  ROLLD                           Roll down mn entries
NEXT                               Repeat main loop
S  →ARRY                           Convert stack to array
»  »  »

                                     Checksum: # 31021d   Bytes: 225
```

To modify PIV so that it does exact arithmetic, we replace $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, and $\boxed{\div}$ by programs PLUS, SBTR, TMES, and DVD for adding, subtracting, multiplying, and dividing fractions. These four programs call two utility programs GCD (greatest common divisor) and CNCL (cancel). The purpose of GCD and CNCL is to reduce fractions to their simplest form.

To use GCD, two nonzero integers are put on the stack. GCD returns the largest positive integer dividing both integers without remainder. The program is a minor modification of William C. Wickes GCD program, given in his *HP 48 Insights, Part I.*

```
       «  WHILE  DUP2  MOD  DUP  0  ≠                       GCD
REPEAT  ROT  DROP  END  ROT  DROP2  ABS  »
```

To use CNCL, two nonzero integers are put on the stack. CNCL removes any common factors they may have and returns a rational number to the stack in the form (a, b). CNCL calls GCD.

```
«  DUP2  GCD  DUP  4  ROLLD  /  3                           CNCL
   ROLLD  SWAP  /  SWAP  R→C  »
```

Each of PLUS, SBTR, TMES, and DVD takes two complex/rational numbers from the stack and returns one complex/rational number to the stack.

```
≪ → U V ≪ U RE V IM ∗ V RE U IM              PLUS
   ∗ + U IM V IM ∗ CNCL ≫ ≫
```

```
≪ C→R SWAP NEG SWAP R→C PLUS ≫               SBTR
```

```
≪ → U V ≪ U RE V RE                          TMES
  ∗ U IM V IM ∗ CNCL ≫ ≫
```

```
≪ C→R SWAP R→C TMES C→R DUP IF 0 <           DVD
   THEN R→C NEG ELSE R→C END ≫
```

The necessary modifications to PIV are as follows. We must replace ∗ by TMES, / by DVD, − by SBTR, 0 by $(0, 1)$, and 1 by $(1, 1)$. There are six modifications in all. Referring to the program PIV given above, there are three changes on line 10 (∗, /, −), one on line 16 (/), one on line 17 (0), and one on line 18 (1). If you choose to make these modifications and want to keep both PIV and, say, PIVR, then use RCL to get a copy of PIV on the stack, use EDIT to make the modifications, and then store the result as PIVR.

EXAMPLE 3. Try PIVR on the small system

$$\begin{aligned}(2/3)x + (1/5)y &= 1/7\\ (-4/3)x + (9/5)y &= 2/3\end{aligned}$$

SOLUTION. After putting the elements of the augmented matrix on the stack, enter the size of this matrix, and then run RMAT. Use PIVR to pivot on, say, first the {2 1} position and then the {1 2} position.

```
2  SPC  3  →V2
Similarly, enter  1/5,  1/7,
  −4/3,  9/5,  and  2/3
{ 2 3 }  ENTER  RMAT
{ 2 1 }  ENTER  PIVR
{ 1 2 }  ENTER  PIVR
```

The solution can be read from the final output of PIVR. We find $x = 13/154$ and $y = 100/231$. To view the entire matrix, press ▼. Use the cursor keys to highlight various elements in the matrix. The highlighted element can be viewed in the command line.

EXAMPLE 4. Solve the system with augmented matrix

$$\begin{bmatrix} -9 & 9 & -96 & 32 & 174 \\ 0 & 1 & -25 & 25 & 249 \\ 1 & 1 & 20 & 20 & 280 \\ 18 & 9 & 51 & 17 & 315 \end{bmatrix}$$

SOLUTION. Put all of the entries of the augmented matrix on the command line, separating them by SPC, and then press ENTER. Key in { 4 5 } and press ENTER. Use RMAT and then PIVR. Try pivots {1 1}, {2 2}, {3 3}, and then {4 4}. The result should agree with

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 12 \end{bmatrix}$$

Exercises 6.3

In the following problems use either PIV or PIVR.

A.1 Find the solution to the system

$$\begin{aligned} 5x - 2y + z &= -9 \\ x + 3y - 4z &= 0 \\ 3x + y + z &= 1 \end{aligned}$$

A.2 Find the solution to the system

$$\begin{aligned} 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 + 4x_2 + 3x_3 &= 1 \\ -x_1 + 3x_2 + 2x_3 &= -5 \end{aligned}$$

A.3 Find the solution to the system

$$\{\ x - y + 2z = 2,\ 2x - y - 3z = 15,\ x + z = 7\ \}$$

A.4 Find the solution to the system

$$\{\ x_1 + 2x_2 - 3x_3 = -17,\ 2x_1 - x_2 = 6,\ 5x_1 + 3x_2 + x_3 = 9\}$$

A.5 Use PIV in finding the solution to the system with augmented matrix

$$\begin{bmatrix} 1.21 & 2.30 & -5.17 & 6.32 \\ -2.87 & -3.59 & 4.91 & 5.82 \\ 1.53 & 2.33 & -5.25 & 6.01 \end{bmatrix}$$

B.1 Describe the full solution set of the system

$$\begin{aligned} x + y + z &= 2 \\ x - y - 2z &= 3 \end{aligned}$$

B.2 Describe the full solution set of the system

$$\{\ 2x - y + z - w = 0,\ x - z + w = 1,\ 3x + y + 2z - 2w = -1\ \}$$

B.3 Describe the full solution set of the system with augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 1 & -2 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

B.4 Show that the system

$$\{\ -3x_1 + x_2 + 4x_3 = 1,\ x_1 + x_2 + x_3 = -1,$$
$$-2x_1 + x_3 = -1,\ x_1 + x_2 - 2x_3 = 0\ \}$$

has no solution.

B.5 Describe the full solution set of the system

$$\{\ x - y - z + w = -2,\ x + y - z + w = 1,\ x + 4y - z - w = 5\ \}$$

B.6 Find the full solution set of the system with augmented matrix

$$\begin{bmatrix} 2 & 1 & -3 & 4 & 0 & 2 \\ 1 & -2 & 1 & -1 & 3 & 10 \\ 3 & 9 & -12 & 15 & -9 & -24 \end{bmatrix}$$

B.7 Determine by hand if the homogeneous system (right-hand-sides all zeros)

$$\{\ x + y + 2z = 0,\ x + 3z = 0,\ -2z = 0\ \}$$

has any solution other than $x = y = z = 0$.

B.8 Use PIV or PIVR in determining if the homogeneous system (right-hand-sides all zeros) with coefficient matrix

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 3 \\ 2 & 2 & 5 & 0 \end{bmatrix}$$

has any non-trivial (the zero solution is often called the trivial solution) solutions.

B.9 Use PIV in determining if the homogeneous system (right-hand-sides all zeros) with coefficient matrix

$$\begin{bmatrix} 0.7 & 1.3 & 2.9 & -5.1 \\ 3.1 & -2.7 & 1.6 & -4.8 \\ 4.1 & -9.3 & -5.5 & 5.7 \end{bmatrix}$$

has any non-zero solutions.

C.1 In this problem we examine the question of how to choose pivots in the context of an example. We show that it is neither always possible nor desirable to choose pivots along the diagonal of the augmented matrix. Note that the two systems of equations in (1) are very nearly identical.

$$\begin{aligned} 0x + y &= 1 \qquad & (1.0 \times 10^{-13})x + y &= 1 \\ x + y &= 2 \qquad & x + y &= 2 \end{aligned} \tag{1}$$

The solution of the system on the left is $x = 1$ and $y = 1$. Note that we cannot pivot on the $\{1\ 1\}$ position. Not only would PIV fail since a division by 0 would be attempted, it is in fact not possible to eliminate x from the second equation by adding to the second equation a multiple of the first equation. However, we may pivot on any one of the positions $\{1\ 2\}$, $\{2\ 1\}$, or $\{2\ 2\}$. For the system on the right, the exact solution (you should find it by hand calculation), is $x = 1/(1 - 1.0 \times 10^{-13}) \approx 1$ and $y = (1 - 2.0 \times 10^{-13})/(1 - 1.0 \times 10^{-13}) \approx 1$. If we use PIV and pivot first on the $\{1\ 1\}$ position and then on $\{\ 2\ 2\}$, we find the solution to be $x = 0$ and $y = 1$. What has happened? We show in (2) the results of hand calculation (on the left) and HP 48 calculation, pivoting on the $\{1\ 1\}$ position. We have denoted 1.0×10^{-13} by s.

$$\begin{matrix} 1 & 1/s & 1/s \\ 0 & (s-1)/s & (2s-1)/s \end{matrix} \qquad\qquad \begin{matrix} 1 & 1/s & 1/s \\ 0 & -1/s & -1/s \end{matrix} \tag{2}$$

When PIV calculated the entry in the $\{2\ 2\}$ position of the new matrix it first multiplied the entries in the $\{2\ 1\}$ and $\{1\ 2\}$ positions, divided the product by the pivot, and then subtracted the result from the entry in the $\{2\ 2\}$ position, that is, $1 - (1 \cdot 1)/s$. The HP 48 did the subtraction $1 - 10^{13}$ and got -10^{13} as the difference. The HP 48 is unable to calculate sums and differences to more than 12 significant digits without special programming. All calculators and computers are subject to a similar limitation. The HP 48 got -10^{13} instead of the correct $1 - 1/s$ and $2 - 1/s$ in the $\{2\ 2\}$ and $\{2\ 3\}$ positions. These errors lead in the next step to the incorrect solution $x = 0$ and $y = 1$. To reduce the potential loss of accuracy in choosing pivots it is usually best to avoid pivoting on (relatively) small entries. One strategy is to choose a column and then pivot on the entry having the largest absolute value. Try the given system again, using first $\{2\ 1\}$ and then $\{1\ 2\}$ as pivots.

6.3 SOLVING EQUATIONS WITH THE DIVIDE KEY

The second method of solving systems of equations $AX = B$ is related to the Gauss-Jordan algorithm. The algorithm underlying the second method does not calculate the inverse of A, but factors the coefficient matrix into the Crout LU decomposition using partial pivoting. To use this algorithm, put B and A on the stack and press $\boxed{\div}$. The resulting vector X is more accurate than $X = A^{-1}B$, where A^{-1} is calculated using the $\boxed{1/x}$ key. We use Example 4 of §6.2 to illustrate the method. (We have relabeled Example 4 as Example 1.) See problem C.1 for an elementary discussion of the LU decomposition.

EXAMPLE 1. Solve the system

$$\begin{aligned} -9x + 9y - 96z + 32w &= 174 \\ y - 25z + 25w &= 249 \\ x + y + 20z + 20w &= 280 \\ 18x + 9y + 51z + 17w &= 315 \end{aligned} \tag{1}$$

SOLUTION. Equation (1) is equivalent to the matrix equation $AX = B$, where

$$A = \begin{bmatrix} -9 & 9 & -96 & 32 \\ 0 & 1 & -25 & 25 \\ 1 & 1 & 20 & 20 \\ 18 & 9 & 51 & 17 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 174 \\ 249 \\ 280 \\ 315 \end{bmatrix}$$

We wish to solve for X. As a memory aid, the solution of $ax = b$ can be written as $x = b/a$. The arithmetic b/a would be done on the stack by entering b first. To solve $AX = B$ we enter B first, followed by A.

```
[ ]  174  SPC  249  SPC
280  SPC  315  ENTER
-9  SPC  9  SPC
-96  SPC  32  SPC
0  SPC  (and so on)
51  SPC  17  ENTER
{4 4 }  →ARR  ÷
```

To view the solution vector X press ▼. Use the cursor keys to look at the entries of X. We find the solution $X = [\,1\ -1\ 2\ 12\,]$, with small rounding errors. Please leave X on the stack so that we may demonstrate the program CLEAN, which we give below.

To facilitate viewing such solution vectors as X in Example 1 we may use the following short program.

```
« 9 RND STD »
```

Store this program as, say, CLEAN. If we run CLEAN with the result of Example 1 on the stack, the vector [1 −1 2 12] is returned. CLEAN "removes" small rounding errors. It would be unwise to trust CLEAN too far. In any case, the results of CLEAN should be checked in the original problem statement.

Exercises 6.3

Use the divide key in solving the following problems, which were taken from Exercises 6.3. Use CLEAN if you wish.

A.1 Find the solution to the system

$$\begin{aligned} 5x - 2y + z &= -9 \\ x + 3y - 4z &= 0 \\ 3x + y + z &= 1 \end{aligned}$$

A.2 Find the solution to the system

$$\begin{aligned} 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 + 4x_2 + 3x_3 &= 1 \\ -x_1 + 3x_2 + 2x_3 &= -5 \end{aligned}$$

A.3 Find the solution to the system

$$\{\ x - y + 2z = 2,\ 2x - y - 3z = 15,\ x + z = 7\ \}$$

A.4 Find the solution to the system

$$\{\ x_1 + 2x_2 - 3x_3 = -17,\ 2x_1 - x_2 = 6,\ 5x_1 + 3x_2 + x_3 = 9\}$$

B.1 Solve the system in Example 1 by calculating $X = A^{-1}B$. Enter and store the matrices A and B. Put A on the stack and press $\boxed{1/x}$. This returns A^{-1} to the stack. Now put B on the stack and press $*$. The result is X. You may wish to use CLEAN.

C.1 Fill in the details of the following outline of an algorithm for decomposing a square matrix A into the product of two matrices L and U, where L is lower triangular (all elements above the diagonal are 0) and U is upper triangular (all elements below the diagonal are 0). We use the coefficient matrix A in Example 1 of §6.2.

If we were able to write $A = LU$ for a system $AX = B$, we could then solve for X by solving the system

$$LY = B \quad \text{for } Y \text{ and then the system} \quad UX = Y \quad \text{for } X. \tag{2}$$

This is useful in that if we must solve $AX = B$ for many different choices of B, then once L and U have been found, we may solve, for each B, the simple systems in (2) instead of forming an augmented matrix and going though the entire Gauss-Jordan algorithm.

The matrices L and U are formed by reducing A to upper triangular form by row operations and recording the multipliers used in the reduction. We show the reduction of A in (3), where we have recorded the multipliers (which are not part of the matrices shown) in square brackets.

$$\begin{array}{ccc} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -1 & -1 \end{array} \quad \longrightarrow \quad \begin{array}{cccc} & 1 & 1 & 2 \\ {[-2]} & 0 & -3 & -3 \\ {[-1]} & 0 & -2 & -3 \end{array} \quad \longrightarrow$$

$$\begin{array}{cccc} & 1 & 1 & 2 \\ & 0 & -3 & -3 \\ {[-\frac{2}{3}]} & 0 & 0 & -1 \end{array} \tag{3}$$

The matices L and U are given by

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -(-2) & 1 & 0 \\ -(-1) & -\left(-\frac{2}{3}\right) & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Verify that the determinant of A is the product of the diagonal elements of U.

Chapter 7

Integration

7.0 PREVIEW

Many applications of calculus depend upon the evaluation of integrals of the form

$$\int_a^b f(x)\,dx \tag{1}$$

If, for example, the shape and density of an object are specified by functions, its center of mass is specified by the values of several integrals of the form (1). The functions describing the shape and density may be given as formulas or as tables of values. If the function f is given as a formula and if the formula is not too complex, (1) may be evaluated by using the fundamental theorem of calculus. Otherwise, we must approximate (1) in some way, perhaps using numerical integration. Numerical integration is widely used in applied mathematics.

If acceleration data for an object are given, either by formula or as a table of values, one integration will give the velocity of that object at any time t and a second integration will give its coordinate position. The object can be moving in a line, in a plane, or in space. Suppose, for example, a bead is sliding on a straight wire and experiences acceleration $a(t)$ at any time t. Suppose further that at time $t = 0$ the coordinate position of the bead is x_0 and its velocity is v_0. From this information we may calculate the velocity $v(t)$ and position $x(t)$ at any time t. These are given by

$$v(t) = \int_0^t a(\tau)\,d\tau + v_0, \quad \text{and} \quad x(t) = \int_0^t v(\tau)\,d\tau + x_0 \tag{2}$$

Definite integrals are also used in finding areas, surface areas, lengths of curves, and volumes. In statistics, integrals are used to compute probabilities, expected values (means), and standard deviations.

We begin with two simple numerical integration rules. They serve as a review of the definition of the definite integral using upper and lower sums and are part of a numerical integration package, to be started in §7.1 and completed in §7.4. We give a brief introduction to symbolic and numerical integration on the HP 48 in §7.2. Applications of the integral to areas, arc lengths, surface areas, and volumes are given in §7.3. In §7.4 we give programs for the midpoint, trapezoid, and Simpson's rules. These numerical integration algorithms provide a good background for junior- and senior-level courses in numerical analysis. Improper integrals are discussed in examples and exercises. In §7.5 we give programs to relieve some of the labor associated with "partial fraction calculations." These depend upon Chapter 6.

7.1 LOWER AND UPPER SUMS

The definite integral (1) of a function f on the interval $[a, b]$ is often defined in terms of lower and upper sums, that is, in terms of sums of the areas of inscribed and circumscribed rectangles. Although this approach is applicable to any bounded function f, we restrict our review to continuous functions which are either increasing or decreasing on $[a, b]$. In fact, we consider only increasing functions since everything we say about an increasing function can be said about decreasing functions, with at most a sign change. This follows from the fact that if f is decreasing, then $-f$ is increasing.

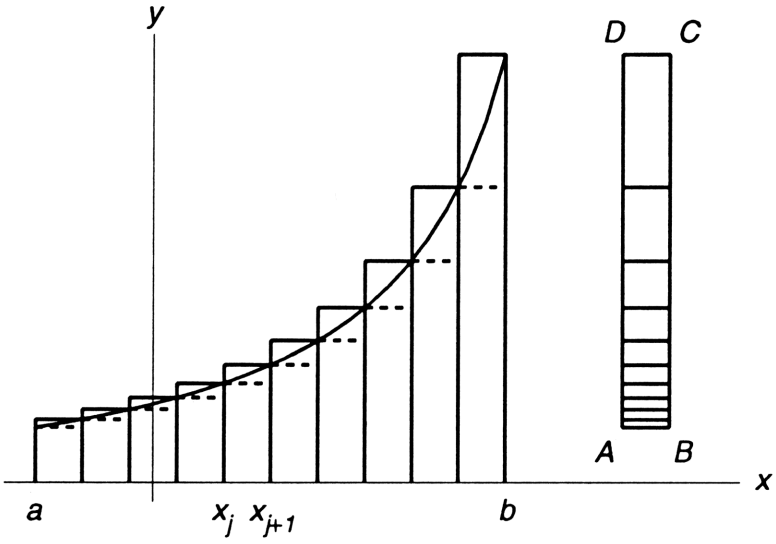


Figure 1

We show in Fig. 1 an increasing function f defined on an interval $[a, b]$. A *partition* or *subdivision* $P_n = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ of $[a, b]$ is shown. The subdivision shown is *regular* in that all subintervals $[x_j, x_{j+1}]$ have equal length h, where

$$h = x_{j+1} - x_j = (b - a)/n, \qquad j = 0, 1, \ldots, n - 1 \tag{3}$$

The common length h is called the *stepsize*.

Having chosen P_n, the lower sum L_n is the sum of the areas of the inscribed rectangles (shown in Fig. 1 with dotted tops), while the upper sum U_n is the sum of the areas of the circumscribed rectangles. A careful look at Fig. 1 shows that

$$L_n = h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)] = h\sum_{j=0}^{n-1} f(a+jh) \qquad (4)$$

and

$$U_n = h[f(a+h) + f(a+2h) + \cdots + f(a+nh)] = h\sum_{j=0}^{n-1} f(a+(j+1)h) \qquad (5)$$

The thin rectangle floating on the right side of Fig. 1 is the difference between U_n and L_n. In each subinterval $[x_j, x_{j+1}]$ the difference between the circumscribed and inscribed rectangles is the small rectangle with dotted bottom, at the top of the column above the subinterval. If each of these small rectangles is translated to the right and vertically aligned, the result is rectangle $ABCD$. Its area is $h[f(b) - f(a)]$. We use this result in (6), below.

The number $\int_a^b f(x)\,dx$ is between L_n and U_n and may be approximated by their average $A_n = (L_n + U_n)/2$.

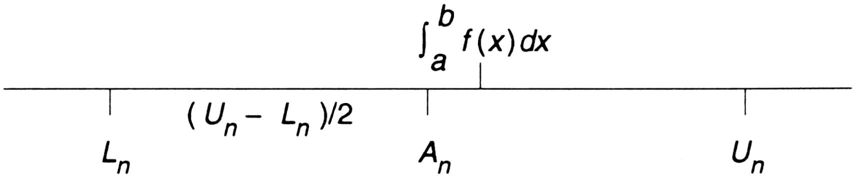


Figure 2

It is easy to see from Fig. 2 that A_n is within a distance of $(U_n - L_n)/2$ from $\int_a^b f(x)\,dx$, that is

$$\left| \int_a^b f(x)\,dx - A_n \right| \le (U_n - L_n)/2 = \frac{h}{2}[f(b) - f(a)] \qquad (6)$$

For suppose the numbers L_n and U_n mark the location of towns on a straight highway. Suppose also that your pet dog Barker is lost, but is known to be on the highway, somewhere between the towns. Barker can not be further from the halfway point than half the distance between the towns. This should make (6) "easy to see." The equality in (6) follows from the comments made earlier about rectangle $ABCD$. We note again that (6) was derived on the assumption that f is increasing.

EXAMPLE 1. We use (4), (5), and (6) to find an approximation to the value of $\int_0^1 \sqrt{x}\,dx$. We also bound the difference between the approximation and the integral. We take $n = 10$. Before calculating L_{10} and U_{10} we note from the equality in (6) that if we calculate L_{10} first, the value of U_{10} may be calculated with very little extra labor. We have

$$U_{10} = L_{10} + h[f(1) - f(0)] = L_{10} + ((1-0)/10)[1-0] = L_{10} + 0.1$$

SOLUTION. From (4) we have

$$L_{10} = h\sum_{j=0}^{10-1} \sqrt{0+jh} = 0.1(\sqrt{0} + \sqrt{0.1} + \cdots + \sqrt{0.9})$$

$$= 0.610509341706$$

and

$$U_{10} = L_{10} + 0.1 = 0.610509341706 + 0.1 = 0.710509341706$$

From these results we have

$$A_{10} = (L_{10} + U_{10})/2 = 0.660509341705$$

From (6) we may calculate how far off A_{10} can be. We have

$$\left|\int_0^1 \sqrt{x}\,dx - A_{10}\right| \leq 0.1[1-0]/2 = 0.05$$

For this integral, whose value is 2/3, the error bound 0.05 is substantially larger than the actual error.

Example 1 gives a hint of the amount of work required to obtain approximate values of integrals. If we must find the value of an integral $\int_a^b f(x)\,dx$ for which there is no known antiderivative F for the integrand f, we must use some kind of numerical approximation. In the remainder of this section we give programs for two of the simplest algorithms. In §7.4 we supplement these with other programs to give a numerical integration package. The package was first written by Professor Tom Tucker of Colgate University.

Each of the sums L_n and U_n has the same basic form. For each there is a starting value s of x. We calculate $f(s)$, increase s by the stepsize h, calculate $f(s+h)$, add it to $f(s)$, increase $s+h$ by $h, \ldots,$ continuing until we have calculated $f(s) + f(s+h) + \cdots + f(s + (n-1)h)$. We multiply the completed sum by h. For L_n (recall that for the moment we have restricted ourselves to increasing functions), $s = x_0 = a$. For U_n, $s = x_1 = a + h$.

SUM

Input: s	*Output:* $h\sum_{j=1}^{n} f(s+(j-1)h)$
`« 0 1 N`	Initialize stack: s, $\Sigma(=0)$; set up loop from 1 to n
`START OVER`	Stack → s, Σ, s
`F +`	Stack → $s, \Sigma' = \Sigma + f(s)$
`SWAP H +`	Stack → $\Sigma', s' = s + h$
`SWAP NEXT`	Stack → s', Σ'; repeat or end loop
`SWAP DROP H * »`	Stack → $h\Sigma$
	Checksum: #31955d Bytes: 65.5

The first program is called SUM. Its purpose is to take s from the stack and calculate

$$h[f(s) + f(s+h) + \cdots + f(s+(n-1)h)] = h\sum_{j=1}^{n} f(s+(j-1)h)$$

The program is written under the assumption that f, n, and h are stored as F, N, and H in the same directory as SUM. It assumes that f is written in program style and relies upon stack arithmetic for speed.

As an alternative to SUM, we give a program using the built-in Σ function. This program takes s from the stack and uses the stored values of n and h. The heart of the program is the algebraic expression `'Σ(J=1,N,F(S+(J-1)*H))'`, which if put on the stack and evaluated, would give $\sum_{j=1}^{n} f(s+(j-1)h)$. The program is

```
« → S « 'Σ(J=1,N,F(S+(J−1)*H))' EVAL H * » »
```

This program though simpler is slower. Please store one of these programs under the name SUM.

In the next several paragraphs we give two utility programs and two programs using SUM. All of this, together with SUM, should be put into a separate directory, perhaps called INTX. Later, we will suggest a convenient order for listing the programs on the INTX menu.

Among the purposes of the integration package is to calculate approximations to integrals both rapidly and conveniently, using easily understood algorithms For this it is useful to standardize variables and have a convenient way of storing them. We use the variables f, a, and b (F, A, and B) to specify the integral $\int_a^b f(x)\,dx$. Use DEF to define and store f as F.

In calculating an approximation to $\int_a^b f(x)\,dx$ we use the variables n and h (N and H). Since $h = (b-a)/n$, it need not be entered.

We use the program ABST to store a and b. The program NSTH stores n and calculates h. These programs are very straightforward.

```
« 'B' STO 'A' STO »                ABST
```

and

```
« 'N' STO B A − N / 'H' STO »      NSTH
```

The objects to be stored are entered in the order listed in the program name. For ABST, we put a on the stack first, followed by b, and then press $\boxed{\text{ABST}}$. The second program takes n from the stack, stores it, and then calculates and stores h. The program NSTH should be executed after ABST since it uses a and b in calculating h.

The INTX menu now contains SUM and the two utility programs ABST and NSTH. In Example 2 we use them to compute the lower sum L_n and the upper sum U_n for the integral $\int_a^b f(x)\,dx$ of an increasing function. These were defined in (4) and (5). We worked through a specific case in Example 1. For an increasing function, the lower sum starts with f evaluated at a, which is the left boundary of the first subinterval, and continues evaluating f at left boundaries. The upper sum starts with f evaluated at $a+h$, the right boundary of the first subinterval, and continues with the right boundaries. The programs for L_n and U_n are wonderfully simple.

```
« A SUM »                          LRECT
« A H + SUM »                      RRECT
```

Store these on INTX as LRECT and RRECT (for "left rectangle" and "right rectangle").

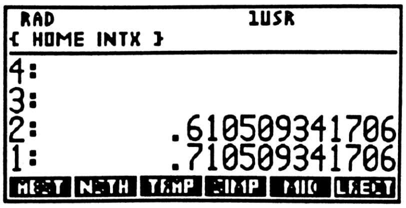
EXAMPLE 2. In Example 1 we calculated

$$L_{10} = 0.610509341706 \qquad \text{and} \qquad U_{10} = 0.710509341706$$

for the integral $\int_0^1 \sqrt{x}\,dx$. Using LRECT and RRECT, repeat the calculation of L_{10} and U_{10}.

SOLUTION.

'F(X)= $\sqrt{}$X'
DEF
0 SPC 1 ABST
10 NSTH
LRECT RRECT



```
RAD                1USR
{ HOME INTX }
4:
3:
2:       .610509341706
1:       .710509341706
```

The lower sum L_{10} and upper sum U_{10} are on the stack. We used STD mode.

If we wish to change the value of n, we key in, say, 50 and press $\boxed{\text{NSTH}}$. Upon running LRECT and RRECT again you should find (after a longer wait than for $n = 10$)

$$L_{50} = .656095342214 \quad \text{and} \quad U_{50} = .676095342214$$

We may calculate A_{50} by pressing the $\boxed{+}$, 2, and $\boxed{\div}$ keys. We obtain

$$A_{50} = (L_{50} + U_{50})/2 = .666095342215$$

To measure how well A_{50} approximates $\int_0^1 \sqrt{x}\,dx$ we use (6).

$$\left| \int_0^1 \sqrt{x}\,dx - A_{50} \right| \le |U_{50} - L_{50}|/2 = .01$$

For this simple integral we can easily show that the approximation A_{50} is within 0.0006 of the value of the integral. Using this knowledge, which in general we cannot know, we note that the error bound $(U_{50} - L_{50})/2$ is very conservative.

If we wish to find A_n to within 0.001 of $\int_0^1 \sqrt{x}\,dx$, we may use the error bound in (6) to find n. We have

$$\left| \int_0^1 \sqrt{x}\,dx - A_n \right| \le (U_n - L_n)/2 = (\sqrt{1} - \sqrt{0})/(2n) < 0.001 \qquad (7)$$

Everything up to the < symbol holds because of (6). We ourselves impose the condition "< 0.001." The least integer n satisfying the inequality $1/(2n) < 0.001$ is $n = 501$. This is the value of n needed to *guarantee* that A_n is within 0.001 of the integral. This is quite a bit of calculation for modest accuracy. Later we discuss more efficient numerical integral algorithms.

Exercises 7.1

In problems A.1–A.4 use the programs ABST, NSTH, LRECT, and RRECT in calculating the approximation A_n to the integral $\int_a^b f(x)\,dx$. The value of n is given in each problem. You may assume that f is increasing on $[a, b]$.

A.1 Calculate the value of the integral $\int_2^5 \sqrt{2x-3}\,dx$ by finding an antiderivative and evaluating it at 2 and 5. Next, calculate L_5, U_5, and A_5 by hand, using your HP 48 for no more than square roots and arithmetic. Finally, calculate L_5 and U_5 using LRECT and RRECT. If you use these programs without clearing the stack in between, you may calculate A_5 with three keystrokes.

A.2 Calculate A_{10} for the integral $\int_0^1 \sin\sqrt{x}\,dx$.

A.3 Calculate A_{10} for the integral $\int_1^5 (3 - \dfrac{x+2}{x\sqrt{x+1}})\,dx$

A.4 The length of the graph of $f(x) = e^x$, $0 \le x \le 1$ is given by the integral

$$\int_0^1 \sqrt{1 + (f'(x))^2}\,dx = \int_0^1 \sqrt{1 + e^{2x}}\,dx$$

Calculate A_{10}.

In the remainder of the A problems the function f is either increasing or decreasing. If it is decreasing, we may do either of two things. Since $-f$ is increasing if f is decreasing and $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$, we may work with the function $-f$, continue to restrict ourselves to increasing functions, and remember to take the negative of our final result. A second way is to note that if f is decreasing, then L_n and U_n are obtained from RRECT and LRECT, respectively. The approximation A_n and the error bound $(U_n - L_n)/2$ are calculated as before.

A.5 The time taken for a spherical water tower to drain is given by the value of the integral

$$\frac{32}{\sqrt{64.32}} \int_{60}^{70} \frac{400 - (H-60)^2}{\sqrt{H}}\,dH$$

Calculate A_{20}.

A.6 Calculate the length of the first quadrant portion of the curve with equation $4x^2 + y^2 = 1$. The integral obtained is an improper integral for the reason that the denominator is 0 at one end of the interval. Transform the improper integral into a proper integral with the trigonometric substitution $2x = \sin\theta$. To obtain the length within 0.01 it is sufficient to calculate A_{40}. Except for special cases, these integrals—called *elliptic integrals*—cannot be calculated in terms of elementary functions. They have been studied extensively and their values tabulated. They are important in many physical problems.

A.7 Cheney and Kincaid, in *Numerical Mathematics and Computing*, introduce a chapter with the following problem. In electrical field theory it is proved that the magnetic field induced by a current flowing in a circular loop of wire has intensity

$$H(x) = \frac{4Ir}{r^2 - x^2} \int_0^{\pi/2} \sqrt{1 - \left(\frac{x}{r}\sin\theta\right)^2}\,d\theta$$

where I is the current, r the radius of the loop, and x the distance from the center to the point at which the magnetic intensity is being computed, where $0 \le x \le r$.

This integral is an elliptic integral (see problem A.6). If we wish to evaluate it we must use a numerical algorithm or look it up in a table of elliptic integrals. Show that for $I = 15.3$, $r = 120$, and $x = 84$, the value of $A_{23} \approx 1.36$. This value of n was chosen so that $|H(84) - A_{23}| < 0.01$.

A.8 The function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is called the error function. Approximate erf(1) using $n = 15$.

B.1 The function

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt \qquad -\infty < x < \infty$$

occurs in optics and is called the *sine integral*. It is understood that for $t = 0$ the integrand—call it f—has the value 1, which is the limit of $\sin t/t$ as $t \to 0$. Show that f is decreasing on $[0, \pi]$. Use (6), as exemplified in Example 2 and adapted to the decreasing case, to choose the number n so that A_n is within 0.001 of Si(1). Using this value of n, calculate A_n. For f key in

'F(T)=IFTE(T≠0,SIN(T)/T,1)'

and press DEF.

B.2 Using the ideas in problem B.1, find to within 0.001 the value of the integral

$$\int_0^1 \frac{e^x - 1}{x}\, dx$$

Look up EXPM in the Owner's Manual. Does its use make a difference here?

B.3 Using a formula in (2), compute $v(1)$ if the acceleration $a(t)$ of an object is given by $a(t) = \sqrt{1 + t^4}$ for $0 \le t \le 1$. Take $v_0 = 1.5$. Choose n so that your result is within 0.001 of $v(1)$.

B.4 Assume that f is increasing and use (4), (5), and (6) to reduce the calculation needed to find A_n by noting that if U_n is known, then L_n can be found without calculating a new sum. What happens if f is decreasing?

B.5 Justify using 0.66 ± 0.06 as an approximation to the integral in Example 1.

C.1 Verify that $n = 40$ is correct in problem A.6.

C.2 Verify that $n = 23$ is correct in problem A.7.

C.3 Verify that $n = 15$ is correct in problem A.8.

7.2 SYMBOLIC AND NUMERICAL INTEGRATION ON THE HP 48

Most integrals $\int_a^b f(x)\, dx$ for calculating the length of a curve must be done *numerically*. For such integrals it is known that no antiderivative F for f, where F is a finite combination of elementary functions, can exist and, therefore, such integrals cannot be done *symbolically*. Even for integrals which can be done symbolically, we may choose numerical integration, simply as a matter of expediency. However, for integrals occurring in a symbolic calculation or depending upon a parameter, we may wish to seek a symbolic, exact solution. In subsequent calculations such a solution can lead to significant insights or simplifications. For integrals depending upon a parameter, a symbolic solution can be evaluated by simple substitution, avoiding repeated numerical integration for different values of the parameter.

The purpose of this section is to provide a brief introduction to the built-in symbolic and numerical integration. At the present time, the kinds of functions that can be done symbolically by the HP 48 is quite limited. The HP 48 can symbolically integrate polynomials. It can symbolically integrate each of the integrals

$$\int_a^b \sin x\,dx, \qquad \int_a^b \ln x\,dx, \qquad \text{and} \qquad \int_0^x \arcsin t\,dt, \tag{1}$$

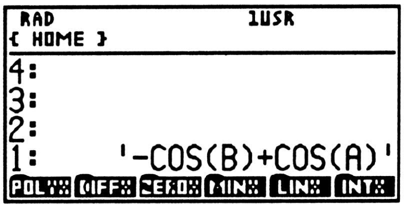
as we outline in Example 1. However, it will not integrate symbolically such elementary integrals as

$$\int_a^b x\sin(cx+d)\,dx \qquad \text{or} \qquad \int_a^b \sin^2(cx+d)\,dx$$

EXAMPLE 1. Integrate symbolically each of the integrals in (1).

SOLUTION. Do these integrations in the HOME directory. Purge any existing values of A, B, and X so that these variables will be treated symbolically. We do $\int_a^b \sin x\,dx$ first. We may enter the "data" a, b, $\sin x$, and x (the variable of integration) with the EquationWriter or using algebraic or stack entry. We start with algebraic entry.

```
'∫(A,B,SIN(X),X)'
EVAL
EVAL
```

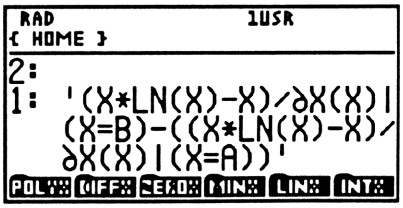


```
RAD                 1USR
{ HOME }
4:
3:
2:
1:     '-COS(B)+COS(A)'
```

After the first EVAL, the HP 48 returns a "closed-form" expression without an integral sign, signifying that it found an antiderivative for the integrand $\sin x$. The second EVAL evaluates the antiderivative at the limits a and b. This gives $\cos a - \cos b$.

We illustrate stack entry with the integral $\int_a^b \ln x\,dx$. We put the data a, b, $\ln x$, and x on the stack and then press $\boxed{\int}$. As output we show the stack just prior to the one EVAL needed for this form of entry.

```
A  ENTER  B  ENTER
'LN(X)'  ENTER
X  ENTER
∫
```

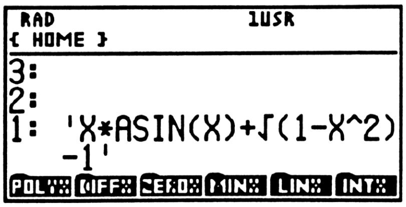


```
RAD                 1USR
{ HOME }
2:
1: '(X*LN(X)-X)/∂X(X)|
   (X=B)-((X*LN(X)-X)/
   ∂X(X)|(X=A))'
```

Press $\boxed{\text{EVAL}}$ to obtain $b \ln b - b - (a \ln a - a)$.

We may interpret the value of the integral $\int_0^x \arcsin t \, dt$ as the area beneath the graph of $y = \arcsin t$ from $t = 0$ to $t = x$. In this integral t is the integration variable and x is a constant.

0 ENTER X ENTER
'ASIN(T)' ENTER
T ENTER
$\boxed{\int}$ EVAL



```
RAD                    1USR
{ HOME }
3:
2:
1: 'X*ASIN(X)+√(1-X^2)
   -1'
```

The value of the integral is

$$A(x) = x \arcsin x + \sqrt{1 - x^2} - 1$$

We may find the area beneath the graph of $y = \arcsin t$ from $t = 0$ to $t = 0.5$ by evaluating $A(0.5)$. Before using the SOLVR to do this, make a copy of $A(x)$ by pressing ENTER. Using the SOLVR we easily find $A(0.5) \approx 0.128$.

Now return to the VAR menu, drop the SOLVR result, and purge 'X'. This leaves $A(x)$ on the stack. Next differentiate $A(x)$ with respect to x by entering 'X' and pressing $\boxed{\partial}$. After using COLCT on the ↰SYMBOLIC menu we obtain $\arcsin x$. This is what we started with. Is this a surprise? If so, review the Fundamental Theorem of Calculus.

The HP 48 can integrate polynomials symbolically, provided they are fully expanded. To integrate

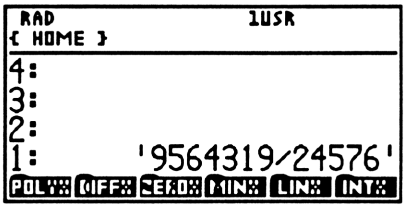
$$\int_{\frac{1}{4}}^{\frac{9}{2}} x^2 \left(\tfrac{5}{8}x^2 + \tfrac{5}{4}x + 1\right) dx, \tag{2}$$

for example, we must write the integrand as a sum of terms of the form $a_j x^j$.

EXAMPLE 2. Integrate symbolically the integral (2).

SOLUTION. We enter the integrand in expanded form. We may enter fractions in decimal form, calculating them mentally or on the stack, as required, or as an algebraic expression 'a/b'.

.25 ENTER 4.5 ENTER
'5*X^4/8+5*X^3/4+
X^2' ENTER X
ENTER $\boxed{\int}$ EVAL
↰SYMBOLIC NXT →Q



```
RAD                    1USR
{ HOME }
4:
3:
2:
1:        '9564319/24576'
```

The "best-guess" fraction returned by pressing $\boxed{\to\text{Q}}$, which is found on the ↰SYMBOLIC NXT menu, is correct.

In the next example we give a simple function for which we must use numerical integration. Neither the HP 48 nor any conceivable computer can integrate this function symbolically.

In using the HP 48 to approximate an integral $\int_a^b f(x)\,dx$, we may set an *accuracy factor* W to shorten the calculation. If we need an approximation to only 2 or 3 significant figures, we can save time by setting W appropriately. The HP 48 returns both an approximation A to the integral and a number called the "uncertainty of integration." The latter is stored on the VAR menu as IERR. The error, that is the absolute value of the difference between A and the true value of the integral, is almost certainly less than IERR. The algorithm used by the HP 48 to calculate a numerical value for $\int_a^b f(x)\,dx$ is a very good one and is extremely unlikely to fail.

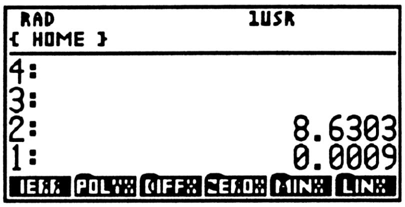
EXAMPLE 3. Calculate the arc length of the graph of $y = x^3$, $0 \le x \le 2$, to 3 significant figures.

SOLUTION. The arc length s of the graph of $y = f(x)$, $a \le x \le b$, is

$$s = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

Thus we must calculate $\int_0^2 \sqrt{1+9x^4}\,dx$. We use algebraic entry and try EVAL. The HP 48 returns our entry as a signal that it cannot do this integral symbolically. To approximate s to 3 significant figures, we try setting the accuracy factor to 0.0001 by 4 FIX and then press $\boxed{\to\text{NUM}}$.

'∫(0,2,√(1+9*X^4),X)'
ENTER EVAL
↰MODES FMT 4 FIX
VAR →NUM $\boxed{\text{IERR}}$



```
RAD                    1USR
{ HOME }
4:
3:
2:                   8.6303
1:                   0.0009
```

We may state with confidence that to 3 significant figures the arc length s is 8.63.

Exercises 7.2

A.1 Find symbolically the area beneath the graph of $f(x) = \sqrt{x+1}$, $0 \le x \le 1$.

A.2 Find symbolically the integral $\int_0^b \sqrt{x+1}\,dx$. Use →Q on your answer.

A.3 Find symbolically the area beneath the graph of $f(x) = \tan x$, $0 \le x \le 1.5$.

A.4 Find symbolically the integral $\int_0^b \tan x\,dx$.

A.5 Find symbolically the area beneath the graph of $f(x) = \arcsin x$, $0 \le x \le 1$.

A.6 Find symbolically the integral $\int_0^b \arcsin x\,dx$.

A.7 Find an antiderivative of $\tan x/\cos x$ by replacing x by t and integrating from 0 to x. Terms not involving x can be dropped.

A.8 Find an antiderivative of $\arctan x$ by replacing x by t and integrating from a to x. Terms not involving x can be dropped.

A.9 Calculate the arc length of the graph of $y = x^4$, $0 \le x \le 1$ to 3 significant figures.

A.10 Approximate the integral $\int_0^{\pi/2} \sqrt{\sin x}\, dx$ to 3 significant figures.

A.11 Find to three significant figures the area beneath the semi-circle with equation $y = \sqrt{1 - x^2}$ and above the line with equation $y = 1/3$.

B.1 Can the following integrals be done symbolically by the HP 48?

$$\int_a^b \sin(cx + d)\, dx, \qquad \int_a^b \tan(cx + d)\, dx \qquad \text{and} \qquad \int_a^b \ln(cx + d)\, dx$$

7.3 APPLICATIONS OF THE INTEGRAL

In the preceding sections, we have provided calculator/computer activities designed to strengthen both your understanding of integrals and your effectiveness in dealing with them.

What *use* are integrals? A chapter in your calculus text provides the answer. There you will find integration formulas for areas, volumes, arc length, centroids among other things. How do applications arise and how should you and your HP 48 deal with the formulas?

Generally, applications arise as follows. Suppose you are trying to solve a difficult problem (e.g., trying to find the length of a curve) and discover that you can break up the problem into a large number of smaller problems (e.g., you could break up the curve into a large number of small arcs). Suppose, also, that you can approximate solutions to the smaller problems in such a way that the sum of the approximations gives a good approximation to the answer to the main problem (e.g., you could approximate the length of each small arc with the length of a line segment and add them up to get a good approximation of the total length). Finally, suppose that the approximation gets better and better the more you subdivide the main problem. When you pass to the limit, you get an integration formula.

How should you and your HP 48 deal with integration formulas? The bulk of this section is devoted to that question.

Integration Formulas

In your book you will find several formulas like the following:

$$\text{AREA UNDER THE CURVE} = \int_a^b f(x)\, dx \tag{1}$$

$$\text{AREA BETWEEN TWO CURVES} = \int_a^b |f(x) - g(x)|\, dx \tag{2}$$

$$\text{ARC LENGTH} = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{3}$$

$$\text{ARC LENGTH} = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{4}$$

$$\text{SURFACE AREA} = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx \tag{5}$$

$$\text{VOLUME} = \int_a^b \pi f(x)^2\, dx \tag{6}$$

$$\text{CENTROID} = \left(\frac{\int_a^b x(f(x) - g(x))\, dx}{\int_a^b (f(x) - g(x))\, dx}, \frac{\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\, dx}{\int_a^b (f(x) - g(x))\, dx} \right) \tag{7}$$

What is important about formulas like these? In the first place, it is important that you understand the derivation process because such understanding will enhance your general understanding of the applicability of the integral.

Should you memorize all of the integration formulas in your book? The answer is that you shouldn't memorize any of them, because the HP 48 can do that for you. In fact, we will show you in this section how to convert such formulas into calculator keys that will make it easy for you to work many application problems. Having said that, we must add a few words of caution.

First, most people still see merit in working problems "by hand" and your instructor may well want you to do it that way. If so, you can put the programs of this section to good use as answer-checkers for problems you work by hand.

Second, even though the HP 48 usually does a marvelous job on integration problems, things can go wrong. One thing you have to worry about is *time*. The HP 48 can take an extraordinary amount of time to work some innocent looking problems and you need to be on the lookout for time-saving devices. Another thing you have to worry about is *accuracy*. Usually there's a trade-off between time and accuracy. In any case, you should always question the accuracy of the calculator and always look for ways to check the reasonableness of your answers.

Third—and most important—whether you are applying integration formulas or their program equivalents, be sure to apply them *correctly*!

On Applying Formulas Incorrectly

The mistake made by most beginners is that they think (or at least hope) that to find so-and-so, all you have to do is plug the right things into the so-and-so formula. Suppose, for example, you want to find the length of the portion of the parabola $x = y^2$ that lies between $(0, 0)$ and $(4, 2)$. If you blindly shove the data "$x = y^2$ between $(0, 0)$ and $(4, 2)$" into either formula (3) or formula (4), disaster will almost surely strike. Even if you get lucky in the shoving process, you'll find that such luck will be short-lived when you try to work even the slightest variation of the same problem. See Example 1 for correct ways to set up this problem.

On Applying Formulas Correctly

The trouble with the above formulas is that they are *incomplete*. What is needed is an accompanying explanation or a picture corresponding to each formula. For example, in (3) it is important to know that the formula refers to a curve of the type $y = f(x)$, $a \leq x \leq b$, and in (4) that the formula refers to a curve defined parametrically by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$.

One thing that helps most people is to *associate pictures with formulas*. In this section, we will help you do this by including with each program we present a corresponding picture.

> **ADVICE:** *Always be sure that the problem you are trying to solve matches the picture shown before you start pressing keys!*

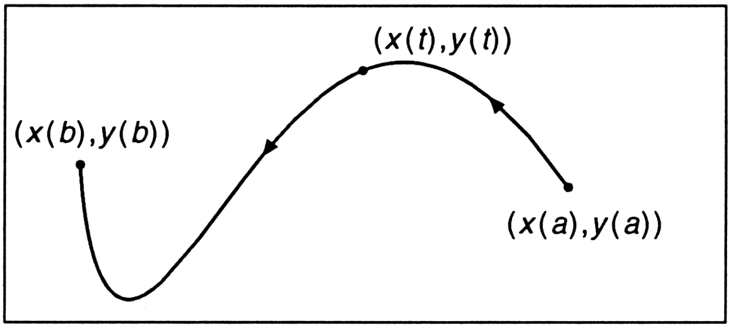
The programs in this section are based on the following simple integration procedure. To calculate $\int_a^b f(x)\,dx$, key in the following:

$$a \quad b \quad f(\mathrm{X}) \quad \mathrm{X} \quad \int \quad \boxed{\to\mathrm{NUM}}$$

This procedure combines antidifferentiation and numerical integration. Here's how it works. First, the HP 48 tries to antidifferentiate $f(\mathrm{X})$. If it succeeds, it applies the *Fundamental Theorem of Calculus* to get the exact answer. If it fails, it tries to do a numerical integration with relative accuracy of 10^{-11}. The only trouble is that sometimes it takes a long time to complete the task. You can always speed things up by asking for less accuracy, as described in the preceding section. Simply enter k FIX for the value k of your choice. The resulting error bound will be less than 10^{-k} times the value of the integral.

If you decide to enter the programs of this section in your calculator, you will probably want to open a new directory APPS for this purpose.

The program LENGTH (see box) is based on formula (4).



LENGTH

Inputs: x(T), y(T), a, b	*Output: Length of curve shown*
≪ 4 ROLL T ∂ SQ 4 ROLL T ∂ SQ + √ T ∫ →NUM ≫	Calculates formula (4) Checksum: #3748d Bytes: 64

EXAMPLE 1. Find the length of the parabola $x = y^2$ that lies between $(0, 0)$ and $(4, 2)$.

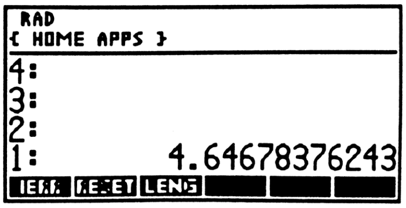
SOLUTION. Start by drawing a picture of the problem. See Fig. 3(a).

Does this picture match the model picture for LENGTH (see box above program)? The answer is no. However, we can easily reformulate it so that it does. All we have to do is parametrize the curve as follows:

$$x(t) = t^2,\ y(t) = t,\ 0 \le t \le 2$$

The picture then becomes as shown in Fig. 3(b) and we may apply LENGTH to obtain a quick and easy solution.

T SQ T 0
2 $\boxed{\text{LENGTH}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:          4.64678376243
```

Calculation time: about 13 seconds
Error bound: 5×10^{-11} (press IERR)

For an alternate but not so quick solution, see Exercise B.7.

We turn now to the problems posed and left unanswered in § 3.1. These problems all involve the polynomial $p(x) = x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x$. Store $p(x)$ under the name P:

0 −6 2 9 −3 −3 1 $\boxed{\text{POLY}}$ P $\boxed{\text{STO}}$

This function will be referred to throughout Example 2.

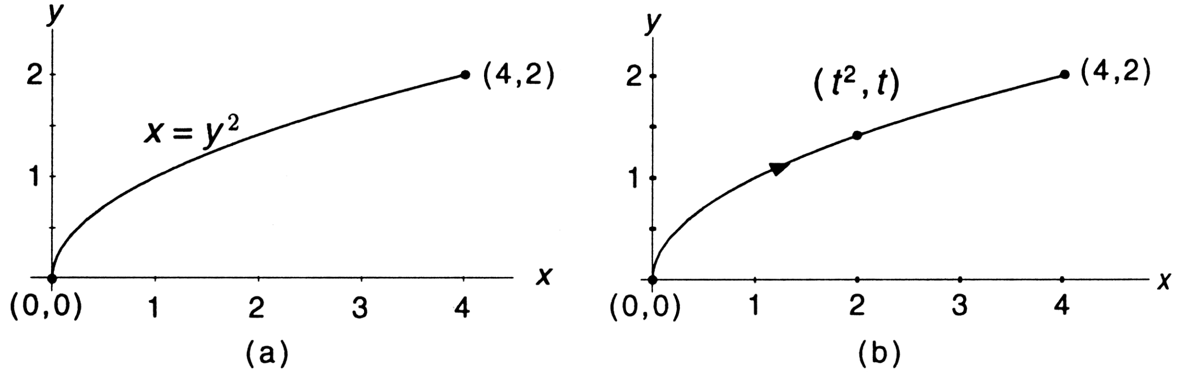


Figure 3

EXAMPLE 2A. (cf. Question 5, § 3.1) Find the area of the shaded region in Fig. 4(a).

SOLUTION. The problem clearly fits the picture model Fig. 4(b) for formula (1) with $a = -1$, $b = 0$, and $f(x) = p(x)$. (For determination of the zeros of $p(x)$, see Example 1C, § 3.1.)

Thus we have

$$\text{AREA} = \int_{-1}^{0} (x^6 - 3x^5 - 3x^4 + 9x^3 + 2x^2 - 6x)\, dx,$$

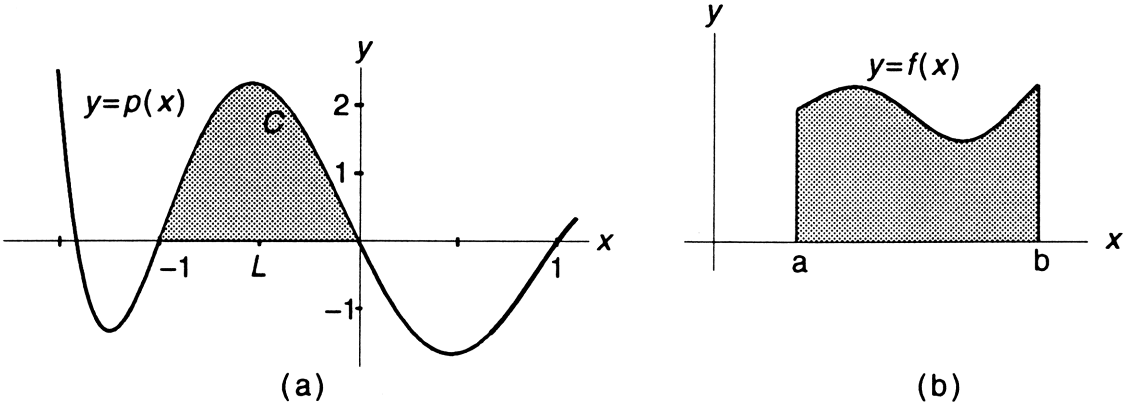
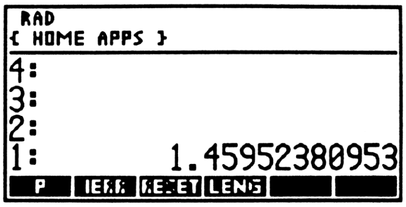


Figure 4

and evaluation by the HP 48 is routine.

−1 0 P X
∫ $\boxed{\to\text{NUM}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:          1.45952380953
```

Calculation time: about 20 seconds

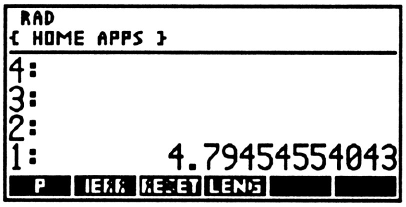
✓Point to note

Here the calculator gives no integration error because the integrand is a polynomial. Even so, there is still a small round-off error. The precise result (which you can easily check by hand) is613/420 = 1.45952380952 ⋯.

EXAMPLE 2B. (cf. Question 6, § 3.1) Find the perimeter of the shaded region in Fig. 4(a).

SOLUTION. Clearly, the perimeter is equal to $L + C = 1 + C$. See Fig. 4(a). The problem of finding C fits the model for LENGTH with $x(t) = t$, $y(t) = p(t)$, $a = -1$, and $b = 0$. So with $p(T)$ stored under the name P, we may obtain L as follows.

T P T FEVAL
−1 0
$\boxed{\text{LENGTH}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:          4.79454554043
```

Calculation time: 3 minutes
Error bound: 5×10^{-11}

Thus, the perimeter is approximately equal to 5.79454554043. Is this a reasonable answer? One way to get a rough check on the answer is to approximate it by the perimeter of the triangle with vertices $(-1,0)$, $(0,0)$, and $(-.5,2.3)$ (noting that, by Example 1D, § 3.1, the maximum occurs at $(-0.537896525, 2.31353350649)$); this gives $C \approx 2\sqrt{2.3^2+.5^2} \approx 4.71$ which shows that the answer is indeed reasonable.

EXAMPLE 2C. (cf. Question 7, § 3.1) Find the "center" of the shaded region in Fig. 4(a).

SOLUTION. This is sort of a trick problem because, there is no previously agreed upon mathematical meaning of the word "center". The reason for this is not that the idea is unimportant; rather, there are *several* meaningful interpretations of this concept. What do *you* think the "center of a region" ought to mean?

One interpretation is that of *centroid*, and that is the interpretation we will take here. A second interpretation is considered in Exercise C.4.

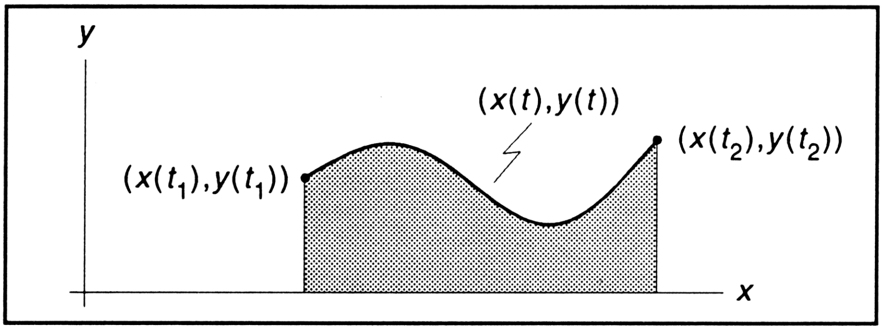
We leave it to the reader to write an HP 48 program corresponding to formula (7) and to verify that the answer to Example 2C is $(-0.516313214\cdots, 0.905807352\cdots)$. See Exercise B.9.

About Area Under a Parametrically Defined Curve

By making a change of variable $x = x(t)$, we obtain a parametric version of formula (1):

$$\text{AREA} = \int_a^b y\,dx = \int_{t_1}^{t_2} y(t)x'(t)\,dt \tag{8}$$

Here the numbers t_1 and t_2 correspond to the left- and right-hand endpoints of the x-interval *in that order.* Note that t_1 could be larger than t_2. The program AREA (see box) is based on formula (8).



AREA

Inputs: $x(T)$, $y(T)$, t_1, t_2	Output: *Area of region shown*

```
« 4 ROLL T ∂
  4 ROLL * T ∫ →NUM »
```

Calculates formula (4)

Checksum: #17421d Bytes: 47.5

Next we turn to the problems posed but left unanswered in § 3.2.

EXAMPLE 3A. (cf. Question 4, § 3.2) Compare the areas enclosed by the two loops of the curve $x = t^2$, $y = t^5 - 6t^3 + 8t$, $-2.2 \le t \le 2.2$. See Fig. 5.

SOLUTION. By symmetry (see Example 2C, § 3.2), we may concentrate on finding the areas of S_1 and S_2. To apply the model for AREA, we need to know exactly which t-values correspond to which points. Observe the "time flow". At time $t = -2.2$, the "particle" begins its journey at the point A. From point A, it moves upward to the left, crosses the x-axis for the first time at time $t = -2$; traverses the arc C_2 from $t = -2$ to $t = -\sqrt{2}$, crosses the x-axis a second time at $t = -\sqrt{2}$; etc. Finally, at time $t = 2.2$, it reaches the point B, its destination.

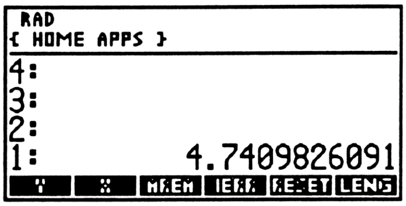
Clearly, the regions S_1 and S_2 fit the model for AREA separately. Before applying the program, store $x(t)$ and $y(t)$ under the names X and Y. We will refer to these functions throughout Example 3.

T SQ X $\boxed{\text{STO}}$

T 5 ^ 6 T 3 ^ $*$ − 8 T $*$ + Y $\boxed{\text{STO}}$

Now apply AREA.

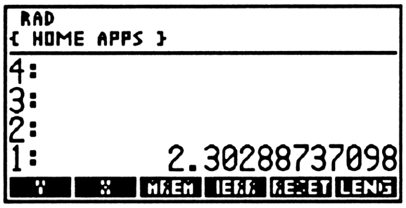
X Y 0 $\sqrt{2}$
$\boxed{\text{AREA}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:              4.7409826091
```

Calculation time: 30 seconds
Error bound: 5 × 10^{-11}

X Y $-\sqrt{2}$ -2
$\boxed{\text{AREA}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:             2.30288737098
```

Calculation time: 1 minute
Error bound: 3 × 10^{-11}

It follows that the areas of the top-shaped and football-shaped regions are 9.48 and 4.61, approximately. Since the ratio of these areas is ≈ 2.1, the area of the football-shaped region is a little less than half the area of the top-shaped region. This answers the question posed in § 3.2.

✓Point to note

Even though the integrands here are polynomials, the calculator fails to recognize that fact and resorts to numerical integration.

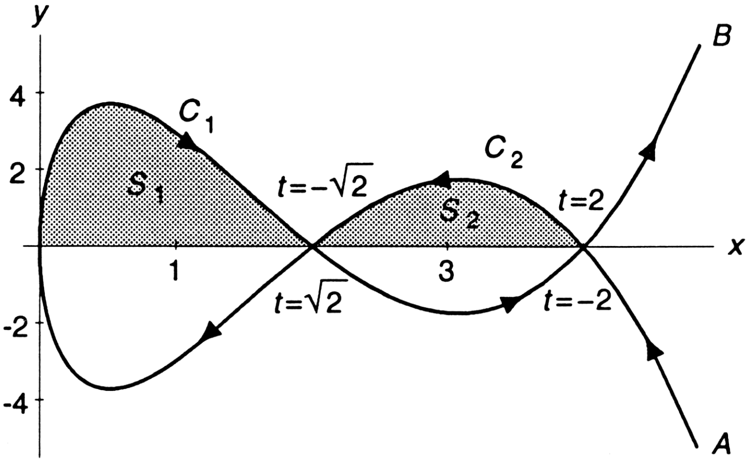
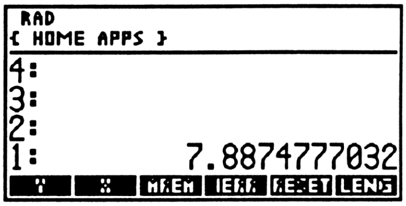


Figure 5

EXAMPLE 3B. (cf. Question 5, § 3.2) Compare the perimeters of the regions of Example 3A.

SOLUTION. By symmetry, the problem reduces to finding the lengths of the upper arcs C_1 and C_2 of S_1 and S_2, respectively. See Fig. 5. Clearly, each of these arcs fits the model for LENGTH and we can obtain the lengths of C_1 and C_2 as follows.

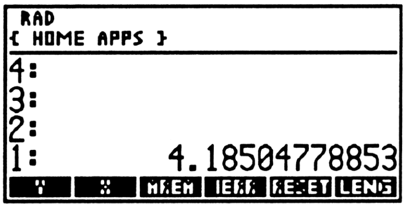
X Y 0 $\sqrt{2}$
$\boxed{\text{LENGTH}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:           7.8874777032
```

Calculation time: 1 minute, 53 seconds
Error bound: 8×10^{-11}

X Y −2 $-\sqrt{2}$
$\boxed{\text{LENGTH}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:          4.18504778853
```

Calculation time: 1 minute
Error bound: 5×10^{-11}

It follows that the perimeter of the top-shaped region ≈ 15.77, the perimeter of the football-shaped region ≈ 8.37, and the ratio of perimeters is about 1.88 : 1.

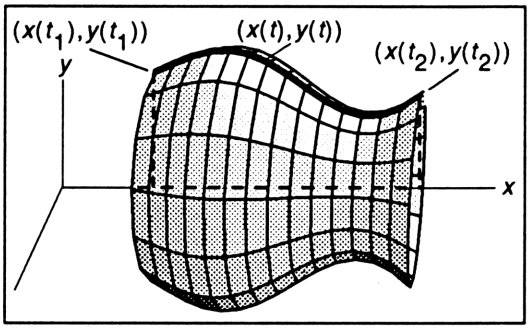
About Volume and Surface Area of a Solid of Revolution generated by a Parametrically Defined Curve

By making the change of variable $x = x(t)$, we obtain the following parametric versions of formulas (5) and (6):

$$\text{VOLUME} = \int_{t_1}^{t_2} \pi y(t)^2 x'(t)\, dt \tag{9}$$

$$\text{SURFACE AREA} = \left| \int_{t_1}^{t_2} 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt \right| \tag{10}$$

As with formula (8), it is important to understand that t_1 and t_2 are the t-values corresponding, respectively, to the left- and right-hand endpoints of the x-interval and that it may happen that t_1 is larger than t_2. Of course, the result should turn out positive. The programs VOLUME and SURFACEAREA are based on formulas (9) and (10). See boxes.



VOLUME	
Inputs: $x(T)$, $y(T)$, a, b	*Output: Volume of solid shown*

```
≪ 4 ROLL T ∂
  4 ROLL SQ * π
  * T ∫ →NUM ≫
```

Calculates formula (9)

Checksum: #53538d Bytes: 57

SURFACEAREA	
Inputs: $x(T)$, $y(T)$, a, b	*Output: Surface area of surface shown*

```
≪ 4 ROLL 4 ROLL
  → X Y ≪
  2 π * Y *
  X T ∂ SQ
  Y T ∂ SQ
  + √ *
  T ∫ →NUM ABS ≫ ≫
```

Calculates formula (10)

Checksum: #239d Bytes: 114

EXAMPLE 3C. (cf. Question 8, § 3.2) Rotating the curve of Example 3A about the x-axis gives what appears to be a 3-dimensional top and a 3-dimensional football. Find and compare their volumes and surface areas. See Fig. 6.

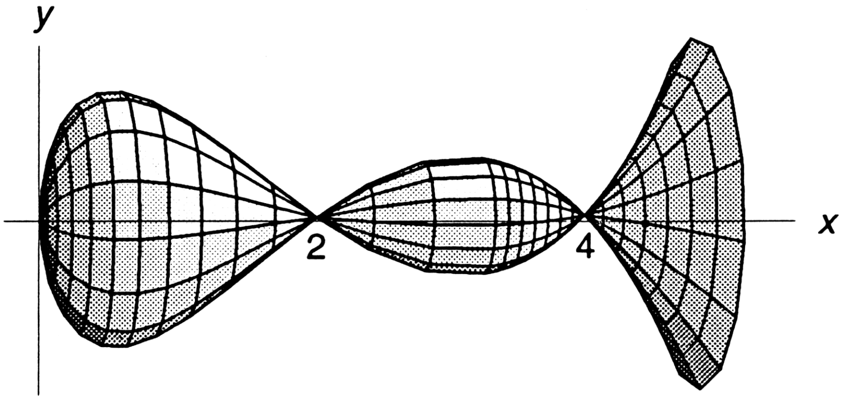
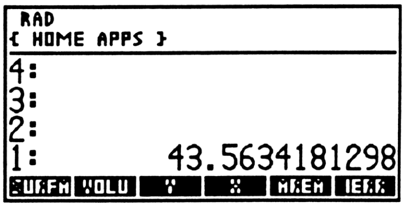


Figure 6

SOLUTION. Clearly, the model for VOLUME and SURFACE AREA applies to both 3-dimensional objects. Thus, we may obtain the volumes and surface areas as follows:

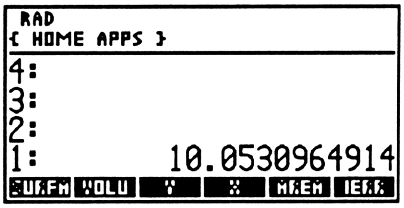
X Y 0 $\sqrt{2}$
$\boxed{\text{VOLUME}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:        43.5634181298
```

Calculation time: 28 seconds
Error bound: 5×10^{-10}

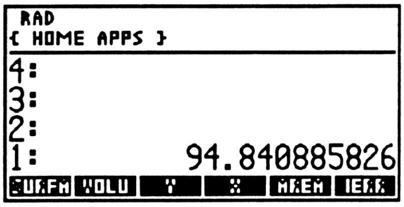
X Y $-\sqrt{2}$ -2
$\boxed{\text{VOLUME}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:        10.0530964914
```

Calculation time: 28 seconds
Error bound: 10^{-10}

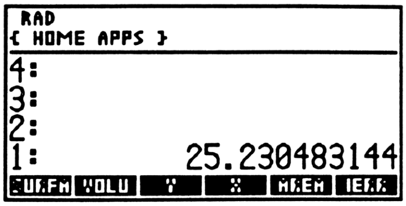
X Y 0 $\sqrt{2}$
$\boxed{\text{SURFACEAREA}}$



```
RAD
{ HOME APPS }
4:
3:
2:
1:         94.840885826
```

Calculation time: 3 minutes
Error bound: 10^{-9}

X Y $-\sqrt{2}$ -2
$\boxed{\text{SURFACEAREA}}$



```
 RAD
{ HOME APPS }
4:
3:
2:
1:          25.230483144
```

Calculation time: 1 minute, 35 seconds
Error bound: 3×10^{-10}

It follows that the volume of top-shaped region ≈ 43.5634181298, the volume of football-shaped region ≈ 10.0530964914, the surface area of the top-shaped region ≈ 94.840885826, and the surface area of the football-shaped region ≈ 25.230483144. Thus, the ratio of volumes and surface areas are ≈ 4.33333333337 and 3.75898017032, respectively.

About Limitations

As you may suspect and may wish to confirm, the above answer of 4.33333333337 is really $4\frac{1}{3}$ *exactly*. This is not obvious and it would have been nice had the calculator told us. Thus, once again, we see evidence of HP 48 power accompanied by limitations. To be fair to Hewlett-Packard Co., this is only what should be expected. No matter how good the technology gets, there will *always* be limitations.

One such limitation is the time one is willing to wait to get a result. We now give examples of integration problems that cause time difficulties for the HP 48, then we show how to combine traditional methods—specifically, "substitution" and "integration by parts"—with HP 48 power to alleviate the difficulties.

EXAMPLE 4. Evaluate

$$\int_0^1 \frac{\cos x}{\sqrt{x}}\, dx$$

to 4 decimal place accuracy.

SOLUTION. The HP 48 takes over two hours to work this problem directly, obtaining the answer 1.8090. A far more efficient way to work the problem is to first make the substitution $u = \sqrt{x}$. This will transform the given improper integral into the following proper integral (see your textbook for the definition and discussion of *improper integrals*):

$$\int_0^1 2\cos u^2\, du$$

Now put the HP 48 to work. It will take only 3 seconds to obtain the required degree of accuracy! Moreover, in about 6 seconds, we obtain the answer of 1.8090484758 with error $< 2 \times 10^{-11}$.

Why is the first integral so hard for the HP 48 and the second one so easy? One might think that the difference is explainable in terms of improper and proper integrals. Example 5 below shows that this is not the case.

EXAMPLE 5. Evaluate

$$\int_0^{2/\pi} \sin(1/x)\,dx$$

to 4 decimal place accuracy.

SOLUTION. As with Example 4, the HP 48 takes over two hours to work the problem. Note that this integral exists as a Riemann integral because the integrand is bounded and continuous everywhere except at 0. The trouble here seems to lie more with the oscillatory behavior of the function $\sin(1/x)$ than it does with the fact that $\sin(1/x)$ is undefined at the origin. Indeed, if you replace the lower limit by .001, you'll find that it still takes about an hour.

Again, we can greatly improve on things by using elementary integration techniques. We start by making the substitution $t = 1/x$. This transforms the given proper integral into the following improper integral:

$$\int_0^{2/\pi} \sin(1/x)\,dx = \int_{\pi/2}^{\infty} t^{-2}\sin t\,dt$$

Next we integrate by parts three times to obtain the following:

$$\begin{aligned}\int_0^{2/\pi} \sin(1/x)\,dx &= -2\int_{\pi/2}^{\infty} t^{-3}\cos t\,dt\\ &= \frac{16}{\pi^3} - 6\int_{\pi/2}^{\infty} t^{-4}\sin t\,dt\\ &= \frac{16}{\pi^3} + 24\int_{\pi/2}^{\infty} t^{-5}\cos t\,dt.\end{aligned}$$

Now break up the last integral into two pieces:

$$\int_{\pi/2}^{\infty} t^{-5}\cos t\,dt = I_1 + I_2, \quad \text{where}$$

$$I_1 = \int_{\pi/2}^{100} t^{-5}\cos t\,dt \quad \text{and}$$

$$I_2 = \int_{100}^{\infty} t^{-5}\cos t\,dt.$$

For the first part, the HP 48 takes less than a minute to obtain the result -0.01464 with error $< 2 \times 10^{-7}$ (using 5 FIX).

For the second part, we obtain the following approximation:

$$|I_2| = \left| \int_{100}^{\infty} t^{-5} \cos t \, dt \right|$$

$$\leq \int_{100}^{\infty} |t^{-5} \cos t| \, dt$$

$$\leq \int_{100}^{\infty} t^{-5} \, dt = \frac{1}{4} \times 10^{-8}$$

Thus,

$$\int_0^{2/\pi} \sin(1/x) \, dx \approx \frac{16}{\pi^3} + 24 \times -0.0146419 \approx 0.1646,$$

where the total error is less than

$$24 \times (2 \times 10^{-7} + .25 \times 10^{-8}) \leq 5 \times 10^{-6},$$

which gives the desired degree of accuracy.

Exercises 7.3

A.1 Find the area under the curve $y = \tan^{-1} x$ from $x = 0$ to $x = 1$.

A.2 Find the area between the curves $y = x + 3$ and $y = x^2 + 4x + 3$.

A.3 Find the length of the hyperbola $xy = 1$ from $(1, 1)$ to $(2, \frac{1}{2})$.

A.4 Find the length of one arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

A.5 Find the length of the cardioid $r = 1 + \sin\theta$.

A.6 Find the length of the Lissajous curve $x = 2\sin 2t$, $y = 2\cos 3t$, $0 \leq t \leq 2\pi$.

A.7 (a) Find the length of the curve $x = 2\sin 3t$, $y = 2\cos 2t$, $0 \leq t \leq 2\pi$

(b) Find the length of the curve $x = 2\sin 3t$, $y = 2\cos 2t$, $\pi/2 \leq t \leq 3\pi/2$

(c) Explain the connection between (a) and (b).

A.8 Find the area under one arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

A.9 Find the area enclosed by the loop of the curve $x = t^2$, $y = t^3 - 3t$.

A.10 Find the perimeter of the loop of the curve $x = t^2$, $y = t^3 - 3t$.

A.11 Find the volume of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y = x^3$ about the x-axis.

A.12 Find the surface area of the solid of revolution obtained by rotating the region bounded by the curves $y = x^2$ and $y = x^3$ about the y-axis.

A.13 Find the volume of a doughnut having inner radius 1 and outer radius 3.

A.14 Find the area of the region bounded by $y = \sin x$ and $y = \cos x$ between two consecutive intersection points. Does it matter which two you take?

A.15 Find the perimeter of the region bounded by $y = \sin x$ and $y = \cos x$ between two consecutive intersection points. Does it matter which two you take?

A.16 Find the centroid of the region bounded by $y = x^2$ and $y = x^3$.

A.17 Find the centroid of the region bounded by $y = \sin x$ and $y = \cos x$, $x \in [0, \pi/4]$.

A.18 Find the volume enclosed by the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{2} + \dfrac{z^2}{4} = 1$.

A.19 Find the surface area of the ellipsoid $\dfrac{x^2}{9} + \dfrac{y^2}{2} + \dfrac{z^2}{4} = 1$.

Exercises 1–3 refer to the parametric curve $x = 6\sin 2t$, $y = 1.5\cos 3t$, $0 \leq t \leq 2\pi$, discussed in Example 3 of § 3.2.

B.1 Find the area of (a) one of the "hearts"; (b) one of the "boomerangs"; (c) the big "football".

B.2 Find the perimeter of (a) one of the "hearts"; (b) one of the "boomerangs"; (c) the big "football".

B.3 If you spin the big "football" about the x-axis, you'll get a solid that resembles a 3-dimensional football. Find its volume and surface area.

B.4 Find the area of one of the "ears" in the bunny-shaped curve $x = \cos t - \sin 2t$, $y = \sin 3t$, $0 \leq t \leq 2\pi$.

B.5 Find the length of the double spiral $r = \pm\theta$, $0 \leq \theta \leq 6\pi$ (see Fig. 4, § 3.2).

B.6 Write an HP 48 program corresponding to formula (3); also provide a model picture to accompany the program.

B.7 Use the solution to B.6 to find (an approximation for) the length of the parabola $x = y^2$ between $(0, 0)$ and $(4, 2)$. Why do you think the HP 48 has trouble with this calculation?

B.8 Write an HP 48 program corresponding to formula (7); also provide a model picture to accompany the program.

B.9 Use the solution to B.8 to find the centroid of the region in Fig. ?.

C.1 Find numbers a and b such $\int_a^b P(x)\,dx = 1$ where $P(x)$ is your personal polynomial as defined in A.11, § 0.7.

C.2 Compare the areas in B.1 and generalize.

C.3 Find the areas and lengths of the nine loops in Example 5, § 3.2.

C.4 Consider horizontal lines that pass through the shaded region in Fig 4(a). Exactly one of these will divide the region into two equal areas. Find it. Similarly, there is exactly one vertical line that divides the region into two equal areas. Find it. Compare the intersection of these two lines with the centroid of the region (see Example 2C). Is this a reasonable interpretation of "center"? Explain.

7.4 MIDPOINT, TRAPEZOID, AND SIMPSON'S RULES

In §7.1 we discussed two "rules" for approximating $\int_a^b f(x)\,dx$, namely, the "left rectangle rule" and the "right rectangle rule." For increasing functions these two rules correspond to summing the areas of the inscribed and circumscribed rectangles. We asked then that you start entering an integration package into your calculator. The heart of the package—the program that does the actual arithmetic–was called SUM, which with ABST, NSTH, LRECT, and RRECT will be used to complete the integration package.

This package is much simpler than the HP 48's built-in numerical integration algorithm. For this reason it is easier to understand, particularly the control of error. And, as remarked earlier, it provides an introduction to numerical integration.

We begin by giving geometric or algebraic derivations of the midpoint, trapezoid, and Simpson's rules. This is followed by a brief discussion of the accuracy of each of the three rules. Finally, we give the programs MID, TRAP, and SIMP and work through several examples.

Midpoint Rule

We show in Fig. 7 a typical subinterval in a Riemann sum for $\int_a^b f(x)\,dx$. The interval $[a, b]$ has been divided into n equal subintervals with the points $a = x_0, x_1, \ldots, x_n = b$, so

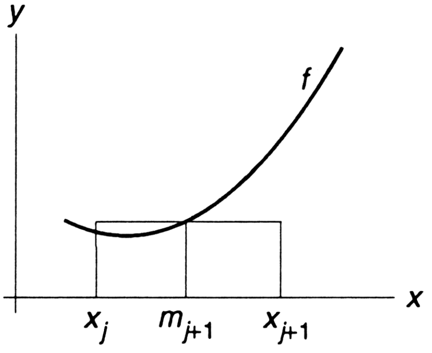


Figure 7

that each subinterval has length $(b-a)/n = h$. We choose to evaluate f at the midpoint m_{j+1} of the subinterval $[x_j, x_{j+1}]$. In terms of area, the midpoint rule approximates the area beneath the graph of f on the interval $[x_j, x_{j+1}]$ by the rectangle whose height is $f(m_{j+1})$. The Riemann sum corresponding to these choices is denoted by M_n, where

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{j=0}^{n-1} f(m_{j+1})(x_{j+1}-x_j)\\ &\approx h\sum_{j=0}^{n-1} f(m_{j+1}) = M_n \qquad (1)\end{aligned}$$

Trapezoid Rule

The trapezoid rule is based on a different approximation to the area beneath the graph of f on $[x_j, x_{j+1}]$. Instead of the rectangle with height $f(m_{j+1})$ and width h we use the trapezoid shown in Fig. 8, with height h and bases $y_j = f(x_j)$ and $y_{j+1} = f(x_{j+1})$. (Usually trapezoids are drawn so that their bases are horizontal; here the bases are perpendicular to the x-axis.) The area of this trapezoid is its height times the average of its two bases, that is, $h(y_j + y_{j+1})/2$. The corresponding Riemann sum T_n is

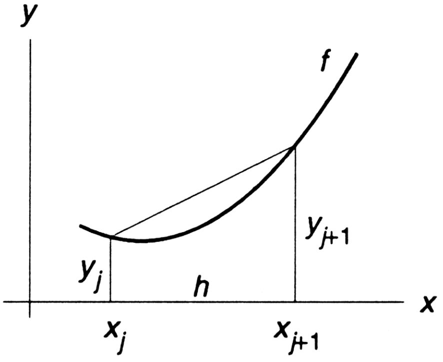


Figure 8

$$\int_a^b f(x)\,dx \approx \sum_{j=0}^{n-1} \frac{f(x_j)+f(x_{j+1})}{2}(x_{j+1}-x_j) = \frac{h}{2}\sum_{j=0}^{n-1}(y_j+y_{j+1}) = T_n \qquad (2)$$

In (3) we rewrite this sum in two ways, first in the traditional form of the trapezoid rule, and then in a form convenient for calculation with the program SUM.

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{2}\,(y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n)\\ &\approx h\left(y_0 + y_1 + \cdots + y_{n-1} + \frac{y_n - y_0}{2}\right) = T_n \qquad (3)\end{aligned}$$

Simpson's Rule

To describe the simple idea on which Simpson's rule is based we reinterpret the trapezoid rule. Instead of thinking about the trapezoid rule as an approximation to the area beneath the graph of f by several trapezoids, we may think about it as an approximation in which we replace f by a different, simpler function g and then integrate g instead of f. The simpler function g is constructed on the subintervals. The graph of g on $[x_j, x_{j+1}]$ is the straight line joining (x_j, y_j) and (x_{j+1}, y_{j+1}). We may approximate $\int_{x_j}^{x_{j+1}} f(x)\,dx$ by $\int_{x_j}^{x_{j+1}} g(x)\,dx$. The latter integral may be calculated exactly.

Simpson's rule is based on a different choice of approximating function. Instead of defining g using the line through two successive points (x_j, y_j) and (x_{j+1}, y_{j+1}), we define g using the parabola through three successive points. For the trapezoid rule, g is a polynomial of degree one; for Simpson's rule, g is a polynomial of degree two. We are fitting f with parabolas instead of straight line segments. Since three successive points come from two adjacent intervals the number n of intervals in the subdivision must be even. For this reason, as well as reasons which will become clear later, we change our use of n for Simpson's rule. We subdivide the interval $[a, b]$ into $2n$ subintervals with the subdivision $\{a = x_0, x_1, \ldots, x_{2n-1}, x_{2n} = b\}$.

We show in Fig. 9 three successive points x_{j-1}, x_j, and x_{j+1} of a subdivsion. We wish to approximate $\int_{x_{j-1}}^{x_{j+1}} f(x)\,dx$ by $\int_{x_{j-1}}^{x_{j+1}} g(x)\,dx$, where g is the quadratic function whose graph passes through the points (x_{j-1}, y_{j-1}), (x_j, y_j), and (x_{j+1}, y_{j+1}). We take advantage of the fact that we may translate f to the left or right without changing the value of the integral. We show the interval $[x_{j-1}, x_{j+1}]$ translated to $[-h, h]$ in the left sketch in Fig. 9. We have relabeled the ordinates as y_{-1}, y_0, and y_1.

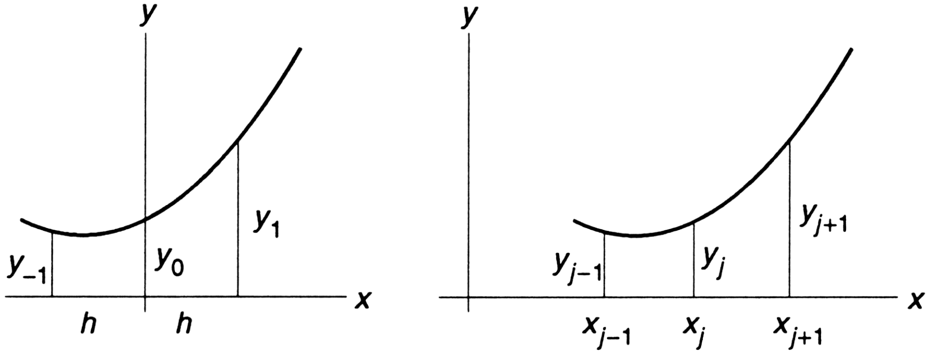


Figure 9

Let the parabola through the points $(-h, y_{-1})$, $(0, y_0)$, and (h, y_1) have the equation $y = ax^2 + bx + c$. Simpson's rule is based on the approximation

$$\int_{x_{j-1}}^{x_{j+1}} f(x)\,dx \approx \int_{-h}^{h} (ax^2 + bx + c)\,dx = \frac{2}{3}ah^3 + 2ch \tag{4}$$

The last calculation is exact. Note that $\frac{2}{3}ah^3 + 2ch$ does not depend upon b. We find that $c = y_0$ from the condition that the point $(0, y_0)$ must lie on the parabola. We obtain the equations (which we can solve for a and b)

$$y_{-1} = ah^2 - bh + c$$
$$y_1 = ah^2 + bh + c$$

from the conditions that the points $(-h, y_{-1})$ and (h, y_1) must also lie on the parabola. Adding these equations we find $y_{-1} + y_1 = 2ah^2 + 2c$. From this result and (4) we have

$$\int_{x_{j-1}}^{x_{j+1}} f(x)\,dx \approx \frac{h}{3}(2ah^2 + 2c + 4c) = \frac{h}{3}(y_{-1} + y_1 + 4y_0) = \frac{h}{3}(y_{j-1} + y_{j+1} + 4y_j) \qquad (5)$$

Adding the approximations for all n pairs of subintervals gives Simpson's rule:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots\right] \\ &\approx \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + \cdots + y_{2n-1})\right. \\ &\qquad\qquad \left. +2(y_2 + y_4 + \cdots + y_{2n-2}) + y_{2n}\right] = S_n \end{aligned} \qquad (6)$$

When we use M_n, T_n, or S_n to approximate $\int_a^b f(x)\,dx$, several kinds of errors may arise. First, the calculator values for functions or real numbers are themselves usually not exact. For most problems, these errors will not affect our results. The more serious errors are implicit in the approximations to f made in the midpoint, trapezoid, and Simpson's rules. Fortunately, there are reliable bounds for this kind of error. We state these without proof, but outline an argument for one of these bounds in problem C.2. The notations M_n, T_n, and S_n used in (7)–(9) were given in (1), (3), and (6).

$$\left|\int_a^b f(x)\,dx - M_n\right| \le \frac{M(b-a)^3}{24n^2}, \quad \text{where} \quad |f'(x)| \le M \quad \text{for all } x \text{ in } [a,b] \qquad (7)$$

$$\left|\int_a^b f(x)\,dx - T_n\right| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where} \quad |f''(x)| \le M \quad \text{for all } x \text{ in } [a,b] \qquad (8)$$

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{M(b-a)^5}{2880n^4}, \quad \text{where} \quad |f^{(4)}(x)| \le M \quad \text{for all } x \text{ in } [a,b] \qquad (9)$$

We illustrate several of these bounds in the examples just following a brief discussion of the programs MID, TRAP, and SIMP, which we use to calculate M_n, T_n, and S_n. Each program uses the program SUM discussed in §7.1. Recall that SUM assumes that a, b, f, and n are stored under the names A, B, F, and N, respectively, and that f is written in program style. SUM has one input, s, which is the "starting point," that is, the value of x at which f is first evaluated. The output of SUM is the sum $f(s)+f(s+h)+\cdots+f(s+(n-1)h)$ multiplied by h, that is, $h\sum_{j=1}^{n} f(s+(j-1)h)$.

We give the programs with brief, incomplete explanations, leaving the details to several problems.

The program MID has no inputs and outputs one number, M_n. The starting point for MID is $(x_0 + x_1)/2 = a + h/2$.

```
« A H 2 / + SUM »                                MID
```

The program TRAP has no inputs and outputs one number, T_n. We base TRAP on (3). We use SUM to calculate $h(y_0 + y_1 + \cdots + y_{n-1})$, which is the value returned by LRECT, and then modify this value to match (3).

```
« LRECT B F A F - 2 / H * + »                    TRAP
```

The program SIMP has no inputs and outputs one number, S_n. In using Simpson's rule it is important to keep in mind that the number S_n is based on $2n$ subdivisions. **The program for S_n assumes, however, that n is stored as** N. The notation S_n may help you recall this convention. SIMP uses the programs SUM, MID, and TRAP, following the formula (which we ask you to verify in a problem)

$$S_n = \frac{2M_n + T_n}{3} \tag{10}$$

The programs SUM, MID, and TRAP use the value of n stored under N. Thus, they, in effect, assume the points of the subdivision are $a = x_0, x_2, \cdots, x_{2n-2}, x_{2n}$. The starting point of MID, however, is $(x_0 + x_2)/2$, which is x_1.

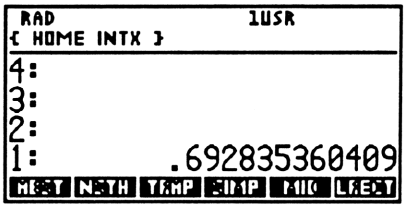
```
≪  MID  2  *  TRAP  +  3  /  ≫                    SIMP
```

Before we give an example we suggest a convenient ordering of the now completed integration package. We prefer the order ABST, NSTH, TRAP, SIMP, MID, LRECT, RRECT, SUM, N, A, B, H, F. To order the programs in the INTX directory, first store arbitrary values under the names N, A, B, H, and F. The purpose of this is to establish a place for these variables and thereby prevent them from being inserted at the head of the INTX menu every time they are stored. Next, form the list { FABST NSTH ⋯ H F } by pressing { } and then the white keys beneath the names FABST, NSTH ... on the INTX menu. Finally, press $\boxed{\text{ORDER}}$ on the ↰MEMORY DIR menu.

EXAMPLE 1. Use the fact that $\int_1^2 1/x\,dx = \ln 2$ and the midpoint formula to estimate $\ln 2$.

SOLUTION. Suppose we decide to use the midpoint formula with $n =$ 10. We calculate M_{10} using MID and then bound the error afterwards. Go to the INTX menu.

```
'F(X)=1/X'
DEF
1  SPC  2   ABST
10  NSTH
MID
```



```
RAD                    1USR
{ HOME INTX }
4:
3:
2:
1:           .692835360409
```

To bound

$$\left| \int_1^2 1/x\,dx - M_{10} \right|$$

we use (7). For this we need to find an "upper bound" M for $|f'(x)|$ on $[1,2]$. Since $f'(x) = -1/x^2$, we see that $M = 1$ is the least value of M for which $|f'(x)| \le M$ on $[1,2]$. We have, then,

$$\left| \int_1^2 \frac{1}{x}\,dx - M_{10} \right| \le \frac{M(b-a)^3}{24n^2} = \frac{1}{24 \cdot 10^2} = \frac{1}{2400} = 0.000416\cdots$$

The actual error (which, usually, we do not know) is $|\ln 2 - 0.69283\cdots| = 0.00031\cdots$.

EXAMPLE 2. Use the trapezoid rule to approximate the value of the integral

$$\int_0^2 e^{-x^2}\,dx$$

to within 0.001 of its value.

SOLUTION. We use (8) to determine the number n of subdivisions required to achieve this accuracy. We first find an upper bound on $|f''(x)|$ as x varies over $[0, 2]$. We have

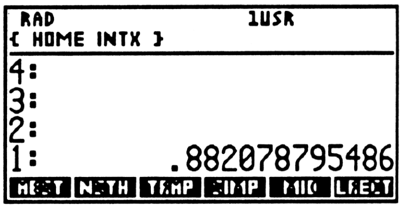
$$\begin{aligned} f(x) &= e^{-x^2} \\ f'(x) &= -2xe^{-x^2} \\ f''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} \\ &= 2e^{-x^2}\left(2x^2 - 1\right) \end{aligned}$$

From the last expression, it is not difficult to see that for all x in $[0, 2]$, $|f''(x)| \leq 2 \cdot 1 \cdot 7$. The factors of 1 and 7 in this bound are the maximum values of e^{-x^2} and $2x^2 - 1$ on $[0, 2]$. It follows that we may take $M = 14$ in (8). (This value of M can be lowered , but at a cost. We return to this point later.) Since we wish to approximate the integral within 0.001, it follows from (8) that it is sufficient to choose n so that

$$\frac{14(2-0)^3}{12n^2} \leq 0.001$$

Use the SOLVR in showing that the least integer n satisfying this inequality is $n = 97$.

```
'F(X)=EXP(-X^2)'
DEF
0  SPC  2  ABST
97  NSTH
TRAP
```



```
RAD                    1USR
{ HOME INTX }
4:
3:
2:
1:           .882078795486
```

After 5 seconds or so, TRAP returns $.88207\cdots$, which is, by our choice of n, within 0.001 of the true value of the integral.

It is known that $\int_0^2 e^{-x^2}\,dx = 0.8820813907\cdots$. We see, then, that our approximation was calculated to more accuracy than we required. Perhaps we could have gotten by with a smaller value of n, found by bounding the size of $|f''(x)|$ less crudely. Indeed, as we ask you to show in an exercise, M can be taken as small as 2. If we had used this value of M in (8) we would have found that $n = 37$ is sufficient for the accuracy 0.001. Ultimately, we must balance the time it would have taken us to obtain a smaller upper bound M against the decrease in calculation time due to a smaller value of n. In this case, the decrease in time amounts to two or three seconds. As an alternative to the method we used here to bound the size of $|f''(x)|$, we may use symbolic differentiation and graphing to obtain a value of M. We go through this as part of the next example.

EXAMPLE 3. The improper integral

$$\int_0^\infty \sqrt{x}e^{-x^2}\,dx \tag{11}$$

is "improper" since the upper limit is not a real number. This particular improper integral is *convergent* since the integrand decreases sufficiently fast as $x \to \infty$ that the area beneath its graph is finite. (It converges by the comparison test, using the facts that $\sqrt{x}e^{-x^2} \le e^{-x}$, for all $x \ge 0$, and $\int_0^\infty e^{-x}\,dx$ converges.) We wish to calculate the value of this integral to within, say, 0.001.

SOLUTION. After a change of variable (let $x = u^2$) to simplify the integrand, we split the integral into two parts by finding a number b such that the last integral (to which we refer as the "tail") in

$$\int_0^\infty \sqrt{x}e^{-x^2}\,dx = \int_0^\infty 2u^2e^{-u^4}\,du = \int_0^b 2u^2e^{-u^4}\,du + \int_b^\infty 2u^2e^{-u^4}\,du$$

is small. We use the trapezoid rule to approximate $\int_0^b 2u^2e^{-u^4}\,du$. We must choose n for the trapezoid rule and b for the tail so that

$$\left|\int_0^\infty 2u^2e^{-u^4}\,du - T_n\right|$$
$$\le \left|\int_0^b 2u^2e^{-u^4}\,du - T_n + \int_b^\infty 2u^2e^{-u^4}\,du\right| < 0.001 \tag{12}$$

We do this by choosing n and b so that

$$\left|\int_0^b 2u^2e^{-u^4}\,du - T_n\right| < 0.0005 \qquad \text{and} \qquad \int_b^\infty 2u^2e^{-u^4}\,du < 0.0005$$

To bound the tail we use the fact that $u^2e^{-u^4} < u^2e^{-u^3}$ for $u > 1$. We have

$$\int_b^\infty 2u^2e^{-u^4}\,du < \int_b^\infty 2u^2e^{-u^3}\,du = \tfrac{2}{3}e^{-b^3}$$

The inequality $\frac{2}{3}e^{-b^3} < 0.0005$ is satisfied for $b = 2.7$.

To find n so that $\left|\int_0^b 2u^2e^{-u^4}\,du - T_n\right| < 0.0005$, we use (8). To choose M, we use the HP 48 to differentiate $f(u) = 2u^2e^{-u^4}$ twice and to graph the result. We show the graph of f'' on the interval $[0, 2.7]$ in Fig. 10.

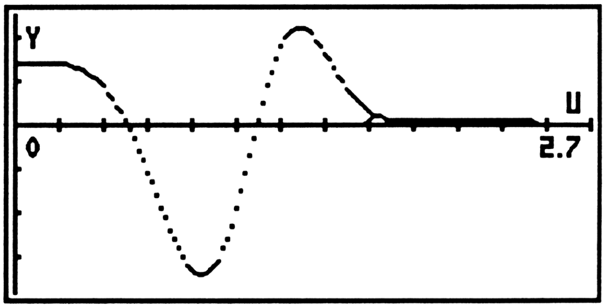


Figure 10

Positioning the cursor on the low point of the graph, we find the coordinates there to be $(0.9, -10.3)$, more or less. We take $M = 10.3$. This leads to $n = 58$. Using TRAP we find $T_{58} = 0.6127\cdots$. From our calculations and (12), we know that the integral (11) is within 0.001 of $0.6127\cdots$.

Exercises 7.4

A.1 Calculate $\ln 3 = \int_1^3 (1/x)\,dx$ using MID. Take $n = 20$. How accurate is your value compared with that returned by the HP 48 built-in LN?

A.2 Calculate $\int_0^1 e^{-x^2}\,dx$ to within 0.001 using TRAP. Follow the outline in Example 2, using an overestimate of M.

A.3 Calculate $\sin^{-1} 0.5$ to within 0.0001 by using TRAP. Use the fact that

$$\sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt, \quad |x| < 1$$

Take $n = 18$. Compare the calculated value with the well known value of $\sin^{-1} 0.5$.

A.4 Calculate π to within 0.0001 by using TRAP on the integral $\int_0^1 4/(1+x^2)\,dx$. Take $n = 82$. Compare your value with the HP 48 built-in $\boxed{\pi}$.

A.5 Recalculate to within 0.01 the length of the first quadrant portion of the curve with equation $4x^2 + y^2 = 1$ using (a) MID and (b) TRAP. See problem A.6 in Exercises 7.1. It is given that $|f'(\theta)| \le 0.5$ and $|f''(\theta)| \le 2$ for θ in $[0, \pi/2]$, where $f(\theta) = 0.5\sqrt{1 + 3\sin^2\theta}$.

A.6 Referring to problem B.1 in Exercises 6.1, use TRAP to calculate Si(1) to within 0.001. It is given that $|f''(x)| \le 1/3$ for x in $[0, 1]$.

A.7 By using symbolic integration and evaluation, show that Simpson's rule gives exact results for the integral $\int_{-1}^2 (5x^3 - x^2 + 3x + 5)\,dx$

A.8 Referring to problem A.7 in Exercises 6.1, use TRAP to calculate the value of $H(84)$ to within 0.01. Use $n = 5$.

A.9 Show by hand or HP 48 calculator that

$$\int_a^b (x-a)(x-b)\,dx = \frac{(a-b)^3}{6}$$

This result is needed in problem C.2.

A.10 The problem described here is taken from *The Aircraft Sidestep Maneuver*, from a series on applied mathematics produced at Oklahoma State University. When an aircraft is making a landing approach in bad weather, the guidance system in use may require the pilot to align his aircraft with the runway just after the aircraft emerges from fog. The required maneuver is called the sidestep manuever. Since the time in which to complete the manuever safely may be limited, close analysis of the flight path is necessary. Using elementary mechanics (discussed in detail in the booklet), it can be shown that the coordinates $x(t)$ and $y(t)$ of the aircraft at any time t (in seconds) are

$$x(t) = \int_0^t v\cos\left(\frac{g}{kv}\ln|\sec kt|\right)dt \qquad \text{and} \qquad y(t) = \int_0^t v\sin\left(\frac{g}{kv}\ln|\sec kt|\right)dt,$$

where

$$v = 202.54 \quad \text{feet/sec} \quad (\text{speed} = 120 \text{ knots})$$
$$k = 0.0349066 \quad \text{radians/sec} \quad (2°/\text{sec, rate of bank})$$
$$g = 32.2 \quad \text{feet/sec/sec} \quad (\text{acceleration of gravity})$$

Given that the absolute values of the second derivatives of the integrands are bounded by 1.2 on the interval $[0, 7.5]$, calculate $x(7.5)$ and $y(7.5)$, accurate to 0.5 feet.

B.1 In the derivation of the trapezoid rule, verify that (3) follows from (2).

B.2 In the derivation of Simpson's rule, verify that (6) follows from (5).

B.3 Show that T_n may be calculated by the shorter program TRAP2:

```
« LRECT RRECT + 2 / »
```

Give a reason why we chose instead to use TRAP. Support your answer with some empirical evidence.

B.4 Verify (10), which connects Simpson's rule to the midpoint and trapezoid rules.

B.5 Give clear, concise explanations of the programs MID, TRAP, and SIMP.

B.6 Verify the statement in Example 2 that the maximum of $|f''(x)|$ on $[0, 2]$ is 2.0.

B.7 Show that in problem A.1 the error bound (7) gives $|\ln 3 - M_{20}| \leq 0.00084$. Show that the actual error is less than 0.00037.

B.8 Verify that in problem A.3, $|f''(x)| \leq 3.1$ for x in $[0, 0.5]$. Taking $M = 3.1$, show that the error bound 0.0001 is correct with $n = 18$.

B.9 Verify that in problem A.4, $|f''(x)| \leq 8$ for x in $[0, 1]$. Taking $M = 8$, show that the error bound 0.0001 is correct with $n = 82$.

B.10 (Continuation of problem A.10) Verify the accuracy of Table 1, taken from the *The Aircraft Sidestep Manuever.* Use 0.1 for the error.

Table 1

t (sec)	x (ft)	y (ft)	t (sec)	x (ft)	y (ft)
0.0	0.0	0.0	4.0	810.0	12.0
0.5	101.3	0.0	4.5	911.1	17.1
1.0	202.5	0.2	5.0	1012.2	23.5
1.5	303.8	0.6	5.5	1113.2	31.3
2.0	405.1	1.5	6.0	1214.0	40.7
2.5	506.3	2.9	6.5	1314.7	51.7
3.0	607.6	5.1	7.0	1415.1	64.6
3.5	708.8	8.1	7.5	1515.3	79.5

B.11 In problem A.6 we stated that $|f''(x)| \leq 1/3$ for x in $[0,1]$. Show that this is true by showing that

$$f(x) = \frac{\sin x}{x} = \frac{x - (1/3!)x^3 + (1/5!)x^5 - \cdots}{x}$$
$$= 1 - (1/3!)x^2 + (1/5!)x^4 - \cdots$$

and then differentiating the last series term-by-term. Also, use Simpson's rule to recalculate Si(1) to within 0.001.

B.12 Verify the value of $n = 5$ in problem A.8.

B.13 In problem A.7 you showed that Simpson's rule gives exact results for a specific third degree polynomial. Infer from (9) that this result holds for all third degree polynomials.

B.14 Fill in the details in Example 2.

B.15 The exponential integral function is defined as

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \qquad x > 0$$

Sketch a graph of E_1 by calculating and plotting $E_1(x)$, for $x = 0.2, 0.4, \ldots, 1.2$. For graphing, finding $E_1(x)$ within 0.1 is sufficient. Begin by finding b such that the tail $\int_b^\infty e^{-t}/t\,dt < 0.05$. Next, find a single value of n for, say, the trapezoid rule, that is suitable for all of the x-values.

B.16 The gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad x > 0$$

This function generalizes the factorial function in that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x > 0 \qquad \text{and} \qquad \Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2\ldots$$

Verify that $\Gamma(3/2) \approx 0.886$ (actual value $\sqrt{\pi}/2$) by calculating the value of the improper integral. To use TRAP for this purpose, we may rewrite the integral using a substitution ($t = w^2$) and integration by parts. Verify that

$$\Gamma(3/2) = \int_0^\infty t^{1/2}e^{-t}\,dt = \int_0^\infty 2w^2e^{-w^2}\,dw = \int_0^b e^{-w^2}\,dw + \int_b^\infty e^{-w^2}\,dw$$

Take as given (see problem B.17) that for $b = 2.8$, $\int_b^\infty e^{-w^2}\,dw < 0.001/2$. Next, find n so that the trapezoid approximation T_n to the second integral is within .001/2 of its value. Finally, calculate $\Gamma(3/2)$ to within 0.001. Note that the gamma function is built-in to the HP 48. Find ! on the MTH PROB menu. For x a positive integer, ! returns $x!$. For non-integral x, ! returns $\Gamma(x+1)$. To calculate $\Gamma(3/2)$, enter 0.5 and press $\boxed{!}$.

B.17 Referring to problem B.16, verify the statement about b by noting that for $b \geq 1/2$,

$$\int_b^\infty e^{-w^2}\,dw \leq \int_b^\infty 2we^{-w^2}\,dw = e^{-b^2}$$

B.18 The normal probability function is

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

If x is a random variable having a normal distribution, then the probability $P(x_0)$ that x is less than a given value x_0 is given by

$$P(x_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-x^2/2}\, dx$$

Use Simpson's rule in calculating a table of values of $P(x_0)$, accurate in the second decimal, for $x_0 = -2.0, -1.8, \ldots, 0.0, 0.2, \ldots, 2.0$. Use ∂ and PLOT in choosing M on a suitable interval. Use symmetry and the fact that $\int_{-\infty}^{\infty} Z(x)\, dx = 1$.

C.1 Fill in the details of the following argument for the error bound (8) for the trapezoid rule. We use the phrase "a polynomial p interpolates a function f on a subdivision $\pi = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$" if p agrees with f at the points of π, that is, $p(x_i) = f(x_i)$, for $i = 0, 1, \ldots, n$.

Suppose that p is a polynomial of degree n and p interpolates a function f on a subdivision π. Suppose further that f and its first $n+1$ derivatives are continuous. Under these circumstances, for each $x \in [a, b]$ there is a corresponding number c in (a, b) for which

$$e(x) = f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) w(x), \tag{13}$$

where $w(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.

In proving this result note that the function e measures the closeness of the interpolating polynomial p to f. The number $e(x)$ is the error we would commit in using $p(x)$ to approximate $f(x)$. Let $x \in [a, b]$ be given. If it happens that x is one of the points in π, c may be chosen at will since both sides of (13) are zero. Suppose, then, that x is not in π. Define the function F on $[a, b]$ by

$$F(t) = f(t) - p(t) - Cw(t), \qquad \text{where} \quad C = \frac{f(x) - p(x)}{w(x)}$$

It is clear that F has $n + 1$ continuous derivatives. Also,

$$F(x) = F(x_0) = F(x_1) = \cdots = F(x_n) = 0,$$

so that F is zero for at least $n + 2$ points of $[a, b]$. By Rolle's Theorem, it follows that F' must be zero for $n + 1$ points and F'' must be zero for at least n points of $[a, b]$. If we continue using Rolle's Theorem in this way, we see that the $(n + 1)$st derivative of F must be zero at least once in $[a, b]$. Let such a point be c. We have

$$0 = f^{(n+1)}(c) - p^{(n+1)}(c) - \frac{f(x) - p(x)}{w(x)} w^{(n+1)}(c)$$

Since $p^{(n+1)}(c) = 0$ and $w^{(n+1)}(c) = (n + 1)!$, we may rewrite this result as

$$e(x) = f(x) - p(x) = \frac{w(x) f^{(n+1)}(c)}{(n + 1)!} \tag{14}$$

For the trapezoid rule we use straight line approximations to f on each subinterval. We use (14) with $n = 1$. To minimize fussy notation, we first consider the subdivision $\pi = \{x_0, x_1\}$. Letting M be a number such that $|f''(x)| \leq M$ for all $x \in [a, b]$, we have from (13) that

$$\int_a^b e(x)\,dx = \int_a^b f(x)\,dx - \int_a^b p(x)\,dx = \int_a^b \frac{w(x)f^{(2)}(c)}{2}\,dx$$

Letting $T = \int_a^b p(x)\,dx$, we have with the help of the result in problem A.9

$$\left|\int_a^b f(x)\,dx - T\right| \leq \left|\frac{M}{2}\int_a^b (x-a)(x-b)\,dx\right| = \frac{M}{12}(b-a)^3$$

From this result it follows that for a subdivision $\{x_0, x_1, \ldots, x_n\}$,

$$\left|\int_a^b f(x)\,dx - \frac{h}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n)\right| \leq \frac{M(b-a)^3}{12n^2},$$

which is the result given in (8).

7.5 PARTIAL FRACTION CALCULATIONS

"Partial fractions" is the name given to one of the standard methods of integration. It is a technique for splitting a rational function—a ratio of two polynomials—into a sum of simpler rational functions. Splitting a rational function into a sum of simpler fractions is labor-intensive, particularly if it came from a practical problem. In this section we remind you of the partial fractions algorithm and use the results of Chapter 6 to help you with the associated calculations.

A typical partial fractions problem starts life in the form $R(x) = p(x)/q(x)$, where p and q are polynomials and the degree of p is less than that of q. First we must factor the denominator into a product of linear or quadratic factors, having no zeros in common. For problems arising from applications, where most polynomials do not have rational roots, the factorization step requires a significant amount of calculation. We assume here that the factorization of q has been done, leaving us with integrands of the form

$$R(x) = \frac{p(x)}{g_1(x)^{m_1}\, g_2(x)^{m_2} \cdots g_n(x)^{m_n}} \tag{1}$$

where $m_1, \ldots, m_n$ are positive integers and each of $g_1(x), \ldots, g_n(x)$ is either of the form $rx + s$ or $ax^2 + bx + c$. The quadratic factors must be *irreducible*, that is, $b^2 - 4ac < 0$, so that their zeros are complex numbers.

It is a result from algebra that fractions of the form (1) can be decomposed into a sum of $m_1 + \cdots + m_n$ terms. The first m_1 terms, which correspond to the factor $g_1(x)^{m_1}$, have the form

$$\frac{w_1(x)}{g_1(x)^1} + \frac{w_2(x)}{g_1(x)^2} + \cdots + \frac{w_{m_1}(x)}{g_1(x)^{m_1}} \tag{2}$$

If $g_1(x) = rx + s$, then the polynomials $w_1, \ldots, w_{m_1}$ are constants $a_1, \ldots a_{m_1}$; if $g_1(x) = ax^2 + bx + c$, then the polynomials $w_1, \ldots, w_{m_1}$ are polynomials of degree 1, that is, $w_1(x) =$

$b_1x + c_1, \ldots, w_{m_1} = b_{m_1}x + c_{m_1}$. The constants $a_1, \ldots, a_{m_1}$ or $b_1, \ldots, b_{m_1}, c_1, \ldots, c_{m_1}$ are "unknowns" to be determined. The remaining factors

$$g_2(x)^{m_2}, \ldots, g_n(x)^{m_n}$$

of the denominator of (1) are handled similarly.

To decompose $R(x)$ into groups of fractions of the form shown in (2), we must determine the unknown coefficients. We illustrate the usual procedures with a specific fraction. We give in (3) the fraction $R(x)$ we wish to decompose and the form the decomposition must take according to (2).

$$\frac{8x^3 - 22x^2 + 45x + 249}{(x^2 - 6x + 25)(2x - 1)^2} = \frac{b_1x + c_1}{x^2 - 6x + 25} + \frac{a_1}{2x - 1} + \frac{a_2}{(2x - 1)^2} \tag{3}$$

The quadratic factor $x^2 - 6x + 25$ is irreducible since $(-6)^2 - 4 \cdot 1 \cdot 25 = -64 < 0$.

One way of determining the unknown coefficients in (3) is to multiply both sides of (3) by the factored denominator of $R(x)$ and write the result as a polynomial in x. We show these two steps in

$$\begin{aligned} 8x^3 - 22x^2 + 45x + 249 = {} & (b_1x + c_1)(2x - 1)^2 + a_1(x^2 - 6x + 25)(2x - 1) \\ & + a_2(x^2 - 6x + 25) \end{aligned} \tag{4}$$

and

$$\begin{aligned} 8x^3 - 22x^2 + 45x + 249 = {} & (4b_1 + 2a_1)x^3 + (-4b_1 + 4c_1 - 13a_1 + a_2)x^2 \\ & + (b_1 - 4c_1 + 56a_1 - 6a_2)x + (c_1 - 25a_1 + 25a_2) \end{aligned} \tag{5}$$

Recalling that two polynomials are equal for all values of x if and only if their coefficients are the same, we may "equate coefficients" in (5) to obtain the system of equations

$$\begin{aligned} 4b_1 \qquad\quad + \ 2a_1 \qquad &= 8 \\ -4b_1 + 4c_1 - 13a_1 + \quad a_2 &= -22 \\ b_1 - 4c_1 + 56a_1 - \ 6a_2 &= 45 \\ c_1 - 25a_1 + 25a_2 &= 249 \end{aligned} \tag{6}$$

The solution of (6) is $b_1 = 1$, $c_1 = -1$, $a_1 = 2$, and $a_2 = 12$. We may now rewrite (3) as

$$\frac{8x^3 - 22x^2 + 45x + 249}{(x^2 - 6x + 25)(2x - 1)^2} = \frac{x - 1}{x^2 - 6x + 25} + \frac{2}{2x - 1} + \frac{12}{(2x - 1)^2} \tag{7}$$

We give two algorithms for solving (6) after describing a second way of determining the unknown coefficients in (3).

The first way—sketched above—depends upon doing the algebra required to rearrange (4) to get (5) and then solving the system of equations (6). The second way depends upon the condition that (4) hold for all values of x. From this we obtain equations in the unknowns b_1, c_1, a_1, and a_2 by replacing x by several numerical values. We then solve a system of equations.

For hand calculation it is best to choose values of x so that as many terms as possible in (4) are zero. For example, if we replace x by 1/2 in (4) we obtain (after some arithmetic) $267 = 89a_2/4$. This gives $a_2 = 12$. The other factor in the denominator of $R(x)$ has the complex zeros $3 \pm 4i$. These may be used in the same way as the real zero 1/2, but the arithmetic is more difficult. We return to this in problem B.1.

Rather than use complex numbers, we replace x by four distinct real numbers, choosing 1/2 as one of the numbers. Letting $x = 1/2, \ -1, \ 0, \ 1$, we obtain the equations (in which a_2 has been replaced by 12)

$$\begin{aligned} -9b_1 + 9c_1 - 96a_1 &= -210 \\ c_1 - 25a_1 &= -51 \\ b_1 + \ c_1 + 20a_1 &= 40 \end{aligned} \tag{8}$$

The solution of (8) is $b_1 = 1$, $c_1 = -1$, and $a_1 = 2$. With the value of a_2 determined above, this is the same solution we found earlier.

It is the second method—replacing x by several numerical values—that we use to generate a system of equations $AX = B$ for finding the partial fraction coefficients.

We discussed in Chapter 6 two methods for solving systems $AX = B$, namely Gauss-Jordan pivoting and the divide key $\boxed{\div}$. Depending upon your preference between these methods, we give two programs to generate the matrices needed to finish the job of solving partial fraction problems. We give PSUB for Gauss-Jordan pivoting and MSUB for the divide key. Although these programs are connected to an integration technique, they make use of PIV or PIVR. We therefore suggest putting them on the LINX menu.

We start with a rational function $R(x)$ of the form (1). We decompose $R(x)$ by rewriting it using (2). Specifically, for each factor $g_j(x)^{m_j}$ we must write out a sum of terms of the form (2). An example is given in (3). Next, multiply both sides of the decomposition by the denominator of $R(x)$, distributing it over the terms of the right side and removing any common factors. This results in an equation similar to (4). Enter the result into your HP 48. For (4) you would enter

```
'8*X^3-22*X^2+45*X+249=(B1*X+ C1)*(2*X-1)^2
  +A1*(X^2- 6*X+25)*(2*X-1)+A2*(X^2-6*X+25)'        (9)
```

This is the most tedious part, but must be done accurately if the results are to be useful. Do not key in this expression just yet. We refer to such expressions as (9) as EQ.

We put *EQ* and a list *V* of the unknowns on the stack. In this case, the list of unknowns is { B1 C1 A1 A2 }. The program PSUB takes *EQ* and *V* as input and returns the augmented matrix $[A|B]$. The program MSUB has the same input but returns the matrices A and B, separately. Each program prompts for x values. After PSUB or MSUB, we use either PIV or the $\boxed{\div}$ key to solve the system $AX = B$.

The program PSUB is the main program. MSUB calls PSUB and then uses the augmented matrix output of PSUB, splitting it into matrices A and B. The list of variables is $V = \{v_1, \ldots, v_n\}$.

We illustrate the use of PSUB with a simple partial fractions problem.

EXAMPLE 1. Find A and B in the partial fraction decomposition

$$\frac{1}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

PSUB

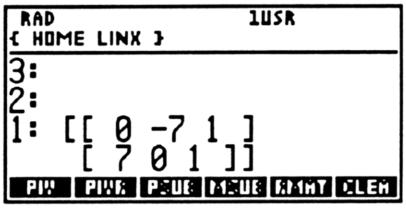
Inputs: EQ, V	*Outputs: The augmented matrix* $[A\|B]$
« DUP SIZE → EQ V N	Find n; EQ, V & N are local variables
« 1 N FOR I CLLCD	Start main loop & clear screen
"ENTER X" PROMPT	Halt for data entry
'X' STO	Store current value of x
V DUP OBJ→ 1 SWAP	Stack: $V, v_1, \ldots, v_n, 1, n$
START 0 SWAP STO NEXT	Set all v_i to 0
EQ →NUM DUP → RHS	Evaluate EQ with all $v_i = 0$ to get right-hand-side b_i; dup & store b_i
« 1 N FOR K	Set up inner loop; on i th pass
SWAP DUP K GET 1 SWAP	spread i th row of augmented
STO EQ →NUM ROT SWAP	vertically on stack;
OVER − NEG DUP 4 ROLLD	details of stack manipulations
− NEXT	are left to a problem
DROP2 RHS » NEXT	End outer loop
N DUP 1 + 2 →LIST →ARRY	Convert augmented matrix spread
V 'X' + PURGE » »	on stack to matrix form
	Checksum: # 24742d Bytes: 256.5

SOLUTION. After multiplying by $(x-2)(x+5)$, the decomposition equation becomes

$$1 = A(x+5) + B(x-2)$$

Enter this equation first and the list {A B} of variables second.

'1=A∗(X+5)+B∗(X−2)'
ENTER { A B }
ENTER PSUB
(PSUB asks: ENTER X)
5 $\boxed{+/-}$ CONT
(PSUB asks: ENTER X)
2 CONT



```
RAD                    1USR
{ HOME LINX }
3:
2:
1: [[ 0 -7 1 ]
    [ 7 0 1 ]]
```

PSUB returns one of the augmented matrices

$$\begin{bmatrix} 0 & -7 & 1 \\ 7 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 0 & 1 \\ 0 & -7 & 1 \end{bmatrix}$$

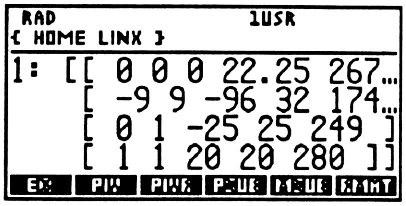
depending on the order in which −5 and 2 are entered. We do not need PIV to solve this simple system. We see that

$$\frac{1}{(x-2)(x+5)} = \frac{1/7}{x-2} - \frac{1/7}{x+5}$$

EXAMPLE 2. Find b_1, c_1, a_1, and a_2 in the partial fraction decomposition (3). The decomposition equation is given in (4) and (9).

SOLUTION. Carefully enter (9). Store it as EQ. (This is not necessary, but it saves having to re-enter (9) in case of error.)

EQ { B1 C1 A1 A2 }
ENTER PSUB
(PSUB asks: ENTER X)
.5 CONT
1 [+/−] CONT
0 CONT 1 CONT



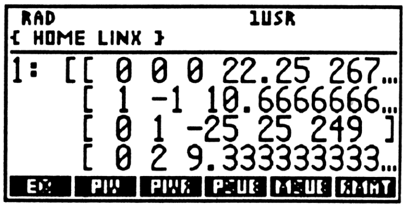
```
RAD                 1USR
{ HOME LINX }
1: [[ 0 0 0 22.25 267…
    [ -9 9 -96 32 174…
    [ 0 1 -25 25 249 ]
    [ 1 1 20 20 280 ]]
 EQ  PIV  PIVR PSUB MSUB RMAT
```

PSUB returns the augmented matrix

$$\begin{bmatrix} 0 & 0 & 0 & 22.25 & 267 \\ -9 & 9 & -96 & 32 & 174 \\ 0 & 1 & -25 & 25 & 249 \\ 1 & 1 & 20 & 20 & 280 \end{bmatrix} \tag{10}$$

We use PIV to solve the system of equations with (10) as augmented matrix. We pivot on the positions {2 1}, {4 2}, {3 3}, and {1 4}. We show the input and result of the first step. We start with (10) on the stack, as returned by PSUB.

{2 1 } ENTER
PIV



```
RAD                 1USR
{ HOME LINX }
1: [[ 0 0 0 22.25 267…
    [ 1 -1 10.6666666…
    [ 0 1 -25 25 249 ]
    [ 0 2 9.333333333…
 EQ  PIV  PIVR PSUB MSUB RMAT
```

Continue to pivot, using {4 2}, {3 3}, and {1 4} as successive pivot positions. The result should be the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 12 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1.99999\cdots \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \tag{11}$$

You may remove small rounding errors from this result by using the program CLEAN. In any case, we infer from (11) the solution $b_1 = 1$, $c_1 = -1$, $a_1 = 2$, , $a_2 = 12$.

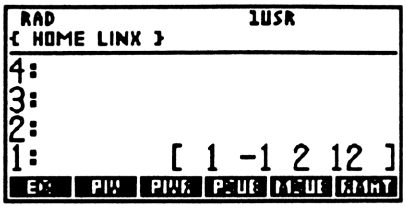
To use the [÷] key instead of PIV we use the program MSUB:

```
« PSUB TRN OBJ→ 2 GET → N « N →ARRY
  → B « { N N } →ARRY TRN B SWAP » » »
```

This program has the same input as PSUB but instead of returning the augmented matrix it returns B on level 2 and A on level 1. Key in MSUB and store it on the LINX menu. A convenient order of LINX is PIV, PIVR, PSUB, MSUB, RMAT, and CLEAN. The order of the remaining utility programs is not important.

Recall EQ to the stack and key in the variables {B1 C1 A1 A2} once more. Run MSUB, using the x values .5, −1, 0, and 1. MSUB returns matrices A and B. You may view B by pressing SWAP. After viewing and before [÷], press SWAP again.

EQ { B1 C1 A1 A2 }
ENTER MSUB
(MSUB asks: ENTER X)
(MSUB asks: ENTER X)
.5 CONT
1 $\boxed{+/-}$ CONT
0 CONT 1 CONT
$\boxed{\div}$



```
RAD                    1USR
{ HOME LINX }
4:
3:
2:
1:            [ 1 -1 2 12 ]
```

Exercises 7.5

In A.1–A.4 use PSUB followed by PIV or PIVR, or use MSUB followed by the $\boxed{\div}$ key.

A.1 Decompose the fraction

$$\frac{-26+23x}{(3+x)(-7+4x)}$$

A.2 Decompose the fraction

$$\frac{213-56x+38x^2-x^3}{(1+x)(-9+2x)(3+x^2)}$$

A.3 Decompose the fraction

$$\frac{7+29x+55x^2+21x^3+2x^4}{(5+x)^3\,(3+x+x^2)}$$

A.4 Decompose the fraction

$$\frac{7+69x+19x^2+30x^3+4x^4+3x^5}{(1+x)^2(5+x^2)^2}$$

B.1 In the decomposition of (3) we remarked then that we could have used the complex zeros $3 \pm 4i$ in generating a system of equations. We outline here one way of using such zeros. We begin with a simpler problem. Suppose we wish to decompose the rational function $(2x^2-2x-2)/((x-3)(x^2+1))$. The zeros of the denominator are 3, i, and $-i$. For entry into the HP 48, a complex number $a+ib$ is entered as $(a\ b)$ or (a, b). Enter the decomposition equation and run PSUB as usual, except enter 3, $(0,1)$, and $(0,-1)$. You should obtain the matrix on the left.

$$\begin{bmatrix} (10,0) & (0,0) & (0,0) & (10,0) \\ (0,0) & (-1,-3) & (-3,1) & (-4,-2) \\ (0,0) & (-1,3) & (-3,-1) & (-4,2) \end{bmatrix} \qquad \begin{bmatrix} (1,0) & (0,0) & (0,0) & (1,0) \\ (0,0) & (1,0) & (0,0) & (1,0) \\ (0,0) & (0,0) & (1,0) & (1,0) \end{bmatrix}$$

Run PIV as usual, pivoting on, say, the {1 1}, {2 2}, and {3 3} positions. You should obtain the matrix on the right, from which the solution $A=1$, $B=1$, and $C=1$ may be read. Returning now to (4), use PSUB with the x values .5, $3+4i$, $3-4i$, and 0. The values .5 and 0 are for the repeated factor $2x-1$. After obtaining the augmented matrix, pivot on {4 1}, {1 2}, {2 3}, and {3 4} to obtain the solution.

C.1 Explain to a friend how MSUB works and then write out your explanation.

Chapter 8

Sequences & Series

8.0 PREVIEW

Sequences and series. Two simple but rich ideas. A chapter in your textbook, bearing a similar title, will lead you through a study of these two ideas, starting with basic definitions and ending with work on power series, Maclaurin series, and Taylor series. As you page through this chapter and contemplate its study, you might get the idea that the work amounts merely to learning a lot of isolated facts and mastering certain "tests". While there's an element of truth to this, it nevertheless is the study itself that deserves the focus of your attention. As you go through the study, you will find that the material is quite different from what you have previously experienced and you will need to take time to enjoy the scenery. It will stir your imagination, arouse your curiosity, and impress upon you the logic, elegance, power, and beauty of mathematics. When you are done, you'll have both appreciation for the subject and its far-reaching "tools" like those which enable you to go back and forth between functions and their power series representations.

Having said all that, let's face the fact that the subject is hard for most students. The HP 48 supplementary material contained herein won't change that. However, as you will see, the HP 48 will unlock some doors that will tend to put things down to earth and make the material easier to comprehend and appreciate. What's more, it will add a little fun to your study.

One problem that many students have with sequences and series—for that matter, with mathematics in general—is that they approach it from an almost purely *symbolic* point of view, neglecting *numerical* and *geometric* considerations. They seem to think that all one has to do is shove symbols around the right way and things will turn out o.k. Sometimes that works, sometimes it doesn't. When it doesn't work, the symbol shover can look pretty silly to those who have a broader perspective.

Before machines like the HP 48 came along, there was an excuse for not going far with numerical and geometric aspects of the subject: numerical work takes time and geometric analysis depends on numerical work. Now, thanks to the HP 48 and other such machines, students no longer have an excuse for failing to understand what goes on numerically and geometrically.

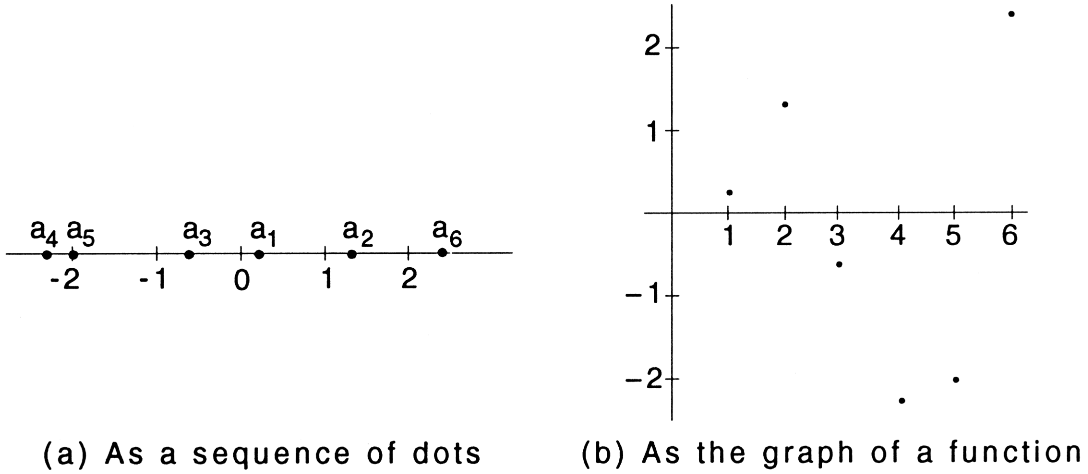
In this chapter we will show you how to use the HP 48 to enhance your numerical and geometric understanding of sequences and series. Section 8.1 is devoted to sequences, 8.2 to series, and 8.3 to power series representations. Finally, in section 8.4, we will show you how to turn the HP 48 into a movie machine, which, among other things, will enable you to "see" and better appreciate the idea of convergence of power series.

8.1 SEQUENCES

By a *sequence* (of numbers) is meant any ordered infinite succession of numbers a_1, a_2, a_3, $\cdots$. For example, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\cdots$; or 1, -2, 3, -4, $\cdots$; or even 1, 1, 1, $\cdots$. Note that a sequence always has infinitely-many *terms* (the three dots indicate that it goes on forever) and that the terms of a sequence need not be distinct (i.e., different from one another). A sequence is often denoted by curly brackets around its nth term. For example, $\{\frac{1}{n}\}$ denotes the sequence 1, $\frac{1}{2}$, $\frac{1}{3}$, $\cdots$; $\{\frac{n}{n+1}\}$ the sequence $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\cdots$; and $\{1+(-1)^n\}$ the sequence 0, 2, 0, 2, $\cdots$.

Geometric Interpretations

Geometrically, two ways to view a sequence of numbers are: (a) as a sequence of dots on a number line; and (b) as the graph of the function $f(n) = a_n$ whose domain is the set of natural numbers. These two ways are indicated in Fig. 1 for a sequence that starts with the terms 0.2, 1.3, -0.6, -2.3, -2, and 2.4.



(a) As a sequence of dots (b) As the graph of a function

Figure 1 Geometric Interpretations of a Sequence

In this section and the next we will develop some HP 48 programs to help you analyze and make conjectures concerning sequences and series. You may also want to develop some programs on your own. With this in mind, we suggest that you open a new directory, say SS, for these special programs.

The first program, which we call DOTS, enables you to view a sequence of numbers as a sequence of dots (actually tickmarks) on a number line without the burden of a lot of hand calculation. To run it, enter the Nth term of a sequence (use N instead of n), the endpoints l and r of the interval you want to view, a reference point c (to be identified with a long tickmark), and the number m of dots you want plotted. Note that an important feature of the output is that the dots will be plotted *dynamically* thus captivating the order aspect of a sequence. You'll find it helpful to accompany DOTS with the program INPUTS from § 3.2.

DOTS

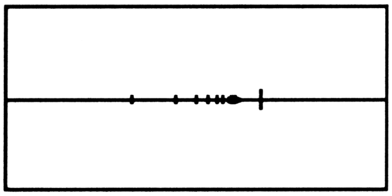
Inputs: $a(N)$, l, r, c, m	Output: Dots corresponding to $a_1, a_2, \cdots, a_m$ on the interval $[l, r]$ with point c specially marked.

```
« → A L R C M            Introduces local variables
« RESET {#0 #0} PVIEW
  -6.5 6.5 FOR I I 0      Draws x-axis without tickmarks
  R→C PIXON .1 STEP
  L R + 2 / 0 R→C         Sets PPAR so that viewing
  CENTR R L -             portion of the x-axis corresponds to
  13 / *W                 the interval [l, r]
  -.3 .3 FOR I C I        Puts a special tickmark at x = c
  R→C PIXON .1 STEP
  1 M FOR I I 'N'         Makes tickmarks at
  STO A EVAL DUP          the points a1, a2, ..., am
  .1 R→C PIXON -.1
  R→C PIXON NEXT
  'N' PURGE A L R C       Purges the name N; stores the
  4 →LIST 'IN' STO        list {a, l, r, c} under the name IN
  GRAPH » »

                          Checksum: #22378d  Bytes: 373.5
```

EXAMPLE 1. Represent the sequence $\{\frac{n}{n+1}\}$ as a sequence of dots.

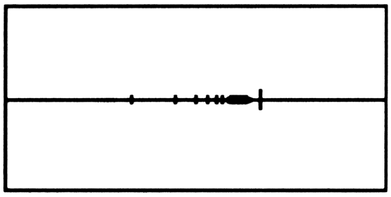
SOLUTION. To figure out an appropriate viewing interval and reference point, note that $0 \leq \frac{n}{n+1} \leq 1$ for all $n = 1, 2, \cdots$. Thus, one reasonable choice (there are others) would be: $l = 0$, $r = 1.5$, and $c = 1$. What value of m should we choose? We will try $m = 10$ with the idea in mind that we may want to try a larger value of m later.

N N 1 +
/ 0 1.5 1
10 DOTS



To see 20 dots, press ON, then:

INPUTS 20 DOTS



Note that the long tickmark to the right represents $x = 1$. To check approximate values represented by the other marks shown, use the + cursor and (x, y).

The Vocabulary of Sequences

Note that from watching the dots being plotted in the above example, it is clear that they march steadily to the right. Such a sequence is called *increasing.* Check your calculus textbook or ask your instructor for precise definitions of the words *increasing*, *decreasing*, *nonincreasing*, *nondecreasing*, and *monotone.*

You should not confuse the above type of geometric observation—no matter how convincing—with a *proof.* (To *prove* that the sequence $\{\frac{n}{n+1}\}$ is increasing, you would verify that $\frac{n}{n+1} < \frac{n+1}{n+2}$ for all natural numbers n. Can you do this?)

Other important adjectives for sequences are *bounded*, *unbounded*, *bounded above*, *bounded below*, *convergent*, and *divergent.* Again, you should check with your text or your instructor for the exact meanings of these words. From a geometric point of view: a sequence is *increasing* if the dots march steadily to the right; *decreasing* if the dots march steadily to the left; *bounded* if all of the dots stay inside of some finite interval; *bounded above* if there is a "barrier" to the right so that all of the dots stay to the left of that barrier; *bounded below* if there is a "barrier" to the left so that all of the dots stay to the right of that barrier; *convergent* if the dots "cluster" at exactly one point. We emphasize that these interpretations are heuristic and are meant to motivate and explain not replace the precise definitions you will find in your book.

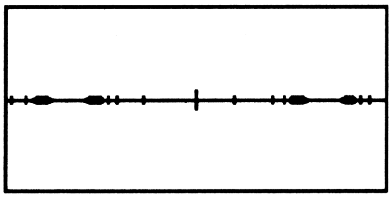
EXAMPLE 2. Use the program DOTS to make a geometric analysis of the sequence $\{a_n\}$ defined by

$$a_n = (-1)^n \left(1 + \frac{\sin \frac{n\pi}{2} + \cos \frac{n\pi}{2}}{\sqrt{n}}\right)$$

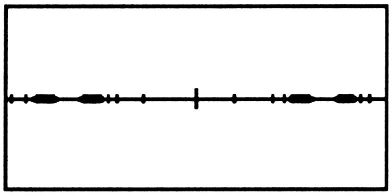
for $n = 1, 2, 3, \cdots$.

SOLUTION.

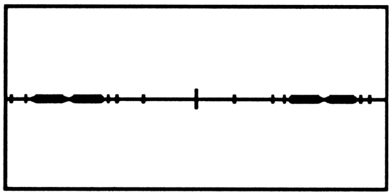
−1 N ^ 1 N π
∗ 2 / SIN
$\boxed{\text{ARG}}$ COS
+ N √ / +
∗ −1.5 1.5 0
50 $\boxed{\text{DOTS}}$



$\boxed{\text{ON}}$ $\boxed{\text{INPUTS}}$
100 $\boxed{\text{DOTS}}$



$\boxed{\text{ON}}$ $\boxed{\text{INPUTS}}$
1000 $\boxed{\text{DOTS}}$

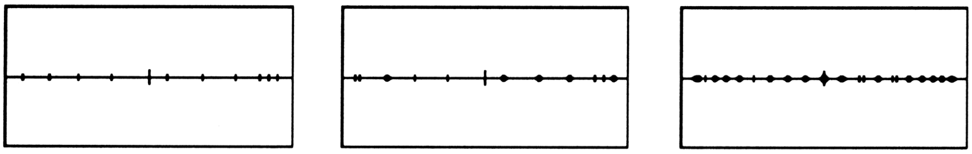


EXAMPLE 3. Use DOTS to make a geometric analysis of the sequence $\{\sin n\}$.

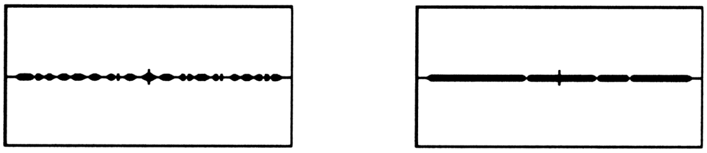
SOLUTION. The dot sequences shown in Fig. 2 correspond to $l = -1.1$, $r = 1.1$, $c = 0$, and $m = 10, 20, 50, 100, 200$, respectively. What conclusions can we draw?

For one thing, it is clear that the sequence is bounded by 1 ($|\sin n| \leq 1$ for all n); also, it is clearly *not monotone* (it skips all over the place). Concerning the question of convergence, it *appears* to cluster at every single number between -1 and 1 (can that really happen?); it surely does *not* appear to be convergent. This conjecture turns out to be right on target. In other words, it *is* true that the sequence $\{\sin n\}$ clusters at every number between -1 and 1. More precisely, it can be shown that if λ is *any* number between -1 and 1 inclusive, then there exists a subsequence of $\{\sin n\}$ that converges to λ. (This result is not easy to prove.)

The second program GR makes it possible to view a sequence of numbers as a function defined on the set $\{1, 2, 3, \cdots\}$ of natural numbers without a lot of hand calculation. To run it, enter the Nth term of the sequence and the length m of the interval $[0, m]$ you want to view.



(a) 10 terms (b) 20 terms (c) 50 terms



(d) 100 terms (e) 200 terms

Figure 2 A bounded sequence with infinitely-many cluster points?

EXAMPLE 4. Represent the sequence $\{\frac{n}{n+1}\}$ as the graph of a function.

SOLUTION. As in Example 1 we start with $m = 10$.

GR

Inputs: $a(N)$, m	Output: Points $(1, a(1))$, $(2, a(2))$, $\cdots$, $(m, a(m))$

```
« → A M « ERASE {#0 #0}
  PVIEW M 12 / DUP *W
  5.5 * 0 R→C CENTR
  DRAX
  1 M FOR I I I 'N' STO
  A EVAL R→C PIXON NEXT
  'N' PURGE
  A GRAPH RESET » »
```

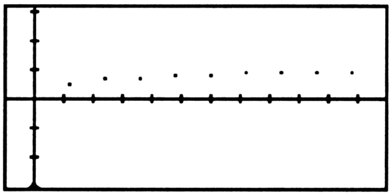
Introduces local variables; sets appropriate PPAR for a good view of the interval; draws the axes

Plots points $(1, a(1))$, $(2, a(2))$, $\cdots$, $(m, a(m))$

Puts $a(N)$ on stack

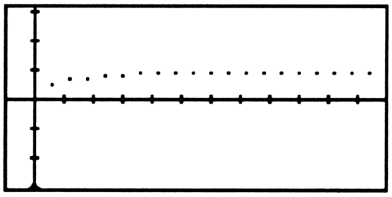
Checksum: #3231d Bytes: 191.5

$\boxed{\text{RESET}}$ N N 1 + / 10 $\boxed{\text{GR}}$



To improve to $m = 20$ all we have to enter is 20 $\boxed{\text{GR}}$ because the program was nice enough to leave the Nth term of the sequence on the stack.

20 $\boxed{\text{GR}}$

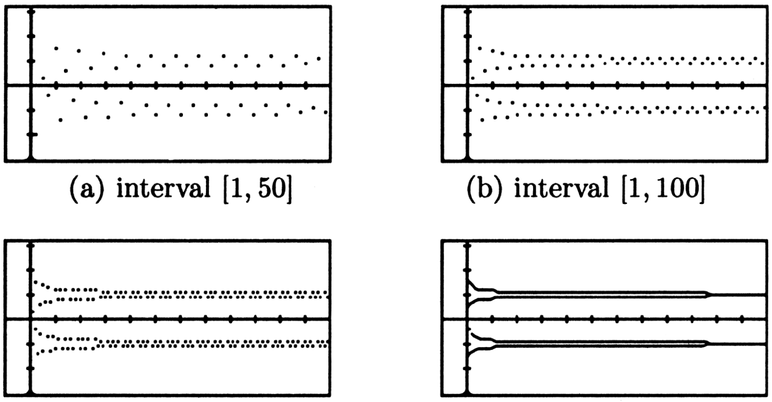


All the sequence terminology mentioned earlier (increasing, decreasing, convergent, divergent, etc.) can again be described in terms of this new graphical model. For example, "bounded above" means that there is a horizontal line such that the graph stays below that line and "convergent" means that the dot sequence that forms the graph has exactly one horizontal asymptote.

What can we say about the sequence $\{\frac{n}{n+1}\}$ based on our scant graphical findings? For one thing, it *appears* that the sequence is increasing (or at least nondecreasing) because the graph is going up as we go to the right. For another thing, it appears (as it should) that the sequence is bounded above by 1. In fact, it may even appear that the sequence converges to 1 (as in fact it does).

EXAMPLE 5. Use the program GR to make a geometric analysis of the sequence $\{a_n\}$ of Example 2.

SOLUTION. We obtain graphs corresponding to $m = 50$, 100, 200, and 500. See Fig. 3. From these graphs, it is apparent that the sequence has cluster points at ± 1.



(a) interval $[1, 50]$ (b) interval $[1, 100]$

(c) interval $[1, 200]$ (d) interval $[1, 500]$

Figure 3 Graph of the sequence $a_n = (-1)^n \left(1 + \frac{\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2}}{\sqrt{n}}\right)$

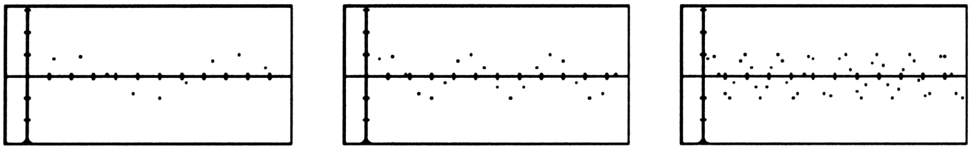
EXAMPLE 6. Use GR to make a geometric analysis of the sequence $\{\sin n\}$.

SOLUTION. The graphs in Fig. 4 corresponding to $m = 10$, 20, 50, 200, and 1000 shed some light on our earlier conjecture that the sequence clusters at *every* number λ between -1 and 1 (see Example 3). From Fig. 4(e) it is apparent that, asymptotically, the graph gets arbitrarily close to any line $y = \lambda$ where $-1 \leq \lambda \leq 1$.

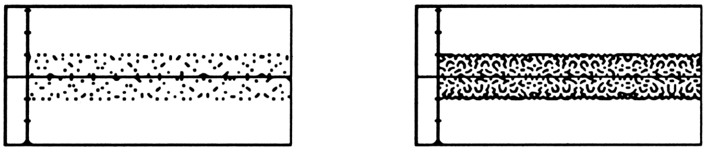
We turn now to numerical aspects. The program NXT (see box) exhibits the terms of a sequence starting with any initial value of N. [The name NXT is acceptable to your calculator even though NXT is already on the keyboard (a Category I key).]

NXT	
Inputs: none	*Output:* a_{N+1}
≪ N 1 + 'N' STO A EVAL ≫	Increments N by 1 Evaluates a_{N+1} Checksum: #4616d Bytes: 46

To run NXT, do the following in order:
1. Store the Nth term of the sequence under the name A.
2. Store a starting value of N under the name N.
3. Press NXT as many times as you'd like to see terms.



(a) interval [1, 10] (b) interval [1, 20] (c) interval [1, 50]



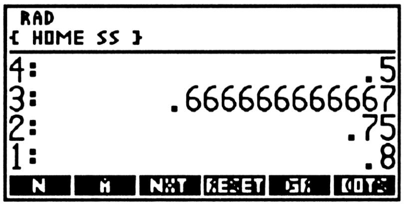
(d) interval [1, 200] (e) interval [1, 1000]

Figure 4 Graphs of the sequence $\{\sin n\}$

EXAMPLE 7. Use NXT to generate the first 28 terms of the sequence $\{\frac{n}{n+1}\}$.

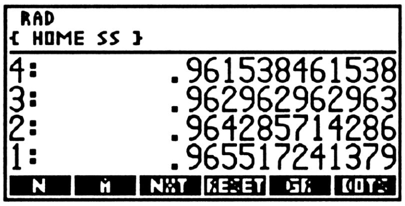
SOLUTION.

N N 1 + /
A $\boxed{\text{STO}}$ 0 N
$\boxed{\text{STO}}$ $\boxed{\text{NXT}}$ $\boxed{\text{NXT}}$
$\boxed{\text{NXT}}$ $\boxed{\text{NXT}}$



```
RAD
{ HOME SS }
4:                .5
3:     .666666666667
2:               .75
1:                .8
```

$\boxed{\text{NXT}}$ ··· $\boxed{\text{NXT}}$
(24 more times)



```
RAD
{ HOME SS }
4:     .961538461538
3:     .962962962963
2:     .964285714286
1:     .965517241379
```

As you watch the first 28 or so terms scroll by, certain things become evident. For example, it becomes evident that the numbers are getting larger, i.e., that the sequence is increasing; also that they are staying less than 1, i.e., the sequence is bounded above by 1; perhaps you may even suspect that the numbers are getting closer and closer to 1, i.e., the sequence is converging to 1. Again, we emphasize that all of this is just evidence, not proof.

The program SEQUENCE (see box) is more elaborate and user-friendly than NXT. It allows the user to scroll through the sequence page-by-page, pausing or quitting at will, and not having to worry about using up the calculator's memory on an extra long stack.

> To run SEQUENCE, enter the Nth term of the sequence, a starting value of N, then press $\boxed{\text{SEQU}}$.
> To pause, press 1;to continue press 1; to quit press 0.

Note that because of the fairly complicated loop structure of this program,

SEQUENCE

Inputs: $a(N)$, m	*Output: Numbers $a(m)$, $a(m+1)$, $\cdots$*

```
≪  'N'  STO  'A'  STO           Stores input under names A and N
  CLLCD                          Clears the display
  DO                             Begins first DO-UNTIL-END loop
    DO  1  7  FOR  I             Begins second DO-UNTIL-END loop;
      NXT  I                     displays seven consecutive
      DISP  NEXT                 values of the sequence, and continues
    UNTIL  KEY                   to display values in blocks of seven
    END                          until a key is pressed
  UNTIL
    IF  92  ≠                    Pauses if key other than 0 is pressed;
    THEN                         continues to pause until another key
      DO                         is pressed
      UNTIL  KEY
      END  92  ≠                 Exits if key 0 is pressed; continues
        1  0  IFTE               if key other than 0 is pressed
    ELSE  1                      Exits if first key pressed is 0
    END
  END  A  N                      Returns Nth term of sequence and
  {  A  N  }  PURGE  ≫           last value of N to stack

                                   Checksum: #15007d   Bytes: 181
```

we have presented it in staggered form so that the interested reader can easily identify the components of each loop. For example, the UNTIL that goes with the first (outermost) DO is located directly under it six lines later. If you have trouble understanding this program, don't worry about it. This is one program you can safely use as a "black box".

SEQUENCE makes it possible to obtain useful numerical data about sequences quickly and smoothly. The data shown in Fig. 5 can be obtained in a matter of seconds. It is for the same three sequences discussed in Examples 1–6. By studying the data, see if you can rediscover evidence of monotonicity, boundedness, and convergence.

In the next section, we will modify SEQUENCE slightly to obtain a similar program for infinite series.

Exercises 8.1

A.1 For each of the following sequences, use the programs DOTS and GR to make conjectures concerning monotonicity and convergence.

(a) $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ (b) $\left\{\dfrac{\ln n}{\sqrt{n}}\right\}$ (c) $\left\{\dfrac{2+(-1)^n}{\sqrt{n}}\right\}$

```
.990196078431
.990291262136
.990384615385
.990476190476
.990566037736
.990654205607
.990740740741
```

```
.990825688073
.990909090909
.990990990991
.991071428571
.991150442478
.991228070175
.991304347826
```

```
.991379310345
.991452991453
.991525423729
.991596638655
.991666666667
.99173553719
.991803278689
```

```
-1.09950371905
.90098524571
-.901467072196
1.0980580675
-1.09759000724
.90287141379
-.903326351091
```

```
1.09622504488
-1.09578262858
.904653741022
-.905084200403
1.09449111825
-1.09407208682
.906341418899
```

```
-.906749519231
1.09284766907
-1.0924500327
.907942538192
-.908330150243
1.09128709297
-1.09090909093
```

```
.452025787178
.994826791358
.622988631442
-.321622403163
-.970535283537
-.727142500081
.184781744561
```

```
.926818505418
.816742606636
-4.42426780851E-2
-.864551448611
-.889995604367
-9.71819058932E-2
.784980388681
```

```
.945435334025
.236661393364
-.689697940935
-.981952169044
-.371404101438
.580611184212
.998815224724
```

Figure 5 Sequence scrolls for (A) $\{\frac{n}{n+1}\}$; (B) $\{(-1)^n\left(1+\frac{\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}}{\sqrt{n}}\right)\}$; and (C) $\{\sin n\}$. Terms shown are for n = 100–121.

A.2 For each of the following sequences, use the programs DOTS and GR to make conjectures concerning cluster points.

(a) $\{(-1)^n\tan^{-1}\sqrt{n}\}$ (b) $\{\tan n\}$ (c) $\left\{\tanh\frac{1}{n}+\sin\frac{n\pi}{2}\right\}$

A.3 Use the program SEQUENCE to make conjectures about monotonicity and convergence for each of the following sequences.

(a) $\left\{n\sin\frac{1}{n}\right\}$ (b) $\left\{n\sin^{-1}\frac{1}{n}\right\}$ (c) $\left\{\frac{1000\sin n}{n}\right\}$

(d) $\left\{\left(\frac{n+1}{n}\right)^n\right\}$ (e) $\{n(e^{1/n}-1)\}$ (f) $\{2\tan^{-1}n\}$

A.4 Use any method to make conjectures concerning boundedness, monotonicity, and convergence for each of the following sequences.

(a) $\{(-1)^n\}$ (b) $\left\{\frac{1}{\sqrt{n}}\right\}$ (c) $\left\{\frac{1}{n^{0.1}}\right\}$

(d) $\left\{\frac{(-1)^n}{\sqrt{n}}\right\}$ (e) $\{\cos n\pi\}$ (f) $\left\{\cos\frac{n\pi}{4}\right\}$

(g) $\{\cos n\}$ (h) $\left\{\frac{\cos n}{\ln(n+1)}\right\}$ (i) $\left\{\cos\frac{\pi}{n}\right\}$

(j) $\left\{2+\frac{1}{n}\right\}$ (k) $\left\{\frac{n}{1+\sqrt{n}}\right\}$ (l) $\left\{\frac{4n+1}{5n-2}\right\}$

(m) $\left\{\frac{4n+\sqrt{n}}{5n-2}\right\}$ (n) $\left\{\frac{\ln^2 n}{\sqrt{n}}\right\}$ (o) $\left\{\frac{\ln\ln n}{\ln(n+1)}\right\}$

(p) $\{n^2-n\}$ (q) $\left\{\frac{2^n}{n!}\right\}$ (r) $\left\{\frac{1.01^n}{n^5}\right\}$

B.1 Use any method to make conjectures concerning convergence of the following sequences.

(a) $\left\{\frac{n^{100}}{2^n}\right\}$ (b) $\left\{\frac{100^n}{n!}\right\}$ (c) $\left\{\left(1+\frac{2}{n}\right)^n\right\}$ (d) $\{n(2^{1/n}-1)\}$

B.2 Given $a_n = n^{\frac{1}{\log\log(n+1)}}$ where log denotes $\log_{10}$.

(a) Use GR to obtain a graph of the sequence $\{a_n\}$ showing that the values of a_n decrease until n reaches a certain value N_0 then increase after that.

(b) Use SEQUENCE to determine the point (N_0, a_{N_0}).

C.1 Formulate generalizations of each of the conclusions in B.1 in which the constants 2 and 100 are replaced by arbitrary constants and give proofs of the generalized statements.

C.2 Let $a_n = n^{\frac{1}{\ln(n+1)}}$ for $n = 1, 2, \cdots$.

(a) Use the HP 48 to formulate a conjecture.

(b) Prove that the sequence $\{a_n\}$ is bounded and monotone.

(c) Find the limit.

C.3 Let $a_1 = 1$ and $a_{n+1} = \sin a_n$ for $n = 1, 2, \cdots$ (so that $a_2 = \sin 1$, $a_3 = \sin(\sin(1))$, etc.)

(a) Use the SIN key successively to generate the terms of this sequence and to make a conjecture.

(b) Prove that the sequence $\{a_n\}$ is bounded and monotone.

(c) Find the limit.

C.4 (a) Write a simple program to generate the terms of the sequence $\sqrt{2}$, $\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\sqrt{2}}}, \cdots$.

(b) What do you conjecture?

(c) Prove your conjecture.

8.2 SERIES

Pretend that you have the naiveté of a sixth grader, that you're good at arithmetic (including long division), and that you know how to use an HP 48 calculator. (You might also want to keep in the back of your mind your recently acquired knowledge of sequences.) You're about to get a "long addition" assignment. The assignment: carry out the following operations:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=? \tag{1}$$

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=? \tag{2}$$

$$1-1+1-1+1-1+\ldots=? \tag{3}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots=? \tag{4}$$

Just to make sure there is no misunderstanding about your task, you're supposed to know *absolutely nothing about infinite series* and your job is simply to "add them all up". The three dots mean that the process is to go on indefinitely and the pattern is to continue. For instance, to carry out long addition problem (1), after you add 1, $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$, you must then add $\frac{1}{16}$, $\frac{1}{32}$, and so on, adding all numbers of the form $\frac{1}{2^n}$, $n = 0, 1, 2, \cdots$.

Take your time with the above problems as it will be time well spent. Be especially sure to take time to make a conjecture about Problem (1). When you have finished, compare your conclusions with the ones below.

Analysis of Problem (1): $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=?$

Most people when given the above instructions will just start adding and see what happens. Their work would go something like this: Let's see now, $1+\frac{1}{2}$ is 1.5; then $+\ \frac{1}{4}$ gives 1.75; then $+\ \frac{1}{8}$: $1.75+.125 = 1.875$; then $+\ \frac{1}{16}$: $1.875+0.0625 = 1.9375$; etc. Thanks to RPN, all these calculations are easy to perform with the HP 48. The simple procedure is to key in the next power of two, invert it, then add. We can easily automate the procedure with the following program:

```
« S 2 N ^ INV + 'S' STO N 1 + 'N' STO S »
ADD1 STO
```

To run the program, enter 0 N STO 0 S STO, then press ADD1 repeatedly to obtain as many "partial sums" as you want. The second column of Table 1 shows the first 20 partial sums.

Observe that the partial sums form a sequence. What can we say about this sequence? For one thing, it is obviously increasing. Also it *appears* to be bounded above by 2. Maybe it even converges to 2. Further evidence supporting the limit of 2 can be obtained by looking at additional partial sums. If you do this, you will find that the HP 48 gives 1.99999999999 as the 37th partial sum and then gives 2 for all sums beyond that. (Can that really be true? What do you think?)

We can add geometric credence to the answer of 2 as follows. Start with a piece of string 2″ long, cut it in half, and throw away one of the 1″ pieces. Now cut in half the remaining 1″ piece and throw away half of that. At this stage you will have thrown thrown away $1+\frac{1}{2}=1\frac{1}{2}$ inches of string and have left $\frac{1}{2}$ inch. Continue the process, at each stage cutting in half what you have left and throwing away half of that. After N cuts, you will have thrown away $1+\frac{1}{2}+\cdots+(\frac{1}{2})^{N-1}$ inches of string and be left with $(\frac{1}{2})^{N-1}$ inch.

Even though these arguments are heuristic in nature, they should convince you that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$$

exactly!

Before continuing with the other "long addition problems", let's be more efficient on how we put the HP 48 to work on such problems. What we need is a generalization of ADD1 to generate *sequences* of partial sums. The program NXTΣ, modeled after NXT for sequences, fits the bill. See program box.

NXTΣ

Inputs: none	*Output: next partial sum*

```
« S N 1 + 'N'
  STO A EVAL
  + DUP 'S' STO »
```

Increments N by 1
Evaluates next term to be added
Forms and stores $(N+1)$st partial sum

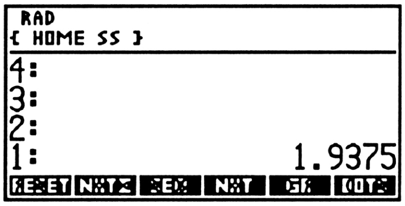
Checksum: #36424d Bytes: 68.5

To run NXTΣ, do the following in order:
1. Store the Nth term under the name A.
2. Initialize both N and S to zero.
3. Press NXTΣ repeatedly to obtain partial sums.

✓Points to note

1. The HP 48 has a built-in command Σ (GREEN TAN) for evaluating any sum of the form
$$\sum_{n=P}^{Q} a_n = a_P + a_{P+1} + \cdots + a_Q$$
To use it, enter n, P, Q, a_n in that order and press Σ. For example, here's how you could evaluate $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$.

```
RESET
N 1 5 2
N 1 − ^
INV Σ
```



```
RAD
{ HOME SS }
4:
3:
2:
1:                1.9375
```

2. Since Σ is a special operation key, the HP 48 won't allow direct entry of 'NXTΣ'. You can do it indirectly as follows: enter the list {NXT Σ}, use ◀ to delete the space between NXT and Σ, then press OBJ→ DROP.
3. NXTΣ is more general than ADD1 so you can purge ADD1.

Analysis of Problem (2): $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = ?$

As with Problem 1, we begin by calculating partial sums. Observe that the typical term to be added is the reciprocal of a perfect square, i.e., the nth term is $\frac{1}{n^2}$. Thus, we can use NXTΣ as follows:

INPUT: $\boxed{\text{RESET}}$ N SQ INV A $\boxed{\text{STO}}$ 0 N $\boxed{\text{STO}}$ 0 S $\boxed{\text{STO}}$
$\boxed{\text{NXT}\Sigma}$ $\boxed{\text{NXT}\Sigma}$ $\cdots$

OUTPUT: Table 1, column 3

Table 1 Partial sums

N	Problem (1)	Problem (2)	Problem (3)	Problem (4)
1	1	1	1	1
2	1.5	1.25	0	1.5
3	1.75	1.36111111111	1	1.83333333333
4	1.875	1.42361111111	0	2.08333333333
5	1.9375	1.46361111111	1	2.28333333333
6	1.96875	1.49138888889	0	2.45
7	1.984375	1.51179705216	1	2.59285714286
8	1.9921875	1.52742205216	0	2.71785714286
9	1.99609375	1.53976773117	1	2.82896825397
10	1.998046875	1.54976773117	0	2.92896825397
11	1.9990234375	1.55803219398	1	3.01987734488
12	1.99951171875	1.56497663842	0	3.10321067821
13	1.99975585938	1.57089379818	1	3.18013375513
14	1.99987792969	1.575995839	0	3.25156232656
15	1.99993896485	1.58044028344	1	3.31822899323
16	1.99996948243	1.58434653344	0	3.38072899323
17	1.99998474122	1.58780674105	1	3.43955252264
18	1.99999237061	1.5908931608	0	3.4951080782
19	1.99999618531	1.5936632439	1	3.54773965715
20	1.99999809266	1.5961632439	0	3.59773965715

What can we say about these partial sums? Clearly, they are increasing. Is there an upper bound? From what we see in Table 1, it seems safe to conjecture that 2 is an upper bound. If so, then the sequence must converge to something (remember: a bounded monotone sequence converges). Could the answer be 1.6? What do you think? If you look at a few more partial sums, you'll see that 1.6 is ruled out because the terms get larger than that. A look at longer partial sums doesn't seem to help much. For example, according to the HP 48, the 500th partial sum is 1.64293606562, the 1000th partial sum is 1.64393456674, and the 10,000th partial sum is 1.64483407191. Surprisingly, the answer to this problem turns out to be $\pi^2/6 = 1.64493406685\cdots$!

Analysis of Problem (3): $1 - 1 + 1 - 1 + 1 - ... = ?$

This one looks pretty easy. In fact, we don't even need a calculator to analyze it. Clearly,

$$1 - 1 + 1 - 1 + 1 - \cdots = (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0.$$

But wait a minute. Here's another way of looking at it:

$$1 - 1 + 1 - 1 + 1 - \cdots = 1 + (-1 + 1) + (-1 + 1) + \cdots = 1 + 0 + 0 + \cdots = 1$$

Hmmm. Is the answer 0? Or is it 1? What do you think? Hint: look at column 4 of Table 1. We'll return to this question later.

Analysis of Problem (4): $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = ?$

Here we see that the typical terms to be added are the reciprocals of natural numbers. The nth term is clearly $\frac{1}{n}$. To get the HP 48 to do the work for us, do the following:

INPUT: $\boxed{\text{RESET}}$ N INV A $\boxed{\text{STO}}$ 0 N $\boxed{\text{STO}}$ 0 S $\boxed{\text{STO}}$
$\boxed{\text{NXT}\Sigma}$ $\boxed{\text{NXT}\Sigma}$ $\cdots$
OUTPUT: Table 1, last column

What can we say about this sequence of partial sums? Clearly, it is increasing. Is it bounded above? From what we see it seems safe to conjecture that 10 is an upper bound. If so, then the sequence of partial sums must converge to something. What do you think it is? After seeing what happened in Problem (2), you couldn't be blamed for being cautious. A look at more partial sums doesn't seem to shed much light. For example, the 500th partial sum (according to the HP 48) is 6.792823412997, the 1000th partial sum is 7.48547086047, and the 10,000th partial sum is 9.78760603576.

Surprisingly, the answer to this problem turns out to be ∞! In other words, the sequence of partial sums is *not* bounded above. By the way, your calculator would never be able to discover this fact. Why? See Exercise C.1.

About Sequences and Series

Problems 1–4 are examples of infinite series problems. As you will read in your book, an *infinite series* is any expression of the form

$$a_1 + a_2 + a_3 + \cdots$$

The word *series* is often used in place of infinite series. Note that outside of the realm of mathematics, "sequence" and "series" are often used synonymously. In mathematics, they are completely different concepts. Since this is a common source of confusion, let's carefully compare the two concepts:

- A *sequence* is an ordered infinite succession of numbers a_1, a_2, a_3, $\cdots$. There is a first number, a second number, and so on. They are separated by commas, not plus signs. The terms themselves may or may not arise from additions.

- A *series* is an expression of the form $a_1+a_2+a_3+\cdots$. Nothing more, nothing less. It is important to observe that *associated with any series are two important sequences.* These are:
  - (i) the sequence of terms: $a_1,\ a_2,\ a_3,\ \cdots$; and
  - (ii) the sequence of partial sums: $a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \cdots$.

The adjectives *convergent* and *divergent* apply both to sequences and series. Other adjectives like *bounded* and *monotone* apply only to sequences. In view of Problems (1)–(4), it should come as no shock that a series is said to converge (to L) if its sequence of partial sums converges (to L). Likewise, a series is said to diverge if its sequence of partial sums diverges.

The Big Question

Given an infinite series, the big question is does it converge or doesn't it? It may come as a surprise that the main question is *not* what is the sum? The reason for this is that Problem (2) above is far more typical than Problem (1). In other words, it is typical that convergent series converge to strange numbers.

Let us now discuss Problems (1)–(4) from a more sophisticated point of view. The series in Problem (1) is an example of a convergent *geometric series.* As we suspected, it does indeed converge to 2. Geometric series are discussed in your textbook. They are the simplest type of series and are easily programmed. See Exercises A.11–12.

The series in Problem (2) is an example of a convergent *p-series* (with $p = 2$). You will find a discussion of *p*-series in your book following the "integral test". There are many proofs of the fact that this series converges to $\pi^2/6$. Some of these proofs claim to be "elementary" but none claims to be "easy".

Does the series in Problem (3) converge to 0 or 1? The answer is neither. It doesn't even converge. Why? Because the sequence of partial sums is 1, 0, 1, 0, 1, 0, $\cdots$ which clearly diverges.

The series in Problem (4) is called the *harmonic series.* It is a very special series, partly because it diverges so slowly. It is a *p*-series with $p = 1$ and is often useful as a counterexample. To get an idea of how slowly it diverges, the 100,000th partial sum is just a little bigger than 12. Also see Exercise C.1.

A User-Friendly Series Program

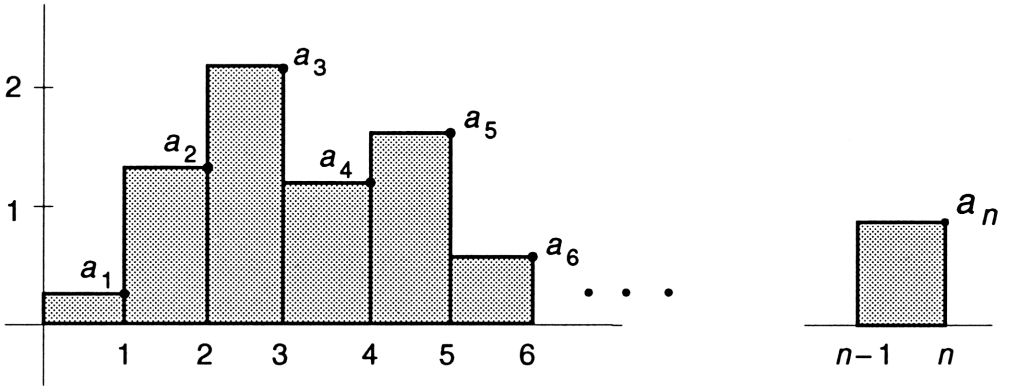
The program SEQUENCE can easily be modified to obtain a user-friendly program for scrolling through the partial sums of any series as follows:

1. Press $\boxed{\text{SEQU}}$ $\boxed{\text{RCL}}$ $\boxed{\text{EDIT}}$;
2. Replace 'N' $\boxed{\text{STO}}$ by 0 'N' $\boxed{\text{STO}}$;
3. Insert 0 'S' $\boxed{\text{STO}}$ after 0 'N' $\boxed{\text{STO}}$;
4. Replace NXT by NXTΣ;
5. Press $\boxed{\text{ENTER}}$;
6. Type SERIES $\boxed{\text{STO}}$.

To run SERIES enter the Nth term of the series
and press $\boxed{\text{SERIES}}$. To pause, press 1;
to continue press 1; to quit press 0.

A Geometric Interpretation of Series

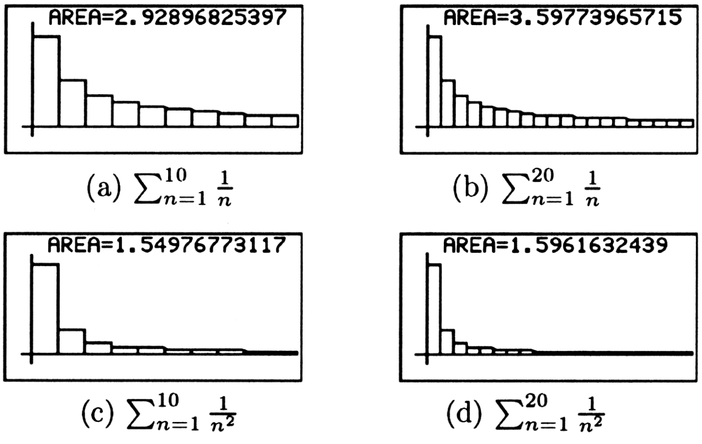
At the beginning of this chapter, we promised to give both numerical and geometric enhancement to both sequences and series. How can infinite series be viewed *geometrically*? One way is as "area under a curve". The "curve" here is the step function whose value is equal to a_n on the interval $(n-1, n)$. To see the truth of this, note that, for $a_n > 0$, the nth rectangle has dimensions $1 \times a_n$ and, therefore, the nth rectangle has area equal to a_n. See Figs. 6 and 7.



(a) Σa_n = Area under curve

Area = a_n

Figure 6 Geometric Interpretation of a Series



AREA=2.92896825397

(a) $\sum_{n=1}^{10} \frac{1}{n}$

AREA=3.59773965715

(b) $\sum_{n=1}^{20} \frac{1}{n}$

AREA=1.54976773117

(c) $\sum_{n=1}^{10} \frac{1}{n^2}$

AREA=1.5961632439

(d) $\sum_{n=1}^{20} \frac{1}{n^2}$

Figure 7 Partial Areas Corresponding to Partial Sums

Exercises 8.2

For problems A.1–A.10 begin by regarding each as a "long addition" problem. (a) Use SERIES or NXTΣ to calculate several partial sums; and (b) make a guess at the answer. Then (c) use textbook methods to check for convergence.

A.1 $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

A.2 $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

A.3 $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$

A.4 $\cos 1 + \cos \frac{1}{2} + \cos \frac{1}{3} + \cos \frac{1}{4} + \cdots$

A.5 $\sin 1 + \sin \frac{1}{2} + \sin \frac{1}{3} + \sin \frac{1}{4} + \cdots$

A.6 $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$

A.7 $1 - \frac{\pi^2}{3!} + \frac{\pi^4}{5!} - \frac{\pi^6}{7!} + \cdots$

A.8 $1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \cdots$

A.9 $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \frac{1}{16} + \cdots$

A.10 $\frac{1}{6} + \frac{1}{20} + \frac{1}{42} + \frac{1}{72} + \frac{1}{110} + \cdots$

A.11 Write a program that takes as input two numbers a and r, with $|r| < 1$, and gives as output the sum of the geometric series $a + ar + ar^2 + \cdots$.

A.12 Use the result of problem A.11 to sum the following series:

(a) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$

(b) $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots$

(c) $\frac{3}{16} - \frac{9}{64} + \frac{27}{256} - \frac{81}{1024} + \cdots$

(d) $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \frac{1}{16} - \cdots$

B.1 Prove your conjecture in A.8. $\left[\textit{Hint: } \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\right]$

C.1 Let S_n be the nth partial sum of the harmonic series, i.e., $S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$

for $n = 1, 2, \cdots$.

(a) Prove that $S_{2^n - 1} \le n$ for all $n = 1, 2, \cdots$.

(b) Explain why the HP 48 thinks that $S_n \le 40$ for all n.

(c) As a practical matter, explain why the HP can't calculate $S_{10^{12}}$. [*Hint:* how long does it take the HP 48 to calculate S_{1000} ?]

8.3 POWER SERIES

What's so special about *power series*? The answer is that they are the natural generalization of polynomials and that they are "nice" functions. Recall that a *polynomial* is any function of the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N,$$

where N is a nonnegative integer and $a_0, a_1, a_2, \cdots, a_N$ are real numbers. A *power series* is any expression of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots ,$$

where $a_0, a_1, a_2, \cdots$ are real numbers. Of course, one big difference between power series and polynomials is that you have to worry about convergence of the power series. However, as you can see from your textbook, the convergence situation is quite simple for power series: corresponding to any given power series, there will be an interval of values of x inside of which you have convergence and outside of which you have divergence. This interval, called the interval of convergence, will always have the form $(-R, R)$ for some R, $0 \le R \le \infty$. The power series may or may not converge at the endpoints $\pm R$, but that is a relatively minor consideration.

Power series are "nice". You can add, subtract, multiply, or even long-divide them just like they were long polynomials—provided you stay inside the intervals of convergence of the power series involved; you can also differentiate or integrate power series term-by-term as many times as you'd like—provided you do so inside the interval of convergence. In particular, a power series is of *Class* $C^{(\infty)}$, i.e., has derivatives of all orders, inside its interval of convergence.

There is a rich supply of power series. To get an idea of this, you can form your own personal power series based on your birthdate as follows. Let AB/CD/EFGH represent the month/date/year of your birthday. For example, if you were born on Mar. 8, 1973, then $\mathrm{A}=0$, $\mathrm{B}=3$, $\mathrm{C}=0$, $\mathrm{D}=8$, $\mathrm{E}=1$, $\mathrm{F}=9$, $\mathrm{G}=7$, and $\mathrm{H}=3$. Form your personal power series as follows:

$$\begin{aligned}\mathrm{P}(x) &= \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \cdots + \mathrm{H}x^7 \\ &+ 10\mathrm{A}x^8 + 10\mathrm{B}x^9 + \cdots + 10\mathrm{H}x^{15} \\ &+ 100\mathrm{A}x^{16} + 100\mathrm{B}x^{17} + \cdots + 100\mathrm{H}x^{23} \\ &+ 1000\mathrm{A}x^{24} + 1000\mathrm{B}x^{25} + \cdots \\ &+ \cdots\end{aligned}$$

What is the interval of convergence of your personal power series? As a check on your answer, the HP 48 gives R = .749894209332. Now how did the HP 48 know your birthday?! See Exercise B.2.

The class of power series is so rich, you might get the idea it would contain, in disguise, some old friends like $e^x, \sin x$, and $\cos x$. Indeed, it does. In fact, this is the whole idea of Maclaurin series and, more generally, Taylor series. Briefly, the following is true for all "normal" $C^{(\infty)}$ functions $f(x)$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!} + \cdots$$

in $(-R, R)$ where R is the radius of convergence of the power series on the right-hand side of the equality.

You might have wondered back in § 1.1 how we knew that $\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$. The mystery unravels once you look at the Maclaurin series expansion for $\sin x$:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

As you may guess, $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$ would give even a better approximation to $\sin x$ near 0.

Generally speaking, it is highly desirable to approximate $C^{(\infty)}$ functions by polynomials because, as we have pointed out previously, polynomials are conceptually so basic and simple. Such approximations can generally be obtained by truncating the Maclaurin series or Taylor series. Of course, one must always worry about approximation errors, but that will not be our main concern here. We will concern ourselves primarily with the easier problem of finding truncations of Maclaurin series and Taylor series.

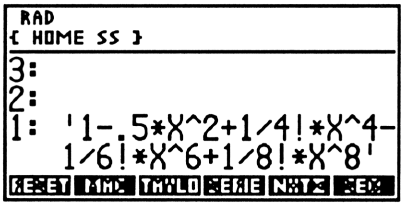
You may have noticed that the HP 48 has a built-in command called TAYLR (located in the SYMBOLIC menu). A more appropriate name for this command would be MAC since the built-in program really calculates partial sums of Maclaurin series not Taylor series. However, as the HP 48 manual points out, Taylor series can be obtained through a process that amounts to making a translation and following it with the inverse translation. The programs MAC and TAYLOR (see boxes) clarify and simplify the situation.

MAC	
Inputs: $f(x)$, N	*Output: Nth degree polynomial truncation of Maclaurin series for $f(x)$*
`« → F N «`	Introduces local variables
`F X N TAYLR » »`	Uses built-in TAYLR program
	Checksum: #24531d Bytes: 50

EXAMPLE 1. Find the 8th degree polynomial truncation of the Maclaurin series for $\cos x$.

SOLUTION.

X COS 8 MAC

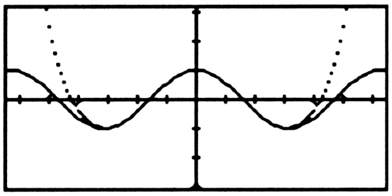


```
RAD
{ HOME SS }
3:
2:
1:  '1-.5*X^2+1/4!*X^4-
    1/6!*X^6+1/8!*X^8'
```

TAYLOR

Inputs: $f(x)$, c, N	*Output: Nth degree Taylor polynomial for $f(x)$ about c*
≪ → F C N ≪	Introduces local variables
X C + 'X' STO	Stores $X + C$ under name X
F EVAL 'X' PURGE	Forms $f(X + c)$; purges name X
N MAC	Finds Maclaurin poly. for $g(X) = f(X + c)$
X C − 'X' STO	Stores $X - C$ under name X
EVAL	Evaluates Mac poly. at $X - c$
'X' PURGE ≫ ≫	Purges name X
	Checksum: #48502d Bytes: 133

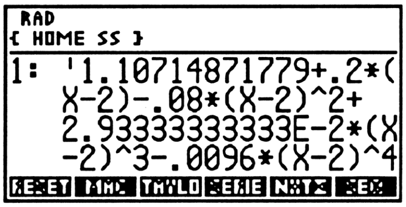
RESET X COS
= ↰ EQ
DRAX DRAW



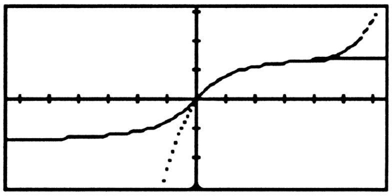
EXAMPLE 2. Find the 5th degree Taylor polynomial for $\tan^{-1} x$ about $x = 2$.

SOLUTION.

X ATAN 2
5 TAYLO


```
RAD
{ HOME SS }
1: '1.10714871779+.2*(
   X-2)-.08*(X-2)^2+
   2.93333333333E-2*(X
   -2)^3-.0096*(X-2)^4
```

RESET X
ATAN =
↰ EQ
DRAX DRAW



Exercises 8.3

A.1 (a) Find the 6th degree polynomial approximation for the Maclaurin series of $(1 + x^2)^{-1}$, then press EVAL to combine constants. What do you suspect? Confirm your suspicions by expanding $(1 + x^2)^{-1}$ as a geometric series.

(b) Make a sketch showing both $y = (1 + x^2)^{-1}$ and the polynomial approximation on the interval $[-1, 1]$.

A.2 (a) Find the 4th degree Taylor polynomial approximation for $(1+x^2)^{-1}$ about $x = 2$.

(b) Make a sketch showing both $y = (1 + x^2)^{-1}$ and the approximating polynomial near $x = 2$.

A.3 (a) Find the 6th degree polynomial that approximates the Maclaurin series for $e^{0.2x} \sin 2x$.

(b) Make a sketch showing both $y = e^{0.2x} \sin 2x$ and the approximating polynomial.

A.4 (a) Find the 5th degree Taylor polynomial for $\ln x$ about $x = 1$. Based on this information, what would you guess is the Taylor series expansion of $\ln x$ about $x = 1$?

(b) Make a sketch showing both $y = \ln x$ and the approximating polynomial.

B.1 Find a polynomial $p(x)$ (of as small degree as possible) which approximates $\sin x$ on the interval $[-\pi, \pi]$ so closely that the HP 48 can't tell the difference. Put another way, how could you design a SIN key for the HP 48?

B.2 Explain why *all* "personal power series", as defined in this section, have radius of convergence equal to $0.1^{1/8} \approx 0.749894209332$.

8.4 HP MOVIES

You can make moving pictures on the HP 48 just like movie-producers make animated cartoons, i.e., by putting pictures together in quick succession. This can be done by the following 3-step process: (1) encode the pictures to be used; (2) assign names to the codes; and (3) use the names in a movie-making program.

To encode a picture in the graphics environment, press STO; to encode a picture inside a program, use PICT RCL. You assign names to codes like you would assign names to anything else; simply use STO.

Now suppose you have stored a sequence of pictures under the names S1, S2, $\cdots$, SN and want to put them together to make a movie. The program MOVIE makes it easy. See box.

EXAMPLE 1. Encode the curves $y = \sin(x - \pi(n-1)/2)$, $n = 1, 2, 3, 4$, and store them under the names S1, S2, S3, S4, respectively. Then make a movie with S1, S2, S3, S4 as its frames and show the movie 10 times. Describe the effect.

SOLUTION. To encode and store S2 do the following:

$$\boxed{\text{RESET}}\quad \text{X}\quad \pi\quad 2\quad /\quad -\quad \text{SIN}\quad \boxed{\Lsh}\quad \boxed{\text{EQ}}$$

MOVIE

Inputs: *n*, *w*, *t*	*Output: Movie consisting of frames S1, S2, ···, SN with w seconds between frames repeated t times.*

```
«  →  N  W  T                       Introduces local variables
  «  {#0  #0}  PVIEW                Accesses graphics environment
    1  T  FOR  J                    Begins repetition loop
    1  N  FOR  I                    Begins frames loop
    "S"  I  →STR  +  OBJ→           Forms SI
    PICT  {#0  #0}  ROT  REPL       Shows frame SI
    W  WAIT
    NEXT  NEXT  »  »                Ends loops
```

Checksum: #8960d Bytes: 170.5

$\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$ $\boxed{\text{STO}}$ $\boxed{\text{ON}}$ S2 $\boxed{\text{STO}}$

Similarly, you can encode and store S1, S3, and S4. To make the required 4-frame movie and show it 10 times, enter: 4 .1 10 $\boxed{\text{MOVIE}}$. The effect will be a sine curve marching steadily to the right. See Fig. 8(a) for a "filmstrip".

EXAMPLE 2. Make an HP 48 movie showing partial sums of the Maclaurin series for $\sin x$ converging to $\sin x$.

SOLUTION. The program CODE encodes and stores frames corresponding to the first ten distinct partial sums. Before running CODE, purge any of the names S1, ···, S10 which are on the VAR menu. Be alert to the fact that to run CODE requires about 12.5 Kbytes of memory. (You can check how much memory you have available by pressing MEM in the MEMORY directory.)

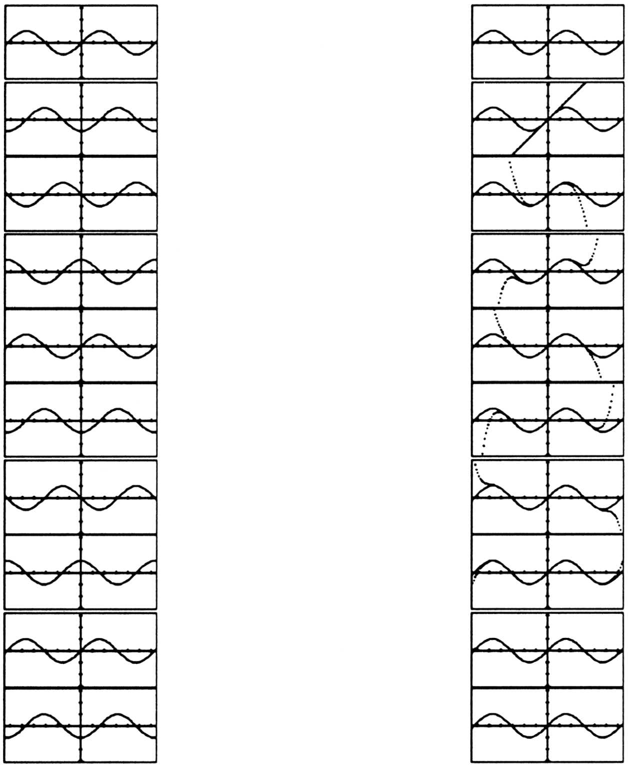
After you have run CODE, enter: 10 1 1 $\boxed{\text{MOVIE}}$ to see convergence of the Maclaurin series to $\sin x$. See Fig. 8(b) for a "filmstrip".

Exercises 8.4

A.1 Make a 2-frame movie showing a robot waving its arms. [*Hint:* for the artwork, use DOT+ and DOT− as described in § 0.6.]

A.2 Encode the curves $y = 3\sin\frac{3x}{n+1}$, $n = 1, 2, 3, 4, 5$, and store them under the names S1, S2, S3, S4, S5, respectively. Then make a movie with S1, S2, S3, S4, S5 as its frames. Before running it, see if you can predict the outcome.

CODE	
Inputs: none	*Output: Frames S1, ···, S10*
`« RESET DRAX`	Preliminaries
`X SIN STEQ DRAW`	Graphs the curve $y = \sin x$
`PICT RCL 'S1' STO`	Encodes graph of $\sin x$; stores it as S1
`1 9 FOR I ERASE`	Begins FOR-NEXT loop; clears display
`DRAX X SIN 2 I * 1 -`	Sets up argument for MAC program
`MAC STEQ DRAW`	Graphs (2I – 1)st degree Mac. poly.
`PICT RCL S1 +`	Encodes graph of Mac. poly. and $\sin x$
`"S" I 1 + →STR +`	Stores code as S(I+1);
`OBJ→ STO NEXT »`	ends loop
	Checksum: #27815d Bytes: 150.5



(a) (b)

Figure 8 Filmstrips

A.3 Make a movie showing a tangent line moving along an interesting curve. Describe the action on a section of the curve that is (a) concave upward; (b) concave downward. Describe what happens near an inflection point. (Note that part of the exercise is to figure out what is "interesting" and what is not.)

A.4 Make a movie showing a sequence of secant lines approaching the tangent line to the curve $y = \frac{1}{8}x^2$ at the point $(4, 2)$.

A.5 Make a movie showing partial sums of the Maclaurin series for $1.5 \cos 2x$ converging to $1.5 \cos 2x$.

B.1 Make a movie showing the circle of curvature rolling along a cardioid.

B.2 Make a movie showing a cycloid being generated by a rolling circle.

C.1 Make a movie showing a sequence of Riemann sums converging geometrically and numerically to the integral

$$\int_1^7 [0.3\sqrt{x+3}\sin(x+3) + 2.5]\, dx.$$

Note that (6, 2) CENTR will center the curve so that there is enough space for a message of the form "AREA = $\cdots$". Use numerical integration to check your result.

Chapter 9

Functions of Several Variables

9.0 PREVIEW

In this chapter we illustrate some ways the HP 48 can serve as companion in the study of functions of several variables. The reader may wish to open a new directory $\boxed{\text{XYZ}}$, say, for the purpose of assembling the programs which will be developed in this chapter. Many of these can be used as "double-check programs" for problems worked by hand. Others, like LEVEL (below), perform tasks that no one could reasonably do by hand.

9.1 FUNCTIONS OF TWO VARIABLES

The first step towards understanding functions of several variables is to understand functions of *two* variables and the first step towards understanding functions of two variables is to understand them geometrically. There are two ways.

Representation as a Surface

The best way to visualize a function of two variables is as a surface in an (x, y, z)-coordinate system. This is analogous to visualizing a function of one variable as a curve in an (x, y)-coordinate system. The surface corresponding to a function $f(x, y)$ consists of those and only those points which have the form $(x, y, f(x, y))$. You may think of a surface as a mountain made out of tinted glass. It's always there, but you may look through it to see what's on the other side.

Conceptually, surface plotting is simple. All you have to do is make a dot corresponding to each and every point $(x, y, f(x, y))$. For example, to plot the surface corresponding to the function

$$f(x, y) = 3\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{25}} ,$$

you might start by taking $x = 1$, $y = 2$, calculate $f(1, 2) \approx 2.3$, and then make a dot at (1, 2, 2.3). See Fig. 1(a). You could continue in this way as long as you like, plotting more and more dots, and you might think the more dots the better. However, this is not the case.

Fig. 1(b) illustrates what happens when *all* dots are plotted. By contrast, Fig. 1(c) which shows only the traces and a few vertical sections is obviously superior. Why? The reason is one that all artists and architects know about: it takes tricks to draw three-dimensional objects on a two-dimensional object like a sheet of paper. Common tricks employed by artists, architects, as well as sophisticated computer software like *Mathematica*, include: deliberate omissions, use of color, shading, perspective, and different viewpoints.

In calculus, you need not be an artist and you need not know a lot of tricks. You need only know how to plot points and draw *traces* and *sections*. If you are unsure of the meanings of these words, check your book for clarification.

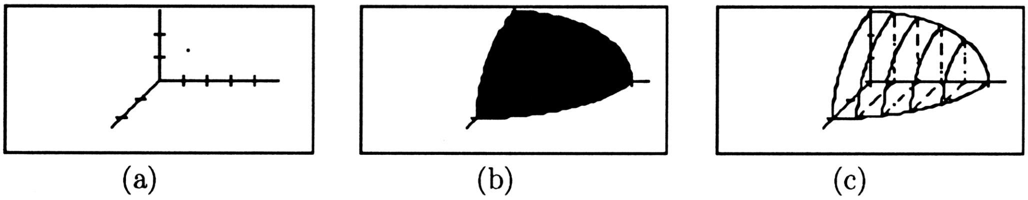

(a) (b) (c)

Figure 1

The HP 48 has a built-in 3D plot package with limited capabilities—too limited for our purposes. We encourage readers to experiment with this package, especially with "wireframe" and "Y-Slice" plots. The plots shown in Figure 2 are Wireframe plots. All other 3D plots in this chapter were produced by *Mathematica*.

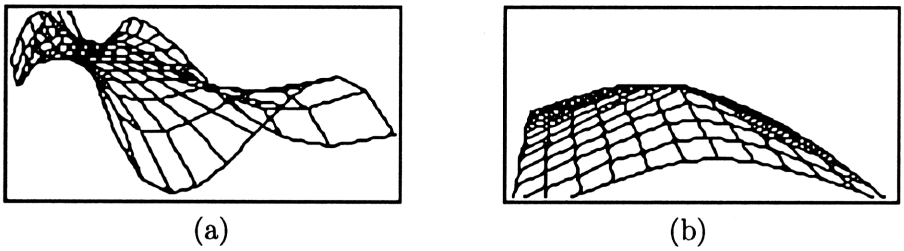

(a) (b)

Figure 2

Representation as a Set of Level Curves

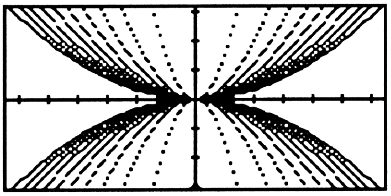
The second way of visualizing functions of two variables is through the use of *level curves*. Level curves of a function $f(x, y)$ are curves of the form $f(x, y) = c$ where c is a constant. [See § 3.3 for a discussion of *curves defined by an equation*.] The idea of visualizing a function by looking at its level curves is the same as that of visualizing a mountainous region by looking at a topographic map of the region. Thus, one thinks of level curves as being in the (x, y)-plane of an (x, y, z)-coordinate system. As with topographic maps, level curves are normally shown for a set of *equally spaced* values of c. In this way, one can tell, for example, that the surface is steep where level curves are bunched together and gradual where they are spread out.

Graphing level curves with the HP 48 is simple provided you can either solve $f(x, y) = c$ for y in terms of x or for x in terms of y or if you can represent the curves parametrically. [See § 3.1–3.3 for a detailed discussion of curve plotting.] We will illustrate these three cases with examples, then consider the general case.

EXAMPLE 1. Sketch the level curves of $f(x, y) = \dfrac{3x^2}{y}$ corresponding to $c = \pm 2, \pm 6, \pm 10, \cdots, \pm 38$.

SOLUTION. Clearly, $f(x, y) = c$ is equivalent to $y = \dfrac{3x^2}{c}$ and we may generate the required level curves as follows.

```
RESET  ≪ {#0 #0}
PVIEW DRAX −38
38 FOR C 3 EQ
STO C / X SQ
∗ DRAW 4
STEP ≫ EVAL
```



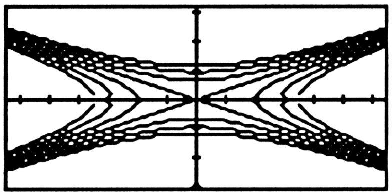
What can be said about the surface $z = f(x, y)$ by looking at these level curves? Before much of anything can be said, one needs to know which curves go with which values of c. This can be done either by keeping track of the curves as they are drawn or by studying the defining equation after they are drawn. No matter how you do it, it should be clear to you that downward parabolas correspond to negative values of c and upward parabolas to positive values of c; it should also be clear that broad parabolas correspond to the large values of $|c|$ and narrow ones to small values of $|c|$.

Now imagine taking a walk along the y-axis (which corresponds to level $c = 0$). What would the terrain look like, say, from the point $(0, 100)$? Looking away from the origin, it would be like being in a gentle valley—much like a plain. Looking towards the origin, one would see something like a box canyon with extremely high walls around it. From the negative y-axis, it would be like being on a mountain ridge, gently sloping far away from the origin but increasingly steep-sided close to the origin.

EXAMPLE 2. Sketch the level curves of $f(x, y) = x^2 - 9y^2$ corresponding to $c = 0, \pm 4, \pm 8, \pm 12$.

SOLUTION. Recalling that CONIC can be used to graph arbitrary second degree equations (see § 3.3), we may generate the required curves as follows.

```
RESET  « {#0 #0}
PVIEW  DRAX
-12  12  FOR  C
X  SQ  9  Y  SQ
*  -  C  =  EQ
STO  CONIC  DRAW
4  STEP  PICTURE
»  EVAL
```

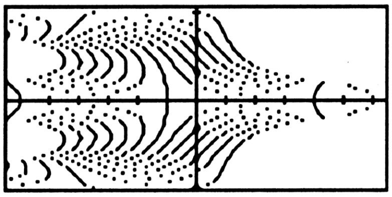


Can you tell which curve goes with which value of c? (Assume standard plot parameters.) What can you say about the surface $z = f(x,y)$ by looking at these level curves? In particular, how would you describe the scenery from the point of view of walking along the x-axis? along the y-axis? along the lines $y = \pm x/3$? See Exercise A.4.

EXAMPLE 3. Sketch the level curves of $f(x,y) = y + x \sin y$ corresponding to $c = 0$, $\pm.5$, ±1, ±1.5, ±2, ±2.5, ±3.

SOLUTION. Clearly, $f(x,y) = c$ is equivalent to $x = (c-y)/\sin y$, and we may represent these curves parametrically by taking $y = t$, $x = (c-t)/\sin t$, then use PARA (see § 3.2) to generate the required curves as follows.

```
RESET  « {#0 #0}
PVIEW  -3  3  FOR
C  C  T  -  T
SIN  /  T  -6.25
6.25  .1  PARA  .5
STEP  »  EVAL
```



Can you tell which curve goes with which c? (Cf. Example 4(a) below.)

✓Point to note

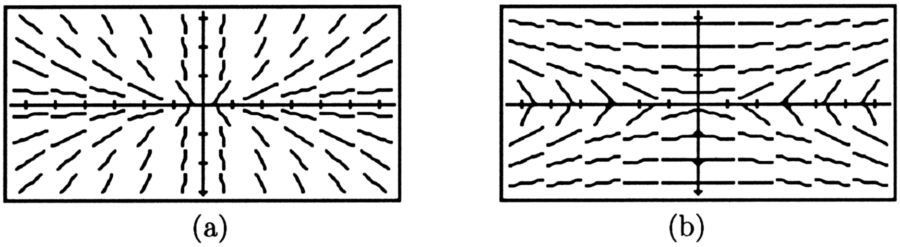
It is necessary to press ON after each of the above level curves is drawn. This is because the program PARA ends with the command PICTURE. If you wish to have the curves drawn continuously, simply delete PICTURE from PARA, and after the curves are drawn, press ◀.

How can we sketch level curves for functions like $f(x,y) = x \cos y - y \sin(x/2)$ where it is difficult or even impossible to solve for one variable in terms of the other one? In other words, how can we sketch a general curve of the form $f(x,y) = 0$? (Cf. § 3.3.) The program LEVEL, based on the ideas of § 4.5, provides an answer. It works well for most functions you are likely to encounter in a calculus course. However, there are limitations and to understand these limitations you need to be aware of the fact that the program samples "only" half of the possible x-values and makes only four "guesses" in the y-direction for each x. Refinements can be made by modifying the step sizes in the program or by adjusting the plot parameters.

If you use this program, be prepared to do something else while it's running as it is generally very time-consuming!

✓Point to note

If you want to get a quick check on the general form of level curves, use Ps-Contour in the 3D plot package. "Pseudo-Contour" plots for Examples 1 and 2 are shown in Figure 3. For further details, see your Owner's Manual.



(a) (b)

Figure 3

LEVEL	
Inputs: $f(X,Y)$, $\{c_1, \cdots, c_2\}$	*Output: Level curves* $f(x,y) = c_i$ $(i = 1, \cdots, n)$
« 1 *H 'PPAR' RCL OBJ→ 6 DROPN OBJ→ ROT OBJ→	Puts the endpoints of the viewing rectangle (determined by PPAR) on the stack
→ F L B D A C «	Introduces local variables
{#0 #0} PVIEW DRAX 1 L SIZE FOR	Enters graphics environment, draws axes
K A B FOR I I 'X' STO	Begins loops on c, x, and y (in that order)
C D FOR J F L K GET −	Forms function $f(x,y) - c_i$
EVAL Y J IFERR ROOT	Fixes x, looks for zeros of $f(x,y) - c_i$
THEN DROP DROP DROP ELSE DROP	Neglects any root calculation error
F L K GET − EVAL ABS .00001 < IF THEN X Y	Tests value to make sure it really is a zero and not an extremum
R→C PIXON END 'Y'	Plots a point corresponding to a zero
PURGE END D C − 4 / STEP B A − 65 / STEP 'X' PURGE NEXT F L » »	Ends loops, cleans up VAR menu
	Checksum: # 59221d Bytes: 397

EXAMPLE 4. For each of the following, sketch level curves of the given function for the given values of c:

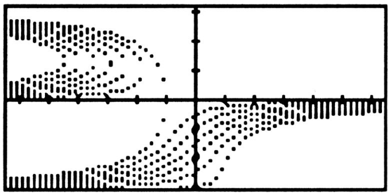
(a) $y + x\sin y$; $c = -3, -2.5, -2, -1.5, -1, -.5, 0$.

(b) $x\cos y - y\sin(x/2)$; $c = 0, \pm 2, \cdots, \pm 6$.

(c) $x\cos y - y\sin(x/2)$; $c = 0, \pm 1, \cdots, \pm 6$.

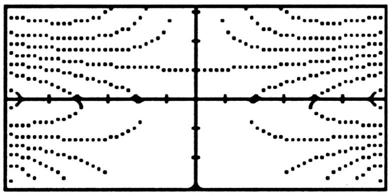
SOLUTION.

(a)
```
[RESET]  Y  X  Y
SIN  *  +
{-3  - 2.5  - 2
-1.5  - 1  - .5  0}
[LEVEL]
```



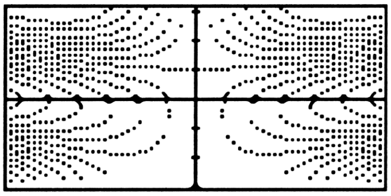
Compare this result with Example 3.

(b)
```
[RESET]  X  Y  COS
*  Y  X  2  /
SIN  *  -
{-6  - 4  - 2  0  2  4  6}
[LEVEL]
```



Can you tell which level curve goes with which c? Hint: look at the intercepts.

(c)
```
X  Y  COS  *
Y  X  2  /
SIN  *  -
{-5  - 3  - 1  1  3  5}
[LEVEL]
```



About Evaluation of Functions of Two or Three Variables

As with functions of one variable, there are different evaluation schemes to choose from for functions of several variables. We will use two analogues of FEVAL (see § 0.4) called F2EVAL and F3EVAL. Other evaluation methods are considered in the exercises.

EXAMPLE 5. Find:

(a) $g(2,1)$ and $g(1,2)$ if $g(x,y) = 3x^2 - 4y^3$.

(b) $f(1,2,3)$ if $f(r,s,t) = e^{-rst} + \sqrt{\ln(2r - s + 6t)}$.

F2EVAL

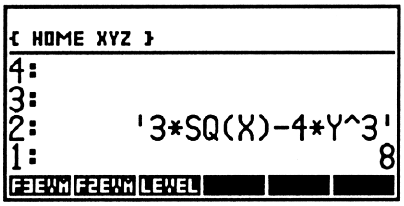
Inputs: $f(X,Y)$, a, b	*Output:* $f(X,Y)$, $f(a,b)$
`« 'Y' STO 'X' STO`	Stores a, b under names X, Y, respectively
`DUP`	Makes a second copy of $f(X,Y)$
`EVAL`	Evaluates f at (a,b)
`{X Y} PURGE »`	Purges the names X and Y
	Checksum: # 31488d Bytes: 66

F3EVAL

Inputs: $f(X,Y,Z)$, a, b, c	*Output:* $f(X,Y,Z)$, $f(a,b,c)$
`« 'Z' STO 'Y' STO` `'X' STO`	Stores a, b, c under names X, Y, Z resp.
`DUP`	Makes a second copy of $f(X,Y,Z)$
`EVAL`	Evaluates f at (a,b,c)
`{X Y Z} PURGE »`	Purges the names X, Y and Z
	Checksum: # 27367d Bytes: 82.5

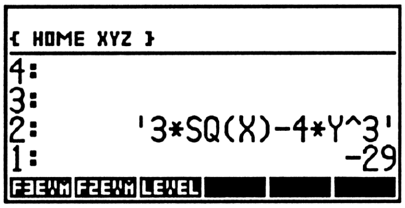
SOLUTION.

(a) 3 X SQ * 4
Y 3 ^ * −
2 1 F2EVAL



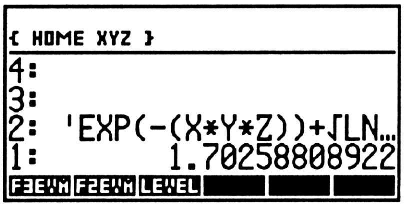
```
{ HOME XYZ }
4:
3:
2:       '3*SQ(X)-4*Y^3'
1:                     8
```

DROP 1 2
F2EVAL



```
{ HOME XYZ }
4:
3:
2:       '3*SQ(X)-4*Y^3'
1:                   -29
```

(b) X Y * Z *
+/− EXP 2 X
* Y − 6 Z
* + LN √ +
1 2 3 F3EVAL



```
{ HOME XYZ }
4:
3:
2: 'EXP(-(X*Y*Z))+√LN…
1:        1.70258808922
```

✓Point to note

The programs F2EVAL and F3EVAL assume that the variables are X, Y and X, Y, Z, respectively. It is always easy to make an adjustment in case other variable symbols are used as in Example 5(b). Alternatively, F2EVAL and F3EVAL can easily be modified to accept any variable symbols. See Exercise B.4.

Exercises 9.1

A.1 Use the method of Example 1 to sketch the level curves $f(x,y) = 3x + 2y$ corresponding to levels 0, ±1, ⋯, ±6.

A.2 Use the method of Example 2 to sketch the level curves of $f(x,y) = x + \frac{1}{4}y^2$ corresponding to levels 0, ±1, ±2, ±3, ±4.

A.3 Parametrize an arbitrary level curve of $f(x,y) = \frac{1}{4}x^2 + y^2$ and use PARA to generate the curves corresponding to $c = 1, 2, \cdots, 10$.

A.4 Make a hand sketch of the level curves of Example 2, showing which levels correspond to which curves. How would you describe the scenery from the point of view of walking along the x-axis? the y-axis? the lines $y = \pm\frac{x}{3}$?

A.5 Use F2EVAL to calculate $f(3,2) - f(2,3)$ where $f(x,y) = 170x^3 - 41xy^2 + 16y^5 - 19$.

A.6 Use F3EVAL to calculate $g(\sqrt{2}, \pi/4, \pi/5\sqrt{3})$ where $g(r,\theta,\phi) = re^{-\theta} + \sin\phi$.

B.1 Sketch the level curves of $f(x,y) = e^{0.1x^2}\sin y$ for $c = 0, \pm0.5, \pm1, \pm2, \pm3$. What's special about the cases $c = \pm1$?

B.2 Sketch the level curves of $f(x,y) = \sin x \sin y$ for $c = 0, \pm0.2, \pm0.5, \pm0.8, \pm1$. [*Suggestion*: use both of your HPs.]

B.3 Sketch the level curves of $f(x,y) = x\sin y - y\sin x$ for $c = -4, -2, 0, 2, 4, 6$. Use $(6,3)$ as center.

B.4 Modify the programs F2EVAL and F3EVAL to accept arbitrary independent variables. Rework Exercise A.5 and A.6 using these programs. (Cf. Exercise B.4, § 0.4.)

B.5 Use SOLVE (GREEN 7) to rework Example 5, then press $\boxed{\text{RESET}}$ to clean up the VAR menu.

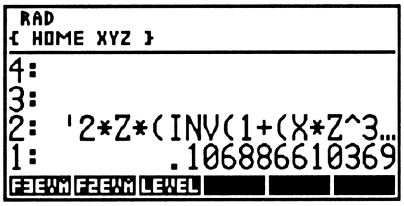
9.2 PARTIAL DERIVATIVES, DIRECTIONAL DERIVATIVES & GRADIENTS

Partial derivatives can easily be calculated for many functions on the HP 48 using the following procedure: enter the function of your choice and the variable of your choice (in that order), then press the derivative key ∂ (GREEN SIN).

EXAMPLE 1. Find $f_{yz}(1,2,3)$ for $f(x,y,z) = z^2 \arctan \frac{xz^3}{y}$.

SOLUTION.

```
Z  SQ  X  Z
3  ^  *  Y  /
ATAN  *  Y  ∂
Z  ∂  1  2  3
[F3EVAL]
```



```
RAD
{ HOME XYZ }
4:
3:
2:  '2*Z*(INV(1+(X*Z^3…
1:        .106886610369
```

About Directional Derivatives

The directional derivative of a function $f(x,y)$ at (x_0, y_0) in the direction of a unit vector $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$ is defined as:

$$f'_{\boldsymbol{v}}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t} \qquad (1)$$

Like derivatives of a function of one variable, directional derivatives represent slopes of tangent lines. The number $f'_{\boldsymbol{v}}(x_0, y_0)$ is the slope of the tangent line to the surface $z = f(x,y)$ at $(x_0, y_0, f(x_0, y_0))$ in the direction determined by $\boldsymbol{v}$. Note that the directional derivatives $f'_{\boldsymbol{i}}$ and $f'_{\boldsymbol{j}}$ are the same as the partial derivatives f_x and f_y, respectively. For further details, see your textbook.

If you apply L'Hospital's rule together with the chain rule to (1), you will obtain the following formula for directional derivatives:

$$f'_{\boldsymbol{v}}(x_0, y_0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b$$

This leads to the program DD2 for calculating directional derivatives of functions of two variables. See box. Note that to apply DD2, *the input vector need not be a unit vector*. A slight modification gives a program for calculating directional derivatives of functions of three variables. See Exercises A.7–8.

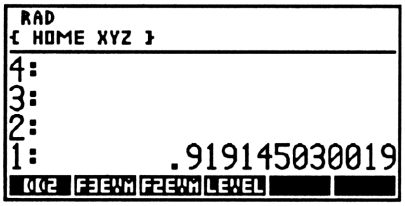
EXAMPLE 2. Find the directional derivative of $f(x,y) = e^x \cos y$ at $(0, \frac{\pi}{4})$ in the direction $5\boldsymbol{i} - 2\boldsymbol{j}$.

DD2

Inputs: $f(X,Y)$, $\boldsymbol{v} = [a,b]$, x_0, y_0	*Output:* $f'_{\boldsymbol{v}}(x_0, y_0)$
« → F V X0 Y0	Introduces local variables
« V V ABS / V→	Normalizes $\boldsymbol{v}$
→ A B «	Intro. loc. vars. for the components of $\boldsymbol{v}$
F X ∂ A * F Y	Calculates the directional derivative
∂ B * +	
X0 'X' STO Y0	Stores x_0 in X, y_0 in Y
'Y' STO	
EVAL {X Y}	Evaluates the dir. der. at (x_0, y_0)
PURGE » » »	
	Checksum: # 62651d Bytes: 171.5

SOLUTION.

X EXP Y COS
* [5 -2] 0 π
4 / $\boxed{\text{DD2}}$



```
 RAD
{ HOME XYZ }
4:
3:
2:
1:          .919145030019
```

About Gradients

From the algebraic point of view, gradients are trivial. The gradient of a function f is defined by $\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j}$ for functions of two variables and by $\nabla f = f_x\boldsymbol{i} + f_y\boldsymbol{j} + f_z\boldsymbol{k}$ for functions of three variables.

The significance of gradients comes from their geometric interpretations. There are three main interpretations:

(1) If f is a function of two variables, then ∇f is perpendicular to all of the level curves of f. In particular, if a curve is defined by $f(x,y) = 0$ and if (x_0, y_0) is any point on the curve, then $\nabla f(x_0, y_0)$ is normal (i.e., perpendicular) to the curve at (x_0, y_0).

(2) If f is a function of three variables, then ∇f is perpendicular to all of the level surfaces of f. That is, if c is any number and if $f(x_0, y_0, z_0) = c$, then $\nabla f(x_0, y_0, z_0)$ is perpendicular to the tangent plane to the surface $f(x,y,z) = c$ at (x_0, y_0, z_0). In particular, if a surface is defined by $f(x,y,z) = 0$ and if (x_0, y_0, z_0) is any point on the surface, then $\nabla f(x_0, y_0, z_0)$ is normal to the surface at (x_0, y_0, z_0).

(3) If f is a function of two variables, then ∇f represents the direction of steepest ascent and $-\nabla f$ the direction of steepest descent. Moreover, $|\nabla f|$ *is* the steepest ascent (= the maximum directional derivative) and $-|\nabla f|$ the steepest descent (= the minimum directional derivative).

The programs DEL2 and DEL3 find the gradient of an arbitrary function f of two and three variables, respectively.

DEL2

Inputs: $f(X,Y)$, a, b	*Output:* $\nabla f(a,b)$

```
« 'Y' STO 'X' STO →
  F «
  F 'X' ∂ F 'Y' ∂ 2
  →ARRY {X Y}
  PURGE » »
```

Stores a, b under names X, Y, respectively

Calculates ∇f
Cleans up VAR menu

Checksum: # 22866d Bytes: 109

DEL3

Inputs: $f(X,Y,Z)$, a, b, c	*Output:* $\nabla f(a,b,c)$

```
« 'Z' STO 'Y' STO
  'X' STO → F «
  F 'X' ∂ F 'Y'
  ∂ F 'Z' ∂ 3
  →ARRY {X Y Z}
  PURGE » »
```

Stores a, b, c under names X, Y, Z, resp.

Calculates ∇f

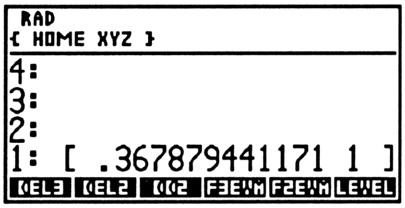
Cleans up VAR menu

Checksum: # 42555d Bytes: 142

EXAMPLE 3. Find a vector perpendicular to the curve $xe^{-y^2} + y\cos x = 1$ at $(0,1)$.

SOLUTION.

```
X Y SQ [+/-]
EXP * Y X
COS * + 0 1
[DEL2]
```

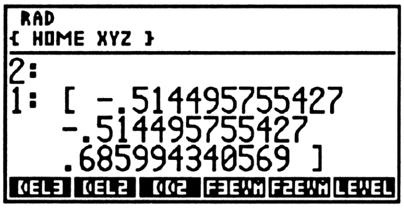


```
RAD
{ HOME XYZ }
4:
3:
2:
1: [ .367879441171 1 ]
```

EXAMPLE 4. Find the inner unit normal to the ellipsoid $3x^2 + y^2 + 2z^2 = 14$ at the point $(-1,-3,2)$.

SOLUTION.

```
3 X SQ * Y
SQ + 2 Z
SQ * + -1
-3 2 [DEL3]
[ENTER] ABS /
```



```
RAD
{ HOME XYZ }
2:
1: [ -.514495755427
     -.514495755427
     .685994340569 ]
```

This is the outer unit normal. Why? The inner unit normal points in the opposite direction:

ANSWER: [.514495755427 .514495755427 − .685994340569]
(= $0.514495755427\,\boldsymbol{i} + 0.514495755427\,\boldsymbol{j}$
$-0.685994340569\,\boldsymbol{k}$).

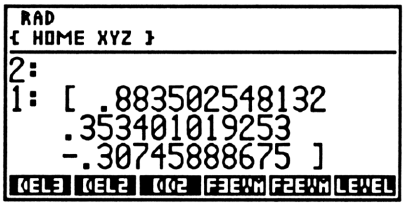
EXAMPLE 5. Suppose that a mountain has equation $z = 3 - 0.05x^2 + 0.03y^2$ and that a skier is skiing down the fall line (the line of direct descent down the mountain). Find the direction pointed by the skier's skis as a unit vector in three-dimensional space at the point $(3, 2, 2.43)$.

SOLUTION. The direction of steepest descent as a vector in the xy-plane is given by $-\nabla f$. Therefore, a unit vector in that direction is given by $\boldsymbol{u} = -\nabla f/|\nabla f|$. The slope of the surface in the direction $\boldsymbol{u}$ is $m = -|\nabla f|$, the steepest descent. It follows that the direction pointed by the skier skis is

$$\boldsymbol{u} + m\boldsymbol{k} = -\frac{\nabla f}{|\nabla f|} - |\nabla f|\boldsymbol{k}.$$

We may normalize and evaluate this expression at (3, 2, 2.43) as follows. (Note that ARG = GREEN EEX.)

```
3  .05  X  SQ  *
–  .03  Y  SQ
*  –  3  2  0
DEL3  +/–
ENTER  ABS  /
ARG  SWAP
DROP  [0  0  1]
*  –  ENTER
ABS  /
```



```
RAD
{ HOME XYZ }
2:
1: [ .883502548132
     .353401019253
    -.30745888675 ]
```

Exercises 9.2

Exercises 1–6 are routine. We suggest that you first work these problems by hand, then check your results with the HP 48.

A.1 Find $f_x(2, 5)$ where $f(x, y) = 4 - 3x^3 - 6y^2$.

A.2 Find $\left.\dfrac{\partial z}{\partial y}\right|_{(3,-1)}$ where $z = \sqrt{5x^2 + 3y^2 + 1}$.

A.3 Find $g_{rts}(1, -1, 1)$ where $g(r, s, t) = 5r^3t - 2s^2t + \ln(r^2 + s^2 + t^2)$.

A.4 Find the directional derivative of $f(x, y) = 2xy^3$ at the point $(-1, 2)$ in the direction $2\,\boldsymbol{i} - 3\,\boldsymbol{j}$.

A.5 Find the gradient of $f(x,y) = 125(x^2+y^2)^{-1/2}$ at the point $(4,3)$.

A.6 Find a unit vector perpendicular to the surface $x^3z + 2xy^2 - yz^3 = 9$ at the point $(1,2,-1)$.

A.7 Write an HP 48 program, DD3, to calculate directional derivatives of functions of three variables. [*Hint*: either (1) recall and modify DD2 or (2) use the formula $f'_{\boldsymbol{v}}(x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0)\cdot\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}$ and the program DEL3.]

A.8 Use DD3 from Exercise A.7 to find the directional derivative of the function $f(x,y,z) = \dfrac{x-z}{y+z}$ at the point $(1,2,3)$ in the direction of the point $(3,-7,-4)$.

9.3 TANGENT PLANES AND NORMAL LINES

In this section, we obtain programs to generate equations of tangent planes and normal lines to a given surface at a given point. We consider the following two situations:

(1) The surface is defined by $z = f(x,y)$;
(2) The surface is defined by $g(x,y,z) = 0$.

It is important to be able to deal with both situations and it is essential to distinguish one from the other before you start applying a formula or a program. An example of (1) is the paraboloid $z = x^2+y^2$ $(f(x,y) = x^2+y^2)$; an example of (2) is the sphere $x^2+y^2+z^2 = 4$ $(g(x,y,z) = x^2+y^2+z^2-4)$. Note that (2) is more general than (1) because any equation of the form $z = f(x,y)$ can be rewritten in the equivalent form $f(x,y) - z = 0$.

About Planes and Lines

To find the equation of a plane, two things are needed: a point on it and a vector perpendicular to it. If the point is (x_0,y_0,z_0) and the vector is $A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$, then the equation of the plane is:

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

To find the equation of a line in three-dimensional space, two things are needed: a point on it and a vector in the direction of it. If the point is (x_0,y_0,z_0) and the vector is $A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$, then the equation of the line in parametric form is:

$$x = x_0 + At,\ \ y = y_0 + Bt,\ \ z = z_0 + Ct,\ \ -\infty < t < \infty.$$

The programs PLANE and LINE are based on the above formulas.

✓Point to note

The HP 48 has a built-in command called LINE that has nothing to do with the program we are calling LINE here. (The HP 48 command is for drawing a line in two-dimensional space.) Because of this, the HP 48 won't let you name the program in uppercase letters, but it's o.k. to use lowercase letters. Enter this as follows:

$$\boxed{\alpha}\quad\boxed{\alpha}\quad\boxed{\Lsh}\quad\boxed{\alpha}\quad \text{LINE}\quad\boxed{\alpha}\quad\boxed{\text{STO}}$$

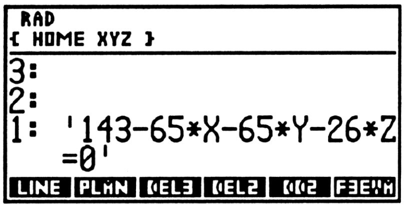
PLANE	
Inputs: x_0, y_0, z_0, $\boldsymbol{v} = [a\ \ b\ \ c]$	*Output: equation of the plane through* (x_0, y_0, z_0) *and perpendicular to* $\boldsymbol{v}$
≪ V→ → X0 Y0 Z0 A	Introduces local variables
B C ≪ A X X0 −	Forms the expression
∗ B Y Y0 − ∗ +	$a(x-x_0)+b(y-y_0)+c(z-z_0)$
C Z Z0 − ∗ +	
EXPAN EXPAN COLCT	Simplifies expression
0 = ≫ ≫	Forms equation
	Checksum: # 64567d Bytes: 135.5

LINE (see Point to note, above)	
Inputs: x_0, y_0, z_0, $\boldsymbol{v} = [a\ \ b\ \ c]$	*Output: parametric equations of the line through* (x_0, y_0, z_0) *in the direction* $\boldsymbol{v}$
≪ V→ → X0 Y0 Z0	Introduces local variables
A B C ≪ X X0	Forms parametric representation
A T ∗ + =	in list form
Y Y0 B T ∗ +	
= Z Z0 C	
T ∗ + = 3	
→LIST ≫ ≫	
	Checksum: # 36251d Bytes: 143

EXAMPLE 1. Find the equation of the plane through the points A = (−1, 2, 3), B = (4, −5, 8), and C = (2, 3, −7).

SOLUTION. Since $\boldsymbol{v} = \overrightarrow{AB} \times \overrightarrow{CB} = [5\ \ -7\ \ 5] \times [2\ \ -8\ \ 15]$ is a vector perpendicular to the plane (why?), we may obtain the equation as follows.

−1 2 3 [5 −7 5]
[2 −8 15] CROSS
[PLANE]



```
RAD
{ HOME XYZ }
3:
2:
1: '143-65*X-65*Y-26*Z
   =0'
```

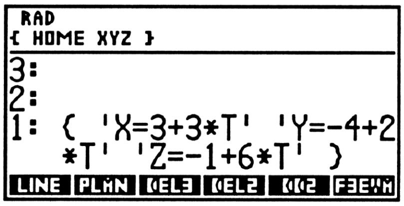
✓Point to note

CROSS is in MATH VECTR.

EXAMPLE 2. Find the equation of the line through the points A = (3, −4, −1) and B = (6, −2, 5).

SOLUTION. Clearly, $\overrightarrow{AB} = [3\ \ 2\ \ 6]$ is a vector in the direction of the line, so we may obtain the equation as follows.

3 −4 −1
[3 2 6] $\boxed{\text{LINE}}$



```
RAD
{ HOME XYZ }
3:
2:
1: { 'X=3+3*T' 'Y=-4+2
   *T' 'Z=-1+6*T' }
```

How can we find the tangent plane to a surface at a given point (x_0, y_0, z_0)? Since we are given a point, all we need is a vector perpendicular to the plane. Likewise, to find the normal line to the surface at (x_0, y_0, z_0), all we need is a vector in the direction of the normal line. Obviously, we are talking about one and the same vector. How can we find it?

Interpretation (2) of the gradient in the previous section gives the answer: if a surface is defined by $g(x, y, z) = 0$, then a normal vector is given by $\nabla g(x_0, y_0, z_0)$; if a surface is defined by $z = f(x, y)$, a normal vector is given by $f_x(x_0, y_0)\boldsymbol{i} + f_y(x_0, y_0)\boldsymbol{j} - \boldsymbol{k}$. Consult your book for additional details.

The programs TN2 and TN3 are based on the above considerations. Use TN2 if the surface is defined by $z = f(x, y)$; use TN3 if it is defined by $g(x, y, z) = 0$.

✓Points to note

1. To apply TN2 you need to have TN3 in the same directory.
2. To apply TN3, PLANE and LINE should be in the same directory.

TN2	
Inputs: $f(X, Y)$, x_0, y_0	*Output: equations of tangent plane and normal line to the surface* $z = f(x, y)$ *at* $(x_0, y_0, f(x_0, y_0))$.

```
« → F X0 Y0                  Introduces local variables
  « F Z -
    X0 Y0 X0 'X'             Puts z = f(x,y) in form f(x,y) - z = 0
    STO Y0
    'Y' STO F                Calculates f(x0,y0)
    EVAL {X Y}
    PURGE TN3 » »            Applies TN3

                               Checksum: # 57561d  Bytes: 128
```

TN3

Inputs: $g(X,Y,Z)$, x_0, y_0, z_0	*Output: equations of tangent plane and normal line to the surface* $g(x,y,z)=0$ *at* (x_0,y_0,z_0).

```
« 4 DUPN DEL3          Calculates ∇g
  5 ROLL
  DROP 4 DUPN          Applies PLANE
  PLANE 5
  ROLLD line »         Applies LINE

                          Checksum: # 63229d   Bytes: 63.5
```

EXAMPLE 3. Find the equations of the tangent plane and normal line to the surface $z = \sin x \sin 3y$ at $(\pi/6, \pi/4)$.

SOLUTION.

```
INPUT:    X  SIN  3  Y  *  SIN  *  π
          6  /  π  4  /  TN2
OUTPUT:   '.865956483959 + .612372435697*X
          −1.06066017178*Y − Z = 0'
          {'X = .52359877559 + .612372435697*T'
          'Y = .785398163398 − 1.06066017178*T'
          'Z = .353553390594 − T'}
```

EXAMPLE 4. Find equations of the tangent plane and normal line to the paraboloid $x = 5 - y^2/9 - z^2/16$ at $y = 2, z = -2$.

SOLUTION. This equation is equivalent to $x + y^2/9 + z^2/16 - 5 = 0$. Thus, we can generate the required equations as follows:

```
INPUT:    X  Y  SQ  9  /  +  Z  SQ  16  /  +  5
          −  5  4  9  /  −  4  16  /  −  2  −2  TN3
OUTPUT:   ' − 5.69444444445 + X + .444444444444*Y
          −.25*Z = 0'
          {'X = 4.30555555556 + T'
          'Y = 2 + .444444444444*T'
          'Z = −2 − .25*T'}
```

Exercises 9.3

Exercises 1–8 below are routine. We suggest that you first work these problems by hand, then check your results using the HP 48.

A.1 Find an equation of the plane passing through the point $(1,-2,3)$ and having normal vector $\boldsymbol{n} = [2,-3,4]$.

A.2 Find parametric equations for the line that passes through the point $(5, -2, 1)$ and parallel to the vector $\boldsymbol{v} = [1, 2, 3]$.

A.3 Find parametric equations of the line that passes through the points $(5, -1, 2)$ and $(3, 0, -7)$.

A.4 Find an equation of the plane passing through the points $(1, -2, 3)$, $(2, 0, -1)$ and $(-1, 4, 0)$.

A.5 Find parametric equations for the line through the point $(-3, 2, -5)$ and parallel to the x-axis.

A.6 Find an equation of the plane through $(-3, 2, -5)$ and parallel to the xz-plane.

A.7 Find an equation of the tangent plane to the surface $z = \sqrt{4y^2 - x^2}$ at the point $(8, -5, 6)$.

A.8 Find parametric equations for the normal line to the ellipsoid $3x^2 + 5y^2 + 2z^2 = 19$ at the point $(-2, 1, -1)$.

For Exercises 1–5 below, (a) write HP 48 programs to perform the indicated tasks; (b) test your programs on examples or exercises from your calculus book.

B.1 Input: $[a_1,\ a_2,\ a_3]$, $[b_1,\ b_2,\ b_3]$

Output: parametric equations for the line that passes through the points (a_1, a_2, a_3) and (b_1, b_2, b_3)

B.2 Input: $[a_1,\ a_2,\ a_3]$, $[b_1,\ b_2,\ b_3]$

Output: projection of $a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ on $b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

[You may assume that $[b_1,\ b_2,\ b_3] \neq \mathbf{0}$.]

B.3 Input: $[a_1,\ a_2,\ a_3]$, $[b_1,\ b_2,\ b_3]$, $[c_1,\ c_2,\ c_3]$

Output: area of the triangle with vertices (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3).

B.4 Input: x_0, y_0, z_0, A, B, C, D

Output: distance from point (x_0, y_0, z_0) to plane $Ax + By + Cz = D$.

B.5 Input: $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$

Output: volume of the parallelepiped determined by $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$.

B.6 As everyone knows, three points determine a plane. More precisely, three *noncollinear* points determine a plane. The following program, PLN3, takes as input three points in vector (bracket) form and gives as output an equation of the plane determined by them.

```
≪ → A B C ≪ A V→ B A - C A -
            CROSS PLANE ≫ ≫
```

(a) Use PLN3 to rework Example 1 and Exercise A.1.

(b) What would happen if you were to apply PLN3 to three *collinear* points?

9.4 MAX-MIN PROBLEMS

For a differentiable function $f(x)$ of one variable, one way to solve max-min problems is to carry out the following 2-step procedure:

Step 1. Solve the equation $f'(x) = 0$ for x.
Step 2. Classify each solution as a maximum, a minimum, or neither.

Step 1 can be done by any of the zero-finding methods discussed in Chapter 4; Step 2 is usually just a matter of looking at the graph of $f(x)$ or other information available. If you have any doubts about the classification, you can always use one of the traditional methods, e.g., the second derivative test. For further details, see your text.

For functions of two variables the procedure is similar:

Step 1. Solve the two equations $f_x = 0$, $f_y = 0$ for x and y.
Step 2. Classify each solution as a maximum, a minimum, or neither.

Solving the system in Step 1 is somewhat more difficult than solving the equation $f'(x) = 0$ and special methods are needed. One such method is the *method of steepest descent.*

The Steepest Descent Algorithm

The *method of steepest descent* is a numerical algorithm for locating a relative minimum of a function, starting with a reasonably good guess at its location. It is easily modified to deal with the case of a relative maximum and can be extended to higher dimensions.

The method is a two-dimensional analogue of the following observation. In searching for a relative minimum of a function of one variable, we may start with an estimate x_1 and look at the sign of $f'(x_1)$. If $f'(x_1) < 0$, we search to the right of x_1 since the function is decreasing near x_1. If $f'(x_1) > 0$, we search to the left of x_1. Note that, in either case, we search in the direction of *descent* of the function $f(x)$.

For functions of two variables, things are more complicated because there are infinitely-many directions to take into account, but the idea is similar: we start with a guess (x_1, y_1) and search in the direction of steepest descent. Recall that the direction of steepest descent is given by $-\nabla f$ (see § 9.2).

To be more specific, from the first guess (x_1, y_1) let us proceed in the direction $-\nabla f$ until we reach the point where we can descend no farther. Let the coordinates of this low point be (x_2, y_2). We then recalculate the direction of steepest descent at (x_2, y_2) and head in this new direction until we reach the point where we can descend no farther. And so on. In this way we obtain a sequence $\{(x_n, y_n)\}$ of points which evidently converges to the desired minimum point.

The HP 48 program EXTREME is based on the above algorithm, and can be used to locate either *maxima* or *minima.* As with any other iterative procedure, reliability depends on the accuracy of the first guess. As always, things can go wrong and you would be flirting with danger if you failed to have your other HP engaged.

EXAMPLE 1. Figure 4(a) (produced with LEVEL) shows the level curves of

$$f(x, y) = x^4 + 3x^2y + 5y^2 + x + y$$

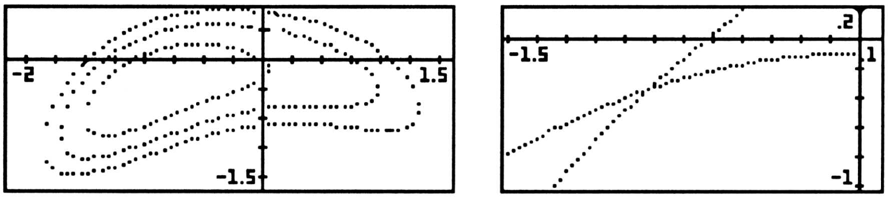
EXTREME

Inputs: $f(X,Y)$, x_n, y_n	*Output:* $f(X,Y)$, x_{n+1}, y_{n+1}
« → F X1 Y1	Introduces local variables
« F X1 Y1 DEL2 DUP	Calculates the direction of steepest ascent $[a,b] = \nabla f/\lvert\nabla f\rvert$ at the given point (x_n, y_n)
[0 0] ≠ IF THEN DUP ABS / V→ →	Begins IF-THEN-ELSE-END; allows for two possibilities: $[a,b] \neq 0$ and $[a,b] = 0$
A B « X1 A T * + 'X' STO Y1 B T * + 'Y' STO F EVAL 'T' ∂ 'T' 0 ROOT	If $[a,b] \neq 0$, program finds extreme point (x_{n+1}, y_{n+1}) of $g(t) = f(x_n + at, y_n + bt)$ near $t = 0$ by solving for the zero of $g'(t)$ near $t = 0$ and puts $f(x,y)$, x_{n+1}, y_{n+1} on stack
DROP F X1 A T * + Y1 B T * + { T X Y } PURGE » ELSE	Cleans up directory
DROP F X1 Y1 END » »	If $[a,b] = 0$, program returns $f(x,y)$, x_n, y_n to stack
	Checksum: # 60914d Bytes: 329

corresponding to c = 0, 1, 2. Figure 4(b) (also produced with LEVEL) shows the intersection of the two curves

$$\begin{cases} f_x = 4x^3 + 6xy + y = 0 \\ f_y = 3x^2 + 10y + 1 = 0 \end{cases}.$$

Figure 4(a) suggests that elevations decrease inwardly and that f has a relative minimum somewhere inside the inner loop. Make an "educated guess" at the location of the minimum point, then use EXTREME to refine your guess.



(a) $f(x,y) = 0, 1, 2$

(b) $f_x(x,y) = 0,$
$f_y(x,y) = 0$

Figure 4

SOLUTION. From Figure 4(a), a reasonable guess would be $(-1, -0.5)$; from Figure 4(b), $(-0.87, -0.33)$.

We apply EXTREME starting with the very rough first guess of $(-1, 0)$.

```
INPUT:    X  4  ^  3  X  SQ  *  Y
          *  +  5  Y  SQ  *  +
          X  +  Y  +  -1  0  EXTREME

OUTPUT:   'X^4 + 3*SQ(X)*Y + 5*SQ(Y) + X + Y'
          -.797109531689
          -.270520624414
```

Notice that the output is arranged so that EXTREME may be easily applied again. We need only continue executing EXTREME to obtain a sequence $\{(x_n, y_n)\}$ of approximations. The values in Table 1 strongly suggest that the points (x_n, y_n) are approaching a limiting value, that the numbers $f(x_n, y_n)$ are approaching a relative minimum, and that the length of the gradient is approaching 0 (as it should).

For this particular function it is not difficult to find the minimum using direct methods (you are asked to do this in Exercise B.4). The direct method yields the answers $(x_0, y_0) = (-0.88632420664\cdots, -0.33567117978\cdots)$ and $f(x_0, y_0) = -0.83257874487\cdots$.

Table 1 Successive Approximations to a Relative Minimum of $f(x, y) = x^4 + 3x^2y + 5y^2 + x + y$

n	(x_n, y_n)	$f(x_n, y_n)$	$\|\nabla f\|$
1	$(-1, 0)$	0	5
2	$(-.797109531689, -.270520624414)$	$-.813663897902$	.334907620642
3	$(-.885563139165, -.336860830024)$	$-.832564710439$	.019927599878
4	$(-.886407055843, -.335735607788)$	$-.832578727064$	.000339453236
5	$(-.886323218811, -.335672730015)$	$-.832578744851$	.000025944422
6	$(-.886324317078, -.335671265659)$	$-.832578744877$	.000000452436
7	$(-.886324205332, -.335671181851)$	$-.832578744878$	.000000034544
8	$(-.886324206794, -.335671179901)$	$-.832578744878$	.0000000006
9	$(-.886324206642, -.335671179787)$	$-.832578744876$	.000000000042
10	$(-.886324206645, -.335671179784)$	$-.832578744877$	.000000000022

The Second Derivative Test

One way to classify the critical points of a function of two variables is to use the *second derivative test.* For details, see your text. The program SDTST (see box) does the routine calculation for you.

✓Points to note

1. The output is really a white lie based on another white lie. We are telling the machine that (x_{10}, y_{10}) is a critical point when in fact it is only an *approximation* to a critical point.

SDTST	
Inputs: $f(X,Y)$, a, b where (a,b) is a critical point.	*Output: (message)* "MAXIMUM POINT", "MINIMUM POINT", "SADDLE POINT", "TEST FAILS", *or* "NOT A CRITICAL POINT!"

```
≪ → F A B ≪ F X ∂
  X ∂ DUP F Y ∂ Y
  ∂ * F X ∂ Y ∂ SQ
  - F X ∂ SQ F Y
  ∂ SQ + A 'X' STO
  B 'Y' STO EVAL
  IF .00001 >
  THEN CLLCD "NOT A
  CRITICAL POINT!" 3 DISP
  3 FREEZE DROP DROP
  ELSE EVAL SWAP EVAL
  → D C ≪
  D 0 ==
  IF THEN CLLCD
  "     TEST FAILS" 3 DISP
  3 FREEZE ELSE D 0
  < IF THEN CLLCD
  "     SADDLE POINT" 3
  DISP 3 FREEZE ELSE
  C 0 > IF THEN CLLCD
  "    MINIMUM POINT"
  3 DISP 3 FREEZE
  ELSE CLLCD
  "    MAXIMUM POINT"
  3 DISP 3
  FREEZE END END
  END ≫ END F
  {X Y} PURGE ≫ ≫
```

Introduces local variables
Calculates f_{xx} and $f_{xx}f_{yy} - f_{xy}^2$ at (a,b)
Calculates $f_x^2 + f_y^2$ at (a,b)

Checks that point is critical point

Introduces local variables for $f_{xx}f_{yy} - f_{xy}^2$ and f_{xx}.
Sets up IF-THEN-ELSE-END branches for the various combinations of signs

Checksum: # 54213d Bytes: 518

The machine, in turn, tells us that it is a minimum point. Such white lies are not apt to cause trouble provided $f_{xx}f_{yy} - f_{xy}^2$ is continuous which is generally the case.

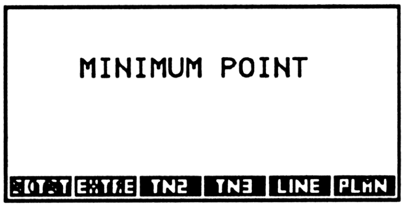
2. The program will accept critical points or good approximations of critical points, but not points that are way off. In such cases, it will tell you: "NOT A CRITICAL POINT!" This feature gives the program a second use, namely as a double-checker of hand-calculated critical points.

3. To assure that the Checksum and Bytes of your program match the ones given in the box, put exactly five spaces each in front of TEST FAILS" and SADDLE POINT", and four spaces each in front of MINIMUM POINT" and MAXIMUM POINT".

EXAMPLE 2. Use the program SDTST to test the point $(x_{10}, y_{10}) = (-.886324206645, -.335671179784)$ of Example 1. (See Table 1.)

SOLUTION.

X 4 ^ 3 X
SQ $*$ Y $*$
+ 5 Y SQ $*$
+ X + Y +
−.886324206645
−.335671179784
$\boxed{\text{SDTST}}$


MINIMUM POINT

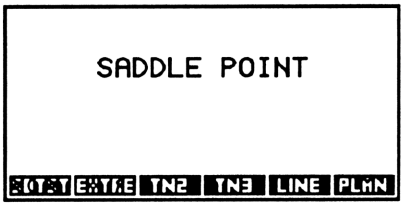
EXAMPLE 3. Find and classify the critical points of $f(x, y) = 3xy - x^3 - y^3$.

SOLUTION. Here the critical points (0, 0) and (1, 1) are easily found by solving the system:

$$\begin{cases} f_x = 3y - 3x^2 = 0 \\ f_y = 3x - 3y^2 = 0 \end{cases}$$

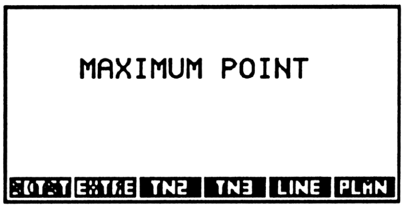
We classify them using SDTST.

3 X $*$ Y $*$
X 3 ^ − Y
3 ^ − 0 0
$\boxed{\text{SDTST}}$


SADDLE POINT

Notice that when you press ON, you find $f(x, y)$ on the stack so that you can reapply the test to (1, 1):

1 1 $\boxed{\text{SDTST}}$


MAXIMUM POINT

Exercises 9.4

For Exercises 1–5, find and classify the critical points for each function. We recommend that you work these problems first by hand, then check your results with the HP 48.

A.1 $x^2(x-1)^2 + y^2$

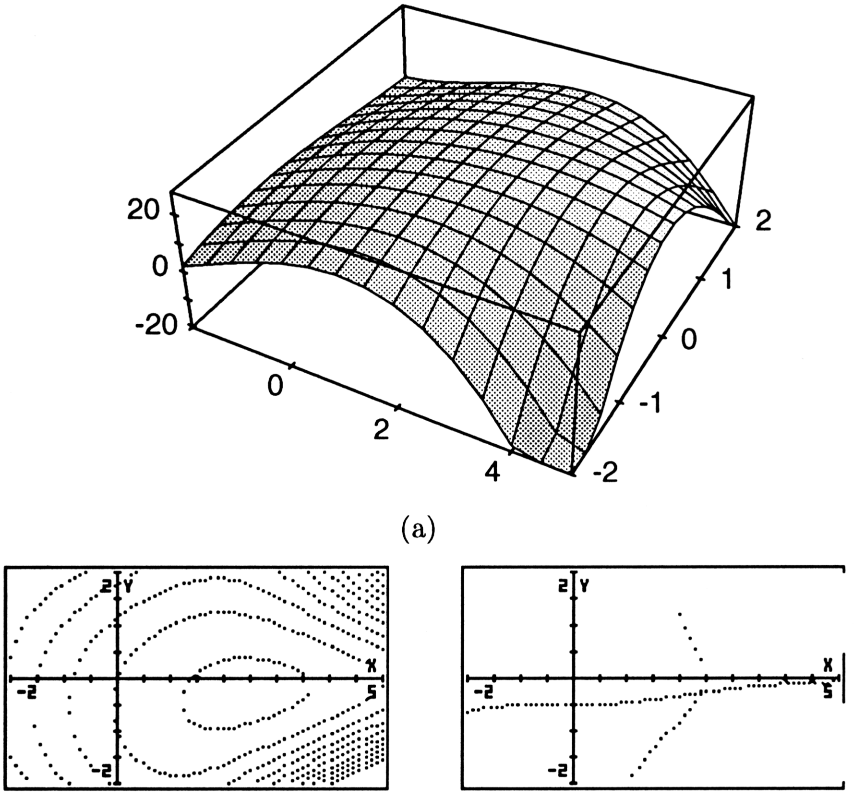
A.2 $2x^2 + y^2 - xy^2$

A.3 $8x^3 - 3xy - y^3$

A.4 $(x-1)e^x(y^2-2y)$

A.5 $2xy^2 - x^2y + 4xy$

A.6 This problem is concerned with the location of the maximum point of the function $f(x,y) = 20 + 5x - x^2 - 2y - 2y^2 + 0.1x^2y^3 - 0.1x^3y^2$ for $-2 \le x \le 5$, $-2 \le y \le 2$. Figure 5 shows (a) the surface, (b) some level curves, and (c) the curves $f_x = 0$, $f_y = 0$. Use all three parts of Fig. 5 to make an "educated guess" at the location of the maximum point, then use EXTREME to refine your guess. Apply EXTREME repeatedly until there is no change, then calculate f and $|\nabla f|$ at the point you found. Finally, use SDTST to verify that it is a maximum point. (Parts (a) of Figures 5–8 were produced by *Mathematica*.)



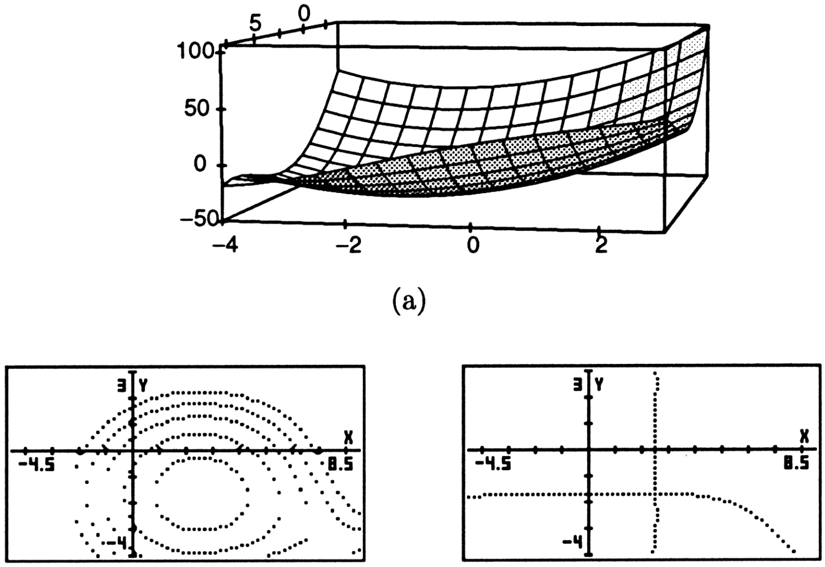
(a)

(b)

(c)

Figure 5

A.7 This problem is concerned with the location of the minimum point of the function $f(x,y) = 2x^2 + 3y^2 - 10x + 10y - 0.001x^4y^2 + 0.001x^2y^4 - 25$ for $-4 \le x \le 8$, $-4 \le y \le 3$. Figure 6 shows (a) the surface, (b) some level curves, and (c) the curves $f_x = 0$, $f_y = 0$. Use all three parts of Fig. 6 to make an "educated guess" at the location of the minimum point, then use EXTREME to refine your guess. Apply EXTREME repeatedly until there is no change, then calculate f and $|\nabla f|$ at the point you found. Finally, use SDTST to verify that it is a minimum point.



(a)

(b) (c)

Figure 6

For Exercises 1–3, find and classify the critical points for each function. Use any method.

B.1 $2x^2 - 5x + 8 - 2x^2y + 6xy - xy^2$

B.2 $x^2 - \sin x + 2y^3 - 3y^2$

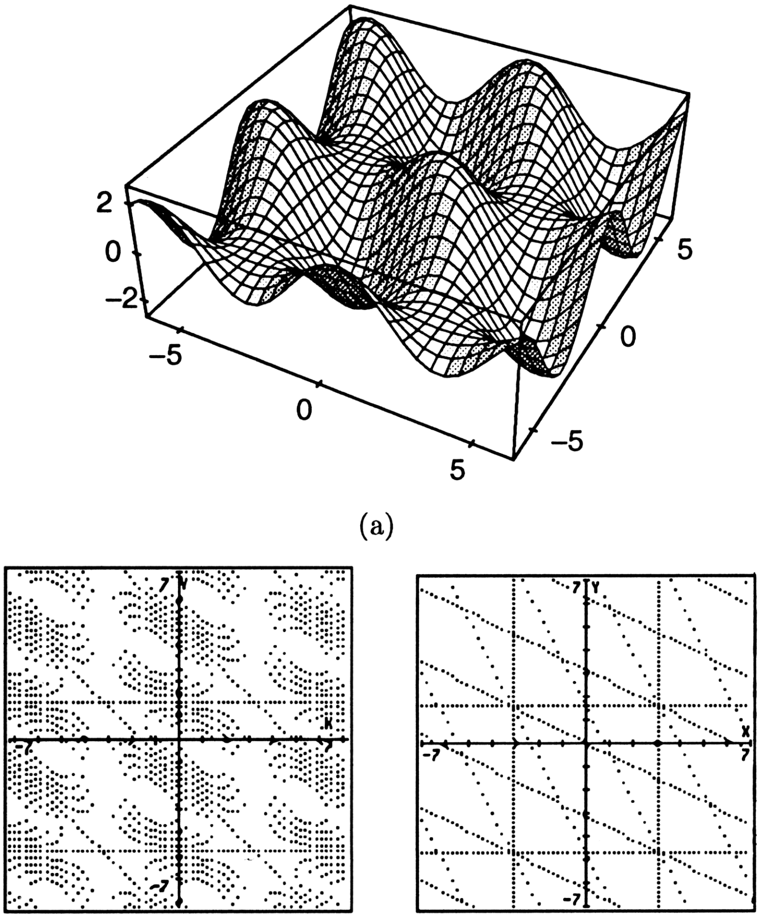
B.3 $(2x^3 - 4x - 1)(y^2 - 1)$

B.4 Obtain the result of Example 1 by using direct methods to solve the system:

$$\begin{cases} f_x = 4x^3 + 6xy + y = 0 \\ f_y = 3x^2 + 10y + 1 = 0 \end{cases}$$

B.5 This problem concerns the location of extrema of the function $f(x, y) = \cos(x + y) + \sin x + \cos y$ for $-2\pi \leq x \leq 2\pi$, $-2\pi \leq y \leq 2\pi$. Figure 7 shows (a) the surface, (b) some level curves, and (c) the curves $f_x = 0$, $f_y = 0$. Locate at least one maximum point and at least one minimum point as follows: (1) select a high spot and a low spot in Fig. 7(a) (your choice); (2) locate the corresponding inner loops in Fig. 7(b); (3) locate the corresponding critical points in Fig. 7(c) and estimate their coordinates; (4) apply EXTREME (repeatedly) to your estimates to obtain 12-digit approximations to the extreme points; (5) use SDTST to check your conclusions.

B.6 This problem concerns the location of extrema of the function $f(x, y) = x \sin y - y \sin x$ for $0 \leq x \leq 15$, $0 \leq y \leq 15$. Figure 8 shows (a) the surface, (b) some level curves, and (c) the curves $f_x = 0$, $f_y = 0$. Locate at least one maximum point and at least one minimum point as follows: (1) select a high spot and a low spot in Fig. 8(a) (your choice); (2) locate the corresponding



(a)

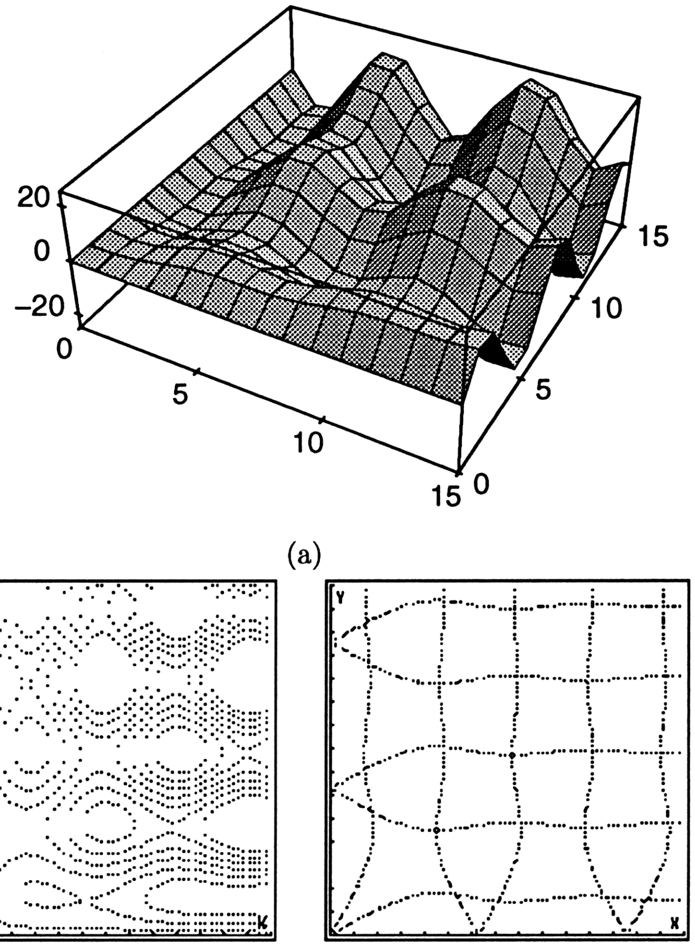
(b) (c)

Figure 7

inner loops in Fig. 8(b); (3) locate the corresponding critical points in Fig. 8(c) and estimate their coordinates; (4) apply EXTREME (repeatedly) to your estimates to obtain 12-digit approximations to the extreme points; (5) use SDTST to check your conclusions.

C.1 (a) The curves in Fig. 7(c) resemble straight lines. Show that they are indeed straight lines and determine their equations.

(b) Find exact formulas for all maxima and minima of the function $f(x, y) = \cos(x + y) + \sin x + \cos y$.



(a)

(b)

(c)

Figure 8

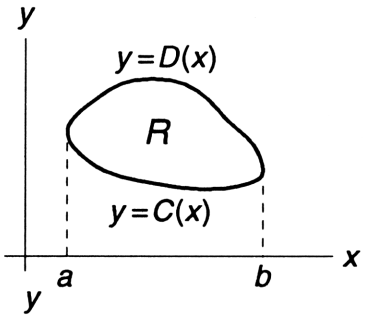
9.5 DOUBLE INTEGRALS, TRIPLE INTEGRALS & LINE INTEGRALS

In this section we show how to use the HP 48 to obtain good approximations for double, triple, and line integrals. Since all such integrals ultimately reduce to single integrals and since the HP 48 is an excellent integrator, it shouldn't come as a surprise that the HP 48 is an effective tool for dealing with these more general integrals.

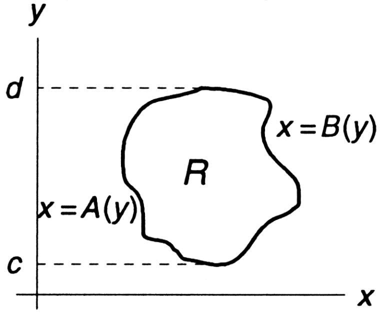
About Double Integrals

In your calculus book you will find a discussion about the connection between double integrals and iterated single integrals. It is important that you understand the difference and know what is going on geometrically. Typically, double integrals reduce to iterated single integrals of one of the two types below. Consult your text for additional explanation. We again remind you of the importance of understanding the connection between formulas and pictures (cf § 6.3).

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{C(x)}^{D(x)} f(x,y)\,dy \right) dx$$



$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{A(y)}^{B(y)} f(x,y)\,dx \right) dy$$



The HP 48 programs DBLY and DBLX correspond to the above formulas. See boxes. Use DBLY if the first (i.e., inside) integration to be carried out is with respect to y; use DBLX if the first integration is to be carried out with respect to x.

The HP 48 programs DBLY and DBLX correspond to the above formulas. See boxes. Use DBLY if the first (i.e., inside) integration to be carried out is with respect to y; use DBLX if the first integration is to be carried out with respect to x.

DBLY	
Inputs: a, b, $C(X)$, $D(X)$, $f(X,Y)$	*Output:* $\int_a^b \left(\int_{C(x)}^{D(x)} f(x,y)\,dy\right) dx$
« 5 DUPN 5 →LIST 'IN'	Stores input data in a list for easy recall
STO 'Y' ∫ 'X' ∫ →NUM STD »	Attempts symbolic integration; settles for a numerical approximation if symbolic integration is not possible
	Checksum: # 4444d Bytes: 70.5

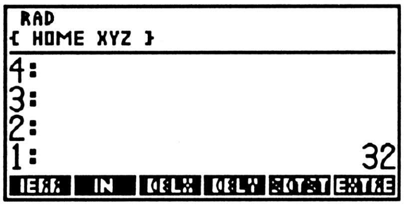
EXAMPLE 1. Evaluate $\displaystyle\int_0^2 \left(\int_{-1}^{3} (5 - 3x + 2y)\,dx \right) dy.$

DBLX

Inputs: c, d, A(Y), B(Y), f(X,Y)	*Output:* $\int_c^d \left(\int_{A(y)}^{B(y)} f(x,y)\,dx\right)\,dy$
≪ 5 DUPN 5 →LIST 'IN' STO 'X' ∫ 'Y' ∫ →NUM STD ≫	Stores input data in a list for easy recall Attempts symbolic integration; settles for a numerical approximation if symbolic integration is not possible Checksum: # 20818d Bytes: 70.5

SOLUTION.

STD 0 2 −1
3 5 3 X ∗
− 2 Y ∗ +
$\boxed{\text{DBLX}}$



```
RAD
{ HOME XYZ }
4:
3:
2:
1:                    32
```

Calculator time: about 20 seconds.
Error bound (press IERR): 0.00000000032.

✓Point to note

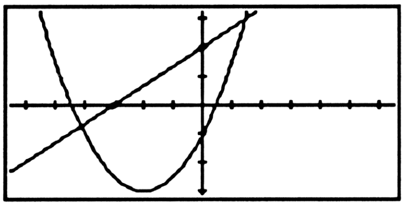
As with single integration, you can specify k-digit relative accuracy (for the k of your choice) by setting the display mode to k FIX. This is generally a good idea because if you leave the calculator in standard (STD) mode, it will attempt to obtain 12 digit accuracy. Generally, application of the programs DBLY and DBLX is quite time-consuming and one is wise to begin with modest accuracy demands. Unless both integrand and limits of integration are very simple (as in the preceding example), begin with either 2 FIX or 3 FIX. You will then have an idea of how long it will take to obtain better accuracy. The following short program INPUTS (see § 3.2) makes it easy to experiment with time and accuracy.

≪ IN OBJ→ DROP ≫ INPUTS $\boxed{\text{STO}}$

EXAMPLE 2. Find $\iint_R \sin xy\,dA$ where R is the region bounded by the curves $y = \frac{1}{2}x^2 + 2x - 1$ and $y = \frac{2}{3}x + 2$.

SOLUTION. The first step in any such problem is to *draw a picture of the region.* Enter, store, and graph the boundary curves as follows:

$\boxed{\text{RESET}}$ X SQ
2 / 2 X *
+ 1 − C
$\boxed{\text{STO}}$ 2 3 /
X * 2 + D
$\boxed{\text{STO}}$ C D =
$\boxed{\text{↰}}$ $\boxed{\text{EQ}}$ $\boxed{\text{DRAX}}$ $\boxed{\text{DRAW}}$



Clearly,

$$\iint\limits_R \sin xy \, dA = \int_a^b \left(\int_{C(x)}^{D(x)} \sin xy \; dy \right) dx,$$

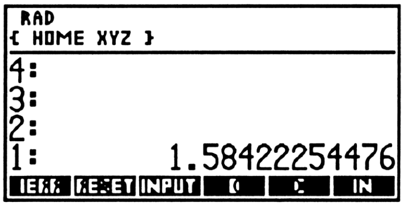
where $C(x) = \frac{1}{2}x^2 + 2x - 1$ and $D(x) = \frac{2}{3}x + 2$. What are a and b? These values can be obtained by hand (see Exercise A.14), but it's easier and more fun to obtain them using ISECT while in the graphics environment. Do the following:

1. Press $\boxed{\text{FCN}}$.
2. Move the graphics cursor (+) to the approximate location of right-most intersection point and press $\boxed{\text{ISECT}}$.
3. Move the graphics cursor (+) to the approximate location of left-most intersection point and press $\boxed{\text{ISECT}}$.
4. Press $\boxed{\text{ON}}$ twice to return to the stack.
5. Enter RE $\boxed{\text{SWAP}}$ RE.

The result is shown in Fig. 9(b). (Fig. 9(a) was produced as a "truth plot". For details, see your HP 48 manual.) *Question*: How do you suppose the calculator knows how to find intersection points? See Exercise A.15.

With a and b already on the stack, we may carry out the required integration as follows:

$\boxed{\text{RESET}}$ 2 FIX
C D X Y *
SIN $\boxed{\text{DBLY}}$

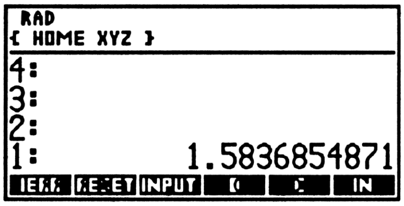


```
RAD
{ HOME XYZ }
4:
3:
2:
1:          1.58422254476
```

Calculation time: about 1.5 minutes.
Error bound: 0.024.

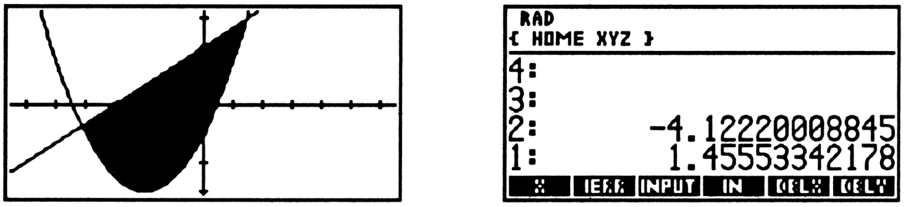
Now use $\boxed{\text{INPUTS}}$ to obtain better accuracy as follows:

6 FIX $\boxed{\text{INPUTS}}$
$\boxed{\text{DBLY}}$



```
RAD
{ HOME XYZ }
4:
3:
2:
1:          1.5836854871
```

Calculation time: about 6 minutes.
Error bound: 0.0000024.



```
RAD
{ HOME XYZ }
4:
3:
2:       -4.12220008845
1:        1.45553342178
```

(a) region R (b) intersection points

Figure 9

About Triple Integrals

Like double integrals, triple integrals typically reduce to iterated single integrals. There are six integration formulas corresponding to the six ways of ordering dx, dy, and dz: $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, and $dz\,dy\,dx$. The formula corresponding to $dz\,dx\,dy$ is

$$\iiint\limits_R g(x,y,z)\,dV = \int_{y_1}^{y_2}\left(\int_{x_1(y)}^{x_2(y)}\left(\int_{z_1(x,y)}^{z_2(x,y)} g(x,y,z)\,dz\right)dx\right)dy$$

where R is the three-dimensional region

$$R = \{(x,y,z) : y_1 \le y \le y_2,\ \ x_1(y) \le x \le x_2(y),\ \ z_1(x,y) \le z \le z_2(x,y)\}.$$

You can think of R as the solid caught between the surfaces $z = z_1(x,y)$ and $z = z_2(x,y)$ and above the two-dimensional region $\{(x,y) : y_1 \le y \le y_2,\ \ x_1(y) \le x \le x_2(y)\}$.

The program TPLZX corresponds to the above formula. See box. Similar programs can easily be written for the other five situations. Alternatively, variables can be changed to match those in TPLZX.

EXAMPLE 3. Find $\iiint\limits_R \ln(x+y+z)\,dA$ where R is the hexahedron bounded by the planes $z = 0$, $y = 1$, $y = 2$, $y = x$, $x = 0$, and $z = x + y$.

TPLZX	
Inputs: a, b, $x_1(Y)$, $x_2(Y)$, $z_1(X,Y)$, $z_2(X,Y)$, $g(X,Y)$	*Output:* $\int_a^b \left(\int_{x_1(y)}^{x_2(y)} \left(\int_{z_1(x,y)}^{z_2(x,y)} g(x,y,z)\,dz\right) dx\right) dy$

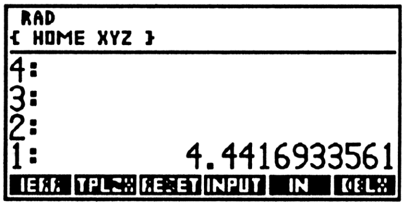
`« 7 DUPN 7 →LIST 'IN'`
`STO 'Z' ∫ 'X' ∫`
`'Y' ∫ →NUM STD »`
	Stores input data in a list for easy recall
	Attempts symbolic integration; settles for a numerical approximation if symbolic integration is not possible
	Checksum: # 44864d Bytes: 83.5

SOLUTION. Think of R as the solid between the planes $z = 0$ and $z = x + y$ and above the trapezoid $1 \le y \le 2$, $0 \le x \le y$. See Fig. 10. Thus:

$$\iiint_R \ln(x+y+z)\,dV = \int_1^2 \left(\int_0^y \left(\int_0^{x+y} \ln(x+y+z)\,dz\right) dx\right) dy$$

We approximate this integral using TPLZX. (You may want to check this by hand.)

2 FIX 1 2
0 Y 0 X Y
+ X Y + Z
+ LN $\boxed{\text{TPLZX}}$

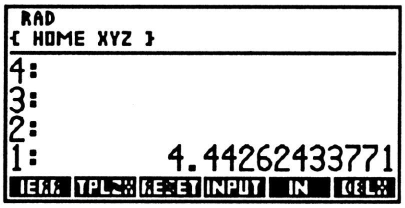


```
RAD
{ HOME XYZ }
4:
3:
2:
1:          4.4416933561
```

Calculation time: about 4.5 minutes.
Error bound: 0.05

Use $\boxed{\text{INPUTS}}$ to obtain more accuracy:

3 FIX $\boxed{\text{INPUTS}}$
$\boxed{\text{TPLZX}}$



```
RAD
{ HOME XYZ }
4:
3:
2:
1:         4.44262433771
```

Calculation time: about 20 minutes.
Error bound: 0.005

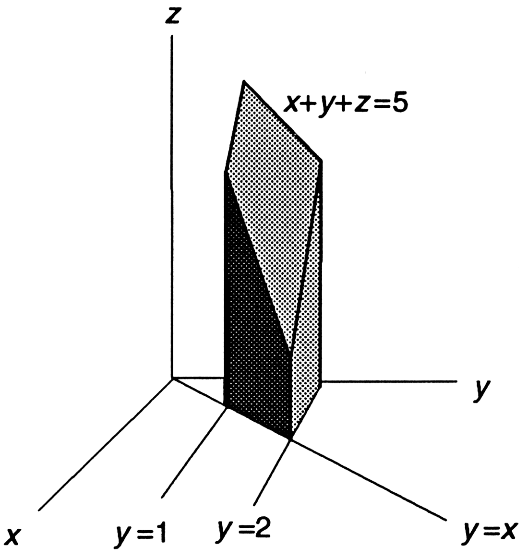


Figure 10

About Line Integrals

Line integrals are really curve integrals. There are two kinds:

$$(1)\quad \int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$(2)\quad \int_C f(x,y)\,ds$$

Here, C represents an arbitrary curve in the (x,y)-plane. An integral of type (1) does not have a simple geometric interpretation; one of type (2) can be thought of as the area of the "curtain" between the curve C and the curve $z = f(x,y)$ where (x,y) is restricted to C (e.g., see Example 5).

Evaluation of line integrals is easy once a parametrization of C is known. If a parametrization is given by $x = x(t)$, $y = y(t)$, $a \le t \le b$, then we have

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]\,dt \quad \text{and}$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

The programs LINXY and LINS correspond to the above formulas.

EXAMPLE 4. Find $\int_{C_i} 2x(y+1)\,dx + x^2\,dy$, $i = 1, 2, 3$, where

(a) C_1: $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$;

(b) C_2: $x = (4\pi/\sqrt{3})\sin(t/3)$, $y = 2\sin(t/2)$, $0 \le t \le 2\pi$; and

LINXY	
Inputs: $P(X,Y)$, $Q(X,Y)$, $x(T)$, $y(T)$, a, b	*Output:* $\int_C P(x,y)\,dx + Q(x,y)\,dy$

```
« 6 DUPN 6 →LIST 'IN'        Stores input data in a list for
                             easy recall
  STO → F G U V A            Defines local variables
  B «
  U 'X' STO V 'Y'            Forms the integrand
  STO A B
  F EVAL U 'T' ∂ *
  G EVAL V 'T' ∂ *
  + 'T' ∫ →NUM               Performs the integration
  { X Y } PURGE » »

                             Checksum: # 2076d   Bytes: 204.5
```

LINS	
Inputs: $f(X,Y)$, $x(T)$, $y(T)$, a, b	*Output:* $\int_C f(x,y)\,ds$

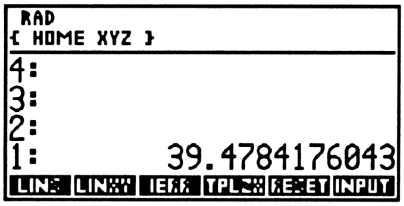
```
« 5 DUPN 5 →LIST 'IN'        Stores input data in a list for
                             easy recall
  STO → F U V                Defines local variables
  A B «
  U 'X' STO V 'Y'            Sets up the integral
  STO A B
  F EVAL U 'T' ∂ SQ
  V 'T' ∂ SQ + √
  * 'T' ∫ →NUM               Performs the integration
  { X Y } PURGE » »

                             Checksum: # 62802d   Bytes: 197
```

(c) C_3: $x = 2t$, $y = -2\sin^2 3t$, $0 \le t \le \pi$.

Figure 11 shows the curves C_1, C_2, and C_3. Except for starting and ending at the same places, these curves apparently have little in common.

SOLUTION.

(a)
```
2 X * Y 1
+ * X SQ
T T SIN -
1 T COS -
0 2 π *
[LINXY]
```



```
RAD
{ HOME XYZ }
4:
3:
2:
1:        39.4784176043
```

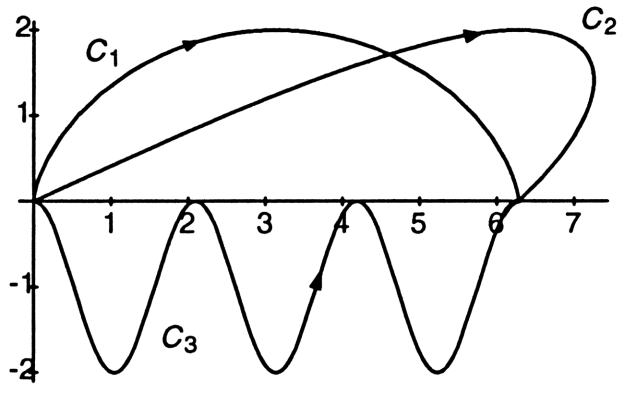
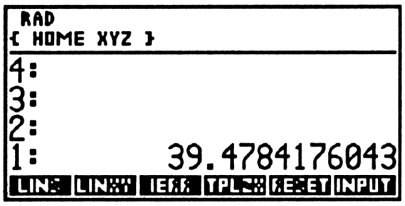


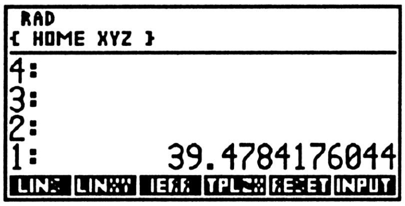
Figure 11

(b) $\boxed{\text{INPUTS}}$ $\boxed{\text{DROP}}$
$\boxed{\text{DROP}}$ $\boxed{\text{DROP}}$
$\boxed{\text{DROP}}$ 4 π $*$
3 $\sqrt{\ }$ / T 3
/ SIN $*$ 2
T 2 / SIN
$*$ 0 2 π $*$
$\boxed{\text{LINXY}}$



```
RAD
{ HOME XYZ }
4:
3:
2:
1:          39.4784176043
```

(c) $\boxed{\text{INPUTS}}$ $\boxed{\text{DROP}}$
$\boxed{\text{DROP}}$ $\boxed{\text{DROP}}$
$\boxed{\text{DROP}}$ 2 T $*$
2 3 T $*$ SIN
SQ $*$ $\boxed{+/-}$
0 π $\boxed{\text{LINXY}}$



```
RAD
{ HOME XYZ }
4:
3:
2:
1:          39.4784176044
```

Calculation times: about 1 minute.
Error bounds: approximately 10^{-9}.

Based on the above findings, what do you suspect? How could you obtain more evidence? See Exercise A.16.

EXAMPLE 5. Find the surface area of the solid formed by the intersection of the three right circular cylinders $x^2+y^2=1$, $x^2+z^2=1$, and $y^2+z^2=1$.

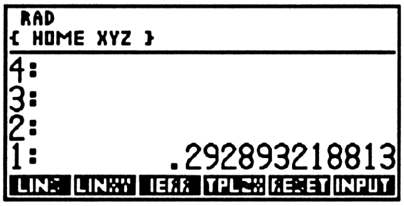
SOLUTION. Figure 12(a) shows the surface of interest. It consists of twelve congruent pieces each of which may be thought of as being further subdivided into four smaller pieces, as indicated in Fig. 12(b). We may use line integration to find the surface area of the small shaded piece since this is the "curtain" between the curve C: $x=\cos t$, $y=\sin t$, $0 \le t \le \pi/4$ and the curve $z=\sqrt{1-x^2}$, $x^2+y^2=1$.

The area is given by:

$$SA = \int_C \sqrt{1 - x^2}\, ds,$$

and can be evaluated as follows (it's also easy to do by hand):

1 X SQ $-$ $\sqrt{\ }$
T COS T SIN
0 π 4 / $\boxed{\text{LINS}}$



```
 RAD
{ HOME XYZ }
4:
3:
2:
1:          .292893218813
```

Calculation time: about 15 seconds.
Error bound: 10^{-11}.
Total area ≈ 48 × .292893218813 ≈ 14.058874503.

Exercises 9.5

Exercises 1–9 are routine integration problems. We suggest that you first work these problems by hand, then check your results with the HP 48.

A.1 $\displaystyle\int_{-2}^{3}\int_{-1}^{2}(4xy^2 - y)\,dx\,dy$

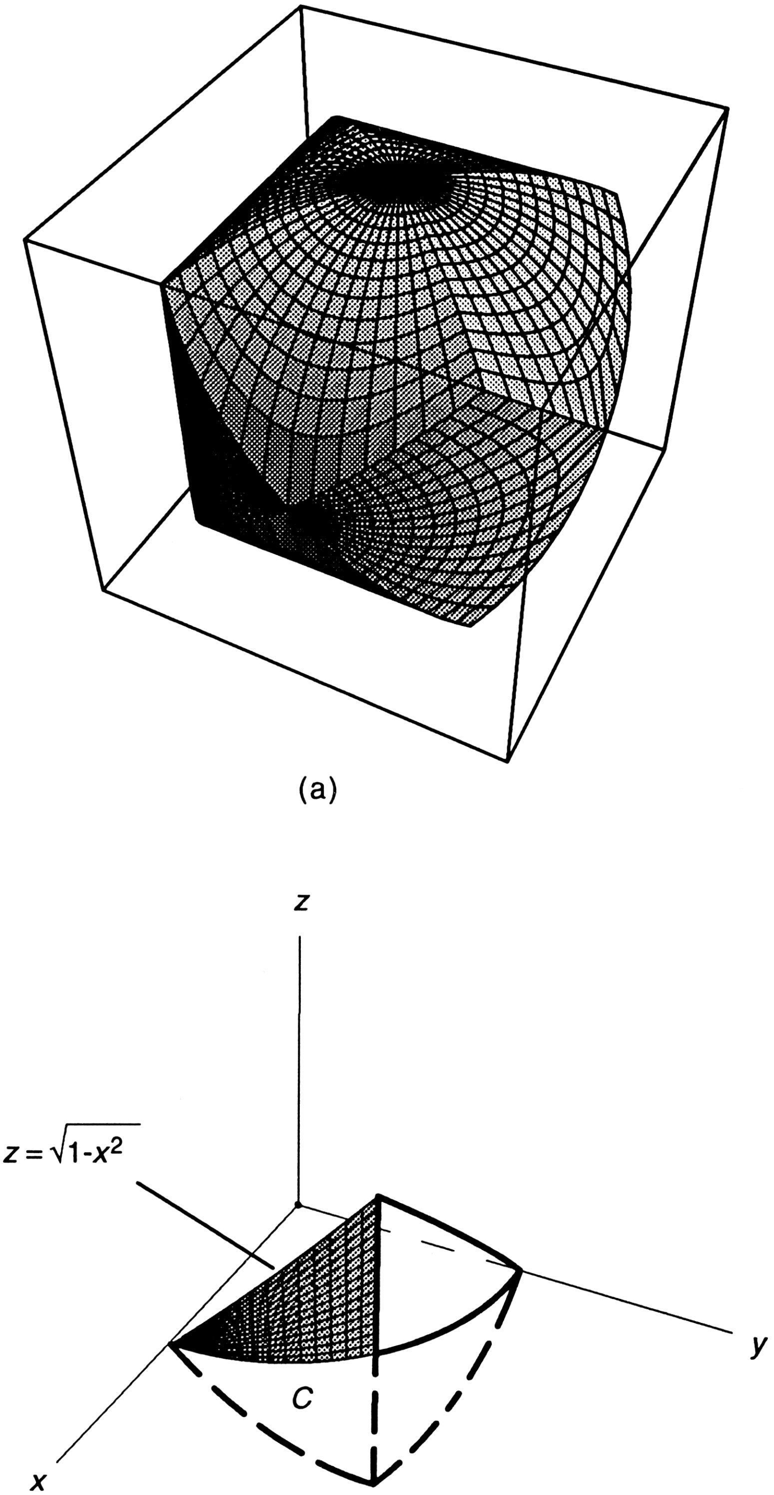
A.2 $\displaystyle\int_{0}^{\pi/3}\int_{0}^{\pi/6}(x\sin y - y\sin x)\,dy\,dx$

A.3 $\displaystyle\int_{-1}^{2}\int_{3x}^{x^2}(2x^3 - 3y)\,dy\,dx$

A.4 $\displaystyle\iint_R (7x - 4y + 1)\,dA$ where R is the region bounded by the curves $y = x^2$ and $y = \frac{1}{2}x + 1$.

A.5 $\displaystyle\iint_R (x + y + 1)\,dA$ where R is the region bounded by the curves $y = \sqrt{2x}$, $y = \sqrt{x+1}$, and $y = 0$.

A.6 $\displaystyle\int_{0}^{1}\int_{2}^{3}\int_{-1}^{2}(2x - 3y + z)\,dz\,dx\,dy$



(a)

(b)

Figure 12

A.7 $\displaystyle\iiint_R z\,dV$ where R is bounded below by the square $1 \le x \le 2$, $2 \le y \le 3$ and above by the plane $x + y + z = 5$.

A.8 $\displaystyle\int_C (x+2y)\,dx + x^2\,dy$ where C is the curve defined by $x = t^2$, $y = 3t$, $0 \le t \le 1$.

A.9 $\displaystyle\int_C y\,dx - x^2\,dy$ where C is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$ oriented in the counterclockwise sense.

A.10 Use DBLX or DBLY to find the area between the curves $x = y^2$ and $y = x^2$.

A.11 Approximate $\displaystyle\iint_T e^{x^2+y^2}\,dA$, where T is the triangle bounded by $y = x$, $y = 0$, and $x = 1$. Give an error bound for your answer.

A.12 Approximate $\displaystyle\iint_S \cos(x^2y)\,dA$, where S is the square $0 \le x \le \pi/2$, $0 \le y \le \pi/2$. Give an error bound for your answer.

A.13 Find the area of the surface bounded below by the curve

$$x = \cos y,\ 0 \le y \le \pi/2,\ z = 0$$

and above by the curve

$$z = \sin^2(\pi x)\cos^2 y,\ x = \cos y,\ 0 \le y \le \pi/2.$$

Give an error bound for your answer.

A.14 Show that the exact values of a and b in Example 2 are $(-4 \pm \sqrt{70})/3$.

A.15 How do you think the HP 48 finds intersection points? [*Hints*: (1) How would *you* do it?; (2) If you move the graphics cursor (+) to any point on the vertical line through an intersection point and press ISECT, the result is the same as if you were to place the cursor on the intersection point.]

A.16 (a) Give an example of a curve C_4 from $(0,0)$ to $(2\pi, 0)$ different from C_1, C_2, and C_3 in Example 4;

(b) Calculate $\displaystyle\int_{C_4} 2x(y+1)\,dx + x^2\,dy$;

(c) What's going on?

Answers

CHAPTER 0 ANSWERS

Chapter 0 Section 0.1

A.1(a) $\boxed{5}\ \boxed{8}\ \boxed{\text{SPC}}\ \boxed{3}\ \boxed{7}\ \boxed{\times}$; 2146 **A.1(b)** $\boxed{'}\ \boxed{5}\ \boxed{8}\ \boxed{\times}\ \boxed{3}\ \boxed{7}\ \boxed{\text{EVAL}}$; 2146 **A.2(a)** $\boxed{2}\ \boxed{\sqrt{x}}\ \boxed{3}\ \boxed{\sqrt{x}}\ \boxed{+}$; 3.14626436994 **A.2(b)** $\boxed{'}\ \boxed{\sqrt{x}}\ \boxed{2}\ \boxed{+}\ \boxed{\sqrt{x}}\ \boxed{3}\ \boxed{\text{EVAL}}$; 3.14626436994 **A.3(a)** $\boxed{2}\ \boxed{\text{SPC}}\ \boxed{3}\ \boxed{y^x}\ \boxed{3}\ \boxed{\text{SPC}}\ \boxed{2}\ \boxed{y^x}\ \boxed{-}$; −1 **A.3(b)** $\boxed{'}\ \boxed{2}\ \boxed{y^x}\ \boxed{3}\ \boxed{-}\ \boxed{3}\ \boxed{y^x}\ \boxed{2}\ \boxed{\text{EVAL}}$; −1 **A.4(a)** $\boxed{2}\ \boxed{\text{SPC}}\ \boxed{3}\ \boxed{\text{SPC}}\ \boxed{4}\ \boxed{y^x}\ \boxed{y^x}$; 2.41785163923 E 24 **A.4(b)** $\boxed{'}\ \boxed{2}\ \boxed{y^x}\ \boxed{\text{↰}}\ \boxed{\div}\ \boxed{3}\ \boxed{y^x}\ \boxed{4}\ \boxed{\text{EVAL}}$; 2.41785163923 E 24 **A.5(a)** $\boxed{\text{↰}}\ \boxed{\text{SPC}}\ \boxed{\text{SPC}}\ \boxed{1}\ \boxed{2}\ \boxed{\div}\ \boxed{\text{SIN}}\ \boxed{\text{↰}}\ \boxed{\text{SPC}}\ \boxed{\text{SPC}}\ \boxed{1}\ \boxed{2}\ \boxed{\div}\ \boxed{\text{COS}}\ \boxed{-}\ \boxed{\text{↱}}\ \boxed{\text{EVAL}}$; −.707106781187 **A.5(b)** $\boxed{'}\ \boxed{\text{SIN}}\ \boxed{\text{↰}}\ \boxed{\text{SPC}}\ \boxed{\div}\ \boxed{1}\ \boxed{2}\ \boxed{\blacktriangleright}\ \boxed{-}\ \boxed{\text{COS}}\ \boxed{\text{↰}}\ \boxed{\text{SPC}}\ \boxed{\div}\ \boxed{1}\ \boxed{2}\ \boxed{\text{↱}}\ \boxed{\text{EVAL}}$; −.707106781187
A.6 −.482300884956 **A.7** 1.2 **A.8** 29.1897158284 **A.9** 25439.6230976
A.10 2.61803398875 **A.11** .037550723392 **A.12** 1.96157056081
A.16 .564705882352

Chapter 0 Section 0.2

A.1(a) 5 **A.1(b)** 151 **A.1(c)** BEEP (+ Error: Too Few Arguments) **A.1(d)** no change **A.1(e)** 5 **A.2(a)** 0 **A.2(b)** BEEP (/ Error: Infinite Result) **A.2(c)** 0 **A.2(d)** 1 **A.3(a)** 0 **A.3(b)** 0 **A.4(a)** 5 **A.4(b)** 0 **A.4(c)** Level 1: 6.68950291345E198 **A.5(a)** 9 **A.5(b)** 3 **A.5(c)** 9 **A.5(d)** 1 **A.5(e)** BEEP (SWAP Error: Too Few Arguments) **A.6(a)** 1 **A.6(b)** −1 **A.6(c)** −1 **A.6(d)** 1 **A.6(e)** −1 **A.6(f)** −1 **A.7(a)** 2 **A.7(b)** −2 **A.7(c)** −2 **A.7(d)** .25 **A.7(e)** (1.99999999999, −8.48119329392E − 12) **A.7(f)** 1.14159265359 **A.8(a)** You would get 10! = 3628800 **A.8(b)** You would get $1 + \cdots + 10 = \frac{10(11)}{2} = 55$

B.1(a) 16 **B.1(b)** 16 **B.1(c)** 7.62559748498E12 **B.1(d)** 19683; $a^{(a^a)} = (a^a)^a$ only if $a = 1$ or $a = 2$ **B.2(a)** $n!$ **B.2(b)** $\frac{n(n+1)}{2}$

Chapter 0 Section 0.4

A.1 ≪ → X1 Y1 X2 Y2 ≪ X1 X2 + 2 / Y1 Y2 + 2 / R→C ≫ ≫ **A.2** ≪ + 2 / ≫ **A.3** ≪ → X1 Y1 X2 Y2 ≪ Y2 Y1 − X2 X1 − / ≫ ≫ **A.4** ≪ − C→R SWAP / ≫ **A.5** ≪ → M B ≪ Y M X * B + = ≫ ≫ **A.6** ≪ → M A ≪ Y M X * M A * − = ≫ ≫ **A.7** ≪ → A B M ≪ Y M X *

B M A * − + EXPAN COLCT = ≫ ≫ **A.8** ≪ → X1 Y1 X2 Y2 ≪ Y Y2 Y1 − X2 X1 − / X X1 − * Y1 + EXPAN EXPAN COLCT = ≫ ≫ **A.9** ≪ X SWAP = ≫ **A.10** ≪ → X1 Y1 X2 Y2 ≪ IF X1 X2 ≠ THEN X1 Y1 X2 Y2 SLIN ELSE X1 VLIN END ≫ ≫ **A.11** ≪ → X1 Y1 X2 Y2 ≪ X1 X2 + 2 / Y1 Y2 + 2 / X1 X2 − Y2 Y1 − / PTSL ≫ ≫ **A.12** ≪ A B ≪ Y A B / NEG X * EXPAN = ≫ ≫ **A.13** ≪ → L W H ≪ 2 L W * L H * + H W * + * ≫ ≫ **A.14** ≪ → R ≪ π R SQ * 2 π * R * ≫ ≫ **A.15** ≪ * ≫ **A.16** 0, .785398163397, 1.5706963268, 1.57079631679, 1.57079632679 **A.19** .707106781188, 1.09454090923, 1.0029994985, 1.000006

B.1 ≪ → A B ≪ IF B 0 ≠ THEN A B A B / NEG PTSL ELSE X A = END ≫ ≫ **B.2** ≪ → A B C ≪ B SQ 4 A * C * − √ → S ≪ B NEG S + 2 A * / B NEG S − 2 A * / ≫ ≫ ≫ **B.3** ≪ → A B C ≪ B 2 A * / NEG C B SQ 4 A * / − R→C ≫ ≫ **B.4** ≪ → 3 DUPN DROP DUP 4 ROLLD ROT SWAP STO EVAL SWAP PURGE ≫

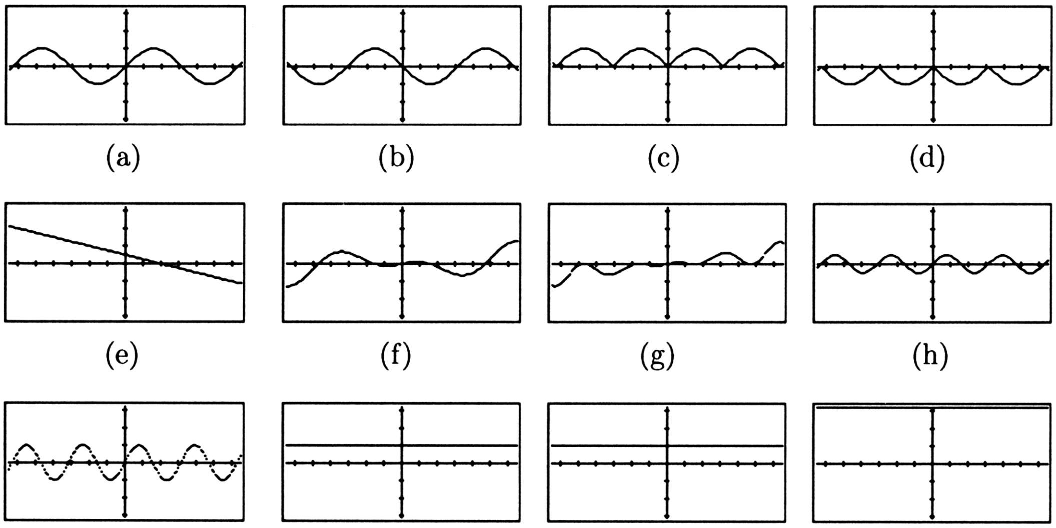
Chapter 0 Section 0.5

A.1 The HP won't accept the name HOME because it is a built-in command; it accepts hOME because it regards it as a different name. Before you can purge a directory name, you must first purge the contents of that directory.

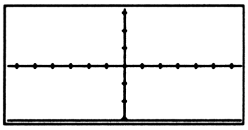
B.1 ≪ → N D1 D2 ≪ HOME D1 EVAL N RCL N PURGE HOME D2 EVAL N STO ≫ ≫

Chapter 0 Section 0.6

A.1



(a) (b) (c) (d) (e) (f) (g) (h) (i) (j) (k) (l)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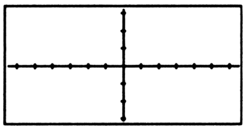
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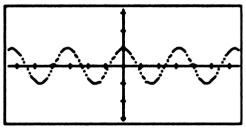
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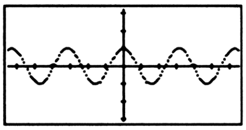
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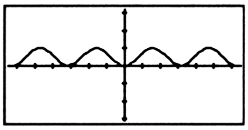
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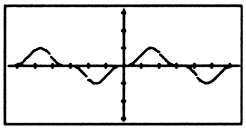
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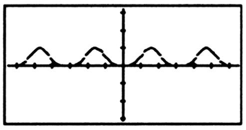
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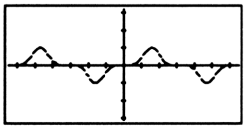
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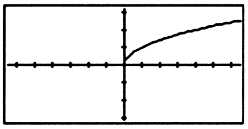
A.2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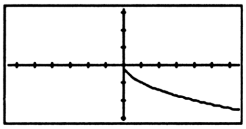
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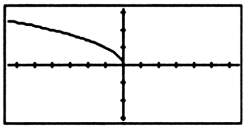
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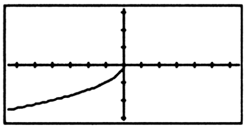
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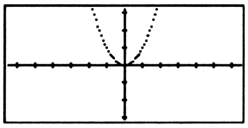
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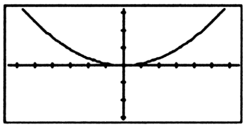
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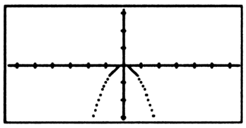
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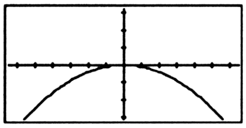
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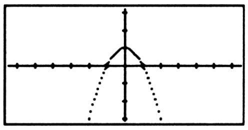
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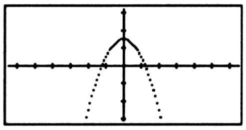
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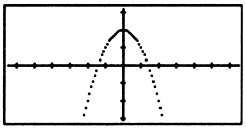
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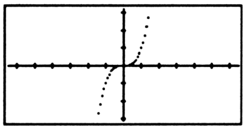
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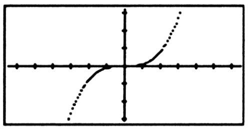
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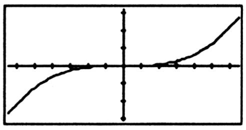
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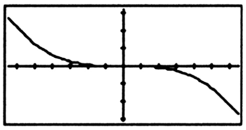
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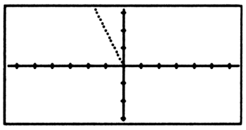
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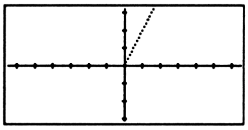
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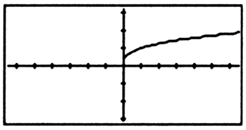
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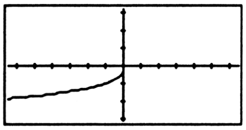
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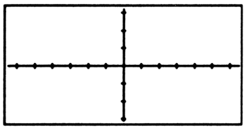
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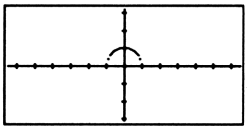
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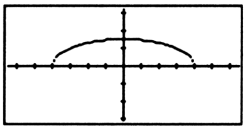
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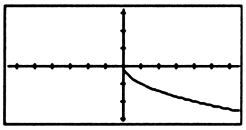
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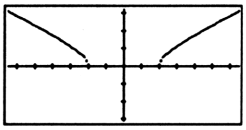
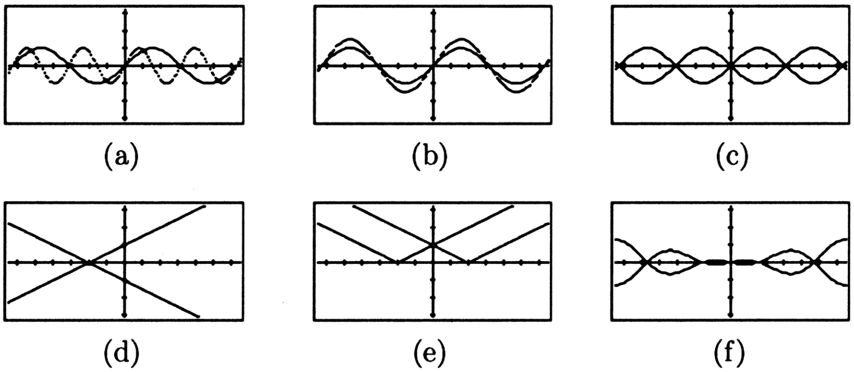
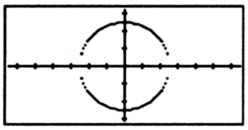

A.6

A.7


(a) (b) (c)

(d) (e) (f)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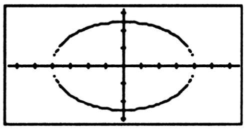
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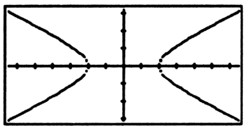
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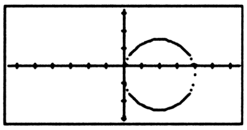
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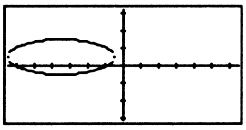
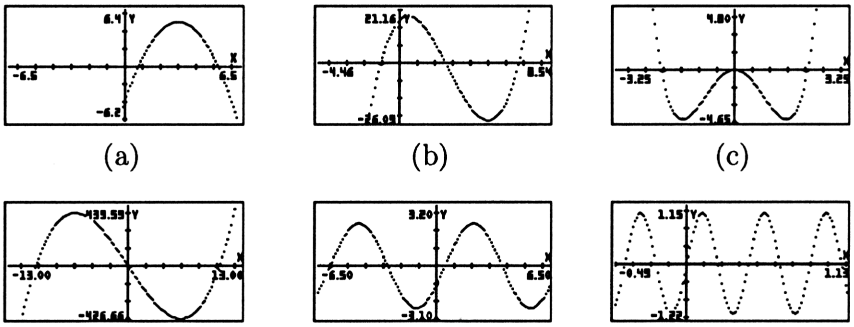

A.12

A.13



(a) (b) (c)

(d) (e) (f)

Chapter 0 Section 0.7

A.1 ≪ 1 100 FOR I I NEXT ≫ **A.2** ≪ 1 100 FOR I 101 I − NEXT ≫ **A.3** ≪ → N ≪ 1 N FOR I I SQ NEXT N →LIST ≫ ≫ **A.4** ≪ → N ≪ 1 N FOR I I I ^ NEXT N →LIST ≫ ≫
A.5 ≪ → N ≪ 1 N FOR I I ! NEXT N →LIST ≫ ≫
A.6 ≪ DRAX −1 5 FOR X X 2 R→C PIXON .1 STEP GRAPH ≫
A.7 ≪ DRAX 0 3 FOR T T T R→C PIXON .1 STEP GRAPH ≫
A.9 It doesn't have a degree because it is not a polynomial. **A.10** POLYGON
A.11 Enter 'IFTE ($x < 0$, −2, 3)' STEQ DRAW

B.1 ≪ → N ≪ 0 0 N 1 − FOR I 2 I ^ INV + DUP NEXT DROP N →LIST ≫ ≫ **B.2** ≪ → N ≪ 0 1 N FOR I I INV +

DUP NEXT DROP N →LIST ≫ ≫ **B.3** ≪ → A B C ≪ DRAX A B FOR X X C R→C PIXON .1 STEP GRAPH ≫ **B.4** ≪ DRAX 0 1 FOR T 8 T ∗ 3 − 3 T ∗ R→C PIXON .01 STEP GRAPH ≫ **B.5** ≪ → N ≪ 0 N 1 − FOR I N I − √ NEXT N →LIST ≫ ≫

CHAPTER 1 ANSWERS

Chapter 1 Section 1.1

A.3 Using an HP timing program, FEVAL took 0.18 seconds, FOFX took 0.19 seconds, and program style evaluation took 0.12 seconds. **A.5** \$143.56; \$144.50; \$144.99; \$145.49; \$145.50. The difference between daily and hourly compounding is negligible. **A.6** Yes. Compounding \$0.01 once a year at 4% gives \$$1.08\times 10^{15}$. **A .7** $h\circ h(w) = w,\ w\neq -1$; $h\circ f(x) = (1-\sqrt{x-2})/(1+\sqrt{x-2}),\ x\geq 2$; $h\circ g(y) = -(1+y^2)/(3+y^2),\ -\infty < y < \infty$; $f\circ h(w) = \sqrt{(-1-3w)/(1+w)},\ -1 < w \leq -1/3$; $f\circ f(x) = \sqrt{\sqrt{x-2}-2},\ x\geq 6$; $f\circ g(y) = |y|,\ -\infty < y < \infty$; $g\circ h(w) = (3w^2+2w+3)/(1+w)^2,\ w\neq -1$; $g\circ f(x) = x,\ x\geq 2$; $g\circ g(y) = 4y^2+y^4+6,\ -\infty < y < \infty$.

B.1 $S = 0.56$, $T = 0.79$; $t_1 = 16.2$. **B.3** $n = 50 :\ 3.126 \leq \pi \leq 3.157$; $n = 100 :\ 3.134 \leq \pi \leq 3.149$. **B.4** ≪ − ABS ≫, with inputs (x_1, y_1) and (x_2, y_2) on the stack; output is the distance between these points. **B.5** Destination point is (23.3, −3.1). A program for both HP28 and HP48, using basic definitions and C→R and R→C conversions: ≪ DEG → P R T ≪ P C→R T SIN R ∗ + SWAP T COS R ∗ + SWAP R→C RAD ≫ ≫. An alternative for the HP28 is: ≪ DEG R→C P→R + RAD ≫. An alternative for the HP48 is: ≪ DEG −19 SF −15 CF −16 SF →V2 + −16 CF RAD ≫. **B.6** Approximately (0.84, 0.71). The program ≪ DUP SQ R→C (2,0) − ABS ≫ facilitates the calculations. We have not assumed knowledge of minimization techniques based on differentiation.

C.1 ≪ → M1 M2 ≪ M2 M1 DUP2 − 3 ROLLD ∗ 1 + / ATAN DUP IF 0 < THEN π + END π / 180 ∗ ≫ ≫.

Chapter 1 Section 1.2

A.6 −2.8 and −1.1. **A.7** After RESET try 2∗H. (1.5, −6.2). **A.9** (1.2, 12.8). **A.10** Try −3, 11 for XRNG and −50, 20 for YRNG. −2.0, 2.7, (5.1,−23.3), 9.3. **A.11** (0.8, 1.8). **A.12** −1.3, 2.2, 4.0.

B.1 (3.2, 1.0), (1.9, 2.0), (0.9, 0.0). **B.2** $(0.8 \pm 2n\pi, 1.6), (5.5 \pm 2n\pi, -1.2)$. **B.3** The graph of $x/2 - \sin x$ is a variation about the graph of $x/2$ by $\sin x$. **B.5** Try .2∗H and .1∗W. **B.6** For $x \leq 0$, $f(x) = x+2$; for $0 \leq x \leq 1$, $f(x) = 2$; for $x \geq 1$, $f(x) = 2(2-x)x$. So, two lines and a parabola. **B.7** [0, 2). **B.8** Use −1, 7 for XRNG and −2, 2 for YRNG. (0.8, 0.3). **B.9** Try −3, 3 for XRNG and −4, 0 for YRNG. Local maxima at −1.1, .37, 1.3. **B.10** A difficult graph, partly due to scale.

It has an x-intercept at 4, horizontal asymptote of $y = 3/64$, local minima at $x = 0, 4$, and local maxima at $x \approx -0.37, 0.26$. Try 0.2,0.3 for XRNG and then use AUTO.

C.1 Partial answer: The inequality shows that for $0 < x \leq 0.22$, the HP values of $f(x)$ are incorrect. Tracing the calculation with $x = 0.22$, $1 - \cos 0.22^6$ has only 4 significant figures on the HP. Actual value, with 8 significant figures, is 6.4275013×10^{-9}. At the division, the number in the fifth decimal place is incorrect. At $x = 0.1$, calculator value is 0 while the true value is within 10^{-13} of 0.5. **C.3** $x \approx 9.2, y = 1/\sqrt{3}$. **C.4** Try $-2, 2$ for XRNG and $-10, 0$ for YRNG. There are local maxima near $-1, .8, 1.2$.

Chapter 1 Section 1.3

A.1(a) 0.75 **A.1(b)** 2.5 **A.1(c)** 0 **A.1(d)** 0.5 **A.1(e)** 0.5 **A.1(f)** 0.209986 **A.2** 44.4 hours. **A.3** 4.5×10^{-9} is slope of AB; $R(4) = 4.3 \times 10^{-9}$; slope of AB is close to slope of tangent line. **A.4** 10.5283 hours. **A.5** 6.8 per hour. **A.6** Approximately 3 million dollars.

B.1(a) 0.5 **B.1(b)** 1 **B.1(c)** 2 **B.1(d)** ≈ 4.9. Exact value is $\sqrt{2}(16/3 - 4\sqrt{2}/3$. **B.3** $2.7726 \cdot 5 \cdot 10^{-23} 2^{4t}$ **B.4** Approximately 0.59 meters.

Chapter 1 Section 1.4

A.1 $3, 5, -2$ **A.2** $3, 2/3, 0.2, -5$ **A.3** $2, 0.2, -1/7, -9/5$ **A.4** $2, -2, 5, -7, (-1 \pm \sqrt{3}i)/2$ **A.5** $-5, 2, 5$ **A.6** $2/3, 1/2, \pm i$ **A.7** $-5, (-1 \pm \sqrt{5}i)/2$

A.9 -1.87938524157, 0.347296355334, 1.53208888624

A.10 -2.53208888624, -1.34729635533, 0.879385241572

A.11 -3.86080585311, -2.25410168837, 0.114907541477

A.12 1.36523001341, $-2.68261500671 \pm 0.358259359924i$

A.13 $h = \sqrt{3}a$ **A.14** 6.52703644666cm; other zeros out of range.

A.15 0.415774556783, 2.29428036028, 6.28994508294

A.16 0.263560319718, 1.41340305911, 3.59642577104, 7.08581000586, 12.6408008443

A.17 -0.507787629558, 0.132300820777, 0.708820142114

A.18 ± 1.65068012389, $\pm$ 0.524647623275

A.19 0, $\pm$ 0.442930458136, $\pm$ 0.798214220989

A.20 ± 0.339981043585, ± 0.861136311594 **A.21** $-3/4, 10/9, 2 \pm 3i$

A.22 $-3/7, \pm 7i, 5/2 \pm i/2$ **A.23** $1, -7/2, -3/5, (-1 \pm \sqrt{7}i)/4$

A.24 $(-1.32693251066, -1.31889412187)$,
$(-0.400224818095, 0.579466878998)$, $(0.470331690691, -0.90233641238)$,
$(1.44005098004, 7.29398366658)$, $(2.31677465803, 276.118613325)$

A.25 43.9846696021 **A.26** $-1/7, 8/7, 8/7, (-1 \pm 3\sqrt{3}i)/2$

A.27 $1/7, \sqrt{3}, \sqrt{3}, -\sqrt{3}, -\sqrt{3}$ **A.28** $2/9, 2/9, 4/9, (1 \pm i)/9$

B.2 $-2 - 3i$, $(5 \pm \sqrt{3}i)/2$ **B.3** In increasing order: 3.14159265359, 31.4159265359, 314.159265359, 3141.59265359. In decreasing order: 3141.59265359, 314.159265565, 31.4159244735, 3.1415945115.

B.5 Polynomial is $-\lambda^3 + 9\lambda^2 - 18\lambda + 6$; 0.415774556784, 2.29428036027, 6.28994508294

C.1(c) The costs of the traditional, factored, and preconditioned forms are $10001(4f_1 + 3f_2)$, $10001(2f_1 + 3f_2)$, and $20003f_1 + 30005f_2$. **C.2** Cost of preconditioning is $5f_1 +$

$8f_2$; cost of one evaluation is $5f_1 + 7f_2$. **C.3** For traditional, factored, and preconditioned the costs are $10001(12f_1+7f_2)$, $10001(6f_1+7f_2)$, and $5f_1+8f_2+10001(5f_1+7f_2)$, respectively. Simplifying and assuming that $f_1 > 2f_2$, which is conservative, the preconditioned is cheapest.

CHAPTER 2 ANSWERS

Chapter 2 Section 2.1

A.1 The difference quotient strongly resembles $-\sin x$, which is the derivative of $\cos x$.
A.2 The derivative is $\tan x$ is $\sec^2 x$. **A.3** 1.5 meters/sec $\approx 1.4\pi/3$ meters/sec
A.4 They are nearly identical graphs. They differ most at $x = 0.1$, where their values are 1.31 and 1.58.

B.3 ≪ DUP2 + 4 ROLL SWAP FEVAL SWAP 4 ROLL FEVAL SWAP 4 ROLLD − SWAP / ≫ Try '√X' 'W' 'H' as input.
B.4 ≪ → X H 'F(X+H)−F(X))/H' ≫ Store as DQ.
B.5 ≪ → X H 'F(X+H)−F(X−H))/(2*H)' ≫ Store as SDQ.

C.1 To extend the program, repeat each step, replacing DQ by SDQ, and inserting SQ just after H. The regular decrease with h of the second and fourth columns suggests the third and fifth. The accuracy of the symmetric difference is better by an order of magnitude, that is, by a power of 10. **C.2** $\Delta_1(h) = -h/(8c^{3/2}) \approx -h/8$. $\Delta_2(h) = (h^2/32)(1/c_1^{5/2} + 1/c_2^{5/2}) \approx h^2/16$, where $1-h < c_2 < 1 < c_1 < 1+h$

Chapter 2 Section 2.2

A.2 (a) -17 (b) 1.5 (c) 0.710775 (d) 0.0551344 (e) 0.118597 (f) 0.348259
A.8 -0.117535 inches/hour **A.9** (a) $-4xy/(3y^2 + 2x^2)$ (b) $-x^2/y^2$ (c) $-(1 + y^2)/(2xy+3y^2-1)$ (d) $\cos(x+y)/(1-\cos(x+y)$ (e) $(3+2\sqrt{y-3x})/(1+4y\sqrt{y-3x})$ (f) $-(2xy^2+y)/(3x^2y+2x)$ (g) $-\dfrac{8x\sqrt{1+y}\sqrt{1-x\sqrt{1+y}}+2(1+y)}{x+8y\sqrt{1+y}\sqrt{1-x\sqrt{1+y}}}$
(h) $\dfrac{\sqrt{y}\,(1-2\sqrt{xy}-3xy)}{\sqrt{x}\,(x+2x\sqrt{xy}+2\sqrt{y})}$
A.10 (a) $y - y_0 = 0.69308(x - x_0)$ (b) $y - y_0 = -0.27328(x - x_0)$ (c) $y - y_0 = -0.80122(x - x_0)$ (d) $y - y_0 = 1.91682(x - x_0)$ (e) $y - y_0 = 1.91068(x - x_0)$ (f) $y-y_0 = -0.44425(x-x_0)$ (g) $y-y_0 = -1.50389(x-x_0)$ (h) $y-y_0 = -0.452658(x-x_0)$
A.11 (a) 2.04667; 2.04718 (b) 1.63852; 1.63763 (c) 0.648859; 624047 (d) 0.220728; 0.236928

B.1 The 27 are: ABS, ACOS, ACOSH, ALOG, ARG, ASIN, ASINH, ATAN, ATANH, CONJ, COS, COSH, EXP, EXPM, INV, LN, LNP1, LOG, SIN, SINH, SQ, +, −, *, /, ^, √. **B.4** ⟨⟨ → F X0 H ⟨⟨ F DUP X0 'X' STO EVAL SWAP 'X' ∂ H * + 'X' PURGE ⟩⟩ ⟩⟩
B.5 Use the equations given in the answer to **A.11**

C.1 ⟩⟩ → G X1 Y1 ⟨⟨ G IMPL X1 'X' STO Y1 'Y' STO EVAL { X Y } PURGE ⟩⟩ ⟩⟩

Chapter 2 Section 2.3

A.1 At $(1,-2)$, $\alpha = 108.43^\circ$; at $(3,-4)$, $\alpha = 45^\circ$ **A.2** At $(2,0)$, $\alpha = 108.43^\circ$ **A.3** For line, $\alpha = 42.00^\circ$; for sine curve, $\alpha = 45^\circ$ **A.4** At $(-1,0)$, $\alpha = 85.24^\circ$; at 1, $\alpha = 97.13^\circ$; at 5, $\alpha = 87.61^\circ$ **A.5** At $(-1,1)$, $\alpha = 135^\circ$; at $(1,3)$, $\alpha = 71.57^\circ$ **A.6** θ and $180 - \theta$; $\alpha_{1/4} = \arctan(\frac{1}{2}\tan\theta)$; $\alpha_{1/2} = 0$; $\alpha_{3/4} = 180 - \arctan(\frac{1}{2}\tan\theta)$ **A .7** Angle of incidence = 2.86°. Slope of reflected ray is 0.1003. **A.8** At $(1,3)$, $\theta_{12} = 153.43^\circ$; at $(-1,1)$, $\theta_{12} = 90^\circ$ **A.9** (a) Vertex at $(1,-2)$; focus at $(7/4,-2)$; opens to right. (b) $(y+2)^2 = -4 \cdot 11(x-3)$ (c) The squares of the distances are $(h+p-x)^2$ and $(x-h+p)^2$. Rewrite the latter using the equation of the parabola. (d) 1/80 meter **A.10** (a) Center $(1,0)$; vertices $(2,0)$ and $(0,0)$; foci at $(1+\sqrt{2},0)$ and $(1-\sqrt{2},0)$ (b) Center $(-1,2)$; vertices $(-1 \pm \sqrt{3},2)$; foci $(-1 \pm 5/sqrt5/6,2)$ (c) $(x-7/2)^2/(3/2)^2 - (y+3)^2/2^2 = 1$ **A.11** $(x+1.9)^2/1.1^2 - y^2/3.2 = 1$

B.1 Assuming the calculator stays in MODE RAD, simply add π / 180 * after PURGE. **B.3** First, observe that it is no loss of generality to set $h = k = 0$. Then use Fig. 7, extending the horizontal ray and the ray reflected to the focus to the other side of the parabola. The proof given in (7) and (8) can be used. **B.4** Minimum required diameter = 0.2503; diameter of dark spot = 0.0626; diameter of smallest hole = 0.0156 **B .5** Minimum required diameter = 0.256411; diameter of dark spot = 0.02736; diameter of smallest hole = 0.02679 **B .6** $H = -12$ **B.7** Copy from A.8, but after first + add DUP 0 IF == THEN DROP2 90 ELSE . Also, insert END after last * **B.8** ≪ → X ≪ X INC1 SWAP X INC1 – DUP 0 IF < THEN 180 + END ≫ ≫

C.1 ≪ → F X ≪ 'F(Z) ' 'Z ' ∂ X 'Z' STO EVAL ATAN DUP IF 0 < THEN π + END 'Z ' PURGE ≫ ≫ **C.3** Minimum required diameter = 0.747 feet; diameter of hole = 0.116 feet

Chapter 3 Section 3.1

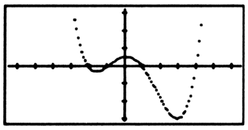
A.1 *Zeros*: −2.05150078168, −0.902312625486, 0.942787602586, 4.01102580458; *Extrema*: (−1.56970497975, −4.50858331328) rel. min., (0.107532983742, 7.10864271358) rel. max., (2.962171996, −41.0375593999) abs. min.; *Inflection Points*: (−0.822875655532, 0.83300524427), (1.82287565553, −20.3330052441) **A.2** *Zeros*: −0.282750613112, 1.38095330959; *Extremum*: (0.828981543581, −6.28484236891) abs. min.; *Inflection Points*: (0.4, −6.024), (0.666666666684, −6.18518518521) **A.3** *Zeros*: −0.25, 0, 1; *Extrema*: (−0.25, 0) rel. max., (−0.12845892868, −0.00440129472746) rel. min., (0, 0) rel. max., (0.778458928682, −2.27205870527) rel. min.; *Inflection Points*: (−0.2, −0.00192), (−0.0561862178477, −0.00200396157286), (0.556186217848, −1.42768353842) **A.4** *Zeros*: 0.0910514044938, 9.02292317841; *Extremum*: (6.73637714341, −719.684974608) abs. min.; *Inflection Points*: (0.037346992108, .590111006562), (4.46265300791, −431.430388788) **A.5** *Zeros*: −0.370603316959, 0.381936710524; *Extremum*: (0, −3) abs. min.; *Inflection Points*: (6, 54429), (10, 99997) **A.6** *Zeros*: ±1.32287565553, 1.5; *Extrema*: (−0.632782218537, 2946.69399461) abs. min., (1.38278221858, 1.06899461) rel. max., (0, 0) rel. max., (1.5, 0) rel. min.; *Inflection Points*: (1.44221865525, 0.52883894), (−0.0577813447567, −1743.86217227)

A.8 $y = -12.5 + 14.0625x$ **A.9** $y = -9.86960440115 + 6.28318530722x$

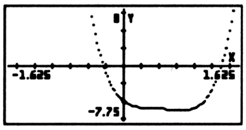
A.10 $y = -3.1049049982 + 1.6380450483x$ **A.11** f is the one that goes through the origin.

B.1 Clearly, $q(1) > 0$ and $q(2) > 0$. Also, by Example 2, $q(1.68989794855) < 0$.

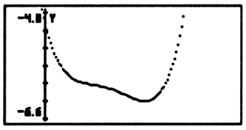
B.2 The calculator flashes the word EXTREMUM, then tells you that there is a ROOT at 1.68989743975.



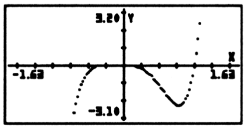
A.1



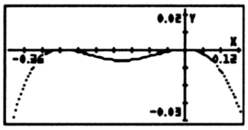
A.2



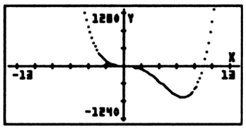
A.2



A.3



A.3



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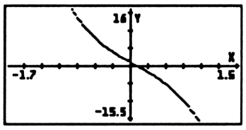
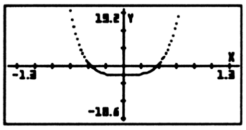
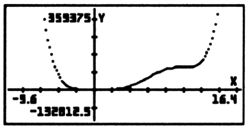


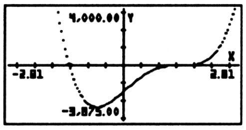
A.4



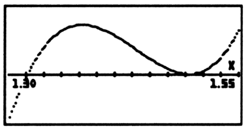
A.5



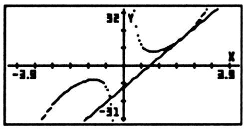
A.5



A.6



A.6



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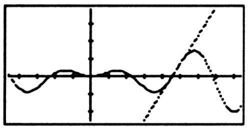
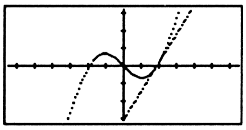
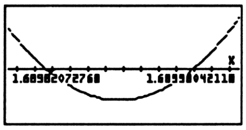


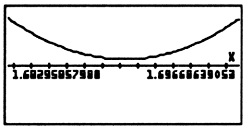
A.9



A.10



B.1



B.2

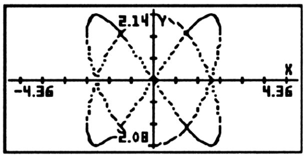
Chapter 3 Section 3.2

A.1 x-intercepts: 0, $\pm\sqrt{3}$; y-intercepts: 0, $\pm\sqrt{2}$; relative maxima: $(\pm\sqrt{3}, 2)$, $(0, 2)$; relative minima: $(\pm\sqrt{3}, -2)$, $(0, -2)$; inflection point: $(0, 0)$ **A.2** x-intercepts: $\pm\sqrt{2}$; y-intercepts: -1, 2; relative maximum: $(0, 2)$; relative minimum: $(\pm 2, -2)$; inflection point: none **A.3** x-intercepts: ± 5.44053852576, -0.02324974005; y-intercepts: -3.16153529132, 0.014939095, 2.6465961961; relative maxima: $(\pm 3.2227527617, 3.16227766017)$; relative minima: $(\pm 0.11774819461, -3.16227766017)$; inflection point: $(\pm 1.5081375479, 0.939021878471)$ **A.4** x-intercept: 0; y-intercepts: 0, -2; maxima: $(\pm 1.12054722171\ 1.63417691117)$; minima: $(\pm 1.6654928338, -0.7082509852)$, $(0, -2)$ **A.5** x-intercepts: ± 3.99822137144; y-intercepts: -2, 1, 2.5; maximum: $(\pm 1.93649167309, 2.6875)$; minimum: $(0, -2)$ **A.6** x-intercepts: 0, $\pm 2\pi$, $\pm 4\pi$; y-intercept: 0; maxima: $(\pm\pi, 2)$, $(\pm 3\pi, 2)$; minima: $(0, 0)$, $(0, \pm 2\pi)$, $(0, \pm 4\pi)$ **A.7** y-intercept: 0.6; maxima: $(\pm\pi, 1.4)$, $(\pm 3\pi, 1.4)$; minima: $(0, 0.6)$, $(\pm 2\pi, 0.6)$, $(\pm 4\pi, 0.6)$; inflection points: $(\pm 0.7926734425133 + 2n\pi, 0.84)$, $(\pm 5.49051188205 + 2n\pi, 0.84)$, $n = 0, 1$ **A.8** x-intercept: $\pm(\frac{\pi}{3} - \sqrt{3})$, $\pm(\frac{7\pi}{3} - \sqrt{3})$, $\pm(\frac{5\pi}{3} + \sqrt{3})$, $\pm(\frac{11\pi}{3} + \sqrt{3})$; y-intercept: -1, 1.6380450483; maxima: $(\pm\pi, 3)$, $(\pm 3\pi, 3)$; minima: $(0, -1)$, $(\pm 2\pi, -1)$, $(\pm 4\pi, -1)$ **A.9** x-intercepts: $-1/2$, 1; y-intercept: 1; maximum: $(0.3, 1.125)$; minima: $(-1, -2)$, $(1, 0)$ **A.10** x-intercepts: -3, ± 6; y-intercepts: ± 1.5; maxima: $(3.5584219849, 2.64025888957)$, $(-5.0584219849, 0.5535130947)$; minima: $(3.5584219849, -2.64025888957)$, $(-5.0584219849, -0.5535130947)$; inflection points: $(-1.01963065989, \pm 0.97578211674)$ **A.11** x-intercepts: -1.54508497188, -5, 4.04508497188; y-intercept: 1.5; maxima: $(-5, 0)$, $(-1.80189805015, 2.45126536676)$; minima: $(-3.4685647168, -1.06237647786)$, $(5, -3)$;

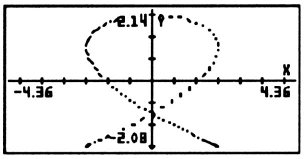
inflection point: (0.833333333335, 0.69444444446) **A.12** x-intercepts: −1.54508497188, ±5, 4.04508497188; y-intercepts: ±1.5; maxima: (−4.11690685633, 2.85894167805), (4.7089265974, 0.335004984445), (1.07464692556, 1.82397326262); minima: (−4.11690685633, −2.85894167805), (1.07464692556, −1.82397326262), (4.7089265976, −0.335004984445); inflection points: (−2.3158490886, ±1.042844254), (3.48576835002, ±0.48415270692) **A.13** x-intercepts: 0, ±4.7552825815, ±2.93892626147; y-intercept: 0; maxima: (−4.88314795858, 1.8239732626), (2.8374421468, 2.8589416781); minima: (−2.8374421468, −2.8589416781), (4.88314795858, −1.8239732626); inflection points: (±4.33012701893, ±1.299038106) **A.14** x-intercepts: 0, ±1; y-intercepts: ±0.8660254038, 0; maxima: $(1\pm\frac{1}{\sqrt{2}}, 1)$; minima: $(-1 \pm \frac{1}{\sqrt{2}}, -1)$; inflection point: (0, 0) **A.15(b)** absolute minimum point = (1.43653739005, −3.05478406218) **A.15(c)** $y = 1.83279002684x - 9.44382890683$ **A.16(a)** x-intercepts: −2, 1, 6; y-intercepts: ±0.8965754721, ±3.34606521495 **A.16(b)** $-3.34606521495 \le x \le 15.0625$, $-5.078125 \le y \le 5.078125$ **A.17(b)** self-intersection point: (0, 3), corresponding t-values: $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$ **A.18(b)** (±2, 1.5), (0, 2.4515479395) **A.19** $y = \pm\sqrt{3}x$ **A.20** (5, 0), (−1, 0), (−1.125, ±0.992156741645) **A.21** horizontal distance ≈ 6.2976, vertical distance = 6; therefore, even though it may appear to be a circle, it is not a circle. **A.22** (±2.2, 0), (0, −1.2), (0, 3.2), $(\pm a, \pm a)$, and $(\pm a, \pm a)$, where $a = 1.1\sqrt{2} \pm 0.5$

B.1 ≪ INPUTS DROP DROP → X Y ≪ Y T ∂ T ∂ X T ∂ * Y T ∂ X T ∂ T ∂ * − ≫ ≫ **B.2** left-most point = (6.25, −9), vertex = (−5.25, −12) **B.3** An ellipse. **B.4** Circle with center (0, 0), radius $\sqrt{2}$. **B.5(a)** (−5.25, −1.625) **B.5(b)** $y = 0.3x - 0.05$ **B.5(c)** No; T passes through the point (0, −0.005). **B.8** $\pm\sqrt{(9 \pm \sqrt{41}/5}$ **B.9** $y = \pm x$

C.4(a) (3.95836368579, −131.428064266), (5.52035400795, 137.57728216), (7.08618705687, 47.013081025), (8.65380430391, −43.349356455), (10.2223920291, −133.601548836), (11.7915653248, 136.212947959), (13.3611188323, 46.070813981), (14.9309330796, −44.04154769), (16.5085005167, −134.566131488), (0, 0) **C.4(b)** (3.926990817, −135), (5.4977871438, 135), (7.0685834706, 45), (8.6393797974, −45), (10.2101761242, −135), (11.780972451, 135), (13.3517687778, 45), (14.9225651046, −45), (16.4933614314, −135)



A.1



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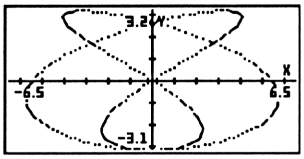

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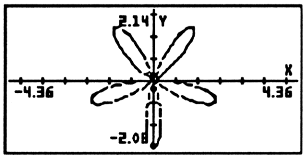

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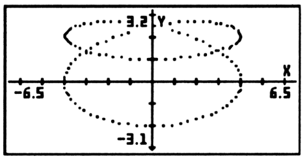
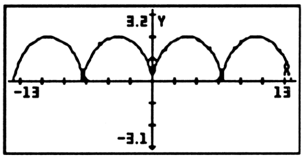
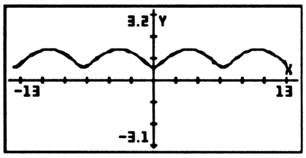


A.5



A.6



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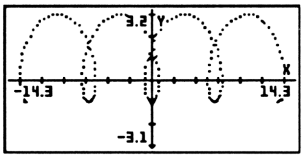

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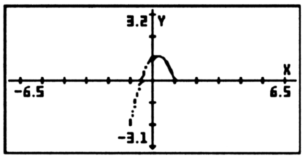

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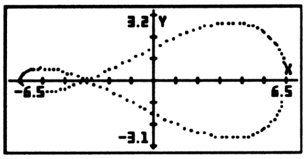

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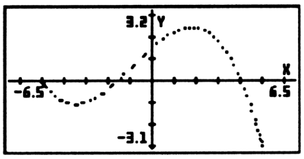

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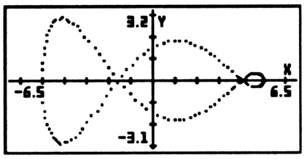

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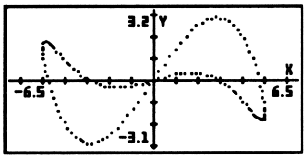
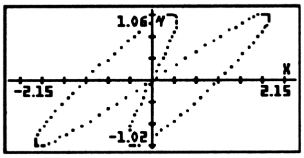


A.13



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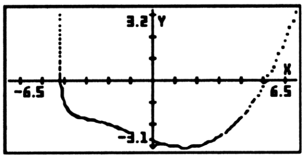

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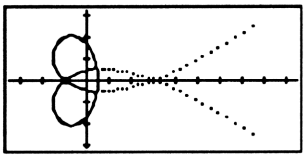
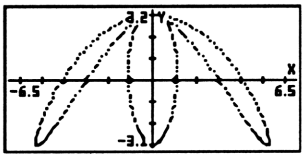


A.16



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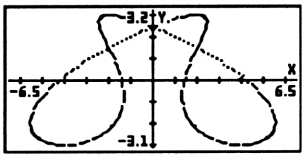

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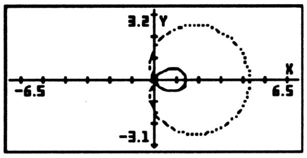

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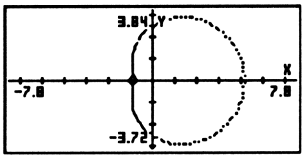

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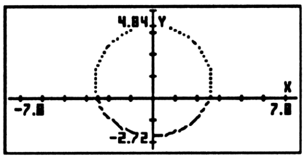
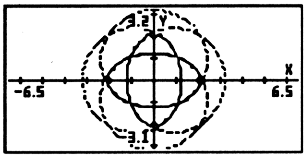
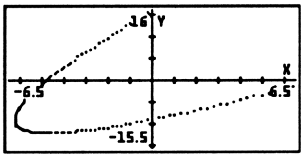


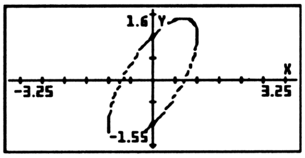
A.21



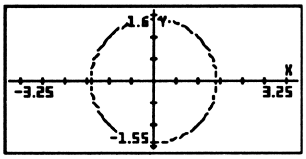
A.22



B.2



B.3



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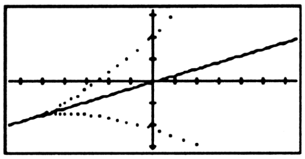
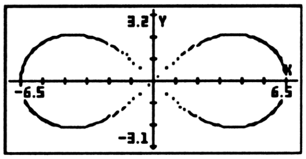
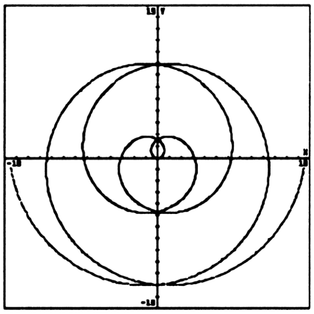


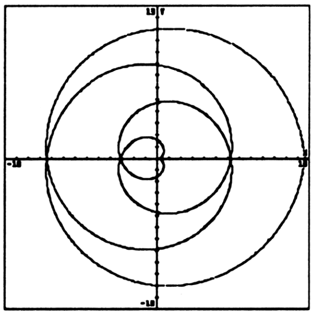
B.5



B.9



A.23(b)

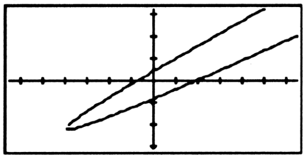


A.23(c)

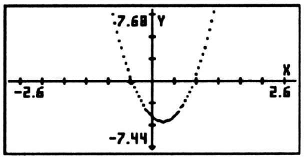
Chapter 3 Section 3.3

A.1(c) vertex: (0.19677398, −4.45424742); focus: (0.19677398, −4.43635887); directrix: $y = -4.47213596$ **A.1(d)** vertex: (−3.896, −2.168); focus: (−3.88, −2.16); directrix: $y = -2x - 10$ **A.2(c)** vertices: (±4.71404521, 0); foci: (±3.67784856, 0); center: (0, 0) **A.2(d)** vertices: (±3.33333333, ±3.33333333); foci: (±2.60063166, ±2.60063166); center: (0, 0) **A.3(c)** vertex: (−1.44152858, 0.10667311); focus: (0.18292998, 0.10667311); directrix: $x = -0.21176055$ **A.3(d)** vertex: (0.080761235, 0.071165840); focus: (0.19022869, −0.093035343); directrix: $y = \frac{2}{3}x + 0.25450451$ **A.4(c)** vertices: $(\pm\sqrt{3/5}, 0)$; foci: $(\pm\sqrt{6/5}, 0)$; center: (0, 0); asymptotes: $y = \pm x$ **A.4(d)** vertices: (±0.73484692, ±0.24494897); foci: (±1.03923048, ±0.34641016); center: (0, 0); asymptotes: $y = 0.5x$, $y = -2x$ **A.5(c)** vertices: $(0, \pm\sqrt{2})$; foci: (0, ±1); center: (0, 0) **A.5(d)** vertices: (±1.13137085, ±0.84852814); foci: (±0.8, ±0.6); center: (0, 0) **A.6(c)** center: (1.4, −2); vertices: $(1.4 \pm \sqrt{3.6}, -0.2)$; foci: $(1.4 \pm \sqrt{6}, -0.2)$; asymptotes: $y + 0.2 = \pm\sqrt{2/3}(x - 1.4)$ **A.6(d)** center: (1, −1); vertices: (−0.51789328, 0.13841996), (2.51789328, −2.13841996); foci: (−0.95959179, 0.46969385), (2.95959179, −2.46969385); asymptotes: $y = 0.04124145x - 1.04124145$, $y = 3.04124144 - 4.04124143x$ **A.7** $y = \frac{5}{6}x^2 - \frac{7}{6}x - 1$; $x = -\frac{5}{2}y^2 - \frac{1}{2}y + 2$

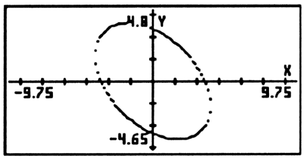
B.1 ≪ C→R 4 ROLL C→R 5 ROLL C→R → X3 Y3 X1 Y1 X2 Y2 ≪ Y1 Y3 − X1 X3 − / Y2 Y1 − X1 X2 − / + X3 X2 − / → A ≪ Y1 Y2 − X1 X2 − / X1 X2 + A ∗ − → B ≪ Y1 B X1 ∗ − A X1 SQ ∗ − Y SWAP A X SQ ∗ B X ∗ + + EXPAN COLCT ≫ ≫ ≫ ≫ **B.2** $y = \frac{1}{4}x^2 - 1$; $0.47140452x^2 - 0.94280904xy + 0.47140452y^2 - 1.41421356y - 1.88561808317 = 0$ **B.3** ≪ OBJ→ DROP DROP SWAP DROP → F ≪ F F X ∂ θ X STO EVAL 2 / ROT EVAL ROT EVAL 'X' PURGE → A C B ≪ B 2 A ∗ / NEG B SQ 4 A ∗ / C − NEG R→C DUP θ 4 A ∗ INV R→C + DUP IM 2 A ∗ INV − Y SWAP = ≫ ≫ ≫ **B.5** For rectangular plots, the HP 48 fills in pixels uniformly with respect to the x-axis. For parametric plots, it fills in pixels uniformly with the parametrization interval. As a result, in places where the graph is very steep, rectangular plots produce relatively few pixels while parametric plots will produce many more for a small step size.



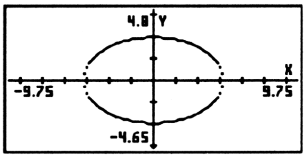
A.1



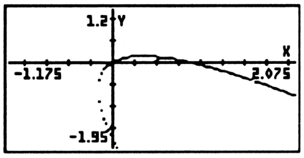
A.1



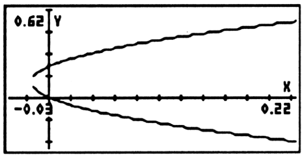
A.2



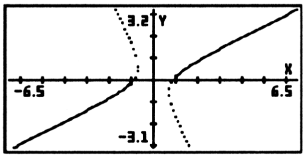
A.2



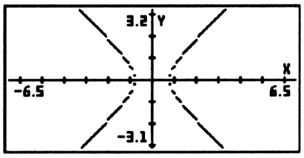
A.3



A.3



A.4



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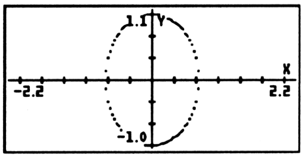
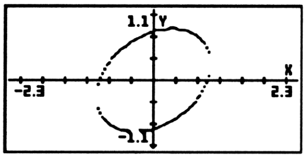
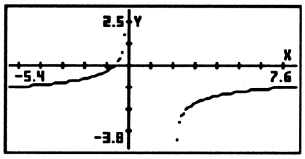


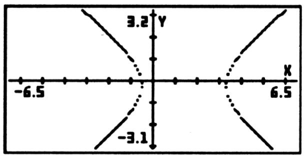
A.5



A.5



A.6



A.6

CHAPTER 4 ANSWERS

Chapter 4 Section 4.1

A.1 $[2.0, 2.1]$ **A.2** $[0.3, 0.4]$, $[4.0, 4.1]$ **A.3** $[0.9, 1.0]$ **A.4** $[0.5, 0.6]$
A.5 $[0.8, 0.9]$ **A.6** $[-0.9, -0.8]$, $[0.8, 0.9]$, $[4.0, 4.1]$ **A.7** $[0.1, 0.2]$, $[3.1, 3.2]$
A.8 $[1.4, 1.5]$

B.1 Use `'IFTE(X<2,-(2-X)^(1/3)+2*X^2-15,(X-2)^(1/3)+2*X^2-15)'`, $[-2.9, -2.8]$, $[2.6, 2.7]$ **B.2** $[0.000, 0.001]$, $[0.004, 0.005]$, $[0.013, 0.014]$
B.3 Consider the function $h(x) = f(x) - g(x)$, $x \in I$

Section 4.2

A.1 2.236 **A.2** −0.578 **A.3** −2.89, 2.66. You may wish to enter the function in the form

`≪ → X 'IFTE(X≥2,(X-2)^(1/3)+2*X^2-15,-(2-X)^(1/3)+2*X^2-15)' ≫`

A.4 11.84 cubic meters **A.5** 0.400, 4.000 **A.6** 0.0007, 0.0047, 0.0131.

B.1 For $\varphi_{23} = 20°$, $\varepsilon = 4.48$; for $\varphi_{23} = 30°$, $\varepsilon = 2.40$; for $\varphi_{23} = 40°$, $\varepsilon = 1.53$; for $\varphi_{23} = 50°$, $\varepsilon = 1.07$ **B.2** 74.68, 467.99, 1310.39

Section 4.3

A.1 1.7100 **A.2** 0.7391 **A.3** 4.4934 **A.4** 75.96° **A.5** 33.31°, 68.43°, 78.26° **A.6** −0.98 **A.7** 1.875, 4.694, 7.855, 10.996 **A.9** 2.0945

B.1 In order: 519.1,570.0,620.1,671.3,721.8,772.3,822.7,873.0,923.3. For initial guesses, observe that $x \approx \sqrt{((L - 300\pi)/2)^2 + 100^2}$. From trigonometry, $L = 2(\sqrt{x^2 - 100^2} + 200\pi - 100\cos^{-1}(100/x)$

C.1 `≪ → X ≪ 'X-F(X)/DF(X)' EVAL X ≫ ≫` **C.4** $-0.4956 \pm 1.3101i$ and $2.9956 \pm 4.0003i$

Section 4.4

A.1 126.632 **A.2** 0.416, 2.294, 6.290 **A.3** Intersection at $\theta_1 \approx 0.0133$. The limits of integration are from θ_1 to $\theta_1 + \pi$. The area is ≈ 9.88. **A.4** 2.472

B.1 See Fig. 5

C.2 $\alpha = 24.5$. The two largest roots come together as α decreases. For example, at $\alpha = 24.59$, the zeros are 0.2636, 1.4239, 3.3479, 9.48473, and 10.0699.

Section 4.5

A.1 y is determined as a function of x since g changes sign and $g'(y) > 0$ for $0 < y < \pi/2$ and $x \geq 0$, where $g(y) = x \tan y + y^3 - 4$; $f(1000) \approx 3.12$. It appears that y is bounded as $x \to \infty$. To preserve equation as $x \to \infty$, the term $x \tan y \to 0$. We guess that $y \to \pi$. See Fig. A.1

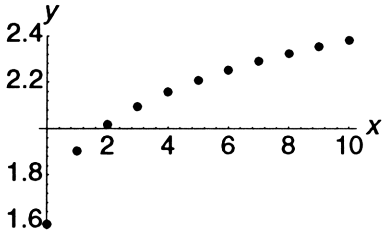


Fig. A.1

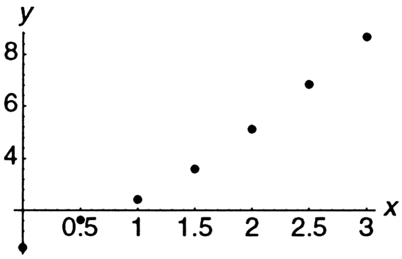


Fig. A.2

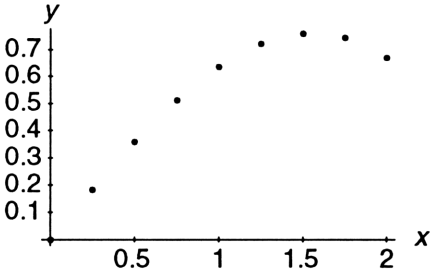


Fig. A.3

A.2 y is determined as a function of x since g changes sign and $g'(y) \geq 0$, where $g(y) = -x^3 + 6xy + 3y^2 - 8y\sqrt{xy} - 1$. Note that $g'(y) = 6(x + y - 2\sqrt{xy}) \geq 0$ for all $x, y \geq 0$. See Fig. A.2 **A.3** Maximum at $(1.5, 0.75)$, approximately. Solving $1 = 10\cos(x - y/9)$ and $x + 11y - 10\sin(x - y/9)$ simultaneously we find $x \approx 1.5542$ and $y \approx 0.763132$. See Fig. A.3. **A.4** $y - 0.1583 = -0.4563(x - 1.25)$ **A.5** $(3.995, -0.04157)$. This may be estimated as at the end of Example 2, though it would be tedious to obtain four significant figures. Alternatively, from setting the numerator of y' equal to zero and the original equation we find the equations $-3xy(xy^2+2)-1 = 0$ and $x^2y(xy^2+3)+x-2 = 0$. Solve the first for xy^2 and use the result in the second equation. Solve the result for y, finding $y = (2 - 2x/3)/x^2$. Substitute this in the first equation and simplify to obtain $3x^4 - 100x^3/9 - 8x^2 + 24x - 24 = 0$. One zero is 3.99542. **A.6** Factoring the positive exponential from $f'(x)$, it is not difficult to observe that the remaining factor is > 0 for $-1 < x < 1$. Using the SOLVR, $x \approx 0.784 = f^{-1}(0.9)$. By interpolation, $f^{-1}(0.9) \approx .786$.

B.1 Let $x \in [0, 3]$ be given. Define $g(y) = e^{0.01y}(0.1x + y) - e^{-x}(1 + 2x - 0.2x)$. Then $g(0) < 0$ & $g(y) > 0$ for sufficiently large y. So, by IMV, $g(y) = 0$ for at least one y. For $y \geq 0$, $g'(y) > 0$. So, at most one y such that $g(y) = 0$.

CHAPTER 5

Section 5.1

A.1 Shortest distance = 1.228 **A.3** $x_{\text{min}} \approx 8.1802$, $f(x_{\text{min}}) \approx -0.030050$ **A.4** $x_{\text{min}} \approx 3.2923$, $f(x_{\text{min}}) \approx -1.7939$, $x_{\text{max}} = 2\pi$, $(f(x_{\text{max}} \approx 2.5066$ **A.5** $x_{\text{min}} = 3$, $f(x_{\text{min}}) \approx 0.35714$, $x_{\text{max}} \approx 0.59607$, $(f(x_{\text{max}} \approx 1.1184$ **A.6** Local minima at 0.0000 and 1.0650; local maxima at 0.26900 and 1.4000 **A.7** $x_{\text{max}} = 1.0000, f(x_{\text{max}}) = 0.73576$

Section 5.2

A.1 0.78 **A.2** Using $a = 0.0$, $b = 0.5$, and $-f$, $x_{\text{max}} \approx 0.37$ **A.3** -1.57 **A.4** 9.42 at $x_{\text{max}} \approx 2.47$
B.4 If x is outside the interval & is moving away from the interval, each distance is increasing
C.1 If $a \leq u < v \leq x_{textmin}$ and $f(u) \leq f(v)$, then f would have a local minimum in $[a, x_{\text{min}}]$, which is impossible.

Section 5.3

A.1 $x_{\text{min}} \approx 0.528$; $f(x_{\text{min}}) \approx -0.086$ **A.2** $x_{\text{min}} \approx 2.718$; $f(x_{\text{min}}) \approx 0.632$
A.3 From a plot of f on $[-1.5, 1.5]$, with the help of Z-BOX, f is unimodal on $[-1.5, -0.3]$ and $[0, 1.5]$. The function $-f$ is unimodal on $[-0.3, 0]$. Using GSS on these intervals and calculating f at -1.5 and 1.5 we find $x_1 \approx -0.570$ is a local minimum point and $f(x_1) \approx 0.00866$, $x_3 \approx 0.791$ is an absolute minimum point and $f(x_3) \approx -0.551$, $x_2 \approx -0.222$ is a local maximum point and $f(x_2) \approx 0.0419$, $x_0 \approx -1.5$ is an absolute maximum point and $f(x_0) \approx 3.4125$, and $x_4 \approx 1.5$ is a local maximum point and $f(x_4) \approx 2.2125$
A.4 $\theta \approx 0.936$ **A.5** $x_{\text{min}} \approx 133.333$, $f(x_{\text{min}}) \approx 250$

B.1 It takes $2.5n$ function evaluations in BMM to get $(1/2)^n$ factor. So, to achieve 10^{-5}, BMM needs $2.5(5\ln 10)/\ln 2 \approx 41.5$. Similarly, GSS requires $(-5\ln 10)ln((-1+\sqrt{5})/2) \approx 23.9$ **B.2** 14.1 feet **B.3** See problem A.5. If the angles are measured from the normal to the mirror, then they are ≈ 0.927 (radians) or 53.1°

C.1 From $BE^2 = (AB/2)^2 + AB^2$ and $BE = AB/2 + AH$, we have $AH^2 + AB \cdot AH = AB^2$. From $AB^2 - AB \cdot BH = AB(AB - AH) = AB \cdot BH$ we get $AH^2 = AB \cdot BH$.
C.2 A solution procedure is to generate $y = g(x)$ values with the program

```
≪ → X ≪ X 'W' STO '4*W^2*Y^3-2*W^3*Y^2+9*W*Y-1'
                    'Y' X ROOT ≫ ≫,
```

which may be stored as G. Next, store

```
≪ → X ≪ X SQ X G SQ + ≫
```

as F and then use GSS, perhaps with $E = 0.001$

CHAPTER 6 ANSWERS

Chapter 6 Section 6.2

A.1 $[-58/53, 138/53, 89/53]$ **A.2** $[3, -2, 2]$ **A.3** $[8, 4, -1]$
A.4 $[53/43, -38/43, 236/43]$ **A.5** $[-1.4760\cdots, -6.6033\cdots, -4.5055\cdots]$

B.1 $[5/2 + (1/2)k, -1/2 - (3/2)k, k]$, for any choice of k **B.2** $[1/4, -1/4, -3/4 + k, k]$, for any choice of k **B.3** $[2 - 5r - s, -2 + 4r, r, s]$, for any choice of r, s **B.4** Pivoting on the {1,1},{2,2},and {3,3}-positions, for example, gives a matrix with bottom row $[0, 0, 0, 13]$, which shows no solution **B.5** $[-3/4 + k, 3/2, k, 1/4]$, for and choice of k **B.6** $[14/5 + r - (7/5)s - (3/5)t, -18/5 + r - (6/5)s + (6/5)t, r, s, t]$, for any choice of r, s, t **B.7** No **B.8** $[k, -k, 0, 0]$ is a solution, for any k **B.9** $\approx [-1.6740r + 3.3801s, -1.3294r + 2.1030s, r, s]$, for all r, s

Chapter 6 Section 6.3

A.1 $[-1.0943\cdots, 2.60377\cdots, 16792\cdots]$ **A.2** $[3, -2, 2]$, with CLEAN
A.3 $[8, 4, -1]$ **A.4** $[2, -2, 5]$, with CLEAN

CHAPTER 7 ANSWERS

Chapter 7 Section 7.1

A.1 $7^{3/2}/3 - 1/3$; $L_5 = 5.32813\cdots$; $U_5 = 6.31558\cdots$; $A_5 = 5.82186\cdots$ **A.2** $A_{10} = 0.31117\cdots$ **A.3** $A_{10} = 8.11352\cdots$ **A.4** $A_{10} = 2.00503\cdots$ **A.5** Decreasing; $A_{10} = 1818.96\cdots$ **A.6** Increasing (after changing variables); $A_{40} = 1.21105\cdots$ **A.8** Decreasing; $0.84239\cdots$

B.1 Let $n = 80$; $\mathrm{Si}(1) \approx 0.946$ **B.2** Increasing; let $n = 360$; 1.318; no difference in final outcome **B.3** $n = 208$; $v(1) \approx 1.5 + 1.089 = 2.589$ **B.4** $A_n = L_n + (h/2)[f(b) - f(a)]$. If f is decreasing, $A_n = U_n + (h/2)[f(b) - f(a)]$.

Chapter 7 Section 7.2

Purge a or b from the current and higher directories.
A.1 $1.2189\cdots$ **A.2** $\frac{2}{3}(b+1)^{3/2} - 2/3$ **A.3** $2.6487\cdots$ **A.4** $-\ln(\cos(b))$
A.5 $0.5707\cdots$ **A.6** $b \arcsin(b) + \sqrt{1 - b^2} - 1$ **A.7** $1/\cos(x)$ **A.8** $x \arctan(x) - \ln(1 + x^2)/2$ **A.9** 1.60 **A.10** 1.20 **A.11** 0.917

B.1 No.

Chapter 7 Section 7.3

A.1 $\frac{\pi}{4} - \frac{1}{2}\ln 2$ ($\approx$ 0.438824573118) **A.2** 4.5 **A.3** 1.13209039331 **A.4** 8 **A.5** 8 **A.6** 30.5787478555 **A.7(a)** 30.4184213031 **A.7(b)** 15.2092106276 **A.7(c)** The curves in (a) and (b) generate the same set of points. The difference is that curve (a) generates this set twice. **A.8** 9.42477796077 **A.9** 8.31384387632 **A.10** 10.7350606813 **A.11** 0.179519580206 ($\approx$ $(2\pi)/35$) **A.12** 11.2498439727 **A.13** 39.4784176044 **A.14** $2\sqrt{2}$ ($\approx$ 2.82842712474); no. **A.15** 7.64039557806; no. **A.16** (0.6, 0.34285714286) **A.17** (0.267303498964, 0.603553390598) **A.18** 50.2654824576 **A.19** 67.672872654.

B.1(a) 10.8 **B.1(b)** 21.6 **B.1(c)** 21.6 **B.2(a)** 15.639015605 **B.2(b)** 26.6261172656 **B.2(c)** 21.9742033213 **B.3** Volume $\approx$ 41.3206167666$\cdots$ Surface area $\approx$ 70.0107085546 **B.4** 5.26110432798 **B.5** 359.435746536 **B.6** $\ll$ ROT X ∂ SQ 1 + $\surd$ X $\int$ →NUM $\gg$ Model picture is Fig. 4(b) without shading. **B.7** 4.6465 The curve has a vertical tangent at (0, 0). **B.8** $\ll$ → F G A B $\ll$ A B X F G − $*$ X $\int$ →NUM A B 0.5 F SQ G SQ − $*$ X $\int$ →NUM R→C A B F G − X $\int$ →NUM / $\gg$ $\gg$

Chapter 7 Section 7.4

A.1 $M_{20} = 1.09824\cdots$ **A.2** $M = 2 \cdot 1 \cdot 1, n = 13, T_{13} = 0.74646\cdots$ **A.3** $T_{18} = 0.52364\cdots, \pi/6 = 0.52359\cdots$ **A.4** $T_{82} = 3.14156\cdots$ **A.5** $M_3 = 1.21111\cdots, T_9 = 1.21105\cdots$ **A.6** $T_6 = 0.94538\cdots$ **A.7** 35.25 **A.8** $T_5 = 1.24167\cdots$ **A.10** $n = 10$ for trapezoid rule, $x(75) \approx 1515.22, y(7.5) \approx 79.83$

B.3 TRAP2 calls SUM twice, adding nearly the same numbers. **B.4** S_n is based on $2n$ subdivisions and $h = (b-a)/(2n)$. Since n is stored, the number $(b-a)/n = 2h$ is stored under H and used in M_n and T_n. Result follows from

$$M_n = f\left(\frac{x_0 + x_2}{2}\right)2h + \cdots = 2h[f(x_1) + \cdots]$$

$$T_n = \tfrac{1}{2}2h[y_0 + 2(y_2 + \cdots + y_{2n-2}) + y_{2n}] = h[y_0 + 2(y_2 + \cdots) + y_{2n}]$$

B.6 Graphically, $f''(0) = -2$ and $|f''(x)| \le 2$ elsewhere. Analytically, maximize the function f'' by finding the zeros of $f'''(x) = -8x(x^2-1)e^{-x^2}$ and checking endpoints of $[0, 2]$. **B.12** Graphically, $|f''(x)| \le f''(\pi/2) = 0.686\cdots$, say, $M = 0.7$. This gives error less than $0.00905 < 0.01$. **B.15** Take $b = 3$, $M = 25 > |f'(0.2)| = 24.56\cdots$, $n = 31$. $E_1(0.2) = 1.2, E_1(0.4) = 0.7, E_1(0.6) = 0.4, E_1(0.8) = 0.3, E_1(1.0) = 0.2, E_1(1.2) = 0.1$ **B.16** $M = 2, n = 86, \Gamma(3/2) \approx 0.886$ **B.18** Calculate $P(0) = 0.5, P(0.2), \ldots, P(2.0)$. $P(0.2) = 0.5 + (1/\sqrt{2\pi})\int_0^{0.2} e^{-x^2/2}\,dx$, etc. Also, $P(-0.2) = 1 - P(0.2)$, etc. Include $(1/\sqrt{2\pi})$ as part of integrand for calculation of M. For all of the values $0.0, 0.2, \ldots, 2.0$, we may take $M = 0.4$. Then $n = 2$. $P(0.2) = 0.58, P(0.4) = 0.66, P(0.6) = 0.73, P(0.8) = 0.79, P(1.0) = 0.84, P(1.2) = 0.88, P(1.4) = 0.92, P(1.6) = 0.95, P(1.8) = 0.96, P(2.0) = 0.98$.

Chapter 8 Section 8.1

A.1(a) not monotone, converges to 0 **A.1(b)** decreasing for $x \geq 8$, converges to 0. **A.1(c)** not monotone, converges to 0. **A.2(a)** $\pm\pi/2$ **A.2(b)** all real numbers **A.2(c)** 0, ± 1 **A.3(a)** increasing, converges to 1. **A.3(b)** decreasing, converges to 1. **A.3(c)** not monotone, converges to 0. **A.3(d)** increasing, converges to e. **A.3(e)** decreasing, converges to 1. **A.3(f)** increasing, converges to π. **A.4(a)** bounded, not monotone, not convergent. **A.4(b)** bounded, decreasing, convergent to 0. **A.4(c)** bounded, decreasing, convergent to 0. **A.4(d)** bounded, not monotone, convergent to 0. **A.4(e)** bounded, not monotone, not convergent. **A.4(f)** bounded, not monotone, not convergent. **A.4(g)** bounded, not monotone, not convergent. **A.4(h)** bounded, not monotone, convergent to 0. **A.4(i)** bounded, increasing, convergent to 1. **A.4(j)** bounded, decreasing, convergent to 2. **A.4(k)** unbounded, increasing, divergent. **A.4(l)** bounded, decreasing, convergent to 0.8 **A.4(m)** bounded, decreasing, convergent to 0.8. **A.4(n)** bounded, not monotone, convergent to 0. **A.4(o)** bounded, not monotone, convergent to 0. **A.4(p)** not bounded, increasing, divergent. **A.4(q)** bounded, decreasing, convergent to 0. **A.4(r)** unbounded, not monotone, divergent.

B.1(a) converges to 0. **B.1(b)** converges to 0. **B.1(c)** converges to e^2. **B.1(d)** converges to $\ln 2$. **B.2(b)** (435, 1729426.51524).

Chapter 8 Section 8.2

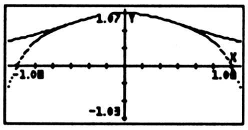
A.1 $\ln 2$ **A.2** $e - 1$ **A.3** π **A.4** ∞ **A.5** ∞ **A.6** -1 **A.7** 0 **A.8** 2 **A.9** ∞ **A.10** $1 - \ln 2$ **A.11** $\ll$ $\rightarrow$ A R $\ll$ A 1 R $-$ / $\gg$ $\gg$. **A.12(a)** 0.75 **A.12(b)** $\frac{1}{9}$ **A.12(c)** $\frac{3}{28}$ **A.12(d)** $1 - 0.5 = 0.5$

C.1(b) consider the series $1 + 10^{-13} + 10^{-14} + 10^{-15} + \cdots$

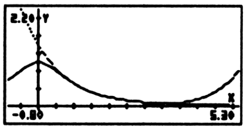
Chapter 8 Section 8.3

A.1(a) $1 - x^2 + x^4 - x^6$ **A.2(a)** $0.2 - 0.16(x-2) + 0.088(x-2)^2 - 0.0384(x-2)^3 + 0.01312(x-2)^4$ **A.3(a)** $2x + 0.4x^2 - 1.293x^3 - 0.264x^4 + 0.24013x^5 + 0.0515608x^6$ **A.4(a)** $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$

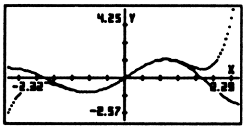
B.1 $x - \dfrac{x^3}{3!} + \cdots - \dfrac{x^{23}}{23!}$ will guarantee that the error is $< \dfrac{x^{25}}{25!} < 10^{-12}$.



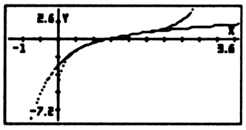
A.1



A.2



A.3



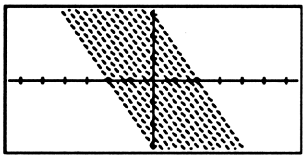
A.4

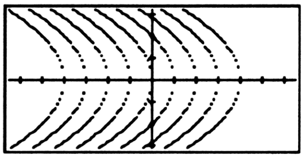
Chapter 9 Section 9.1

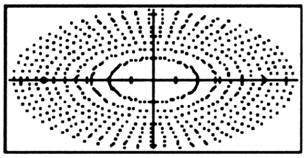
A.4 Starting at (0, 0) and walking along the x-axis in either direction would be an increasingly steep climb. Along the y-axis in either direction would be an increasingly steep

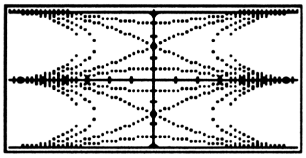
descent. Along the lines $y = \pm\frac{x}{3}$, you would theoretically stay on level ground; in fact, footing would be difficult with a wall to your right and a steep drop-off to your left.
A.5 100 **A.6** 0.999649728498

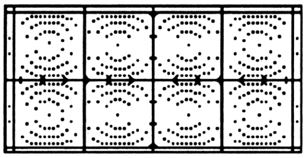
B.1 The level curves for $C = \pm 1$ intersect the interval $(-\pi, \pi]$ of the y-axis at exactly one point.


A.1


A.2


A.3


B.1


B.2

Chapter 9 Section 9.2

A.1 -36 **A.2** $-\frac{3}{7} = -0.\overline{428571}$ **A.3** $-\frac{16}{27} = -0.\overline{592}$ **A.4** $\frac{104}{\sqrt{13}} \approx$ 28.8444102037 **A.5** $[-4, -3] = -4\boldsymbol{i} - 3\boldsymbol{j}$ **A.6** $(5\boldsymbol{i} + 9\boldsymbol{j} - 5\boldsymbol{k})/\sqrt{131}$
A.8 0.0449211581301

Chapter 9 Section 9.3

A.1 $2x - 3y + 4z = 20$ **A.2** $x = 5 + t,\ y = -2 + 2t,\ z = 1 + 3t$ **A.3** $x = 3 + 2t,\ y = -t,\ z = -7 + 9t$ **A.4** $30x + 11y + 2z = 14$ **A.5** $x = -3 + t,\ y = 2,\ z = -5$
A.6 $y = 2$ **A.7** $4x + 10y + 3z = 0$ **A.8** $x = -2 - 12t,\ y = 1 + 10t,\ z = -1 - 4t$

Chapter 9 Section 9.4

A.1 Minima at (0, 0) and (1, 0); saddle point at (0.5, 0) **A.2** Minimum at (0, 0); saddle points at (1, ±2) **A.3** Maximum at (−1/4, 1/2); saddle point at (0, 0) **A.4** Maximum at (0, 1); saddle points at (1, 0) and (1, 2) **A.5** Minimum at (4/3, −2/3); saddle points at (0, 0), (0, −2), and (4, 0) **A.6** (2.42954732764, −0.27210776694) **A.7** (2.53559500795, −1.66972010972), −45.8960394318, 5.7×10^{-11}

B.1 Minimum at (2/3, 7/3); saddle points at (0, 1), (0, 5), and (2, 1) **B.2** Minimum at (0.450183611295, 1); saddle point at (0.450183611295, 0) **B.3** Maximum at $(\sqrt{2/3}, 0)$; minimum at $(-\sqrt{2/3}, 0)$; saddle points at (1.52568712087, ±1), (−0.258652022505, ±1), and (−1.2670350936, ±1)

Chapter 9 Section 9.5

A.1 62.5 **A.2** -0.0049208730168 **A.3** 12 **A.4** -0.0730133288
A.5 $\frac{8}{15}\sqrt{2}+\frac{1}{2}\approx 1.25424723326$ **A.6** 12 **A.7** 7/12 **A.8** 5.1 **A.9** -20π
A.10 1/3 **A.11** 1.0696750649; error bound 10^{-6} **A.12** 1.63808113709; error bound 2×10^{-6} **A.13** 0.240477289185; error bound 3×10^{-12}

Index

Program Index

Trouble-Prevention & Trouble-Shooting

1. If the hourglass annunciator ⧖ won't go away, try pressing $\boxed{\text{ON}}$; if that doesn't work, press $\boxed{\text{ON}}$ and $\boxed{\text{C}}$ at the same time. You could also "reset the memory" so that the calculator goes back to being like it was when you first took it out of the box. Just press ON, A, and F simultaneously. Think twice before you do this. Also, don't tell enemies or practical jokers how to do this!
2. If you get hopelessly tangled up, press $\boxed{\text{ON}}$ one or more times. Press $\boxed{\text{VAR}}$ to get to your variables menu. Press $\boxed{\text{HOME}}$ to get to your home directory.
3. If the annunciator ((•)) comes on and if you haven't set alarms, you have to change the batteries. Be sure the calculator is OFF when you change batteries. See the Owner's Manual for additional instructions.
4. If the calculator won't allow RPN entry of algebraic expressions like X + 1, clear flag −3: −3CF $\boxed{\text{ENTER}}$.
5. If the calculator puts a number on the stack when you press a letter key (like X), it probably means that that letter is on your VAR menu and should be purged. If that doesn't work, check the VAR menu in higher directories.
6. If all numbers are shown with, say, two decimals, you are in 2 FIX mode. To get into standard mode, press $\boxed{\text{STD}}$ in the MODES menu ($\boxed{\Lsh}$ $\boxed{\text{CST}}$ $\boxed{\text{FMT}}$).
7. If you're having trouble graphing, try purging the plot parameters (key in 'PPAR' $\boxed{\text{PURGE}}$ or press $\boxed{\text{RESET}}$).
8. Troubles in graphing expressions including trig functions often disappear by going to RADians mode. RAD should be in the upper left-hand corner of the display.
9. If you get vertical lines when you graph a function, go to page 2 of the PLOT menu ($\boxed{\Lsh}$ 8 $\boxed{\text{NXT}}$ $\boxed{\text{FLAG}}$) and toggle to CNC■, then redraw the graph.
10. To quickly access the graphics environment from the stack, press ◄. To leave the graphics environment, press $\boxed{\text{ON}}$.
11. To get rid of the menu in the graphics environment, press MENU (page 2 of EDIT); to get the menu back, press any white key. [There is no simple way to get rid of the menu in the stack environment.]
12. To trouble-shoot a program, insert HALT in your program, then use SST to single-step your way through the program. Or put input data on stack and name of program, then $\boxed{\text{PRG}}$ $\boxed{\text{NXT}}$ $\boxed{\text{RUN}}$ $\boxed{\text{DBUG}}$ $\boxed{\text{SST}}$ $\boxed{\text{SST}}$, ...
13. If the HALT symbol doesn't go away, press $\boxed{\text{KILL}}$.
14. If R ∡ Z or R ∡ ∡ appears in the upper left-hand corner of the display, it means you are in polar (cylindrical) or spherical mode. To get rid of either, press: $\boxed{\text{MTH}}$ $\boxed{\text{VECTR}}$ $\boxed{\text{NXT}}$ $\boxed{\text{RECT}}$. To get rid of R ∡ Z, press $\boxed{\Rsh}$ $\boxed{\text{POLAR}}$.
15. If the graphics cursor seems to temporarily delete pixels as you move it around the viewing window, press $\boxed{+/-}$ in the graphics menu ($\boxed{\text{EDIT}}$ $\boxed{\text{NXT}}$).
16. If you are having trouble storing an object, the name you are trying to store it under is probably being used for something else. The calculator won't let you store an object under a built-in name; however, you can fool it by using lowercase.
17. If you have trouble entering a program, check your spacing. Put spaces around → only if it is being used to define local variables. Be careful not to confuse 0 with O.
18. To get numerical approximations for π and e (and to get $(0, 1)$ for i), use →NUM. To automate this, key in −2SF ENTER (to go back, key in −2CF ENTER).

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