HP48 COURSE NOTES

48

Introduction to the HP-48

The HP-48 is not just another graphing calculator, but a true "computer in your hand," with a powerful WYSIWYG (What You See Is What You Get) symbolic algebra system, a spreadsheet-like matrix environment, and unit management. This remarkable software comes with its own hardware that includes a port for file exchange via cable connection to either IBM or Macintosh computers (using the KERMIT protocol), and a two-way infrared communications system for wireless transfer from calculator to printer or another calculator. The expandable version of the HP-48 also includes two RAM/ROM expansion slots (like disk drives on a computer),

In this paper we attempt to give a brief tour of some of the functionality of the HP-48. This short guide is not intended to be a substitute for the owner's manuals. However, we hope that the novice can use it to quickly sample and get a feel for the machine. For those who desire more detailed information, particularly on programming, we list several references at the end.

NOTES:

Helvetica font indicates a labelled key on the HP-48 keyboard. (Examples: [MTH] or [ENTER] or [SIN] .) A boxed expression in the HP-48's own font indicates one of the six white keys directly below the screen. The label for the key appears along the bottom of the screen once the appropriate menu is activated. (Example: [STEQ] is found through the [⌐] [PLOT] or [⌐] [SOLVE] menus.)

48.1 GENERAL INFORMATION

1. Busy signal:

Whenever the annunciator (little hourglass) is on at the top of the screen, the calculator is "busy" with a calculation. Pressing [ON] while the annunciator is on will abort the computation and return you to the stack.

2. Directory:

The calculator's memory is organized in a tree structure consisting of directories, subdirectories, subsubdirectories, etc. Pressing [UP] moves you up one level in the directory structure. Pressing [HOME] moves you immediately to the top level of the directory.

3. Shift Keys:

The three keys immediately above the [ON] key are shift keys:

[⌐] in blue [¬] in orange [α] for letters

They are associated with the corresponding colored labels or letters around the other keys. When one of the shift keys has been pressed, its own label is displayed at the top of the screen. [⌐] and [¬] toggle on and off. Pressing [α] twice in a row locks the calculator in alpha mode, a third press unlocks it.

NOTE: In this guide, we will not indicate the shift key except for special characters not labelled on the keyboard.

4. Entering an object onto the stack:

When you type, the characters appear on the *command line* until you press [ENTER] or an operation key which forces automatic entry.

2 [ENTER] 1: 2

5. Deleting characters and objects:

The "backarrow" key is a *backspace* and will delete the last character you typed on the command line. If nothing is on the command line, then this key drops the last entry off the stack. [DROP] also drops the last entry off the stack. [CLR] drops all entries off the stack (but does not clear anything stored in memory).

6. Numeric display: [FIX]

[MODES] 4 [FIX] fixes the numeric display to 4 decimal places.

[STD] returns to the default setting of 12 significant digits.

Try this: [π] [→NUM] [MODES] 4 [FIX] [STD] 3 [FIX] 4 [FIX] etc.

7. Radian mode:

The default setting for the HP-48 is degree mode for angle measure. [RAD] toggles between radian mode and degree mode. (RAD is displayed at the top of the screen when you are in radian mode, but DEG is *not* displayed when you are in degree mode.)

8. Storing information in memory:

3 [ENTER] W [STO]

stores 3 under the variable name W.

9. Recalling stored information from memory:

W [ENTER] 1: 3

OR: [VAR] [W]

10. Purging stored information from memory

[VAR] ['] [W] [PURGE] makes W disappear from the [VAR] menu.

48.2 RPN ARITHMETIC

Like most HP calculators, the HP-48 utilizes *Reverse Polish Notation*: First one enters the objects to be operated on, then one applies the operation. The answers shown for the following examples are for the standard ([STD]) numeric display.

1. Change sign or enter a negative number:

2 [+/−] [ENTER] 1: -2

(Note: pressing [−] makes the calculator attempt to perform subtraction.)

2. Add: $26 + 82$

26 [ENTER] 82 [+] 1: 108

3. Subtract: $86 - 32$

86 [ENTER] 32 [−] 1: 54

4. Multiply: $62 \cdot 45$

62 [ENTER] 45 [×] 1: 2790

5. Divide: $85 \div 20$

85 [ENTER] 20 [÷] 1: 4.25

6. Powers: $(42)^5$

42 [ENTER] 5 [y^x] 1: 130691232

7. Square root: $\sqrt{20}$

20 [$\sqrt{x}$] 1: 4.472135955

8. Square: $(25)^2$

25 [x^2] 1: 625

9. **Reciprocal:** $\frac{1}{85}$

85 [$1/x$] 1: 1.17647058824 E-2

(which is equivalent to 0.0117647058824)

10. **Combinations of arithmetic operations:**

a. $5(3+4)$

5 [ENTER] 3 [ENTER] 4 [+] [×] 1: 35

b. $6\sqrt{10}$

6 [ENTER] 10 [$\sqrt{x}$] [×] 1: 18.973665961

c. $5+2^3$

5 [ENTER] 2 [ENTER] 3 [y^x] [+] 1: 13

d. $(25+7)^{4/5}$

25 [ENTER] 7 [+] 4 [ENTER] 5 [ENTER] [÷] [y^x] 1: 16

11. **Trigonometric values:**

a. sin 2 (radian mode)

2 [SIN] 1: .909297426826

b. arctan 1

1 [ATAN] 1: .785398163397

12. **Logarithms and exponentials:**

a. log 2

2 [log] 1: .301029995664

b. e^{10}

10 [e^x] 1: 22026.4657948

48.3 UNITS

Units of measurement can be attached to numerical quantities by selecting the appropriate unit from the [UNITS] menu. Unit management is automatically taken care of by the calculator.

2 [UNITS] [LENG] [FT]	1:	2_ ft
3 [IN]	2:	2_ ft
	1:	3_ in
[+]	1:	27_ in

Compound units such as mi/hr are handled without any problems.

To simply convert from one unit to another, press the orange shift key and then the desired unit.

[⌝] [YD]	1:	0.75 _ yd

Special constants

Special constants such as π and e are displayed in symbolic form provided [SYM] (under [MODES]) is toggled "on" (indicated by a lit square on the soft key).

To find π in decimal and rational form:

[π] [→NUM] [→Q]

Use [FIX] to control the number of digits used in the numerator and denominator of the rational form.

To find $\arcsin .5$ as a rational multiple of π,

.5 [ENTER] [ASIN]	1:	.523598775598
[ALGEBRA] [NXT] [→Qπ]	1:	'1/6*π'

To find e in decimal:

[α] [⌝] [E] [ENTER] [→NUM] [→Q]

NOTE: lower case letters are activated with [α] [⌝]; Greek letters are activated with [α] [⌜]; other special symbols (like ∞) may also be accessed with one or the other of these shift combinations.

48.4 ALGEBRAIC NOTATION AND EVALUATING EXPRESSIONS

The [′] key is used to denote symbolic expressions. It is called the "tick" key and is located as the left-most key in the third row from the top.

1. Using algebraic notation for arithmetic

a. Evaluate $'2 + 3'$

[′] 2 [+] 3 [ENTER]	1:	'2+3'
[EVAL]	1:	5

b. Evaluate $'7(4 + 9) - 15'$

[′] 7 [×] [()] 4 [+] 9 [▶] [−] 15 [ENTER]	1:	'7*(4+9)-15'
[EVAL]	1:	76

Note the effect of the [▶] key. Pressing [()] again opens another set of parentheses rather than closing the first set.

2. Decimal to fraction and back:

[ATTN] 2.5 [ENTER] [→Q]	1:	'5/2'
[EVAL]	1:	2.5
[′] 58 [÷] 3 [ENTER]	1:	'58/3'
[EVAL]	1:	19.3333333333
[→Q]	1:	'58/3'

48.5 SOLVE APPLICATION

The [SOLVE] application allows you to work with expressions and functions numerically.

For example, suppose we have stored the expression $x^2 - 3$ under the variable name S:

['] X [y^x] 2 [−] 3 [ENTER] S [STO]
S [ENTER] 1: 'X^ 2-3'

. Evaluating stored variables using the SOLVER

To evaluate S at $x = 3$: [VAR] [S] [SOLVE] [STEQ] [SOLVR] 3 [X] [EXPR=]

1: EXPR: 6

To evaluate S at $x = 15$: 1 5 [X] [EXPR=] to obtain

1: EXPR: 222

NOTE: The letters "EXPR:" are simply labels for your convenience, and are not recognized by the calculator for computation purposes. For example, you could press [+] to obtain the sum 228.

If you press the orange shift key and then [X], the HP-48 will use the current value of X as a "seed" to find the closest root of the expression.

48.6 EQUATION WRITER

The Equation Writer uses the screen as a "blackboard" to write expressions in usual textbook format. To activate, press [⬏] [EQUATION] .

The most important key to remember while using the Equation Writer application is [▶] . Use it whenever you want to proceed to the next "component" of an expression. For example, you press [▶] whenever you wish to leave a denominator or get "outside" a radical sign.

If you wish to simply typeset an expression that is already on the stack, press the down arrow key.

1. To write $x^2 - 2$ in the Equation Writer and store in the SOLVER

X [y^x] 2 [▶] [−] 2	$X^2 - 2$
[ENTER]	1: 'X^ 2-2'
[SOLVE] [NEW] Y1 [ENTER]	
	Current equation:
	Y1: 'X^ 2-2'

(NOTE: when naming an equation or a function, the calculator is locked in alpha mode.)

2. To enter $.5x + 1$ in the Equation Writer and store in the PLOTTER:

[EQUATION] .5 X [+] 1	$.5X + 1$
[ENTER]	1: '.5*X+1'
[PLOT] [NEW] Y2 [ENTER]	
	Plot type: FUNCTION
	Y2: '.5*X+1'

48.7 THE CATALOG

1. To display a catalog of entered functions:

[PLOT] (or [SOLVE]) [CAT]

```
Y2:          '.5*X+1'
EQ:              'Y2'
Y1:          'X^ 2-2'
```

NOTES: [NEW] and [CAT] are both accessed through either [SOLVE] or [PLOT] [NEW] is used to put an equation or function in the catalog. [VAR] shows [Y2] and [Y1] as available through user defined soft keys. Press each in turn to obtain them on the stack.

2. To remove functions from the catalog:

[PLOT] (or [SOLVE]) [CAT] [NXT]

Move the cursor to indicate the function to be removed and press [PURGE] .

48.8 ALGEBRA

All of the following assume that [X] has been purged from the [VAR] menu.

1. Expanding expressions and collecting terms

To expand $(4x+1)^2$, enter the Equation Writer by pressing [EQUATION]

[( )] 4 X [+] 1 [▶] [y^x] 2 [ENTER]	1:	'(4*X+1) ^ 2'
[ALGEBRA] [EXPA] [EXPA] [COLCT]	1:	'1+16*X^ 2+8*X'

[▼] gets $1+16X^2+8X$ back into the Equation Writer.

2. Solve linear equations

Solve $3x - 1 = 0$:

[EQUATION] 3 X - 1 [ENTER] ['] X [ENTER]	2:	'3*X-1'
	1:	'X'
[ALGEBRA] [ISOL]	1:	'X=.333333333333'
[→Q]	1:	'X=1/3'

3. Solve Quadratic equations

Solve $x^2 - x - 6 = 0$.

'X^ 2-X-6' [ENTER] [ENTER] 'X [ENTER]	3:	'X^ 2-X-6'
	2:	'X^ 2-X-6'
	1:	'X'
[ALGEBRA] [QUAD]	1:	'X=(1+s1*5)/2'

NOTE: $s1$ is the calculator's symbol for ± 1.

[ENTER] 1 ['] [α] [⌐] S 1 [STO] [EVAL]	2:	'X^ 2-X-6'
	1:	'X = 3'
[SWAP] 1 [+/−] ['] [α] [⌐] S 1 [STO] [EVAL]	3:	'X^ 2-X-6'
	2:	'X = 3'
	1:	'X = -2'

∴ the solutions to $x^2 - x - 6 = 0$ are $x = 3$ and $x = -2$.

48.9 CALCULUS

(Reminder: Purge X from the [VAR] menu.)

1. Differentiation

In the Equation Writer, enter $\frac{d}{dx}(x^5 + 3x^2 - 15)$:

[∂] X [▶] X [y^x] 5 [▶] [+] 3 X [y^x] 2 [▶] [−] 15 [ENTER]

1: '∂X(X^ 5+3*X^ 2-15)'

Press [EVAL] repeatedly, observing the results. Continue until all ∂X's are gone. Then press [ALGEBRA] [COLCT] until the result no longer changes.

Duplicate the result by pressing [ENTER] . To evaluate the derivative at $x = 2$:

2 [ENTER] 'X [STO] [EVAL] 1: 92

To take the derivative of $\sin x^2$ without seeing the chain rule unfold step-by-step, first purge the value stored in X: 'X [PURGE]

'SIN (X ^ 2 [ENTER] 'X [ENTER] 2: 'SIN(X ^ 2)'

1: 'X'

[∂] 1: 'COS(X ^ 2)*(2*X)'

2. Integration

Evaluate the definite integral $\int_0^1 \frac{1}{1+t^2} dt$

[EQUATION] [∫] 0 [▶] 1 [▶] 1 [÷] 1 [+] T [y^x] 2 [▶] [▶] [▶] T [ENTER]

to obtain the display 1: ' ∫(0,1,1/(1+T^ 2),T)'

Press [ENTER] to duplicate this entry. Now press [EVAL] [EVAL] :

1: .785398163397

[ALGEBRA] [NXT] [— Qπ] 1: '1/4 * π'

Now let's evaluate the indefinite integral $\int_0^x \frac{1}{1+t^2} dt$

[SWAP] [EDIT] gets our definite integral onto an editing line. Move the cursor so that it is flashing over the upper limit 1. Press [DEL] (delete) and type X in its place. Finally, press [ENTER] . To check the integral in the Equation Writer, press [▼] . Press [ON] to return to the stack. Now press [EVAL] [EVAL] :

1: 'ATAN(X)'

If the calculator returns the original integral to you, it means that it was unable to find an antiderivative for the integrand. To numerically integrate a definite integral, press [— NUM] . The accuracy of the computation is governed by how many decimal places have been specified using [FIX] .

48.10 GRAPHING ENVIRONMENT

The Graphics tools on the HP-48 can be subdivided into two basic environments:

1) graphing (or plotting), and

2) drawing (or constructing).

To view the graphics screen at any time, press [GRAPH] . To return to the stack, press [ATTN] .

Plotting Parameters

To set up graphing with the default (usual) parameters, i.e., with the center of the screen at the origin, with coordinate axes shown, with each axis mark representing one unit, and each pixel valued at 0.1 unit, we need to make sure any different plotting parameters have been purged. To do this, press [PLOT] [PLOTR] [NXT] [RESET] . The default viewing window is $[-6.5, 6.5] \times [-3.1, 3.2]$.

[PLOT] Glossary

The [PLOT] menu is one of the more extensive (some of the menu keys have little folder tabs indicating that they bring up menus themselves). Here we briefly describe most of the keys found under the [PLOT] menu.

[PLOTR] brings up a plotting parameter screen, showing you the current equation, independent and dependent variables, and the x and y ranges of the viewing window. [PLOTR] also brings up its own menu which will be described below.

[PTYPE] displays the menu of graph types you can select.

[NEW] stores a newly typed in expression into the catalog with a name of your choosing. If you do not choose a name, then the expression is given the name EQ.

[EDEQ] puts the expression on the command line for editing. After editing is complete, [ENTER] replaces the old expression with the new one. To abort the editing process, press [ATTN] .

[STEQ] stores an expression under the name EQ.

[CAT] displays the catalog of stored expressions from which any may be selected for graphing.

Note that the last four keys on this menu are exactly the same as those under the [SOLVE] menu.

PLOTR submenu

The keys used most often under the PLOTR submenu are described here.

ERASE erases the existing graphics screen.

DRAW plots the graph within the parameters displayed.

AUTO computes the values of the expression for the x interval and sets the y range automatically to show a complete graph over the given domain.

XRNG sets a new range for x: press **4** +/− ENTER **5** ENTER XRNG to set the x-range to $[-4, 5]$

YRNG sets a new range for y.

INDEP selects a new independent variable from the expression to be graphed on the horizontal axis: ' **A** INDEP defines "A" as the independent variable.

DEPN selects a new dependent variable.

RESET resets the plotting parameters to their default settings.

Interactive zooming features

Once you've drawn a graph, you have an interactive menu available.

ZOOM brings up a zooming feature submenu which provides for selective zooming in or out by factors on one or both axes.

a. XAUTO selects x-axis zoom with automatic y-axis scaling:
Pressing XAUTO **2** doubles width of x interval, y interval autoscaled.

b. X selects x-axis zoom with no change in y.

c. Y selects y-axis zoom with no change in x.

d. XY selects zoom factor for *both* axes.

e. EXIT takes you back to the top level of the interactive menu.

Z-BOX sets diagonally opposite corners for a new viewing rectangle. Use cursor keys to fix the two corners by pressing Z-BOX . (The box is not normally shown. If a box is desired, press Z-BOX for one corner, NXT BOX for the other corner, then NXT NXT Z-BOX for "zooming in" .

CENT regraphs the function with the cursor's position as the new center of the Viewing Rectangle with the same interval radii for the x and y axes.

COORD displays the coordinates of the cursor in the lower left corner of the screen. Also available through + when in graphics environment.

48.11 FUNCTION GRAPHING EXAMPLES

1. Plotting functions

Enter the function $x^2 - 2$ named Y1 as detailed earlier, then press ATTN ⌐ PLOT CAT . Use ▼ to select then PLOTR to obtain

```
Plot  type:  FUNCTION
Y1:  ' X^ 2-2  '
Indep:   ' X '
x:              -6.5   6.5
y:              -3.1   3.2
```

If these are not the plotting parameters you see, press NXT RESET .

PREV ERASE DRAW displays the graph of $y = x^2 - 2$.

(NOTE: ERASE clears the graphics screen; otherwise, the new graph will be plotted over whatever is on the existing screen.)

The cursor keys now move a small crosshairs around the screen.

COORD displays the coordinates of the cursor in the lower left corner of the screen (the soft key designators also have disappeared). Pressing COORD again or - removes the coordinates.

The + toggles the menu labels on and off. Press the cursor keys several times while the coordinates are displayed. (The cursor moves much faster when the coordinates are not displayed.)

LABEL displays left-right and top-bottom endpoints on any axes that happen to be in the viewing rectangle. They cannot be removed without pressing ATTN ERASE DRAW .

Reset Y1 for graphing by pressing PLOT . If Y1 is the function indicated, press PLOTR ; if not, select Y1 through CAT , than press PLOTR , then press AUTO . The calculator will accept the values of x and rescale the y values for a complete graph of Y1 within the viewing rectangle. To find out what these y values are, press ATTN PLOT PLOTR . They can also be displayed on the graph by pressing LABEL − (this works only if the y-axis is in the viewing window).

To graph two functions simultaneously, enter them as the two sides of an equation F(X) = G(X).

Interactive function tools

The [FCN] contains several interactive tools for working with functions directly in the graphics environment.

[ROOT] snaps the cursor to the nearest root, displays its value, and records it on the stack with the label "Root:" .

[INTER] snaps the cursor to the nearest intersection point, displays its value, and records it on the stack with the label "Isect:" .

[SLOPE] calculates the derivative of the function at the cursor location, displays it, and records it on the stack with the label "Slope:" .

[AREA] first marks the lower limit of integration. The second time it is pressed, it computes the definite integral of the function from the first mark to the current cursor position, displays the value, and records it on the stack with the label "Area:" .

[EXTR] snaps the cursor to the nearest extremum, displays its coordinates, and records it on the stack with the label "Extr:" .

[EXIT] leaves the function folder.

Pressing [NXT] shows [F(X)] , which computes the function's value at the cursor location, displays it, and records it on the stack with the label "F(x):" .

[F'] computes the function's derivative symbolically, then graphs it, followed by the graph of the original function.

If more than one function is entered into EQ, then [NXEQ] allows you to cycle through the list.

Interactive drawing tools

[DOT +] leaves a trail for the cursor.

[DOT -] erases a trail for the cursor.

[LINE] draws a line from the last position point to the existing position and marks the existing position as the new reference point.

[TLINE] (for "toggle" line) draws a line from the existing position to the most previous marked reference point but does not change the reference point.

[BOX] draws a box using marked points as diagonal corners.

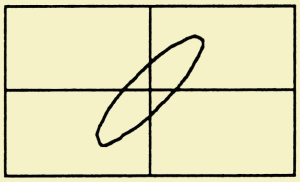
[CIRCL] draws a circle using the marked point as center and distance to current point as radius.

48.12 OTHER TYPES OF GRAPHS

1. [PLOT] [PTYPE] [CONIC]

Through the equation writer, enter $4x^2 - 3x \cdot y + y^2 - 4 = 0$ (Note: there must be a multiplication sign between x and y to distinguish it from a single variable with the name xy.) in the catalog with any appropriate name, e.g. EL.

Press [ERASE] [DRAW] to obtain



2. [PTYPE] [POLAR]

Through the equation writer, enter $R = \sin 2.5\theta$:

[α] R [◥] = [SIN] 2.5 [◤] F [ENTER] [NEW] P [ENTER]

```
Plot type: POLAR
P: ' R = SIN (2.5 * θ )'
Indep: ' X '
x:          -6.5    6.5
y:          -3.1    3.2
```

Change the Independent variable to θ:

['] [α] [◤] F [INDEP]

Set the plotting range for θ at approximately $0 \le \theta \le 4\pi$:

[◥] [{ }] [α] [◤] F [SPC] 0 [SPC] 12.6 [ENTER] [INDEP]

To plot the Polar graph, press [DRAW]

3. [PLOT] [PTYPE] [PARA]

First, let's reset the plotting parameters at their default values using [RESET].

To plot $x(t) = t^3$ and $y(t) = t^2$ for $1 \leq t \leq 2$, we first enter the pair of coordinate functions in the complex form $x(t) + iy(t)$:

['] [α] T [y^x] 3 [+] [α] [⌐] I [X] [α] T [y^x] 2 [STEQ]

Now, we enter the parameter T as the independent variable along with its starting and ending values:

{ T 1 2 [ENTER] [INDEP]

```
Plot type:  PARAMETRIC
EQ:  ' T^ 3 + i*T^ 2 '
Indep:  { T 1 2 }
x:             -6.5    6.5
y:             -3.1    3.2
```

Finally, erase the graphics screen and draw the curve: [ERASE] [DRAW]

4. [PLOT] [PTYPE] [TRUTH]

Through the equation writer enter $y < x + 2$ by pressing

[α] Y [α] [⌐] 2 [α] x [+] 2 [ENTER] [NEW].

Name it J.

Now enter $y^2 < 9 - x^2$ by pressing

Y [y^x] 2 [α] [⌐] 2 9 [−] X [y^x] 2 [ENTER] [NEW].

Name it K.

Finally, enter J *and* K in the equation writer by pressing

J [PRG] [TEST] [AND] K [ENTER] [NEW].

Name it JK.

Set the plotting parameters at their default values and press [DRAW].

This takes several minutes to plot completely.

Press [NXT] [NXT] [KEYS] to see the complete truth set.

TICAP HP 48 Workshop Examples

Based on inputs of the TICAP committee, developed by Dr. Tom Dick of Oregon State University*

The following conventions are used in this document:

- Keys are shown enclosed in straight brackets [ ]. Thus, [SIN] means press the sine key.
- Soft keys are shown in curly brackets { }. Thus, {SEC} means press the top row key under the legend "SEC" in the display.
- The orange colored left-shift key is referred to as [ORANGE]. Press [ORANGE] then press the key below the orange legend that you wish to access. For instance, to access PLOT which is printed in orange above the eight key, press the orange key then the eight key. This instruction would be shortened to [ORANGE] [PLOT] or simply to [PLOT] in this document.
- The blue colored right-shift key is referred to as [BLUE]. It works just like [ORANGE] except it provides access to the blue legends above the keys. In many examples this shifted key is omitted from the keystrokes but it still must be pressed to access the blue commands.
- To key in alphabetic characters, first press the [α] key then press the key to the left of and slightly above the alphabetic character. To key in X, you would press [α] then the sixth key in the fourth row. To lock alphabetic mode on press [α] twice. You can also hold the [α] key down while you are keying alphabetic characters.
- You may wish to press [BLUE] [CLR] (5th row, 6th column) to clear the stack from time to time. This will get rid of clutter and make the examples easier to follow.
- When the HP 48 beeps at you and you are stuck, press [ON] (1st key, bottom row). the [ON} key says ATTENTION! to the HP 48 and gets it out of special modes such as the graphics display mode. When in doubt, press [ON].
- When you see the "^" character it means the [y^x] key.

*Typographical, and possibly other errors, were contributed by Dennis York of Hewlett-Packard.

I. Functions & Limits

Example 1 Defining your own functions.

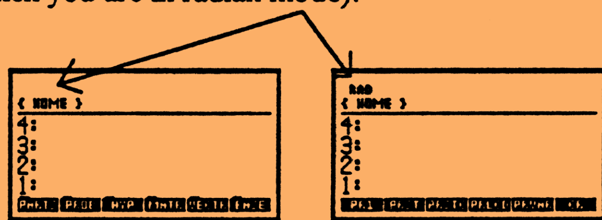
To define your own functions, use the [ORANGE] [DEF] key:

'SEC(X) = 1/COS(X)' [ORANGE] [DEF]
'CSC(X) = 1/SIN(X)' [ORANGE] [DEF]
'CTN(X) = 1/TAN(X)' [ORANGE] [DEF]

> ☛ Note: when typing, use [▷] to move outside parenthesis.

These three trigonometric functions now appear under the [VAR] menu, and you can evaluate or graph them like any built-in function.

Try them out. [ORANGE] [RAD] toggles between radian and degree mode (RAD shows at the top of the screen when you are in radian mode).



1) In degree mode:

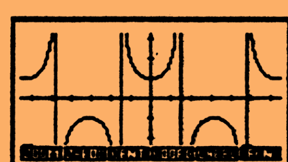
60 {SEC}   returns   1:   2

2) Now, toggle back to radian mode.
To graph y=sec(x), activate[ORANGE] [PLOT] menu
'SEC(X) {STEQ}
{PLOTR}

> ☛ Note: if the plot type is not FUNCTION, press [PLOT] {PTYPE} {FUNC} {PLOTR}

{ERASE}
{DRAW}

> ☛ Note: to set x and y ranges at default values, press [NXT] {RESET} [NXT] [NXT] before graphing.



3) To see how the calculator stores your function as a program, recall it by pressing [ON] [VAR] [BLUE] {SEC} and see:

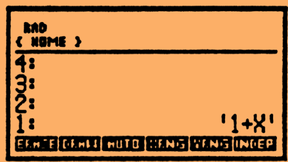
1:      << X '1/COS(X)' >>

I. Functions & Limits

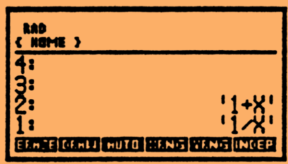
Example 2 Investigating limits numerically.

The SOLVER is a nice tool for investigating limits. It allows you to evaluate a function for a sequence of inputs very easily. Try this:

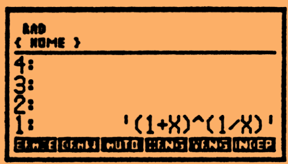
'1 + X' [ENTER]



'1 / X' [ENTER]



[y^x]



We've built the expression $(1+x)^{1/x}$ on the stack.

To investigate the function numerically, we send it to the SOLVER:
[SOLVE] {STEQ} {SOLVR}

To evaluate the expression for any value x, use the white {X} key to input the value and the dark {EXPR=} key to see the output result.

For instance,

1 {X} {EXPR=} returns 1: EXPR: 2

Now, investigate $\lim_{x \to 2} (1+x)^{1/x}$

by trying	x	=	2.1,	2.01,	2.001, ...
			1.9,	1.99,	1.999, ...

Then, investigate $\lim_{x \to 0} (1+x)^{1/x}$

by trying	x	=	0.1,	0.01,	0.001, ...
			-0.1,	-0.01,	-0.001 ...

[BLUE] [CLR] to clear the display.

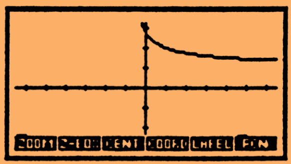
I. Functions & Limits

Example 3 Investigating limits graphically.

Now, let's examine $\lim_{x \to 0} (1 + x)^{1/x}$ graphically.

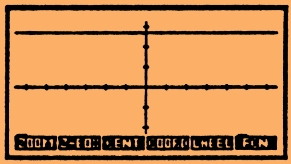
The {STEQ} supplies functions to both the SOLVER and the PLOTTER, so to graph $y = (1 + x)^{1/x}$, all we need do is:

[PLOT] {PLOTR} {ERASE} {DRAW}



To investigate the behavior near x=0, we zoom in horizontally. Let's try a factor of 10^6:

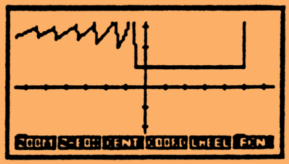
Press {ZOOM}{X} .000001 [ENTER].



This picture is graphical evidence (but not proof) that our function has a limit as $x \to 0$.

Example 3 Investigating limits graphically (continued)

Let's zoom in even further!
{ZOOM} {X} .000001 [ENTER]

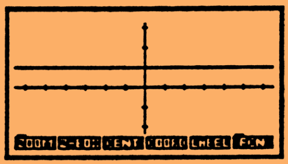


Wow! What is happening here?

We're seeing the spectacular effects of round-off error. Since the calculator carries 12 significant digits of precision, we will have run into difficulties in evaluating $(1+x)^{1/x}$ when x is near 10^{-12}. (Remember, we zoomed in by 10^6 twice.) Indeed, if we zoom in by 10^6 once more, $1+x$ will be rounded to exactly 1.

Let's see what happens.

{ZOOM} {X} .000001 [ENTER]



Press [ON] to leave plot.

The moral is that our graphing calculators are wonderful exploratory tools, but we will need to make our students aware of their limitations so that they use them intelligently.

II. Graphs & Continuity

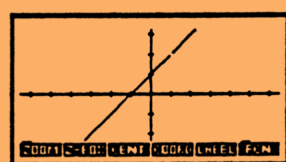
Example 1 Removable Discontinuities

When $\lim_{x \to a} f(x)$ exists, but is not equal to $f(a)$, we say $f(x)$ has a removable discontinuity at x=a (since we could "fix" the function by redefining $f(a)$) Graphically, a removable discontinuity may show up as a "hole" in the graph of the function. For instance, let's reset the plot parameters to their default settings ([PLOT] {PLOTR} [NXT] {RESET}) and then plot

$$y = \frac{x^2 - 1}{x - 1}$$

'X [y^x] 2 - 1 [ENTER]	1:	'X^2-1'
'X - 1 [ENTER]	2:	'X^2-1'
	1:	'X-1'
[÷]	1:	'(X^2-1) / (X-1)'

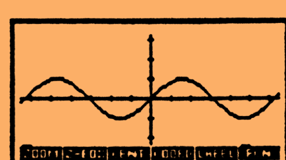
[PLOT] {STEQ} {PLOTR} {ERASE} {DRAW}



In this case we can see the "hole" at (1,2). However, if we graph $y = \cos(x)\tan(x)$:

[ON] 'COS(X) [ENTER]	1:	'COS(X)'
'TAN(X) [ENTER]	2:	'COS(X)'
	1:	'TAN(X)'
[X]	1:	'COS(X)*TAN(X)'

[PLOT] {STEQ} {PLOTR} {ERASE} {DRAW}



We see a sine curve with no holes, since none are located precisely at pixel locations. Press [ON] to leave plot.

II. Graphs & Continuity

Example 2 Non-removable Discontinuities

Two examples of non-removable or essential discontinuities are vertical asymptotes and jump discontinuities. How these appear on the graphics screen depend on two things:

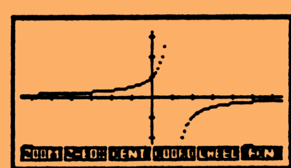
1) Whether the calculator is in connected mode and draws lines between successively plotted points.
2) Whether the discontinuity occurs precisely at a pixel location or not.

The toggle for connected modes is found under the [MODES] menu. Press [MODES] [NXT].

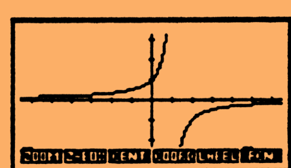
{CNC▪} means the calculator is in connected mode
{CNCT} means it's not

Turn connected mode off and try graphing $y = \frac{1}{1-x}$

[PLOT] '1/(1-X) {STEQ} {PLOTR} {ERASE} {DRAW}

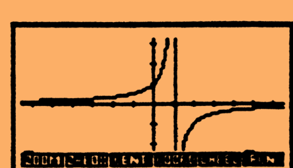


Now, turn connected mode back on and graph again.
[ON] [MODES] [NXT] {CNCT} [BLUE] [PLOT] {ERASE} {DRAW}



Let's edit the expression slightly.

Press [ON] [PLOT] {EDEQ} and use the cursor keys and the delete key [DEL] to change the expression to '1/(1.01-X)' Then press [ENTER] {PLOTR} {ERASE} {DRAW}. Since the asymptote does not fall exactly on a pixel, and the calculator is in connected mode, the vertical asymptote appears to be drawn in.

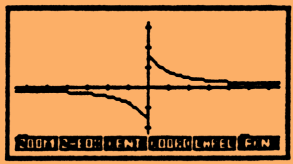


II. Graphs & Continuity

Example 2 Non-removable Discontinuities (continued)

To see an example of a function with a jump discontinuity:

[ON] [PLOT] 'ATAN (1/X) {STEQ} {PLOTR} {ERASE} {DRAW}.



Press [ON] to leave plot.

Here's how to create piecewise-defined functions. For example, to define

$$f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 & x > 1 \end{cases}$$

We make use of the IFTE (If Then Else) as follows:

'IFTE (X≤1,X+1,X^2)'

☛ Note: You can find ≤ under [PRG] {TEST}.

For functions built up from three or more pieces, we can nest the IFTE's:

'IFTE (X≤1,X+1,IFTE(X≤2,X^2,2-X))'

is the expression needed to define

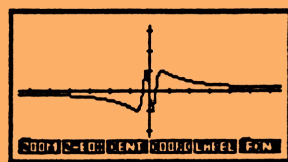
$$f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 & 1 < x \le 2 \\ 2-x & x > 2 \end{cases}$$

II. Graphs & Continuity

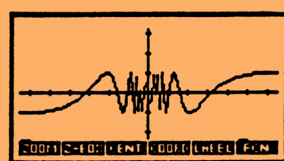
Example 3 Other Types of Function Behavior.

Not all discontinuities can be classified as holes, jumps, or asymptotes. Try these examples (be sure you're in radian mode):

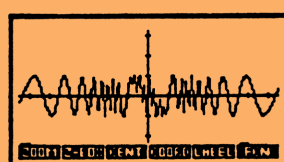
[PLOT] 'SIN(1/X) {STEQ} {PLOTR} {ERASE} {DRAW}.



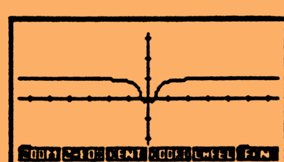
Now, zoom in horizontally to see some wild oscillatory behavior:
{ZOOM} {X} .1 [ENTER]



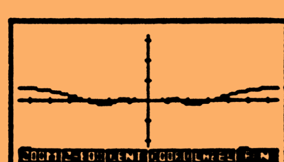
Repeated zooms do not tame this beast.



Reset your plot parameters to default settings
[ON] [NXT] {RESET}
and try [PLOT] 'X*SIN(1/X) {STEQ} {PLOTR} {ERASE} {DRAW}.



Zoom in horizontally to see how we've "damped" those oscillations.
{ZOOM} {X} .1 [ENTER]



Press [ON] to leave plot.

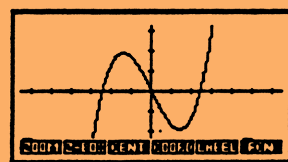
III. The Derivative

Example 1 Local Slope

We can appreciate the idea of a derivative measuring local slope by simply zooming in on the graph of a differentiable function.

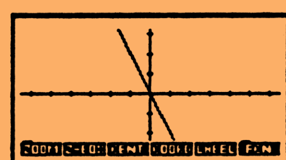
Consider $f(x) = \frac{x^3}{3} - 2x$ and suppose we want to approximate $f'(0)$ graphically.

[PLOT] 'X^3/3-2*X' {STEQ} {PLOTR} [NXT] {RESET} [NXT] [NXT] {ERASE} {DRAW}



Now, let's zoom in with equal scale factors both horizontally and vertically:

{ZOOM} {XY} .1 [ENTER]



Under magnification, the graph appears straight. This local linearity is a fundamentally important property of differentiable functions. The slope of the line, we say, appears to be -2 and that should be an approximation of $f'(0)$.

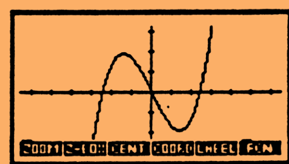
III. The Derivative

Example 2 Graphical Relationships

When we think of the process of evaluating the local slope at each point along the graph, we are describing the derivative function. The behavior of a function and its derivative are closely related, and we can examine this relationship graphically.

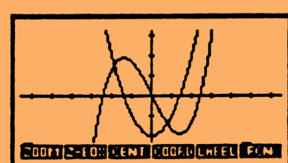
Let's zoom back out to see our graph at the original settings.

{ZOOM} {XY} 10 [ENTER]



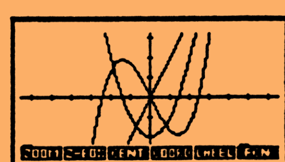
Now press {FCN} [NXT]

Pressing the {F'} key will graph the derivative of the function along with the original function.



Now we can see how the sign of f' is related to the increasing and decreasing behavior of f.

Press {FCN}[NXT]{F'} again, and we'll see the graph of the second derivative.



Now we can see how the sign of f' is related to the concavity of the graph of f.

Critical points and inflection points, the first and second derivative tests can all be discussed graphically now.

Press [ON] to leave plot.

III. The Derivative

Example 3 Chain Rule

The HP 48 knows the rules of differentiation, including the chain. This allows it to differentiate virtually any "closed form" function encountered in first-year calculus.

We'll illustrate two ways of differentiating on the HP. First, we'll purge any existing value of x with 'X' [PURGE]

1) To differentiate $\sin(x^2)$ with respect to x:

'SIN(X^2)' [ENTER]	2:	'SIN(X^2)'
'X' [ENTER]	1:	'X'
[∂]	1:	'COS(X^2)*(2*X)'

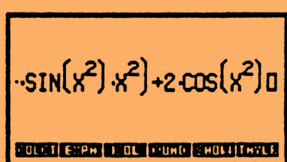
To take the second derivative, simply differentiate with respect to x again.

'X' [ENTER] [∂]	1:	'-(SIN(x^2)*(2*X)*(2*X)) +COS(X^2)*2'

To simplify a bit,

[ALGEBRA] {COLCT}	1:	'-(4*SIN(X^2)*X^2)+ 2*COS(X^2)'

To see the display in textbook format, press [∀] to "typeset" the expression in the equation writer (this takes a little while).



[ON] [ON] returns you to the stack.

2) To see the chain rule applied step-by-step, use this method
'∂X(SIN(X^2))' [ENTER]

Repeated presses of the [EVAL] key shows the differentiation step-by-step.

3) Try this. Press [RAD] to toggle to degree mode, and differentiate $\sin(x)$.

'SIN(X)' [ENTER}		
'X' [ENTER]		
[∂]	1:	'COS(X)*(π/180)'

IV. Applications of the Derivative

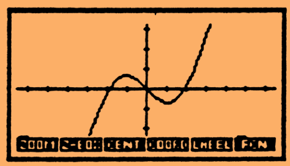
Example 1 Newton's Method

The solver is a nice tool for iteration. We can use it to illustrate Newton's method for root-finding. The iteration formula applied to a function f is

$$x - \frac{f(x)}{f'(x)}$$

Consider $f(x) = x - 2\sin(x)$ as an example. First, let's plot the function (reset the plot parameters to default settings if necessary and make sure that radians mode is set).

[PLOT] 'X-2*SIN(X) {STEQ} {PLOTR} {ERASE} {DRAW}



We see that $x=0$ is a root and there are two other roots evident near $x=\pm 2$.

Now, let's build the iteration formula.

[ON] 'X' [ENTER] [ENTER] [PURGE] to get rid of any X variable that might be substituted into our symbolic result and to put an extra copy of 'X' on the stack for later.

'X-2*SIN(X)' [ENTER] [ENTER]	3: 'X' 2: 'X-2*SIN(X)' 1: 'X-2*SIN(X)'
'X' [ENTER] [∂]	3: 'X' 2: 'X-2*SIN(X)' 1: '1-2*COS(X)'
[÷] [-]	1: 'X-(X-2*SIN(X))/ (1-2*COS(X))'

Move to the SOLVER
[SOLVE] {STEQ} {SOLVR}

Enter the initial seed with the white {X} key. Now by pressing {EXPR=} [ENTER] {X} repeatedly, we obtain the sequence of results. Try starting with initial seed of 0.91. Then try 0.92, 0.93, etc. to see the sensitivity of Newton's Method.

IV. Applications of the Derivative

Example 2 Maximizing the Volume of a Box.

Squares are cut from the four corners of a 10X15 rectangle and the "tabs" are folded up to create an open top box. Determine the maximum volume of such a box.

The objective function in this problem is

$$V(x) = x(10-2x)(15-2x)$$

Where x is the side length of one of the corner squares cut from the rectangle.

The calculator provides for a variety of strategies:

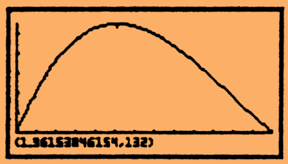
1) Plot $V(x)$ over the interval $0 \le x \le 5$
'X' [ENTER]
'10-2*X' [ENTER]
'15-2*X' [ENTER] [×] [×]

1: 'X*(10-2*X)*(15-2*X)'

[PLOT] {STEQ} {PLOTR} 0 [SPC] 5 {XRNG} {ERASE} {AUTO}

☞ Note: {AUTO} is an auto-scaling feature.

Use the cursor keys and {COORD} to approximate the maximum.

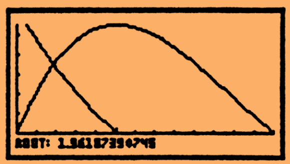


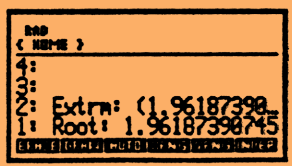
2) Press [+] {FCN} and then {EXTR}. This feeds the cross hair coordinates as a seed to a root finder for the first derivative.

3) We could also plot $V'(x)$

[NXT] {F'} and locate its root graphically using the cursor keys and {COORD}

4) Use the {ROOT} key in the {FCN} menu to find the root.





IV. Applications of the Derivative

Example 3 Moving Particle.

Problem: A particle moves along a straight line. It's position at time t is $s(t) = t^3 - 2t^2 - t$

1) Find the average velocity from t=1.99 to t=2.01. First, let's define the position function:

'S(T) = T^3-2*T^2-T' [ENTER] [DEF].

Now we can evaluate the average velocity as

$$\frac{S(2.01) - S(1.99)}{2.01 - 1.99}$$

2.01 [VAR] {S}	1:	-1.969599
1.99 {S}	2:	-1969599
	1:	-2.029601
[-]	1:	.060002
.02 [÷]	1:	3.0001

2) Find the instantaneous velocity at time *t*=2. First, purge any existing value of *t*.

'T' [PURGE]		
'S(T)' [ENTER]		
'T' [ENTER] [∂]	1:	'3*T^2-2*(2*T)-1'

If you like, you can simplify using [ALGEBRA] {COLCT}

Define the velocity function

'V(T)' [ENTER] [SWAP] [=] [DEF]

and evaluate it at t=2

2 [VAR] {V}	1:	3

Example 3 Moving Particle (continued)

3) Find the acceleration at t=2.

Differentiate V:

'V(T)' [ENTER] 'T' [ENTER] [∂]   1:   '3*(2*T)-4'

Define the acceleration function

'A(T)' [ENTER] [SWAP] [=] [DEF]

and evaluate it at t=2

2 [VAR] {A}   1:   8

4) What's the shape of the graph of *s(t)* at t=2? The answers to 2) and 3) suggest a positive slope and concave up shape.

To verify by graphing, we must change the independent variable to T

[PLOT] 'S(T)' {STEQ} {PLOTR}
[NXT] {RESET} [NXT] [NXT] 'T' {INDEP} {DRAW}

Press [ON] to exit plot.

V. The Integral

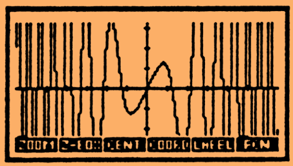
Example 1 Estimating Integrals Graphically.

A graphing calculator allows us to make a ball park estimate of a definite integral's value visually. For instance, given

$$\int_0^1 2x\cos(x^2)dx$$

We could first graph the function.

[PLOT] '2*X*COS(X^2)' {STEQ} {PLOTR} [NXT] {RESET} [NXT] [NXT] {DRAW}



Based on the graph, we might estimate the value of the definite integral to be slightly less than 1.

In fact,

$$\int_0^1 2x\cos(x^2)dx = \sin(x^2)\ \Big]_{x=0}^{x=1} = \sin(1) - \sin(0) \approx 0.8415$$

As an interesting extra credit exercise, use the {AREA} soft key under {FNC} to integrate the curve from 0 to 1. You may want to zoom in on the region of interest with {Z-BOX}. Use the {COORD} soft key to display the cursor coordinates as you position the cursor to x=0, press {AREA}and to x=1 and press {AREA}.

Press [ON] to exit plot.

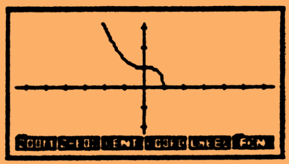
Example 2 Estimating Integrals - Using Built-in Integrator.

Now, let's work with a less standard integral and apply our estimation strategy again.

$$\int_0^1 \sqrt{1-x^3}\,dx$$

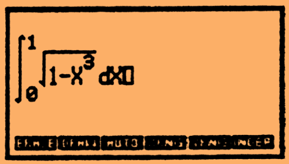
This cannot be evaluated with the usual paper-and-pencil techniques. However, we can still graph the function.

[PLOT] '$\sqrt{}$(1-X^3)' {STEQ} {PLOTR} {ERASE} {DRAW}



The region of interest looks similar to one-quarter of the unit circle, so we might guess that π/4≈0.7854 would be close. If we use the built-in HP integration routine, we can see the EQUATION WRITER in action.

[ON] [EQUATION] [∫] 0 [>] 1 [>] [√x] 1 - X [y^x] 3 [>] [>] [>] X



[ENTER] [⇨NUM] gives us 1: .841309263195

V. The Integral

Example 3 Numerical Integration Techniques

We calculated $\int_0^1 \sqrt{1 - x^3}\,dx = .841309263195$ with the HP's numerical integrator. Let's compare the result with those obtained by some standard numerical integration methods.

> ☛ Note: You'll need the integration programs RECL, RECC, RECR, TRAP and SIMP beamed to you by infrared transfer. With the exceptions of RECL and RECR, which are derivatives of RECC, they are listed below for reference.

RECC - Computes the integral of *F(X)*, from *A* to *B*, using *N* rectangular elements. On entry, *N* is in level 3, A is in level 2, and *B* is in level 1. Returns the integral based on evaluating each rectangle in the center. *F(X)* must be defined. To modify this program so that the elements are evaluated on the left, change the summation to H*Σ(I=0,N-1,F(A+I*H)). To modify the program so that elements are evaluated on the right, change the summation to H*Σ(I=1,N,F(A+I*H)). You could also include all three summations and generate all three results for comparison.

« → N A B	Take variables *N, A, B* from stack
« '(B-A)/N' EVAL → H	Compute *H*.
'H*Σ(I=0.5,N,F(A+I*H))' »	Compute integral, mid estimate.
»	
[ENTER] 'RECC' [ENTER] [STO]	Checksum: # 56540d

TRAP - Computes the integral of *F(X)*, from *A* to *B*, using *N* trapezoidal elements. On entry, *N* is in level 3, *A* is in level 2, and *B* is in level 1. *F(X)* must be defined.

« → N A B	Take variables from stack
« '(B-A)/N' EVAL → H	Compute element width, *H*
'H/2*(F(A)-F(B)+2*Σ(I=1,N,F(A+I*H)))' »	Compute trapezoidal sum
»	Save in 'TRAP'
[ENTER] 'TRAP' [ENTER] [STO]	Checksum: # 29955d

SIMP - Computes the integral of *F(X)*, from *A* to *B*, using Simpson's Rule. On entry, the even number of elements *N* is in level 3, *A* is in level 2, and *B* is in level 1. *F(X)* must be defined.

« → N A B	Take variable *A, B* from stack
« '(B-A)/N' EVAL → H	Compute element width, *H*
'H/6*(F(A)+F(B)+4*F(B-H/2)+	
2*Σ(I=1,N-1,F(A+I*H)+2*F(A+I*H-H/2)))' »	Compute Simpson's Summation
»	Save in 'SIMP'
[ENTER] 'SIMP' [ENTER] [STO]	Checksum: # 31730d

Using 20 subdivisions for each method, you'll find

(left-rectangle sum)

'F(X)=√(1-X^3)' [ENTER] [DEF]

20 [ENTER] 0 [ENTER] 1 {RECL}   1:   .86229589309

(right-rectangle sum)

20 [ENTER] 0 [ENTER] 1 {RECR}   1:   .812295893085

(mid-point rule)

20 [ENTER] 0 [ENTER] 1 {RECC}   1:   .842480395875

(trapezoidal rule: average of left and right rectangles)

20 [ENTER] 0 [ENTER] 1 {TRAP}   1:   .837295893085

(Simpson's rule: weighted average of midpoint and trapezoidal)

20 [ENTER] 0 [ENTER] 1 {SIMP}   1:   .840752228275

VI. Sequences & Series

Example 1 Investigating Sequences Numerically.

The SOLVER is also handy for exploring a sequence's behavior. For example, to investigate.

$$\left\{(1+\frac{1}{n})^n\right\}_{n=1}^{\infty}$$

We enter the formula for the Nth term into the SOLVER

[SOLVE] '1 + 1/N' [ENTER] 'N' [ENTER] [y^x] {STEQ} {SOLVR}

Now we can evaluate the Nth term for any integer value N.

10	{N}	{EXPR=}	1:	EXPR:2.593742460
100	{N}	{EXPR=}	1:	EXPR:2.704813829
10000	{N}	{EXPR=}	1:	EXPR:2.7181459268
10[EEX]20	{N}	{EXPR=}	1:	EXPR:1

What happened? Remember our discussion of round-off errors from before?

Example 2 Order of Approximation

For x close to 0, cos(x) is close to 1, but how close? The order of an approximation gives us an indication.

To understand the idea, let's calculate cos(x)-1 for a sequence of values x. This time we'll put the function in our catalog under the name F.

[SOLVE] 'COS(X)-1' {NEW} F [ENTER] {SOLVR}

.05 {X} {EXPR=}	1:	EXPR: -.001249739605
.1 {X} {EXPR=}	1:	EXPR: -.004995834722
.2 {X} {EXPR=}	1:	EXPR: -.019933422159
.4 {X} {EXPR=}	1:	EXPR: -.078939005997

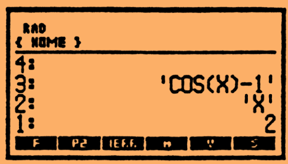
When x gets twice as close to 0, cos(x) gets about four (2^2) times as close to 1. We describe this by saying the approximation is of order 2.

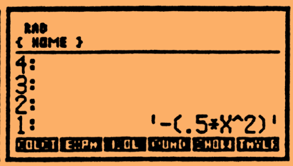
VI. Sequences & Series

Example 3 Taylor Polynomials

Now, let's graph cos(x)-1 and some of its Taylor polynomial approximations. For a second-degree Taylor polynomial approximation about x=0.

[VAR] {F} 'X' [ENTER] 2 [ENTER]
[ALGEBRA] {TAYLR}





[PLOT] {NEW} P2 [ENTER]

We've added the polynomial to our catalog under the name P2.

Now repeat for fourth-degree and sixth-degree Taylor polynomial approximations, adding them to the catalog under the names P4 and P6, respectively. Now let's graph the original function F and its three approximations.

We'll plot a list of functions as follows:

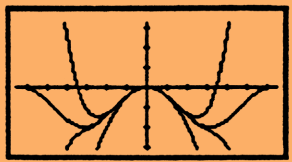
{PLOT} {CAT} gives us access to our catalog.

Use the cursors to point the arrow at F and press {EQ+}

Now	point at P2,	press {EQ+}
	point at P4,	press {EQ+}
	point at P6,	press {EQ+}

You should see the list { F P2 P4 P6 } at the top of screen.

Now press {PLOTR} {ERASE} {DRAW} and watch each of the four functions plotted, one after another. Press [-] to "take away" the soft keys for a better view.



DEPARTMENT OF MATHEMATICS

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HP-48SX CALCULATOR USE AGREEMENT

I do hereby agree that my borrowing of the HP-48SX pocket calculator #_____ from the Department of Mathematics is for my instructional use in MTH _____ during _____term, _____.

I expressly understand that I am responsible to the Department of Mathematics for the return of the calculator which I check out.

In the event the calculator is not returned in good working condition, I understand that I will be billed for its replacement and that future registration and transcripts may be withheld until this bill is paid.

______________________	______________________
Social Security Number	Signature
______________________	______________________
local phone number	Printed Name
______________________	______________________
______________________	______________________
______________________	______________________
local address	permanent home address

Calculator checked out on ______________________________________
date

- -

HP-48SX calculator #_____ received in good working order on this date

______________________ by ______________________
Date Signature of instructor

Documentation for programs

RSUM (Store function in EQ)

inputs:
- A — left endpt
- B — right "
- N — partition size
- R — real #: 0 = left, .5 = mid, 1 = right

output: value

TRAP (Store function in EQ)

inputs:
- A
- B — as above
- N

output: value

SIMP (Store function in EQ)

inputs:
- A
- B
- N (1/2 # of partition points)

output: value

SLPF (slopefield)

input: function of x and y

output: slopefield

Euler

inputs:
- x_0
- y_0
- Δx
- derivative $f'(x,y)$

output: plots solution to $\frac{dy}{dx} = f'(x,y)$ starting at (x_0, y_0)

2.1 EXPLORING LIMITS NUMERICALLY

When we write

$$\lim_{x \to a} f(x) = L$$

it means that the outputs $f(x)$ can be made arbitrarily close (or even equal) to a specific real number L simply by requiring that the inputs x are sufficiently close (but *not* equal) to the real number a. When this is true, we say that the limit exists and $f(x) \to L$ as $x \to a$.

By computing $f(x)$ for a sequence of inputs, $x = x_1, x_2, \ldots$, approaching a from the left ($x < a$), and for another sequence approaching a from the right ($x > a$), we can gather evidence (but not proof) whether or not

$$\lim_{x \to a} f(x)$$

exists and, if it exists, what the limiting value is. If both sequences seem to stabilize on the same value L, then this is supporting evidence that the limit exists and $\lim_{x \to a} f(x) = L$.

EXAMPLE 1 Numerically investigate $\lim_{x \to 0} f(x)$ (if it exists) if f is the function

$$f(x) = \frac{\sqrt{|x|}}{|\sqrt{x+1} - 1|}.$$

Solution We will explore the limit by computing $f(x)$ for x sampled closer and closer to 0 (see Table 2.1).

x $\xrightarrow{f}$	$f(x)$
.1	6.478902473
.01	20.04987607
.001	63.26136854
.0001	200.0040001
.00001	632.4555320
.000001	2000.000000

x $\xrightarrow{f}$	$f(x)$
-.1	6.162277666
-.01	19.94987434
-.001	63.22973312
-.0001	199.9948001
-.00001	632.4555320
-.000001	2000.000000

Table 2.1 $f(x) = \dfrac{\sqrt{|x|}}{|\sqrt{x+1} - 1|}$ sampled close to 0.

As the inputs x get closer and closer to 0 both from the left and from the right, the outputs $f(x)$ appear to get bigger and bigger without stabilizing. This is *evidence* that the limit does not exist. ■

EXAMPLE 2 Numerically investigate $\lim_{x\to 0} f(x)$ (if it exists) if f is the function

$$f(x) = \frac{|x|}{|\sqrt{x+1}-1|}.$$

Solution Again we can explore the limit by computing $f(x)$ for x sampled closer and closer to 0 (see Table 2.2) .

x	$\xrightarrow{f}$ $f(x)$	x	$\xrightarrow{f}$ $f(x)$
.1	2.048808848	-.1	1.948683298
.01	2.004987562	-.01	1.994987437
.001	2.000499875	-.001	1.999499875
.0001	2.000049999	-.0001	1.999949999
.00001	2.000005000	-.00001	1.999995000
.000001	2.000000500	-.000001	1.999999500

Table 2.2 $f(x) = \frac{|x|}{|\sqrt{x+1}-1|}$ sampled near 0.

From both directions, $x > 0$ and $x < 0$, the outputs $f(x)$ seem to be stabilizing toward 2. This is evidence that $\lim_{x\to 0} f(x) = 2$. ■

We examine a sequence of values in order to find evidence of a "trend." Simply taking a single sample very near the limit point does *not* give very good evidence whether the limit actually exists. The limiting value does not have to be infinite for the limit not to exist.

EXAMPLE 3 Numerically investigate $\lim_{x\to 0} f(x)$ (if it exists) if f is the function

$$f(x) = \cos(1/x).$$

Solution Again we can explore the limit by computing $f(x)$ for x sampled closer and closer to 0 (see Table 2.3) .

x	$\xrightarrow{f}$ $f(x)$	x	$\xrightarrow{f}$ $f(x)$
0.1	- 0.839071529076	-0.1	- 0.839071529076
0.01	0.862318872288	-0.01	0.862318872288
0.001	0.562379076291	-0.001	0.562379076291
0.0001	- 0.952155368259	-0.0001	- 0.952155368259
0.00001	- 0.999360807438	-0.00001	- 0.999360807438
0.000001	0.936752127533	-0.000001	0.936752127533
0.0000001	- 0.907270386182	-0.0000001	- 0.907270386182
0.00000001	- 0.363385089356	-0.00000001	- 0.363385089356

Table 2.3 $f(x) = \cos(1/x)$ sampled near 0.

From both directions, $x > 0$ and $x < 0$, the values seem to be not to be stabilizing at all even though they are confined to values between -1 and 1. This is evidence that the limit does not exist. ■

EXAMPLE 4 Numerically investigate $\lim_{x \to 3} f(x)$ (if it exists) if f is the function

$$f(x) = \frac{2x - 6}{\sqrt{x^2 - 6x + 9}}.$$

Solution Again we can explore the limit by computing $f(x)$ for x sampled closer and closer to 3 (see Table 2.4) .

x $\xrightarrow{f}$	$f(x)$	x $\xrightarrow{f}$	$f(x)$
3.1	2	2.9	-2
3.01	2	2.99	-2
3.001	2	2.999	-2
3.0001	2	2.9999	-2
3.00001	2	2.99999	-2
3.000001	2	2.999999	-2

Table 2.4 $f(x) = \dfrac{2x - 6}{\sqrt{x^2 - 6x + 9}}$ sampled near 3.

From the right ($x > 3$) the values seem to stabilize at 2. From the left ($x < 3$) the values seem to stabilize at -2. Although it appears that there is a limiting value both from the left and from the right, these one-side limits are not the same. This is evidence that the limit does not exist. We could summarize this paricular behavior with the notation

$$\lim_{x \to 3^-} f(x) = -2 \quad \text{and} \quad \lim_{x \to 3^+} f(x) = 2,$$

but we emphasize that $\lim_{x \to 3} f(x)$ does not exist. ■

Machine precision and limits

The number of digits a machine allocates to representing a number is called its precision. Whenever the machine completes one step of a computation the result is rounded to this number of digits. If a machine has precision 10 its arithmetic operations are referred to as 10-digit arithmetic. It should be noted that in this machine arithmetic the usual rules of arithmetic (associativity of addition, etc.) are not exact and instead can best be understood in terms of the repeated rounding of exact arithmetic operations.

Since a machine computation only carries a limited number of digits, the accuracy of the computed limit may actually decrease as you sample very close to the limit point. This is especially true of difference quotients.

EXAMPLE 5 Consider

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{where} \qquad f(x) = \frac{x}{54321},$$

that is to say

$$\lim_{x \to 3} \frac{\frac{x}{54321} - \frac{3}{54321}}{x - 3}.$$

A sequence of sample values computed with 10-digit arithmetic is shown in Table 2.5.

x	$f(x)$
3.1	0.00001840908673
3.01	0.00001840908673
3.001	0.0000184090867
3.0001	0.000018409087
3.00001	0.00001840909
3.000001	0.0000184091
3.00000001	0.00001841
3.000000001	0.0000184

Table 2.5 Sampled values of $\frac{f(x) - f(3)}{x - 3}$ for x near 3.

If we combine terms in the expression being computed we find that

$$\begin{aligned} \frac{f(x) - f(3)}{x - 3} &= \frac{\frac{x}{54321} - \frac{3}{54321}}{x - 3} \\ &= \frac{\frac{x-3}{54321}}{x - 3} \\ &= \frac{1}{54321} \qquad \text{for } x \neq 3 \\ &= 0.00001840908673 \qquad \text{to 10 digits.} \end{aligned}$$

In this case, then, the limit as computed by the sequence of values is less and less accurate as x approaches 3. ■

EXERCISES

Find numeric evidence that the indicated limits do or do not exist.

1. $\lim_{x \to 0} 2 + \dfrac{(x+1)^3 - (x+1)^2}{x}$

2. $\lim_{x \to 0} \dfrac{(x+1)^6 - (x+1)^4}{x}$

3. $\lim_{x \to 0} \dfrac{|x^2|}{|\sqrt{x^2+4} - 2|}$

4. $\lim_{x \to 0} \dfrac{|x^2|}{|\sqrt{x+x^2+4} - 2 - 0.25x|}$

5. $\lim_{x \to 0} x \ln(|x|)$

6. $\lim_{x \to 0} x \ln(|x|)^{10}$

7. $\lim_{x \to 0} x \cdot 2^{(1/|x|)}$

8. $\lim_{x \to 0} x^{10} \cdot 2^{(1/|x|)}$

For each of the following limits, find evidence which suggests either that the limit exists or doesn't exist.

9. $\lim_{x \to 0} \dfrac{\sqrt{1-x^4}}{1-x^2}$

10. $\lim_{x \to 0} \dfrac{\sin(x^2)}{x^2}$

11. $\lim_{x \to 0} \dfrac{1}{x} - \dfrac{1}{\sin(x)}$

12. $\lim_{x \to 1} \dfrac{|2x^2 + x - 1|}{3|x+1| - 0.5}$

13. $\lim_{x \to 1} (x-1)^2 \cdot \ln((x-1)^2)$

14. $\lim_{x \to 2} \dfrac{1}{x-2} + \dfrac{x-1}{2-x}$

15. $\lim_{x \to 4} \text{Sign}(x-4) + 15/25$

16. $\lim_{x \to -3} (x+3) \cdot \text{Sign}(x+3)$

17. $\lim_{x \to -2} \dfrac{x^3 - 5x + 8}{24x + 48}$

18. $\lim_{x \to -12} \dfrac{x^3 + 10x^2 + 16x + 8}{5x + 5}$

19. $\lim_{x \to -3} \dfrac{x^4 - 12x^2 + x - 2}{30x - 90}$

20. $\lim_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4}$

21. $\lim_{x \to -3} [\dfrac{|x+3|}{2x+6+1}]$

22. $\lim_{x \to 3} [\dfrac{2x^3 - x^2 - 12x - 9}{16x - 48 - 3/2}]$

23. $\lim_{x \to -1} \dfrac{x^2 - x + 1}{5x - 5}$

24. $\lim_{x \to 2} \dfrac{2x^2 - 5x + 2}{5x^2 - 7x - 6}$

2.2 EXPLORING LIMITS GRAPHICALLY

With machine graphics, we might be able to estimate the value of a limit by graphing the function f over an interval containing the "target" a, and examining the behavior of the graph near $x = a$. In fact, we can approach a in the sense of zooming in by rescaling the horizontal axis.

EXAMPLE 6 Estimate graphically

$$\lim_{x \to 0} \frac{\sin x}{x}.$$

Solution Figure 2.1 shows two machine-generated plots of the graph of $y = \dfrac{\sin x}{x}$.

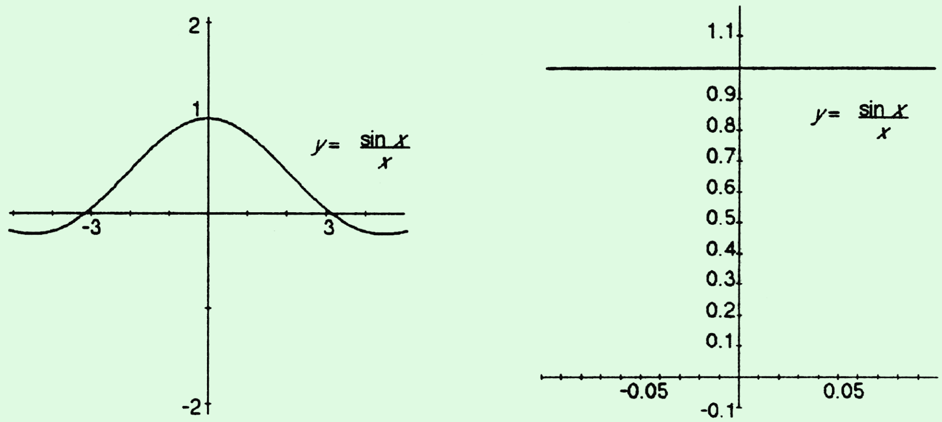


Figure 2.1 Graphs of $y = \dfrac{\sin x}{x}$.

The first appears to indicate an output value of 1 at $x = 0$ (though we know that 0 is not in the domain of the function). To investigate this further, in the second graph we show a close-up of the graph after we scaled the horizontal axis to plot between $x = -0.1$ and $x = 0.1$. We see that the graph of $y = \dfrac{\sin x}{x}$ resembles the horizontal line $y = 1$. This reinforces our first impression, so we could say that ***graphically,***

$$\lim_{x \to 0} \frac{\sin x}{x} \textit{ appears to be } 1.$$

■

Definition 1

The function f **has the limit** L **at** $x = a$, written

$$\lim_{x \to a} f(x) = L \qquad \text{or} \qquad f(x) \longrightarrow L \text{ as } x \longrightarrow a$$

if and only if the following condition holds: Given any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Let's examine this condition more closely. Think of the positive number ϵ (the Greek letter "epsilon") as a desired function *output error tolerance*. The statement $|f(x) - L| < \epsilon$ is just another way of saying that the function output $f(x)$ needs to be within ϵ of the number L. Now, think of the positive number δ (the Greek letter "delta") as the *input error tolerance* required to guarantee our desired output accuracy. The condition $0 < |x-a| < \delta$ means x is within δ of a but $x \neq a$. Hence, if

$$a - \delta < x < a \quad \text{or} \quad a < x < a + \delta,$$

then we must have

$$L - \epsilon < f(x) < L + \epsilon.$$

The formal limit definition says that given *any* positive output error tolerance, we can always find a corresponding positive input error tolerance that guarantees the desired output accuracy. We can think of the definition as providing a universal error tolerance test that L must pass in order to be called the limit of $f(x)$ as $x \longrightarrow a$.

Graphical interpretation of the formal limit definition

Graphically, the requirements of the formal definition of limit correspond to the graph of $y = f(x)$ being forced to lie between the horizontal lines $y = L - \epsilon$ and $y = L + \epsilon$ provided the inputs x are between the vertical lines $x = a - \delta$ and $x = a + \delta$. The only exception allowed would be at the actual value $x = a$ itself. We could also restate the formal definition of limit in machine graphical terms. Suppose we are given a value $\epsilon > 0$ as our output tolerance. First, we center the screen at $x = a$, and we scale the *vertical* axis so that the vertical range runs from $L - \epsilon$ to $L + \epsilon$.

Once the vertical axis has been scaled in this way, our challenge is to rescale the *horizontal axis* so that the graph of $y = f(x)$ enters from the left and leaves only from the right (with the possible exception of a hole or jump at a). We are not allowed to tamper with the vertical scaling at all; we must achieve the well behaved graph through horizontal scaling only. If we are successful, then the distance from a to the edge of the graphing window is playing the role of δ in the formal definition. To say that

$$f(x) \longrightarrow L \text{ as } x \longrightarrow a$$

we must be able to achieve this graph goal for any given $\epsilon > 0$.

EXAMPLE 7 In the second graph of Figure 2.1, the vertical axis was first scaled so that any output between 0.995 and 1.005 is machine plotted on the same horizontal line $y = 1$. We can think of this line as a very long, thin viewing window with a vertical range corresponding to $\epsilon = .005$. The fact that the graph appears identical to this horizontal line over the interval $(-0.1, 0.1)$ and in particular enters this long, thin viewing window from the left and leaves only from the right suggests that $\delta = 0.1$ is a sufficiently small input tolerance to guarantee that the outputs $\frac{\sin x}{x}$ are within $\epsilon = .005$ of 1. ■

Since a machine-generated graph is only a finite collection of dots, a "hole" in the graph may or may not appear. For example, the hole might occur *between* two adjacent plotted values and go undetected. In this example, we do not see the hole in the graph of $y = \frac{\sin x}{x}$ because the y-axis itself fills it in.

EXAMPLE 8 Graphically determine an estimate for $\lim_{x\to 0} f(x)$ if f is the function

$$f(x) = \frac{(x+1)^3 - (x+1)^2}{x}$$

Solution Graphing f with a reasonable viewing window of $[-1, 1] \times [-2, 2]$ gives the graph shown in Figure 2.2.

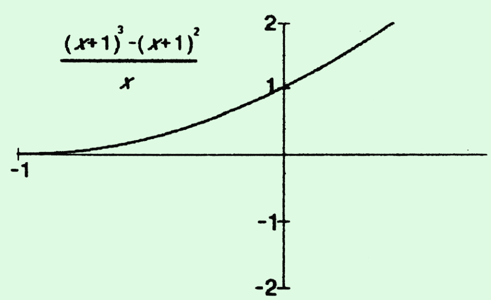


Figure 2.2 Graph of $f(x) = \frac{(x+1)^3 - (x+1)^2}{x}$ with window $[-1, 1] \times [-2, 2]$.

From this graph, it appears that the limit is $L = 1$. If our initial vertical range did not include the portion of the graph near 0, we could have expanded the vertical range until it was visible. ■

EXAMPLE 9 With the function as above, and $\epsilon = 0.1$, and assuming $L = 1$ (our candidate limit), graphically determine a δ so that

$$0 < |x| < \delta \implies |f(x) - 1| < \epsilon.$$

Solution We need to set the vertical range corresponding to our given ϵ, namely

$$[L - \epsilon, L + \epsilon] = [1 - 0.1, 1 + 0.1] = [0.9, 1.1].$$

Then we need to zoom in horizontally until the graph remains entirely within the window as it is plotted from left to right. If we set the viewing window to $[-.1, .1] \times [.9, 1.1]$, we obtain the graph shown in Figure 2.3.

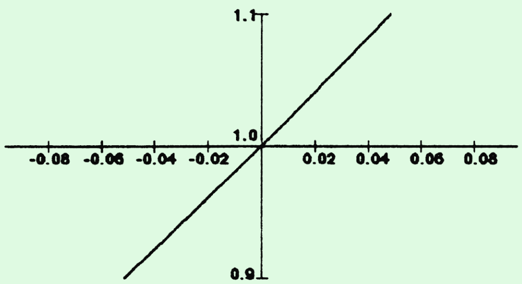


Figure 2.3 Graph of $f(x) = \dfrac{(x+1)^3 - (x+1)^2}{x}$ with window $[-0.1, 0.1] \times [0.9, 1.1]$.

In this graph, you can see that the function graph leaves the top and bottom of the window indicating that our chosen δ (in this case $\delta = 0.1$) was too large. We can see, however, that if we choose $\delta = 0.04$ the condition would be satisfied. Setting the horizontal range to $[-0.04, 0.04]$ and graphing again gives the graph shown in Figure 2.4.

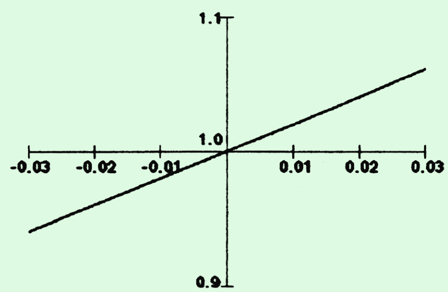


Figure 2.4 Graph of $f(x) = \dfrac{(x+1)^3 - (x+1)^2}{x}$ with window $[-0.03, 0.03] \times [0.9, 1.1]$.

This suggests that if we take $\delta = 0.04$ then

$$0 < |x| < \delta \implies |f(x) - 1| < \epsilon.$$

Notice that any smaller value δ works just as well. ■

EXERCISES

For each of the following limits, find evidence which suggests either that the limit exists or doesn't exist. If it seems to exists, find an estimate for δ *corresponding to* $\epsilon = 0.01$

1. $\lim_{x\to 0} \frac{\sqrt{1-x^4}}{1-x^2}$

2. $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$

3. $\lim_{x\to 0} \frac{\sin(x^2)}{x}$

4. $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin(x)}$

5. $\lim_{x\to 1} \frac{|2x^2+x-1|}{3|x+1|-0.5}$

6. $\lim_{x\to 1} (x-1)^2 \cdot \ln((x-1)^2)$

7. $\lim_{x\to 2} \frac{1}{x-2} + \frac{x-1}{2-x}$

8. $\lim_{x\to 4} \text{Sign}(x-4) + 15/25$

9. $\lim_{x\to -3} (x+3) \cdot \text{Sign}(x+3)$

10. $\lim_{x\to -2} \frac{x^3-5x+8}{24x+48}$

11. $\lim_{x\to -12} \frac{x^3+10x^2+16x+8}{5x+5}$

12. $\lim_{x\to -3} \frac{x^4-12x^2+x-2}{30x-90}$

13. $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$

14. $\lim_{x\to -3} [\frac{|x+3|}{2x+6+1}]$

15. $\lim_{x\to 3} [\frac{2x^3-x^2-12x-9}{16x-48-3/2}]$

16. $\lim_{x\to -1} \frac{x^2-x+1}{5x-5}$

17. $\lim_{x\to 2} \frac{2x^2-5x+2}{5x^2-7x-6}$

2.3 EXPLORING CONTINUITY GRAPHICALLY

We say a function is **continuous at a point** when the limit of a function matches the output at that point. Using the notation of limits, we have the following definition.

Definition 2

> The function f is **continuous at** $x = a$ if and only if
>
> $$\lim_{x \to a} f(x) = f(a).$$

More explicitly, there are three requirements for a function f to be continuous at $x = a$:

1) $\lim_{x \to a} f(x)$ **must exist.**
2) $f(a)$ **must be defined.**
3) **These values must match.**

Put simply, if the function f is continuous at $x = a$ then we can predict the *correct* output value $f(a)$ on the basis of the outputs in a neighborhood of a.

The formal definition of continuity is sometimes expressed using ϵ's and δ's directly instead of the limit requirement:

Definition 3

> The function f is **continuous at** $x = a$ if and only if the following condition holds: Given any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. If f is continuous at all real numbers, then we simply say that f is **continuous**.

Graphically speaking, given any vertical scaling on a viewing window centered at $(a, f(a))$, it must be possible to rescale the graph horizontally so that the function's graph stays in view from the left edge of the screen to the right.

A function is continuous on the open interval (a, b) if f is continuous at every value on (a, b). Graphically, a function is continuous on an interval if there are no "breaks" in the graph of the function.

While graphing is an aid to determining whether a function is continuous, you will still need to use your knowledge of domain and limits in order to determine it conclusively.

Low resolution of the screen can hide breaks in the graph which will not appear on the screen. A too large viewing window can also make jumps appear in the graph of a continuous function. If a function has a discontinuity at an irrational number (or any other number with no machine representation) then the discontinuity may not show up graphically. In this case, even if we observe discontinuous behavior, it cannot show up at the "right" place since the right place cannot be represented.

While we will point out examples of these problems, they are the exceptional cases. Generally, you can get a very good idea from a graph whether or not a function is continuous.

Types of discontinuities

Some of the more common kinds of discontinuities are the hole, the jump, the skip, the pole, and the oscillator. A hole discontinuity occurs where the function has a limit at some point but is not defined there. This is illustrated in Figure 2.5.

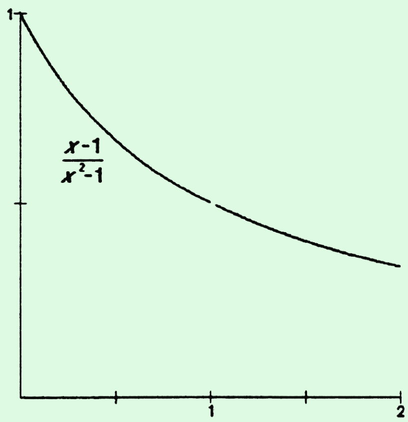


Figure 2.5 A function with a hole discontinuity at 1.

A jump discontinuity occurs when the left- and right-hand limits both exist, but are not equal to each other. This is illustrated in Figure 2.6.

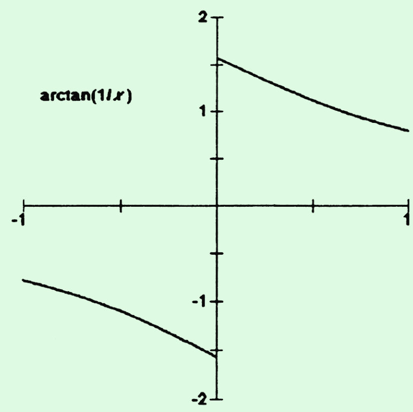


Figure 2.6 A function with a jump discontinuity at 0.

A skip discontinuity occurs when a limit exists but is not equal to the function value. This is illustrated in Figure 2.7.

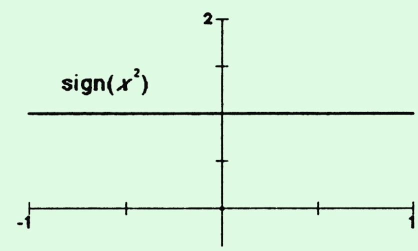


Figure 2.7 A function with a skip discontinuity at 0.

A pole discontinuity occurs when a limit is infinite. This is illustrated in Figure 2.8.

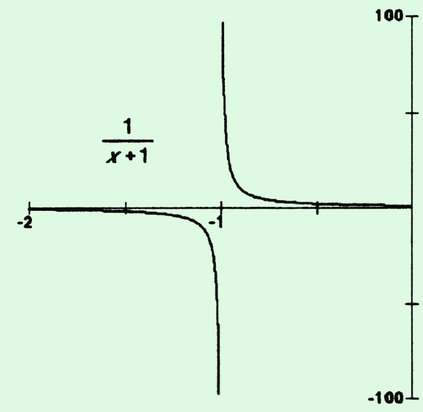


Figure 2.8 A function with a pole discontinuity at -1.

An oscillator discontinuity occurs when the function is bounded but the limit still does not exist. This is illustrated in Figure 2.9.

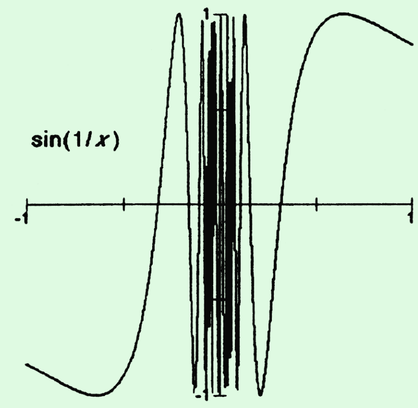


Figure 2.9 A function with an oscillator discontinuity at 0.

EXAMPLE 10 Investigate the continuity near 1 of the function

$$f(x) = \begin{cases} 1.7 & \text{for} \quad |x-1| > 10^{-15} \\ 1.7 & \text{for} \quad x = 1 \\ \textit{undefined} & \text{otherwise} \end{cases}$$

Solution Graphing this function with a viewing window of $[0,2] \times [0,2]$ results in the picture shown in Figure 2.10.

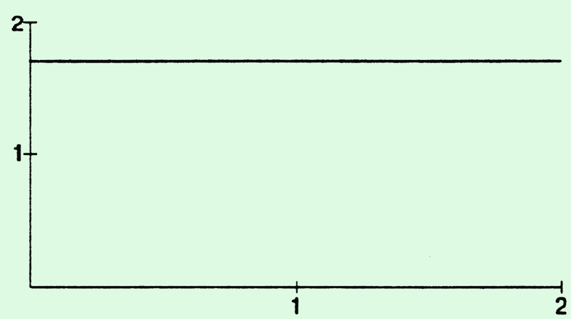


Figure 2.10 Graph of f with viewing window $[0,2] \times [0,2]$.

The graph seems perfectly flat near 1. If $1+5\cdot10^{-16}$ (1.0000000000000005) is not a number representable on the machine, the graph will continue to look flat no matter how much we zoom in. Even so, it is clear from the definition that there is a region near 1 in which the function is undefined. In particular, f is *not* continuous at 1. ■

EXAMPLE 11 Investigate the continuity of

$$f(x) = \sqrt{\frac{(|25000x| - 1)^2 - 1}{25000}}$$

near input 0.

Solution Selecting a viewing window of $[-1, 1] \times [-1, 1]$ gives the graph shown in Figure 2.11.

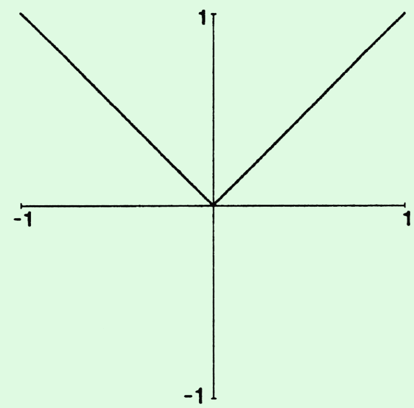


Figure 2.11 Graph of $f(x) = \sqrt{\frac{(|25000x| - 1)^2 - 1}{25000}}$ with a viewing window of $[-1, 1] \times [-1, 1]$.

If we zoom in by a factor of 1000, we get the graph in Figure 2.12.

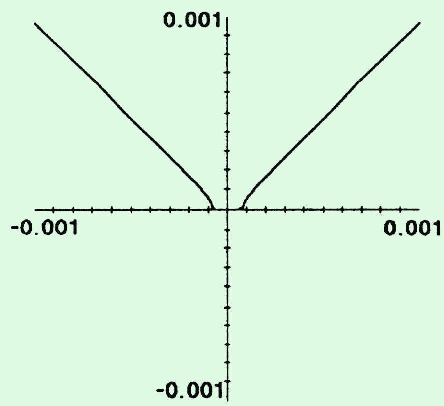


Figure 2.12 Zooming in on the graph of f.

We can verify the observation of a gap by noting that if x is small enough (but not 0), then

$$-1 < |25000x| - 1 < 1.$$

This means that $(|25000x| - 1)^2 - 1$ is negative and

$$\sqrt{(|25000x| - 1)^2 - 1}$$

is undefined. ■

EXAMPLE 12 Determine if

$$f(x) = \frac{0.5\cos(x)}{\cos(x)}$$

is continuous on $(-5, 5)$.

Solution First examine the graph on the interval $(-5, 5)$ as shown in Figure 2.13.

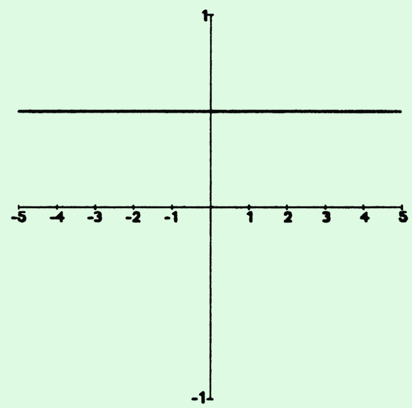


Figure 2.13 Graph of $f(x) = \dfrac{0.5\cos(x)}{\cos(x)}$ with a viewing window $[-5, 5] \times [-1, 1]$.

Note that the graph appears to be a flat line with a pixel lit in every column. Even zooming in vertically shows no additional features. However, this function is *not* continuous on this interval. Every point at which the denominator is zero will be undefined. The graph of the denominator $y = \cos(x)$ is shown in the same viewing window in Figure 2.14.

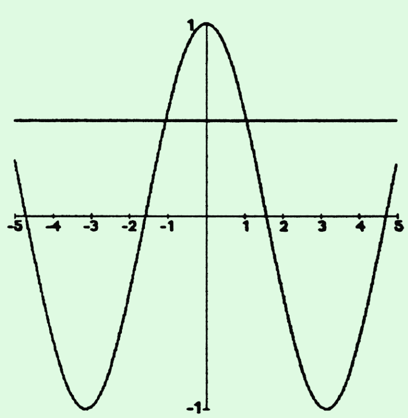


Figure 2.14 Graph of f and the denominator $y = \cos(x)$.

3.2 ESTIMATING DERIVATIVES GRAPHICALLY

The key property of a function that is linear at an input x_0 is that its graph is straight over some neighborhood of x_0. Most functions are not locally linear in this strict sense, but, remarkably, many functions are differentiable at most of their inputs. If we examine the graph of a function at such a point under sufficient magnification, it should look like a straight line.

Using machine graphics, if we zoom in far enough with equal scaling vertically and horizontally on the graph of a function having a derivative at that point, then the graph will appear straight. The slope of this straight line should be reasonably close to the derivative at that point.

> **To estimate the derivative value $f'(x_0)$ *graphically*, zoom in on the function's graph at the point $(x_0, f(x_0))$ until the graph appears straight. The slope of this line will be an approximation to the value $f'(x_0)$.**

EXAMPLE 6 Estimate the local slope of the graph $y = \sin(x)$ at the input $x = 0$.

Solution Graphing f and zooming in near input 0 we find that the graph looks like a straight line with slope 1 (Figure 3.1).

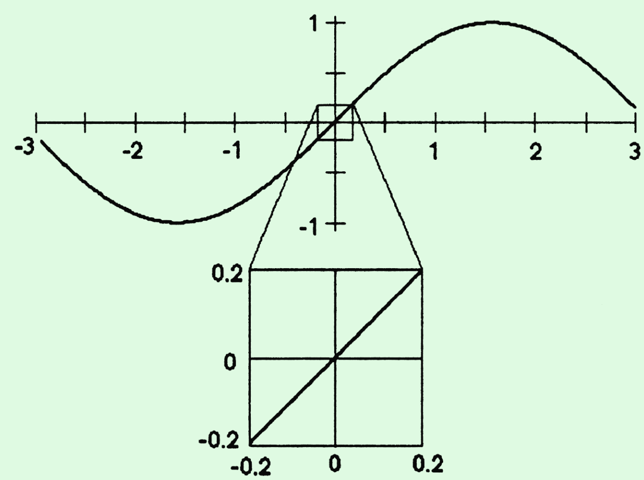


Figure 3.1 Zooming in on the graph of $f(x) = \sin(x)$ near 0.

Therefore, we would estimate the slope of the graph of $y = \sin(x)$ at $x = 0$ to be 1. ■

Realize that the numerical and graphical estimation methods are essentially equivalent—to compute the slope of the function's graph, we must calculate a difference quotient for two points. In a close-up window of the graph, the two points will necessarily be close together.

> **With the bounds on the precision of machine computation, keep in mind that two points chosen** *too close* **together could result in** *worse*, **not better accuracy in the calculation of the difference quotient.**

EXAMPLE 7 Find the local slope of the absolute value function $f(x) = |x|$ graphically at the inputs $x_0 = 2, -3$, and 0.

Solution Graphing $y = |x|$ we find that the graph looks like a straight line with slope 1 at $x = 2$. Similarly, at $x = -3$, we find that the graph looks like a straight line with slope -1. Therefore, the local slope of f at 2 is 1, and the local slope at -3 is -1 (see Figure 3.2).

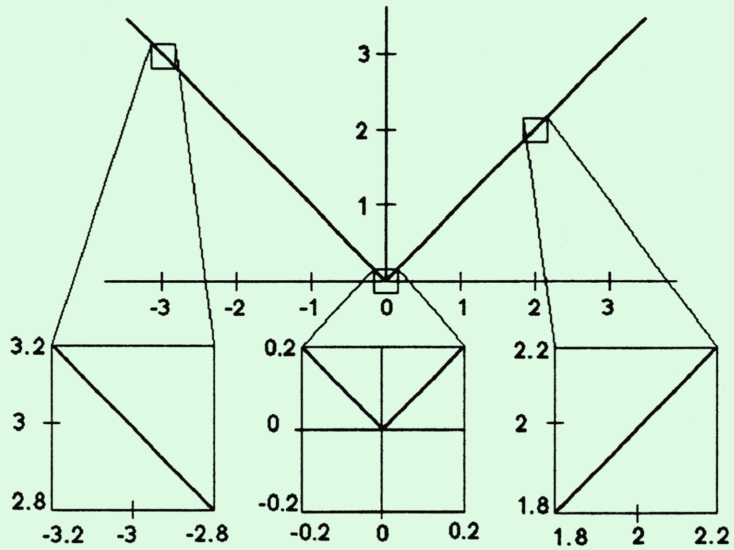


Figure 3.2 Zooming in on the graph of $f(x) = |x|$.

At $x = 0$, we find that the graph doesn't look like a straight line. Moreover, no matter how much we zoom in, there is always a sharp corner in the graph at the origin. This corresponds to the fact that the derivative $f'(0)$ is *undefined*. ■

EXAMPLE 8 Graphically estimate the derivative $f'(1)$ (if it exists) if f is the function

$$f(x) = \arcsin(\frac{1}{\sqrt{1+x^2}}).$$

Solution Graphing f in a region around 1 yields the graph shown in Figure 3.3. Even with this rather large-scale view you can see that near input 1 the graph has a negative slope (it slants downward right to left.)

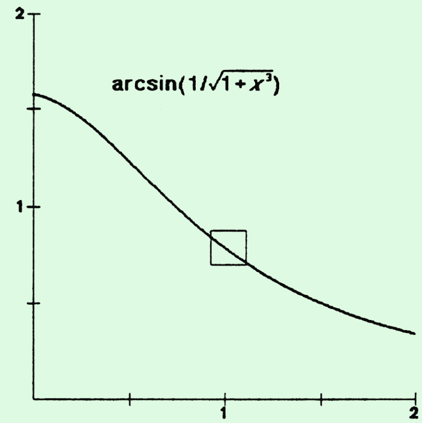


Figure 3.3 Graphing $f(x) = \arcsin(1/\sqrt{1+x^2})$ with a square viewing window.

Zooming in near input 1 yields the graph shown in Figure 3.4. The graph goes down about 7.3 units for 10 units left to right. The slope $f'(1)$ is therefore about $\frac{-7.3}{10} = -0.73$.

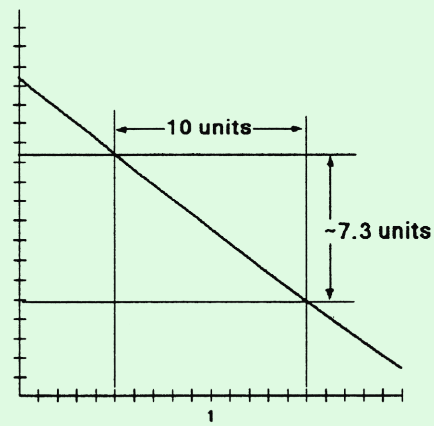


Figure 3.4 Zooming in on f near 1.

■

If the graph of f cannot be made to "straighten out" by zooming in near a, this is evidence that f is not differentiable at a. This can happen by the same mechanisms which can cause f to be discontinuous — jumps, skips, etc. There are also additional ways in which a function can fail to be differentiable.

EXAMPLE 9 Graphically estimate $f'(0)$ (if it exists) if f is

$$f(x) = |x|^{3/5}.$$

Solution As shown in Figure 3.5, zooming in on the graph of f near 0 results in the cusp, evident at normal scaling, that becomes sharper and sharper as we zoom in. Since this graph does not look like a straight line as we zoom in on the origin, this is evidence that f is not differentiable at $x = 0$.

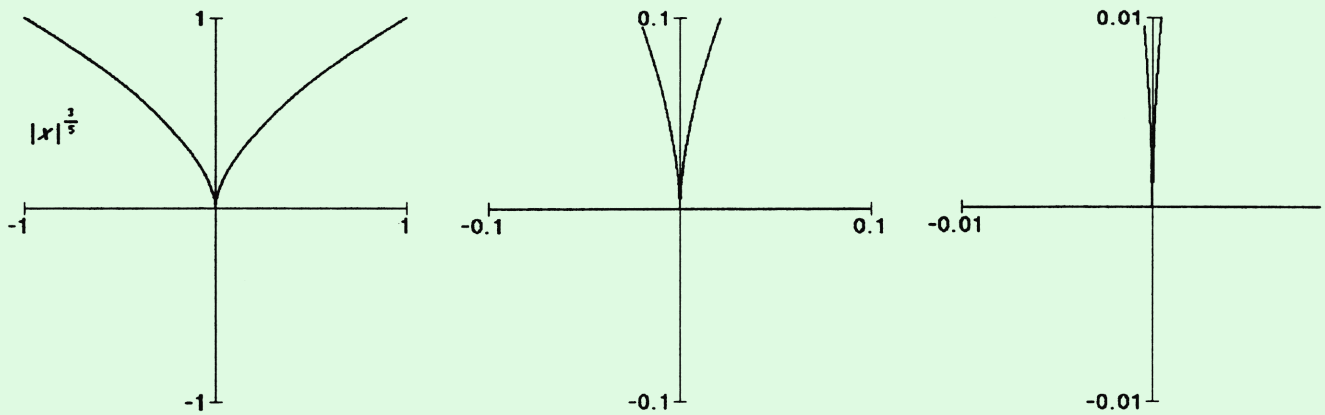


Figure 3.5 Zooming in on the graph of $f(x) = |x|^{3/5}$

■

EXAMPLE 10 Graphically estimate $f'(0)$ (if it exists) if f is

$$f(x) = x \sin(1/x).$$

Solution As shown in Figure 3.6, zooming in on the graph of f near 0 exhibits continued oscillatory behavior with no hint of settling down to a straight line. This is evidence that the function is not differentiable at 0.

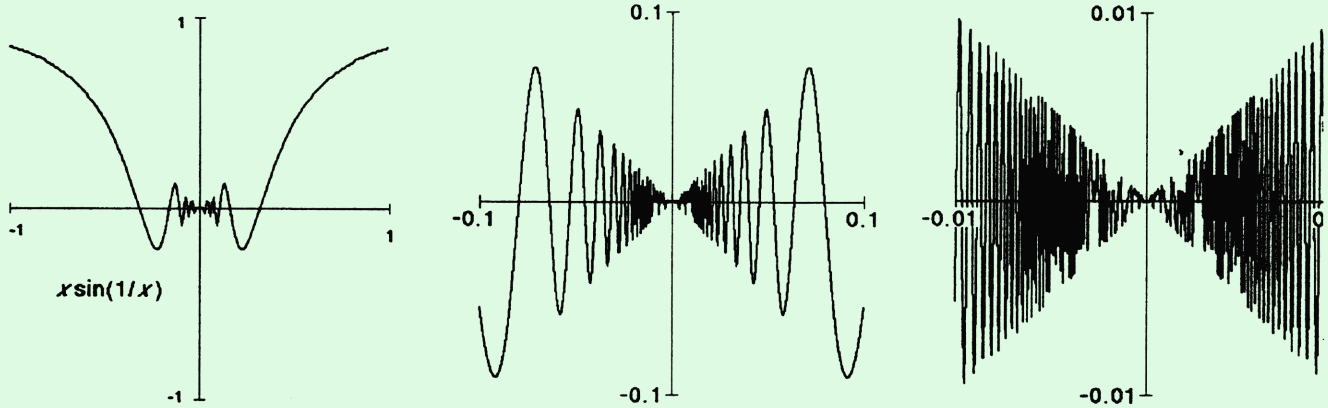


Figure 3.6 Zooming in on the graph of $f(x) = x\sin(1/x)$

■

EXERCISES

In exercises 1-10, zoom in on the graph of the indicated function at the point $(1, 1)$ *to estimate its derivative at* $x = 1$ *graphically.*

1. $x \longmapsto x^{2/3}$
2. $x \longmapsto x^{3/2}$
3. $x \longmapsto x^{4/3}$
4. $x \longmapsto x^{3/4}$
5. $x \longmapsto x^{-1/2}$
6. $x \longmapsto x^{-3}$
7. $x \longmapsto \arctan x$
8. $x \longmapsto \operatorname{arccot} x$
9. $x \longmapsto 2^x$
10. $x \longmapsto \log_{10} x$

In exercises 11-20, zoom in on the graph of the indicated function to estimate its derivative graphically at $x = \pi/4$ *and* $x = 2\pi/3$.

11. $x \longmapsto \sin x$
12. $x \longmapsto \cos x$
13. $x \longmapsto \tan x$
14. $x \longmapsto \sec x$
15. $x \longmapsto \csc x$
16. $x \longmapsto \cot x$
17. $x \longmapsto \sin x^2$
18. $x \longmapsto \sin^2 x$
19. $x \longmapsto \sin^2 x + \cos^2 x$
20. $x \longmapsto \sin 6x$

In exercises 21-40, graphically estimate the indicated derivative (if it exists.)

21. $f'(1)$ $f(x) = x^2 \exp(x)$

22. $f'(3.81)$ $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$

23. $f'(1)$ $f(x) = \sin(\sin(\sin(x)))$

24. $f'(1)$ $f(x) = ((x+1)^3 + 1)^3$

25. $f'(2)$ $f(x) = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}}$

26. $f'(3.5)$ $f(x) = \ln(x+1)$

27. $f'(-19)$ $f(x) = \dfrac{1}{x^2 + x + 1}$

28. $f'(1)$ $f(x) = \sqrt{x^2 - 2x + 1}$

29. $f'(0)$ $f(x) = \sin(|x|)$

30. $f'(0)$ $f(x) = \cos(|x|)$

31. $f'(1.5)$ $f(x) = \cos^2(|x|) + \sin^2(x)$

32. $f'(2)$ $f(x) = |x^3 - 8|$

33. $f'(2)$ $f(x) = |x^2 - 4|$

34. $f'(2)$ $f(x) = |x^2 - 4|^{\frac{1}{3}}$

35. $f'(0)$ $f(x) = \ln(1/\sqrt{x^2 + 0.00001})$

36. $f'(0)$ $f(x) = x\ln(1/\sqrt{x^2 + 0.00001})$

37. $f'(1)$ $f(x) = \ln(1/\sqrt{x^2 + 0.00001})$

38. $f'(1)$ $f(x) = \ln(x)$

39. $f'(1)$ $f(x) = \dfrac{\ln(x)}{1-x}$

40. $f'(0)$ $f(x) = \tan(|x|)$

5.2 NUMERICAL TECHNIQUES OF INTEGRATION

The Second Fundamental Theorem of Calculus allows us to evaluate a definite integral provided we can find an antiderivative for the integrand:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F' = f$.

If a machine such as a calculator or computer is capable of finding an antiderivative F for the function f, then the machine can simply use the Second Fundamental Theorem to evaluate the definite integral.

Sometimes it is difficult or even impossible to find a nice closed form formula for the antiderivative F. Of course, the First Fundamental Theorem guarantees that we can always find an antiderivative for a continuous function, but that puts us back where we started—trying to calculate the signed area under the graph of f.

In this section, we examine some of the numerical approximation techniques that can be used in evaluating definite integrals. The key idea to keep in mind is that a definite integral

$$\int_a^b f(x)\,dx$$

is *defined* to be the limiting value of Riemann sums

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x,$$

where the interval $[a, b]$ is partitioned into subintervals of length Δx, one input x is chosen from each subinterval, and we sum up the products $f(x)\Delta x$ for all the subintervals.

One way to approximate the value of a definite integral is to simply evaluate such a Riemann sum for a particular subinterval size Δx and a particular choice of inputs from those subintervals. This is indeed the strategy in the first three methods of numerical integration we discuss. Other approximation techniques can be thought of as improvements achieved by averaging these results.

Left endpoint, right endpoint and midpoint rules

The left endpoint, right endpoint, and midpoint rules for approximating definite integrals are named for the choice of input we make from each subinterval. The advantage of making these particular selections is that they are easy to "automate." That is, it is not difficult to program a machine to make these selections and then carry out the computation. Let's make the procedures for carrying out these techniques explicit.

Step 1. Choose a number n of subintervals in the partition of $[a, b]$.

Step 2. Calculate $\Delta x = \dfrac{b-a}{n}$.

Step 3. Locate the n inputs $x_1, x_2, ..., x_n$.

Step 4. Evaluate f at each input x_i and find the Riemann sum:

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x.$$

For the left endpoint rule, our inputs will be

$$x_1 = a, \quad x_2 = a + \Delta x, \quad x_3 = a + 2\Delta x, \quad \ldots, \quad x_n = a + (n-1)\Delta x.$$

For the right endpoint rule, our inputs will be

$$x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad x_3 = a + 3\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b.$$

For the midpoint rule, our inputs will be

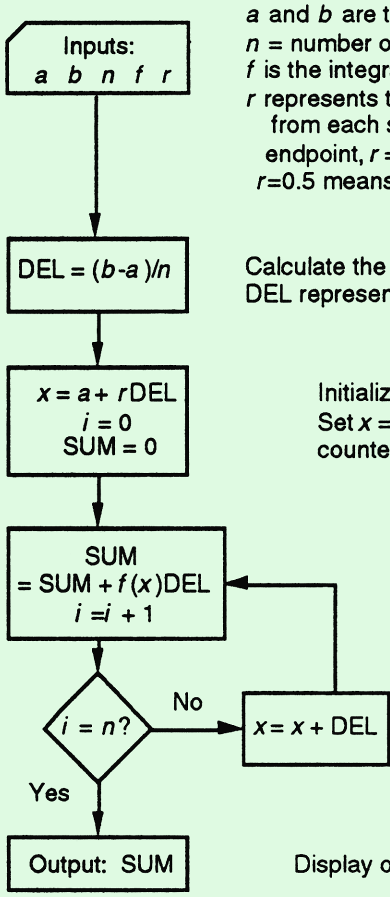
$$x_1 = a + \frac{\Delta x}{2}, \quad x_2 = a + \frac{3\Delta x}{2}, \quad x_3 = a + \frac{5\Delta x}{2}, \quad \ldots, \quad x_n = a + \frac{(2n-1)\Delta x}{2}.$$

Note that our inputs are equally spaced apart under all three rules, so that once we know the first input x_1, the rest are determined:

$$x_2 = x_1 + \Delta x, \quad x_3 = x_2 + \Delta x, \quad x_4 = x_3 + \Delta x, \quad \ldots, \quad x_n = x_{n-1} + \Delta x.$$

Using regular partitions makes it reasonably easy to program these techniques on a programmable calculator or computer. In fact, we could write a general program that would handle all three of these techniques. The syntax of the program will, of course, depend on the particular programming language of your calculator or computer system, but the structure can be described by the following flow chart.

PROGRAM RSUM (Rectangle SUM) for approximating $\int_a^b f(x)\,dx$



Comments

a and *b* are the endpoints of the interval;
n = number of subintervals in the partition;
f is the integrand;
r represents the choice of input from each subinterval (*r*=0 means left endpoint, *r* =1 means right endpoint, *r*=0.5 means midpoint).

Calculate the length of each subinterval: DEL represents Δx.

Initialize for program loop: Set x = first input and set counter *i* and SUM = 0.

This loop calculates the signed area of the rectangles corresponding to each subinterval and adds it to the running total represented by SUM

Display or print final total SUM

Example of usage: Approximate $\int_1^2 (1/x)\,dx$ using the midpoint rule and a partition of size $n = 5$.

Inputs: 1 2 5 1/x 0.5 Output: 0.6919

EXAMPLE 2 Approximate $\int_{.5}^{3.5} \sin^3(x)\,dx$ using each of the three rules for a partition of size $n = 6$.

Solution Here $a = 0.5, b = 3.5, n = 6$, and $f(x) = \sin^3(x)$. The subinterval size is

$$\Delta x = \frac{b-a}{n} = \frac{3.5-0.5}{6} = 0.5.$$

Using $r = 0$ (left endpoint rule) in the *RSUM* program, we obtain $SUM \approx 1.334$.

Using $r = 1$ (right endpoint rule) in the $RSUM$ program, we obtain $SUM \approx 1.257$.

Using $r = 0.5$ (midpoint rule) in the $RSUM$ program, we obtain $SUM \approx 1.325$.

For comparison, a machine computation yields $\int_{.5}^{3.5} \sin^3(x)\,dx \approx 1.315$, so in this case the midpoint rule gave the best approximation. ■

The rectangles whose (signed) areas are represented by the terms in each Riemann sum are illustrated in Figures 5.1 through 5.3. The right and left endpoint rules are sometimes called **right** and **left rectangle rules**.

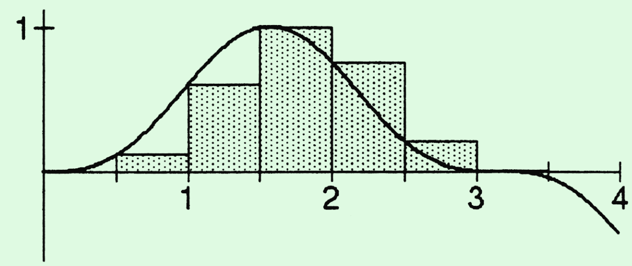


Figure 5.1 Left rectangle approximation of $\int_{.5}^{3.5} \sin^3(x)\,dx$ for a partition of size $n = 6$.

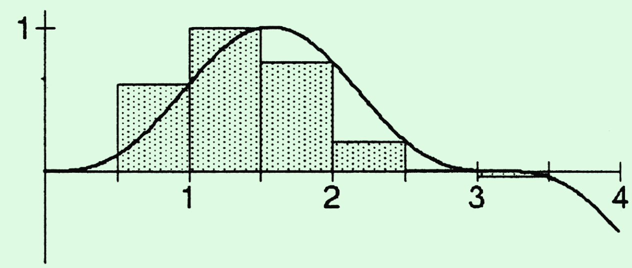


Figure 5.2 Right rectangle approximation of $\int_{.5}^{3.5} \sin^3(x)\,dx$ for a partition of size $n = 6$.

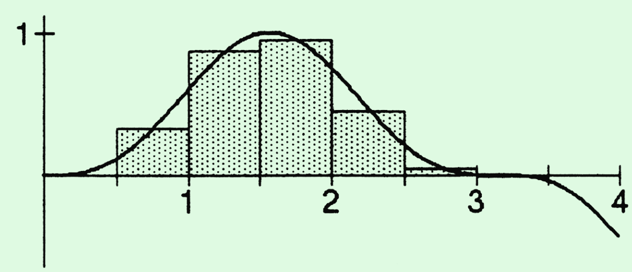


Figure 5.3 Midpoint approximation of $\int_{.5}^{3.5} \sin^3(x)\,dx$ for a partition of size $n = 6$.

Notice that if a function f is *increasing* over a given subinterval, then the area of the left rectangle is an underestimate of the area under the graph, while the area of the right rectangle is an overestimate. In this case, the area of the midpoint rectangle will be between this underestimate and overestimate (but not necessarily closer to the true area than each of the endpoint approximations). The situation is reversed when the function f is *decreasing* over the subinterval. If the function f is not monotonic over a subinterval, then the relationship between the true area and the areas of the left, right, and midpoint rectangles cannot be determined without closer examination.

As the partition becomes finer and finer (in other words n is chosen larger and larger) one expects that these Riemann sum approximations should get closer and closer to the actual value of the definite integral. In general, this must be true as long as the definite integral exists. However, for certain choices of n, it is possible that the particular sampling of inputs obtained by using the left, right, or midpoint rules may give a *terrible* approximation. Since all three of these methods sample at regular spaced intervals, a function that is periodic or has some graphical symmetry will be approximated badly for certain partition sizes n. Indeed, it is possible for a smaller value of n to produce a better result than a larger value in some cases.

Trapezoidal Rule and Simpson's Rule

The next two techniques we discuss approximate the value of a definite integral by using *averages* of Riemann sums. These techniques generally produce better results than the right, left, and midpoint techniques.

The **trapezoidal rule** for a partition of size n is simply the average of the results obtained by using the right and left endpoint rules for partitions of size n.

Once you have a program like $RSUM$, it is easy to calculate the trapezoidal approximation:

First, calculate $RSUM$ with $r = 0$. Call the result $LEFT$. Next, calculate $RSUM$ with $r = 1$. Call the result $RIGHT$. Finally, average the results: $(0.5)(LEFT + RIGHT)$. Call the result $TRAP$.

EXAMPLE 3 Approximate $\int_{0.5}^{3.5} \sin^3(x)\,dx$ using the trapezoidal rule for a partition of size $n = 6$.

Solution We have already determined the results of the right and left endpoint rules for this definite integral for a partition of size $n = 6$. Therefore the trapezoidal rule approximation is $TRAP = (0.5)(1.334 + 1.257) = 1.296$. ■

The idea behind the trapezoidal rule is that over intervals for which f is monotonic (either increasing or decreasing,) then one endpoint rule will give an underestimate and the other endpoint rule will give an overestimate. Hence, averaging the results from these two endpoint rules should give a better approximation to the value of the definite integral. The trapezoidal rule derives its name from the fact that the average of the areas of the right and left rectangles can be thought of as the area of a trapezoid with the same base (see Figure 5.4).

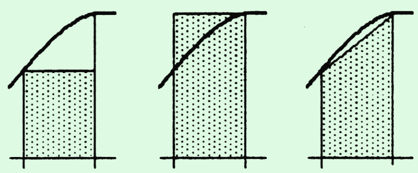


Figure 5.4 Trapezoidal rule is the average of the left and right endpoint rules.

The easiest way to remember the trapezoidal rule is as the average of the left and right rectangle rules. Figure 5.5 illustrates the trapezoidal approximation for the definite integral of the previous example.

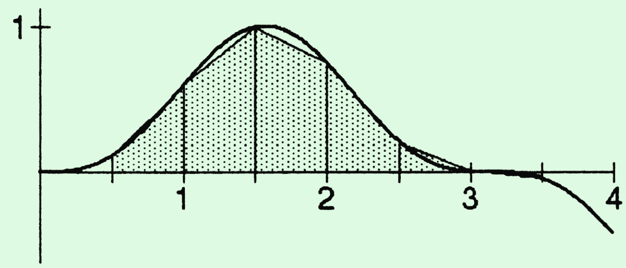


Figure 5.5 Trapezoidal approximation of $\int_{.5}^{3.5} \sin^3(x)\,dx$ for a partition of size $n = 6$.

An even better approximation to the value of a definite integral can be obtained by taking a *weighted* average of the midpoint and trapezoidal rules. The motivation behind this strategy is to take into account the concavity of the function's graph over each subinterval.

Figure 5.6 illustrates the midpoint and trapezoidal approximations over subintervals where the function graph is concave down and concave up.

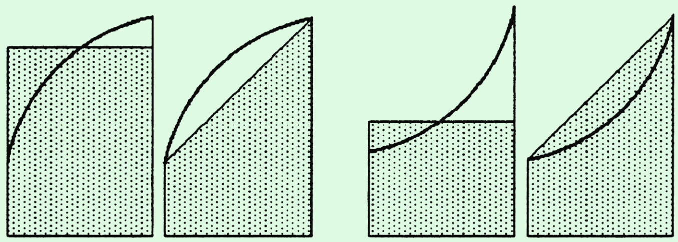


Figure 5.6 Comparing midpoint and trapezoidal approximations.

Look closely at Figure 5.6. Note that for the concave down graph (the two pictures on the left), the midpoint rule overestimates the area while the trapezoidal rule underestimates the area. Of the two estimates, the trapezoidal estimate appears to be approximately twice as far off as the midpoint estimate. For the concave up graph (the two pictures on the right), the midpoint rule now underestimates the area while the trapezoidal rule overestimates the area. Again, the trapezoidal estimate appears to be approximately twice as far off as the midpoint estimate.

This suggests that the value whose distance to the midpoint estimate is half as far as the distance to the trapezoidal estimate is a much better approximation of the actual area under the graph. This is precisely the motivation behind the approximation technique known as **Simpson's rule**, named after the English mathematician Thomas Simpson (1710-61).

The Simpson's rule approximation value can be computed by using a weighted average of the trapezoidal and midpoint estimates, namely

$$\text{Simpson's estimate} = \frac{1}{3}(\text{Trapezoidal estimate}) + \frac{2}{3}(\text{Midpoint estimate}).$$

Again, with the $RSUM$ program available, it is easy to program or calculate the Simpson's rule approximation:

First, calculate $TRAP$ as before.

Next, calculate $RSUM$ with $r = 0.5$. Call the result MID.

Finally, calculate the weighted average $\dfrac{2 \cdot MID + TRAP}{3}$. Call the result $SIMP$.

EXAMPLE 4 Approximate $\int_{0.5}^{3.5} \sin^3(x)\,dx$ using Simpson's Rule for a partition of size $n = 6$.

Solution We have already determined the results of the midpoint rule and trapezoidal rules for this definite integral for a partition of size $n = 6$. The Simpson's rule approximation is

$$\frac{1.296}{3} + \frac{2(1.325)}{3} \approx 1.315.$$

Note that this approximation is accurate to three decimal places of the actual value of the definite integral. ■

Here's another way of thinking about the trapezoidal rule and Simpson's Rule. When we subdivide the interval $[a, b]$ into n subintervals, we obtain the graph of a piecewise linear function by connecting the points $(x_i, f(x_i))$ corresponding to the endpoints of these subintervals. The trapezoidal rule is simply the approximation we obtain when we integrate this piecewise linear function instead of our original function f.

Now, instead of connecting the graph points at the ends of each subinterval with a straight line segment, suppose we find a *parabola* which passes through both of these points *and* the middle graph point. (Three noncollinear points determine a parabola just as two points determine a line; if the two endpoints and the midpoint line up, we can just connect them with a straight line instead of a parabola.) Figure 5.7 illustrates this idea for a given subinterval.

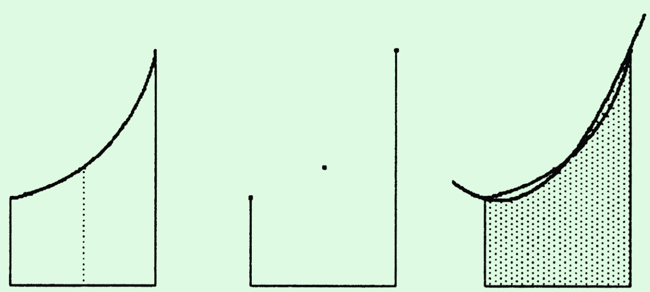


Figure 5.7 Approximating the graph with a parabola through three points.

Notice how closely the parabola approximates the graph. If we connect these parabolic pieces end-to-end for each subinterval, we obtain the graph of a **piece-wise quadratic function**. Simpson's Rule is simply the approximation we obtain when we integrate this piece-wise quadratic function instead of our original function. Figure 5.8 shows the piece-wise quadratic fit for the example we computed earlier. The graph appears virtually identical to the original graph, and consequently, the values of the definite integrals agreed to three decimal places.

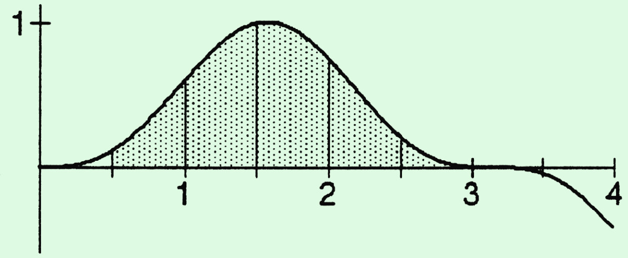


Figure 5.8 Simpson's approximation of $\int_{.5}^{3.5} \sin^3(x)\,dx$ for a partition of size $n = 6$.

On the other hand, Simpson's rule requires us to sample almost twice as many inputs as the trapezoidal rule for the same partition. Indeed, we essentially have a partition of size 12 for this example, since we sample the midpoints of each subinterval as well as the endpoints. The approximation could be improved even more by sampling more inputs and using a piece-wise cubic function, or by using the first derivative values at the endpoints to create a cubic spline approximation.

Summary of the numerical integration techniques

Let's summarize by introducing some convenient notation. Suppose we partition a closed interval $[a, b]$ into n equal-sized subintervals of length $\Delta x = (b-a)/n$. Let's label the function outputs at the n midpoints with odd subscripts:

$$y_1 = f(a + \frac{\Delta x}{2}), \quad y_3 = f(a + \frac{3\Delta x}{2}), \quad y_5 = f(a + \frac{5\Delta x}{2}), \quad \ldots$$

up to

$$y_{2n-1} = f(a + \frac{(2n-1)\Delta x}{2}) = f(b - \frac{\Delta x}{2}).$$

Now let's label the function outputs at the endpoints of the subintervals with even subscripts, starting with 0:

$$y_0 = f(a), \quad y_2 = f(a + \Delta x), \quad y_4 = f(a + 2\Delta x), \quad \ldots$$

up to

$$y_{2n} = f(a + n\Delta x) = f(b).$$

With this labeling scheme, we can write down an explicit formula for each of the rules discussed in this section.

Numerical approximations of $\int_a^b f(x)\,dx$ for a partition of size n

Left rectangle rule:

$$\int_a^b f(x)\,dx \approx L_n = \Delta x(y_0 + y_2 + \cdots + y_{2n-2}).$$

Right rectangle rule: $\int_a^b f(x)\,dx \approx R_n$, where

$$\int_a^b f(x)\,dx \approx R_n = \Delta x(y_2 + y_4 + \cdots + y_{2n}).$$

Midpoint rule:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x(y_1 + y_3 + \cdots + y_{2n-1}).$$

Trapezoidal rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}(y_0 + 2y_2 + 2y_4 + \cdots + 2y_{2n-2} + y_{2n}).$$

Simpson's rule:

$$\int_a^b f(x)\,dx \approx S_{2n} = \frac{2M_n + T_n}{3}.$$

The subscript $2n$ in the notation S_{2n} for Simpson's rule is to indicate that we have effectively subdivided our interval $[a, b]$ into $2n$ subintervals, since we make use of both the endpoints and midpoints of each of our original n subintervals. The explicit formula for Simpson's rule is

$$S_{2n} = \frac{\Delta x}{6}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}).$$

EXERCISES

Using partitions of size $n = 2, 4, 8, 16,$ *and* 32 *find the left, right, and midpoint estimates for the definite integrals in exercises 1-6 by using a RSUM program for your calculator or computer.*

1. $\int_0^1 \sqrt{1+x^2}\,dx$

2. $\int_{-1}^0 3^x\,dx$

3. $\int_{-1}^1 \arctan(x)\,dx$

4. $\int_1^4 \log_2(x)\,dx$

5. $\int_0^1 e^{-x^2}\,dx$

6. $\int_{0.5}^{3.5} \sin(x^3)\,dx$

Using partitions of size $n = 2, 4, 8, 16, 32$ *and the results from exercises 1-6, calculate TRAP and SIMP for exercises 7-12. Calculate each integral with a machine to five decimal place accuracy and compare these values with the five estimates you have obtained.*

7. $\int_0^1 \sqrt{1+x^2}\,dx$

8. $\int_{-1}^0 3^x\,dx$

9. $\int_{-1}^1 \arctan(x)\,dx$

10. $\int_1^4 \log_2(x)\,dx$

11. $\int_0^1 e^{-x^2}\,dx$

12. $\int_{0.5}^{3.5} \sin(x^3)\,dx$

The table of values below are the minute-by-minute speed readings of a car over the second 15 *minutes of its trip. Use this table to answer exercises 13-14.*

time t in minutes	speed V in mph
15	28
16	24
17	20
18	16
19	12
20	9
21	6
22	3

time t in minutes	speed V in mph
23	0
24	1
25	1
26	2
27	3
28	4
29	6
30	8

13. Estimate the distance covered over the interval $[15, 30]$ using $\Delta t = 1$ minutes and the trapezoidal rule.

14. Estimate the distance covered over the interval $[20, 30]$ using $\Delta t = 2$ minutes and Simpson's Rule.

---

15. Using the table of values below for the continuous function f, estimate the definite integral of f over the entire interval using the trapezoidal rule.

x	$f(x)$
1.27319	3.81456
1.27320	3.86714
1.27321	3.90551
1.27322	3.92017
1.27323	4.34405
1.27324	4.66292
1.27325	4.69141
1.27326	4.65674
1.27327	4.61993
1.27328	4.59550
1.27329	4.58799
1.27330	4.52556

16. Use Simpson's rule to find 5 terms of a sequence approximating

$$\int_0^1 \frac{4}{1+x^2}\,dx.$$

Let the number of subdivisions of $[0, 1]$ be $1, 2, 3, 4, 5$. What does the limit of this sequence appear to be? How many decimal places of accuracy does a_5 give? Using the trapezoidal approximation and 50 subdivisions of $[0, 1]$, approximate the same integral. How many decimal places of accuracy do you get?

Exercises 17-20 ask you to estimate

$$\int_0^4 x^2(x-1)(x-2)(x-3)(x-4)\,dx$$

using various approximation techniques.

17. Find the trapezoidal estimates for partitions of size $n = 16$ and $n = 32$.

18. Find the Simpson's rule estimates for partitions of size $n = 4$ and $n = 8$.

19. Expand the polynomial $x^2(x-1)(x-2)(x-3)(x-4)$ and integrate over $[0, 4]$ using the Second Fundamental Theorem of Calculus.

20. Which estimate is better, the trapezoidal estimate for $n = 32$ or Simpson's rule for $n = 8$?

9.2 COMPARING TAYLOR POLYNOMIALS

The nth degree Taylor polynomial approximation p_n to the function f at the point $x = a$ is defined to be

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In this activity, you'll see graphically the convergence of Taylor polynomial approximations. In general the higher the degree n for a Taylor polynomial, the better the approximation. Certainly that is true at the point $x = a$. However, at other points, the story may be different.

Some symbolic algebra systems automatically expand and collect terms in powers of x in polynomial manipulation. This is less than desirable if you wish to have your Taylor polynomial expansion in terms of powers of $(x - a)$ instead.

Here's a way to "fool" your system if it behaves this way. If you desire a Taylor polynomial expansion of the function f about $x = a$, let your system find the Taylor polynomial expansion of $f(t + a)$ about $t = 0$. After the system has finished simplifying, you can just replace every appearance of t by $(x - a)$.

EXERCISES

In exercises 1-10 you are given a function $f(x)$. Find and plot the first five Taylor polynomial approximations to each function ($n = 1, 2, 3, 4$, and 5), as well as the graph of the original function at the given point $x = a$. You are also given the degrees of two of the Taylor polynomials. For these, find a point where the Taylor polynomial of lower degree gives a better approximation than the Taylor polynomial of higher degree, and a point where the Taylor polynomial of higher degree gives a better approximation than the Taylor polynomial of lower degree.

1. $f(x) = \sin(x-3)\exp(x-3)$, degrees 1 and 2, near $x = 3$.

2. $f(x) = \ln(x)$, degrees 3 and 4, near $x = 1$.

3. $f(x) = \dfrac{x+1}{x+9}$, degrees 3 and 4, near $x = 0$.

4. $f(x) = \sqrt{x}$, degrees 2 and 3, near $x = 1$.

5. $f(x) = \dfrac{x^3 + x^2 + x + 1}{x+5}$, degrees 2 and 3, near $x = 1$.

6. $f(x) = \arctan(x)$, degrees 1 and 3, near $x = 0$.

7. $f(x) = x^2 + \sin(x)$, degrees 2 and 3. near $x = 0$.

8. $f(x) = \sqrt{x^2+1}$, degrees 1 and 2, near $x = 1$.

9. $f(x) = \dfrac{1}{x}$, degrees 2 and 3, near $x = 5$.

10. $f(x) = (x^2+1)\exp(x)$, degrees 2 and 3, near $x = 0$.

---

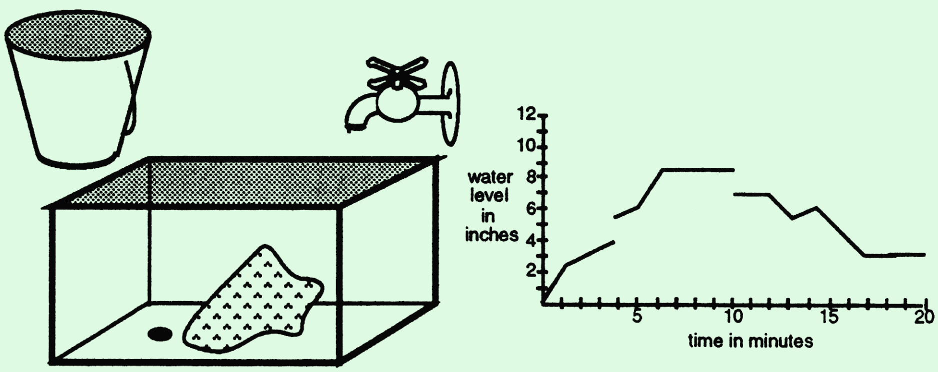
11. Find a function whose degree 2 Taylor polynomial at 0 is everywhere (but at $x = 0$) a better approximation to the function than the degree 1 Taylor polynomial.

12. Find a function whose degree 1 Taylor polynomial at 0 is a better approximation for all *large* x than the degree 2 Taylor polynomial.

13. Find the order of agreement between each Taylor polynomial and the given function at the specified point in exercises 1-10.

14. Explain why the symbolic algebra system strategy discussed above works.

Below is an illustration of an aquarium along with a graph of its water level as a function of time. When the faucet is on, the water level rises at a steady rate. Similarly, when the plug is pulled out, the water level falls at a steady rate (but slower than the faucet's rate). At various times some events happen that affect the water level and/or the rate at which the water level changes. In exercises 16-25 you are asked to identify at exactly what time the given event occurred.



16. The plug is pulled out with the faucet turned off.

17. A large rock is pulled out of the aquarium.

18. The plug is pulled out with the faucet turned on.

19. The plug is put in with the faucet turned off.

20. The plug is put in with the faucet turned on.

21. The faucet is turned on with the plug in.

22. The faucet is turned on with the plug out.

23. A bucket of water is dumped into the aquarium all at once.

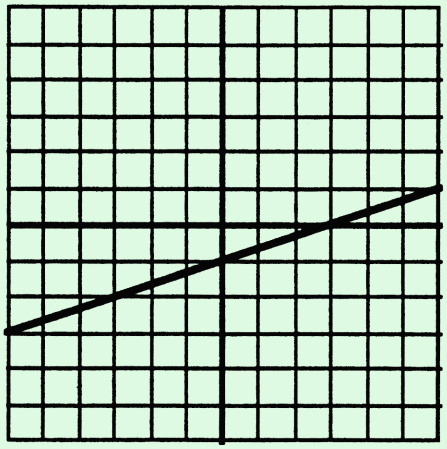
24. The faucet is turned off with the plug in.

25. The faucet is turned off with the plug out.

---

26. Now, assume that the rock is placed back in the aquarium at $t = 20$ minutes and the faucet is turned back on. Suppose that the aquarium is 12 inches deep. When will the aquarium overflow?

21.



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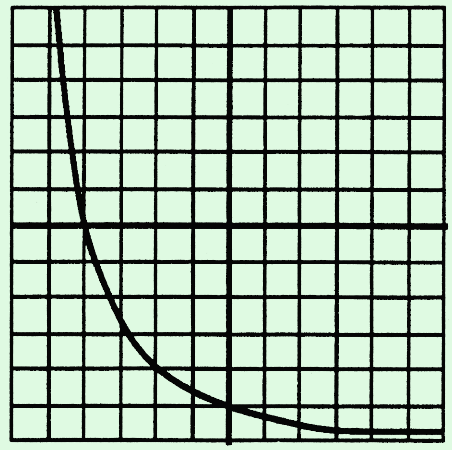

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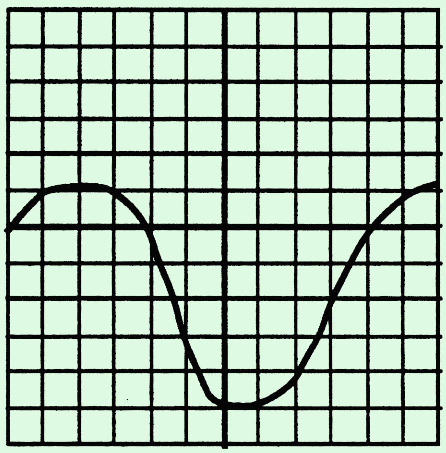

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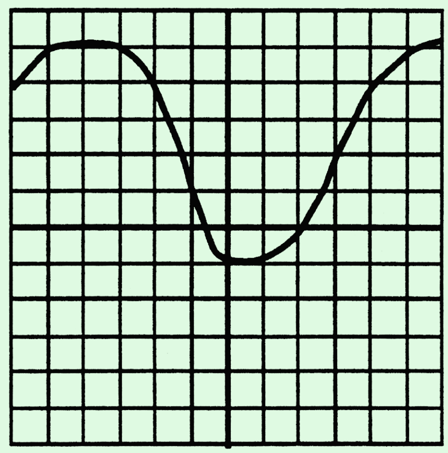

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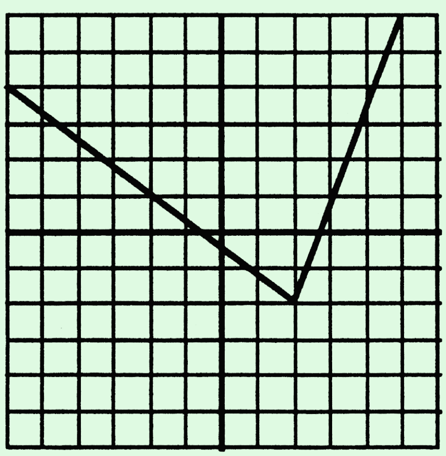

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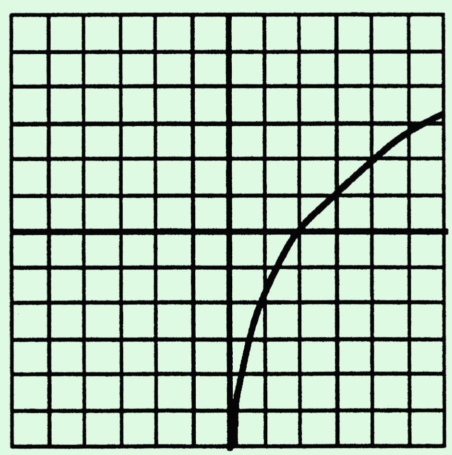
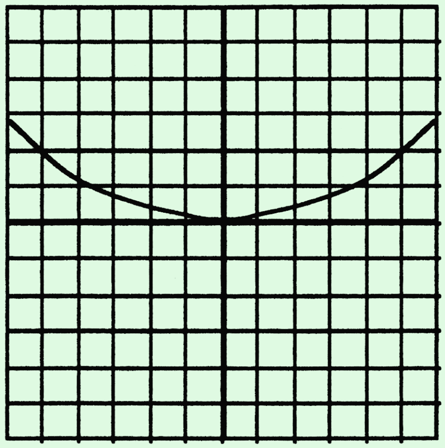
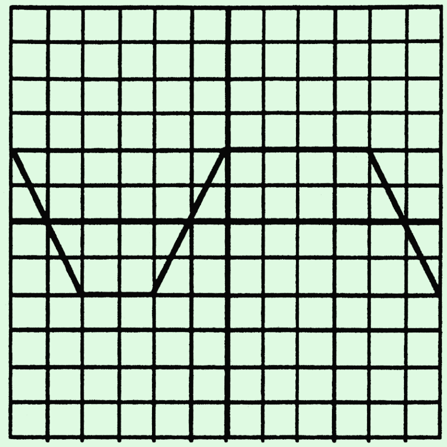


27.



28.



3.5 NEW DERIVATIVES FROM OLD

In this section we will derive more of the basic rules which govern how the derivatives of functions behave when the functions are combined algebraically or through composition. In the last section we saw that the derivative of a sum or difference of two functions is simply the sum or difference of the derivatives, and the derivative of a constant multiple of a function is the same constant multiple of the derivative.

For products, quotients, and compositions of functions, the rules for finding derivatives are slightly more complicated. Once we have these rules, then we will be able to compute the derivative of almost any function built up from the basic functions.

The product rule

The linearity properties for derivatives are very natural. At first glance, the derivative rule for a product is surprising. While it would be easy to remember, let's make it clear:

> **WARNING: The derivative of a product is NOT the product of the derivatives.**

TEST QUESTIONS FOR CHAPTER 7

For exercises 1 and 2, sketch the slope field of the differential equation in the region $[-2, 2] \times [-4, 4]$ and using the slope field, sketch the graph of the solution to the differential equation with the given initial condition in the same region.

1. $\dfrac{y'}{y^3} = x^3 : \quad y(-2) = -1$

2. $y' = 4y : \quad y(1) = -2$

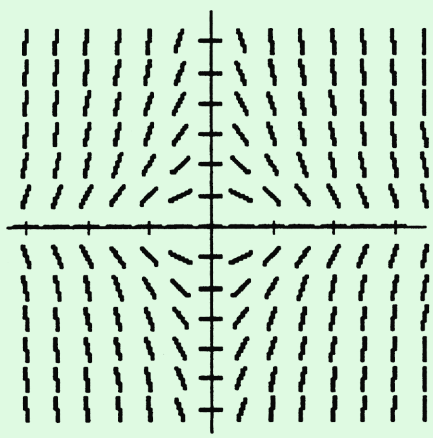
For exercises 3 and 4, solve the differential equations given and sketch the solution to each in the region $[-2, 2] \times [-4, 4]$.

3. $\dfrac{y'}{y^3} = x^3 : \quad y(-2) = -1$

4. $y' = 4y : \quad y(1) = -2$

5. Below is the slope field of the differential equation:

$$\frac{y'}{y} = -2x$$



Sketch the solution of the differential equation with the initial condition: $y(0) = -3$.

6. Solve the following differential equation with the given initial condition:

$$\frac{y'}{y} = -2x; \qquad y(0) = -3.$$

7. Using your calculator or computer, sketch the solution from 6.

8. Solve the following differential equation with the given initial condition:

$$yy' = \cos(x); \quad y(\pi) = -4.$$

9. Below are experimental bacterial population samples taken 1 hour apart:

$$5070, \qquad 4995, \qquad 4921 \qquad \text{and} \qquad 4848.$$

Predict the population 4 hours after the last sample.

10. Suppose an animal grows at a rate proportional to its weight. If the animal weighs 100 pounds at birth and 150 pounds in one month, find its expected weight in one year. When will the animal weigh 1000 pounds?

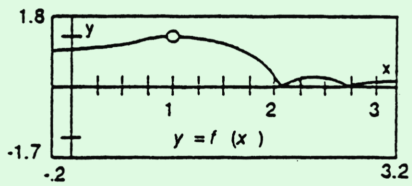
11. Awareness of a company's products often declines exponentially after an advertising campaign ends. A calculator company launches a new product with an advertising campaign. Sales rise to a peak of 1500 units per week. A week after the campaign ends, sales are down to 1300 units per week. If a company decides to run another campaign when sales fall to 65% of their peak level, when will the next campaign begin?

12. A cup of black coffee is poured from a pot, whose contents are at 200° F, into a noninsulated cup in a room at 70°F. After a minute, the coffee has cooled to 190°F. What is the temperature of the coffee after 3 minutes? How long is required before the coffee reaches a drinkable temperature of 150°F?

13. A certain drug has a half-life of 4 hours in the bloodstream. 500 mg is injected initially. Two hours later, 200 mg is administered. How many hours after this second injection will there be 100 mg in the bloodstream?

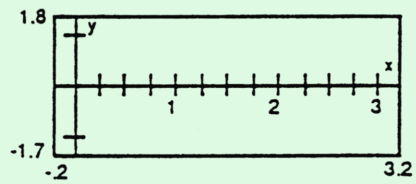
Midterm II

1. The graph of $f : x \longmapsto \left|\frac{\sin(x^2-1)}{x^2-1}\right|$ is shown below.



a. Use the calculator to find $f(0)$.

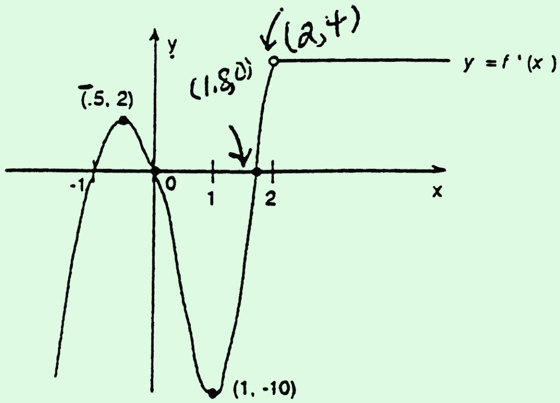
b. Use the calculator to graph $y = f'(x)$ within the same viewing window as $y = f(x)$ above and sketch below.



c. Evaluate $f'(0)$.

d. Find an equation of the line tangent to the curve at $x = 0$.

2. Below is the graph of $y = f'(x)$ (**the derivative of** $f : x \longmapsto f(x)$).



Over which intervals is the **original function** f

a. increasing?

b. decreasing?

Over which intervals is the graph of the **original function** f

c. concave up?

d. concave down?

Find the x-coordinates of the

e. local minimum points of the **original function** f.

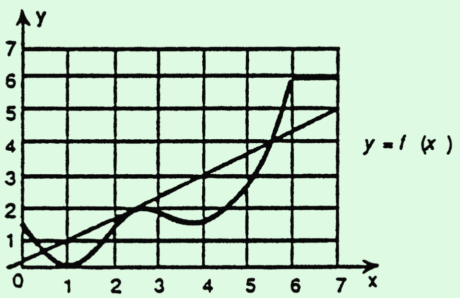
f. local maximum points of the **original function** f.

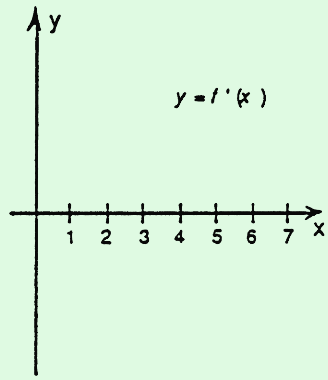
3. The table below contains information with regard to the height in inches, $h : t \longmapsto h(t)$, of a spider on the wall at various times between 1 second and 2 seconds inclusive.

input	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
output	3.64	3.65	3.69	3.76	3.85	3.94	4.02	4.09	4.14	4.17	4.17

Estimate the instantaneous speed of the spider at 1.3 seconds.

4. Below is the graph of f and the line tangent to f at $x = 2.5$.

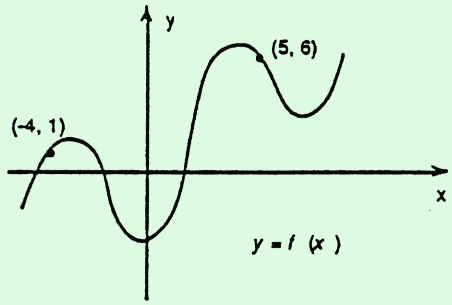


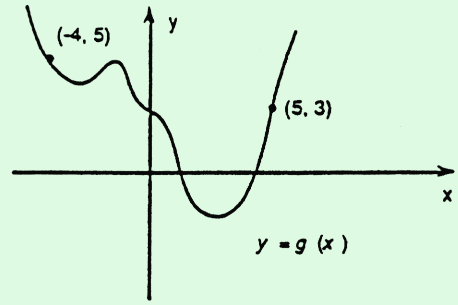


a. Find $f'(2.5)$ and find the equation of the normal line at $x = 2.5$. Sketch this normal line on the axes above.

b. Sketch the derivative of f over the interval $[0, 7]$ using the axes above.

5. Using f and g given by the graphs and information from the table below, compute the derivatives indicated in a. - c.





f '(-4) = 3
g '(-4) = -7
f '(5) = -6
g '(5) = 8

a. $F'(-4)$ where $F : x \longmapsto f(x)g(x)$.

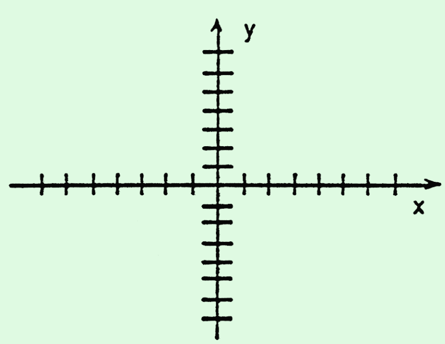
b. $F'(5)$ where $F : x \longmapsto [3f(x) + g(x)]^3$.

c. $F'(-4)$ where $F : x \longmapsto \dfrac{f(g(x))}{g(x)}$.

6. a. Sketch a function, f, satisfying the information given in the two tables below.

For $-\infty < x < -4$,	$f'(x) < 0$ and $f''(x) < 0$.
For $-4 < x < 0$,	$f'(x) < 0$ and $f''(x) > 0$.
For $0 < x < 3$,	$f'(x) < 0$ and $f''(x) < 0$.
For $3 < x < \infty$,	$f'(x) > 0$ and $f''(x) > 0$.

x	$f(x)$
-4	undefined
0	-1
3	-2



b. Find all critical points of f.

c. Find all inflection points of f.

7. The position of a car after the brakes have been applied is given by $s : t \longmapsto -11t^2 + 88t$ where s is in **feet** and t is in **seconds**. There is a brick wall 170 feet measured from a point where the brakes were initially applied. If the car hits the wall traveling at a speed less than 10 mph, the fender will withstand the impact of the collision and there will be little or no damage.

a. Will this car suffer any major damage? Explain why or why not.

b. How far would the car travel beyond the wall if the wall were not there?

c. What is the acceleration at any time t?