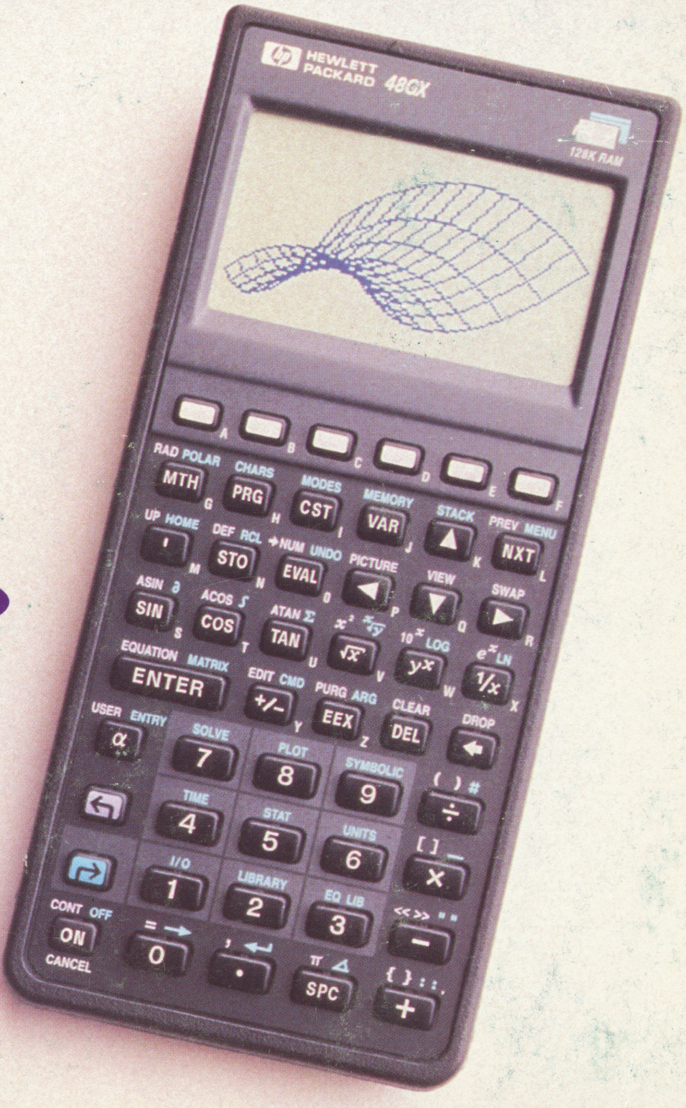
CALCULUS INVESTIGATIONS

WITH THE

HP-48G/GX



DONALD R. LATORRE

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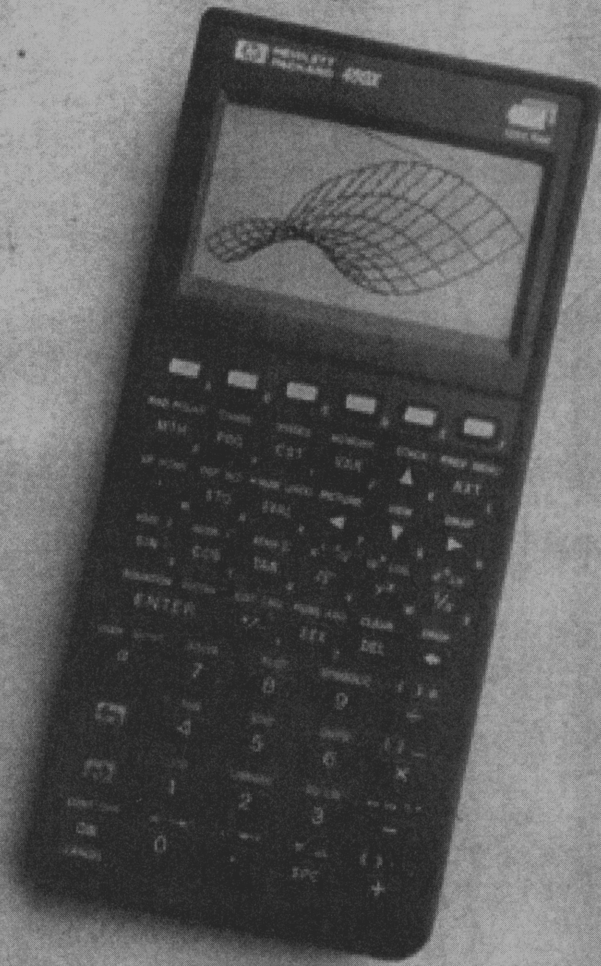
Clemson University



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CHARLES RIVER MEDIA, INC.
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617-871-4184
617-871-4376 (FAX)
chrivmedia@aol.com

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CONTENTS

PREFACE

It is not too surprising that the current movement directed towards reform in the teaching and learning of calculus is occurring at a time when hand-held technology is, literally, *invading* our mathematics classrooms. Indeed, these two movements have a strong element of casual interaction.

The so-called "calculus reform" movement is generally recognized to date from the conference *Toward a Lean and Lively Calculus*, held at Tulane University in January 1986. There was already substantial evidence of widespread dissatisfaction with the way that calculus was being taught and with the results of that teaching, but the Tulane conference was the first to legitimatize that concern.

Graphics calculators were not yet an issue in January 1986, and the discussion on hand-held technology focused primarily on calculators with numerical "solve" and "integrate" keys — calculators such as Hewlett-Packard's HP-15C and units by Texas Instruments, Casio and Radio Shack. Although the role of symbolic manipulation programs (*smp's*) in calculus was often at the forefront of the discussion, the conference ultimately recommended syllabi for Calculus I and Calculus II that assumed that all students would have access only to numerical integrate and numerical solve keys. But, a separate syllabus for a microcomputer based "Calculus II – Computer Alternative", clearly envisioned the more powerful changes that smp's could facilitate.

The January 1987 release of the Hewlett Packard HP-28C calculator gave the mathematics teaching community its first glimpse of the dynamic changes possible with sophisticated hand-held technology. For not only was the HP-28C a graphics calculator, but it was also the first calculator to process symbolic objects — expressions, programs, strings, etc. — with many of the same commands and operations used on numbers. Since 1987, the HP-28C has evolved through the HP-28S and the HP-48S/SX units into the current HP-48G/GX series units. As the most powerful and sophisticated calculators available, the -48G/GX units offer students and teachers unprecedented opportunity to bring graphical, numerical, and symbolic processing into the teaching and learning of calculus.

This book is a textbook supplement for undergraduate courses in single variable calculus. It presents appropriate pedagogical uses of, and teaching code for, the Hewlett Packard HP-48G/GX graphics calculators. It is intended to help students and instructors incorporate these powerful devices as a tool for the interactive learning of single variable calculus, and is independent of any particular textbook. The chapters survey the main topics of the subject and include activities that have been carefully designed to engage students in a modern, technology enhanced study of the material.

FEATURES

Outline of the Book

No two instructors and no two textbooks approach single variable calculus alike. Therefore, I have organized the material into independent chapters that address main topics:

- Functions and Graphs
- Derivatives
- Integrals
- Series

These four chapters are preceded by an introductory chapter on Getting Started with the HP-48G/GX and followed by an appendix on the organization of our HP-48G/GX Teaching Code for Calculus.

Teaching Code for Calculus

The Teaching Code is a collection of special-purpose HP-48G/GX programs, each one addressing a specific aspect of the course. A complete listing of the teaching code appears on the inside back cover. The code is readily available from the author for downloading to an HP-48G/GX from a microcomputer.

Pedagogy

The material is an outgrowth of the extensive classroom use of the HP-48 calculators (and before that, the HP-28 units) at Clemson University in teaching single variable calculus since 1987. Starting with an early pilot course taught by my colleague John Kenelly with the HP-28C in 1987, Clemson has been at the forefront of the move to graphics calculators and now teaches over 100 classes each year in which every student is required to have their own HP-48G/GX unit. The university is strongly oriented towards science and engineering, and our mainstream calculus is populated by students from a variety of fields: the chemical, physical and biological sciences, mathematical and computer sciences, all engineering fields, secondary mathematics education, architecture, accounting and economics, and a few liberal arts students. We do not teach an abstract, proof-oriented course. Instead, our instructors concentrate on explanations, examples, classroom discussions, and calculator activities to generate interest and enthusiasm for learning calculus. For beginning students of calculus, proofs are not as important as "convincing evidence".

HOW TO USE THIS BOOK

At Clemson, the material in this book is used to supplement whatever textbook we are using at the particular time. If the use of technology is to be of any real significance in the learning process, then it must not be used as an occasional "add-on" to the course. Rather, it must become an integral part of the teaching and learning process. Therefore, we require our students to use their HP-48G/GX units on a regular, almost daily basis. We have found that the calculators bring a unique, personal dimension to the use of technology.

Whenever it is appropriate, homework assignments can be made from this book; sometimes in addition to assignments from the main textbook, sometimes in lieu of such assignments. My own personal teaching style allows free and unrestricted use of the calculators on all tests and exams. There is ample opportunity for me to assess my students learning of both concepts and procedures, so the technology poses no threat. On the contrary, it has helped my students to place in proper perspective much of what has traditionally occupied their predecessors in courses in single variable calculus: excessive attention to routine, algebraically intensive procedures for finding derivatives and antiderivatives. I have found students to be overwhelmingly enthusiastic about the use of the calculators as a tool to help them learn.

ACKNOWLEDGMENTS

I want to express my deep appreciation to a number of people.

First, to my colleague John Kenelly. It was John who first foresaw the potential for Clemson with the HP units in calculus and it was he who led our initial efforts in this area. On the national level, John's leadership with regard to the use of graphics calculators in calculus is unsurpassed and his contributions towards shaping the calculus reform movement are immense.

Secondly, to my colleague Gil Proctor. Not only for the material in this book dealing with improper integrals and the ratio test, but for his enthusiastic and untiring support of my personal efforts to incorporate graphics calculators throughout the first two years of the undergraduate program at Clemson. Gil and John have been the wellspring of many productive ideas in Clemson's move towards a technology-based curriculum.

To both John Kenelly and Tom Tucker (of Colgate University) for their numerical integration routines, which date back to the early days of the HP-28C. To Bill Wickes and Charlie Patton of Hewlett Packard. To Bill for his brilliant code for the program FTC dealing with the Fundamental Theorem of Calculus and to Charlie for his code TAYLAT to speed up Taylor polynomial calculations (composed extemporaneously in my office in 1993!). To Robert Simms of Clemson for his very creative programs GRECT, GSEQ and GPS that enable students to visualize rectangle approximations to integrals and the graphs of infinite sequences. To Jim Nicholson, for his truly pioneering efforts in using the HP units in single variable calculus and for his many ideas that permeate this book. To Cynthia Harris for her creative use of the HP's in class after class, and for inspiring many of us to become better teachers. To Bill Beckwith, Lin Dearing, Charlie Harden and Paul Holmes for their willingness to take risks and assume leadership roles in helping to institutionalize hand-held technology at Clemson. And, with great admiration to April Haynes, who with considerable skill and patience, produced by word-processing everything that appears between the covers of this book.

Finally, I wish to gratefully acknowledge the support of the National Science Foundation (NSF USE-9153309) during the preparation of this book. I am also indebted to the Fund for the Improvement of Postsecondary Education (FIPSE) for their early support of my ventures into teaching with hand-held technology.

Clemson University *Don LaTorre*

July, 1995

1 GETTING STARTED WITH THE HP-48G/GX

This brief chapter is intended to provide new users with a basic introduction to the HP-48G/GX calculator and its operation. It is no substitute for the User's Guide, but should help you get started quickly.

Notation

To help you recognize calculator keystrokes and commands, we shall adopt certain notational conventions.

- With the exception of the six white keys on the top row, keys will be represented by helvetica characters in a box: [ENTER], [EVAL], [STO], etc.
- Shifted keys on the 48G/GX may occasionally have the key name in a box preceded by the appropriate shift as in [↰] [CST]. But ordinarily, we will not show the shift.
- Menu keys for commands on various menus will show the key name *in outline form* in a box, as in [ROOT] or [TANL].
- Calculator operations and commands that appear in programs or in the text material will be in helvetica characters, e.g., DUP SWAP INV.

On, Off and Contrast

Press the [ON] key (bottom left of the keyboard) to turn the unit on. Press [↱] [OFF] to turn it off. The [OFF] key is the right-shifted (green) version of the

ON key. With the calculator on, hold down the ON key and press + to darken the display contrast or − to make it lighter.

Stack Display Screen

When you first turn on a factory fresh HP-48G series calculator, you will be looking at the *stack display screen*. To remove any objects from the screen that may remain from previous use, press the ON key several times then the DEL key (on the same row of keys as ENTER). Above the horizontal line near the top of the screen you will see {HOME}, indicating that you are in your HOME directory. Immediately below are levels 1-4 of the *stack*. Like lines on a piece of paper, the stack is a sequence of temporary storage locations for numbers and the other kinds of objects used by the calculator such as algebraic expressions, arrays, equations, and programs.

Just below level 1 are six menu boxes. Normally, these menu boxes will have labels in them that reflect the operation of the *six white menu keys* beneath them. If you press the MTH key near the top left of the keyboard, the labels will show that the first page of the MTH menu contains the six *submenus* VECTR, MATR, LIST, HYP, REAL , and BASE; the NXT key (same row as MTH) will turn you to the second page of the MTH menu and another NXT will cycle you back to the beginning. Return to the previous page with PREV (the left shifted NXT key). The small horizontal tabs above the labels in the MTH menu indicate that each of the boxes contains a submenu (a file, folder or *subdirectory* in HP parlance). Open the HYP (= hyperbolic) submenu by pressing the white menu key beneath it to access the special commands for working with hyperbolic functions. Press MTH to return to the MTH menu at any time.

Similarly, the PRG key opens the PRG (= Program) menu where you may use the white menu keys to access the various submenus of commands for use in writing programs. An extremely important key is the [VAR] key. It opens the VAR (= Variables) menu, *which is where you look to find the objects that you have created and stored into the memory of the machine.* For routine calculations on the stack, it does not matter which menu labels are active. Simply press [CST] to make them all blank.

Keyboard

The keyboard of an HP-48G series calculator may at first appear to be somewhat intimidating. But, like the control panel of any high-performance device, it enables you to control and to monitor a vast array of operations. The number entry keys are bordered on the right by [+], [−], [×], and [+]; and on the left by [ON], [↱], [↰], and [α]. The right-shift key [↱] (green) and the left-shift key [↰] (purple) are color coded to many of the keyboard labels, and the [α] key is used to obtain alphabetical characters.

Adjacent to [ENTER] is [+/−] for changing signs, then [EEX] for entering exponents, [DEL] for deleting characters (and clearing the stack), and [⇦] for backspace-and-delete (and dropping objects from level 1). The [SIN], [COS], [TAN], and [√x] keys are just above, as are [y^x] (for obtaining powers) and [1/x] (for reciprocals and matrix inverses). Above the trig function keys are ['] (tick), for entering algebraic expressions, and [STO] and [EVAL] for storing and evaluating objects. The four cursor keys [△], [▽], [◁] and [▷] control the movement of the cursor when it is active.

Applications and Command Menus

You will notice that some keys have both purple and green labels printed above them (like the [α] key), but many have only one of the two (like the [7], [8] and [▷] keys).

The keys that have only green labels above them represent *applications*, e.g., I/O, PLOT, SOLVE, SYMBOLIC. The right-shifted version of an application key invokes a specially designed user-interface that lets you interact directly with the named application, often through the use of *input forms*, which are the HP equivalent of the familiar computer "dialogue boxes". Alternatively, the left-shifted version of an application key gives you access to the various commands on the *command menu* that is associated with the particular application. The commands may be included in programs or executed directly from the keyboard while viewing the stack display screen.

Display Settings

It is best to keep the calculator's angle mode set to radians in order to work with trigonometric functions. Press [↰] (purple) [MTH] to toggle between radian mode and degree mode. When radian mode is set, the message RAD appears at the top left of the stack display screen.

To display numbers in standard form, set your unit to STD display mode (the default setting) by pressing [↰] [MODES] (the left-shifted [CST] key), opening the FMT (= Format) menu and checking to see that the left-most menu box reads [STD□]. The small box next to STD indicates that STD mode is active. If the menu simply reads [STD] press the associated white menu key to activate STD mode. Now press [↱] [MODES] to interact with the main MODES screen. You should see that the number format is highlighted and set to Std, and that the angle measure is set to Radians. Press the [▽] twice to highlight the coordinate system field (it should read Rectangular, by default). To see how to change such a field, press the white menu key beneath [CHOOS], use [▽] to highlight Polar and press [OK]. You have just changed to polar coordinates. Now change back to rectangular coordinates. When the display is set to show only a fixed number of digits to the right of the decimal point, say with 3 FIX to display only three such digits, the numerical calculations are still executed internally to the full 12- or 15-digit precision of the machine. Only the display is affected. By resetting to STD mode, you will display full 12-digit precision. Unless stated otherwise, we will assume throughout this book that the display mode is set to STD and that the coordinate system is set to RECTANGULAR.

The √ by BEEP means that the beeper is turned on (to alert you of syntax errors, alarms, etc.). To activate the clock, highlight the clock field and press the [√] key. Leave the fraction mark (FM,) unchecked so that decimal points, rather than commas, will appear in decimal numbers like 123.45. Exit this screen by pressing [OK]. Notice that the time and date now appear above the horizontal line. If you wish to modify the time or date, press [↱] [TIME] (the green shifted [4] key) and proceed as above.

Symbolic Execution Mode

The HP-48G/GX is a third generation *symbolic* calculator, which means that you can apply operations and functions to symbolic expressions and obtain symbolic results. For example, you can enter the symbolic expressions for x^2 and for $sin\ x$, then press the [+] key to obtain the symbolic result $x^2 + sin\ x$. Most other calculators are numerical calculators, capable of applying functions and operations only to numerical objects to obtain numerical results.

Symbolic execution mode is controlled by a system *flag* (flag -3). In the default state, flag -3 is clear and the HP-48 is in Symbolic Execution Mode. In this mode, the symbolic constants (e, i π, MAXR, and MINR) and functions with symbolic arguments will evaluate to symbolic results. But if flag -3 is set, Numerical Results mode is active and the symbolic constants and functions with symbolic arguments will evaluate to numbers.

We *strongly recommend* that you keep your HP-48G/GX in Symbolic Execution Mode. If you go to the MODES menu with the [↰] [CST] keys and open the [MISC] submenu, the SYM menu key should read [SYM□]. The small box that appears next to SYM indicates that Symbolic Execution Mode is active. If no box appears in this key, simply press the [SYM] key to change it to [SYM□].

Numerical Calculations

Simple numerical calculations are done on the stack. The idea is this: put inputs on the stack and then execute commands that use the inputs. To enter -12.34, begin by pressing the appropriate number keys and the decimal point key (bottom row, center), then use [+/−] to change the sign. Notice that the typing starts at the bottom left of the display screen, below level 1 of the stack, on the *command line*. Press [ENTER] to put -12.34 on level 1. Now enter 56.789; notice that [ENTER] inserts it onto level 1, moving -12.34 up to level 2. Press [+] to compute the sum. To recapture the stack before you added, press [↱] [UNDO] (the right-shifted EVAL key). Now subtract 56.789 from -12.34 with [−], then use [UNDO] and swap positions with [SWAP] (the right cursor key [▷]; no need to press [↰] now). Now subtract again to get 69.129. Take the square root with [√x], then cube the result with 3 [Y^x]. You should have 574.765129278.

To edit this result, press the [▽] (down cursor) key, use the right cursor key to move the cursor over the 9, delete the 9 with [DEL] and press 3 [ENTER]. Now use [↱] [LN] (the right-shifted [1/x] key) to obtain the natural logarithm. To multiply by π, press [↰] [π] (π is obtained with the left-shift [SPC] key) then [×]. Notice the symbolic result '6.35396147609 * π' on level 1, enclosed in tick marks. To convert this to a numerical result, use [↰] [→NUM] (the left-shift [EVAL] key). Now drop the 19.9615586945 from level 1 with [⇦]. The

[⇦] key drops objects from level 1; the adjacent key (labeled CLEAR in purple) clears the entire stack. Normally, you need not left-shift these keys; shifting is required only when the command line is active.

n^{th} Roots

With a real or complex number on stack level 1, the [$\sqrt{x}$] key will return its square root. If the number is real and negative, say -3, then the [$\sqrt{x}$] key will return a complex number whose real part is zero: (0, 1.73205080757) for the square root of -3.

To take the n^{th} root of a real number x for $n > 2$ we can calculate $x^{1/n}$: 2 ^ (1/3) is 1.25992104989. But when n is odd and x is negative, this procedure will always return a complex number: (-8) ^ (1/3) is (1, 1.73205080757). This result is the *principal* cube root of -8, certainly not the *real* cube root that we expected. To obtain the real n^{th} root of a negative number for an odd value of n, use the XROOT key [$\sqrt[x]{y}$], which is the right-shifted [$\sqrt{x}$] key. For example, to obtain the real cube root of -8, simply enter -8 and then 3 (the desired root); press [$\sqrt[x]{y}$] to obtain -2.

Data Entry

When keying a sequence of real numbers into the command line, say 1.1, 2.2 and 3.3, you must separate the numbers with spaces or commas for proper recognition, as in 1.1 2.2 3.3 or 1.1, 2.2, 3.3. *We recommend that you use spaces for ease of use.* For consistency we will show commas, but you should always interpret them as spaces. You need not insert commas or spaces between a real number and a complex number (an ordered pair), or between two complex numbers, because the calculator recognizes parentheses as object delimiters. *Unless we specify otherwise, all examples and exercises in this book assume the calculator is set to **STD** display mode.*

Algebraic Expressions

Algebraic expressions must be typed in beginning with a ' (tick) mark using the ['] key. Alphabetical characters are obtained by first pressing [α] and then the desired key. *Note that alphabetical characters appear in white letters to the lower right of the keys on the top four rows.* To produce, say 'S', press ['] followed by [α] [SIN] [ENTER]. Lower case characters are obtained by the sequence ['] [α] [↰], then the character key. For example, ['] [α] [↰] [D] [ENTER] puts 'd' on level 1. (Thus [α] *left-shift* will give *lower case*).

To enter the algebraic expression 'SIN(X)', press ['] [SIN] [α] [1/x] [ENTER]. Notice the location of the cursor after each keystroke; after [1/x] the cursor is still inside the right parenthesis. To move it outside, use the right cursor key [▷]. But, pressing [ENTER] does it all for you. As a more complicated example, try 'COS(X ∧ 2)/(2 ∗ X ∧ 3)'. The keystroke sequence is:

['] [COS] [α] [1/x] [Y^X] 2 [▷] [÷] [↰] [÷] 2
[×] [α] [1/x] [Y^X] 3 [ENTER].

Yes, it is necessary to insert the ∗ in 2 ∗ X ∧ 3; if you forget, when you press [ENTER], an Invalid Syntax message will appear and you can then correct your typing. If things are not going well on the command line, remember that the [⇦] key will backspace and delete. Finally, if you get desperate, press [ON] (sometimes, more than once) to cancel what is taking place and then start over.

Stack Manipulation

We often need to manipulate the stack. For example, to duplicate one or more levels, to copy an object from a higher level down to level 1, or to otherwise rearrange the stack. Complete details can be found in chapter 3 of the HP-48G series

User's Guide, but we will survey the basics here. This survey should suffice for most purposes.

To make a duplicate copy of the object on level 1, simply press [ENTER]. This executes the DUP command, which duplicates level 1. We have already commented on the obvious keyboard commands DROP (the [⇦] key), CLEAR (the [DEL] key), SWAP (the [▷] key), and UNDO (the [↱] [EVAL] key). Although the DROP, CLEAR and SWAP keys are labelled in purple, it is not necessary to use the purple [↰] key except when the command line is active.

The best way to understand the other stack commands is to begin with your stack arranged like this:

4: 'S'
3: 'T'
2: 'U'
1: 'V'

Now press the [▵] key to engage the *interactive* stack. The interactive stack is an environment that lets you interact with the stack and is active when the dark pointer ▶ appears at the left of the screen. You exit the interactive stack with [ENTER] or [ON] (either one will work). So arrange your stack as in the above illustration and then press the [▵] key. The commands that are most often used are PICK, ROLL, ROLLD, →LIST and (on the next page) DUPN and DRPN.

Move the pointer up to level 3 and press [PICK] [ENTER]. The command PICK copies the content of level 3 to level 1. Use [⇦] to DROP the 'T' from level 1.

Now move the pointer back to level 3 and press →LIST ENTER. Notice that the contents of levels 1-3 were put into a list (lists use curly braces). Now restore the stack to its original state with UNDO.

The commands DUPN and DRPN (on the next page) are almost self-evident. With the pointer situated on level *N*, DUPN will duplicate the first *N* levels of the stack while DRPN will drop the first *N* levels. Try using DRPN with the pointer at level 3. Press ENTER to exit, then use UNDO to restore everything.

The last two commands, ROLL and ROLLD are extremely useful. With the pointer specifying the number *N* of levels, ROLL will push (*roll*) the stack upward, causing the object on level *N* to fall down to level 1. Try using 4 ROLL to rearrange the stack:

```
4: 'S'                  4: 'T'
3: 'T'   4 ROLL ☞       3: 'U'
2: 'U'                  2: 'V'
1: 'V'                  1: 'S'
```

(The 'S' *rolled* off the top level and fell down to level 1)

The command ROLLD (*roll down*) is just the opposite: it pulls the specified number of objects down, causing the level 1 object to move to the top level. Restore the current stack to its original state with 4 ROLLD. Now use CLEAR to clear the stack.

RPN

RPN stands for *Reverse Polish Notation*, the type of logic used by almost all Hewlett Packard calculators. The essence of RPN is this: first provide the inputs, then execute commands that operate on the inputs. When we did our earlier calculations on the stack, we were using RPN entry. Thus, to add -12.34 and 56.789 in

RPN we input -12.34 and 56.789, then executed the command +. In fact, we built -12.34 using RPN: input 12.34, then press [+/−]. Notice how this differs from the *algebraic entry logic* employed by most other types of calculators. Algebraic entry requires that we type in -12.34 + 56.789 from left-to-right and then press an [ENTER] or [=] key. To produce a numerical result for $\sqrt{ln\ 2.3}$ on the HP-48 using algebraic entry we type

' √ LN 2.3 EVAL

But to obtain this using RPN, we do

2.3 LN √

RPN is an especially powerful logic for constructing the algebraic expressions that we encounter in a beginning study of calculus. Expressions such as

$$\sqrt{1 + cos^2(x^3)} \qquad \text{or} \qquad (1+x)^{2/3} + \frac{2x+1}{\sqrt{x^2-4}} \ .$$

Consider the first of these two. Superficially, it is simply the square root of one plus the cosine squared of x^3. But it is important that we understand this expression mathematically, *from inside out,* as follows: start with x and cube it, take the cosine of x^3 and square the result, then add 1 and take the square root. RPN entry corresponds exactly to this way of thinking:

'X' 3 ∧ COS SQ 1 + √ .

A more complex example is provided by the second of the above two expressions. First, try entering this expression using direct algebraic entry (remember to start with a ' (tick) mark); what did you find out? Now use RPN entry as follows: begin by putting the three main components '(1 + X) ∧ (2/3)', '2 * X + 1', and '√(X ∧ 2 − 4)' on the stack in this order (you can use either direct algebraic entry or RPN for any of them); now press [÷] to build the quotient, then [+] to obtain the sum.

This last example clearly illustrates why RPN is the preferred method for entering complicated expressions onto the stack. Most users tend to develop their own style, often using direct algebraic entry to build simple components and then RPN to produce the more complicated final results. Of course, all programs on the HP-48 must be written in RPN. For example, the program

« SQ SWAP SQ + √ »

uses RPN logic to take two inputs from the stack, say x and y, and then returns the result $\sqrt{x^2 + y^2}$.

Memory Management

The HP-48 can manipulate and store many types of *objects*, such as real and complex numbers, algebraic expressions, vectors and matrices, lists, graphics, programs, and text. Any of these objects can be placed on the stack, but to be saved in the calculator's memory it must be given a name and stored. When you store an object, it is stored as a *variable* in *user memory* (that part of the calculator's memory that you, the user, have access to) and is accessible through the VAR menu. The variables that you create in this way are called *global variables* to distinguish them from other kinds of variables that the HP-48 uses (e.g., *local variables* – that are created within and used entirely by a program, and *system variables* – that are used by the calculator's operating system). You can think of a global variable as a named storage location containing an object.

For example, suppose that you wish to create a variable named TRY1 containing a program that will accept numbers x and y as inputs and calculate $\sqrt{x^2 + y^2}$. Here is the program:

« SQ SWAP SQ + √ »

To build the program, press [←] [−] to get the program delimiters « », then use [←] [√x] [←] [▷] [←] [√x] [+] [√x] [ENTER]. Now put the name 'TRY1' on the stack and press the [STO] key. If you press the VAR key you will see that the leftmost menu key is labeled [TRY1]. To run the program with inputs 1 and 2, put 1 and 2 on the stack and then press [TRY1] to see the result 2.2360679775. In fact, you need not actually enter the inputs onto the stack: simply press 1 [SPC] 2, then [TRY1] to get the result. The HP-48 recognizes spaces as object separators and TRY1 will take the inputs directly from the command line. We will often use this shortcut with our programs.

To delete a variable from user memory, put its name on stack level 1 and execute the command PURGE. The [PURGE] key is the left-shifted [EEX] key. To purge variable TRY1, press ['] (tick), [TRY1] [ENTER], then [PURGE].

To organize the variables that you create, you can put them into files (or *directories*). Whenever you create a variable and store it, it is stored in the current directory. If you are using a factory fresh HP-48 then your current directory is the HOME directory, indicated by the list { HOME } at the top left of the stack display screen. The name of the current directory always appears as the rightmost name in the list that begins with HOME, as above. To create a subdirectory named CALC in which you can store any variables that you may need in a study of calculus, begin by putting the name 'CALC' on stack level 1. Now press [←] [MEMORY] (the left-shifted [VAR] key), open the DIR submenu and execute the command CRDIR (create directory). If you then open the VAR menu you will see the [CALC] directory on the left. The short bar above the label is suggestive of the tab on a file folder, and reminds you that CALC is a subdirectory. Press [CALC] to open this directory and notice the list {HOME CALC} at the top of your screen, indicating that the current directory is now CALC. This directory is presently empty, containing no variables.

To return to the parent directory HOME, you need only go up one level in the directory tree. The commands UP and HOME, executed by shifting the ['] (tick) key appropriately, send you up one level or, alternatively, send you directly to HOME.

A few final comments about storing and purging variables from directories. The same variable can exist in different directories, often containing different objects. For example, whenever you use the PLOT application, copies of the reserved variables EQ (the "equation") and PPAR (the plot parameters) are stored into the current directory. Likewise, whenever you use the SOLVE application, a copy of EQ and the "unknown" variable are stored in the current directory. In this way, EQ and, say, 'X' can appear in different directories with different contents. Since the contents of EQ and PPAR are automatically updated whenever the PLOT application is used, it is usually not important to purge them. On the other hand, a variable like 'X', which is the default independent variable for graphing, should be purged from the current directory immediately after it is used. Keep in mind, also, that when you purge 'X' from a particular directory it may continue to exist in an "ancestral" directory where it may cause trouble later on. For example, suppose that CALC is the current directory, that no variable 'X' is stored in CALC, but that the parent directory HOME contains the variable 'X' in which the value 2 is stored. Suppose further that you wish to take the symbolic derivative of a function f with respect to the independent variable 'X'. Because 'X' appears in the parent directory, the derivative will be automatically evaluated at the value $x = 2$. This is because the HP-48 always searches *upward* in the directory tree in search of variables; it does *not* search for variable in directories that are on the same level as, or below, the current directory. And, having found that HOME contains variable 'X' with the value 2, the derivative at $x = 2$ was returned. Had the calculator found no value for 'X', it would have treated 'X' symbolically, as was desired. Moral: purge all 'X''s.

2 FUNCTIONS: EVALUATION AND GRAPHING

Beginning calculus is a study of the behavior of functions: their variation, rates of change, limiting behaviors. We shall thus begin with a brief look at how functions can be represented, evaluated and graphed on the HP-48G/GX calculators.

2.1 FUNCTION EVALUATION

Evaluating with SOLVR

The basic idea of a function F of a single variable x is simple enough: for each value of the input variable x we obtain exactly one output value $F(x)$. The HP-48G/GX units have a built-in environment that is ideal for the evaluation of functions, the SOLVR. Although the SOLVR is designed to solve equations, the format of its menu makes it convenient for evaluating functions. With the function on level 1, press [↰] [SOLVE] to access the SOLVE application, open the ROOT subdirectory and then load the function on level 1 into EQ by pressing [↰] [EQ] (remember: left-shift will *load*). Now press the [SOLVR] key. To evaluate the function stored in EQ at a number (or variable), simply key in the number (or variable), press [X] then [EXPR=]. For example, to numerically investigate the behavior of the function $f(x) = \frac{x+2}{2x+1}$ as x approaches 0 we can proceed as follows.

Put '(X + 2) / (2 * X + 1)' on level 1 and press [↰] [SOLVE] [ROOT], then [↰] [EQ] [SOLVR]. To find $F(.01)$, press [EEX] 2 [+/−] then [X] [EXPR=] to see 1.97058823529.

To find $F(.0001)$, press [EEX] 4 [+/−] then [X] [EXPR=] to see 1.99970005999.

To find $F(.000001)$, press [EEX] 6 [+/−] then [X] [EXPR=] to see 1.99999700001.

To find $F(.00000001)$, press [EEX] 8 [+/−] then [X] [EXPR=] to see 1.99999997.

Clearly, we can see that $f(x)$ is approaching 2 as we let x approach 0 from the right. What happens to $f(x)$ as we let x approach 0 from the left? Experiment to find out.

When using the **SOLVR**, if you store an equation, say 'expression 1 = expression 2', in **EQ** instead of a single expression, pressing [EXPR=] for a given value of X will return two values, one for the left side of the equation and one for the right side. This provides a convenient way to compare the outputs of two functions at various input values of X.

You should be aware that whenever you use the calculator's **SOLVE** application, the last value for X is stored under the variable name 'X' in user memory. You need not make explicit use of the **SOLVR** for this to occur: pressing [ROOT] on the **FCN** submenu automatically activates the **SOLVR** (as do the commands **ISECT**, **EXTR** and **F'** which appear as menu keys on the **FCN** submenu). You can see this variable by pressing [VAR] to go to the **VAR** menu, where you will see the menu key [X]. Press [X] to recall the value stored for X. Our recommendation is that before going on to the next application you *immediately* purge this variable to avoid trouble later on. Purge by pressing ['] [X] [PURGE].

User-Defined Functions

Another way to represent functions on the HP-48G/GX is by creating *user-defined functions*. In HP-48 parlance, a user-defined function is simply a short program that captures the essence of the formal way that we define a function by an equation like $F(x) = 2\ sin\ x + sin\ 4x$. Here, F is the name of the function, x is the input variable, and the expression to the right of the = sign is an algebraic description of the desired output for a given input x.

The user-defined function that represents this mathematical function is the program « → X '2 * SIN(X) + SIN(4 * X)' » stored in the global variable F. The DEFINE command lets you create a user-defined function directly from an equation. For the example at hand, simply enter the equation 'F(X) = 2 * SIN(X) + SIN(4 * X)' onto level 1 of the stack and press [↰] [DEF]. If you access the VAR menu with the [VAR] key, you will see the label [F] appearing above a white menu key; this identifies F as the name of the user-defined function. To verify that the variable named F actually contains the above program, you can recall the contents of variable F by pressing [↱] [F]; press [DROP] when you've finished viewing the program.

To evaluate this function, enter the desired input and press the menu key [F]. For example, put 'T ^ 2' on level 1 and press [F] to see '2 * SIN(T ^ 2) + SIN(4 * T ^ 2)'. Likewise, press 4 [F] to see $2\ sin\ 4 + sin(4 * 4)$ evaluated as -1.80150830728. Note that you can enter the equation 'F(X) = expression in X' directly or by first entering 'F(X)', then the 'expression in X' and pressing [↰] [=]. In either case the [DEF] key automatically creates the user-defined function from the equation.

User-defined functions of two or more variables are constructed in the same way. For instance, to represent $G(s, t) = s/t^2$ enter 'G(S, T) = S/T ^ 2' and press [↰] [DEF].

To evaluate G we input a value for S, then for T. Try it for yourself: $G(2, 3) = .222222222222$.

Piecewise-defined functions often occur in applications and are introduced early in calculus to illustrate the ideas of one-sided limits and points of discontinuity. The best way to represent them on the HP-48G/GX is to use the IFTE command, found on the second page of the PRG BRCH menu. The IFTE command is an abbreviation for the "if ... then ... else ... end" construction and executes one of two procedures that you specify, according as a "test clause" is true or false. The IFTE command takes three arguments: a test argument and two procedural arguments, as in IFTE (test, procedure 1, procedure 2). You should interpret this as "If *test clause is* true, then *execute procedure 1* else *execute procedure 2*".

To represent the function $p(x) = \begin{cases} x^2 - 2x & x < 0 \\ 1 - x^2 & 0 \le x \end{cases}$, the desired expression is 'IFTE(X < 0, X ^ 2 – 2 * X, 1 – X ^ 2)'. Begin with a tick, then go to the second page of the PRG BRCH menu and press [IFTE], followed by the three required arguments separated by commas ([α] [↰] 2 will produce the < symbol or you can go to the PRG TEST menu), then [ENTER]. This expression can now be treated like any other function. For instance, with 'IFTE(X < 0, X ^ 2 – 2 * X, 1 – X ^ 2)' displayed on stack level 1, enter 'P(X)', then press [SWAP], [=] and finally [DEF] to create a user-defined function. Try evaluating $P(X)$ using values to the left and right of 0 to discover the behavior of the function p as x approaches 0.

The general construction for a piecewise-defined function with two pieces like

$$f(x) = \begin{cases} f_1(x) & x \le a_1 \\ f_2(x) & a_1 < x \end{cases}$$

is IFTE($x \le a_1, f_1(x), f_2(x)$).

For three or more pieces, you can nest the IFTE commands:

$$\text{for } f(x) = \begin{cases} f_1(x) & x < a_1 \\ f_2(x) & a_1 \le x < a_2 \\ f_3(x) & a_2 \le x \end{cases},$$

use IFTE($X < a_1, f_1(X)$, IFTE($X < a_2, f_2(X), f_3(X)$)).

Activity Set 2.1

1. What happens to values of $f(x) = \dfrac{\sin x}{x}$ as x approaches 0? Use the SOLVR to find out. Let x approach 0 through values $x = 10^{-2}, 10^{-3}, \ldots, 10^{-6}$, then their negatives. Press [EEX] 2 [+/–] to input 10^{-2}, etc.

2. What happens to values of $f(x) = \dfrac{\cos x - 1}{x}$ as x approaches 0? Use the SOLVR to find out. Let x approach 0 through values $x = 10^{-2}, 10^{-3}, \ldots, 10^{-6}$, then their negatives. Press [EEX] 2 [+/–] to input 10^{-2}, etc.

3. What happens to values of $f(x) = xe^x$ as x approaches $-\infty$? Use the SOLVR to find out. Let x approach $-\infty$ through values x = -1, -10, -1,000 and -10,000.

4. Repeat Activities 1-3, but this time with a user-defined function for $f(x)$.

5. (a) Evaluate $f(x) = (1 + 1/x)^x$ for $x = 10^2, 10^4, \ldots, 10^{11}$. Then make a conjecture about what happens to $f(x)$ as $x \to \infty$.

   (b) Now evaluate $f(x)$ for $x = 10^{12}$. Can you explain the result?

6. Investigate the behavior of the following function as $x \to 0$:

$$f(x) = \frac{x + |1 - \sqrt{x+1}|}{|1 - \sqrt{x+1}|}.$$

7. (a) Use the **IFTE** command to build an expression for

$$f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ \cos x & \text{if } x \geq 0 \end{cases}.$$

(b) Evaluate $f(x)$ for a sequence of values that approaches 0 from the left; what does $f(x)$ approach?

(c) Now evaluate $f(x)$ for a sequence of values that approaches 0 from the right; what does $f(x)$ approach?

(d) In view of your results in (b) and (c), does $\lim_{x \to 0} f(x)$ exist?

8. The *greatest integer* function, often denoted by $\lfloor x \rfloor$, is defined by

$$\lfloor x \rfloor = \text{the greatest integer} \leq x.$$

It is executed on the HP-48G/GX by the **FLOOR** command (a menu key appears on the third page of the **MTH REAL** menu). Use the **FLOOR** command to calculate $\lfloor x \rfloor$ for each of the following values of x:

(a) π^e (b) $-\pi^e$ (c) e^π (d) $(\sqrt{\pi})^{10}$

9. The *least integer* function, often denoted by $\lceil x \rceil$, is defined by

$$\lceil x \rceil = \text{the least integer} \geq x.$$

It is executed on the HP-48G/GX by the **CEIL** command (a menu key appears on the third page of the **MTH REAL** menu). Use the **CEIL** command to calculate $\lceil x \rceil$ for each of the following values of x:

(a) π^e (b) $-\pi^e$ (c) e^π (d) $(\sqrt{\pi})^{10}$

2.2 FUNCTION GRAPHING

The single most important application of the HP-48G/GX to a study of calculus is to create visual images of the wide variety of functions under study. More than anything else, the ability to graph quickly and easily adds a powerful new dimension to the traditional analytical approach to calculus. Many of the important aspects of functional behavior — maximum and minimum values, rates of change, etc. — can be effectively displayed by the graph of the function. With the HP-48, graphical representations can be used from the beginning of the course.

To get informative representations of graphs on the HP-48 you must set the viewing window to display the part of the graph that you want to see. The default settings of the plotting ranges for points (x, y) are $-6.5 \le x \le 6.5$ and $-3.1 \le y \le 3.2$, with a common unit scaling of each axis. Since there are 131 columns and 64 rows of pixels, the default settings produce square pixels of size .1 and your visual intuition of slope and area is preserved on the screen. The default settings also work well for trigonometric functions of amplitude 3 or less. You can, of course, change the settings in a variety of ways , some of which will be illustrated in the examples. To accomodate trigonometric graphs, make sure your calculator is set to radians mode. The [↰] [MTH] key will toggle between degrees and radians; when radian mode is set, the message RAD will appear in the top left corner of the screen.

Basic Plotting

Functions are represented graphically as *plots* in the PICTURE environment. The general procedure to produce a plot of a function of a single independent variable is as follows:

- Access the PLOT application;
- Make sure the plot type is set to FUNCTION;

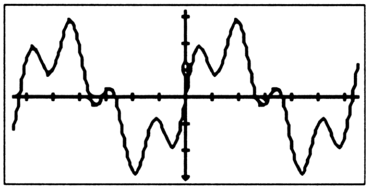
- Enter the expression that defines the function;
- Set the plotting parameters: the independent variable, horizontal and vertical plotting ranges, etc.;
- ERASE (if desired) any previous plots;
- Execute the DRAW command.

The HP-48G/GX calculators allow you to access the PLOT application in two different ways in order to enter a function's expression and to set the plotting parameters: with [↱] [PLOT] to interact directly with the main PLOT screen, or with [↰] [PLOT] to use the various commands on the PLOT menu. We will illustrate both approaches in our first two examples.

EXAMPLE 1. Graph $y = 2 \sin x + sin\ 4\ x$ with the default plotting parameters.

Using the PLOT screen

From the stack display screen, go to the PLOT application with [↱] [PLOT]. The main PLOT screen will show the current plot type, current angle mode, current expression in EQ (if any), the independent variable (*X*, by default), and the current horizontal and vertical display ranges. If the current plot type does not show Function, press [△] [CHOOS], highlight FUNCTION and press [OK]. If necessary, use [▷] and a similar procedure to set the angle mode to RAD. Now highlight the field EQ: and type '2 * SIN(X) + SIN(4 * X)' and press [ENTER]. Notice that when you are using the PLOT screen, you do not have to begin the algebraic expression that defines the function with a tick mark. If the default plotting parameters are current, the independent variable will appear as INDEP: X, the horizontal display range as H-VIEW: -6.5 6.5, and the vertical display range as V-VIEW: -3.1 3.2. If any of these settings appears otherwise, go to the next page of the PLOT menu with NXT, press RESET and highlight Reset Plot and press OK. Once the default plotting parameters are set, return to the previous page with PREV and press ERASE to erase any previous plot. Now press DRAW. You should see a plot like this:



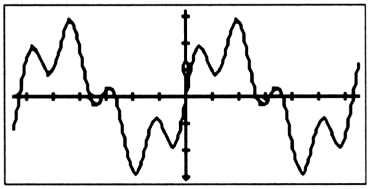
When the plot is complete and the menu labels appear, press TRACE and then use the right and left cursor keys to *trace* along the plot. Press (X, Y) to obtain *coordinate readouts* for the cursor. The *x*: value is the pixel location of the cursor but the *y*: value is the value of the function in EQ computed at the *x*: value. Press + to restore the menu keys. When you have finished viewing the plot, press ON twice to return to the stack environment. You can bring back the plot by using the ◁ key to access the PICTURE environment.

Using the PLOT menu

Enter '2 * SIN(X) + SIN(4 * X)' on level 1 (you will have to start with a tick mark), and press ↰ PLOT to access the PLOT menu. If you do not now read "Ptype: FUNCTION" at the top of your screen press PTYPE and then FUNC. Then use ↰ EQ, to store the expression on level 1 into EQ. Now press PPAR to see the plotting parameters. If you do not now read

Indep:	'X'	
Depnd:	'Y'	
Xrng:	-6.5	6.5
Yrng:	-3.1	3.2
Res:	0	

on your screen, press [RESET] to return your screen to the default settings. Now press [↰] [PREV] to turn back a page and open the PLOT menu with [PLOT]. Press [ERASE] to erase any plot previously drawn, then [DRAX] to draw axes and [DRAW]. You should see the graph we had before:



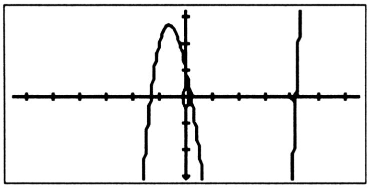
Now trace along the plot with coordinate readouts. When you have finished viewing the plot, return to the stack display screen by pressing [ON]. You can always bring back the plot by using the [◁] key.

Often, in order see more of a plot you can compress or expand the viewing screen vertically or horizontally by using commands from the ZOOM menu, as in the next example.

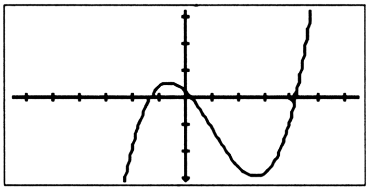
EXAMPLE 2. Graph $y = x^3 - 3x^2 - 5x + 1$.

Using the PLOT screen

Open the PLOT application with [↱] [PLOT]. Since the default settings are current from our previous example, we need only enter the new function. With the EQ field highlighted, type 'X ^ 3 – 3 * X ^ 2 – 5 * X + 1' and press [ENTER]. If you made a mistake in entering the expression, simply highlight the EQ field, press [EDIT], and use the cursor keys and the [DEL] key to correct the entry. Use [OK] to insert the corrected expression into the EQ field. Press [ERASE] and [DRAW] to produce this plot:



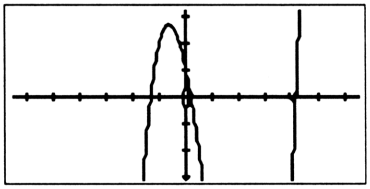
The lower right part of the plot is not visible, so to see more we will zoom out on the vertical axis but leave the x-axis unchanged. Open the **ZOOM** menu, then the **ZFACT** submenu. Set the **H-FACTOR** to 1, the **V-FACTOR** to 5, then press [OK]. Move to the next page and zoom out on the vertical axis with the **VZOUT** command. You will get the following plot:



Use the subtraction key [–] to remove the labels from the bottom of the screen that hide the plot; use [+] to put the labels back. To verify that we have expanded the height of the graphing screen by a factor of 5 activate the coordinate read-out with the menu key [(X, Y)], and use the [Δ] key to move the cursor up to the first tick mark on the *y*-axis. Notice that this tick mark records the zoom factor. The zoom factor 5 was determined by trial and error; a smaller factor failed to show the low point of the graph. Press [ON] to return to the stack display screen when you've finished.

Using the PLOT menu

Begin with 'X ^ 3 – 3 * X ^ 2 – 5 * X + 1' on level 1 of the stack and press [↰] [PLOT] to access the PLOT menu. Then use [↰] [EQ] to load the expression on level 1 into EQ and use [PPAR] to see the current plotting parameters. Since we wish to plot first with the default settings use [RESET] to set the plotting parameters to their default settings. Return to the previous page and open the PLOT menu, then use [ERASE], [DRAX] and [DRAW] to produce this plot:



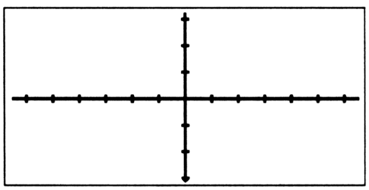
Now zoom out as before to see more of the local behavior.

The above two examples convey the major differences in using the PLOT screen ([↱] [PLOT]) and the PLOT menu ([↰] [PLOT]) to access the PLOT application. Using [↱] [PLOT] lets you interact with the main PLOT screen, and using [↰] [PLOT] provides direct access to the commands on the PLOT menu. Many beginners prefer to interact with the PLOT screen, but more experienced users tend to prefer the menu commands because of their speed and versatility. From here on, we shall leave the choice to you, the reader.

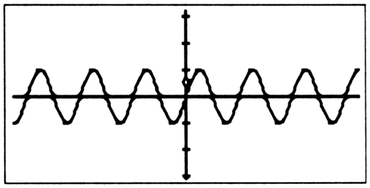
Zoom Operations

As EXAMPLE 2 shows, after producing a plot we may have to modify one or more of the plotting parameters in order to better see some portion of the plot. Here are two examples of plots that require adjustment on the range of x values:

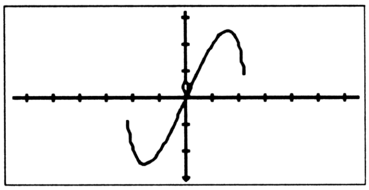
EXAMPLE 3. Graph $y = sin(10\pi x)$ on the default viewing screen. You will see:



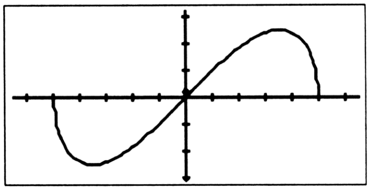
No plot appears. With the default plotting parameters the HP-48 calculates values of y for each of the 131 values of x from x = -6.5 to x = 6.5, .1 unit apart. Since 10π times each of these numbers is an integer multiple of π, the sine function is 0 at each of these values of x. Thus the plot lies along the x-axis. You can see this by turning off the axes and redrawing the plot. To get a better picture we can compress the viewing window in the x direction. Zoom in on the horizontal axis by a factor of 10 (set the **H-FACTOR**, then use [HZIN]) to see:



EXAMPLE 4. If you plot $y = x\sqrt{5 - x^2}$ with the default plot parameters, you will see:



Why does the plot fail to touch the x-axis? From the function, y is 0 when $x = \pm\sqrt{5}$, but these points do not show on the plot. To four decimal places, $\sqrt{5}$ = 2.2361. With the default plotting parameters, the HP-48 will plot a point for x = 2.2; but for $2.3 \leq x$, y is a complex number so no points will be plotted. We can "tie down" the plot to the x-axis by modifying the scale along the x-axis. For example, if we zoom in on x by a factor of 2.2361 we will be rescaling the x-axis so that 5 units on the x-axis is approximately $\sqrt{5}$, and will see the following plot:



The Zoom Menu

Several of the commands on the ZOOM menu are fairly self-evident:

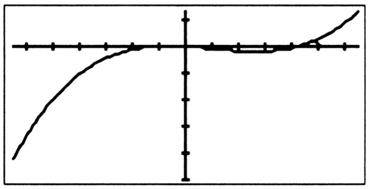
ZIN and ZOUT:	Zoom in or zoom out on both axes according to the ZOOM FACTORS.
HZIN and HZOUT:	Zoom in or zoom out on the horizontal axis according to the H-FACTOR.
VZIN and VZOUT:	Zoom in or zoom out on the vertical axis according to the V-FACTOR.
ZDFLT:	Zoom to the default plotting screen.
ZLAST:	Zoom to the last plotting screen.

But some of the other commands are not so obvious:

ZSQR:	Leaves *Xrng* unchanged but changes *Yrng* so that each pixel is square.
ZDECI:	Leaves *Yrng* unchanged but resets *Xrng* to its default state: -6.5 6.5 . Pixels are .1 unit along the horizontal axis.
ZINTG:	Leaves *Yrng* unchanged but sets *Xrng* to -65 65 so that each pixel along the horizontal axis is 1 unit.
ZTRIG:	Resets *Xrng* so that every 10 pixels equals $\pi/2$ units and resets *Yrng* so that every 10 pixels equals 1 unit.

ZAUTO: Leaves *Xrng* unchanged but rescales the vertical axis by sampling the expression in EQ at 40 equally spaced values across the x-axis plotting range, resets the *Yrng* to include the maximum and minimum sampled values, and then redraws the plot.

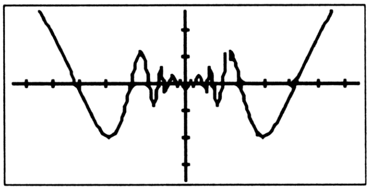
Caution: It is tempting for beginning users of the HP-48G/GX to use the ZAUTO command instead of adjusting the vertical display range in other ways. But *we urge restraint and caution in the use of ZAUTO* because it tends to excessively "flatten" a plot due to the narrow vertical dimension of the display screen. For instance, if we return to the function of EXAMPLE 2, $y = x^3 - 3x^2 - 5x + 1$, and apply the ZAUTO command to the plot obtained with the default plotting parameters, we obtain the following "flattened" plot:



Compare this with the plot we obtained by rescaling the vertical axis with a zoom-out factor of 5. Which would you prefer to see?

The BOXZ application on the ZOOM menu of the HP-48G/GX is especially helpful for zooming in on a particular region of a plot. The basic idea is to capture the region of interest within a small "box", then zoom in on the box. Here's an example.

EXAMPLE 5. Begin by plotting $y = x \sin \frac{1}{x}$ on the default screen. To better see what's happening near the origin, begin by opening the ZOOM menu. Now move the cursor 5 pixels to the left of the origin, then down 3 pixels and open BOXZ file. Now move the cursor 5 pixels to the right of the origin, then 3 pixels above the origin. Notice that the cursor drags a small box that has the origin as its center. Now press ZOOM to zoom in on the box, and obtain the following plot:



Repeat this zooming-in process with BOXZ by moving to a corner of a box 5 pixels to the left and 3 pixels below the origin, then moving to the diagonally opposite corner 5 pixels to the right and 3 pixels above the origin and pressing ZOOM. You should by now be ready to explain the behavior of $y = x \sin \frac{1}{x}$ as x approaches 0.

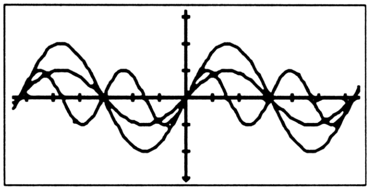
Superimposing Plots

To superimpose the plots of the graphs of two or more functions you can plot them individually without erasing. A better procedure is to put a list {F G H . . . etc.} of the functions $F, G, H, \ldots$ etc. to be graphed into EQ and set the calculator to *sequential plotting mode* (the default mode). When the DRAW command is executed, the functions in the list are plotted sequentially, left-to-right. The following example will illustrate this, both from the PLOT screen and the PLOT menu.

EXAMPLE 6. Superimpose plots of the graphs of $\sin x$, $2 \sin x$ and $\sin 2x$ on the same set of coordinate axes using the default parameters.

(a) **Using the PLOT menu.** Put 'SIN(X)', '2 * SIN(X)' and 'SIN(2 * X)' on the stack press the Δ key to engage the interactive stack. Then move the pointer to level 3 and press →LIST ENTER to build the list

{ 'SIN(X) '2 * SIN(X)' 'SIN(2 * X)' }. Now go to the PLOT menu and store this list into EQ. Open the FLAG menu on the second page of the PLOT menu and make sure that the middle menu key reads [SIMU]. If [SIMU■] appears, toggle off the key. Then return to the PLOT menu, reset the default plotting parameters, and press [ERASE], [DRAX] and [DRAW] to see the plots. Observe how the plots are drawn sequentially from the list.



To draw the plots in the list simultaneously instead of sequentially, go to the second page of the PLOT menu, open the FLAG submenu and toggle on the middle key to show [SIMU■].

(b) **Using the PLOT screen.** Open the PLOT screen and insert the list { 'SIN(X) '2 * SIN(X)' 'SIN(2 * X)' } into the EQ field (note that tick marks are required in the list). Open the OPTS (Options) submenu and make certain that there is no check mark (✔) in the SIMLT field. Return to the previous screen, set the default plotting parameters, then ERASE and DRAW. You should observe the list being plotted sequentially.

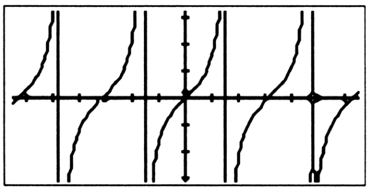
Disconnected Plots

All of our function plots so far have been done in *connected* mode, which means that any spaces between the pixels activated by the plot EQ were filled in with short line segments. But there are times when it is desirable to plot in disconnected mode, so that no filling in will be done. In connected mode, the HP-48 connects adjacent pixels with short line segments and sometimes extraneous lines can appear on the plot.

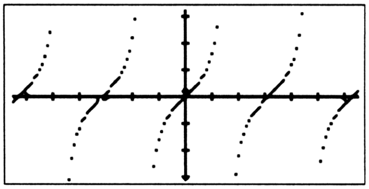
The choice between connected versus disconnected plotting modes is specified by system flag -31. When flag -31 is clear (the default state), plotting is done in connected mode. But if flag -31 is set, then plotting is done in disconnected mode.

If you are using the PLOT screen ([↱] [PLOT]), use the [OPTS] menu key to access the various PLOT OPTIONS. A check (✔) in the CONNECT field indicates that connected plotting mode is active. If you are using the PLOT menu ([↰] [PLOT]), go to the second page of the PLOT menu use the [FLAG] key to access the three flags that are pertinent to basic plotting (AXES, CNCT, and SIMU). If the second menu key shows [CNC□] then connected plotting mode is active. If the key reads [CNCT], then disconnected mode is active; simply press the key to change the status of the flag. Here is an example.

EXAMPLE 7. If you plot $y = tan\ x$ in connected mode using the default screen, you will see



Notice that this plot contains vertical lines that are not part of the graph of $y = tan\ x$. The vertical lines appear because the HP-48 connects adjacent plotted pixels. A graph of the tangent function without these extraneous lines is obtained by plotting in disconnected mode:



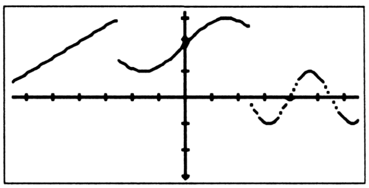
Although the disconnected plot is a little "dotty", it is nevertheless a better representation of the graph of $y = tan\ x$ than the plot above.

Piecewise Plots

Piecewise-defined functions are plotted by putting the defining IFTE expression into EQ and proceeding as usual. To get the plot shown in the next example, use the default viewing screen and the disconnected plotting mode; in the connected mode the calculator will connect the pixels on opposite sides of the two discontinuities and give you an inaccurate representation. To set the HP-48G/GX to plot in disconnected mode, go to the second page of the PLOT menu and open the FLAG submenu. Press the second white menu key so that CNCT appears in the second menu box.

EXAMPLE 8. To plot the graph of $f(x) = \begin{cases} .6x + 4.5 & x < -2.5 \\ 2 + sin\ x & -2.5 \leq x < 2.5 \\ -cos\ 2x & 2.5 \leq x \end{cases}$, use the expression

'IFTE (X < -2.5, .6 * X + 4.5, IFTE (X < 2.5, 2 + SIN(X), -COS(2 * X)))'. This gives the plot:



When you have finished, reset your calculator to plot in connected mode.

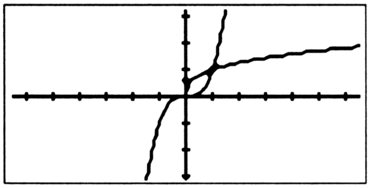
Plotting Inverse Functions

When a function f is one-to-one (different input values produce different output values), it has an inverse function f^{-1} that satisfies

$$f^{-1}(y) = x \quad \text{iff} \quad f(x) = y.$$

Whenever (x, y) is a point on the graph of f then (y, x) will be a point on the graph of f^{-1}. Thus, the graphs of f and f^{-1} will be reflections of one another across the line $y = x$.

To compare the graph of $f(x) = x^3$ with that of its inverse $g(x) = \sqrt[3]{x}$, you can begin by plotting the list { 'X ^ 3' 'X ^ (1/3)' }. Using the default settings you will see:



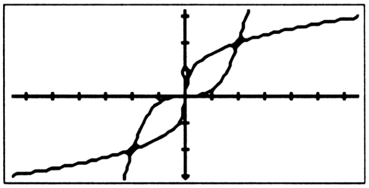
This fails to show the left branch of $g(x) = \sqrt[3]{x}$. The reason is that for each negative value of X, X ^ (1/3) is calculated as the *principal cube root* of x, a complex number, and so no pixel is activated. Although you may at first find this a bit disquieting, the ability of the HP-48G/GX to return complex values for odd roots and for natural logarithms of negative numbers is but one of the many features that makes the unit so appropriate for post-calculus mathematics.

To obtain *real* odd roots of negative numbers, use the XROOT command, given by the [$\sqrt[x]{y}$] key (the [↱] [$\sqrt{x}$] key). To see both branches of the graph of $g(x) = \sqrt[3]{x}$, we must plot the expression 'XROOT(3, X)'. There are two ways to enter this:

(i) Put 'X', then 3 on the stack and press [$\sqrt[x]{y}$].

(ii) Alternatively, go to the Equation Writer with [↰] [ENTER], and enter the expression $\sqrt[3]{x}$ with the keystrokes [↱] [$\sqrt{x}$] 3 [▷] [α] [1/x]. Press [ENTER] to convert this to the expression 'XROOT(3, X)' on stack level 1.

When the list {'X ^ 3' 'XROOT(3, X)'} is plotted with the default screen, then enlarged by a factor of 2 with the [ZOOM] menu, we see the following:



Notice that the original plot and its inverse meet on the line $y = x$ and that the plots are reflections across this line.

To help plot the graph of an inverse function, you can use the following program INV.F.[1] To use INV.F, begin by storing an expression for the original function f in EQ and drawing a "good" plot of EQ, i.e., a plot on which you wish to superimpose a plot of f^{-1}. Then execute INV.F. Since the program uses the expression stored in EQ and the plotting parameters from the reserved variable PPAR, make certain that you produce your original plot from the same user directory in which INV.F resides, preferably, your CALC directory. The program will redraw the original plot of f, overlay the line $y = x$, and then overdraw a plot of f^{-1}. In case f is not a one-to-one function, INV.F will overdraw the *inverse relation* for f.

[1] Thanks to William C. Wickes of Hewlett Packard for suggesting this version that uses parametric plotting.

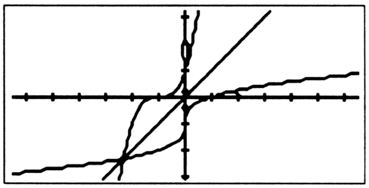
INV.F

Inputs: An expression for a function f, stored in EQ; and the desired plotting parameters, stored in PPAR.

Outputs: Draws, over the plot of $y = f(x)$, plots of the line $y = x$ and of the inverse relation f^{-1} to f.

```
« RCEQ PPAR → eq1 ppar1 « PARAMETRIC eq1 i * 'X' + 'X' i * 'X' +
eq1 'i*X' + 3 →LIST STEQ ERASE DRAX DRAW eq1 STEQ ppar1 'PPAR'
STO FUNCTION PICTURE » »
```

EXAMPLE 9. Plot $f(x) = (x + 1)^3$ with the default viewing window. To see its inverse, clear the graphing screen with [ON] and go to the VAR menu with [VAR]. Press [INV.F] to see plots of f, the line $y = x$, and f^{-1} drawn sequentially:



Parametric Curves

Not every curve in the xy-plane is the graph of a function. For example, a circle is not the graph of a function. More generally, imagine a point P moving in the xy-plane in such a way that its coordinates are given as functions of time t:

$$x = f(t) \text{ and } y = g(t).$$

We call t a *parameter* and call the curve that is traced by the moving point a *parametric curve*.

EXAMPLE 10. The coordinates of a moving point are given by

$$x = 2\,cos2t, \quad y = t - 3\,sin2t \quad \text{for} \quad 0 \le t \le 4.5.$$

Plot the curve and determine the location of the point at $t = 2$.

Using the Parametric Plot Form

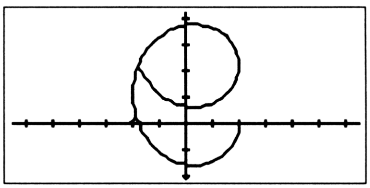
Access the PLOT screen with [↱] [PLOT]. If the plot type does not already read Parametric, open the CHOOS box and select Parametric. Set the angle display mode to Rad (for this example and any others that use trigonometric functions). Parametric plots require that the expression(s) for EQ appear as complex-valued functions: functions like

$$f(t) + i * g(t)$$

where $x = f(t)$ and $y = g(t)$ give the x- and y- coordinates of the moving point. Therefore, enter the following expression into the EQ field:

'2 * COS(2 * T) + i * (T – 3 * SIN(2 * T))'

Now set the independent variable (INDEP:) to 'T', set H-VIEW: -6.5 6.5, and set V-VIEW: -3 6. Open the OPTS submenu and set the independent variable to range from LO: 0 to HI: 4.5. Check AXES and CONNECT. Return to the previous screen with [OK] and ERASE and DRAW to see the following parametric plot. Note that it is traced in a clockwise direction:



To get the *approximate* location of the moving point when t = 2, trace clockwise along the plot with coordinate readouts active. Notice that the screen shows pixel coordinates as t varies. We can only get close to $t = 2$ with $t = 2.01$, and for this value of t the pixel coordinates of the point are approximately (-1.28, 4.31). We can determine the *exact* location of the point when $t = 2$ as follows.

Return to the stack display screen and access the SOLVE menu with [↰] [SOLVE]. Open the ROOT menu, then the SOLVR submenu. Now input the value 2 for [T] and press [EXPR=] to see

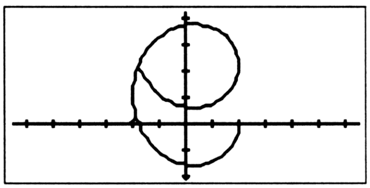
Expr: '-1.30728724173 + i * 4.27040748592.

Thus the exact coordinates of the point when $t = 2$ are

$$x = -1.30728724173 \text{ and } y = 4.27040748592.$$

Using the PLOT menu

Access the PLOT menu with [↰] [PLOT]. If necessary, open the PTYPE menu and press [PARA] to select PARAMETRIC plot mode. Enter '2 * COS(2 * T) + i * (T – 3 * SIN(2 * T))' onto stack level 1 and load it into EQ with [↰] [EQ]. Now open the PPAR submenu. Key in the expression { T 0 4.5 } and touch [INDEP] to specify the independent variable as T with a range from 0 to 4.5. Type -6.5 6.5 and touch [XRNG] to set the *Xrng*, then type -3 6 and touch [YRNG] to set the *Yrng*. Return to the previous page, open the PLOT submenu, and ERASE, DRAX, and DRAW the plot:



You can obtain the coordinates when $t = 2$ as above.

When plotting in PARAMETRIC mode, you are free to specify any variable as the independent variable, not just 'T'. Unless you specify a range of values for that variable, the HP-48G/GX will by default use the values specified by the *Xrng*; this is likely not to be the best choice.

You should recall that the ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has the parameterization

$$x = a \cos t, \quad y = b \sin t \quad \text{for} \quad 0 \le t \le 2\pi.$$

If we take $a = b$ then the circle $x^2 + y^2 = a^2$ has the parameterization

$$x = a \cos t, \quad y = a \sin t \quad \text{for} \quad 0 \le t \le 2\pi.$$

Of course, any function $y = f(x), a \le x \le b$ can be parameterized by

$$x = t, \quad y = f(t) \quad \text{for} \quad a \le t \le b.$$

We shall consider more exotic parametric curves in the activities.

Activity Set 2.2

1. (a) Plot the graph of $y = \frac{\sin x}{x}$ using the default plotting parameters.

   (b) ERASE and then plot the graph of $y = \frac{\sin(-2x)}{x}$.

   (c) ERASE and then plot the graph of $y = \frac{\cos x - 1}{x}$.

2. (a) Plot the list { 'SIN(4 * X)' '-2*SIN(X)' } using the default plotting parameters.

   (b) ERASE and plot the sum of the two functions in the list.

   (c) Overdraw your plot in (b) with the plot of $y = -2 \sin x$.

   (d) ERASE and plot the product of the two functions in the list.

   (e) Overdraw your plot in (d) with the plot of $y = -2 \sin x$.

3. Plot $y = cos(10\pi x)$ on the default plotting screen. Why does the plot look this way? Adjust the screen to make the plot look more like a cosine curve.

4. Set your calculator to degree mode and plot $y = sin(x°)$ using the default screen. Without changing back to radian mode, zoom on X to make the plot look like $sin\ x$, x in radians. When you're done, reset to radian mode.

5. Graph $y = x^3 - 1.3x^2 + .32x - .02$ using the default plotting screen. Examine the behavior of this function near the origin by using **BOXZ** several times.

6. To appreciate how "steep" are the graphs of simple polynomial functions, begin by plotting $y = 34x^3 - 91x^2 - 117x + 54$ on the default screen. Now zoom out along the y-axis as necessary until you can see all the high points and low points (*local extreme points*).

7. Plot $y = cos\ (cos^{-1}x)$ using the default screen. Is the plot what you expected? Now **ERASE** and plot $cos^{-1}(cos\ x)$. Can you explain what you see?

8. Investigate, graphically, the following limit:

$$\lim_{x \to 0} \left(\frac{x + |1 - \sqrt{x+1}|}{|1 - \sqrt{x+1}|} \right)$$

(see Activity 6 in Activity Set 2.1):

9. Graphically investigate the behavior of $f(x) = sin\left(\frac{1}{x}\right)$ near $x = 0$. Begin by plotting on the default screen, then use **BOXZ**. What is your conclusion?

10. (a) Plot $y = \begin{cases} -x & x < 0 \\ sin\ x & 0 \le x < \pi \\ x - \pi & \pi \le x \end{cases}$, using the default plotting screen.

(b) Recall **EQ** to the stack, change it's sign with [+/−] and then overdraw the original plot with this expression.

11. Plot $y = x\sqrt{3 - x^2}$. Adjust the viewing screen to make the plot touch the x-axis at the end points of the domain.

12. Graph $y = x^3 - 9x^2 + 2x + 48$ with the default plotting screen, then zoom out on the vertical axis by a factor of 16 to see the local maximum. Now move the cursor to the point (4, 0), open the **ZOOM** menu and press the menu key CNTR on the second page to relocate the center of the viewing window. You may want to remove the menu key labels to see the local minimum. When you have finished, use **ZLAST** to zoom to the last screen. When the plot is done, use **ZLAST** again.

13. (a) Plot $y = x^2 + \frac{4}{x}$ on the default plotting screen, then zoom out on the vertical axis by a factor of 4. Use **TRACE** to approximate the local minimum value to the right of the origin.

(b) Plot $y = \frac{x^3 - 1}{x - 1}$ on the default plotting screen, then relocate the center of the viewing rectangle at (0, 2). Where is the "hole" in the graph?

14. (a) Use the default plotting screen to plot $f(x) = 2x - 3$, then use the **INV.F** program to plot f^{-1}.

(b) Write an equation for f^{-1}.

(c) **ERASE**, then plot $g(x) = -.6x + 1$ and its inverse. When you've finished, write an equation for g^{-1}.

(d) What is your observation about the slopes of non-parallel lines that are symmetric to the line $y = x$? Prove it.

(e) Is the converse to your observation true?

15. Let $u(x) = x^2 + x + 1$ and $v(x) = \sin x$.

    (a) Plot the composite function $f(x) = u[v(x)]$ on the default plotting screen and compare it with the graph of $v(x)$.

    (b) Now ERASE, plot the composite function $g(x) = v[u(x)]$ on the default plotting screen and compare it with the plot of $u(x)$.

16. Use the default plotting parameters to graph

    (a) $y = x^{2/3}$ (b) $y = 3(x-2)^{2/3} + 1$

17. (a) Use the XROOT command to plot $y = 2\,(x+2)^{2/3} + \dfrac{x-4}{x^2+1}$. Use the default plotting screen.

    (b) Zoom in on both axes by a factor of .6. Trace to obtain an approximation to the local maximum to the left of the origin.

    (c) Now trace to find the approximate location of the local minimum that is nearest to the origin; with the cursor resting at that point, press ENTER to record the coordinates on the stack.

18. The HP-48G/GX command IP will return the integer part of any real number on the stack. Thus, to determine whether a real number X is an integer, we need only test X against IP(X): X is an integer iff X is the same as IP(X). The syntax to test X against IP(X) is 'X == IP(X)', and you can find the == command on the PRG TEST menu. Use these ideas to graph each of the following functions on the default plotting screen.

    (a) $f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ x^2 + 2x - 1 & \text{if } x \text{ is not an integer} \end{cases}$

    (b) $g(x) = \begin{cases} 1 - x & \text{if } x \text{ is an integer} \\ 1 + x & \text{if } x \text{ is not an integer} \end{cases}$

19. Plot the parametric curve traced by a point P moving in such a way that the coordinates are given by the equations

$$x = t - 2 \sin 3t, \quad y = 2 \cos 2t \quad \text{for} \quad 0 \le t \le 6.3.$$

Give the exact location of P when $t = 3$. Use *Xrng*: -3.5 9.5 and *Yrng*: -2 2.

In activities 20 - 25, draw a plot of the indicated parametric curves on the default plotting screen. Go to the **MTH CONS** menu to get a 12-digit approximation for π.

20. $x = 4 \cos t, \quad y = 2 \sin t \quad \text{for} \quad 0 \le t \le 2\pi$

21. $x = 3 \cos t + 2 \cos 3t, \quad y = 3 \sin t - 2 \sin 3t \quad \text{for} \quad 0 \le t \le 2\pi$

22. $x = 2 \cos 3t, \quad y = \sin 7t \quad \text{for} \quad 0 \le t \le 2\pi$
(When done, zoom in using **ZOOM** factors of 2.)

23. $x = 3 \cos^3 t, \quad y = 3 \sin^3 t \quad \text{for} \quad 0 \le t \le 2\pi$

24. $x = \sec t, \quad y = \tan t \quad \text{for} \quad 0 \le t \le 2\pi$

25. $x = 2 \cos t - 1.5 \cos 3t, \quad y = 2 \sin t - 1.5 \sin 3t \quad \text{for} \quad 0 \le t \le 2\pi$

3 DERIVATIVES

Elementary calculus is concerned with the mathematics of continuous change. For example, given a function f whose graph is smooth, the rate of change of f at a point $P = (x, f(x))$ on the graph is given by the intuitive notion of the *slope of the graph* at point P. Calculus provides us with a precise mathematical meaning for this intuitive notion by defining the derivative $f'(x)$ of f at x, then declaring the slope of the graph at P to be the derivative.

3.1 APPROXIMATING SLOPES

Difference Quotients

Given a function f, the *derivative* of f is the function f' given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The geometry is clear enough. The difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

appearing in the definition of f' is the slope of the secant line joining the point $(x, f(x))$ on the graph of f with some nearby point $(x + h, f(x + h))$ on the graph. Thus the derivative can be viewed geometrically as the limiting position of the slopes of nearby secant lines. For a given x, we can approximate $f'(x)$ numerically by evaluating the difference quotient $\frac{f(x+h) - f(x)}{h}$ for suitably small values of h.

A simple way to do this on the HP-48G/GX is to evaluate a user-defined function for the difference quotient:

$$DQ(X,H) = \frac{F(X + H) - F(X)}{H}$$

This procedure requires that we also build a user-defined function F for the given function f.

To illustrate, consider the function $f(x) = (x^2 + 5)^3$. We create a user-defined function F for f: « → X ' (X ^ 2 + 5)^3 ' »; and another, DQ, for the difference quotient: « → X H ' (F(X + H) − F(X))/H ' ». To approximate $f'(2)$, we simply evaluate DQ using input values (2, H) for varying values of H.

H	DQ(2, H)
.001	972.67528
.0001	972.0675
.00001	972.0067
.000001	972
-.001	971.32528
-.0001	971.9325
-.00001	971.9933
-.000001	972

This numerical investigation should convince you that $f'(2) = 972$.

However, you must exercise caution with the numerical computation of difference quotients because they are susceptible to serious *cancellation error* with the finite precision arithmetic used in any machine computation. For example, consider the function

$$f(x) = \frac{\sqrt[3]{1 + cos^2 x}}{x^3}.$$

If you build a user-defined function for f and then evaluate DQ for the following input values (1, H) you will obtain these results:

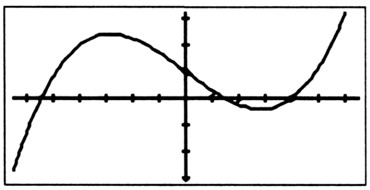
H	DQ(1, H)
10^{-4}	-3.5221718
10^{-5}	-3.522835
10^{-6}	-3.52291
10^{-7}	-3.523

The correct value is $f'(1) = -3.5229074056$, so you can see that we are losing digits with each successive evaluation of the difference quotient.

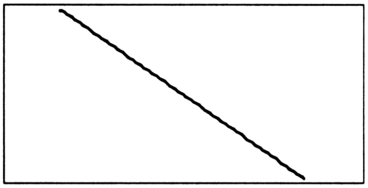
Slopes by Zooming

Because there is a strong element of geometry underlying the definition of the derivative, it is not surprising that graphical investigations can often help in building an understanding of the concepts that surround derivatives. By zooming in on a graph, we can often "see" the slope at a point.

EXAMPLE 1. We wish to "see" the slope of the graph of $f(x) = 2x^3 - 3x + 1$ at the point (-.2, 1.584). Begin by drawing a plot of the graph of $f(x) = 2x^3 - 3x + 1$ on the default screen, then zoom in on the horizontal axis by a factor of 4 to obtain a better view:



Activate TRACE and the coordinate readout (X, Y), trace to the point on the curve where $x = -.2$, reset both zoom factors to 100 with recentering at the crosshairs, and then zoom in. Again, with the cursor resting on the curve at $x = -.2$, zoom in to see the following approximation to the tangent line at $x = -.2$:



To calculate the slope of this line, we choose two points on the line. Trace left to the point P where $x = .20001$ and press **ENTER** to record the precise coordinates on the stack. Then trace right to the point Q where $x = -.19999$ and use **ENTER** to record the precise location. Return to the stack with **ON** and press [−] to calculate the ordered pair $(\Delta x, \Delta y)$, where Δx and Δy are the differences in the x and y coordinates of P and Q, respectively. Use the **C→R** (complex into real) command on the **MTH CMPL** menu to put Δx and Δy on the stack, then **SWAP** and divide to obtain the approximation $\frac{\Delta y}{\Delta x} = -2.76$ to the slope of the curve at $x = -.2$. In this case, we are very accurate: the slope of the curve at $x = -.2$ *is* -2.76.

In the activities that follow, we will use this zooming technique to investigate the slopes of several important functions.

Activity Set 3.1

1. Can you see the slope of $y = \sin x$ at (0, 0)?

   (a) Draw a plot of $y = \sin x$ using the default screen, then zoom in by factors of 100 three times.

   (b) Trace left to the point P on the curve where $x = -.000001$ and record the coordinates on the stack with [ENTER], then trace right to the point Q on the curve where $x = .000001$ and record the coordinate.

(c) Use P and Q to calculate your approximation $\frac{\Delta y}{\Delta x}$ to the slope as in EXAMPLE 1.

(d) Find the slope of $y = \sin x$ at (0, 0) by evaluating difference quotients.

(e) What is the slope of $y = \sin x$ at (0, 0)? Express your answer in mathematical terms as the formal limit of a difference quotient involving the *Sine* function.

2. Can you see the slope of $y = \cos x$ at (0, 0)?

(a) Draw a plot of $y = \cos x$ using the default screen, then zoom in on the horizontal axis by a factor of 100 (no vertical axis zoom). Zoom in again on the horizontal axis by a factor of 100 (no vertical zoom). With the coordinate readout active, trace along the curve to determine its slope at (0, 0).

(b) Find the slope of $y = \cos x$ at (0, 0) by evaluating difference quotients.

(c) Express the slope of $y = \cos x$ at (0, 0) in mathematical terms as the formal limit of a difference quotient involving the *Cosine* function.

3. Use zoom in (with the same horizontal and vertical factors) to estimate the slope of each of the following functions at the point where $x = 1$. Be sure to check (✓) **RECENTER AT CROSSHAIRS** to keep the zooming region centered on the screen, and always make sure that the cursor is resting on the curve at the desired point.

(a) $y = x^{-2/3}$ (b) $y = \frac{1}{x^2} - 2$ (c) $y = \sin(x^2 - 1)$ (d) $y = \sin e^x$

4. Use difference quotients to estimate the following slopes:

   (a) the slope of $y = \sqrt{3x-2}$ at $x = 2$

   (b) the slope of $y = \dfrac{2x}{x^3 - 1}$ at $x = -1$

3.2 DERIVATIVES WITH THE HP-48

It is important that students learn the basic mechanics of finding derivatives without their calculators. However, there are times when it is perfectly natural to use the calculator to take derivatives; for example, when we want to plot a function and its first two derivatives, and then find the roots. Since the plotting and root-finding will be done on the HP-48, we may as well do the differentiation process there also.

The Derivative Function ∂

The HP-48 uses the derivative function ∂ ([∂] is the right-shifted [SIN] key) to perform symbolic differentiation. The differentiation can be executed all at once or in step-by-step fashion following the chain rule. In either case, you must specify the expression that is to be differentiated and also the variable of differentiation. In order to obtain symbolic results, the HP-48 must be set to display symbolic results (the default state) and no numerical value should be stored for the variable of differentiation.

Using the Stack

To perform symbolic differentiation on the stack all at once, the two inputs are specified on the stack:

level 1: 'the expression to be differentiated'

level 2: 'the variable of differentiation'

Then execute the ∂ command with [∂].

EXAMPLE 2. To differentiate $f(x) = sin\ x^2$, arrange the stack as follows:

level 2: 'SIN(X ^ 2)'

level 1: 'X'

then press the [∂] key to see the symbolic result

level 1: 'COS(X ^ 2) * (2 * X)'.

Recall that the chain rule says (in mixed notation) that

$$\frac{d}{dx} f[g(x)] = f'[g(x)]\, g'(x).$$

To perform symbolic differentiation of $f(x) = sin\ x^2$ on the stack in step-by-step fashion following the chain rule, begin with the expression '∂X(SIN(X ^ 2))' on level 1:

level 1: '∂X(SIN(X ^ 2))'.

Press [EVAL] to perform one step of the differentiation and obtain:

level 1: 'COS(X ^ 2) * ∂X(X ^ 2)'.

Press [EVAL] again the perform the second step:

level 1: 'COS(X ^ 2) * (∂X(X) * 2 * X^(2 – 1))'.

Finally, press [EVAL] again to execute the final step:

level 1: 'COS(X ^ 2) * (2 * X)'.

As an alternative to keying in '∂X(SIN(X ^ 2))' directly to stack level 1, you can use the *Equation Writer*. The Equation Writer is an environment that enables you to enter mathematical expressions and text in much the same way they are written by hand. Activate the Equation Writer with [↰] [ENTER], then use the [∂] key to begin the expression. When you see the form

$$\frac{\partial}{\partial \square}$$

respond by entering X and then press the [▷] key to move away from the denominator and obtain

$$\frac{\partial}{\partial x}(\square).$$

Now enter SIN and then X to see

$$\frac{\partial}{\partial x}(\text{SIN}(\text{X}\square\ .$$

Close the two parentheses with [▷] [▷] to obtain

$$\frac{\partial}{\partial x}(\text{SIN}(\text{X}))\ \square.$$

Our use of the [▷] key in this illustration is typical: in the Equation Writer, the [▷] key is used to complete any *subexpression* and move on to the next part. Press [ENTER] to convert the expression into '∂X(SIN(X))' on level 1.

Using the Symbolic Differentiate Screen

Go to the Symbolic application with [↱] [SYMBOLIC], highlight D*ifferentiate* and press [OK]. When the Differentiate screen appears, enter 'SIN(X ^ 2)' into the EXPR field and 'X' into the VAR field. With the result type specified as Symbolic press [OK] to see the result returned all at once to stack level 1:

1: 'COS(X ^ 2) * (2 * X)'.

To obtain the symbolic derivative in step-by-step fashion, return to the Symbolic application and select Differentiate as before. After entering 'X' into the VAR field, press [STEP]. The result of the first step of the differentiation process will appear on level 1:

1: 'COS(X ^ 2) * ∂X(X ^ 2)'.

As before, each succeeding press of the EVAL key will perform another step of the differentiation.

EXAMPLE 3. If you put your calculator in degree mode and take the derivative of a trigonometric function, say $f(x) = \sin x$, you will see 'COS(X) * (π/180)'. Why the factor $\pi/180$? There are several ways to explain this.

When x is measured in radians, we know that $\frac{d}{dx}(\sin x) = \cos x$. Thus by the chain rule, we have $\frac{d}{dx}[\sin(x°)] = \frac{d}{dx}[\sin(\frac{\pi}{180}x)] = \cos(\frac{\pi}{180}x)\frac{d}{dx}(\frac{\pi}{180}x) = \cos(x°)(\frac{\pi}{180})$.

For an explanation at the more fundamental level, recall the derivation of the derivative of the sine function:

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{(\sin x\cos h+\cos x\sin h)-\sin x}{h}\\ &= \lim_{h\to 0}\left(\sin x\cdot\frac{\cos h-1}{h}\right)+\lim_{h\to 0}\left(\cos x\cdot\frac{\sin h}{h}\right)\\ &= \sin x\cdot\left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)+\cos x\cdot\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\end{aligned}$$

Thus, the result depends upon the two limits

$$\lim_{h\to 0}\frac{\cos h-1}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h}.$$

Whether h is measured in radians or degrees, we have

$$\lim_{h\to 0}\frac{\cos h-1}{h} = 0.$$

When h is measured in radians, we have seen (see Activity 1 in Section 3.1) that

$$\lim_{h\to 0}\frac{\sin h}{h} = 1.$$

But when h is measured in degrees,

$$\lim_{h \to 0} \frac{\sin h}{h} = \frac{\pi}{180}$$

(see Activity 1, Section 3.2).

Thus, using degree measure we have

$$\frac{d}{dx}(\sin x) = \sin x \cdot \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) + \cos x \cdot \left(\lim_{h \to 0} \frac{\sin h}{h} \right)$$

$$= \sin x \cdot 0 + \cos x \cdot \frac{\pi}{180}$$

$$= \cos x \cdot \frac{\pi}{180} .$$

Differentiating the XROOT Function

Although the XROOT function is built into the HP-48G/GX, its derivative is *not*. You can, however, differentiate XROOT if you have the following program stored in your HOME directory. It is important that the name *derXROOT* use lowercase letters for *d*, *e*, and *r* followed by XROOT in uppercase letters because this is the syntax recognized by the HP-48's differentiation routine. (*Note*: to obtain lowercase alphabetical characters, use [α] [↰] [D], [α] [↰] [E], etc.)

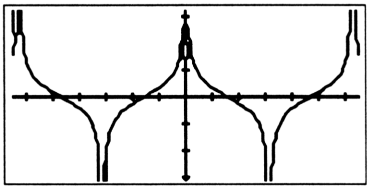
derXROOT

Input: 'XROOT(N, F(X)) on level 1, where $N > 0$ is an integer

Effect: returns $\frac{d}{dx}\left(\sqrt[N]{F(X)} \right)$ on level 1

```
« → n w y z 'INV(n) * XROOT(n, w) ^ (1 – n) * z'»
```

EXAMPLE 4. Plot the derivative of $f(x) = \sqrt[3]{5 \sin x}$ on the default screen. With the program derXROOT in the HOME directory of your 48G/GX, put 'XROOT(3, 5 * SIN(X))' on the stack, enter 'X' and press [∂] to see the derivative '.333333333333 * XROOT(3, 5 * SIN(X))^ – 2 * (5 * COS(X))'. Now plot on the default screen to see



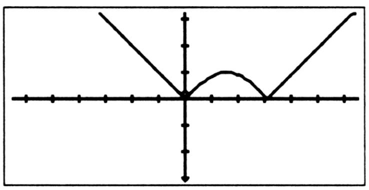
Notice that the derivative is not defined at the values $x = n\pi, n = 0, \pm1, \dots$.

Piecewise Differentiation

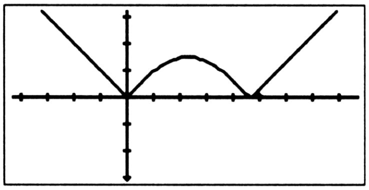
Although the HP-48G/GX will not completely symbolically differentiate a function defined with the IFTE command, it will correctly plot the derivative. Here is an example.

EXAMPLE 5. Find the derivative of $f(x) = \begin{cases} -x & x < 0 \\ \sin x & 0 \le x < \pi \\ x - \pi & \pi \le x \end{cases}$ and then plot both f and its derivative.

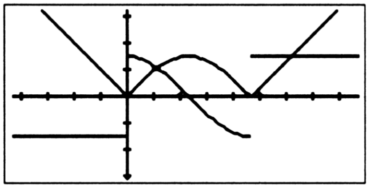
Put two copies of 'IFTE(X < 0, -X, IFTE(X < π, SIN(X), X – π))' on the stack and graph in *disconnected* mode with the default parameters to see



For greater clarity, especially after we overdraw f', trace along the curve to the point where $x = 1.5$, open the ZOOM menu and press [CNTR]. The plot will be redrawn with the point you choose as center. Now **ZOOM** in on both axes by .67 to see



Press [ON] to return to the stack, put 'X' on level 1 and press [∂] to differentiate. This gives 'IFTE(X < 0, ∂X(-X), ∂X(IFTE(X < π, SIN(X), X – π)))'. Notice that the differentiation is not complete. Use of [EVAL] does not change the expression. However, if we plot f' without erasing we see the plot of the derivative superimposed on the plot of f:



Notice that f has local minima at values of x where $f'(x)$ does not exist.

Implicit Differentiation

Implicit differentiation is a technique that is used to obtain the derivative $y' = \frac{dy}{dx}$ when y is *implicitly* defined as a function of x.

For example, the equation $y^3 - xy^2 - y = 5$ implicitly defines y as a function of x. To use implicit differentiation, we think of y as an implicit function of x and apply the chain rule to differentiate both sides of the equation. The resulting equation can then be solved, if necessary for y'.

To implicitly differentiate $y^3 - xy^2 - y = 5$ we proceed as follows:

(i) $3y^2y' - (y^2 + x\,2yy') - y' = 0$ (apply chain rule)

(ii) $(3y^2 - 2xy - 1)y' - y^2 = 0$ (algebra)

(iii) $y' = \dfrac{y^2}{3y^2 - 2xy - 1}$ (solve for y')

The HP-48G/GX cannot remember that y is an implicit function of x. Instead, we must specify that Y depends upon X by using $Y(X)$ instead of simply Y. When the calculator takes the derivative, the symbolic derivative of $Y(X)$ will appear as the expression derY(X, 1). Try it. Put 'Y(X) ^ 2' on level 2, and 'X' on level 1 and press [∂]. You will see 'derY(X, 1) * 2 * Y(X)' returned to level 1, the calculator's version of

$$\frac{d}{dx}(y^2) = 2yy'.$$

To avoid having to type $Y(X)$ in place of Y, and to make the result appear more like what we are accustomed to writing, we can use a short calculator program. The program given below does the following:

- replaces Y with $Y(X)$;
- takes the derivative with respect to X; then
- replaces $Y(X)$ with Y and $derY(X, 1)$ with y' in the resulting expression

IM.y'

Input: level 1: an expression involving X and Y

Effect: differentiates the expression on level 1 with respect to X; returns an expression for the derivative that uses X, Y and y'.

```
« { Y 'Y(X)' } ↑MATCH DROP 'X' ∂ { 'Y(X)' Y } ↑MATCH DROP
  { 'derY(X, 1)' y' } ↑MATCH DROP »
```

Thus, with 'X ^ 2 + Y ^ 2' on level 1, program IM.y' returns '2 * X + y' * 2 * Y'.

EXAMPLE 6. To implicitly differentiate the equation $y^3 - xy^2 - y = 5$ on the HP-48G/GX, put the equation 'Y ^ 3 – X * Y ^ 2 – Y = 5' on level 1 and run program IM.y' to see:

' y' * 3 * Y ^ 2 – (Y ^ 2 + X * (y' * 2 * Y)) – y' = 0'

This is the calculator's version of equation (i) above. You can now isolate y' and then solve for y' as in equations (ii) - (iii).

An alternative to the above way of performing implicit differentiation on the HP-48G/GX, we can use a more advanced result that relates implicit differentiation to partial derivatives. Given a function of two independent variables, say $F(x, y)$, the *partial derivative* F_X *with respect to* x is obtained by regarding y as a constant and taking the derivative with respect to x. For the function $F(x, y) = y^3 - xy^2 - y - 5$ the partial derivative with respect to x is $F_x = -y^2$. Similarly, we obtain the *partial derivative* F_y *with respect to* y by regarding x as a constant and differentiating with respect to y: $F_y = 3y^2 - 2xy - 1$. The following result relates implicit differentiation to partial derivatives:

If the equation $F(x, y) = 0$ *defines* y *as a differentiable function of* x *, then at any point where* $F_y \neq 0$ *we have*

$$\frac{dy}{dx} = \frac{-F_x}{F_y}.$$

Using this result we see that for the example $F(x, y) = y^3 - xy^2 - y - 5$ we have

$$y' = \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{y^2}{3y^2 - 2xy - 1}$$

which agrees with our earlier calculation in (iii). Given an equation $F(x, y) = 0$ that implicitly defines y as a function of x, the following program takes as input the algebraic expression $F(x, y)$ and returns the result $y' = \frac{-F_x}{F_y}$.

y'

Input: level 1: an algebraic expression in terms of X and Y representing $F(x, y)$

Effect: returns to level 1 an algebraic expression of the form $y' = \frac{-Fx}{Fy}$ for the derivative

```
« { X Y } PURGE DUP 'X' ∂ SWAP 'Y' ∂ NEG / 'y' →TAG »
```

EXAMPLE 7. To obtain the derivative y' for the function y of x that is implicitly defined by the equation $y^3 - xy^2 - y - 5 = 0$, put 'Y ^ 3 – X * Y ^ 2 – Y – 5' on level 1 and press [Y'] to see the following result returned to level 1:

1: y': ' - (Y ^ 2) / – (3 * Y ^ 2 – X * (2 * Y) – 1))'

Compare this to our expression in (iii) above.

As the above example illustrates, implicit differentiation usually results in an expression for the derivative y' in terms of x and y, say $y' = G(x, y)$. To evaluate the two-variable function $G(x, y)$ at a particular point (a, b), we can use the following program F.XY.

F.XY

Inputs: level 3: an algebraic expression *F(x, y)* in variables *X* and *Y*

level 2: a real number "*a*"

level 3: a real number "*b*"

Effect: returns the number *F(a, b)* to level 1 and the original expression *F(x, y)* to level 2

```
« 'Y' STO 'X' STO DUP EVAL { X Y } PURGE »
```

EXAMPLE 8. To find the derivatives $y'(-1, 2)$ and $y'(3, -4)$ of the function y implicitly defined by the equation $y^3 - xy^2 - y = 5$, at the points (-1, 2) and (3, -4), put 'Y ^ 3 – X * Y ^ 2 – Y – 5' on level 1 and press [Y'] to obtain the symbolic derivative y'; '-(Y ^ 2 / – (3 * Y ^ 2 – X * (2 * Y) – 1))' on level 1. now press 1 [+/–] [SPC] 2 [F.XY] to see the derivative y'(-1, 2) = 2.666666666667 on level 1 and the original symbolic derivative on level 2. Now SWAP levels 1 and 2 and use 3 [SPC] 4 [+/–] [F.XY] to see $y'(3, -4)$ = .225352112676 on level 1.

Activity Set 3.2

1. (a) Calculate a full precision decimal approximation to $\pi/180$.

   (b) Set your calculator to DEGREE mode. Use the SOLVR to numerically investigate $\lim_{x \to 0} \frac{\sin x}{x}$ in degree mode. Complete the table:

X	SIN(X)/X
.01	
.001	
.0001	
-.01	
-.001	
-.0001	

(c) Keep your calculator in degree mode and do a graphical investigation of $\lim_{x \to 0} \frac{\sin x}{x}$, as follows.

- Plot 'SIN(X)/X' using the default plotting parameters. What do you see? We need to zoom in.

- Put the full precision decimal approximation to $\pi/180$ on level 1. Go to the PLOT menu with [↰] [PLOT], open [PPAR], go to the next page and press [*H] to rescale the vertical axis so that each tick mark represents $\pi/180$. Use [NXT] [PLOT] and [DRAW] to redraw your plot of 'SIN(X)/X'. What is $\lim_{x \to 0} \frac{\sin x}{x}$ in degree mode?

2. For each of the following functions, find the derivative by hand calculation. As a check on your result, use the HP-48G/GX to calculate the symbolic derivative in a single step as follows:

   (i) using the stack

   (ii) using the Symbolic Differentiate Screen.

(a) $y = \dfrac{x}{x^2 + 1}$ (b) $y = \cos\sqrt{x^2 + 1}$

(c) $y = \sqrt[3]{\sin^2 x}$ (d) $y = e^{-x/3}\sin(2x)$

3. (a) Plot $y = \sqrt[3]{\sin^2 x}$ on the default screen.

   (b) Recall **EQ** to the stack, take its derivative with the HP-48 and then overdraw the plot from (a) with a plot of the derivative.

   (c) For what values of x is the derivative undefined? What can you say about the function at these values?

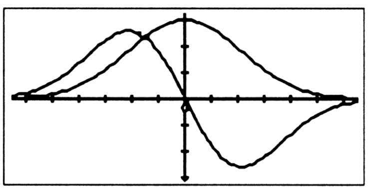
4. (a) Plot $y = \sin|x|$ on the default screen. (Use **ABS(X)** for $|x|$.) By examining the plot, can you tell where the derivative will not be defined?

   (b) Use the HP-48G/GX to take the derivative. The term **SIGN(X)** is interpreted as follows:

$$SIGN(X) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

   Overdraw your plot in (a) with a plot of the derivative. Where is the derivative not defined?

5. The following plot shows two graphs, a function f and its derivative f'. Which plot is f and which is f'?



6. (a) Draw, on the default screen, a plot of $y = \frac{3}{2} \tan^{-1} 2x$. Try hard to visualize a plot of the derivative.

   (b) Confirm (or refute) your visualization efforts by overdrawing a plot of the derivative.

7. Each of the following equations implicitly defines y as a function of x. Use program IM.y' to implicitly differentiate the equation, then solve for the derivative y' by hand.

   (a) $3x^2 - 9y^3 = 17$ (d) $y^3 - \sqrt{x} + \cos xy^2 = 8$

   (b) $x(y^2 + 5x) = 9$ (e) $x^{2/3} + y^{3/2} = 7$

   (c) $xe^y + \sin x^2y - y^2 = 1$ (f) $\cos x = \sin^2 y$

8. Use program y' to obtain the derivative $\frac{dy}{dx}$ of each of the implicitly defined functions in Activity 7, then use program F.XY to evaluate the derivative at the indicated point.

   (a) (5, -3) (c) $(3, \pi/2)$ (e) (8, 9)

   (b) (-2, -3) (d) $(\pi, 2)$ (f) $(\pi/4, \pi/4)$

3.3 USING THE DERIVATIVE

Maxima, Minima and Inflection Points

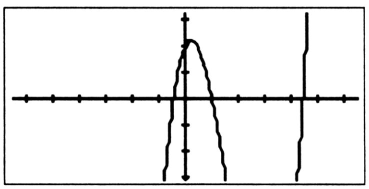
The derivative of a function is the source of considerable information about the behavior of the graph of the function. It can tell us where the graph of the function is increasing and decreasing, help pinpoint the location of local maximum and minimum values on the graph, and show where the graph is concave up and concave down. It is thus advantageous to consider functions and their derivatives from the very beginning of a study of calculus.

A plot of the graph of a function produced on a calculator's screen can often provide valuable information about the behavior of the function. When graphical techniques are carefully combined with an understanding of the derivative as a rate of change, we have a powerful tool for analyzing a function's behavior in considerable detail.

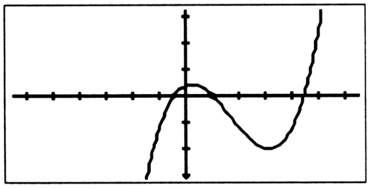
When the DRAW command is executed and the HP-48G/GX draws a plot of the graph of a function, the calculator enters the PICTURE environment and displays the PICTURE menu. In addition to the zoom operations accessible through the ZOOM submenu, the FCN (= function) submenu contains a number of commands that are helpful in analyzing a function's behavior with calculus without leaving the PICTURE environment. Commands such as ROOT (to find roots of equations), ISECT (for finding intersections of curves), SLOPE (for the slope of a graph), AREA (for calculating areas of regions beneath curves), EXTR (for finding extreme points (i.e., local maxima or minima) on curves), F' (to calculate and plot the derivative of a function), and TANL (to plot the line tangent to a curve at a point).

EXAMPLE 9. Find the x-intercepts, local maxima and minima, and any inflection points for the function $f(x) = x^3 - 5x^2 + 2x + 2$.

Use the default parameters in connected mode to obtain the following plot:



Since the plot goes off screen, we zoom out on the vertical axis. A zoom factor of 5 gives the plot:



This plot shows all of the graph between $x = -2$ and $x = 5$. Before proceeding, we pause to consider what calculus tells us about the graph of this function.

The function is a cubic polynomial, so has at most three real roots. Since we see the plot crossing the x-axis three times, all the x-intercepts are displayed. The derivative is a second-degree polynomial and thus has at most two real roots. So the graph of f can have at most two local extreme points, and because we see a high point and a low point, we certainly have all the local extrema displayed.

The second derivative is a nonconstant linear function having one real root, so the graph of f has only one inflection point. Since the graph is concave down at the origin and concave up near $x = 3$, the inflection point lies between these two points.

With the plot still displayed open the **FCN** menu.

To find the x intercepts: Move the cursor to the point to the left of 0 where the plot appears to cross the x-axis and press [ROOT]. You will see a twelve digit approximation to this root displayed at the bottom of the screen:

ROOT: -.449489742783.

This has also been entered onto the stack. Go to the stack and you will see:

1: Root: -.449489742783.

Now return to the graph with [PICTURE] (the [◁] key) and find the other two roots in the same way. When you're done, return to the stack to see all three roots. When you find a root of a function in this way, the HP-48 uses the

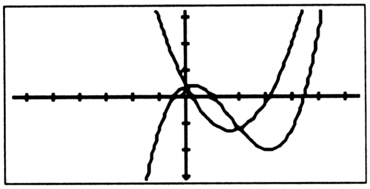
x-coordinate of the cursor as a first approximation for its own ROOT program to numerically approximate the root.

To find the coordinates of the local extrema: Move the cursor to the apparent high point of the graph located above the x-axis just to the right of the origin and press [EXTR]. The message EXTRM: (.213700352153, 2.2088207353) appears below the graph. This point was also entered on the stack. Now move the cursor to the apparent low point of the graph and again press [EXTR]. The new message EXTRM: (3.11963298118, -10.0606725872) appears below the graph and this point was entered on the stack.

When you execute the EXTR command, the HP-48 finds the extreme point by a well-known procedure. It finds the derivative f' of f and then uses the x-coordinate of the cursor as a first approximation for the ROOT program to find a root of f'. Finally, it calculates the value of f at this root and displays the two coordinates.

To find the inflection point: There is no key on the FCN menu to do this so we must use our knowledge of the relation between the function and its derivatives. We know that inflection points occur where the graph of the function changes concavity. And for functions that are everywhere differentiable (such as this one), concavity will change at points where the derivative f' changes its direction, i.e., at points where the graph of f' has a local extremum. We can therefore locate the inflection point on the graph of f by locating the extreme point on the graph of the derivative f'. This can be done in the PICTURE environment.

With the graph of f displayed, go to the second page of the FCN menu and press the [F'] key. This will plot the derivative f' and then replot f.



When you press the [F'] key, EQ becomes a list {f' f} containing f' and f, in this order. The HP-48 commands ROOT, EXTR, etc., apply only to the *first* function in the list, which is now f'. So move the cursor to the apparent low point of f' and press [EXTR]. You will see EXTRM: (1.66666666667,-6.333333333) displayed at the bottom of the screen. This is the low point on the graph of f' and we want to use its x-coordinate as the x-coordinate of the inflection point of f.

To get the y-coordinate of the inflection point P we must evaluate the function f at the x-coordinate of P. Perhaps the easiest way to do this is to use a short program. The following program assumes that EQ is a list { f' f } composed of f' and f in order, and that the coordinates of an extreme point of f' are displayed on stack level 1. With this input, the program returns the corresponding inflection point of f with the tag "Infl". The 1 in the name "INFL1" indicates that the first derivative is used in the process.

INFL1 (Inflection point of f)

Input: level 1: the coordinates (x_0, y_0) of an extreme point of f'

As a stored variable EQ: the list { f' f } consisting of f' and f

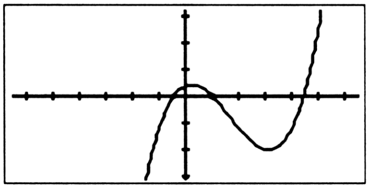
Effect: returns to level 1 the point (x_0, $f(x_0)$) tagged as 'Infl'

```
« RE EQ 2 GET OVER 'X' STO EVAL R→C 'Infl' →TAG 'X' PURGE »
```

With this program in your calculator and the extreme point of f' on stack level 1, press [INFL1] to see

level 1: Infl: (1.66666666667, -3.925925926).

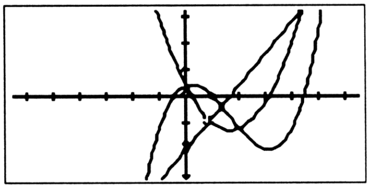
The stack now displays, on levels 6 through 1, the three x-intercepts (roots), the two extreme points and the inflection point of f, all with identifying tags. In the display below, we have set the display mode to show only two decimal places to avoid running off the right of the screen, and have again shown the plot of f to coordinate it with the information about the points of interest:



6:		Root: -0.45
5:		Root 1.00
4:		Root 4.45
3:	Extrm:	(0.21, 2.21)
2:	Extrm:	(3.12, -10.06)
1:	Infl:	(1.67, -3.93)

You will, of course, have to scroll with the [Δ] key to see all six stack levels.

Since the coordinates of extrema can be found from the FCN menu with a single keystroke, it was convenient to find the inflection points of f from the extrema of f'. But another way to find inflection points of f is to find the x-intercepts of the second derivative f'', because f has an inflection point at the values of x where the graph of f'' crosses the x axis. We will now use this method to again locate the inflection point of f. With the graphs of f' and f displayed on the screen, press [F'] again and the calculator will plot f'', then f' and finally f.



EQ is now the list $\{f''\ f'\ f\}$ consisting of f'', f' and f, in this order. Move the cursor to the root of f'' and press [ROOT] to display the message ROOT: 1.66666666667 below the graph. To find the value of f at this x and then display both coordinates as an inflection point, we will use the following program, INFL2. The 2 in the name indicates that the second derivative was used.

INFL2 (Inflection point of f)

Input: level 1: the coordinate x_0 of a root of f''

As a stored variable EQ: the list { f'' f' f } consisting of f'', f' and f.

Effect: returns to level 1 the point (x_0, $f(x_0)$) tagged as 'Infl'

```
« EQ 3 GET OVER 'X' STO EVAL R→C 'Infl' →TAG 'X' PURGE »
```

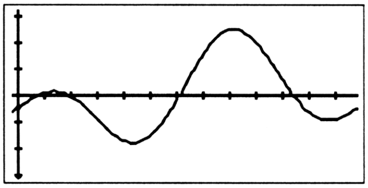
This program assumes that EQ contains a list of f'', f' and f, in this order, and that a root of f'' is displayed on stack level 1. With this input it returns the corresponding inflection point of f, tagged "Infl". With the root of f'' displayed on stack level 1, executing [INFL2] will give:

1: Infl : (1.67, -3.93) (the display was again fixed at two decimal places).

The next example uses a trigonometric function and would not be appropriate without the use of technology because of the difficulty of finding roots. With the HP-48G/GX, the procedure is the same as in EXAMPLE 9.

EXAMPLE 10. Plot the graph of $f(x) = sin(2x) + cos(x + 2)$. Find the x-intercepts and the coordinates of the local extreme points and inflection points.

Since this is a periodic function with period 2π, so it is sufficient to find the desired points on the interval $[0, 2\pi)$. Plot f with the x-range set to $-.1 \leq x \leq 6.29$ and the y-range set to $-2.5 \leq y \leq 2.5$ to see:



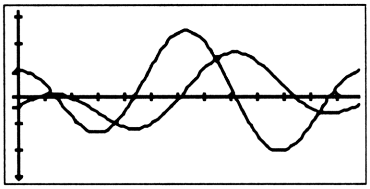
First, the intercepts: Move the cursor to each of the four points between 0 and 2π where the plot appears to cross the x-axis and press ROOT on the FCN submenu at each point. Return to the stack display screen to see:

4:	Root:	.429203673203
3:	Root:	.904129660127
2:	Root:	2.99852476252
1:	Root:	5.09291986492

To find the extreme points: We proceed as in EXAMPLE 9. Move the cursor to each of the four apparent extreme points and press EXTR at each one. Return to the stack display screen and set the display to show two decimal places. You will see these results:

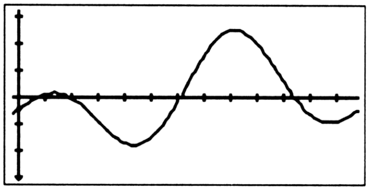
4:	Extrm:	(0.67, 0.08)
3:	Extrm:	(2.14, -1.45)
2:	Extrm:	(4.00, 1.95)
1:	Extrm:	(5.76, -0.77)

To find the inflection points: It should be clear from the plot that there is an inflection point between each consecutive pair of extreme points. We will proceed as before by finding the extreme points of f' and then using program INFL1 to build the inflection points of f. Retrieve the graph of f with PICTURE. When we use F' to plot both f' and f, the high point of f' is off screen, so we zoom out on the vertical axis with a factor of 1.5 to see:



Move the cursor to each of the four extreme points of the graph of f' and press EXTR at each point. Return to the stack display and convert each of the extreme points of f' to inflection points of f with program INFL1. You must do some stack manipulation in order to move the extrema of f' to level 1 in left-to-right order to use with the program. Here's an easy way: with the four extreme points of f' on the stack, press INFL1 to convert the point on level 1 to an inflection point of f. Then press the Δ key to activate the *interactive stack* and then use the Δ key to position the pointer ▶ on level 4 and press ROLLD ENTER. This *rolls* the first four levels of the stack *downward*, moving the first inflection point up to level 4. Now repeat this entire process three more times until all four extreme points are converted and appear in their natural order, left-to-right.

We display again the graph of f and the points that we have found:

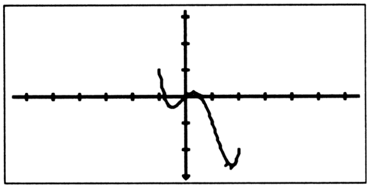


12:	Root:	0.43
11:	Root:	0.90
10:	Root:	3.00
9:	Root:	5.09
8:	Extrm:	(0.67, 0.08)
7:	Extrm:	(2.14, 1.45)
6:	Extrm:	(4.00, 1.95)
5:	Extrm:	(5.76, -0.77)
4:	Infl:	(0.06, -0.35)
3:	Infl:	(1.45, -0.71)
2:	Infl:	(3.09, 0.28)
1:	Infl:	(4.82, 0.64)

In the activities we will examine a function whose inflection points occur where the derivative is not defined. Sometimes we need to find the absolute maximum and absolute minimum values of a function f on a closed interval [a, b].

EXAMPLE 11. Find the absolute maximum and minimum values of the function $f(x) = x^4 - 2x^3 - x^2 + x$ on the interval [-1, 2].

We *could* graph f with $-1 \le x \le 2$ as the x-range but, instead, we will keep the default x-range and *restrict the independent variable* to plot only those points satisfying $-1 \le x \le 2$. The correct plot is

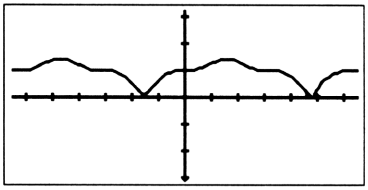


Clearly the absolute maximum value of f on [-1, 2] occurs at the left endpoint $x = -1$, and the absolute minimum value occurs at the minimum point where x is approximately 1.5. To find $f(-1)$, move the cursor to any point whose x-coordinate is -1 and press F(X) on the second page of the PICTURE FCN menu to see the message F(X): 1 displayed at the bottom of the screen. So the absolute maximum value of f on [-1, 2] is 1, occurring when $x = -1$. To find the absolute minimum value of f from the graph, position the cursor near the apparent minimum point whose x-coordinate is close to 1.5 and press EXTR on the PICTURE FCN menu. The message EXTRM: (1.70710678119, -2.66421356232) will appear. Thus the absolute minimum value of f on [-1, 2] is -2.66421356232, occurring when $x = 1.70710678119$. Before moving on to the next example, clear your stack and be sure to reset the independent variable to its default state with RESET.

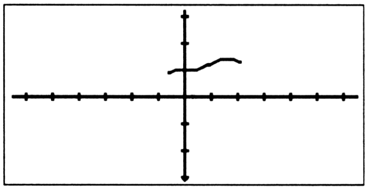
In a situation like that of EXAMPLE 11, if the plot of a function shows an absolute extreme value occurring at the left endpoint of the interval $[a, b]$ but a *is not* a pixel coordinate, say, $a = \sqrt{2}$ or π for example, then you will have to exit the PICTURE environment to evaluate the function at a. Of course, the HP-48 will never evaluate a function at π, only at its 12-digit rational number approximation.

EXAMPLE 12. Find the absolute maximum and absolute minimum values of the function $f(x) = \sqrt{1 + \sin^3 x}$ on the interval $[-\pi/5, 2\pi/3]$.

We first PLOT the graph with the default parameters to see:



Now restrict the independent variable to plot only the interval [$-\pi/5$, $2\pi/3$] and redraw:



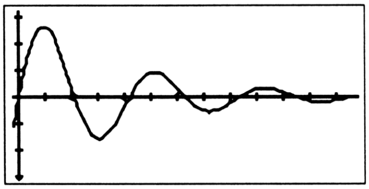
It is clear from the plot that we can use the EXTR command to obtain the absolute maximum:

Extrm: (1.57079632679, 1.41421356237)

We recognize this as the decimal approximation to ($\pi/2$, $\sqrt{2}$). The absolute minimum occurs at the left endpoint of the interval. To evaluate f there, we return to the stack display screen and go to the SOLVE menu with [↰] [SOLVE]. Open the ROOT submenu, then the SOLVR submenu. Build the decimal approximation to $-\pi/5$ with [↰] [π] 5 [÷] [+/−] [→NUM]. Use [ENTER] to make a duplicate copy, then touch [X] and [EXPR=] to see y-coordinate .892706665066. Thus the absolute minimum point of the graph on the interval [$-\pi/5$, $2\pi/3$] is approximately (-.628318530718, .892706665066).

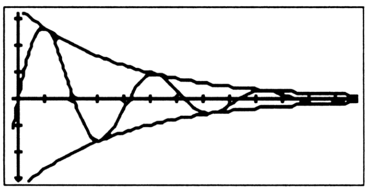
EXAMPLE 13. Plot the graph of $f(x) = 1.7\, e^{-x/2} \sin(3x)$ for $0 \leq x$.

Since we are interested in the graph only for non-negative values of x, we set the x-range as $-.1 \le x \le 6.4$ and the y-range as $-1.55 \le y \le 1.6$. This halving of both ranges retains equal unit distances (number of pixels per coordinate unit) on both axes and produces the graph:



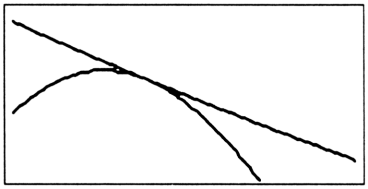
This function (which represents damped harmonic motion) is not periodic and has infinitely many roots, extrema and inflection points for values of $x \ge 0$. We could find any of these that we desired by using the techniques described earlier. But in this example, we will use the HP-48G/GX to analyze another aspect of the function's behavior.

Since $-1 \le sin(3x) \le 1$, the graph of f lies between the graphs of $u(x) = 1.7\, e^{-x/2}$ and $v(x) = -1.7\, e^{-x/2}$, coinciding with the graph of u when $sin(3x) = 1$ and with the graph of v when $sin(3x) = -1$. We can illustrate this by plotting the list { $f\ u\ v$ } using the same plotting parameters that we used for f. Exit the PICTURE environment, recall f to the stack with [↱] [EQ], and use [ENTER] to put a second copy on the stack. Edit the copy on level 1 to read '1.7 * EXP(-X/2)', make a second copy of the newly edited expression and then press [+/−] to change sign. Use the [Δ] key and the →LIST command to build the list { $f\ u\ v$ }. Now store the list in EQ and graph it to see:



The roots of f occur where $sin(3x) = 0$, that is, at the roots of $sin(3x)$. **Question:** *do the extrema of f occur at the extrema of sin(3x), that is, at the points of coincidence of f with u or v?*

We investigate this question both analytically and graphically. Move the cursor to the first maximum point to the right of the y-axis and press EXTR to see EXTRM: (.468549216461, 1.32664626947) at the bottom of the plot screen. If this were the point where $sin(3x) = 1$, then its first coordinate should be $\pi/6$. But $\pi/6 \approx .523598775598$, so the extreme points of f do not coincide with those of $sin(3x)$. We can illustrate this graphically by using BOXZ to zoom in on the region of the graph around the first maximum point to the right of the y-axis:



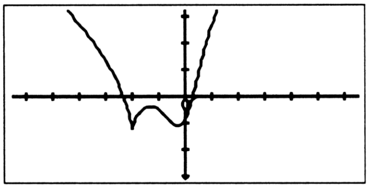
The maximum point of f is clearly seen to be to the left of the point where the graph of f intersects the graph of u. With some analysis of the derivative, you can show that successive extrema of f occur every $\pi/3$ units along the x-axis, as do successive points of coincidence of f with u or v. So the spacing shown between an extreme point and the corresponding point of intersection with one of the bounding graphs is constant.

Caution

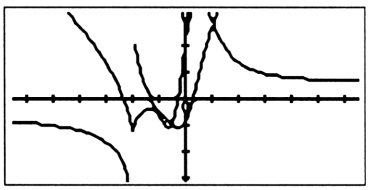
When you execute the EXTR command on the PICTURE FCN menu, the HP-48G/GX takes the derivative of the expression stored in EQ and then finds the value of x closest to the cursor that causes the derivative to evaluate to 0. Thus, if the x-coordinate of the extreme point that you are finding is a root of the derivative, you are using the EXTR command in the way in which it was designed to be used. But, if the extreme value of f does *not* occur at a root of the derivative, you should not use this command.

EXAMPLE 14. Find the roots, extrema and inflection points of the function $f(x) = 2(x + 2)^{2/3} + \frac{x - 4}{x^2 + 1}$.

Put '2 * XROOT(3, (X + 2)^2) + (X – 4)/(X ^ 2 + 1)' on level 1 and plot with the default parameters to see:



We can find the two roots in the usual way, by moving the cursor to each of them and pressing ROOT on the PICTURE FCN menu. We can find the local maximum point near x = -1 and the local minimum near x = -.3 by moving the cursor near these points and pressing EXTR. However, if we move the cursor to the minimum point where x = -2 and press EXTR, we get EXTRM: (-7.52928344591E213, 7.68302819356E142) which is nonsense. From the graph, f clearly has a minimum at $x = -2$ and $f(-2) = 0 + \frac{-6}{5} = -\frac{6}{5}$. The problem is that f has no derivative at x = -2 so the EXTR approach is not appropriate. If we press F' on the PICTURE FCN menu to plot both f and f' we see:



The plot makes it clear that f' does not exist at x = -2. Since the inflection points of f occur at values of x where f' has extrema, we move the cursor near the local minimum of f' to the left of the origin and press EXTR to obtain (-.661278286618, -1.07868129833). Now return to the stack, open the VAR menu and use INFL1 to build the inflection point as

Infl: (-.661278286618, -.81375384108).

Similarly, we find the inflection point that lies to the right of the origin to be

Infl: (.464327883331, .74032208835).

Activity Set 3.3.1

For each of the functions given in Activities 1-18 below, plot the graph and find all local extreme values and inflection points. When a closed interval is given, also find the absolute extreme values on that interval.

1. $f(x) = x^3 - x + 2$
2. $f(x) = x^3 - (1.3)x^2 + (.32)x - .02$
3. $f(x) = x^4 - 2x^3 + 3x - 2$
4. $f(x) = x^5 + 3x^4 - x^3 - 3x^2 - x + 3$
5. $f(x) = \dfrac{4}{2 + x^2}$
6. $f(x) = \dfrac{4}{x^2 - 5}$
7. $f(x) = \sqrt[3]{1 - x^2}$
8. $f(x) = x + 3 \sin x$, on the interval $[0, 2\pi]$.
9. $f(x) = \sin x + 2 \cos(3x)$ on $[0, \pi]$
10. $f(x) = \cos 2x - \sin x$ on $[0, \pi]$
11. $f(x) = \sin(3x) - \cos(2x),\ 0 \le x \le 2\pi$
12. $f(x) = \begin{cases} 1 + x^2 & x < 0 \\ \cos x & 0 \le x < \pi \\ \pi - x & \pi \le x \end{cases}$
13. $f(x) = 3e^{-x^2/4}$
14. $f(x) = x^{x}$
15. $f(x) = \cos(4 \cos^{-1} x)$.
16. $f(x) = 1.5 \tan^{-1} 2x$
17. $f(x) = x^{\sin x},\ 0 \le x \le 2\pi$
18. $f(x) = \dfrac{5}{1 + 5e^{-x}}$

Activities 19-21 are printed with thanks to Jim Nicholson.

19. A telephone company plans to run a new telephone line to a customer whose house is located one mile off the straight road along which the telephone lines are run. The new line must go from a junction box on the road to the customer's house. The junction box that is nearest to the house is three miles down the road from the point on the road that is closest to the house. It costs $100 per mile to run telephone cable along the road and $150 per mile to run cable off the road. What cable route minimizes total costs?

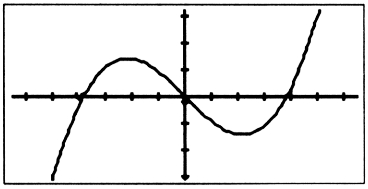
20. The owner of Big Sky Farm wants to build a rectangular paddock using one side of her horse barn as part, or all, of one side of the paddock. Her barn side is 50 feet in length. There is enough material on hand to build 200 feet of paddock fencing. What dimensions will give a paddock with maximum turn-out area?

21. As an afterthought, the owner of Big Sky Farm needs to use enough of her material to repair 70 feet of existing paddock fencing elsewhere, leaving only enough material to build 130 feet of fencing for the new barn paddock. With only 130 feet available, what dimensions will maximize turn-out area?

Newton's Method

The technique known as Newton's method has become a classic topic for inclusion in calculus. It is important because it not only invokes the notion of the derivative to produce a simple geometric procedure for finding roots of many functions, but also because it introduces several important ideas: algorithms, recursion, iteration. And it is especially easy to implement on the HP-48G/GX. The iteration formula for Newton's method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, so we need only iterate the new function $F(x) = x - \frac{f(x)}{f'(x)}$.

EXAMPLE 15. To use Newton's method to find the roots of $f(x) = 3x - 4\sin x$, we first graph f to see how many roots there are and to supply first guesses. The plot below is the result of plotting with the default parameters and then zooming in by a factor of .333 on both axes:



We will now create a user-defined function for $NM(x) = x - \frac{f(x)}{f'(x)}$. An easy way to do this is to put 'NM(X)', 'X', and two copies of '3 * X – 4 * SIN(X)' on the stack, then take the derivative, divide, subtract and equate. The result is 'NM(X) = X – (3 * X – 4 * SIN(X))/(3 – 4 * COS(X))'. Now press [DEF] to create the function NM on the VAR menu.

From the graph, 1.4 appears to be a reasonable first guess, so put 1.4 on the stack and press [NM] to see 1.28871273546 as the next approximation. Press [ENTER] to make a duplicate copy to keep. Now press [NM] again to obtain a second approximation and then [ENTER] to keep a copy. If you repeat this for three more iterations of Newton's method, you will have:

5: 1.28871273546
4: 1.27587035767
3: 1.27569814018
2: 1.27569810928
1: 1.27569810928

Five iterations have given us successive approximations that agree to 11 decimal places.

The above procedure for building a user-defined function to implement Newton's method for a given function f can be automated with a short program. Program NEWTON , given below, takes an expression for $f(x)$ from level 1 of the stack and constructs a user-defined function NM to perform the iteration.

NEWTON

Input: level 1: an expression for *f(x)*

Effect: constructs the user-defined function *NM* to implement Newton's method.

```
« 'X' PURGE 'NM(X)' 'X' ROT DUP 'X' ∂ / – = DEFINE 'X' PURGE »
```

If, for instance, you put '3 * X – 4 * SIN(X)' on level 1 and press [NEWT], you can then execute Newton's method from the menu key [NM] as above.

Newton's method has its limitations. It will obviously not converge if we ever obtain $f'(x_n) = 0$. But there can be other causes for its failure. We shall examine some of these in the next set of Activities.

Roots

You should appreciate Newton's method for what it is: a simple iterative procedure, based upon the geometric interpretation of the derivative as the slope of the tangent line, for finding roots of an equation $y = f(x)$. But it pales in comparison to the more powerful, robust and sophisticated root-finder that is built in to the HP-48G/GX. We called upon this root-finder when we used the [ROOT] key on the FCN submenu of the PICTURE environment to find the roots of a function whose plot was displayed. The location of the cursor on the graphics screen provided the initial guess for the procedure which, like Newton's method, is iterative.

The HP-48's root-finder program is the heart of the HP Solve System, which allows you to use menu keys to obtain a numerical solution to any problem that can be expressed in terms of an equation that includes only one unknown variable. The root-finder can be activated in either of two ways:

- with [↱] [SOLVE] [OK], to gain access to the SOLVE EQUATION screen;
- with [↰] [SOLVE] [ROOT], to gain access to the SOLVE command menu.

No matter which way you activate the HP Solve system, the general procedure for using it is the same:

- enter the equation you want to solve;
- enter values for all known variables;
- optional: enter an initial guess for the unknown variable;
- solve for the unknown variable.

The local procedures that support this general scheme depend upon how you activate the HP Solve system. Here is a worked example in which the HP Solve system is activated and used each way.

EXAMPLE 16. The following equation is often used in financial calculations that involve loan payments:

$$A = P\left[\frac{(r/12)\,(1 + r/12)^n}{(1 + r/12)^n - 1}\right]$$

where: A = monthly payment made at the end of each month

P = total amount of the loan

n = total number of monthly payments

r = annual interest rate (e.g., for 7%, r = .07).

Suppose that you are considering different options associated with buying a new car.

(a) How much would your monthly payments be if you were to borrow $10,000 for 4 years at 7.5% annual interest?

(b) How much could you borrow for 4 years at 7.5% interest if you could only afford to pay $175 per month?

(c) What annual interest rate would you have to obtain in order to borrow $9,000 for 4 years with monthly payments of $215?

Using the SOLVE EQUATION screen

Use [↱] [SOLVE] [OK] to gain access to the SOLVE EQUATION screen. The EQ dialogue box will reflect the contents of the current EQ. Enter 'A = P * (R/12) * (1 + R/12) ^ N / ((1 + R/12) ^ N – 1)' into the EQ field.

(a) Enter .075 into the *R* field, 48 into the *N* field, and 10,000 into the *P* field. *Optional*: as an initial guess for the amount *A* of monthly payment, enter 100. If you make no initial guess, the root finder will use the default guess of 0. Now highlight the *A* field and press [SOLVE] to see the correct monthly payment of $241.79 returned to the *A* field.

(b) With the screen set from part (a), enter 175 into the *A* field. When the *P* field is highlighted, press [SOLVE] to see the correct principal amount $7,237.71 returned to the *P* field.

(c) With the screen set from part (b), enter 9,000 into the *P* field and 215 into the *A* field. Highlight the *R* field and press [SOLVE] to see the correct interest rate of a little over 6.876% returned to the *R* field.

Using the SOLVE command menu

Return to the stack display screen and purge the variables N, R, P, and A from our previous work on this problem; only the EQ *should remain.* Use [↰] [SOLVE] [ROOT] to activate the SOLVE menu. Press [EQ] to verify that the desired expression is present in EQ. Clear the stack (if necessary) and open the [SOLVR]. You will see input boxes for each of the variables A, P, R, N at the bottom of the screen, along with an [EXPR=] box. The top of the screen will display the contents of EQ.

(a) Store the initial guess 100 into variable A with 100 [A], then store the values 10,000 into P, .075 into R, and 48 into N by a similar procedure. Since the value for A is only an initial (and optional) guess, we need to solve for the correct value of variable A. To do this, press [↰] [A]. The message at the top of the screen will say Solving for A. When done, the top of the screen will read Zero (to indicate that an exact root of the equation was found) and show the root on level 1: 241.789019379. Thus the monthly payment would be $241.79.

(b) Now store the value 175 into A and solve for P to see the value 7237.71494872 returned to level 1. Thus, you could only borrow $7,237.71 with $175 monthly payments.

(c) Finally, store the value 9,000 into P, the value 215 into A, and solve for variable R. The correct value is 6.87628494944E-2, so you would have to obtain an annual interest rate of 6.876%. Now press [VAR] and purge the variables used in this problem.

To avoid confusion, you should know that there is another ROOT command on the HP-48G/GX. It appears on the ROOT submenu of the SOLVE menu and is useful for solving in programs. It solves an expression (on level 3) for an unknown (on level 2) using a first guess (on level 1). Try it out now for the function of EXAMPLE 15, $f(x) = 3x - 4 \sin x$, with an initial guess in the vicinity of $x = 1.5$.

Activity Set 3.3.2

1. Use Newton's method to find all roots of the following functions:

   (a) $f(x) = x^3 - 3x^2 - 5x + 15$

   (b) $f(x) = \sin x - 2 \cos 3x$ in the interval $[0, 2\pi]$

   (c) $f(x) = e^x - 2 \cos x$ in the interval $[-2\pi, 1]$

   (d) the *Legendre Polynomial of* degree 3:

   $$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

   (e) the *Chebyshev Polynomial* of degree 4:

   $$T_4(x) = 8x^4 - 8x^2 + 1$$

2. Because Newton's method relies on tangent lines to generate a sequence of successive approximations $x_0, x_1, x_2, \ldots$ to a desired root r, you might expect that the method is somewhat sensitive to the slopes of these tangent lines. Indeed, tangent lines with small slopes often lead us *away* from the root we seek. To see this, try to locate the root $r = 0$ of $f(x) = \sin x$ by Newton's method, using the following initial guesses:

   (a) $x_0 = \pi/2$ [What happens here? Why?]

   (b) $x_0 = 1.6$

   (c) $x_0 = 1.5$

   (d) $x_0 = 1.4$

(e) $x_0 = 1.3$

(f) $x_0 = 1.2$

(g) $x_0 = 1.1$

3. Apply Newton's method to $f(x) = \sqrt[3]{x}$. The graph of f should help you to understand what is happening here.

4. Apply Newton's method to the function

$$f(x) = \begin{cases} -\sqrt{2-x} & \text{for } x < 2 \\ \sqrt{x-2} & \text{for } x \geq 2 \end{cases}.$$

(a) Use any convenient initial guess, say $x_0 = 3$. When program **NM** returns a symbolic result, simply press [EVAL] to evaluate that result and obtain a numerical result.

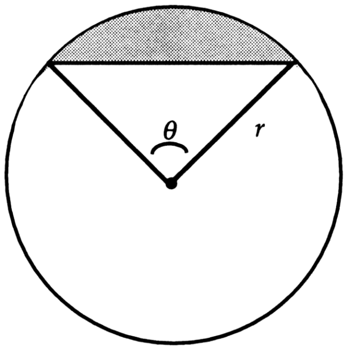
(b) Experiment with several other initial guesses. What is taking place here?

(c) To "see" what is taking place, plot the graph of f on the default plotting screen. Trace along the plot to the point P where $x = 3$ (our first initial guess), open the **FCN** submenu and use [TANL] to plot the tangent line to f at P. Return to [PICT], trace along the plot to the point Q where $x = 1$ (our second guess when $x_0 = 3$), and use [TANL] to draw the tangent line to f at Q. What is apparent about these tangent lines? Return to the stack and examine their slopes.

5. (a) Plot the list { 'EXP (-X ^ 2)' '.75/(1 + X ^ 2)' } on the default screen, then zoom in using zoom factors of 3 to enlarge the plot.

(b) Use the command **ISECT** on the **FCN** submenu to find the points of intersection of the two plots. **ISECT** uses the HP Solve system to produce its results.

6. Find the value for θ (in degrees) that will give the shaded region an area of 1.5 in^2 if r = 4 in. (The command R→D on the MTH REAL menu will convert radians to degrees.)



7. It is well-known that the centroid of the St. Louis arch is in the shape of an inverted catenary (hyperbolic cosine). The outside surface is much thicker at the base than at the top and thus is not a true catenary. Nevertheless, we shall model the outside surface as a catenary having both its height and base equal to 630 ft. Since a catenary hanging above the origin with lowest point at (0, a) has an equation $y = a \cosh \frac{x}{a}$, it is easy to see that an equation for the St. Louis arch is

$$y = 630 + a\,(1 - \cosh \frac{x}{a})$$

for some positive parameter a. To help determine this parameter, we use the fact that the point (315, 0) lies on the arch. Use the HP Solve system to determine the parameter a and then write an equation for the St. Louis arch that is free of unknown parameters. Remember, the HP Solve system only needs an initial guess. The *cosh* command resides on the MTH HYP menu.

Polynomial Approximations

A great deal of calculus is concerned with approximations. Indeed, approximations lie at the heart of the two main ideas of calculus, the derivative and the integral. The derivative is defined as a limit of approximating slopes and the integral is defined as a limit of approximating sums.

Aside from familiar approximations like .3, .33, .333, .3333, ... $\to \frac{1}{3}$ the simplest approximations in calculus occur when we approximate differentiable functions f by their tangent lines at points $x = a$. Recall that the slope of the tangent line of a function f at $x = a$ is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Thus, for values of x close to a, we have

$$f'(a) \approx \frac{f(x) - f(a)}{x - a},$$

so that

(1) $$f(x) \approx f(a) + f'(a)(x - a).$$

The expression on the right hand side of (1) is a linear polynomial in $(x - a)$:

$$P_1(x) = f(a) + f'(a)(x - a)$$

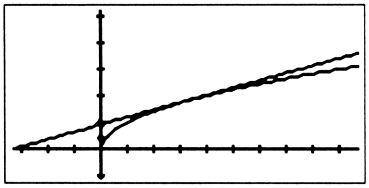
and its graph is the tangent line $y = f(a) + f'(a)(x - a)$ to f at $x = a$. Of all possible linear polynomials in $(x - a)$, $P_1(x)$ is the only one that satisfies the two conditions:

(i) $P_1(a) = f(a)$ [P_1 *and* f agree at $x = a$]

(ii) $P_1'(a) = f'(a)$ [P_1 *and* f *have the same derivative at* $x = a$]

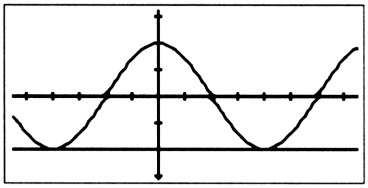
Because conditions (i) and (ii) are enough to completely determine the form of the polynomial $P_1(x)$, we call $P_1(x)$ *the best linear approximation to* f *at* $x = a$. By zooming in enough near the point where $x = a$, the graphs of P_1 and f appear almost identical.

For some functions f, the best linear approximation at $x = a$ can be a good one



The best linear approximation to $f(x) = \sqrt{x}$ at $x = \pi$ is a good fit to the graph for values of x near $x = \pi$.

But for functions f having more *curvature* at $x = a$, the best linear approximation can be poor:



The best linear approximation to $f(x) = \cos x$ at $x = \pi$ is a poor fit to the graph for values of x near $x = \pi$ because of the high degree of curvature.

To account for a higher degree of curvature at a point $x = a$, we need an approximating polynomial whose higher order derivatives are not all zero at that point.

The *best quadratic approximation to the function* f *at* $x = a$ is the quadratic polynomial $P_2(x)$ in $(x - a)$ satisfying the three conditions:

(i) $P_2(a) = f(a)$ [P_2 *and* f *agree at* a]

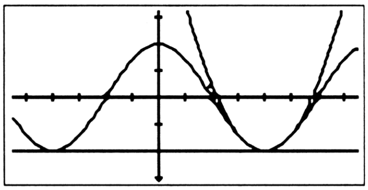
(ii) $P_2'(a) = f'(a)$ [P_2 *and* f *have the same first and second derivatives at* $x = a$]

(iii) $P_2''(a) = f''(a)$

The defining expression is

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a).$$

The plot below shoes the best linear and best quadratic approximations to $f(x) = \cos x$ at $x = \pi$.



The best quadratic approximation to $f(x) = \cos x$ at $x = \pi$
is a better fit than the best linear approximation.

The best linear and quadratic approximations to a function f at $x = a$ are also called the *Taylor Polynomials* of orders 1 and 2 for f at $x = a$. More generally, given a function f whose first n derivatives exist in a neighborhood of $x = a$, the *Taylor Polynomial of order* n *at* $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a) + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

As our last plot suggests, higher order Taylor polynomials at $x = a$ extend the range of values near $x = a$ for which we can expect to get reasonably good approximations to f.

The HP-48G/GX will find Taylor polynomials at $x = 0$ for any function that it can differentiate, and it is easy to write short programs that extend this capability to the more general case of Taylor polynomials at an arbitrary value $x = a$.

Using the Taylor Polynomial Screen

Access the Taylor Polynomial screen with [↱] [SYMBOLIC], highlight Taylor Polynomial and press [OK]. Enter an expression for the function into the EXPR field, say EXPR: 'SIN(X)', the variable of differentiation into the VAR field, VAR: 'X', and the desired order of the Taylor polynomial, say ORDER: 3. With the result set to RESULT: Symbolic, press [OK] to see the Taylor Polynomial at $x = 0$ on stack level 1:

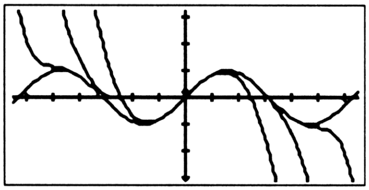
level 1: 'X – 1/3! * X ^ 3'

Using the TAYLR command

The command TAYLR, located on the first page of the [↰] SYMBOLIC menu, requires a threefold input: on level 3 the function f whose Taylor polynomial at $x = 0$ is desired, on level 2 the independent variable, and on level 1 the degree of the desired polynomial. This command produces Taylor polynomials about $x = 0$:

3: 'SIN(X)'
2: 'X'   ☞ 1: 'X – 1/3! * X ^ 3'
1: 3

To efficiently graph plot $f(x) = sin\ x$ and its Taylor polynomials P_3, P_7 and P_{11} of orders 3, 7 and 11 at $x = 0$, begin with 'SIN(X)' on level 1 and press [ENTER] three times to make three additional copies. Build the list {'SIN(X)'} by pressing the [Δ] followed by [→LIST] [ENTER]. SWAP levels 1 and 2, enter 'X' and 3, then press [TAYLR] to build $P_3(x)$: 'X – 1/3! * X ^ 3'. Insert this as the second element of the list with [+]. Now **SWAP** levels 1 and 2 and proceed as before to build $P_7(x)$ and $P_{11}(x)$, adding them to the list as they become available. You can then store the final list into **EQ** and plot with the default viewing screen to see:



Displaying plots is a dramatic way of showing how a function can be approximated by its Taylor polynomials, but you should remember that with the default plotting parameters two plots will coincide for a value of x if their y-coordinates are the same when rounded to one decimal digit; ordinarily, this is not good enough for serious numerical approximations.

To find Taylor polynomials centered about an arbitrary point $x = a$, you can use program TAY.A. Make sure that the independent variable is set to X and that no value is stored for X before using the program.

TAY.A

Input: Level 3: an algebraic expression for a function *f*, in terms of 'X'.

Level 2: the order n of the desired Taylor polynomial.

Level 1: the new center point, *a*.

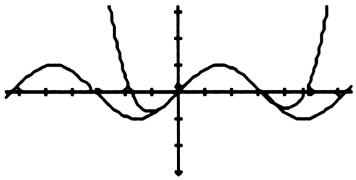
Effect: Returns the Taylor polynomial of order *n* for function *f* at *x* = *a*.

```
« → n a « 'Y' a + 'X' STO EVAL 'Y' n TAYLR 'X' PURGE 'X'
a – 'Y' STO EVAL 'Y' PURGE » »
```

For example, to find the fourth order Taylor polynomial for $f(x) = sin\ x$, at the point $x = 2$, put 'SIN(X)' on the stack, then enter 4 and 2 and press TAY.A to see the calculator's version of

$$0.909297 - 0.416147(x-2) - .454649(x-2)^2 + 0.069358(x-2)^3 + 0.037887(x-2)^4$$

on level 1. (We set the display to show 6 decimal places.) Plot the list containing $sin\ x$ and this polynomial with the default parameters to see:



Notice that the graphs appear to coincide from near $x = -.5$ to near $x = 3.5$, that is, on an interval centered about $x = 2$.

Although TAY.A does the obvious by making a change of variables $X = Y + a$ to translate the Taylor polynomial from $x = 0$ to $x = a$, you should be aware of the fact that the symbolic computations required to calculate higher order Taylor polynomials at points $x = a$ away from $x = 0$ can be substantial. Thus, as a symbolic processor, you may sometimes find the HP-48G/GX not quite up to the task of finding the Taylor polynomials that you desire if you use TAY.A. For example, the HP-48G runs out of memory (32K RAM) before it can produce the Taylor polynomial of order 7 for $f(x) = x^{-1}$ at $x = 2$ and the HP-48GX (with 128K RAM) requires almost 25 minutes to produce this polynomial. The solution is to be a bit more clever in how we approach the symbolics. Program TAYLAT ("Taylor at") is due to Charlie Patton of Hewlett Packard and uses the ↓MATCH and | commands to rearrange the symbolic computations. With TAYLAT, you can produce the Taylor polynomial of order 7 for $f(x) = x^{-1}$ at $x = 2$ in less than 30 seconds.

TAYLAT

Input: Level 4: an expression for a function *f*.

Level 3: the independent variable.

Level 2: the order *n* of the desired Taylor polynomial

Level 1: the new center point *a*.

Effect: Returns the Taylor polynomial of order *n* for function *f*, centered about *x* = *a*.

« → XP VA ORD PT « XP VA VA PT + 2 →LIST ↓MATCH DROP VA ORD TAYLR VA VA PT – 2 →LIST | » »

Activity Set 3.3.3

1. Plot $f(x) = tan^{-1}(x)$ on the default screen. Then overdraw the Taylor Polynomials of orders 1 and 3 for f at $x = 0$.

2. Plot $f(x) = sec\ x$ on the interval $[-\pi, \pi]$. Then overdraw the Taylor Polynomials of orders 2 and 4 for f at $x = 0$.

3. (a) Plot $f(x) = sin\ 2x - 2\ sin\ x$ on the default plotting screen and then overdraw the Taylor polynomials of orders 3, 5, 7 and 9 for f at $x = 0$.

   (b) ERASE the plots from part (a) and plot $f(x) = sin\ 2x - 2\ sin\ x$ using *Xrng*: -2 6 and *Yrng*: -3.1 3.2. Now use TAY.A to overdraw with the plot of the Taylor polynomials of orders 2 and 5 for f centered at $x = 2$.

4. (a) Plot $f(x) = e^{-x^2/2}$ on the default screen, then zoom in with factors of 3.

   (b) Overdraw your plot in (a) with the Taylor polynomials of orders 2 and 4 for f at $x = 0$.

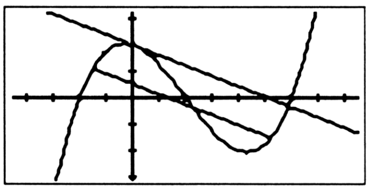
   (c) Now overdraw your plot in (b) with the Taylor polynomial of order 6 for f at $x = 0$. The HP-48G/GX takes quite some time to produce this sixth degree polynomial (approximately 2.5 min), an illustration of how complex the computation of such polynomials can be. To convince yourself, try finding this polynomial by hand.

Discovering the Mean Value Theorem

The Mean Value Theorem is one of the "gems" of elementary calculus. Its statement is simple, its geometric character makes it believable, and it is an extremely useful result. Indeed, the Mean Value Theorem provides the theoretical basis for a host of other theorems that comprise an important part of differential calculus.

What does the Mean Value Theorem say?

Given a function f that is continuous on the closed interval $[a, b]$ and differentiable between a and b, then for some point c between a and b, the tangent line to f at $x = c$ has slope equal to $\dfrac{f(b) - f(a)}{b - a}$.



Geometrically: *At some point c between a and b the tangent line to f at $x = c$ is parallel to the secant line joining $(a, f(a))$ and $(b, f(b))$.*

EXAMPLE 17. In this example we will apply the Mean Value Theorem to the function $f(x) = x^3 - 3x^2 - x + 3$ on the interval [-.75, 2.6] and produce the above plot of the result. Before beginning, purge X from the current directory and its ancestors.

Begin by plotting $f(x) = x^3 - 3x^2 - x + 3$ with *Xrng*: -2.25 4.25 and *Yrng*: -4.65 4.8. Trace left along the plot to the point P where $x = -.75$, press [ENTER] to record the coordinates $(a, f(a))$ of P on the stack, then press [×] to mark the location of the cursor on the plot screen. Now trace right along the plot to the point Q where $x = 2.6$ and again press [ENTER] to record the coordinates $(b, f(b))$ of Q on the stack. Press [NXT], open the EDIT menu, and press [LINE] to draw the secant line joining points P and Q.

To calculate the slope $\dfrac{f(b) - f(a)}{b - a}$ of the secant line, first exit to the stack display screen to see the coordinates of P on level 2 and Q on level 1:

2: (-.75, 1.640625)

1: (2.6, -2.304)

Press [−] to calculate $(a - b, f(a) - f(b))$:

1: (-3.35, 3.944625)

then divide 3.944625 by -3.35 to obtain the slope $m = -1.1775$ (Here is an easy way to do the division without retyping the numbers: open the MTH CMPL (= complex) menu, press [C→R] to separate the ordered pair into its two components, then SWAP and divide.) Leave -1.1775 on level 1.

To find a point c between $x = a$ and $x = b$ where the tangent line to f at $x = c$ has slope equal to -1.1775, recall EQ to the stack and press [ENTER] to make a duplicate copy. Now take the symbolic derivative of EQ, move -1.1775 from level 3 to level 1 with the command 3 ROLL; and then use [↰] [=] to equate -1.1775 to the derivative:

1: '3 * X ^ 2 – 3 * (2 * X) – 1 = -1.1775'

Go to the SOLVE menu with [↰] [SOLVE], open [ROOT] and store the equation on level 1 into EQ. Open [SOLVR]. Since a = -.75 and b = 2.6, we can use $x = 0$ as an initial guess for the root-finder; so put 0 into [X] and then solve for $x = c$ with [↰] [X]. You will see X: 3.00343648694E-2 returned to level 1. Now SWAP the value for c on level 1 with the original $f(x)$ on level 2, restore $f(x)$ as the EQ with the command STEO, open the VAR menu and purge the value stored in [X]. (You should still have the value for c on level 1.)

To build an equation for the tangent line to f at $x = c$, use the following program TAN.L. Recall that the HP-48G/GX has a menu key on the PICTURE FCN menu for drawing the tangent line to a function whose plot appears on the screen. But this built-in feature uses as input the x-coordinate of the cursor, and our value c is not such a point; hence the need for a more general purpose program. At any rate, run program TAN.L now. The program will use the value of c from level 1 and the expression $f(x)$ in EQ to calculate and overdraw a plot of the tanget line to f at $x = c$; for convenience, a copy of the equation of the tangent line is left on level 1 of the stack.

1: '-2.9672865388 – 1.1775 * (X – 3.00343648694E-2)'

TAN.L

Input: Level 1: a real number *c* or a complex number (*c*, *d*)

As the stored variable EQ: an algebraic expression for a function of *f*.

Effect: Calculates an expression for the tangent line to *f* at *x* = *c*, plots the expression on the existing plotting screen, and returns the expression to level 1 of the stack.

```
« DTAG DUP IF TYPE 0 == THEN 'X' STO ELSE C→R DROP 'X'
STO END EQ DUP EVAL EQ 'X' ∂ EVAL 'X' X – * + 'X' PURGE DUP
STEQ DRAW SWAP STEQ PICTURE »
```

Activity Set 3.3.4

1. (a) Plot the function $y = \frac{1}{x^2}$ using *Xrng*: -.5 3 and *Yrng*: 0 4.2.

   (b) Apply the Mean Value Theorem (as in the last Example) to f on the interval $[a, b]$, where a, b are the x-coordinates of points P, Q obtained as follows: trace left along the curve to the point P where $x = .496$, then trace right along the curve to the point Q where $x = 1.44$. As in the last example, overlay plots of the secant line and tangent line on the plot of f.

2. (a) Plot the function $f(x) = \sqrt{x}\ \sin x$ using *Xrng*: 0 3.14 and *Yrng*: 0 1.5.

   (b) Apply the Mean Value Theorem to f on the interval $[a, b]$ where a, b are the x-coordinates of points P, Q determined as follows: trace left along the curve to the point P where $x = .725$, then trace right along the curve to the point Q where $x = 2.15$. Plot the secant line joining P and Q. When you get ready to use the root-finder, use $x = 0$ as your initial guess.

Overdraw plots of the secant and tangent lines. Does the location of the tangent line at $x = c$ surprise you? Now seed the root finder with $x = 1.5$ to find another value for c. Is the tangent line to f at this new $x = c$ the one you expected originally?

3. As Activity 2 shows, there may be more than one value of c between $x = a$ and $x = b$ that meets the conditions of the Mean Value Theorem. Here is a spectacular example.

   (a) Plot $f(x) = \sin x - 2 \cos 3x$ using *Xrng*: 0 6.28 and *Yrng*: -3.1 3.2.

   (b) Apply the Mean Value Theorem to f over the interval $[a, b]$ where a, b are the x-coordinates of points P, Q determined as follows: trace left along the curve to the point P where $x = 2.75$, then trace right along the curve to the point Q where $x = 4.64$. Plot the secant line joining P and Q.

   (c) Now find six values of c between a and b that meet the conditions of the Mean Value Theorem. Do this by using the initial guesses for the root-finder of $x = 1, 2, 3, 4, 5$ and 6. Plot all six tangent lines.

Parametric Differentiation

How can we find the slope of a smooth parametric curve

$$x = f(t), \quad y = g(t), \quad a \leq t \leq b$$

at a point (x_0, y_0) on the curve?

If the coordinate functions f and g are reasonably "well-behaved", then y can be expressed as a function of x, say $y = F(x)$. Then, since y is a function of x and x is a function of t the Chain rule tells us that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} .$$

At a point where $\frac{dx}{dt} \neq 0$ we can then obtain

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}}.$$

In this last equation, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are the rates of change of the coordinates with respect to the parameter t, while $\frac{dy}{dx}$ is the rate of change of y with respect to x — the slope of the curve. In case $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, the slope of the curve is 0, meaning a horizontal tangent line. On the other hand, if $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$ then the curve has a vertical tangent line. The case that both derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are 0 is ruled out when we have a smooth curve.

EXAMPLE 18. Consider the parametric curve given by $x = 2\cos 2t, y = t - 3\sin 2t$ for $0 \leq t \leq 4.5$. We first met this curve in Chapter 2.

The slope of the curve at any value t is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{1 - 6\cos 2t}{-4\sin 2t}.$$

Thus, the slope of the curve at the point where $t = 2$ is

$$m = \frac{1 - 6\cos 2(2)}{-4\sin 2(2)} = 1.62587390888.$$

The point on the curve corresponding to $t = 2$ is (x_0, y_0) = (-1.30728724173, 4.27040748592), so an equation for the tangent line $y = y_0 + m(x - x_0)$ to the curve at this point is

$$y = 4.270470748592 + 1.62587390888(x + 1.30728724713)$$

or

$$y = 1.62587390888x + 6.39589170366.$$

In addition to plotting parametric curves, the HP-48G/GX can calculate the slope of a smooth parametic curve at a point (x_0, y_0) and then overdraw a plot of the tangent line to the curve at this point. Instead of performing all of the calculations on the stack, we can use a program to do most of the work. Program PAR' [= PARametric derivative], given below, will calculate the slope $m = \frac{dy}{dx}$ of a smooth parametric curve at a point (x_0, y_0) and then determine an equation for the tangent line $y = y_0 + m(x - x_0)$.

PAR'

Input: Level 2: a parametric curve '(f(T), g(T)' in terms of the parameter 'T'

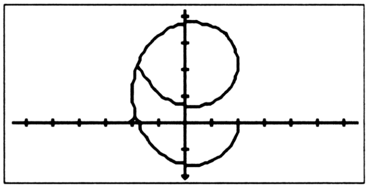
Level 1: a value t_0 of the parameter 'T'

Effect: Calculates the slope $m = \frac{dy}{dx}(x_0, y_0)$ of the curve at (x_0, y_0), where $x_0 = f(t_0)$ and $y_0 = g(t_0)$, and returns to level 1 an expression for the tangent line $y = y_0 + m(x - x_0)$ at (x_0, y_0). Displays the message "VERTICAL TANGENT" in the case of a vertical tangent.

```
« 'T' PURGE SWAP OBJ→ DROP2 OBJ→ DROP2 2 →LIST πLIST i
NEG * COLCT DUP2 2 →LIST 'T' ∂ OBJ→ DROP 5 ROLL →NUM 'T' STO
EVAL 9 RND SWAP EVAL DUP ABS IF 1E-10 ≤ THEN 4 DROPN
"VERTICAL TANGENT" ELSE / 3 ROLLD EVAL SWAP EVAL 3 PICK * –
SWAP 'X' * SWAP + END 'T' PURGE »
```

EXAMPLE 19. we shall apply program PAR' to the parametric curve of EXAMPLE 18.

Begin with a parametric plot of the curve using *Xrng*: -6.5 6.5, *Yrng*: -3 6, and independent variable *T* restricted by { *T* 0 4.5 }. The plot should appear as follows:

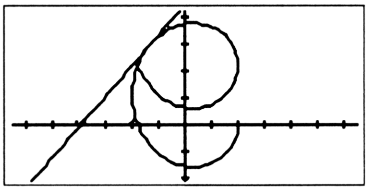


Now recall the EQ to level 1, enter 2 and run program PAR' to see the expression

'1.62587390881 * X + 6.39589170357

for the tangent line when $T = 2$.

To overlay the tangent line on the plot of the curve, change the plot type to FUNCTION, put the above expression into EQ and set the independent variable to *X*. Without erasing the original plot, execute the DRAW command to obtain the following:



Activity Set 3.3.5

1. (a) Plot the parametric curve given by

$$x = t - 2\ sin\ 3t, \quad y = 2\ cos\ 2t, \quad 0 \le t \le 2\pi$$

using *Xrng*: -3.5 9.5 and *Yrng*: -2 2.

(b) Calculate and overdraw plots of the tangent lines to the curve at points corresponding to $t = 0$, $t = \pi/2$, $t = 5\pi/6$, and $t = 7\pi/6$.

2. (a) Plot the parametric curve given by

$$x = 4\ cos\ t, \quad y = 2\ sin\ t, \quad 0 \le t \le 2\pi$$

using the default plotting screen.

(b) Calculate and overdraw plots of the tangent lines to the curve at points corresponding to the following values of t:

(i) $t = 0$ and $t = \pi$

(ii) $t = \pi/2$ and $t = 3\pi/2$

(iii) $t = \pi/4$ and $t = 5\pi/4$

3. (a) Plot the parametric curve given by

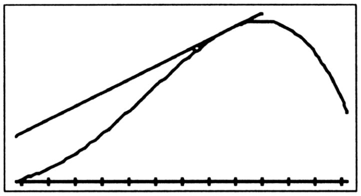
$$x = 3\ cos^3\ t, \quad y = 3\ sin^3\ t, \quad 0 \le t \le 2\pi$$

using the default plotting screen.

(b) Calculate and overdraw plots of the tangent lines to the curve at points corresponding to $t = 0$, $\pi/2$, π and $3\pi/2$; then at points corresponding to $t = 3\pi/4$ and $7\pi/4$.

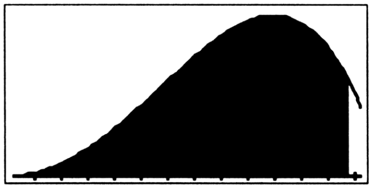
4 INTEGRALS

Calculus is rich in its connections to geometry. The two main ideas of calculus — the derivative and the integral — arose from simple geometric questions: what is the slope of a curve? what is the area beneath a curve?



The derivative $\frac{dy}{dx}$ gives the slope of $y = f(x)$

Figure 1(a)



The integral $\int_a^b f(x)dx$ gives the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$

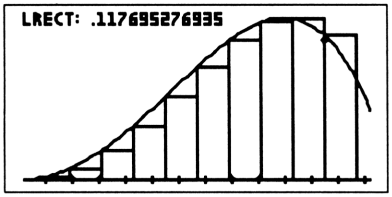
Figure 1(b)

4.1 APPROXIMATING AREA

Rectangle Approximations

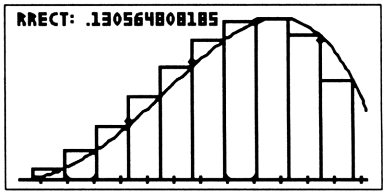
To approximate the area of the region lying between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$ (the shaded region in Figure 1(b)), we can sum areas of rectangles.

Divide the interval $[a, b\,]$ into n equal subintervals of length $h = \frac{b - a}{n}$ with points $a = x_0 < x_1 < \ldots < x_{n-1} = b$. On each subinterval, build a rectangle whose width is h and whose height is given by a value $f(x^*)$ of the function for some x^* chosen within the subinterval. If x^* is always chosen as the left endpoint of the subinterval we build *left rectangles;* if x^* is always chosen as the right endpoint of the subinterval we build *right rectangles*. The sum of the areas of the rectangles is an approximation to the area of the region.



Approximation with 10 left rectangles

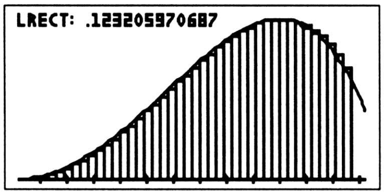
Figure 2(a)



Approximation with 10 right rectangles

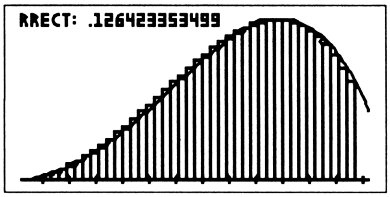
Figure 2(b)

By increasing the number of rectangles we can improve the approximations.



Approximation with 40 left rectangles

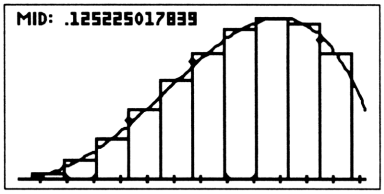
Figure 3(a)



Approximation with 40 right rectangles

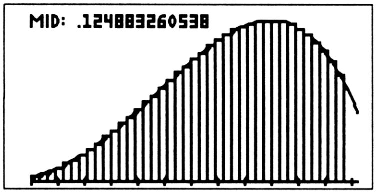
Figure 3(b)

When a graph is increasing, left rectangles will clearly *underestimate* the area, while right rectangles will *overestimate* the area. Exactly the opposite occurs when a graph is decreasing: left rectangles will overestimate and right rectangles underestimate. Thus, a convenient way to balance the errors is to use *midpoint rectangles*, rectangles whose heights are calculated by $f(x^*)$ where x^* is always chosen as the midpoint of each subinterval.



Approximation with 10 midpoint rectangles

Figure 4(a)



Approximation with 40 midpoint rectangles

Figure 4(b)

Since it is impractical to calculate a large number of rectangle areas by hand, we can use the HP-48G/GX. Below we present a sequence of HP-48 programs to do this. The first one, GRECT[1] [= Graphing RECTangles] provides a graphical/numerical interface for rectangle sum approximations. Depending upon your choice, it calls upon programs LRECT [= Left RECTangles], RRECT [Right RECTangles], or MID [= MIDpoint rectangles] to do the numerical calculations. These three programs call upon program SUM to do the actual summing and SUM calls upon program F.val to evaluate the input function f at the appropriate values. Two other utility programs are given: FABSTO, used to store the expression for f and values for a and b ; and NSTO, used to store the number n of subintervals. All of these programs (and two others) can be found in the special INTG subdirectory of the main calculus directory CALC, on the teaching code diskette (available from the publisher). Following a listing of these programs, we will work an example.

---

[1]The GRECT program was written by Robert E. Simms of Clemson Univeristy. We are indebted to him for permission to use it here.

FABSTO

Input: Level 3: an expression for *f(x)*, in terms of 'X'

Level 2: the lower limit of integration, *a*

Level 1: the upper limit of integration, *b*

Effect: stores *f*, *a* and *b* as EQ, *A* and *B*.

```
« 'B' STO 'A' STO STEQ »
```

NSTO

Input: level 1: a positive integer *n*

Effect: stores *n* as the number of subintervals *N* and stores $h = (b - a)/n$ as *H*

```
« 'N' STO B A – N / 'H' STO »
```

GRECT

Input: As stored variables: an expression for *f(x)* in EQ and values for *a* and *b* in *A* and *B*, respectively, from the program FABSTO; a value for *n* in *N* from program NSTO.

Effect: Prompts the user for a rectangle type; based upon the choice, produces an autoscaled plot of the function in EQ, overlays the approximating rectangles on the plot, and calculates the sum of their signed areas; puts the sum on stack level 1 as a tagged object. (By *signed* areas we mean that areas of rectangles lying below the *x*-axis carry a negative (-) sign while areas of rectangles lying above the *x*-axis carry a positive (+) sign.)

Comment: *In order for the program to properly draw the approximating rectangles, the number n of subintervals (rectangles) must be a divisor of 120: N = 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 25, 30, 40, 60 or 120. When n = 120, the entire region beneath the graph from x = a to x = b is shaded because each rectangle is exactly one pixel in width.*

```
« N 120 MIN 1 MAX DUP B A – SWAP / → n h « CLLCD "Select rectangle type 1
for Left 2 for Mid 3 for Right" 1 DISP 7 FREEZE IFERR 0 WAIT THEN DROP ELSE →c «
CASE c 82.1 == THEN 'LRECT' A END c 83.1 == THEN 'MID' A h 2 / + END c 84.1 ==
THEN 'RRECT' A h + END KILL END » 0 0 10 FOR z A B A – 10 / z * + 'X' STO EQ
EVAL NEXT 12 DUPN 1 11 START MAX NEXT 13 ROLLD 1 11 START MIN NEXT DUP2
DUP2 – 2 60 / * 5 ROLLD – 1 60 / * – 3 ROLLD + YRNG B A – 5 120 / * DUP NEG A +
SWAP B + XRNG # 131d # 64d PDIM {# 0d # 0d } PVIEW DRAX DRAW 'X' STO IFERR A
1 n START DUP 0 R→C C→PX SWAP h + DUP 3 ROLLD EQ EVAL R→C C→PX BOX 'X'
h STO+ NEXT DROP PICT { # 0d # 0d } 3 ROLL EVAL DUP 4 ROLLD 1 →GROB REPL 7
FREEZE THEN END 'X' PURGE END » »
```

LRECT

Input: none from the stack

Effect: uses SUM to compute the Riemann sum for the *f*, *a*, *b* and *n* stored, with *f* evaluated at the left end point of each subinterval

```
« A SUM 'lrect' →TAG »
```

RRECT

Input: none from the stack

Effect: uses SUM to compute the Riemann sum for the *f*, *a*, *b* and *n* stored, with *f* evaluated at the right end point of each subinterval

```
« A H + SUM "rrect" →TAG »
```

MID

Input: none from the stack

Effect: uses SUM to compute the Riemann sum for the *f*, *a*, *b* and *n* stored, with *f* evaluated at the midpoint of each subinterval

```
« A H 2 / + SUM "mid" →TAG »
```

F.val

Input: none from the stack

Effect: a utility program used by other programs to evaluate f at a specified number

```
« 'X' STO EQ EVAL »
```

SUM

Input: none from the stack

Effect: a utility program used for computation by each of the Riemann sum programs and by TRAP and SIMP. It takes the initial value of x from the other program, a for LRECT, a + h for RRECT and a + h/2 for MID.

```
« → X « 0 1 N START X F.val + X H + 'X' STO NEXT H * » 'X'
PURGE »
```

EXAMPLE 1. To approximate the area of the region beneath the graph of $f(x) = x^2 - x^4$ over the interval [.05 .9], first with $n = 10$ rectangles and then with $n = 40$ rectangles, arrange the stack like

```
3:  'X ^ 2 – X ^ 4'
2:               .05
1:                .9
```

and press FABST to store 'X ^ 2 – X ^ 4' as EQ, .05 as A and .9 as B. Now enter 10 and press NSTO to store 10 as the number N of rectangles. Press

GRECT ; at the prompt, press 1 to choose left rectangles. You should obtain the plot shown in Figure 2(a). Exit to the stack to see the approximating sum on stack level 1. Press GRECT again, and this time select right rectangles. You should obtain the plot shown in Figure 2(b). Run GRECT again and select midpoint rectangles. You should obtain the plot of Figure 4(a). Return to the stack environment and store 40 into N with 40 NSTO . Select 1 to obtain the plot of Figure 3(a). Running GRECT and selecting 3 and then 2 will produce the plots of Figure 3(b) and Figure 4(b). Clear the stack when you have finished plotting.

Although program GRECT will only plot approximating rectangles for values of n that divide 120, we can use the programs LRECT, RRECT and MID by themselves to obtain rectangle approximations to signed areas for *arbitrary* values of n. As with GRECT, we must first use FABSTO and NSTO to store EQ, A, B and N.

EXAMPLE 2. In EXAMPLE 1 we applied GRECT to $f(x) = x^2 - x^4$ over [.05 .9] to graphically view the approximating rectangles and calculate the sum of their areas for $n = 10$ and $n = 40$. Here we use programs LRECT, RRECT and MID by themselves to simply calculate the approximating sums. Using $n = 100$, 200 and 500 you should verify the following results:

n	LRECT	RRECT	MID
100	.124209601096	.125496554221	.124864054865
200	.124536827979	.125180304542	.124861310615
500	.124731407787	.124988798412	.124860542199

Which column of approximations appears to be the most accurate? The exact area is .124860395833 to twelve decimal places.

Activity Set 4.1.1

1. Consider the function $f(x) = 1/x^2$ over the interval [1, 3].

   (a) Use **GRECT** with n = 15, 30, 60 and 120 rectangles to complete the first four lines of the following table.

n	LRECT	RRECT	MID
15			
30			
60			
120			
200			
500			
1000			

   (b) Now use programs **LRECT**, **RRECT** and **MID** by themselves to complete the last three lines of the table.

   (c) Which of the three columns in the table appears to be producing the best approximations? The exact area is $\frac{2}{3}$ = .666666666666.

2. Consider the function $f(x) = \sqrt{x}\, \sin x$ over the interval $[0, \pi]$.

   (a) Use **GRECT** with n = 15, 30, 60 and 120 rectangles to complete the first four lines of the following table.

n	LRECT	RRECT	MID
15			
30			
60			
120			
200			
500			
1000			

(b) Now use programs **LRECT**, **RRECT** and **MID** by themselves to complete the last three lines of the table.

(c) Which of the three columns in the table appears to be producing the best approximations? The exact area is 2.43532116417 to twelve decimal places.

3. Consider the function $f(x) = 3x - 4 \sin x$ over the interval [-2, 2].

(a) Use program **GRECT** to obtain midpoint rectangle approximations to the signed area over [-2, 2] for n = 5, 20 and 40. Explain your answers.

(b) Now use **GRECT** to compare the left rectangle and right rectangle approximations to the signed area over [-2, 2] for n = 5, 20 and 40.

n	LRECT	RRECT
5		
20		
40		

(i) Explain these results.

(ii) Are these results what you expected? Why don't the left and right rectangle approximations more closely match the results from the midpoint rectangle approximations?

(c) What is the exact signed area of $f(x) = 3x - 4 \sin x$ over [-2, 2]? Why?

4. Consider the function $f(x) = 2 \cos 2x - \sin(x + 2)$ on the interval [0, 4].

(a) Use **GRECT** to obtain left, right and midpoint rectangle approximations to the signed area over [0, 4] for n = 10, 25, and 40. Then use **LRECT**, **RRECT** and **MID** by themselves to obtain approximations for n = 100, 200 and 500.

n	LRECT	RRECT	MID
10			
25			
40			
100			
200			
500			

(b) Which of the three columns in the table appears to be producing the best approximations? The exact signed area is 2.36567536982.

Riemann Sums

Whenever rectangles are used to approximate a region lying between a curve $y = f(x)$ and the x-axis over an interval $[a, b]$, the sum of the areas of the approximating rectangles is given by an expression of the form

(1) $$\sum_{k=1}^{N} f(x_k^*)\ \Delta x_k\ .$$

This sum is based upon a division of the interval $[a, b]$ into N subintervals $[x_0, x_1]$, $[x_1, x_2]$, . . . , $[x_{N-1}, x_N]$ using points $a = x_0 < x_1 < \ldots < x_N = b$. The meaning of the terms in the sum (1) are as follows:

- Δx_k: the width of the k^{th} rectangle
- x_k^*: a point somewhere in the k^{th} subinterval
- $f(x_k^*)$: the height of the k^{th} rectangle
- $f(x_k^*)\ \Delta x_k$: the area of the k^{th} rectangle

To acquire a better understanding of such sums we can use the HP-48G/GX to create user-defined functions for them.

EXAMPLE 3. Given $f(x) = x^2 - x^4$ on the interval [0, 1], create user-defined functions $S(N)$ and $T(N)$ for sums like (1) that use N equally spaced right rectangles and N equally spaced midpoint rectangles.

The key ingredients for the right rectangle sum in HP-48G/GX notation are:

- width of each rectangle (Δx_k): $\frac{1}{N}$
- right endpoint of the k^{th} rectangle (x_k^*): $\frac{K}{N}$
- height of the k^{th} rectangle ($f(x_k^*)$): $\left(\frac{K}{N}\right)^2 - \left(\frac{K}{N}\right)^4$
- area of the k^{th} rectangle: $\left[\left(\frac{K}{N}\right)^2 - \left(\frac{K}{N}\right)^4\right]\left(\frac{1}{N}\right)$

The user-defined function is

$$S(N) = \sum_{K=1}^{N} \left(\left(\frac{K}{N}\right)^2 - \left(\frac{K}{N}\right)^4 \right) \left(\frac{1}{N}\right).$$

If you use the Equation Writer to create this expression, the right-hand side will appear like

$$\sum_{K=1}^{N} \left(\left(\frac{K}{N}\right)^2 - \left(\frac{K}{N}\right)^4 \right) \cdot \left(\frac{1}{N}\right)$$

Remember to use the [▷] key to end each subexpression. If you build the expression on the stack, it will appear as:

1: 'S(N) = Σ (K = 1, N, ((K/N) ^ 2 – (K/N) ^ 4) * (1/N))'

If you build the expression with the Equation Writer, it will be put on the stack when you press [ENTER]. Use [↰] [DEF] to complete the construction. Now evaluate S for values of N = 10, 20, 50, and 100 to obtain the following table:

N	$S(N)$
10	.13167
20	.132916875
50	.13326672
100	.13316667

Suggestion: The *easiest* way to build a complicated user-defined function like this one is to use RPN on the stack. Put 'S(N)' on level 1, then enter 'K', 1, and 'N'. Then put 'K/N' on level 1 and press [ENTER] to make a duplicate copy. Now build

'((K/N) ^ 2 – (K/N) ^ 4) * (1/N)' using RPN. Press [↱] [Σ] to complete the sum. Do [↰] [=] to equate 'S(N)' with the sum, then use [↰] [DEF] to complete the task.

For the midpoint rectangles, the only difference is that we have

- midpoint of the k^{th} rectangle (x_k^*) : $\dfrac{2K-1}{2N}$.

Thus, in this case the user-defined function will be

T(N) = Σ(*K* = 1, *N*, (((2 * *K* – 1)/(2 * *N*)) ^ 2 – ((2 * *K* – 1)/(2 * *N*)) ^ 4) * (1/*N*))'

Evaluating T for N = 10, 20, 50 and 100 we have

N	$T(N)$
10	.13416375
20	.133541484376
50	.133366662
100	.133341666383

For an arbitrary choice of points $x_1 < x_2 < \ldots < x_{N-1}$ between a and b and an arbitrary choice of a point x_k^* in the k^{th} subinterval, the sum (1) is called a *Riemann sum*. Sums constructed from subintervals of equal width, or with a uniform choice of points x_k^* in each subinterval, or both, are special kinds of Riemann sums. But arbitrary Riemann sums work just as well, with no restrictions whatsoever on the widths of the subintervals or the location of the x_k^* 's. The general result is that for a "well-behaved" function on the interval $[a, b]$, e.g., a function f that is continuous on $[a, b]$, Riemann sums have a limit I:

$$\lim_{N\to\infty} \sum_{k=1}^{N} f(x_k^*)\, \Delta x_k = I.$$

The number I is called the *definite integral of f from a to b*, and is denoted by

$$I = \int_a^b f(x)dx.$$

The integral $\int_a^b f(x)dx$ is the *signed area* of the region between the curve $y = f(x)$ and the x-axis over the interval $[a, b]$.

For a given value of n, the midpoint rectangle approximation M_n to an integral $I = \int_a^b f(x)dx$ will be more accurate than the left or right rectangle approximations, L_n or R_n. Upper bounds for the errors are well-known and are related to the first and second derivatives of f. If $|f'(x)| \leq B_1$ and $|f''(x)| \leq B_2$ for all x in $[a, b]$ then

$$|L_n - I| \leq \frac{B_1(b-a)^2}{2n} \text{ and } |R_n - I| \leq \frac{B_1(b-a)^2}{2n} \text{ but } |M_n - I| \leq \frac{B_2(b-a)^3}{24n^2}.$$

Activity Set 4.1.2

1. Consider the region between the curve $y = x^3$ and the x-axis on the interval [0, 4]. We want to create a user-defined function for a Riemann sum like (1) that uses N equally spaced right rectangles.

   (a) Express the following in terms of N:

   - the width of each rectangle (Δx_k):
   - the right endpoint of the k^{th} rectangle (x_k^*):
   - the height of the k^{th} rectangle ($f(x_k^*)$):
   - the area of the k^{th} rectangle $f(x_k^*)\,\Delta x_k$:

(b) Create a user defined function $S(N) = \sum_{K=1}^{N}$ (area of the k^{th} rectangle) on your HP-48G/GX and use it to complete the table below:

N	$S(N)$
10	
50	
100	
200	

2. Repeat Activity 1 using $y = \frac{1}{x}$ on [1, 3].

3. Repeat Activity 1 using $y = e^x$ on [-1, 3].

4. Repeat Activity 1 for $y = \frac{3}{x^2} + x$ on [-3, -1] using N equally spaced *left rectangles.*

Trapezoid and Simpson's Approximations

Recall that for an increasing (decreasing) function, the left rectangle approximation underestimates (overestimates) the area and the right rectangle approximation does exactly the opposite. Thus, midpoint rectangles were introduced as a way of "balancing" the two errors. But instead of using midpoint rectangles, we can simply average the left and right rectangle results.

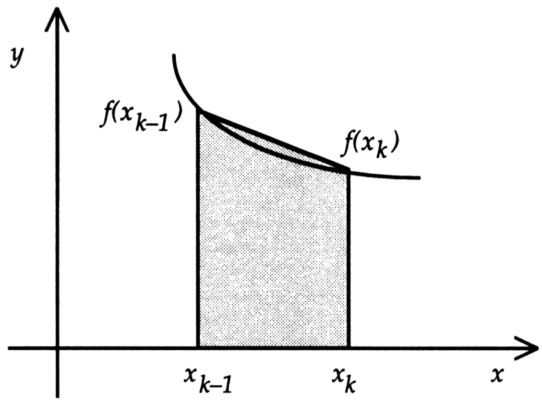
For an evenly spaced division of the interval $[a, b]$ into N subintervals of length Δx we have

$$\text{LRECT} = \sum_{k=1}^{N} f(x_{k-1})\Delta x \quad \text{and} \quad \text{RRECT} = \sum_{k=1}^{N} f(x_k)\Delta x\ .$$

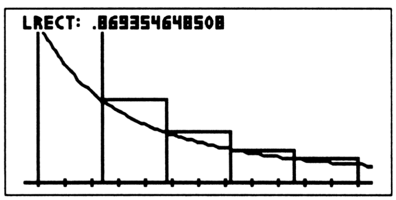
Their average is

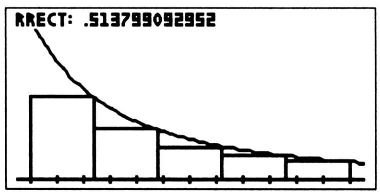
$$\text{Average} = \sum_{k=1}^{N} \frac{[f(x_{k-1}) + f(x_k)]}{2} \Delta x.$$

This summand is the area of a trapezoid sitting on the k^{th} subinterval $[x_{k-1}, x_k]$:

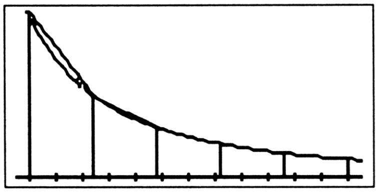


For this reason, the average is called the *Trapezoid approximation.* It is easy to see that it can be a much better approximation than the left and right rectangle approximations. Consider, for example, the case $y = \frac{1}{x^2}$ over the interval [1, 3]. With $n = 5$ we have:



and


But the Trapezoidal approximation looks like this:



TRAP: .69157687073

The following program TRAP appears in the INTG subdirectory of the main CALC directory. Like LRECT, RRECT, and MID, it requires that you first use FABSTO and NSTO.

TRAP

Input: none from the stack

Effect: uses SUM to compute the trapezoidal approximation for the stored quantities *f*, *a*, *b*, and *n*

```
« A SUM B F.val A F.val - 2 / H * + 'X' PURGE "trap" →TAG »
```

EXAMPLE 4. We return to EXAMPLE 2 where we used LRECT, RRECT and MID to calculate approximating sums for the integral $\int_{.05}^{.9} (x^2 - x^4)dx$ using N = 100, 200 and 500.

Applying TRAP we obtain

N	TRAP
100	.124853077658
200	.12485856626
500	.1248601031

Because the Trapezoid approximation simply sums the areas of trapezoids, it is possible to give a formula for the approximation. Given that the interval $[a, b]$ is divided into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ by points $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, let $y_j = f(x_j)$ for $j = 0, 1, \ldots, n$. Then the Trapezoid approximation T_n to $\int_a^b f(x)dx$ is given by

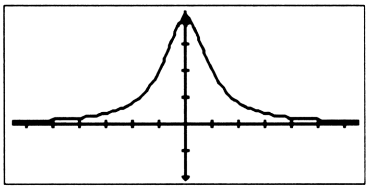
$$T_n = \frac{\Delta x}{2}[y_0 + 2y_2 + \ldots + 2y_{n-1} + y_n].$$

This formula is frequently used in hand calculations for small values of n.

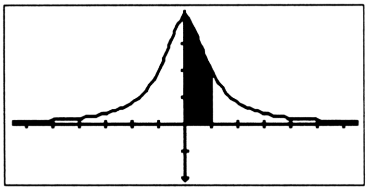
It is well-known that

$$\int_0^1 \frac{4}{1+x^2}dx = \pi .$$

You can verify this with your HP-48G/GX as follows: Draw a plot of $y = \frac{4}{1+x^2}$ using *Xrng*: -6.5 6.5 and *Yrng*: -2.1 4.2.



Move the cursor to (0, 0) and press [×] to mark its location, then reposition the cursor at (1, 0) and press [AREA] on the FCN menu. The message AREA: 3.14159265359 will appear on the lower left of your screen (and on level 1 of the stack). With the origin marked and the cursor still at (1, 0), press [NXT] to return to the FCN menu and press [SHADE] to shade the region whose area is π.



This use of the [AREA] key on the HP-48G/GX requires that the lower and upper limits of integration be given by pixel coordinates.

We shall use this example to compare the errors made by the trapezoid and midpoint approximations. Open the INTG subdirectory and use FABSTO to store '4/(1 + X ^ 2)' into EQ and 0, 1 into A, B respectively. Now enter the following program and store it under the name 'ERROR':

« π →NUM – "error" →TAG ».

For a calculated approximation A on level 1, program ERROR will calculate (A – π) and display it on level 1 with the tag "error".

To compare the Trapezoid and midpoint approximations for N = 50, 100, 150 and 200, proceed as follows:

(i) Use NSTO to store 50 for N. Press [TRAP] [ENTER], then [ERROR] to see

```
2:  trap:    3.1415259869
1:  error:   -.00006666669
```

Now press [MID] [ENTER], then [ERROR] to see

```
2:  mid:     3.14162598694
1:  error:   .00003333335
```

Notice that the magnitude of the error from MID is one-half of that from TRAP.

(ii) - (iv). Repeat step (i) using $N = 100$, 150 and 200 in succession.

The following table summarizes the results:

N	TRAP	ERROR	MID	ERROR
50	3.1415259869	-.00006666669	3.14162598694	.00003333335
100	3.14157598691	-.00001666668	3.1416009869	.00000833331
150	3.1415852461	-.00000740749	3.14159635725	.00000370366
200	3.1415884869	-.00000416669	3.14159473692	.00000208333

The TRAP and MID columns tell us that the trapezoid estimates are too low, while the midpoint estimates are too high. And the two ERROR columns show that the midpoint error is consistently one-half the trapezoid error in magnitude.

This is not surprising if we examine the upper bounds on the errors. We noted earlier that for a given value of n, a bound on the error by the midpoint rectangle approximation M_n to an integral $I = \int_a^b f(x)dx$ is given by

$$| M_n - I | \le \frac{B_2(b-a)^3}{24n^2}, \text{ where } | f''(x)| \le B_2 \text{ for all } x \text{ in } [a, b].$$

For the Trapezoid approximation T_n with n subintervals, a bound on the error is

$$| T_n - I | \le \frac{B_2(b-a)^3}{12n^2} .$$

To get an improved estimate of the integral that balances the errors, we can use a "weighted" average of the trapezoid and midpoint estimates:

$$\text{weighted average} = \frac{1}{3}\text{(trapezoid)} + \frac{2}{3}\text{(midpoint)}.$$

Averaging will tend to balance the low versus high estimates, and we weight the midpoint estimate twice as much because its error is only half that of the trapezoid estimate.

This particular weighted average is known as *Simpson's approximation*. It produces approximations to the integral $\int_a^b f(x)dx$ that are far more accurate than those by the other methods that we have considered. A bound on the error involves the fourth derivative of f on $[a, b]$. If $|f^{(iv)}(x)| \leq B_4$ for all x in $[a, b]$, then Simpson's approximation S_n using n subintervals satisfies

$$|S_n - I| \leq \frac{B_4(b-a)^5}{180n^4} .$$

Because of this, Simpson's approximation produces *exact* results for any integral $\int_a^b f(x)dx$ where $f^{(iv)}(x) = 0$. In particular, it gives exact results for all linear, quadratic and cubic polynomial functions.

The following HP-48G/GX program SIMP resides in the INTG subdirectory of the main CALC directory.

SIMP

Input: none from the stack

Effect: uses MID and TRAP to compute Simpson's approximation for the stored quantities *f*, *a*, *b* and *n*

```
« MID 2 * TRAP + 3 / "simp" →TAG »
```

To appreciate the accuracy of Simpson's approximation, we apply program SIMP to the integral $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$:

N	SIMP	ERROR
5	3.14159261393	-.00000003966
10	3.14159265297	-.00000000062
15	3.14159265354	-.00000000005
20	3.14159265359	0

Like the Trapezoid approximation, there is an easy formula for Simpson's approximation. The formula is based upon dividing the interval $[a, b]$ into an *even number* n of subintervals of equal width $\Delta x = \frac{b-a}{n}$:

$$S_n = \frac{\Delta x}{3}\,(y_0 + 4y_1 + 2y_2 + \dots + 2y_{n-2} + 4y_{n-1} + y_n).$$

Observe, carefully, the pattern 1, 4, 2, 4, 2, ... in the coefficients; and how it ends: 2, 4, 1. This pattern *requires* that n be an even number. This formula for Simpson's approximation to the integral $\int_a^b f(x)dx$ is useful when f is given only in graphical or tabular forms.

Activity Set 4.1.3

In each of the following Activities, use the AREA command on the FCN submenu to obtain an accurate twelve (12) digit approximation to the integral. Then calculate Trapezoid and Simpson's approximations to the integral using the indicated number of subintervals. Keep your numeric display mode set to STD to show full precision.

1. $\int_{1}^{3} \frac{1}{x^2}\,dx$, $n = 50, 100$

2. $\int_{0}^{\pi} \sqrt{x}\ \sin x\ dx$, $n = 100$

3. $\int_{0}^{4} (4\sin x - x)dx$, $n = 100$

4. $\int_{0}^{4} (2\cos 2x - \sin(x+2)dx$, $n = 100$

5. $\int_{0}^{3} \frac{1}{1+x^3}\,dx$, $n = 100$

6. $\int_{-3}^{-1} \left(x + \frac{3}{x^2}\right)dx$, $n = 50, 100$

7. $\int_{0}^{3} e^{-x^2}\,dx$, n = 100

8. $\int_{0}^{1} \sqrt{1+\sec^2 x}\ dx$, $n = 100$

9. $\int_{0}^{2} \sqrt{1+\sin^2 x}\ dx$, $n = 100$

10. $\int_{1}^{2} \sqrt[3]{\left(1+\frac{1}{x^4}\right)}\ dx$, $n = 100, 200$

11. Consider

$$f(x) = \begin{cases} \cos(\pi x^2/2) & x < 1 \\ x^2 - 3x + 2 & x \geq 1 \end{cases}$$

(a) Find the integral $\int_0^3 f(x)dx$ using the AREA key on the FCN submenu.

(b) Now approximate the integral $\int_0^3 f(x)dx$ using Simpson's approximation with $n = 100$ and $n = 200$ subintervals.

4.2 INTEGRATION ON THE HP-48G/GX

Numerical Integration

In the application of calculus to fields such as engineering, physics, probability and statistics there is often a need to obtain fairly accurate estimates of definite integrals. The integrands in question may be simple in appearance, but usually lack elementary, closed-form antiderivatives so that the fundamental theorem of calculus cannot be applied. Simple examples are:

- the standard normal integral $\int_0^z \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$ from probability theory
- the period $T = \int_0^\alpha \frac{2\sqrt{2}\,dy}{\sqrt{\cos y - \cos \alpha}}$ of a simple pendulum
- the electrostatic potential V at a point $P(x,y)$ due to a variable charge density $\lambda(s)$ applied along a straight wire over an interval $[-a, a]$:

$$V = \int_{-a}^{a} \frac{\lambda(s)ds}{\sqrt{(x-s)^2 + y^2}}$$

The HP-48G/GX has a built-in numerical integration routine that uses a Romberg numerical integration technique. The routine is iterative, producing increasingly accurate estimates derived from values of the integrand at points sampled within the interval of integration until three successive estimates agree to within an error tolerance specified by the user. The error tolerance e is specified by setting the numeric display mode as follows: n FIX specifies an error tolerance $e = 10^{-n}$ and STD specifies an error tolerance $e = 10^{-11}$.

For example, setting the numeric display to 5 FIX will specify an error tolerance $e = .00001$. In general, the smaller the error tolerance, the longer the calculation time and the more accurate the result. When the calculation is finished, an estimate of *the error in the result* is given in the variable IERR.

There are two ways to perform a numerical integration on the 48G/GX: with the INTEGRATE Form on the SYMBOLIC Application or on the Stack. We illustrate each way with the integral $\int_0^{\pi} 3x \sin 2x \, dx$. The exact answer and its decimal approximation are

$$\int_0^{\pi} 3x \sin 2x \, dx = -3\pi/2 \approx -4.71238898038.$$

Using the INTEGRATE Form

Open the Symbolic Application with [↱] [SYMBOLIC] and press [OK] to select Integrate. Type in '3 * X * SIN(2 * X)' and use [ENTER] to enter it into the EXPR: field. Then enter 'X' into the VAR: field and 0, π into the LO: and HI: fields. When the RESULT: field is highlighted press [OK] if it reads Numeric, otherwise press [+/–] to change to Numeric. Set the NUMBER FORMAT: field to Std, using the CHOOS box if necessary. Press [OK] to perform the numerical integration.

The result will be shown on stack level 1.

1: -4.71238898038

Press [VAR] then [IERR] to see the error estimate 9.4E-11.

Using the Stack

Arrange the stack as follows:

4:	0
3:	π
2:	'3 * X * SIN(2 * X)'
1:	'X'

Press [↱] [∫] to see the symbolic expression

'∫ (0, π, 3 * X * SIN(2 * X), X)'

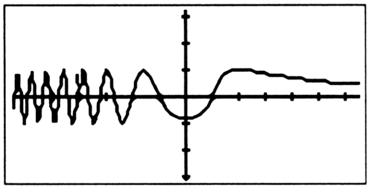
returned to level 1. Use [↰] [→NUM] to see the numerical result -4.71238898038 returned to level 1. Press [VAR] then [IERR] to see the error estimate 9.4E-11.

If you wish, you can use the Equation Writer to key in the integral:

$$\int_0^{\pi} 3\cdot X\cdot SIN(2\cdot X)\ dX$$

Use the [▷] to end each subexpression. When the expression is complete, press [ENTER] to view it on the stack and then [↰] [→NUM] to evaluate. Alternatively, with the expression showing in the Equation Writer press [↰] [→NUM] to bypass the stack and obtain the numerical result.

EXAMPLE 5. As another example, we graph the function $f(x) = \begin{cases} sin(x^2 - 1) & x < 1 \\ sin(\pi/x) & 1 \le x \end{cases}$ with the default parameters and calculate the integral $\int_{-2}^{2} f(x)\,dx$. Enter the function as 'IFTE(X < 1, SIN(X ^ 2 − 1), SIN(π/X))'. Plotting with the default parameters, we see:



To calculate the integral, we first set the numeric display mode to 5 FIX, then press [◁] to again view the plot. Activate coordinate read-out with [(X, Y)], move the cursor to (-2, 0) and press [×] to mark the location. Now move the cursor to (2, 0), press [NXT] to return the menu labels, open the FCN submenu and press [AREA]. In approximately 1 minute, 12 seconds you will see the result .20163 displayed at the bottom left of the screen and on stack level 1. IERR shows the error to be approximately .00003.

To use the numerical integration routine in this way (while viewing a plot of the integrand), the limits of integration must be pixel coordinates. It is also interesting to note that if you perform the same integration on the stack, its execution is a little faster, approximately 1 minute, 8 seconds.

Why did we set the numeric display to 5 FIX instead of asking for full twelve digit precision in STD mode? We actually tried for twelve digit precision, but gave up and interrupted the integration process at the end of one hour. When we seek twelve digit precision, many more points on the integrand are sampled than with five digit precision, and the "break point" $x = 1$ in the definition of the integrand causes problems. How can we obtain twelve digit precision? The trick is to split the integral into two parts at the break point:

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{1} f(x)dx + \int_{1}^{2} f(x)dx.$$

The first integral on the right hand side is found in 36 seconds:

$$\int_{-2}^{1} f(x)dx = -.546976060733.$$

And the second integral on the right hand side takes only 14 seconds:

$$\int_{1}^{2} f(x)dx = .748600792238.$$

Thus, we can add to obtain

$$\int_{-2}^{2} f(x)dx = .201624731505$$

in a little over 50 seconds.

This trick of splitting the integral into several other integrals is standard practice with almost all numerical integration routines on calculators or computers. Obvious separation points are any break points in the definition of the integral (as in the above example), as well as any points where the function is not defined or is non-differentiable.

Activity Set 4.2.1

1. A roller coaster has part of its track in the shape of the curve $y = x + \sin 2x^2$ when plotted using *Xrng*: 0 2 and *Yrng*: 0 3.

   (a) Plot the curve in this viewing window.

   (b) Calculate the area of the region between the track and the x-axis (the ground) over the interval [0, 2].

   (c) Calculate the area of the region between the track and the ground over the interval between the two local maxima.

2. Find the volume of the solid of revolution generated by revolving the curve $y = e^{\sin x}$ around the x-axis over the interval [0, 3].

3. Find the volume of the solid generated by revolving about the y-axis the region between the graph of $y = e^{-x^2}$ and the x-axis over the interval [1/3, 1].

4. Calculate the "arch length" of the St. Louis Arch using the formula from Activity 7 in ACTIVITY SET 3.2.2.

5. Imagine a point P moving along a parametric curve C: $x = f(t)$, $y = g(t)$ in such a way that it traces the curve only once from $t = a$ to $t = b$. Then the length of the curve from $t = a$ to $t = b$ is given by the formula

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\; dt\,.$$

   Find the lengths of the following curves.

   (a) $x = 3\cos^3 t,\ y = 3\sin^3 t$ from $t = 0$ to $t = 2\pi$.

   (b) $x = 3\cos t + 2\cos 3t,\ y = 3\sin t - 2\sin 3t$ from $t = 0$ to $t = 2\pi$.

6. The following formula gives the period T of a simple pendulum of length L that is released from rest at an angle α with the vertical axis (g is the constant acceleration due to gravity):

$$T = \frac{2\sqrt{2}}{\sqrt{g/L}} \int_0^{\alpha} \frac{1}{\sqrt{\cos y - \cos \alpha}}\, dy.$$

Find the approximate period for a pendulum of length $L = 1.5m$ that is released at an angle of $\pi/4$ rad from the vertical axis. (Use 3 FIX.)

Symbolic Integration

Symbolic integration refers to calculating an integral $\int_a^b f(x)dx$ by finding an antiderivative $F(x)$ of the integrand $f(x)$ and then returning a symbolic expression for $F(b) - F(b)$. Because of its restricted memory, the HP-48G/GX can perform symbolic integration for only the following restricted set of integrands:

- All built-in functions that have an antiderivative expressible in terms of built-in functions (except LNP1);
- sums, differences, negatives, linear combinations and other selected patterns of the above functions;
- all derivatives of built-in functions;
- polynomials whose base term is linear.

The HP-48G/GX will not, for example, perform symbolic integration on such simple integrals as

$$\int_a^b x \sin x\, dx, \quad \int_a^b xe^x dx, \quad \text{or} \quad \int_a^b \sin x \cos x\, dx .$$

These integrands are not included in the above list. On the other hand, the HP-48 will perform symbolic integration on the integral

$$\int_a^b \frac{1}{\sin x \cos x}\, dx$$

because the integrand is one of the selected patterns that is built-in. Because of all this, you should not view the HP-48G/GX as a serious symbolic integrator. Nevertheless, we will briefly outline some of its symbolic integration features so that you will be familiar with them.

Whether or not the ∫ function performs numerical or symbolic integration depends upon whether numerical or symbolic execution mode is active. The default state of the HP-48G/GX is for symbolic execution (flag -3 clear). In this state, the ∫ function uses a built-in system of pattern matching and returns a symbolic result (which may be nothing more than the original symbolic input). If you specify numerical results mode by setting flag -3, then the ∫ function will return a numerical result. No matter what the setting of flag -3, you can temporarily achieve numerical results by applying the →NUM command to evaluate an integral.

EXAMPLE 6.

(i) Make certain that your HP-48G/GX is in its default state for symbolic results. With '∫ (0, π, SIN(X), X)' on level 1, [EVAL] returns the following symbolic result.

```
1:  '-COS(X) / ∂ X(X) | (X = π
    ) - (-COS(X) / ∂ X(X) | (X
    = 0 ))'
```

The vertical stroke | is the "where" command, used to substitute values in an expression. You can recognize this result as the HP-48 version of the familiar symbolic expression

$$-\cos x \Bigg|_{x=0}^{x=\pi} .$$

Press EVAL again to effectively substitute the values π and 0 into -COS(X) and obtain the numerical result 2.

(ii) Again, with '∫ (0, π, SIN(X), X)' on level 1 and symbolic results active, press ↰ →NUM to temporarily set numerical results mode and obtain the numerical result 2.

(iii) You can achieve the same symbolic results as in (i) by using the INTEGRATE form in the SYMBOLIC Application. Do ↱ SYMBOLIC OK and then enter 'SIN(X)' into the EQ field, 'X' into the VAR field, and 0 and π into the LO and HI fields. With Symbolic highlighted, press OK to see the same symbolic results as in (i), then use EVAL to effect the substitution and obtain the numerical result 2.

Occasionally, you may want to use your HP-48 to obtain an antiderivative for a function $f(x)$. Recall that Part 1 of the Fundamental Theorem of Calculus tells us that every continuous function $f(x)$ on an interval $[a, b]$ has an antiderivative $F(x)$, namely

$$F(x) = \int_a^x f(t)dt.$$

Therefore, if the HP-48G/GX can symbolically integrate $f(t)$, we can obtain a symbolic expression for the antiderivative $F(x)$.

EXAMPLE 7. To obtain an antiderivative for $f(x) = lnx$, perform the symbolic integration

$$\int_a^x lnt\, dt.$$

Use 'T' for the variable of integration and make certain that the upper limit 'X' is a *formal variable*, i.e., no value for 'X' is stored in the current directory or any of its ancestral directories. Use lowercase 'a' for the lower limit of integration. When the first symbolic result appears, press [PRG] [TYPE] [OBJ→], then [DROP] [DROP] [DROP]. The remaining symbolic result will be

'(T * LN(T) – T) / ∂ T(T) | (T = X)'.

A final [EVAL] will return the desired antiderivative 'X * LN(X) – X'.

Activity Set 4.2.2

In activities 1-10, use your HP-48G/GX to find the indicated antiderivatives.

1. $\int \frac{7}{\sqrt{5x-1}}\, dx$
2. $\int (2x+3)^{-3/2}\, dx$
3. $\int tan^2\, x\, dx$
4. $\int \frac{tan\, x}{cos\, x}\, dx$
5. $\int tan^{-1}(2x+3)\, dx$
6. $\int 3(5x-\pi)^5\, dx$
7. $\int \frac{dx}{(tan\, .5x)(sin\, .5x)}$
8. $\int (x - tanh\, x)\, dx$
9. $\int 2^{x+1}\, dx$
10. $\int x^{s-1}\, dx$

4.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Chief among the significant contributions of Newton and Leibniz to the "invention" of calculus in the 17th century was their clarification of the inverse relationship between differentiation and integration. This relationship, which is the intended focus of the Fundamental Theorem of Calculus, is often obscured by a failure to focus on Part 1 of that Theorem, which asserts that continuous functions have antiderivatives:

$$(\ast) \qquad \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

Traditionally, our calculus textbooks have focused instead on Part 2 of the Theorem, which says that integration "undoes" differentiation, up to a constant:

$$\int_a^x F'(t)dt = F(x) - F(a).$$

Indeed, it is because of their focus on Part 2 that students often come to view integration as simply a search for antiderivatives rather than as the limit of Riemann sums. In retrospect, this has been a somewhat natural occurrence because, in the teaching process, teachers tend to seek out activities that students can *do* to help reinforce their understanding of the theory. And without computing power, the activities that reinforce Part 1 are restricted, for the most part, to purely analytical investigations.

But certainly, the HP-48G/GX provides enough computing power for students to engage in graphical and numerical activities that support Part 1 of the Fundamental Theorem. The midpoint approximation can be used to construct a symbolic expression $F(x)$ that approximates the antiderivative $\int_a^x f(x)dt$, i.e., $F(x) \approx \int_a^x f(t)dt$. This approximation and its derivative F' can then be plotted and we can observe to what extent F' approximates f. Not only does such an activity bring to the fore the mathematical content of Part 1 of the Theorem, but it also reinforces our desired goal of understanding the integral as a limit of approximating sums.

The algebraic formulation of the midpoint approximation using n subintervals of equal length $\Delta x = \dfrac{x-a}{n}$ is

$$\int_a^x f(t)dt \approx \sum_{i=1}^{n} f\left(a + (2i-1)\frac{\Delta x}{2}\right)\Delta x.$$

When f is stored in memory as a user-defined function F, program FTC, given below, takes a and n as inputs and returns an algebraic expression for the midpoint approximation on level 2 and its derivative on level 1.

The program is due to William C. Wickes of Hewlett Packard and is about fifteen times as fast as the original program that we devised for the task. It is a marvel of extremely clever programming, and is written at a level that will not be obvious to a casual HP-48 programmer. We are indebted to Dr. Wickes for his permission to use it here.

FTC (Fundamental Theorem of Calculus)

Input: level 2: the lower limit of integration, *a*

level 1: the number of rectangles, *n*

As a user-defined function *F*: an algebraic expression for *f(x)*.

Effect: Returns to level 2 an expression that is algebraically equivalent to the midpoint approximation

$$\sum_{i=1}^{n} f\left(a+(2i-1)\frac{\Delta x}{2}\right)\Delta x.$$

and to level 1 the derivative of that expression.

```
« 'X' PURGE 'X' 3 PICK – → a n z « a '←i' n / z * + F a '↑↓' n /
z * + F 'X' ∂ { ↑↓ ←i } ↓MATCH DROP → f g « 0 0 .5 n .5 – FOR ←i f
EVAL ROT + SWAP g EVAL + NEXT OVER z * n / 3 ROLLD z * + n / »
» »
```

Consider the elementary function $f(x) = \sin x$. Since f is continuous everywhere, Part 1 of the Fundamental Theorem tells us that the function

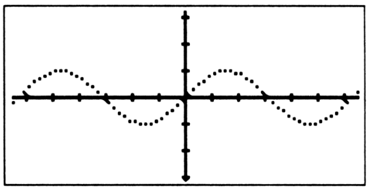
$$G(x) = \int_a^x \sin t \, dt$$

is an antiderivative of $\sin x$ on any interval (a, b). We take $a = 0$ for convenience. Then

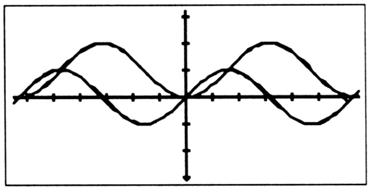
$$G(x) = \int_0^x \sin t \, dt = -\cos t \Big|_0^x = 1 - \cos x$$

and we can readily verify that $G'(x) = \sin x$.

To apply program **FTC**, begin by constructing a user-defined function F for $\sin x$ and then plotting $y = \sin x$ using the default settings for *Xrng* and *Yrng*. To better see what is happening, plot in disconnected mode with the resolution set to .2 (from the **PLOT** menu, use .2 [RES]; from the **PLOT** screen, select [OPTS] and enter .2 into the **STEP** field).



Now run program **FTC** with inputs 0 (for $a = 0$) and 6 (for $n = 6$ rectangles). The program will return to level 2 an expression in 'X' that represents the midpoint approximation to $G(x)$ using $n = 6$ rectangles, and its derivative on level 1. Change the plotting resolution back to its default sate of 0 and then overdraw the derivative on the original plot. Note the close agreement. Now change the resolution back to .2 and overdraw with a plot of the approximate antiderivative. Finally, to see how closely the approximate antiderivative agrees with the known exact antiderivative $1 - \cos x$, change the resolution back to 0 and overdraw a plot of $1 - \cos x$:



Differential Equations and the Fundamental Theorem

Equations that contain derivatives of one or more unknown functions are called *differential equations*. The simplest differential equations have the form $y' = f(x)$ and their general solution is given by $y = \int f(x)dx + C$. Since C is a constant, there are infinitely many solutions. But we can always obtain a particular solution by specifying an *initial condition* that y is required to meet: $y(x_0) = y_0$. Together, the differential equation with an initial condition

$$y' = f(x), \text{ where } y(x_0) = y_0$$

is called an *initial value problem.*

Part 1 of the Fundamental Theorem of Calculus is really an initial value problem. For if we adopt the notation $y(x) = \int_a^x f(t)dt$, then equation ($\star$) of Part 1 of the Theorem reads

$$\frac{dy}{dx} = f(x), \text{ or simply } y' = f(x).$$

Since $y(x)$ represents the (signed) area between the graph of f and the horizontal axis over the interval $[a, x]$, we have the initial condition $y(a) = 0$. Thus, equation ($\star$) of the Fundamental Theorem is really the initial value problem

$$y' = f(x), \quad y(a) = 0.$$

The HP-48G/GX will not only find numerical solutions to initial value problems but will also plot their solutions. To plot a solution to the initial value problem $y'(t) = f(t, y)$, $y(t_0) = y_0$ we go to the [↱] PLOT screen and choose Diff Eq for the plot TYPE. Although the screen will show

PLOT Y'(T) = F(T, Y)

at the top, the default independent variable is 'X', which is fine for our application to the Fundamental Theorem. The application we are referring to, of course, is to simply plot the antiderivative given by Part 1, using its reformulation as an initial value problem.

The special Diff Eq plot screen is designed to let you plot a solution to the general initial value problem $y'(t) = f(t, y)$, subject to the initial condition $y(t_0) = y_0$, over the t-interval $[t_0, t_f]$. For our purposes, we will use the default variable 'X' instead of 'T', and take the initial value of Y to be 0.

EXAMPLE. To illustrate the use of the Diff Eq plotting routine, we will plot an antiderivative $\int_a^x f(t)dt$ for $f(x) = lnx$. For convenience, we take $a = 1$. Thus we want to plot the solution to the initial-value problem

$$y'(x) = lnx, \quad y(1) = 0.$$

Go to the Diff Eq plot screen and set the screen like this:

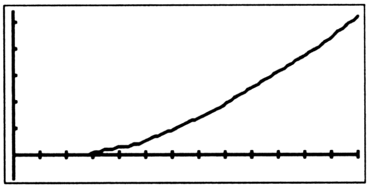
TYPE:	Diff Eq			∡ : Rad
F:	'LN(X)'			
INDEP:	X	INIT:	1	FINAL: 6
SOL:	Y	INIT:	0	_ STIFF

Open OPTS and set the PLOT OPTIONS like this:

TOL:	.000001	STEP:	Dflt	✓ AXES	
H-VAR:	0	H-VIEW:	0	6	
V-VAR:	1	V-VIEW:	-1	6	
H-TICK:	10	V-TICK:	10	✓ PIXELS	

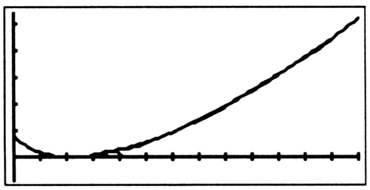
Note: *H-VIEW and V-VIEW correspond to Xrng and Yrng, respectively.*

Press OK to return to the previous screen, then ERASE and DRAW to see a plot of the antiderivative.



The differential equations plotter leaves values stored in X and Y, so you should now purge X and Y from your VAR menu.

In this case, we know a closed-form expression for the antiderivative: $\int lnxdx = xlnx - x + C$. To meet the initial condition $y(1) = 0$, we must choose $C = 1$. If you now overdraw your plot of the initial value solution with a plot of $y = xlnx - x + 1$ (first, be sure to reset the function type to FUNCTION, and the independent variable to 'X' and the dependent variable to 'Y'), you will see that the two plots are in close agreement for $x \geq 1$.



This use of the built-in plotter for numerical solutions to initial value problems is especially helpful for viewing plots of antiderivatives that have no elementary closed-form expressions.

Activity Set 4.3

1. (a) Build a user-defined function F for $y = x \cos x$.

   (b) Plot $y = x \cos x$ in disconnected mode using Resolution .1 (if you are plotting from the PLOT menu) or STEP size .1 (if you are plotting from the PLOT screen), with *Xrng*: 0 6.28 and *Yrng*: -6.3 1.

   (c) Run program FTC with inputs 0 and 6 to construct an approximation to the antiderivative $\int_0^x t \cos t \, dt$ using the midpoint rule with $n = 6$ rectangles.

   (d) Change the resolution (or STEP size) back to 0 and overdraw with a plot of the derivative of your approximate antiderivative. How closely does it appear to approximate $y = x \cos x$?

   (e) Now reset the resolution to .1 and overdraw with a plot of your approximate antiderivative found by FTC in (c).

   (f) Use integration by parts to find an elementary antiderivative of $y = x \cos x$. Choose an initial condition so your antiderivative will pass through (0, 0). Reset the resolution to 0 and overdraw your plot in (e) with this exact antiderivative. How closely do the two plots appear to agree?

2. Repeat Activity 1 using the function $y = x \sin x$. Use *Yrng*: -5.3 2. Zoom out on the vertical axis by a factor of 1.5 after plotting in part (b).

In Activities 3 - 5, the function f is known to have no elementary, closed-form antiderivative. Proceed as in Activity 1, parts (a) - (e). Note any special conditions.

3. Let $f(x) = e^{-x^2}$. Draw the initial plot in disconnected mode with resolution .1 using *Xrng*: -2 2 and *Yrng*: -2 2. Use $a = 0$ and $n = 6$ for program FTC. Reset the resolution to 0 before plotting the results of FTC.

4. Let $f(x) = \sin x^2$. Plot everything in connected mode with default resolution 0 over *Xrng*: 0 6.28 and *Yrng*: -2.5 2.5. Use $a = 0$ and $n = 20$ for program FTC. The higher value for n is needed because of the more frequent oscillations in the graph of f. Notice that the derivative of the approximate antiderivative begins to deviate from f as the oscillations increase in frequency.

5. Let $f(x) = \frac{e^x}{x}$. Draw the initial plot in disconnected mode with resolution .1 using *Xrng*: -6.5 6.5 and *Yrng*: 0 6.3. Use $a = .1$ and $n = 6$ for program FTC. Before plotting the results from FTC, reset to connected mode with resolution 0 and the independent variable restricted to plot only from 0 to 6.5.

6. Use the Differential Equations Plot Screen to plot the following antiderivatives; use the indicated settings for H-VIEW and V-VIEW.

   (a) $\int_0^x e^{-x^2} dx$; H-VIEW: -2 2 and V-VIEW: -2 2

   (b) $\int_0^x \sin x^2 \, dx$; H-VIEW: 0 6.28 and V-VIEW: -2.5 2.5

   (c) $\int_{.1}^x x^{-1} e^x \, dx$; H-VIEW: -6.5 6.5 and V-VIEW: 0 6.3

4.4 IMPROPER INTEGRALS

In applications of calculus we often meet improper integrals like $\int_a^\infty f(x)dx$. The meaning is clear:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx .$$

If the limit is the real number L then we say that the improper integral $\int_a^\infty f(x)dx$ *converges* to L and write

$$\int_a^\infty f(x)dx = L$$

If the limit does not exist (is not a real number), we say that the improper integral $\int_a^\infty f(x)dx$ *diverges*.

Assume that we have a convergent improper integral, say $\int_a^\infty f(x)dx = L$. Then for any value of $N > a$ we have

$$L = \int_a^N f(x)dx + \int_N^\infty f(x) .$$

The second integral in this sum is called the "tail" and if we can choose N so that the tail is "sufficiently small", then we can approximate L with the ordinary integral $\int_a^N f(x)dx$. We measure "sufficiently small" by specifying an acceptable error tolerance $\epsilon > 0$, and then attempt to find a value of N for which

$$\left|\int_N^\infty f(x)dx\right| = \left|L - \int_a^N f(x)dx\right| < \epsilon.$$

Thus, to within the tolerance specified by ϵ, we have $\int_a^N f(x)dx \approx L$. The following result is of some help.

Absolute Comparison Theorem

Suppose that f and g *are continuous functions for* $x \geq a$ *and there is a constant* K *such that* $|f(x)| \leq K\, g(x)$ *whenever* x *is sufficiently large. Then if* $\int_a^\infty g(x)dx$ *converges, so does* $\int_a^\infty f(x)dx$ and

$$\left|\int_a^\infty f(x)dx\right| \leq \int_a^\infty |f(x)|\,dx \leq K\int_a^\infty g(x)\,dx.$$

Two convergent improper integrals that are useful for such comparisons are:

$$\int_a^\infty \frac{1}{x^p}dx = \frac{1}{(p-1)a^{p-1}} \quad \text{for } p > 1.$$

$$\int_a^\infty e^{-cx}\,dx = \frac{1}{ce^{ca}} \quad \text{for } c > 0.$$

(1) Suppose that for sufficiently large x, $|f(x)| \leq \frac{K}{x^p}$ for some $K > 0$ and $p > 1$. Then by the Absolute Comparison Theorem we have

$$\left|\int_N^\infty f(x)dx\right| \leq \int_N^\infty |f(x)|\,dx \leq K\int_N^\infty \frac{1}{x^p}dx = \frac{K}{(p-1)N^{p-1}}.$$

With $N > 0$, to have $\dfrac{K}{(p-1)N^{p-1}} < \epsilon$ we choose $N > \left\{\dfrac{K}{\epsilon}\cdot\dfrac{1}{p-1}\right\}^{1/(p-1)}$.

(2) Suppose that for sufficiently large x, $|f(x)| \le Ke^{-cx}$ for some $K > 0$ and $c > 0$. Then we have

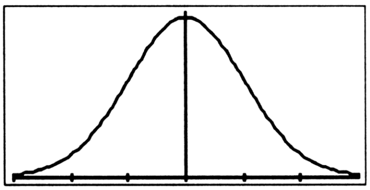
$$\left|\int_N^\infty f(x)dx\right| \le \int_N^\infty |f(x)|\,dx \le K\int_N^\infty e^{-cx}\,dx = \frac{K}{ce^{cN}}.$$

To have $\dfrac{K}{ce^{cN}} < \epsilon$, choose $N > \dfrac{1}{c}\,ln\left(\dfrac{K}{c\,\epsilon}\right)$.

EXAMPLE 8. The improper integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}dx$$

is important in probability theory. A plot of the integrand $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-x^2}$ over the interval [-3, 3] appears below.



Now $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = \lim_{t\to-\infty}\int_t^0 f(x)dx + \lim_{t\to\infty}\int_0^t f(x)dx$. Since $\int_0^t f(x)dx$ is the area of the region between the graph of $y = f(x)$ and the x-axis over the finite interval [0, t], we can interpret the improper integral $\int_0^\infty f(x)dx$ as the area of the "infinite" region between the graph of $y = f(x)$ and the x-axis to the right of $x = 0$.

Thus the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is the area of the entire infinite region between the graph of $y = f(x)$ and the x-axis. By symmetry, to show that $\int_{-\infty}^{\infty} f(x)dx$ converges, it suffices to show that $\int_{0}^{\infty} f(x)dx$ converges.

Now $|f(x)| = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} < \frac{1}{\sqrt{2\pi}} e^{-x/2} < e^{-x/2}$ for $x > 1$. Thus $\int_{0}^{\infty} f(x)dx$ converges because $\int_{0}^{\infty} e^{-x/2}\,dx$ converges. Moreover, we can take $K = 1$ and $c = \frac{1}{2}$ in (2) above, so that

$$N > \frac{1}{c}\, ln\left(\frac{K}{c\,\epsilon}\right) = 2\, ln\left(\frac{2}{\epsilon}\right).$$

With $\epsilon = 10^{-11}$, $N > 2\, ln\left(\frac{2}{10^{-11}}\right) \approx 52.04$ and we can approximate

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \text{ with } \int_{0}^{52.04} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx .$$

Evaluating this last integral with the HP-48G/GX we obtain .500000000001, so

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \approx 1.00000000002 \text{ to within } \epsilon = 10^{-11}.$$

The exact value is 1.

Activity Set 4.4

In each of the following, establish the convergence and then evaluate the improper integral to within the specified tolerance ϵ.

1. $\displaystyle\int_{\pi/4}^{\infty} \frac{\sin x}{x^4}\, dx$; use $\epsilon = .01$

2. $\displaystyle\int_{0}^{\infty} \frac{x}{\sqrt{x^6 + 4}}\, dx$; use $\epsilon = .001$

3. $\displaystyle\int_{0}^{\infty} \frac{xe^{-2x}}{\sqrt[3]{x^3 + 1}}\, dx$; use $\epsilon = .001$

4. $\displaystyle\int_{2}^{\infty} \frac{1}{\sqrt{x^5 - 1}}\, dx$; use $\epsilon = .001$

   (*Hint:* $\dfrac{1}{\sqrt{x^5 - 1}} < \dfrac{2}{\sqrt{x^5}}$ *for what values of* x *?)*

5. $\displaystyle\int_{0}^{\infty} \sqrt{x + 1}\, e^{-2x} dx$; use $\epsilon = .005$

   (*Hint:* $\sqrt{x + 1}\; e^{-2x} = \left(\sqrt{x + 1}\; e^{-(x+1)}\right) e\, e^{-x}$; *what is the maximum value of* $\sqrt{x + 1}\, e^{-(x+1)}$*?)*

5 INFINITE SERIES

Approximations by infinite processes are central to calculus. The concept of the limit of a function

$$\lim_{x \to c} f(x) = L ,$$

which is fundamental to the development of so much in calculus, has its roots in the intuitive notion that as x approaches the number c through an infinite succession of increasingly better approximations

$$x_1, x_2, x_3, \ldots \to c$$

then the corresponding function values are an infinite succession of increasingly better approximations to the limit L :

$$f(x_1), f(x_2), \mathrm{f}(x_3), \ldots \to L .$$

Infinite series, which are expressions of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots ,$$

also represent approximations by an infinite process. Since the core of any such series is the sequence of terms

$$a_1, a_2, a_3, \ldots ,$$

we usually begin a study of series by first considering sequences.

5.1 SEQUENCES

A *sequence* of numbers

(1) $$a_1, a_2, a_3, \ldots$$

is simply an *infinite ordered list*. We often use the compact notation $\{ a_k \}_{k=1}^{\infty}$ to represent the sequence (1), and a_k denotes the k^{th} *term* of the sequence.

More precisely, we can view the sequence (1) as the output values of a function f that is defined only for the positive integers k = 1, 2, 3, ... :

$$\begin{matrix} a_1, & a_2, & a_3, & \ldots \\ f(1), & f(2), & f(3), & \ldots \end{matrix}$$

We can use the HP-48G/GX to calculate and view the terms of a sequence. Program SHO, given below, will calculate and show the consecutive terms of a sequence $\{ a_k \}_{k=1}^{\infty}$ from a specified starting value of k to a specified ending value. This program, and all the others in this chapter, can be found in the SERIES subdirectory of the main CALC directory.

SHO (Show sequence terms)

Input: Level 3-5: expressions for the k^{th} terms of 1-3 sequences $\{f_k\}$, $\{g_k\}$ and $\{h_k\}$ in terms of the variable 'K'

Level 2: a starting value for 'K'

Level 1: an ending value for 'K'

Effect: dynamically displays (every two seconds) the successive terms of 1-3 sequences $\{f_k\}$, $\{g_k\}$ and $\{h_k\}$ from the starting value of k to the ending value, beneath the index k; leaves everything on the stack.

```
« DEPTH → b n d « CLLCD IF d 3 == THEN → f « b n FOR j j
'K' STO K DUP 1 DISP f EVAL DUP 3 DISP 2 WAIT NEXT » 'K' PURGE
ELSE IF d 4 == THEN → f g « b n FOR j j 'K' STO K DUP 1 DISP f
EVAL DUP 3 DISP g EVAL DUP 5 DISP 2 WAIT NEXT » 'K' PURGE
ELSE → f g h « b n FOR j j 'K' STO K DUP 1 DISP f EVAL DUP 3
DISP g EVAL DUP 5 DISP h EVAL DUP 7 DISP 2 WAIT NEXT » 'K'
PURGE END END » »
```

EXAMPLE 1.

(a) Consider the sequence $\{f_k\}_{k=1}^{\infty}$ where $f_k = \frac{1}{k}$. To see the first 25 terms of this sequence, arrange the stack as follows and press [SHO]

```
3: '1/K'
2:    1
1:   25
```

The display will show, in timed two second intervals, the first 25 terms of the index k and the sequence $\{ 1/k \}$. When done, the stack will contain everything that was displayed, so that you can scroll upward with the **[Δ]** and view any particular term.

(b) Consider the two sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, where $f_k = \frac{1}{k}$ and $g_k = \frac{1}{k^2}$.

To see terms 10 through 20 of these two sequences, arrange the stack as follows and run program SHO.

```
4: '1/K'
3: '1/K ^ 2'
2: 10
1: 20
```

The display will show, in timed two second intervals, terms 10 through 20 of each sequence, with sequence $\{f_k\}$ being above sequence $\{g_k\}$ on the display screen, just below the index k. When done, everything is left on the stack for your perusal.

It is helpful to regard the terms of the sequences $\{f_k\}$ and $\{g_k\}$ in EXAMPLE 1 as being sample values from the two ordinary functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, defined for all $x \geq 1$. The graphs of the functions f and g consist of all points in the plane $\left(x, \frac{1}{x}\right)$ and $\left(x, \frac{1}{x^2}\right)$ for $x \geq 1$, respectively. Therefore, the graphs of the sequences $\{f_k\}$ and $\{g_k\}$ will consist of the *discrete* points $\left(k, \frac{1}{k}\right)$ and $\left(k, \frac{1}{k^2}\right)$ for $k = 1, 2, 3, \ldots$.

To plot the graph of a sequence $\{g_k\}_{k=1}^{\infty}$, we can use program GSEQ[1].

---

[1]Program GSEQ and a later one GPS were written by Mr. Robert E. Simms of Clemson University. We are indebted to Mr. Simms for permission to use his programs here.

GSEQ (Sequence Graph)

Input: Level 1: an expression for the k^{th} term a_k of a sequence { a_k } in terms of the variable 'K'

Level 2: the number of discrete points on the graph of the sequence { a_k } that you wish to see.

As a stored variable: program SDRW (below), which scales and plots the points that are created by GSEQ and stored in the variable ΣDAT

Effect: draws coordinate axes and plots, in sequential order, the specified number of discrete points (k, a_k) on the graph of the sequence { a_k }

```
« # 131d # 64d PDIM 0 → eq n k « eq { K k } ↑MATCH DROP 'eq'
STO [ 0 0 ] 1 n FOR k k NEXT n →LIST 1 « DUP 'k' STO eq →NUM 2
ROW→ » DOLIST » OBJ→ 1 + ROW→ 'ΣDAT' STO 1 XCOL 2 YCOL
SDRW 7 FREEZE »
```

SDRW (a utility subroutine)

Effect: Used by GSEQ and GPS to scale and plot the *xy*-data in the matrix ΣDAT

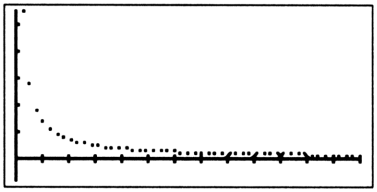
```
« SCLΣ DRAX { # 0d # 0d } PVIEW ΣDAT SIZE 1 GET 1 SWAP FOR
i ΣDAT i ROW- OBJ→ DROP R→C PIXON DROP NEXT »
```

EXAMPLE 2. To plot the first 50 terms of the sequence { a_k } where $a_k = \frac{1}{k}$, arrange the stack as follows:

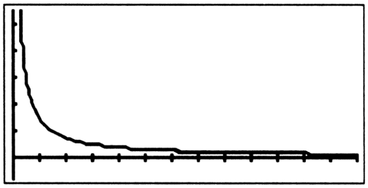
2: '1/K'

1: 50

Now press GSEQ to see the following plot develop:



To graphically verify that the graph of $a_k = \frac{1}{k}$ is simply a discrete sampling of points from the graph of the function $f(x) = \frac{1}{x}$, overdraw the above plot with a plot of the graph of $f(x) = \frac{1}{x}$.



Activity Set 5.1

1. (a) Use program SHO to calculate and view the first 25 terms of the sequence $\{ a_k \}$, where $a_k = \frac{k^2}{2^k}$. Do these terms seem to be approaching a limit?

   (b) Use program GSEQ to plot the first 25 points on the graph of the sequence $\{ a_k \}$. Does the graph suggest that $\lim_{k \to \infty} a_k$ exists?

   (c) Consider the graph in (b) as a discrete sampling of points on the graph of the continuous function $f(x) = \frac{x^2}{2^x}$ for $x \geq 0$. Overdraw your plot in (b) with the graph of f. What does the graph of f say about $\lim_{x \to \infty} \frac{x^2}{2^x}$?

   (d) Use l'Hospital's Rule to analytically find $\lim_{x \to \infty} \frac{x^2}{2^x}$. What is $\lim_{k \to \infty} \frac{k^2}{2^k}$?

2. (a) Use program SHO to calculate and view the first 25 terms of the sequence $\{ a_k \}$ where $a_k = \frac{\ln k}{k}$.

(b) Now use program GSEQ to plot the first 50 point on the graph of $\{ a_k \}$. Does the graph suggest a limit for $\{ a_k \}$?

(c) Overdraw your plot in (b) with the graph of the function $f(x) = \frac{\ln x}{x}$. Does the graph suggest that $\lim_{x \to \infty} \frac{\ln x}{x}$ exists?

(d) Use l'Hospital's Rule to investigate the limit $\lim_{x \to \infty} \frac{\ln x}{x}$. Based on your results, what can you conclude about $\lim_{k \to \infty} \frac{\ln k}{k}$?

3. (a) Use program SHO to calculate and view the first 25 terms of the sequence $\{ a_k \}$ where $a_k = \frac{\sin k}{e^{.05k}}$. Do these numbers convey to you a sense of what is happening to the terms a_k as $k \to \infty$?

(b) Use program GSEQ to plot the first 50 points on the graph of $\{ a_k \}$. What does the plot suggest?

(c) Overdraw your plot in (b) with the graph of $f(x) = \frac{\sin x}{e^{.05x}}$, for $x \geq 0$. What does the new plot suggest about the limit $\lim_{x \to \infty} \frac{\sin x}{e^{.05x}}$ and the limit $\lim_{k \to \infty} \frac{\sin k}{e^{.05k}}$? Is l'Hospital's Rule of any help here?

4. (a) Use SHO to calculate and view the first 25 terms of the sequence $\{ a_k \}$ where $a_k = (-1)^{k+1}\left(\frac{1}{k}\right)$. Do these numbers convey to you a sense of the limit $\lim_{k \to \infty} (-1)^{k+1}\left(\frac{1}{k}\right)$?

(b) Use **GSEQ** to plot the first 50 points on the graph of the sequence $\{ a_k \}$. Is $\lim_{k \to \infty} (-1)^{k+1}\left(\frac{1}{k}\right)$ any more apparent?

(c) Overdraw your plot in (b) with the graphs of a function f and its negative $-f$. Now what can you conclude about the limit of the sequence $\{ a_k \}$?

5. Use program **GSEQ** to investigate the limits of the following sequences $\{ a_k \}$:

(a) $a_k = \dfrac{k}{\sqrt{k} + 1}$ (b) $a_k = \dfrac{\cos k}{\sqrt{k}}$

6. (Just for fun!) Plot the first 300 terms of the graph of the sequence $\{ a_k \}$, where $a_k = \sin k$.

5.2 SERIES

What do we mean by an *infinite series*?

(1) $$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

How can we possibly sum infinitely many numbers?

This is exactly the same kind of question we face when confronted with an improper integral of the form

(2) $$\int_1^{\infty} f(x)dx \ .$$

How can we possibly integrate from 1 to ∞ ?

Infinite series and improper integrals have much more in common that mere superficial appearances. Indeed, they behave in very similar ways.

The improper integral (2) is a limit of ordinary integrals:

$$\int_1^{\infty} f(x)dx = \lim_{t \to \infty} \int_1^{t} f(x)dx \ .$$

In a similar way, the infinite series (1) is a limit of ordinary sums:

(3)
$$\sum_{k=1}^{\infty} a_k = \lim_{N \to \infty} \sum_{k=1}^{N} a_k \ .$$

The ordinary sum $\sum_{k=1}^{N} a_k$ is called a *partial sum* of the series. Indeed, the partial sums of the series form a sequence $\{ s_N \}$, where the terms are:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$\vdots$

$s_N = a_1 + a_2 + \ldots + a_N = \sum_{k=1}^{N} a_k$

If the sequence of partial sums $\{ s_N \}$ has a limit S

$$\lim_{N \to \infty} \{ s_N \} = S \ ,$$

then we call S *the sum of the infinite series* $\sum_{k=1}^{\infty} a_k$ and write

$$S = \sum_{k=1}^{\infty} a_k \ .$$

In this case we also say that the series *converges* to the sum S. If the sequence of partial sums fails to converge to a limit (a real number), then we say that the series *diverges*. It is no wonder that students find infinite series difficult to study. We are combining the terms of one sequence $a_1, a_2, a_3, \ldots$ to form the terms of a new sequence of partial sums $s_1, s_2, s_3, \ldots$, and then have to consider the limit of the sequence of partial sums:

$$S = \lim_{N\to\infty} s_N = \lim_{N\to\infty} \sum_{k=1}^{N} a_k .$$

How can the HP-48G/GX be of use in a study of a subject that is so analytically complex as infinite series?

To begin, we can use the built-in Σ function to quickly calculate partial sums. For example, consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} .$$

Go to the Equation Writer and build the following expression for the N^{th} partial sum $s_N = \sum_{k=1}^{N} \frac{1}{2^{k-1}}$:

$$\sum_{K=1}^{N} \frac{1}{2^{K-1}}$$

Press [ENTER] to throw it onto the stack:

1: 'Σ (K = 1, N, 1/2 ^ (K – 1))'

Open the **SOLVE** application with [↰] [SOLVE], press [ROOT] and load the expression on level 1 into **EQ** with [↰] [EQ] . Now open the [SOLVR] where you will see boxes labeled [K] [N] [EXPR=]. Ignore the box [K] . Put 10 into [N] and press [EXPR=] to see the 10^{th} partial sum

1: Expr: 1.998046875 .

Repeat by putting 20, 30, 35 into [N] and using [EXPR=] to obtain the 20^{th}, 30^{th} and 35^{th} partial sums:

4: Expr: 1.998046875
3: Expr: 1.99999809266
2: Expr: 1.99999999814

1: Expr: 1.99999999995

Is there any doubt that the sequence $\{ s_N \}$ of partial sums is converging to $S = 2$?

$$\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2$$

EXAMPLE 3. The series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ is an example of a *p-series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$, known to converge if and only if $p > 1$.

What is the sum? Using the **SOLVR** as above to calculate partial sums, we have

$s_{100} = 1.08232290538$

$s_{200} = 1.08232319242$

$s_{300} = 1.08232322151$

$s_{400} = 1.08232322861$

$s_{500} = 1.08232323119$

$s_{600} = 1.08232323227$

$s_{700} = 1.08232323295$

$s_{800} = 1.08232323295$

Since these last two partial sums agree to 11 decimal places, we have determined that the sum is $S = 1.08232323295$ to within the precision of the HP-48G/GX.

As an alternative to using the **SOLVR**, you can use the following program **INFSUM**. This program, a modification of one due to William C. Wickes of Hewlett Packard in [1], shows the convergence of the series dynamically, by showing the partial sums as a single number with the last digits changing as more terms are added. The program sums series that begin with initial index $k = 1$, so you will have to make adjustments for series that do not start there. It should only be used with series that are *known to converge*.

INFSUM

Input: the term a_k, for the infinite series $\sum_{k=1}^{\infty} a_k$, in terms of the variable 'K'

Effect: calculates partial sums $\sum_{k=1}^{N} a_k$ until two successive sums agree, displays the last partial sum and the value of n at which agreement was reached

```
« →f « 1 'K' STO f EVAL 2 'K' STO DO DUP f EVAL + DUP 3
DISP 1 'K' STO+ SWAP UNTIL OVER == END K 1 – 'K' PURGE »
```

Continuing with EXAMPLE 3, put '1/K ^ 4' on level 1 and run program INFSUM. You will see the partial sums accumulate dynamically at the top left of the display screen, until two consecutive sums agree to the precision of the HP-48G/GX. This agreement is reached when $N = 669$. Thus to the precision of the machine, the sum is $S \approx 1.08232323295 = \sum_{k=1}^{669} \frac{1}{k^4}$.

EXAMPLE 4. The series $\sum_{k=1}^{\infty} (-1)^{k+1}(1/k^6)$ has terms that alternate in sign. It converges by the alternating series test. As an alternating series, we know that the error made in using any partial sum s_n as the sum of the series is less than the absolute value of the term to be added to get the next partial sum s_{n+1}. For twelve place accuracy, we must take n large enough so that $1/(n + 1)^6 < 5\times10^{-13}$. Using the HP-48 for the calculation, we find that $1/113^6$ is approximately 4.8×10^{-13}. Thus, calculating the 112^{th} partial sum with the SOLVR we obtain .985551091299 as our estimate of the sum, to 12 decimal places. (You can calculate the 113th partial sum to see if you get agreement or run program INFSUM.)

Since the partial sums of a series $\sum_{k=1}^{\infty} a_k$ form a sequence $\{ s_k \}$, it is also helpful to plot the graph of this sequence. Program GPS does that. Like its predecessor GSEQ for sequences, the code is due to Robert E. Simms.

GPS (Graphical partial sums)

Input: Level 2: an expression for the k^{th} term a_k of the series $\sum_{k=1}^{\infty} a_k$, in terms of the variable 'K'

Level 1: the number of partial sums of the series $\sum_{k=1}^{\infty} a_k$ that you wish to plot

Effect: draws coordinate axes and plots, in sequential order, the specified number of points (k, s_k) on the graph of the sequence of partial sums $\{ s_k \}$ for the series $\sum_{k=1}^{\infty} a_k$

```
« # 131d # 64d PDIM 0 0 → eq n s k « eq { K k } ↑MATCH DROP
'eq' STO [ 0 0 ] 1 n FOR k k NEXT n →LIST 1 « DUP 'k' STO eq →NUM
s + DUP 's' STO 2 ROW→ » DOLIST » OBJ→ 1 + ROW→ 'ΣDAT' STO 1
XCOL 2 YCOL SDRW 7 FREEZE »
```

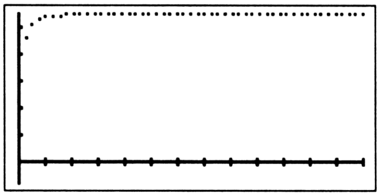
EXAMPLE 5. To show the use of program GPS, we graph the first 50 partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ and $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$.

For the first one, arrange the stack as follows:

2: '1/K ^ 3'

1: 50

Now press GPS to see



Despite the fact that these partial sums appear to level off rather quickly, convergence is *extremely* slow. Program **INFSUM** will approximate the sum S to the full precision of the HP-48 by the 5,849th partial sum

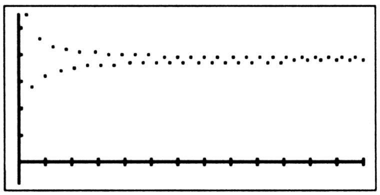
$$\sum_{k=1}^{5849} \frac{1}{k^3} = 1.20205689144 .$$

For the alternating harmonic series, $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}$, arrange the stack like

2: '(-1) ^ (K + 1) * (1/K)'

1: 50

and press GPS to see



Activity Set 5.2.1

1. Consider the series $\sum_{k=0}^{\infty} 10^k/k!$

   (a) Apply a standard test to show that the series converges.

   (b) Plot a graph of the first 25 partial sums of the series.

   (c) Use the **SOLVR** to obtain a 12-digit approximation to the sum of the series.

   (d) Use program **INFSUM** to obtain a 12-digit approximation to the sum. Which partial sum gives a full precision approximation?

   (e) Overdraw your plot in (b) with the sum of the series.

2. Repeat Activity 1 with the following series:

   (a) $\sum_{k=1}^{\infty} 1/k!$ (b) $\sum_{k=1}^{\infty} 5^k/k!$ (c) $\sum_{k=1}^{\infty} (-1)^{k+1}(k/2^k)$

3. Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{1}{k!}\right)$.

   (a) Prove that the series converges.

   (b) Find a value for n so that the n^{th} partial sum will approximate the sum of the series to 12 decimal digits.

   (c) Use the **SOLVR** to calculate the partial sum s_n for the value of n in (b).

4. Prove that the series $\sum_{k=1}^{\infty} (k+1)/k^{10}$ converges. What is the sum to 12 decimal digits?

5. (a) Prove that $\sum_{k=1}^{\infty} k!/(k+2)!$ converges.

(b) Plot a graph of the first 25 partial sums of this series.

(c) Apply program INFSUM to find the sum of the series. Watch closely what takes place. How can you explain this?

Series and Improper Integrals

We mentioned earlier that the connection between series and improper integrals was more than cosmetic:

$$\sum_{k=1}^{\infty} a_k \quad \text{versus} \quad \int_1^{\infty} f(x)dx .$$

Indeed, for series of positive terms we have the *integral test*.

Integral Test

Let $f(x)$ be a continuous, positive, decreasing function for $x \geq 1$ and let $a_k = f(k)$ for $k = 1, 2, \ldots$. Then if either $\int_1^{\infty} f(x)dx$ or $\sum_{k=1}^{\infty} a_k$ converges, both converge. If either $\int_1^{\infty} f(x)dx$ or $\sum_{k=1}^{\infty} a_k$ diverges, both diverge.

In other words, the series $\sum_{k=1}^{\infty} a_k$ and the integral $\int_1^{\infty} f(x)dx$ *converge or diverge together*.

Suppose that we have a convergent pair, $S = \sum_{k=1}^{\infty} a_k$ and $I = \int_1^{\infty} f(x)dx$, as above. In Chapter 4 (Section 4.4.) we saw that

$$I = \int_1^\infty f(x)dx = \int_1^N f(x)dx + \underbrace{\int_N^\infty f(x)dx}_{\text{(the "tail")}}.$$

If we can make the tail of the integral small enough, then we can approximate the improper integral I with the ordinary integral $\int_1^N f(x)dx$: $I \approx \int_1^N f(x)dx$.

Similarly,

$$S = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{N} a_k + \underbrace{\sum_{k=N+1}^{\infty} a_k}_{\text{(the "tail", } R_N)}$$

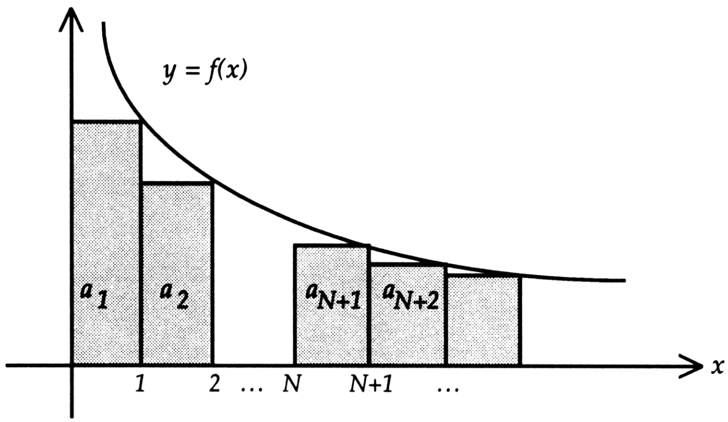
And if we can make the tail of the series small enough, then we can approximate the series with the ordinary sum $\sum_{k=1}^{N} a_k$: $S \approx \sum_{k=1}^{N} a_k$.

As with improper integrals, we measure "small enough" by specifying an acceptable error tolerance $\epsilon > 0$, and then attempt to find a value for N so that

$$\left| \sum_{k=N+1}^{\infty} a_k \right| = \left| S - \sum_{k=1}^{N} a_k \right| < \epsilon .$$

Then, to within the tolerance specified by ϵ, $S \approx \sum_{k=1}^{N} a_k$.

The trick, then, is to relate the tail R_N of the series to the tail of the integral. The following picture tells all:



$$R_N = \sum_{k=N+1}^{\infty} a_k = a_{N+1} + a_{N+2} + \ . \ . \ . \ \leq \int_N^{\infty} f(x)dx$$

(tail of the integral)

Since $f(x) \geq 0$ for all x, the **Absolute Comparison Theorem** for improper integrals applies (see Section 4.4).

(1) Suppose that for sufficiently large x, $f(x) \leq \frac{K}{x^p}$ for some $K > 0$ and $p > 1$. Then by (1) in Section 4.4, we have

$$R_N \leq \int_N^{\infty} f(x)dx \leq \frac{K}{(p-1)N^{p-1}} .$$

Thus, to have $R_N \leq \frac{K}{(p-1)N^{p-1}} < \epsilon$, we choose

$$N > \left\{\frac{K}{\epsilon} \cdot \frac{1}{p-1}\right\}^{1/(p-1)} .$$

(2) Suppose that for sufficiently large x, $f(x) \leq Ke^{-cx}$ for some $K > 0$ and $c > 0$. Then by (2) in Section 4.4., we have

$$R_N \le \int_N^{\infty} f(x)dx \le \frac{K}{ce^{cN}} .$$

To have $R_N \le \frac{K}{ce^{cN}} < \epsilon$, we choose $N > \frac{1}{c} \ln\left(\frac{K}{c\epsilon}\right)$.

EXAMPLE 6. Show that $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^5}$ converges and find its sum S to within $\epsilon = 10^{-6}$.

The natural integral to use is $\int_1^{\infty} \frac{\cos(x)}{x^5} dx$, which converges on comparison with $\int_1^{\infty} \frac{1}{x^5} dx$. Thus, the series $\sum_{k=1}^{\infty} \frac{\cos(k)}{k^5}$ also converges. Since $f(x) = \frac{\cos(x)}{x^5} \le \frac{1}{x^5}$ for $x \ge 1$, to approximate the sum S by $\sum_{k=1}^{N} \frac{\cos(k)}{k^5}$ to within $\epsilon = 10^{-6}$, we need to choose

$$N > \left\{\frac{1}{\epsilon} \cdot \frac{1}{5-1}\right\}^{1/(5-1)} = \left\{\frac{10^6}{4}\right\}^{1/4} \approx 22.36 .$$

Thus $S \approx \sum_{k=1}^{23} \frac{\cos(k)}{k^5} \approx .522820670966$ (Equation Writer, →NUM)

Activity Set 5.2.2

In each of the following, establish the convergence of the series and then find the sum to within the specified tolerance ϵ.

1. $\sum_{k=1}^{\infty} \frac{\sin k}{k^4}$, $\epsilon = .01$

2. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^6 + 4}}$, $\epsilon = .001$

3. $\sum_{k=1}^{\infty} a_k$, where $a_k = \frac{k\, e^{-2k}}{\sqrt[3]{k^3 + 1}}$

4. $\sum_{k=2}^{\infty} \frac{k}{\sqrt{k^5 - 1}}, \ \epsilon = .001$

5. $\sum_{k=1}^{\infty} ln\left(\frac{1}{k^2}\right), \ \epsilon = .0001$

The Ratio Test

The well-known *ratio test* states that for a series $\sum_{k=1}^{\infty} a_k$ of *positive* terms, if the ratios $\frac{a_{k+1}}{a_k}$ approach a limit r,

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$$

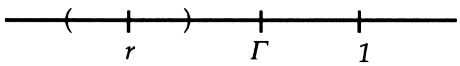
then the series converges for $r < 1$ and diverges for $r > 1$.

Suppose we have a series of positive terms $\sum_{k=1}^{\infty} a_k$ that is known to converge by the ratio test. As in the preceding section, we wish to approximate the sum S of the series by a finite sum $\sum_{k=1}^{N} a_k$ to within a specified tolerance ϵ. We must therefore choose a value for N that will make the absolute value of the tail less than ϵ:

(*) $$\left| \sum_{k=N+1}^{\infty} a_k \right| = \sum_{k=N+1}^{\infty} < \epsilon .$$

Let Γ be any number so that $r < \Gamma < 1$. Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$, we can choose N so that

(1) $$\frac{a_{k+1}}{a_k} < \Gamma \text{ for all } k \geq N.$$



For all $k \geq N$, the ratios $\frac{a_{k+1}}{a_k}$ will lie in the open interval centered at r.

If we can *also* choose N so that

$$(2) \qquad a_N\left(\frac{\Gamma}{1-\Gamma}\right) < \epsilon$$

then we will have our desired result (*). The justification is as follows.

Since we have chosen N so that $\frac{a_{k+1}}{a_k} < \Gamma$ for $k \geq N$ then we have

$$\begin{aligned} a_{N+1} &\leq \Gamma a_N \\ a_{N+2} &\leq \Gamma a_{N+1} \leq \Gamma^2 a_N \\ a_{N+3} &\leq \Gamma a_{N+2} \leq \Gamma^3 a_N \\ &\vdots \end{aligned}$$

etc.

Thus

$$\text{(i)} \quad a_{N+1} + a_{N+2} + \ldots + a_{N+M} < (\Gamma + \Gamma^2 + \ldots + \Gamma^M)\, a_N.$$

Since $|\Gamma| < 1$, the geometric series $\Gamma + \Gamma^2 + \Gamma^3 + \ldots$ converges to $\frac{\Gamma}{1-\Gamma}$. In fact, since the sequence of partial sums of this geometric series is bounded and increasing, we have

$$\text{(ii)} \quad \Gamma + \Gamma^2 + \ldots + \Gamma^M < \frac{\Gamma}{1-\Gamma} \text{ for all } M.$$

Combining (i) and (ii) we have

$$\text{(iii)} \quad a_{N+1} + \ldots + a_{N+M} < a_N\left(\frac{\Gamma}{1-\Gamma}\right).$$

Therefore, we can obtain (*) by choosing N so that, in addition to (1), also $a_N\left(\frac{\Gamma}{1-\Gamma}\right) < \epsilon$.

EXAMPLE 7. Consider the series $\sum_{k=1}^{\infty} \frac{k^{10}}{10^k}$. By the ratio test, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{10}$, so the series converges. We wish to approximate the sum of the series to within $\epsilon = 10^{-4}$. Choose $\Gamma = \frac{1}{2}$. To satisfy condition (1), choose N so that $\frac{1}{10}\left(\frac{k+1}{k}\right)^{10} < \frac{1}{2}$ for all $k \geq N$. This reduces to $\left(1+\frac{1}{k}\right)^{10} < 5$ for $k \geq N$, and the smallest such $N = 6$. To satisfy condition (2) we must *also* choose N so that $\frac{N^{10}}{10^N} < \epsilon = 10^{-4}$. Build a user-defined function for $G(N) = \frac{N^{10}}{10^N}$ and evaluate G for different values of N, starting with the value $N = 6$. We find that $N = 17$ is the first value that gives $G(N) < 10^{-4}$. Now evaluate the sum $\sum_{k=1}^{17} \frac{k^{10}}{10^k}$ on the HP-48G/GX to obtain $S \approx 376.17943$.

Activity Set 5.2.3

Establish the convergence of each of the following series by the ratio test and then find the sum of the series to within the specified tolerance ϵ.

1. $\sum_{k=1}^{\infty} \frac{k}{2^k}$, $\epsilon = 10^{-6}$

2. $\sum_{k=1}^{\infty} \frac{k^3}{e^k}$, $\epsilon = 10^{-6}$

3. $\sum_{k=1}^{\infty} \frac{(k+1)(k+2)}{k!}$, $\epsilon = 10^{-6}$

4. $\sum_{k=1}^{\infty} \frac{10^k k!}{(2k+1)!}$, $\epsilon = 10^{-6}$

5. $\sum_{k=1}^{\infty} \frac{.7^k (k+1)}{k}$, $\epsilon = 10^{-6}$

REFERENCE

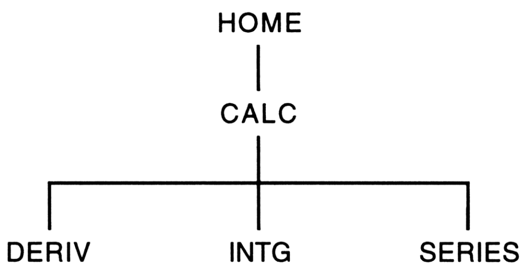
1. Wickes, William C., *HP-48 Insights, Vol. I, Principles and Programming of the HP-48, HP-48G/GX Edition*, Larken Publications, Department PC, 4517 NW Queens Avenue, Corvallis, OR 97330, 1993.

APPENDIX

TEACHING CODE: ORGANIZATION

The special-purpose HP-48G/GX programs for teaching single variable calculus that are contained in this book are called *teaching code*; a listing appears on the inside back cover. The teaching code is readily available on a diskette from the author for downloading to an HP-48G/GX from a microcomputer. This appendix shows how the teaching code is organized in files, or directories.

A factory-fresh HP-48G/GX calculator contains only the built-in HOME directory, indicated by the message { HOME } at the top left of the stack display screen. The teaching code for calculus is stored in a directory called CALC. The CALC directory contains three subdirectories, each one containing teaching code related to a major topic.



- Subdirectory DERIV. Contains the teaching code related to differentiation (see Chapters 2-3): INV.F, derXROOT, IM.y', Y', F.XY, INFL1, INFL2, NEWTON, TAY.A, TAYLAT, TAN.L, and PAR'.

- Subdirectory INTG. Contains the teaching code related to integration (see Chapter 4): FABSTO, NSTO, GRECT, LRECT, RRECT, MID, TRAP, SIMP, SUM, F.val and FTC.

- Subdirectory SERIES. Contains the teaching code related to series (see Chapter 5): SHO, GSEQ, INFSUM, GPS, and SDRW.

To execute any of these programs, open the CALC directory with the [CALC] key, then open the appropriate subdirectory with its menu key. Put the necessary inputs to a particular program on the stack and then execute the name of the program by typing the correct name and using the ENTER key, or *(preferably)* using the appropriate menu key.

You have access to all built-in commands from any CALC subdirectory without exiting from that subdirectory. Simply type the command and press ENTER (be sure to first provide the necessary inputs on the stack), or use the appropriate built-in menu key. If you use a built-in menu key, you can return directly to the subdirectory you are in with the VAR key.

Because the HP-48 needs program derXROOT in order to differentiate the XROOT function, we recommend that you move it from your DERIV subdirectory into your HOME directory, where it sill be accessible from anywhere. Here is how to do that. Open your DERIV subdirectory and recall derROOT to the stack with [↱] [DERXR], then put its name on the stack using ['] [DERXR] [ENTER]. Use [↱] [HOME] to go to HOME. Now press [STO] to store the program in your HOME directory.

To move up from a particular CALC subdirectory to the main CALC directory, use the UP key (the left-shifted ['] key).

To rearrange any of the variables in a subdirectory of CALC (including the CALC directory itself), apply the command ORDER (on the [↰] MEMORY DIR submenu) to a list that contains the names of the variables in the desired order, left-to-right.

And finally, a *word of caution.* With any object on stack level 1, pressing [↰] and then the menu key beneath a particular user-constructed variable (in particular, one of our teaching code programs) will overwrite the contents of that variable with the object from level 1. So be careful; in a hasty moment it is easy to destroy teaching code!

SOLUTIONS

Activity Set 2.1

1.

x	$f(x)$
$\pm 10^{-2}$	.999983333417
$\pm 10^{-3}$	.99999933333
$\pm 10^{-4}$	.999999998333
$\pm 10^{-5}$	.999999999983
$\pm 10^{-6}$	1

2.

x	$f(x)$
$\pm 10^{-2}$	$\mp$ .004999583
$\pm 10^{-3}$	$\mp$ .0005
$\pm 10^{-4}$	$\mp$ .00005
$\pm 10^{-5}$	$\mp$ .000005
$\pm 10^{-6}$	0

3.

x	$f(x)$
-1	-.367879441171
-10	-4.5399297625E-4
-1,000	-5.07595889755E-432
-10,000	0

4. See numbers 1, 2, and 3.

5. (a)

x	$f(x)$
10^2	2.70481382942
10^4	2.71814592683
10^6	2.71828046932
10^8	2.71828181487
10^{10}	2.71828182832
10^{11}	2.71828182845

as $x \to \infty, f(x) \to e$

(b) When $x = 10^{12}, f(x)$ is evaluated as 1 by the HP48. Why? The precision of the HP-48 is 12 decimal digits. With $x = 10^{12}$, $1/x$ = .000000000001 and $1 + 1/x$ is evaluated as 1. Then $1^x = 1$.

6. (a)

x	$f(x)$
10^{-2}	3.00498756295
10^{-3}	3.00049988491
10^{-4}	3.00000400001
10^{-5}	3

as $x \to 0^+, f(x) \to 3$.

x	$f(x)$
-10^{-2}	-.9949874372587
-10^{-3}	-.99499873095
-10^{-4}	-.99995000125
-10^{-5}	-.999994800014
-10^{-6}	-1

as $x \to 0^-, f(x) \to -1$.

7. (a) 'IFTE (X < 0, X ^ 2, COS (X))'

(b)

x	$f(x)$
-10^{-2}	.0001
-10^{-4}	.00000001
-10^{-6}	.000000000001
-10^{-8}	1.E-16
-10^{-10}	1.E-20

as $x \to 0^-, f(x) \to 0$.

(c)

x	$f(x)$
10^{-2}	.999950000417
10^{-4}	.99999995
10^{-6}	1

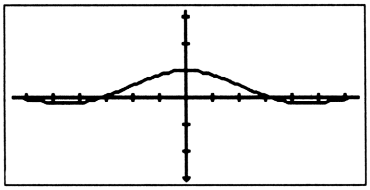
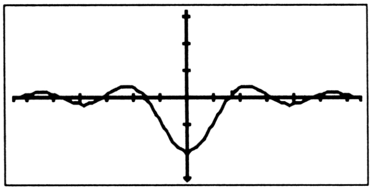
as $x \to 0^+, f(x) \to 1$.

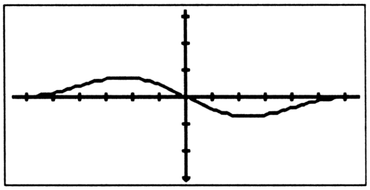
(d) $\lim_{x\to 0} f(x)$ does not exist.

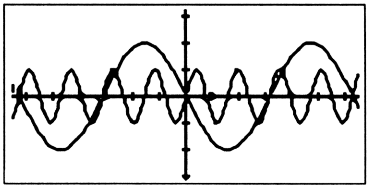
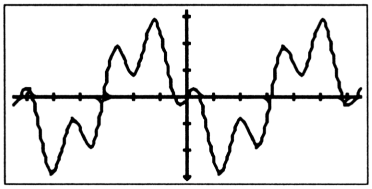
8. (a) 22 (b) -23 (c) 23 (d) 306

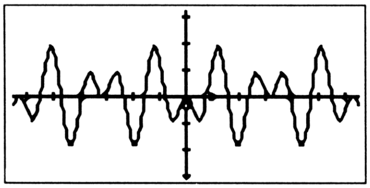
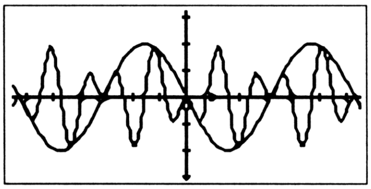
9. (a) 23 (b) -22 (c) 24 (d) 307

Activity Set 2.2

1. (a)  (b) 

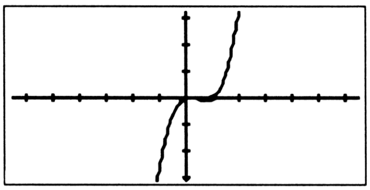
(c) 

2. (a)  (b) 

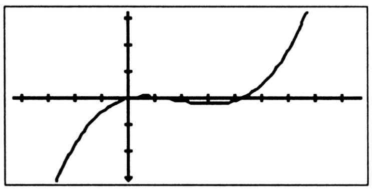
(d)  (e) 

3. HZIN by a factor of 10.

4. Since 1 rad = $180°/\pi \approx 57.2957795131$, HZOUT by this factor.

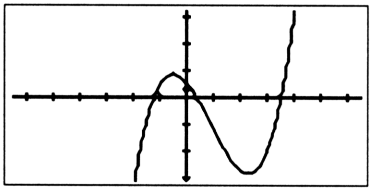
5. 

Original

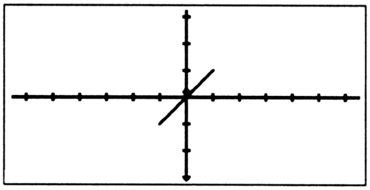


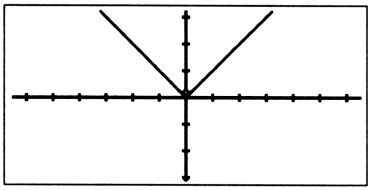
Xrng: -1 2
Yrng: -1 1

6. YZOUT by a factor of 100 to see the plot

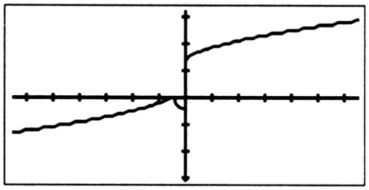


7.



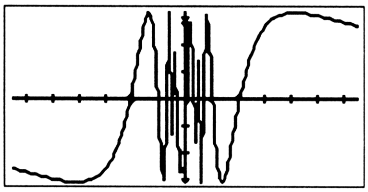


8. Graph with the default PPAR, then ZOUT by a factor of 2 to see



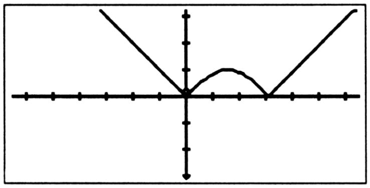
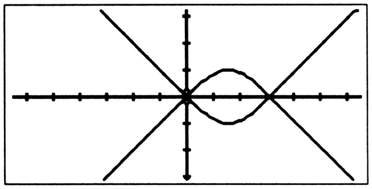
Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, so $\lim_{x \to 0} f(x)$ does not exist.

9. Graph with the default PPAR, then use BOXZ:

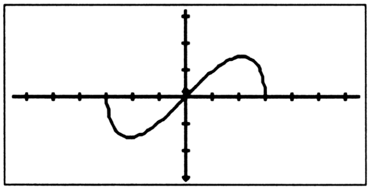


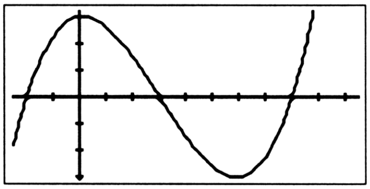
Xrng: -1 1
Yrng: -1 1

Conclusion: $\lim_{x \to 0} f(x)$ does not exist.

10. (a)  (b) 

11. Plot with the default PPAR, then HZIN by a factor of 1.733 to see

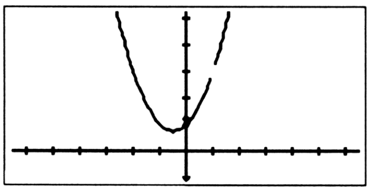


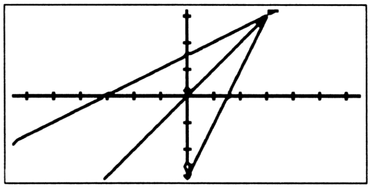
12. The final plot 

13. (a) Trace to x: 1.2 y: 4.77 and press ENTER, then trace to x: 1.3 y: 4.77 and press ENTER. Press ON to return to the stack and see

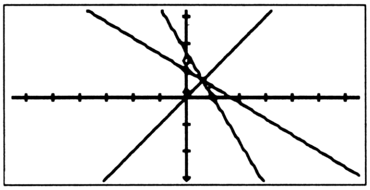
2: (1.2, 4.77333333333)
1: (1.3, 4.76692307692)

The local minimum is approximately the point on level 1.

(b)  The "hole" is at $x = 1$.

14. (a) 

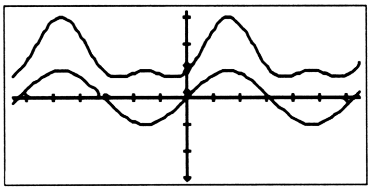
(b) $f^{-1}(x) = \frac{x+3}{2}$

(c) 

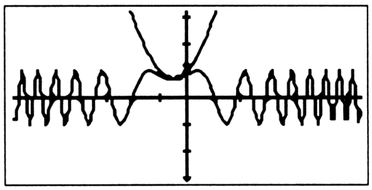
$g^{-1}(x) = \frac{5}{3}(1-x)$

(d) Non-parallel lines that are symmetric to the line $y = x$ have slopes that are reciprocals of one another.

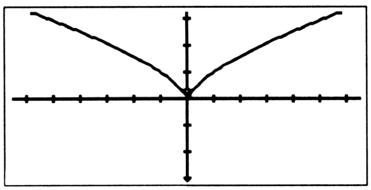
(e) The converse is false.

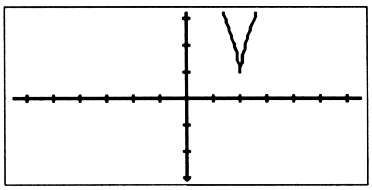
15. (a) 

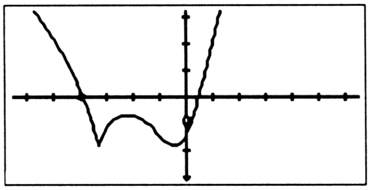
$u[\,v(x)\,]$ and $v(x)$

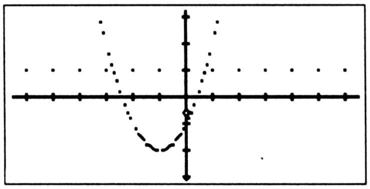
(b) 

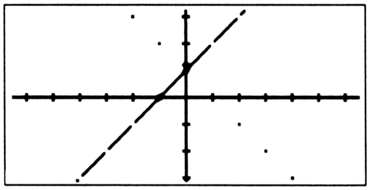
$v[\,u(x)\,]$ and $u(x)$

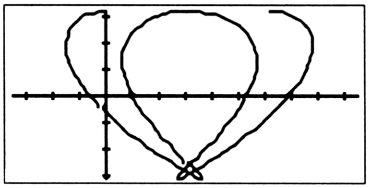
16. (a) 

(b) 

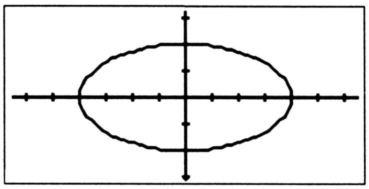
17. (b)  (c) (-.3, -1.09614986962)

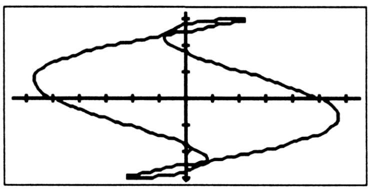
18. (a)  Use disconnected mode.

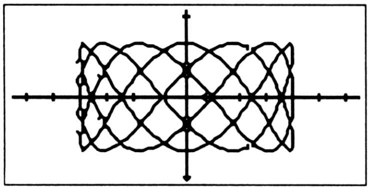
(b) 

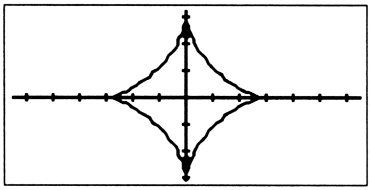
19. 

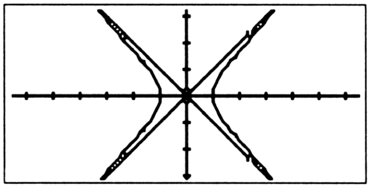
When $t = 3$, $x = 2.17576302952$ and $y = 1.9203405733$.

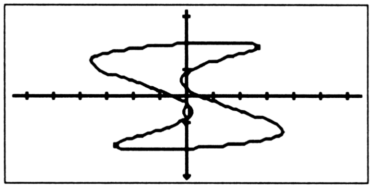
20. 

21. 

22. 

23. 

24. 

25. 

Activity Set 3.1

1. (c) $\frac{\Delta y}{\Delta x} = 1$

(d)

H	$DQ(0,H)$
$\pm 10^{-2}$	.999983333417
$\pm 10^{-4}$	.999999998333
$\pm 10^{-5}$	.999999999983
$\pm 10^{-6}$	1

(e) $\frac{d}{dx}[\,sin\ x\,]_{x\,=\,0} = \lim_{h \to 0} \frac{sin\ (0 + h) - sin\ 0}{h}$

2. (a) After zooming in on the horizontal axis twice by a factor of 100 each time, tracing shows that the y-coordinate remains constant at $y = 1$. Thus the tangent line at $x = 0$ is horizontal with slope 0.

(b)

H	$DQ(0,H)$
$\pm 10^{-2}$	$\mp$ .004999583
$\pm 10^{-3}$	$\mp$ .0005
$\pm 10^{-4}$	$\mp$ .00005
$\pm 10^{-5}$	$\mp$ .000005
$\pm 10^{-6}$	0

(c) $\frac{d}{dx}[\cos x]_{x=0} = \lim_{h \to 0} \frac{\cos(0+h) - \cos 0}{h}$

3. (a) Graph on the default screen; **ZIN** twice by factors of 100 each time. Trace left to x = .99995, then right to 1.00005. Calculate $\frac{\Delta y}{\Delta x}$ = -.66666664. The slope when $x = 1$ is $\frac{-2}{3}$.

(b) Repeat the procedure in part (a), and calculate $\frac{\Delta y}{\Delta x}$ = -2. The slope when $x = 1$ is -2.

(c) Repeat the procedure in part (a), and calculate $\frac{\Delta y}{\Delta x}$ = 1.99999999666. The slope when $x = 1$ is 2.

(d) Repeat the procedure in part (a), and calculate $\frac{\Delta y}{\Delta x}$ = -2.4783497. The slope when $x = 1$ is -2.47834973296.

4. (a)

H	$DQ(2,H)$
10^{-2}	.748598999
10^{-4}	.7499859
10^{-6}	.75
-10^{-2}	.751411548
-10^{-4}	.7500141
-10^{-6}	.75

The slope at $x = 2$ is .75

(b)

H	$DQ(-1,H)$
10^{-2}	.492361819
10^{-4}	.499925
10^{-6}	.5
-10^{-2}	.5073631939
-10^{-4}	.50007498
-10^{-6}	.499999
-10^{-7}	.5

The slope at $x = -1$ is .5

Activity Set 3.2

1. (a) $\pi/180 \approx .0174532925199$

(b)

X	$SIN(X)/X$
.01	.0174532924313
.001	.0174532925191
.0001	.017432925199
-.01	.017432924313
-.001	.017432925191
-.0001	.017432925199

(c) In degree mode, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \dfrac{\pi}{180}$.

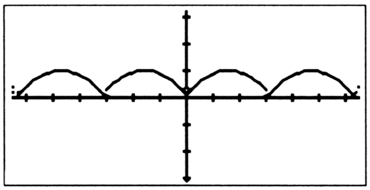
2. (a) $y' = \dfrac{1 - x^2}{(x^2 + 1)^2}$

(b) $y' = \dfrac{-x \sin \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$

(c) $y' = \dfrac{2 \sin x \cos x}{3 \sqrt[3]{\sin^4 x}}$

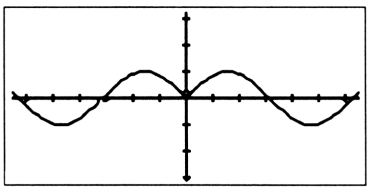
(d) $y' = e^{-x/3}\left(2 \cos 2x - \dfrac{1}{3} \sin 2x\right)$

3. (a)

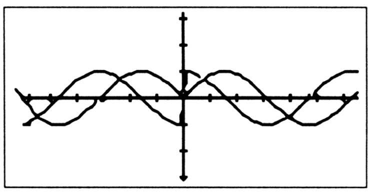


(b) y' is not defined for the values $x = n\pi, n = \pm 1, \pm 2, \ldots$
$y = 0$ for these values of x.

4. (a)

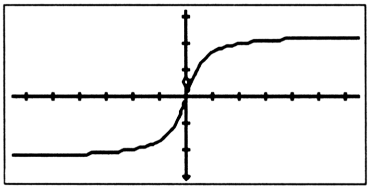
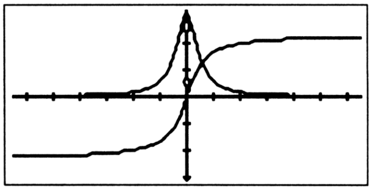


(b)



The derivative is not defined at $x = 0$.

5. A horizontal tangent line to the graph of f will cause the plot of f' to cross the x-axis.

6. (a)  (b) 

7. (a) $y' = \dfrac{2x}{9y^2}$ (b) $y' = \dfrac{-y^2 - 10x}{2xy}$ (c) $y' = \dfrac{-e^y - 2\,xy\cos x^2y}{xe^y + x^2\cos x^2\,y - 2y}$

(d) $y' = \dfrac{\left(y^2 \sin xy^2 + \dfrac{1}{2\sqrt{x}}\right)}{\left(3y^2 - 2xy \sin xy^2\right)}$ (e) $y' = \dfrac{-4}{9x^{1/3}y^{1/2}}$ (f) $\dfrac{-\sin x}{2\sin y\cos y}$

8. (a) $\dfrac{dy}{dx}(5, -3) = .123456790123$ (b) $\dfrac{dy}{dx}(-2, -3) = .91\overline{6}$

(c) $\dfrac{dy}{dx}(3, \pi/2) = -.426089085256$ (d) $\dfrac{dy}{dx}(\pi, 2) = .023507899329$

(e) $\dfrac{dy}{dx}(8, 9) = -7.40740740748E-2$ (f) $\dfrac{dy}{dx}(\pi/4, \pi/4) = .707106781188$

Activity Set 3.3.1

1. local max: (-.577, 2.385)
   local min: (.577, 1.615)
   infl. point: (0, 2)

2. local max: (.149, 2.127)
   local min: (.718, -.090)
   infl. point: (.433, -.044)

3. local max: none
   local min: (-.598, -3.238)
   infl. point: (0, -2) and (1, 0)

4. local max: (-2.459, 21.968) and (-.202, 3.092)
   local min: (-.517, 3.031) and (.778, 1.319)
   infl. point: (-1.875, 14.819), (-.364, 3.061) and (.439, 2.026)

5. local max: $(0, 2)$
   local min: none
   infl. point: $(-.816, 1.5)$ and $(.816, 1.5)$

6. local max: $(0, -.8)$
   local min: none
   infl. point: none

7. local max: $(0, 1)$
   local min: none
   infl. point: $(-1, 0)$ and $(1, 0)$

8. absolute max: $(0, 2\pi)$
   local max: $(1.912, 4.739)$
   local min: $(4.373, 1.544)$
   absolute min: $(0, 0)$
   infl. point: (π, π)

9. absolute max: $(2.068, 2.873)$
   local max: $(.056, 2.028)$
   local min: $(1.018, -1.41)$
   absolute min: $(\pi, -2)$
   infl. point: $(.533, .452)$, $(1.552, .889)$ and $(2.627, .437)$

10. absolute max: $(0, 1)$ and $(\pi, 1)$
    absolute min: $(\pi/2, -2)$
    infl. point: $(.704, -.486)$ and $(2.437, -.486)$

11. absolute max: $(4.712, 2)$
    local max: $(.767, .708)$, $(2.375, .708)$ and $(2\pi, -1)$
    local min: $(0, -1)$
    absolute min: $(3.510, -1.634)$ and $(5.915, -1.634)$

12. local max: none
    local min: none
    infl. point: $(0, 1)$ and $(\pi/2, 0)$

13. local max: $(0, 3)$
    local min: none
    infl. point: $(-1.414, 1.819)$ and $(1.414, 1.819)$

14. local max: none
    local min: $(.368, .692)$
    absolute min: $(-1, -1)$
    infl. point: none

15. absolute max: (-1, 1), (0, 1) and (1, 1)
    absolute min: (-.707, -1) and (.707, -1)
    infl. point: (.-.408, -.111) and (.408, -.111)

16. no local extrema
    infl. point: (0, 0)

17. absolute max: (2.128, 1.898)
    local max: (0, 1) and (2π, 1)
    local min: none
    absolute min: (4.843, .209)
    infl. point: (1.395, 1.388) and (2.916, 1.270)

18. none

19. Run the cable from the junction box along the road to a point .894 miles from the closest point on the road to the house, then run straight to the house.

20. The maximum area is achieved with a square pen of side length 62.5 feet situated on the back of the barn.

21. The maxiumum area is achieved with a rectangular pen 40 × 50 feet in size.

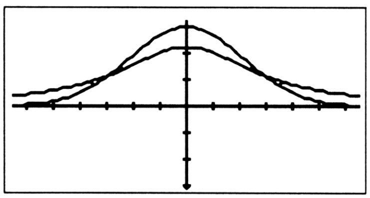
Activity Set 3.3.2

1.	*Starting value*	*Number of iterations*	*Convergence to*
(a)	-2.2	4	-2.2360679775
	2.3	5	between 2.23606797748 and 2.236067897751
	2.9	5	3
(b)	.483	4	.450451159135
	1.79	4	1.74250596672
	2.56	4	2.51943185453
	3.58	5	3.59204381272
	4.88	5	4.8840986203

(c)	-4.71	3	-4.7168606007
	-1.52	4	-1.45367366646
	.552	4	.539785160809
(d)	-.7	5	-.774596669242
	.8	6	.774596669242
(e)	-.9	6	-.923879532511
	-.3	5	-.382683432366

The other two roots are obtained by symmetry.

2. (a) At $x_0 = \pi/2$, the denominator in the iteration formula is equal to 0.
   (b) $\rightarrow$ 31.4159265359 ($\approx 10\pi$) after 8 iterations
   (c) $\rightarrow$ -12.5663706144 ($\approx 4\pi$) after 4 iterations
   (d) $\rightarrow$ 3.14159265359 ($\approx \pi$) after 7 iterations
   (e) $\rightarrow$ -3.14159265359 ($\approx -\pi$) after 6 iterations
   (f) $\rightarrow$ 3.14159265359 ($\approx \pi$) after 6 iterations
   (g) $\rightarrow$ 0 after 6 iterations
3. Newton's method diverges away from 0 for any starting value $x_0 \neq 0$.
4. (a) Starting with $x_0 = 3$, the iterations oscillate between 3 and 1.
   (b) Starting with $x_0 = 2.5$, the iterations oscillate between 2.5 and 1.5; and starting with $x_0 = 1.8$, the iterations oscillate between 1.8 and 2.2. In general, starting with $x_0 = 2 + h$ or $x_1 = 2 - h$, the iterations oscillate between x_0 and x_1 .
   (c) The tangent lines are parallel; they have the same slope.
5. (a)

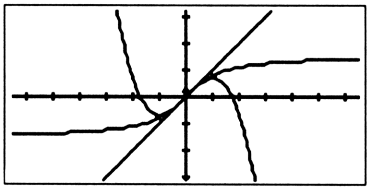


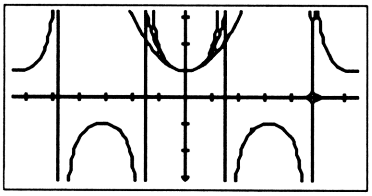
(b) The intersection points are (±.980448246014, .382403569603).

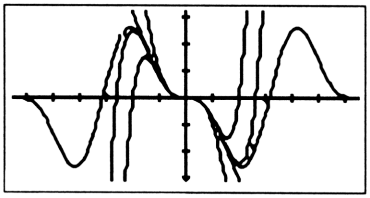
6. $\theta \approx 60.72°$

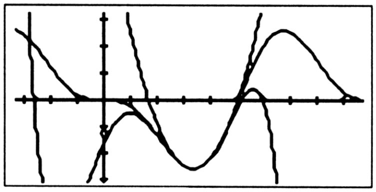
7. a = 127.71148013. An equation for the arch is $y = 630 + 127.7\left(1 - cosh\ \dfrac{x}{127.7}\right)$.

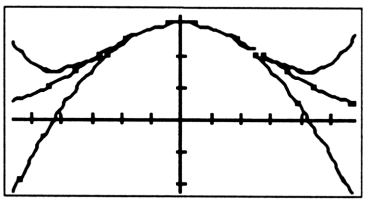
Activity Set 3.3.3

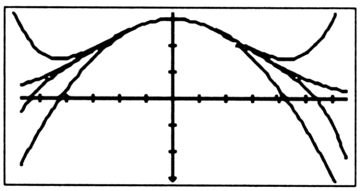
1. 

2. 

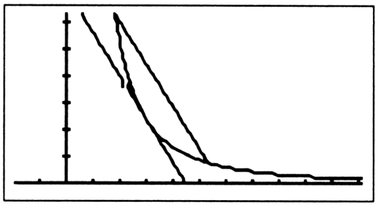
3. (a) 

(b) 

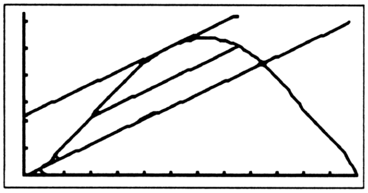
4. (a) 

(b) 

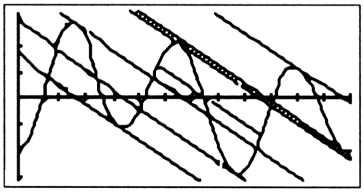
Activity Set 3.3.4

1. 

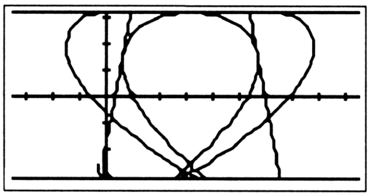
Tangent line: ' 1.53352292311 – 3.7980913324 * (X – .807522931339) '

2. 

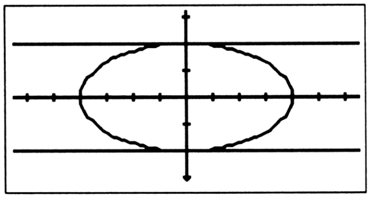
Tangent line: ' 3.01253819463E-2 + .465272415481 * (X – 9.69188952391E-2) '
Tangent line: ' 1.23212271031 + .465272415482 * (X – 1.52177971665) '

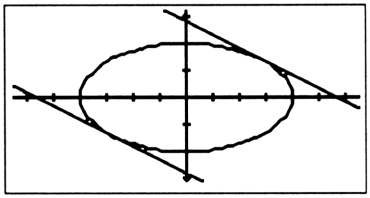
3. 

Activity Set 3.3.5

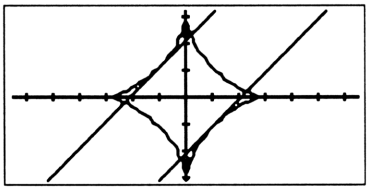
1. 

2. (b) (i) Vertical tangents when $t = 0$ and $t = \pi$

(ii) 

(iii) 

3. Vertical tangents when $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3}{2}\pi$



Tangents when $t = \frac{3\pi}{4}$ and $t = \frac{7\pi}{4}$

Activity Set 4.1.1

1. (a)

n	LRECT	RRECT	MID
15	.728768788418	.6102502699	.665249108684
30	.697008948527	.637749689263	.666310585448
60	.681659767039	.652030137409	.666577539299
120	.674118653051	.659303838236	.666644378016

continued . . .

200	.671127160162	.662238271273	.666658642265
500	.668447012312	.66489145676	.666665382712
1000	.667556197498	.66577841972	.666666345704

(c) The midpoint approximations are the best.

2. (a)

n	LRECT	RRECT	MID
15	2.42832476027	2.42832476027	2.43889647161
30	2.43361061595	2.43361061595	2.43618981423
60	2.43490021509	2.43490021509	2.43553398801
120	2.43521710155	2.43521710155	2.43537360984
200	2.43528393107	2.43528393107	2.43533989579
500	2.4531525233	2.43531525232	2.43532413036
1000	2.43531969458	2.43531969459	2.43532190445

(c) The midpoint approximations are the best.

3. (a) By symmetry, all midpoint approximations are 0.

(b)

n	LRECT	RRECT
5	-1.89024823416	1.89024823416
20	-.47256205854	.47256205854
40	-.23628102927	.23628102927

Activity Set 4.1.2

1. (a) width $= \frac{4}{N}$

right endpoint $= \frac{4K}{N}$

height $= \left(\frac{4K}{N}\right)^3$

area $= \left(\frac{4K}{N}\right)^3\left(\frac{4}{N}\right)$

(b) S(N) = Σ (K = 1, N, (4 * K/N) ^ 3 * (4/N))

N	$S(N)$
10	77.44
50	66.5856
100	65.2864
200	64.6416

2. (a) width $= \frac{2}{N}$

right endpoint $= 1 + \frac{2K}{N} = \frac{2K + N}{N}$

height $= \frac{N}{2K + N}$

area $= \left(\frac{N}{2K+N}\right)\left(\frac{2}{N}\right)$

(b) S(N) = Σ (K = 1, N, N/(2 * K + N) * (2/N))

N	$S(N)$
10	1.0348956599
50	1.0853974528
100	1.0919752503
200	1.09528636265

3. (a) width = $\frac{4}{N}$

right endpoint = $-1 + \frac{4K}{N} = \frac{4K - N}{N}$

height = $e \wedge \left(\frac{4K-N}{N}\right)$

area = $\left(e \wedge \frac{(4K-N)}{N}\right)\left(\frac{4}{N}\right)$

(b) S(N) = Σ (K = 1, N, EXP((4 * K – N)/N * (4/N)))

N	$S(N)$
10	23.923392666
50	20.5168787439
100	20.1146395823
200	19.9154913077

4. (a) width = $\frac{2}{N}$

left endpoint = $-3 + \frac{2(K-1)}{N} = \frac{2K - 3N - 2}{N}$

height = $3 + \left(\frac{2K-3N-2}{N}\right)^2 + \left(\frac{2K-3N-2}{N}\right)$

area = $\left(3 + \left(\frac{2K-3N-2}{N}\right)^2 + \left(\frac{2K-3N-2}{N}\right)\right)\left(\frac{2}{N}\right)$

(b) S(N) = Σ (K = 1, N, 3/((2 * K − 3 * N − 2)/N) ^ 2 + (2 * K − 3 * N − 2)/N * (2/N))

N	*S(N)*
10	-2.44756241499
50	-2.09256321762
100	-2.04647408997
200	-2.0232851862

Activity Set 4.1.3

1. AREA = .666666666667

N	TRAP	SIMP
50	.666923371902	.666666687877
100	.666730858883	.666666667993

2. AREA = 3.71221866457

N	TRAP	SIMP
100	3.71108428728	3.71205883637

3. AREA = -1.38542551654

N	TRAP	SIMP
100	-1.38630748336	-1.38542551069

4. AREA = 2.36567536982

N	TRAP	SIMP
100	2.36496414195	2.36567538512

5. AREA = 1.15444851259

N	TRAP	SIMP
100	1.544445492303	1.5444851427

6. AREA = -2

N	TRAP	SIMP
50	-1.99922988429	-1.99999993636
100	-1.99980742337	-1.999999996

7. AREA = .886207348259

N	TRAP	SIMP
100	.886207292754	.88620734825

8. AREA = 1.58846779582

N	TRAP	SIMP
100	1.58848892823	1.58846779602

9. AREA = 2.50798211423

N	TRAP	SIMP
100	2.50797278189	2.50798211428

10. AREA = 1.08531739235

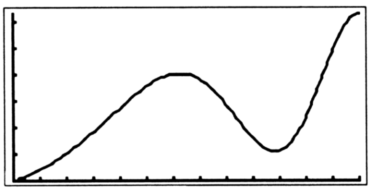
N	TRAP	SIMP
100	1.08532405829	1.0853173924
200	1.08531905887	1.08531739236

11. AREA = 1.446560
(Using 6 FIX)

N	SIMP
100	1.44655963409
200	1.44656012035

Activity Set 4.2.1

1. (a)



(b) AREA: 2.45375670998

(c) AREA: 1.52818183784

2. Volume: 41.1062399578 (units)^2

3. Volume: 1.65549327405 (units)^2

4. The "arch length" is 1493.74 feet

5. (a) 18.001 (b) 40.095

6. Period ≈ 2.554 m

Activity Set 4.2.2

1. $\frac{14}{5}\sqrt{5x-1}$

2. $-(2x+3)^{-1/2}$

3. $tan\ x - x$

4. $sec\ x$

5. $\frac{1}{2}\left[(2x+3)tan^{-1}(2x+3)-\frac{1}{2}ln(1+(2x+3)^2\right]$

6. $\frac{1}{10}(5x-\pi)^6$

7. $-\frac{2}{sin\ .5x}$

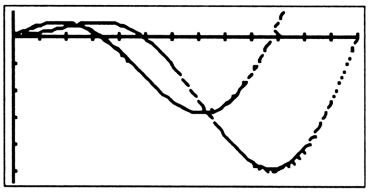
8. $\frac{x^2}{2} - ln\ cosh\ x$

9. $\frac{2^{x+1}}{ln\ 2}$

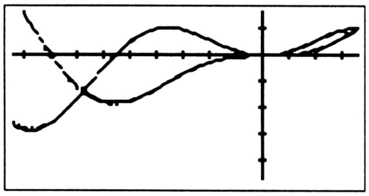
10. If $s - 1 = -1$, $ln\ x$; otherwise $\frac{x^s}{3}$.

Activity Set 4.3

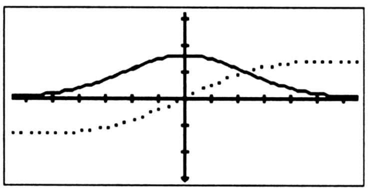
1. The final plot:

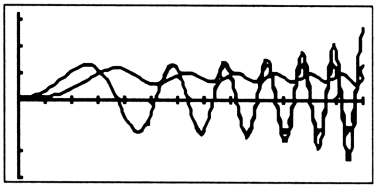


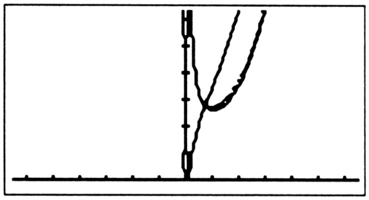
2. The final plot:

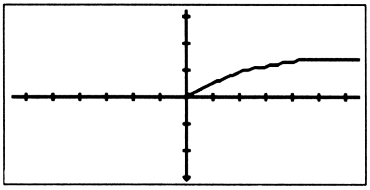
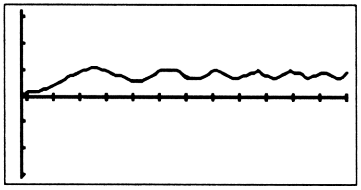


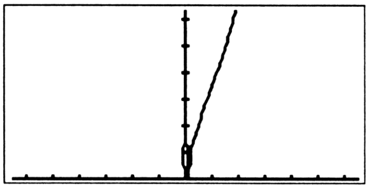
3. The final plot:



4. The final plot: 

5. The final plot: 

6. (a)  (b) 

(c) 

Activity Set 4.4

1. $\left|\frac{\sin x}{x^4}\right| \le \frac{1}{x^4}$. With $K = 1$, $p = 4$ and $\epsilon = .01$ we have $N > 3.22$. Thus, the integral $\approx .56$ to within ϵ.

2. $\left|\frac{x}{\sqrt{x^6+4}}\right| < \frac{1}{x^2}$. With $K = 1$, $p = 2$ and $\epsilon = .001$ we have $N > 1000$. Thus, the integral ≈ 1.112 to within ϵ.

3. $\left|\dfrac{xe^{-2x}}{\sqrt[3]{x^3+1}}\right| < e^{-2x}$. With $K = 1$, $c = 2$ and $\epsilon = .001$ we have $N > 3.2$. Thus, the integral $\approx .198$ to within ϵ.

4. $\left|\dfrac{1}{\sqrt{x^5-1}}\right| < \dfrac{2}{x^{5/2}}$ for $x \geq 2$. With $K = 2$, $p = 5/2$ and $\epsilon = .001$ we have $N > 121.2$. Thus, the integral $\approx .236$.

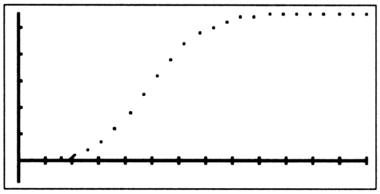
5. Using the hint, for $x \geq 0$ $\left|\sqrt{x+1}\,e^{-2x}\right| < \left(\dfrac{1}{e}\right) ee^{-x}$. Thus $K = 1$ and $c = 1$. For $\epsilon = .005$, $N > 1n200 \approx 5.3$ and the integral $\approx .605$.

Activity Set 5.1

1. (a) - (b). The terms appear to approach 0.
   (c) The graph approaches the x-axis.
   (d) by l-Hospital's Rule, $\lim\limits_{x\to\infty} \dfrac{x^2}{2^x} = 0$. Thus $\lim\limits_{k\to\infty} \dfrac{k^2}{2^k} = 0$.
2. (a) - (b). The terms appear to approach 0.
   (c) The graph approaches the x-axis.
   (d) by l-Hospital's Rule, $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$. Thus $\lim\limits_{k\to\infty} \dfrac{\ln k}{k} = 0$.
3. (a) - (b). The terms appear to approach 0.
   (c) The plot suggests that $\lim\limits_{k\to\infty} \dfrac{\sin k}{e^{.05k}} = 0$.
   The Rule of l-Hospital is of no help.
4. (a) - (b). The terms appear to approach 0.
   (c) Use $f(x) = \dfrac{1}{x}$; $\lim\limits_{k\to\infty} (-1)^{k+1}\left(\dfrac{1}{k}\right) = 0$.
5. (a) $\left\{\dfrac{k}{\sqrt{k}+1}\right\}$ diverges (b) $\left\{\dfrac{\cos k}{\sqrt{k}}\right\}$ converges to 0.

Activity Set 5.2.1

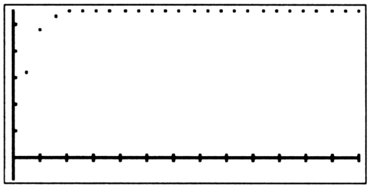
1. (a) Use the ratio test

   (b) 

   (c) sum ≈ 2026.4657948

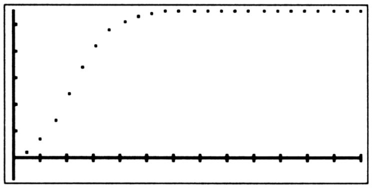
   (d) The 38th partial sum.

2. (a) Use the ratio test.

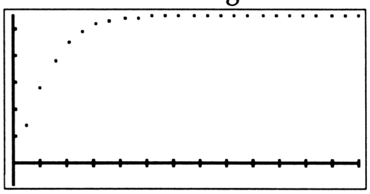
   sum ≈ 1.71828182846
   The 14th partial sum

   (b) Use the ratio test

   sum ≈ 147.413159104
   The 27th partial sum

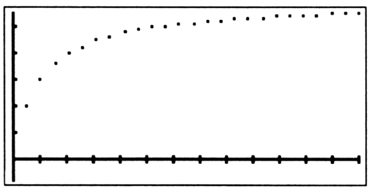
   (c) Use the alternating series test

   sum ≈ 2
   The 42nd partial sum

3. (a) Use the alternating series test. (b) $n = 15$ (c) $S_{15} = 1.71828182846$

4. Sum ≈ 2.00300296795

5. (b) 

(c) For sufficiently large k, $\frac{k!}{(k+2)!}$ is sensed as 1 by the HP-48. From that point on, the partial sums accumulate by 1.

Activity Set 5.2.2

1. $\int_1^\infty \frac{\sin x}{x^4}\,dx$ converges on comparison with $\int_1^\infty \frac{1}{x^4}\,dx$ and $\left|\frac{\sin x}{x^4}\right| < \frac{2}{x^4}$; use $K = 2$ and $p = 4$. With $\epsilon = .01$, $N > 4.05$. Thus the sum $\approx \sum_{k=1}^{5} \frac{\sin k}{k^4} = .90$ to within $\epsilon = .01$.

2. $\int_1^\infty \frac{x}{\sqrt{x^6+4}}\,dx$ converges on comparison with $\int_1^\infty \frac{1}{x^2}\,dx$ since $\left|\frac{x}{\sqrt{x^6+4}}\right| < \frac{1}{x^2}$ for $x > 1$; use $K = 1$ and $p = 2$. With $\epsilon = .001$, $N > 1000$. Thus the sum $\approx \sum_{k=1}^{1001} \frac{k}{\sqrt{k^6+4}} = 1.083$ to within ϵ.

3. Let $f(x) = \frac{xe^{-2x}}{\sqrt[3]{x^3+1}}$. Then $\int_1^\infty f(x)dx$ converges on comparison with $\int_1^\infty e^{-2x}dx$ since $|f(x)| < e^{-2x}$; use $K = 1$ and $c = 2$. With $\epsilon = .0012$, $N > 3.1$. Thus the sum $\approx \sum_{k=1}^{4} a_k = .128$ to within ϵ.

4. $\int_2^\infty \frac{x}{\sqrt{x^5-1}}\,dx$ converges on comparison with $\int_2^\infty \frac{1}{x^{1.2}}\,dx$ since $\frac{x}{\sqrt{x^5-1}} < \frac{1}{x^{1.2}}$ for $x > 1.41$; use $K = 1$ and $p = 1.2$. With $\epsilon = .001$, $N > 5.5$. Thus the sum $\approx \sum_{k=2}^{6} \frac{k}{\sqrt{k^5+1}} = .835$ to within ϵ.

5. $\int_1^\infty ln\left(\frac{1}{x^2}\right)dx$ converges on comparison with $\int_1^\infty \frac{1}{x^2}dx$ since $ln\left(\frac{1}{x^2}\right) < \frac{1}{x^2}$ for $x > 0$; use $K = 1$ and $p = 2$. With $\epsilon = .0001$, $N > 10{,}000$. Thus the sum $\approx \sum_{k=1}^{10,001} ln\left(\frac{1}{k^2}\right) = -164{,}236.277$ to within ϵ.

Activity Set 5.2.3

1. Since $r = \frac{1}{2}$ let $\Gamma = \frac{3}{4}$. The smallest value of N that satisfies (1) is $N = 3$. The smallest value of N that satisfies (2) is $N = 27$. Thus the sum $\approx$ $\sum_{k=1}^{27}\left(\frac{k}{2^k}\right) = 2$ to within $\epsilon = 10^{-6}$.

2. Since $r = \frac{1}{e}$ let $\Gamma = \frac{1}{2}$. The smallest value of N that satisfies (1) is $N = 10$. The smallest value of N that satisfies (2) is $N = 24$. Thus the sum $\approx$ $\sum_{k=1}^{24}\left(\frac{k^3}{e^k}\right) = 6.006512$ to within $\epsilon = 10^{-6}$.

3. Since $r = 0$ let $\Gamma = \frac{1}{2}$. The smallest value of N that satisfies (1) is $N = 3$. The smallest value of N that satisfies (2) is $N = 12$. Thus the sum $\approx$ $\sum_{k=1}^{12}\frac{(k+1)(k+2)}{k!} = 17.027973$ to within $\epsilon = 10^{-6}$.

4. Since $r = 0$ let $\Gamma = \frac{1}{2}$. The smallest value of N that satisfies (1) is $N = 4$. The smallest value of N that satisfies (2) is $N = 14$. Thus the sum $\approx$ $\sum_{k=1}^{14}\frac{10^k k!}{(2k+1)!} = 5.655198$ to within $\epsilon = 10^{-6}$.

5. Since $r = .7$ let $\Gamma = \frac{3}{4}$. The smallest value of N that satisfies (1) is $N = 1$. The smallest value of N that satisfies (2) is $N = 42$. Thus the sum $\approx$ $\sum_{k=1}^{42}\frac{.7^k(k+1)}{k} = 3.537305$ to within $\epsilon = 10^{-6}$.

INDEX

HP-48G/GX TEACHING CODE

derXROOT	Derivative of XROOT
F.val	Utility for SUM
F.XY	Evaluate $F(x,y)$
FABSTO	Store f, a, b
FTC	Fundamental Theorem of Calculus
GPS	Graphical partial sums
GRECT	Graphical rectangles
GSEQ	Sequence graph
IM.y'	Implicit differentiation
INFL1	Inflection point via f'
INFL2	Inflection point via f''
INFSUM	Sum a series
INV.F	Inverse function
LRECT	Left rectangle sum
MID	Midpoint rectangle sum
NEWTON	Newton's Method
NSTO	Store n
PAR'	Parametric derivative
RRECT	Right rectangle sum
SDRW	Utility for GSEQ, GPS
SHO	Show sequence
SIMP	Simpson's rule
SUM	Utility for LRECT, RRECT, etc.
TAN.L	Tangent line
TAY.A	Taylor polynomial at $x = a$
TAYLAT	Taylor polynomial at $x = a$ (alternate version)
TRAP	Trapezoid sum
Y'	Implicit derivative

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