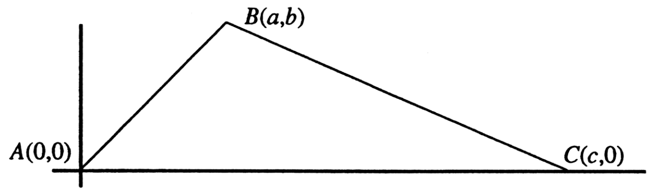
The HP 49G
Quick Start
Activities Guide

Contents

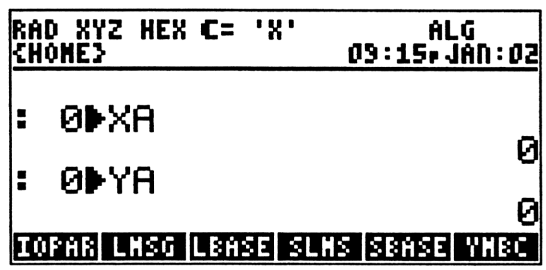
A Coordinate Geometry Proof

Did you realize that you can use your HP 49G to help prove theorems in coordinate geometry? You can! For example: Prove that in any triangle, the segment connecting the midpoints of any two sides is parallel to the third side and had a length equal to $^1/_2$ the third side.



In other words, using the above triangle, show that the segment connecting the midpoints of AB and BC is parallel to AC and has a length equal to $^1/_2$ AC.

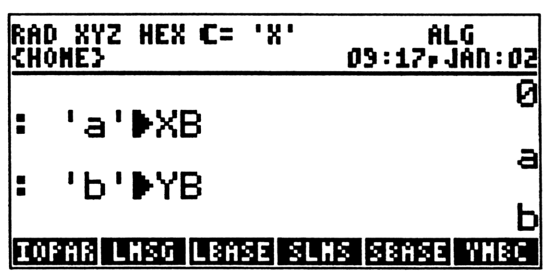
The first step is to store each of the coordinate values of the labeled diagram into its own variable.



To store the *x*- and *y*-coordinates of point *A* into the variables XA and YA, do this: [0] [STO▶] [X] [ALPHA] [A] [ENTER]; and [0] [STO▶] [ALPHA] [Y] [ALPHA] [A] [ENTER].

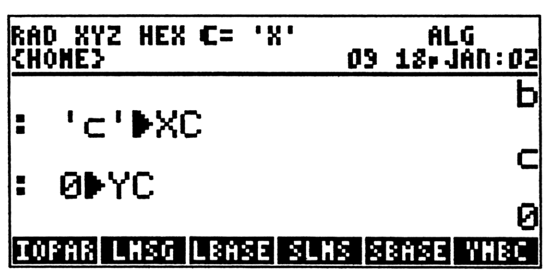
Notice how your inputs echo on the left side, and the results appear on the right.

(Also, notice that if you press [VAR], you will see your variables appear on that menu as you store them.)



Similarly, store the coordinate values of point *B* (*a* and *b*) into the variables XB and YB, as follows:

[↱] ['] [ALPHA] [↰] [A] [▶] [STO▶] [X] [ALPHA] [B] [ENTER]; and [↱] ['] [ALPHA] [↰] [B] [▶] [STO▶] [ALPHA] [Y] [ALPHA] [B] [ENTER].



Finally, store the coordinate values of point *C* (*c* and 0) into the variables XC and YC:

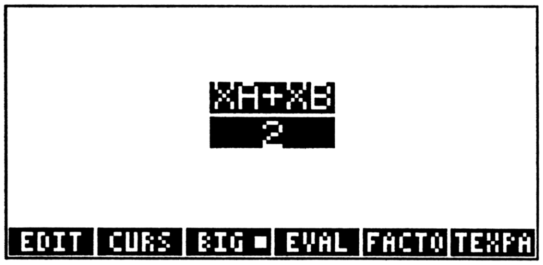
[↱] ['] [ALPHA] [↰] [C] [▶] [STO▶] [X] [ALPHA] [C] [ENTER]; and [0] [STO▶] [ALPHA] [Y] [ALPHA] [C] [ENTER].

Notice in your variable (VAR) menu, as the variable list updates, the newer items appear on the left, while older ones move to the right. To see items other than the most recently stored six, simply press NXT—repeatedly, if necessary, until you see all items and return to the menu's first "page."

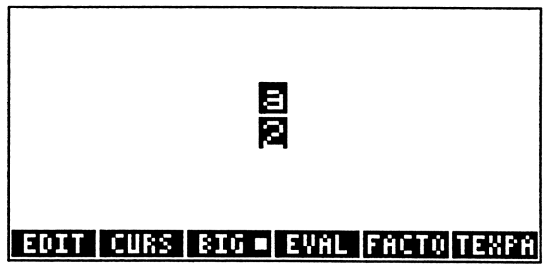
Now that you have all the individual coordinates stored, you can calculate the coordinates of the midpoint of side *AB* of the triangle. Start with the *x*-coordinate:

Press EQW to go into the Equation Writer, then type X ALPHA A + X ALPHA B ▶.

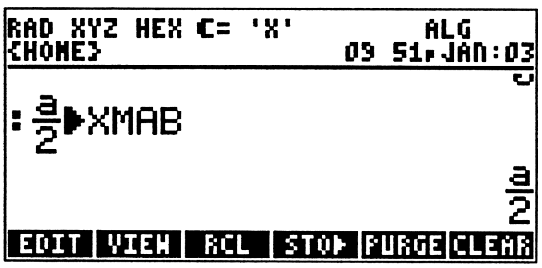
This should produce and highlight this expression:



Now complete the midpoint formula, by typing ÷ 2 ▶.



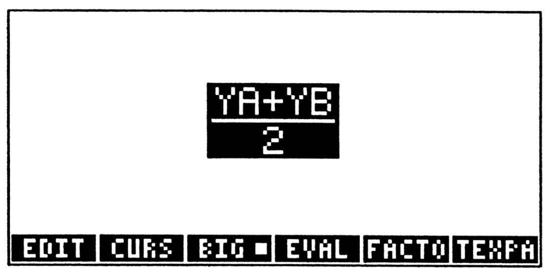
Use ↱ EVAL to perform and simplify the calculation:



Save this result in its own variable name, XMAB ("the X-coordinate of the Midpoint of segment AB"):

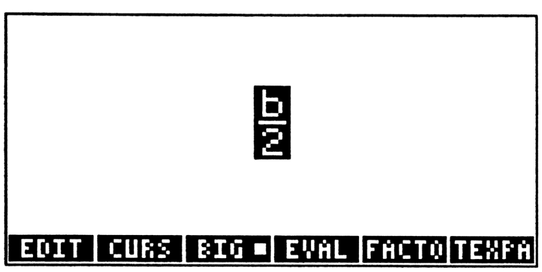
ENTER STO▶ X ALPHA ALPHA M A B ENTER.

Now, go back to the Equation Writer to find the *y*-coordinate of the midpoint similarly:

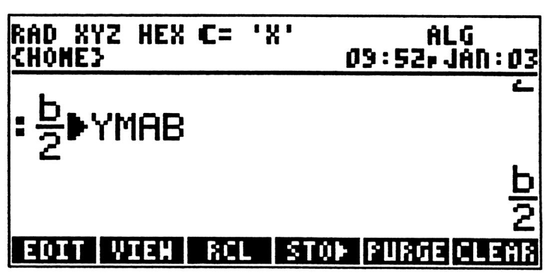


EQW ALPHA Y ALPHA A + ALPHA Y ALPHA B ▶ ÷ 2 ▶...

↱ EVAL...

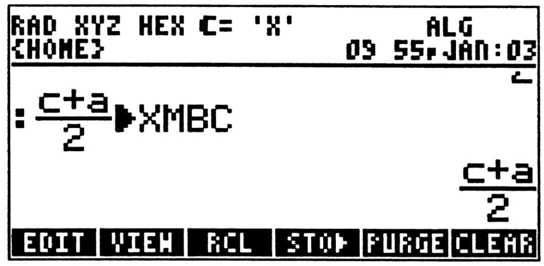


...ENTER STO▶ ALPHA ALPHA Y M A B ENTER.

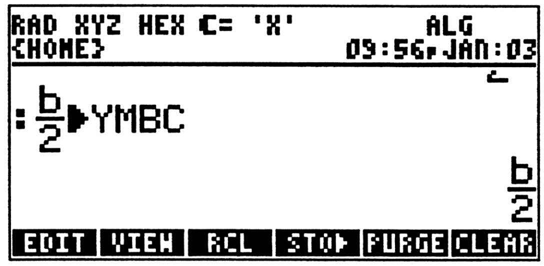


That takes care of the coordinates of the midpoint of *AB*. The next step is to do likewise for the coordinates of the Midpoint of segment BC of the triangle.

[EQW] [X][ALPHA][B] [+] [X][ALPHA][C] [▶] [÷] [2] [▶]
[↱][EVAL] [ENTER] [STO▶][X][ALPHA][ALPHA][M][B][C] [ENTER].



(Again, notice the variables appearing on the VAR menu as they are created.)

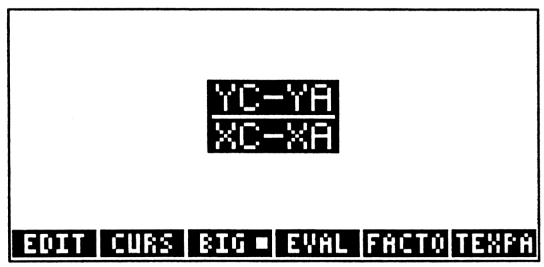


[EQW] [ALPHA][Y][ALPHA][B] [+] [ALPHA][Y][ALPHA][C]
[▶][÷][2] [▶] [↱][EVAL] [ENTER]
[STO▶][ALPHA][ALPHA][Y][M][B][C] [ENTER].

Now that the values are all stored, you can prove the property asserted by the theorem. For example, to demonstrate that the segment connecting the midpoints is parallel to the base, find and compare the slopes.

Start with the slope of the base:

[EQW] [ALPHA][Y][ALPHA][C] [−]
[ALPHA][Y] [ALPHA][A] [▶] [▶] [÷] [X][ALPHA][C]
[−] [X][ALPHA][A] [▶] [▶] [▶].

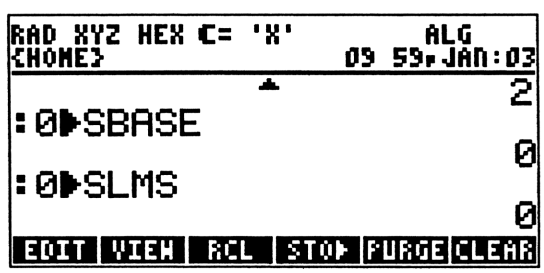


Fnd the result and store it in an aptly named variable:

[↱][EVAL] [ENTER] [STO▶][ALPHA][ALPHA][S][B][A][S][E]
[ENTER].

Now calculate and store the slope of the line connecting the midpoints:

[EQW] [ALPHA][ALPHA][Y][M][B][C][ALPHA] [−]
[ALPHA][ALPHA][Y][M][A][B][ALPHA] [▶] [▶]
[÷] [ALPHA][ALPHA][X][M][B][C][ALPHA]
[−] [ALPHA][ALPHA][X][M][A][B][ALPHA] [▶] [▶] [▶]
[↱][EVAL] [ENTER] [STO▶][ALPHA][ALPHA][S][L][M][S] [ENTER].



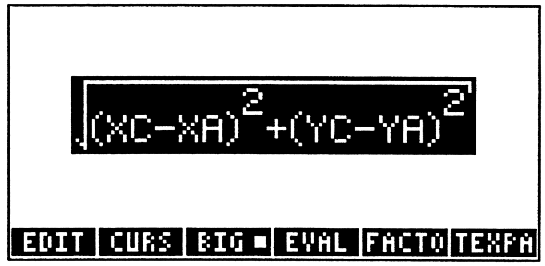
Property confirmed: The two slopes are equal, so those two segments are parallel!

What about the other assertion of the theorem—the relationship of the lengths of these two segments?

Go to the Equation Writer and use the general formula for the distance between two points, plugging in the variables you created.

Try the Base Length first: $\sqrt{(XC - XA)^2 + (YC - YA)^2}$

EQW √x X ALPHA C − X ALPHA A ▶ ▶ y^x 2 ▶ + ALPHA Y ALPHA C − ALPHA Y ALPHA A ▶ ▶ y^x 2 ▶ ▶ ▶.

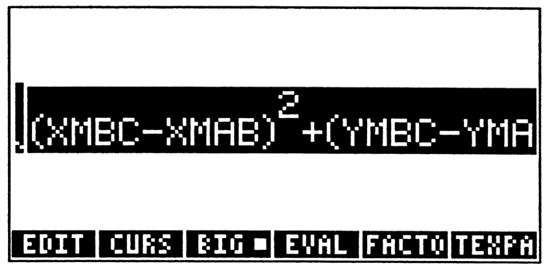


Now calculate and store:

↪ EVAL ENTER STO▶ ALPHA ALPHA L B A S E ENTER.

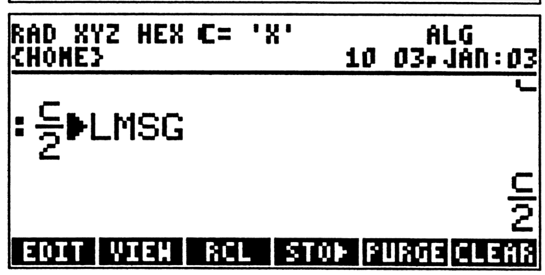
Do likewise for the segment connecting the midpoints:

EQW √x ALPHA ALPHA X M B C ALPHA − ALPHA ALPHA X M A B ALPHA ▶ ▶ y^x 2 ▶ + ALPHA ALPHA Y M B C ALPHA − ALPHA ALPHA Y M A B ALPHA ▶ ▶ y^x 2 ▶ ▶ ▶.



Calculate and store:

↪ EVAL ENTER STO▶ ALPHA ALPHA L M S G ENTER.



To confirm the relationship of the lengths of the two segments, just set up their ratio:

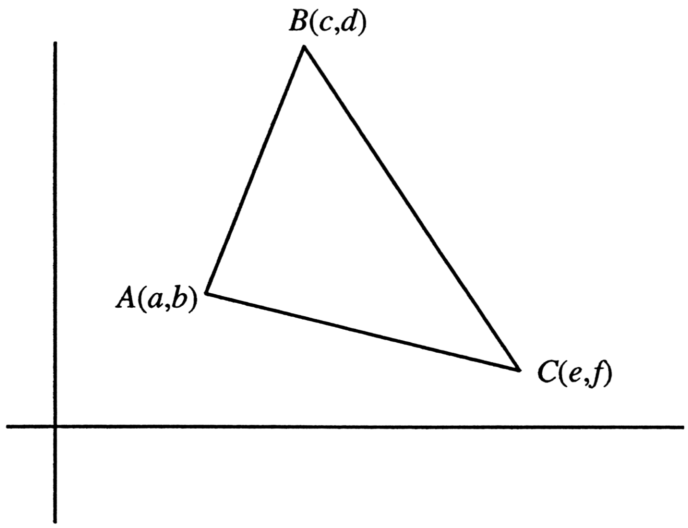
EQW ALPHA ALPHA L M S G ALPHA ÷ ALPHA ALPHA L B A S E ALPHA ▶

Now compute it: ↪ EVAL

Property confirmed: The segment connecting the midpoints of two sides of a triangle is $1/2$ the length of the base!

Follow-Up Activity

The proof you just completed was made simpler by a convenient choice of a triangle whose base sits on the x-axis. But of course, the proof should still hold for a triangle of any orientation, like this:



Use the above general triangle and your HP 49G to re-confirm the proof.

1. What changes might you expect in your calculations?

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2. How might you have to adjust to compare results this time?

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Teacher Notes

With its ability to work with variables and the simplicity of the Equation Writer, the HP 49G affords a great opportunity to connect the properties studied in a Geometry course with higher-level coordinate algebra. This activity walks through a coordinate proof and then provides an extension on the same topic.

While the points raised are fairly simple, the proof shows the value of using ratios to relate things instead of the usual "just show that visually the things are the same, therefore equal."

Skills Used: *Properties of segments connecting the midpoints of a triangle*
Slope of a line segment
Distance formula
Midpoint formula

Skills Introduced: *Storing and Manipulating Variables on the HP49G*
Relating Lengths and Slopes of segments.

Frequently in doing proofs on the calculator like this, more errors arise from labeling than from actual bad computation. Note the careful choice of meaningful variable names—and the use of single-character, lowercase names only for the coordinate values. (Note, too, that while you may choose lengthy names to clarify their contents, names that are too long won't entirely fit in a menu item on the VAR menu, thus compromising their utility as mnemonics.)

The use of the general distance formula is not necessary to find the lengths in the proof, of course. Since you're dealing with horizontal lines, you'd get the same result simply by subtracting the x-coordinates. This is worth some discussion if a student mentions it; if not, it's worth pointing out. The general calculation is definitely necessary in the follow-up activity, however. In fact, that gets rather messy, and the two distances don't necessarily appear to be all comparable upon mere inspection. The power of the machine shows the ratios to be either $^1/_2$ or 2 (depending upon the way they're set up).

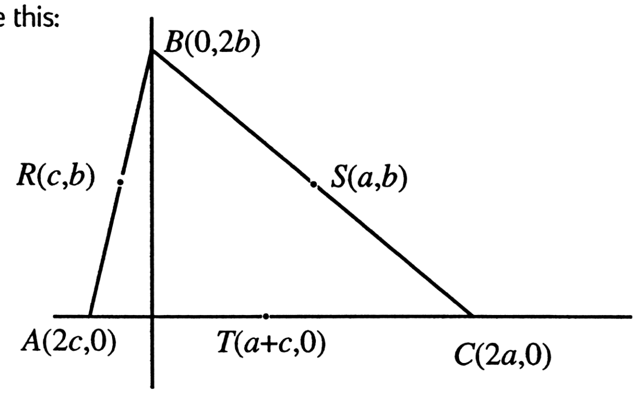
As set up, the proof uses horizontal lines, so the slope values of zero should come as no surprise. Hopefully, at least one student will jump on that point quickly. In the follow-up, the non-zero slopes are still fairly simple and can, with some prodding, be seen intuitively to be equal by the better students. Better for everyone, though, is to set up those slopes in a ratio for comparison. The result should be 1. (If the results come out to be −1, make sure the discussion goes to the fact that with the slope being a difference quotient, the order of the variables is important.)

Another good discussion opportunity arises when the calculator produces an absolute value when calculating the distance results.

Another Coordinate Geometry Proof

Here's another good proof that you can demonstrate with the help of the HP 49G. Show that the perpendicular bisectors of the sides of any triangle meet in a single point.

Choose any arbitrary triangle and orient it so that its base sits on the x-axis. Then name the coordinates of its vertices in a convenient manner, like this:



The equations of the lines forming the perpendicular bisectors of the sides of this triangle are therefore:

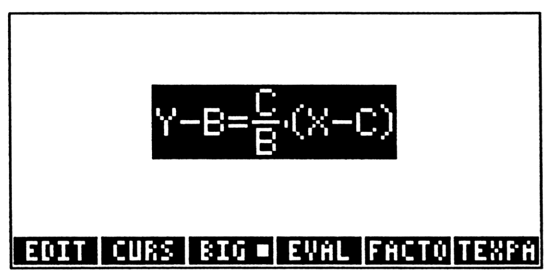
For segment AB: $y - b = c/b\,(x - c)$
For segment BC: $y - b = a/b\,(x - a)$
For segment AC: $x = a + c$

You need to show that these three lines intersect in a single point.

First, before you do anything else, you must purge (erase) the variables A, B, C, X, and Y if they appear on your VAR menu. Press [TOOL] **PURGE** [↰][{}] [↱]['] [ALPHA][A] [▶] [↱][,] [↱]['] [ALPHA][B]..., etc., [ENTER].

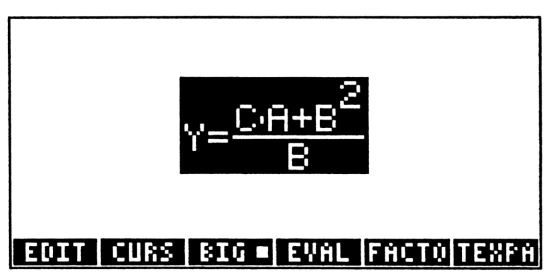
Now, for convenience, type the equation for the perpendicular bisector of segment AC first: [EQW] [X] [↱][=] [ALPHA][A] [+] [ALPHA][C]. Save this for later pasting: [↱][▲] [↱][CUT].

Next, type the equation for the perpendicular bisector of segment AB: [ALPHA][Y] [−] [ALPHA][B] [↱][▲] [↱][=] [ALPHA][C] [÷] [ALPHA][B] [▶] [X] [↰][()] [X] [−] [ALPHA][C] [↱][▲]. You should see this:



Now substitute into this equation the expression for X that you saved (segment AC), and isolate Y:

[↱][ALG] [6] [ENTER] [↱][PASTE] [↱][▲] **EVAL** [+] [ALPHA][B] [↱][▲] **EVAL** **EVAL**. You should see this:



Press [ENTER][ENTER] to send that expression back to the stack. You're going to compare it to the expression for Y that you get from the other equation.

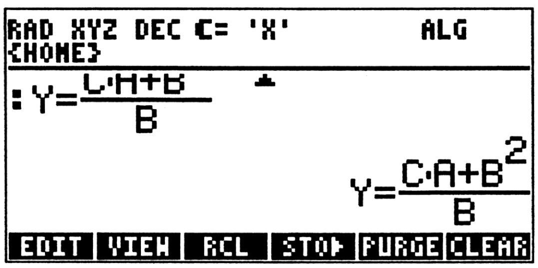
Now go back into the Equation Writer to enter the other equation (for the bisector of BC):

[EQW] [ALPHA][Y] [−] [ALPHA][B] [↱][▲] [↱][=]
[ALPHA][A] [÷] [ALPHA][B] [▶] [×] [↰][()] [X] [−] [ALPHA][A] [↱][▲].

Now do the same substitution for X, and isolate Y:

[↱][ALG][6] [ENTER] [↱][PASTE] [↱][▲] EVAL [+] [ALPHA][B] [↱][▲] EVAL EVAL [ENTER] [ENTER].

Press [▲], and observe that the two equations for Y are identical, proving that the perpendicular bisectors of the sides of a triangle intersect in a single point:



The Definition of Derivative

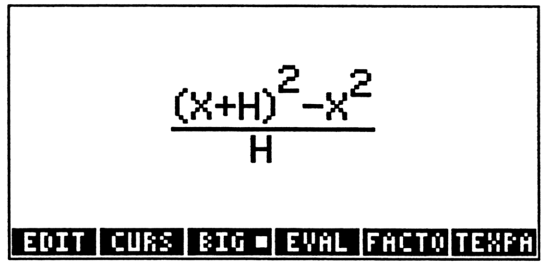
This is a short exercise in the use of the HP 49G's Equation Writer to study limits. For this exercise, you will need to set your machine to ALGebraic mode (via the [MODE] key). You will also need to purge the variable H: [TOOL] [PURGE] [↱] ['] [ALPHA] [H] [ENTER]. Now you're ready to begin.

For any function, f, its derivative, f', is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for all x, if the limit exists.

For the function $f(x) = x^2$, therefore, this definition becomes $f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

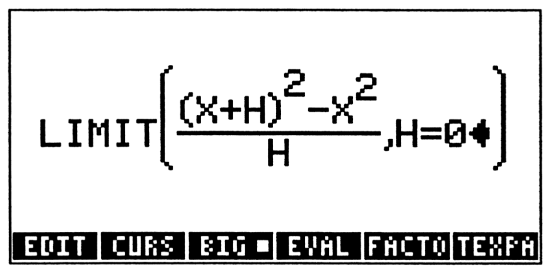
Enter this difference quotient in the Equation Writer:
[EQW] [X] [+] [ALPHA] [H] [↱] [▲] [y^x] [2] [▶] [−] [X] [y^x] [2] [↱] [▲] [÷] [ALPHA] [H].

You should see this:



Next, press [↱] [▲] to highlight the entire expression. Then apply the limit command to the highlighted expression:

[↰] [CALC] [2] [ENTER] [2] [ENTER] [ALPHA] [H] [↱] [=] [0].
You should now see this:



The calculator can now evaluate this limit! Highlight the entire expression ([↱] [▲] again), and [EVAL].... Voila!*

How is this done? Magic? No—people have been evaluating limits long before computer algebra systems. They simply expanded the quotient expression and looked at its behavior as H became very small....

Use [↱] [UNDO] to recover the limit expression. First, expand the $(X+H)^2$ term: [▼] [▼] [▼] [EVAL]. Then highlight the entire numerator, [▲], and [EVAL] to collect like terms. [FACTO] the numerator, yielding

$\lim_{h \to 0} \frac{H(2X+H)}{H}$. Now [▲] to highlight the entire quotient, and [EVAL]: $\lim_{h \to 0} 2(X+H)$

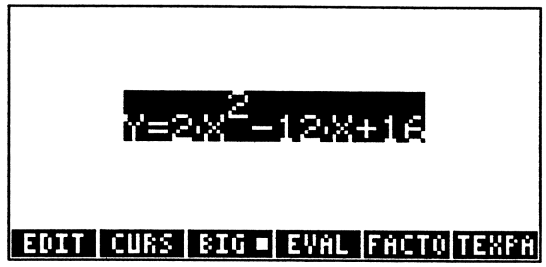
Clearly, as H approaches 0, $(2X+H)$ approaches $2X$ (which you can verify via [↱] [▲] [EVAL].)

That's it: You've shown that if $f(x) = x^2$, its derivative is given by $f'(x) = 2x$.

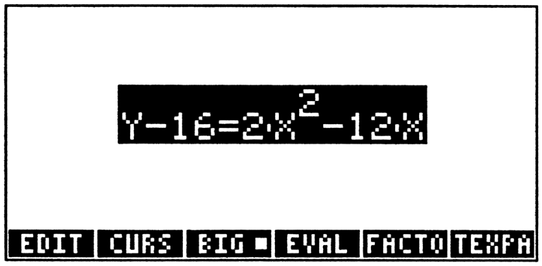
**If you're prompted to choose approximate mode at this point, it means you didn't purge* H *before starting. Respond with* [NO]*, then press* [OK] *when the limit error is announced. Then do the purge procedure given above and begin the exercise again.*

The Standard Equation of a Parabola

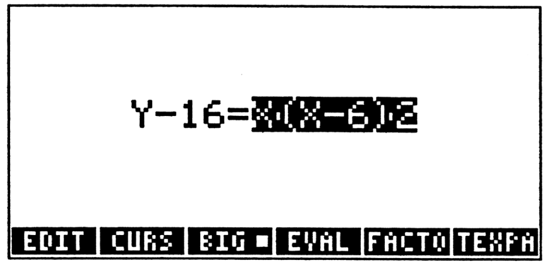
Did you realize that your HP 49G's Equation Writer can help you do algebra? For example, it can help you convert from a parabola's general equation form to the standard form, by completing the square.



First, purge variables X and Y. Then enter an equation, such as $y = 2x^2 - 12x + 16$: [EQW] [ALPHA] [Y] [↱] [=] [2] [X] [y^x] [2] [▶] [▶] [−] [1] [2] [X] [+] [1] [6] [↱] [▲].

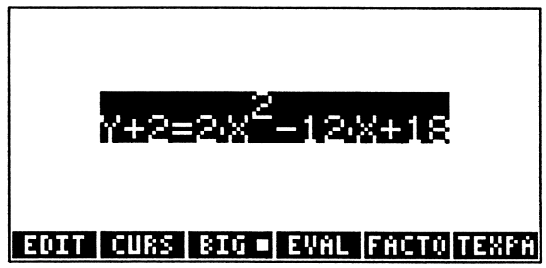


To convert to standard form, the first step is to subtract 16 from both sides: [−] [1] [6] [↱] [▲] **EVAL** **EVAL**.

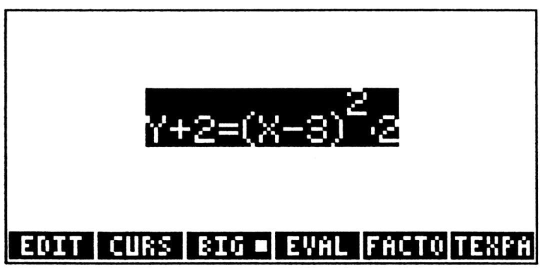


Now highlight and factor the right side: [▼] [▶] **FACTO**.

Next, expand the first two factors: [▼] [↱] [▶] **EVAL**.

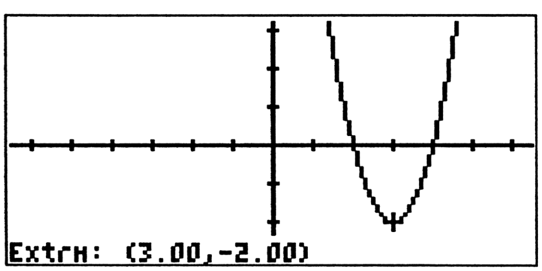


Add $(6/2)^2 \cdot 2$ to both sides: [↱] [▲] [+] [6] [÷] [2] [▶] [y^x] [2] [▶] [×] [2] [↱] [▲] **EVAL** **EVAL**.



Highlight and factor the right side again: [▼] [▶] **FACTO** In this standard form, you can tell by inspection that the vertex of the parabola is at (3,-2).

To verify that graphically, just isolate Y: [▲] [−] [2] [↱] [▲] **EVAL** [▼] [▶] [↱] [COPY] [ON] [↰] [2D/3D]. Set the **Type** to **Function**, highlight the **EQ** field, and [↱] [PASTE] [ENTER].



Set **INDEP** to **X**, then **ERASE** and **DRAW**. Once you're on the finished graph screen, just press **FCN** **EXTR**, and you'll see that the vertex is at (3,-2).

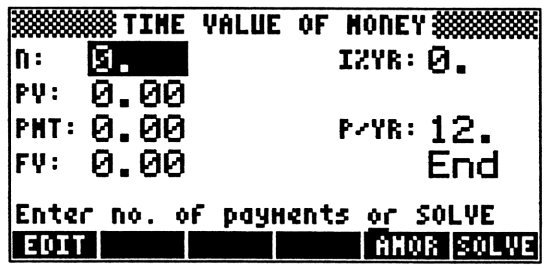
Finance: Rebate or Lower Rate?

The financial calculation tools of your HP 49G can help you make big decisions with little work. Compound interest calculations can seem confusing, but the basic idea is quite straightforward.

For example, suppose you're in the market for a new car. An ad in the local newspaper shows just the make and model you've been looking for—and the price is also fair: $18,000. But the dealership is offering the following choice: They'll either offer you 100% financing (i.e. no down payment) for 4 years, at an annual interest rate of 2.9%; OR they'll give you a $1500 cash rebate which can be immediately applied to your purchase price—in which case you'll have to find conventional financing, currently at 8% for a 4 year car loan.

Which is a better deal? Use your HP49G to see if there is a clear cut good choice. First, since you're going to be dealing with dollars and cents, set the display mode to show only two decimal places: [MODE] [▼] [CHOOS] [▼] [OK] [▶] [2] [OK]. Now you're ready.

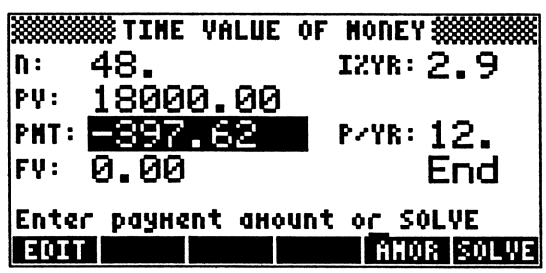
Press [←][FINANCE] to go to the TIME VALUE OF MONEY screen, which has these fields:



n:	Number of payments to be made.
I%YR:	Interest rate (in mixed decimal form).
PV:	Present Value (amount to be financed).
PMT:	Periodic Payment amount.
FV:	Future Value (amount left to be paid at the end of the loan's term).
P/YR:	Number of payments made per year.
End or Begin	The point in each period where the payment occurs.

As in any of the other numerical solvers, you just enter all known data in the appropriate fields, then highlight the item you want to solve for and press [SOLVE]. (And if you exit the screen accidentally, you can get back to it by pressing [←][FINANCE] again. The values you have already entered will still be there.)

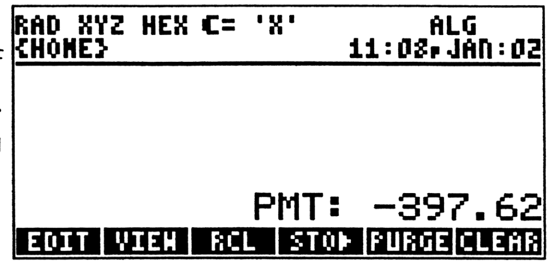
Start with the 2.9% loan of $18,000. Calculate your monthly payment and total cost under that scenario by entering these values in their fields:



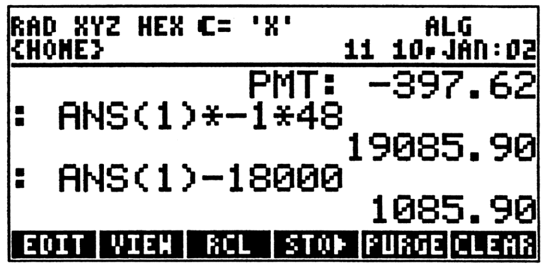
n:	48
I%YR:	2.9
PV:	18000
PMT:	(Skip this field for now—press [▶].)
P/YR:	12
FV:	0 (No balance left at the end.)
End or Begin	Use End—conventional in most loans.

Finally, use the arrow keys to move the highlight back to the PMT field and press [SOLVE]. As shown here, you should get -397.62. This is your "per month" cost for the car under the 2.9% financing option.

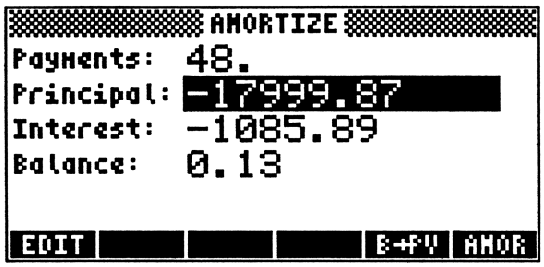
To quickly calculate your total cost over the 4-year term of the loan, just press CANCEL to go back to the home screen. There you'll see that the machine has copied the value you just solved for (in this case, the payment amount):



You can now simply multiply by the number of payments to get the total cost: × +/− 1 (to make it positive) × 4 8 ENTER. Then subtract out the original principal to get the total interest paid: − 1 8 0 0 0 ENTER.

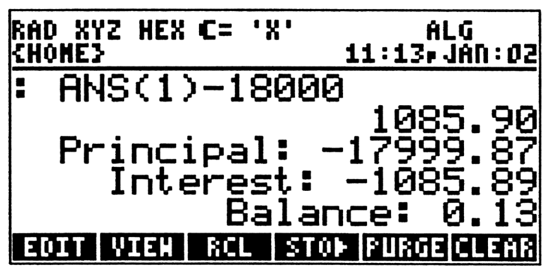


There's another easy way to compute interest (and principal) paid. Press ← FINANCE to return to the **TIME VALUE OF MONEY** screen. All should still be as it was when you calculated the **PMT** amount.



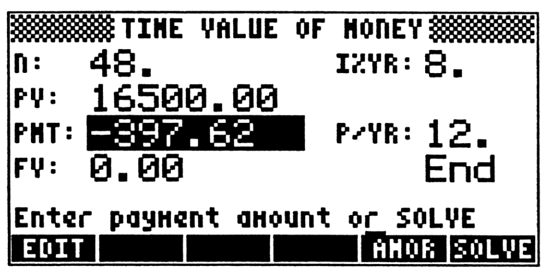
But now press **AMOR** to move to the **AMORTIZE** screen. Here you can enter the number of payments you want to analyze (in this case 48—press 4 8 ENTER), then just press **AMOR** to calculate all the other fields.

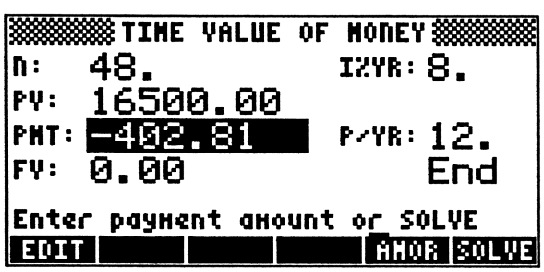
Press ENTER ENTER to return to the stack, where these results have also been copied:



What about the rebate option?

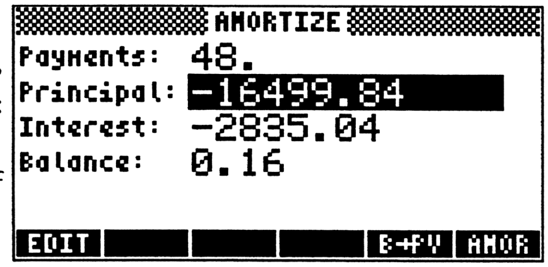
To calculate that scenario's monthly payment and total cost, press ← FINANCE once again, then change only what you need to: At the **I%YR** field, type 8 ENTER. At **PV**, type: 1 8 0 0 0 − 1 5 0 0 ENTER.





Then, with the cursor at the **PMT** field, press **SOLVE**.

Now use the AMORTIZE screen to get the total cost: AMOR, then (at the Payments field): 4 8 ENTER. Then just AMOR:



Now you can go back to the stack and find the total cost of this car by adding those results: 19,335.04

You knew simply by looking at the monthly payment that the 2.9% financing would be cheaper than the rebate, but now you know how much cheaper: $19,085.90 vs. $19,335.04 over the 48-month term, a difference of $249.12, or $5.19 per month.

Follow-Up Activity

1. What if you were to take the rebate against the price of the car, but invest the $1500 at a modest 5% rate for the 48 months?

____

____

____

____

2. What if you were to NOT take the rebate but were to save the $5.19 each month at that same modest 5% rate? Describe how you would calculate the effects of this.

____

____

____

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3. What if the cheaper financing meant you could only get the loan for 3 years instead of 4? What would happen to the total cost of the car? Would you still be able to afford it? Why/Why not?

____

____

____

____

Solutions to the Follow-Up Activity

1. *Assuming an annually compounding savings account, the $1,500 would grow to $1823.26 in the 4 years ($323.26 in interest).*

2. *That would yield $275.15 at the end—an extra $26.03 in interest—merely adding to the advantage of the 2.9% financing choice.*

3. *A 36-month payment would be $522.67 per month, significantly higher than either 48-month option. But because of the more rapid payoff, the total cost of the car would then be only $18,816.08—by far the most cost effective. (But can you afford the higher payment? Food for discussion.)*

Teacher Notes

Skills Used: *Compound growth formula*

Skills Introduced: *Working with the Finance application on the HP49G;*
What Amortization means

A word about rounding: Even while displaying just two decimal places, the HP 49G continues to calculate to its full precision in most areas, including the `TIME VALUE OF MONEY` *screen,. But not so at the* `AMORTIZE` *screen, which is specifically designed to allow for "real-life," in that it actually rounds its calculations based upon the current display setting. This leads to the slight discrepancy between the total costs as calculated from the TVM values (which use all 12 digits of precision) as opposed to amortized numbers (which use only as many decimal places as you specify—usually just 2 digits for dollars and cents, of course). The TVM values will be the more mathematically accurate; the amortized values will better reflect the real world.*

At the TVM screen, the payment for the 2.9% loan of $18,000 is computed to be 397.622841491, which you will see if you look at all the digits. But nobody writes a check every month for that amount. The actual amount would be rounded—in this case, downward: 397.62. That means the loan is being slightly underpaid, so the balance (FV) after the 48th payment is not exactly zero. There's still about thirteen cents owed. You can demonstrate this, by doing "manually" what the `AMORTIZE` *screen does automatically: Enter the number* `-397.62` *(exactly) into the* `PMT` *field and then calculate* `FV`. *Thusthe finance contract in this case would probably stipulate in "47 equal payments of $397.62, with a final payment of $397.75."*

Obviously, this is a great place to discuss rounding significance with the class—and the imprecision of such a precise machine—as well as how finance really works. You might point out that sometimes the final payment will be less—if the mathmatically exact payment is such that the rounding happens to go in the upward direction, so that the loan was slightly overpaid, leaving a credit balance at the end.

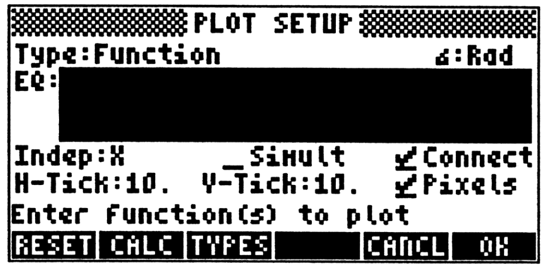
Another point: The application of a state sales tax has an effect on the amount to be financed as well—an issue not addressed in this activity. That opens up other issues to discuss, such as the impact of including the tax as part of the amount to be financed as opposed to paying it up front.

Three Approaches to Limits: Graphical, Numeric and Analytical

The idea of limit is central to calculus. To best understand limits, it helps to consider them in different ways. In this activity, you'll explore limits using three different approaches: numeric, graphical, and analytical.

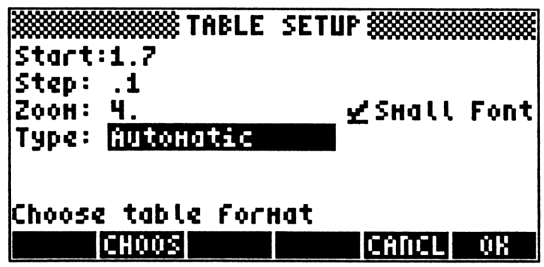
Take a look at the function $f(x) = (2x - 4)/(x^2 - 4)$, for values of x close to 2.

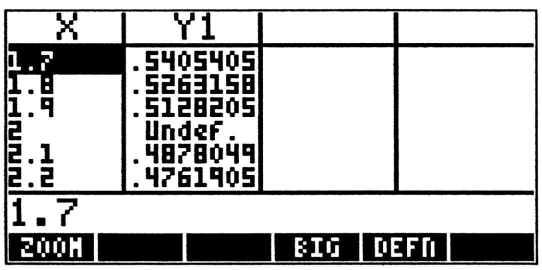
With the calculator in default settings, press ⬅ 2D/3D. If the plot **Type** is something other than **Function**, press **CHOOS**, highlight **Function**, and press **OK**. The **INDEP** field names the independent variable for the graph and table. If it is other than X, highlight it now, and press X ENTER. The **2D/3D PLOT SETUP** screen should now look like this:



Go to the **EQ** field and enter the function: ⬅ Y= NXT **CLEAR** **OK** NXT **ADD** 2 X − 4 ▶ ▶ ÷ X y^x 2 ▶ − 4 ENTER. The screen should appear as shown.

Next, press ⬅ TBLSET and set up the table as shown here:





Finally, press ⬅ TABLE to look at the resulting table, which should resemble this screen. Notice that when $x = 2$, $f(x)$ is not defined. To take a closer look at that region, scroll down the table to highlight the row at $x = 2$. Press **ZOOM** **OK** to zoom in. Notice how the increment between adjacent x-values in the table then reduces by a factor of 4; where before they were 0.1 apart, now they're .025 apart. Zoom in at $x = 2$ a couple more times.

1. Use the calculator's table of values to fill in this table:

x	1.9953125	1.996875	1.9984375	2	2.0015625	2.003125
$f(x)$						

2. What happens to the values of $f(x)$ as x gets close to 2?

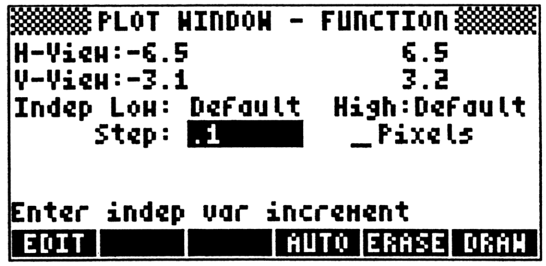
____

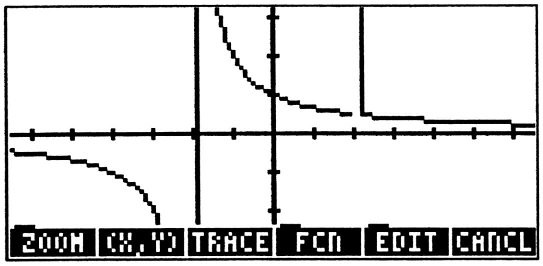
____

____

How does this limit look on a graph? Find out.

Press ← WIN and set your viewing window as shown here:





Now press ERASE DRAW. You should see something like this. [there is a bug in v1.10 that keeps the entire graph from being drawn.]

Press XY TRACE and use the arrow keys to investigate the behavior of the function near $x = 2$.

3. What happens to the graph of $f(x)$ as x gets close to 2?

____

____

____

Can you determine the behavior of this limit analytically—with algebra? Press CANCEL until you get back to the stack. Press ← Y=, highlight the expression (2*X-4)/(X^2-4), and EDIT. This will take you to the Equation Writer. Press ← CALC 2 ENTER 2 ENTER X → = 2, to apply the limit command at x approaching 2. Press → ▲ to highlight the limit expression, then EVAL to evaluate it.

4. What did the calculator give for $\lim_{h \to 2} \frac{2x - 4}{x^2 - 4}$? __________________________

5. How does the calculator's answer compare with your answers from the table and graph?

____

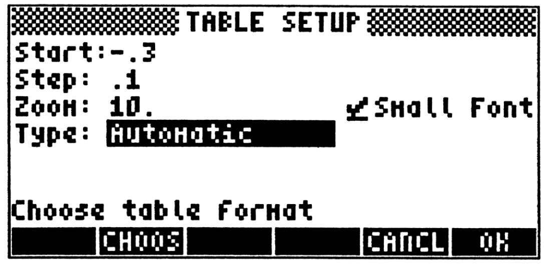
____

6. Press → UNDO to recover your original limit expression. Press ▼ ▼ so that just 2X-4 is selected, then FACTO to factor the numerator. Press ▶ FACTO to factor the denominator. The numerator and denominator have a common factor, so press ▲ EVAL to simplify. What does the original quotient simplify to? ____________________.

7. These two expressions, (2*X-4)/(X^2-4) and its simplified cousin, are equal for all values of X except 2. But when you're taking a limit as X *approaches* 2, you're not concerned with the function's behavior *at* 2. What does the simplified limit evaluate to? ____________________.

Limits at Infinity

What happens to a function of x as x gets arbitrarily large? That's a different sort of limit, one whose "approach" is customarily denoted as $x \to \infty$.



For example, look at $\dfrac{2x-4}{x^2-4}$.

If necessary, press (CANCEL) to get to the stack. Then press (←)(TBLSET) and change your settings as shown:

8. Now press (←)(TABLE) and fill in this table:

x	-0.3	-0.2	-0.1	0	0.1	0.2
$g(x)$						

9. Position the highlight bar on x = 0, and press ZOOM (▽) (ENTER) to zoom out of the table. Doing so multiplies the increment between adjacent table inputs by a factor of 10. Notice that f is undefined at x = -2 and at x = 2. Why? ______

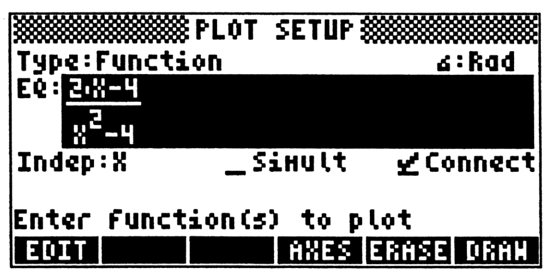
10. With the row at x = 0 highlighted, zoom out of the table three more times, paying attention to how the inputs and outputs change as you do so. Fill in this table of values:

x	-3000	-2000	-1000	0	1000	2000
$g(x)$						

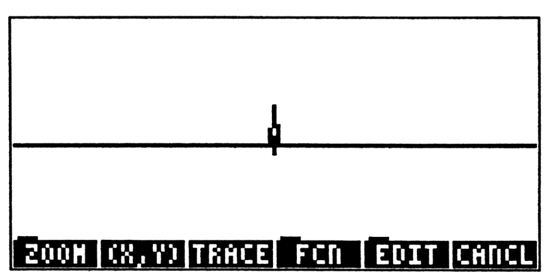
11. Looking at the table, what can you say about $\lim_{x \to \infty} \dfrac{2x-4}{x^2-4}$? ______

______

Explore this limit graphically. Use (←)(WIN) and (←)(2D/3D) to prepare as shown in the screens below. (Be sure to change AXES■ to AXES, as shown, so that the coordinate axes will not be drawn.)



Press ERASE DRAW, then ZOOM ZFACT, set the H-Factor to 10, and OK. Then press (NXT) HZOUT ("Horizontal Zoom Out"). The window is re-sized, multiplying the inter-column distance by 4 (without affecting the inter-row distance). Now zoom out horizontally twice more. You should see this:



12. Except at $x = 0$, the graph looks flat. What does this tell you about the outputs from the function, f?

____

13. What does this say about $\lim_{x \to \infty} \frac{2x - 4}{x^2 - 4}$? ______________________________

14. Press (CANCEL) (←)(Y=) **EDIT** to take your function to the Equation Writer. Then apply the limit command: (←)(CALC) (2) (ENTER) (2) (ENTER) (X) (→)(=) (←)(∞). Press (→)(▲), then **EVAL** to evaluate it.

What does the CAS evaluate $\lim_{x \to \infty} \frac{2x - 4}{x^2 - 4}$ to be? ______________________ How does this compare with your answers to numbers 11 and 13? ______________________

In number 6 above, you found that $\lim_{x \to \infty} \frac{2x - 4}{x^2 - 4} = \lim_{x \to \infty} \frac{2}{x + 2}$. The numerator, 2, stays constant; the denominator, $x + 2$, gets infinitely large with x. The quotient, therefore, approaches 0.

Teacher Notes

This activity can be used early in the chapter on limits. It lets students explore the idea of limits from a multi-representational approach: numerically through a table of values, graphically, and algebraically with the use of the calculator's computer algebra system. Since this activity would usually be covered early in the year, very little familiarity with the calculator is assumed.

Answers

1.

x	1.9953125	1.996875	1.9984375	2	2.0015625	2.003125
$f(x)$	0.5005866	0.5003909	0.5001954	undef	0.4998048	0.4996097

2. *As x approaches 2, it appears that the values of $f(x)$ approach 0.5.*

3. *As x approaches 2, it appears that the vertical coordinates of points on the graph of f approach 0.5.*

4. *0.5*

5. *They are all the same.*

6. $\frac{2}{x+2}$

7. $\frac{2}{2+2}$ *or 0.5.*

8.

x	-0.3	-0.2	-0.1	0	0.1	0.2
$g(x)$	1.176471	1.111111	1.052632	1	0.952381	0.9090909

9. *At $x = 2$ and $x = -2$, the value of $x^2 - 4$ becomes 0, and division by 0 is not defined.*

10.

x	-3000	-2000	-1000	0	1000	2000
$g(x)$	-0.000667	-0.001001	-0.002004	1	0.001996	0.000999

11. *As, it appears the values of $f(x)$ approach 0.*

12. *Since the graph flattens out, the values of f are getting close to one another.*

13. *Outputs getting close to one another offer evidence that the limit exists. Tracing on the graph indicates that the limit is 0.*

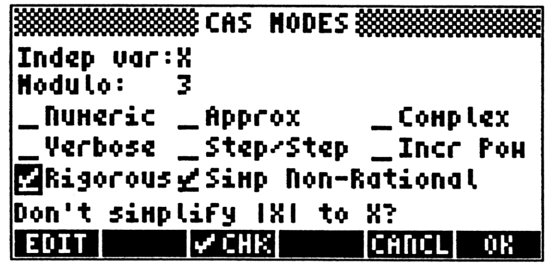
14. *0, the same as indicated from the table and graph.*

Three Approaches to the Derivative: Graphical, Numeric and Analytical

What does a derivative "look like?" How does it behave? Here's an exercise that gives you three different ways to think about a derivative.

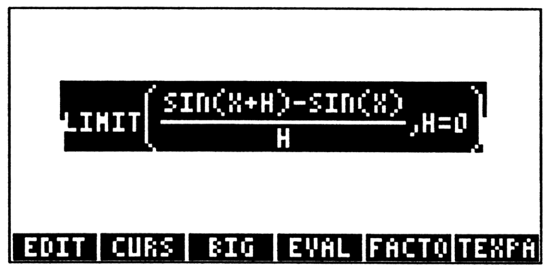
As you know, for any function, $g(x)$, its derivative, $g'(x)$, is defined as $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$ for all x, if the limit exists. For the function $g(x) = \sin(x)$, this definition becomes $g'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$

First purge the variables X and H. Then to evaluate this limit, start by entering it into the HP 49G's Equation Writer and using the symbolic algebra tools.



First, set the modes correctly. Press [MODE] [CAS], and make sure your screen appears as shown here:

Press [OK] [OK] to accept the mode settings, then go to the equation writer, [EQW]. Enter the difference quotient first: [SIN][X] [+] [ALPHA][H] [→][▲] [−] [SIN][X] [→][▲] [÷] [ALPHA][H]. Now highlight the entire expression, [→][▲], then apply the limit command to the highlighted expression: [←][CALC] [2][ENTER] [2][ENTER] [ALPHA][H] [→][=] [0] [→][▲].



If [BIG■] appears in the menu (i.e. if the "big" font is currently selected) change it to [BIG] (press it once) to select the smaller font, so that you can see the entire expression.

The calculator is fully capable of evaluating this limit—press [EVAL].... Voila!

How is this done? Magic? No—people have been evaluating limits long before computer algebra systems. They simply expanded the quotient expression and looked at its behavior as H became very small....

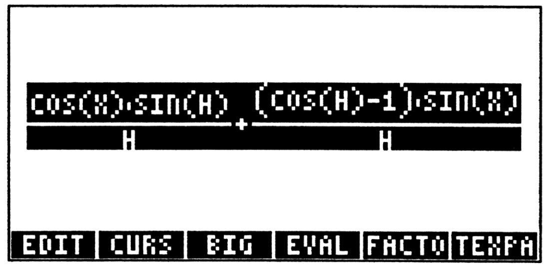
Press [→][UNDO] to recover the limit expression. Now use a trigonometric identity to expand SIN(X+H). Press [▼][▼][▼], which will highlight SIN(X+H). Then press [TEXPA], which will apply the trig identity SIN(A+B)=SIN(A)*COS(B)+COS(A)*SIN(B). Press [▲][▲] to highlight the entire quotient. Now split the fraction into two terms by applying PARTFRAC ([←][ARITH] [2] [ENTER] [1][4] [ENTER]).

Now you have the following limit expression, equivalent to your original:

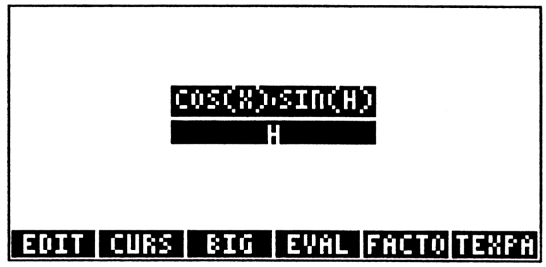
$$\lim_{H \to 0} \frac{COS(X)SIN(H)}{H} + \frac{(COS(H) - 1)SIN(X)}{H}$$

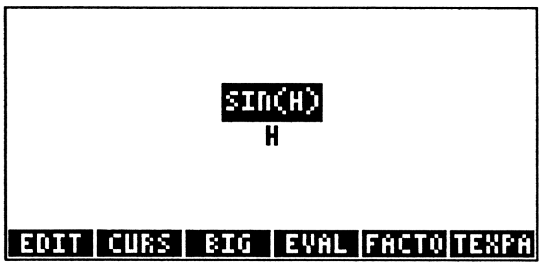
To find the limit of a sum, notice that you can find the sum of the limits (of each addend). Also, note that in the first term, COS(X) does not depend on H, so it can be factored through the limit. Similarly, in the second term, SIN(X) can be factored through. All you really need to do is find how SIN(H)/H and (COS(H)-1)/H behave for H near 0. To do that, use the calculator's table and graphing features.

First, press [↱][COPY] to make a copy of the expression in the limit, as shown here.



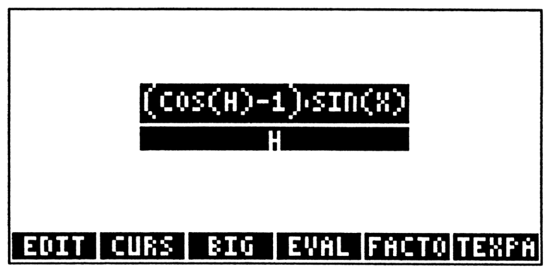
Now press [CANCEL] [↰][2D/3D] and set **Type** to **Function**. Then [↰][Y=] [NXT] [CLEAR] [OK] [NXT] [ADD] [◀][↱][CLEAR]. In the Equation Writer, [↱][PASTE] the expression you copied earlier. Now delete the second addend, [▼] [▶] [↰][DEL]. Your display should now appear like this:

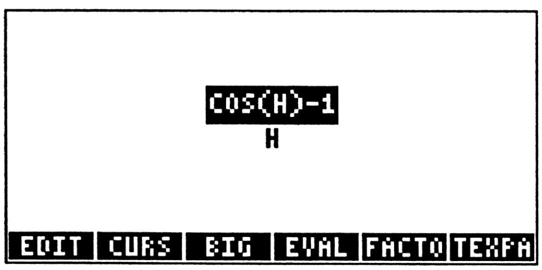




To delete the COS(X) factor, press [▼][▼] [↰][DEL] [↰][DEL]. Your display should appear as shown here.

Press [ENTER] to put that expression into Y1. Then press [ADD] [◀] [↱][CLEAR] [↱][PASTE] to use the original expression, and [▼] [↰][DEL]. Your display should look like this:

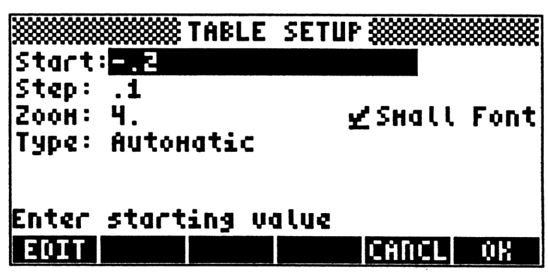




Press [▼][▼] [▶] [↰][DEL] [↰][DEL]. Your display should appear as shown here (left).

Press [ENTER] to put that expression into Y2.

Now press [↰][2D/3D] and set up your plot as shown below, left. (Be sure to change the independent variable to H.) Then press [↰][TBLSET] and set up the as shown below, right.



Press [←][TABLE] DEFN. Press [▽][▽] [▷] to position the cursor on the row with H=0 and in the column with SIN(H)/H. Zoom in to the table twice, by pressing ZOOM OK ZOOM OK, and you'll see outputs for SIN(H)/H and (1-COS(H))/H for values of H near 0.

1. What does it appear $\lim_{H \to 0} \frac{SIN(H)}{H}$ is? __________________________

2. What does it appear $\lim_{H \to 0} \frac{COS(H) - 1}{H}$ is ? __________________________

Go back to your separated, factored limit expression:

$$\lim_{H \to 0} \left(\frac{SIN(X + H) - SIN(X)}{H} \right) = COS(X) \lim_{H \to 0} \frac{SIN(H)}{H} + SIN(X) \lim_{H \to 0} \left(\frac{COS(H) - 1}{H} \right)$$

Replace each limit expression on the right hand side of this equation with your answers from above.

3. What is $\lim_{H \to 0} \left(\frac{SIN(X + H) - SIN(X)}{H} \right)$? __________________________

4. If $g(x) = \sin(x)$, what is its derivative, $g'(x)$? __________________________

Take a graphical and numerical look at this result. Press [←][2D/3D], highlight the INDEP field, and press [X] [ENTER] to change the independent variable to X. Press [←][Y=] [NXT] CLEAR OK [NXT] ADD. Enter the expression (SIN(X+.001)-SIN(X))/.001 for Y1 . (This is the difference quotient for SIN(X), with a small value, .001, substituted for H.) Enter the expression COS(X) for Y2.

Press ERASE DRAW to see the graphs.

5. Trace on the two graphs, observing function outputs for different inputs. What do you notice?

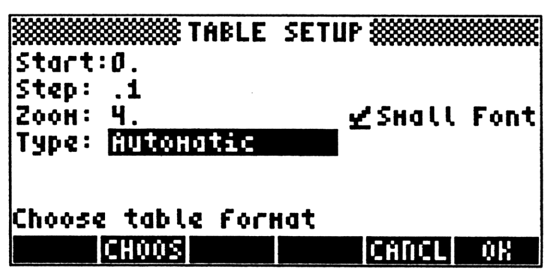
____

____

6. What does this tell you about the derivative of the function $g(x) = \sin(x)$?

____

Press [←][TBLSET], and set up the table as shown here:



Then press [←][TABLE] and look at the resulting table of values.

7. What do you notice about the table of values for the difference quotient and COS(X)?

____

8. What does this numerical evidence tell you about the derivative of $\sin(x)$? ____________________

9. How could you change the definition of Y1 to make the values of Y1 and Y2 even closer?

____

____

____

Teacher Notes

This activity introduces the idea of the derivative function, and introduces connections between the graph of a function and its derivative. Students should know the definition of derivative at a point. They should also be familiar with defining functions and looking at graphs and tables before doing this activity. Most keystrokes are shown, but some familiarity with graphs and tables is assumed.

As a further exploration, students can investigate the difference in behavior between the difference quotient definition of derivative, covered here and the symmetric difference quotient, which states $g'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x - h)}{2h}$

They can do this by looking at the errors in the two approximations—subtracting the exact value of the derivative from the difference quotient (or symmetric difference quotient) approximation.

Answers

1. *1*
2. *0*
3. *cos(x)*
4. *cos(x)*
5. *The two graphs are virtually indistinguishable in the default viewing window.*
6. *This gives graphical evidence that the derivative of sin(x) is cos(x).*
7. *The outputs from the difference quotient are very close, but not exactly equal, to the outputs from cos(x).*
8. *The answer to number 7 gives numeric support to the conclusion that the derivative of sin(x) is cos(x).*

The Derivative and Differentiability

What's the value of the derivative of a function at a point? How does its behavior compare to that of the function itself? When doesn't a derivative exist? You can answer all of these questions by using the difference quotient definition of derivative and your HP 49G calculator.

First, with the calculator in default modes (use the **RESET** option on the [MODE] screens, if necessary), define F(X)=X^3: [←][DEF] [→]['] [ALPHA][F] [←][()] [X] [▶] [→][=] [X][y^x][3] [ENTER]

Then define DQ(X,H)=(F(X+H)−F(X))/H: [←][DEF] [→]['] [ALPHA][D][ALPHA][Q][←][()][X] [→][,] [ALPHA][H] [▶] [→][=] [←][()] [ALPHA][F][←][()][X][+][ALPHA][H] [▶] [−] [ALPHA][F][←][()][X] [▶] [▶] [÷] [ALPHA][H] [ENTER]

DQ(X,H) represents the change in Y with respect to the change in X between the two points (X,F(X)) and (X+H,F(X+H)). In other words, it is measuring the slope of the line through those two points.

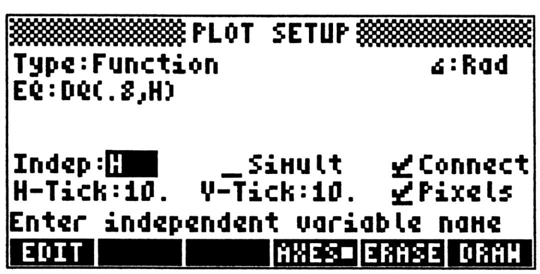
Now press [←][2D/3D], and check the **Type** of the plot. If it is not **Function**, press **CHOOS**, highlight **Function**, and press **OK**. [▼] down to the **INDEP** field and press [ALPHA][H][ENTER] to change the independent variable to **H**. Press [←][Y=] [NXT] **CLEAR** **OK** to clear existing functions in **EQ**, then [NXT] **ADD**.

Key in DQ(.8,H): [▶] [→][CLEAR] [ALPHA][D][ALPHA][Q] [←][()] [.][8][SPC][ALPHA][H] [ENTER]

You will be approximating $\lim_{H \to 0} \dfrac{(.8+H)^3 - .8^3}{H}$

by storing values in H that get closer and closer to 0 and then looking at the corresponding values of DQ(0.8,H).

Press [←][2D/3D]. The screen should look as shown here:



Press [←][TBLSET] and [▼] down to `Type`. Press **CHOOS**, highlight `Build Your Own` and press **OK**.

1. Press [←][TABLE] to go to the table, and type in the values requested for **H** to fill in the table below. (Shortcut: After doing the first column of the table, press [ENTER] to go back to the stack. Press [VAR], and look for a soft key labeled **TAB**. (Press [NXT], as needed, to find it.) Now press [+/−] **TAB** [STO▶] **TAB**. When you go back to the table, all values in the **H** column will have been multiplied by -1! Remember this tip; it will come in handy later in this activity.)

H	*DQ*(0.8,*H*)	*H*	*DQ*(0.8,*H*)
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
0.0001		-0.0001	

2. What do you think $\lim\limits_{H\to 0^+} \frac{(.8+H)^3-.8^3}{H}$ (i.e. the right-hand limit) is? ________________.

3. What do you think $\lim\limits_{H\to 0} \frac{(.8+H)^3-.8^3}{H}$ (i.e. the left-hand limit) is? ________________.

4. Does your answer here agree with the right-hand limit? ______________. It should—if not, check your work! This limit is the derivative of `X^3` at `X=0.8`. It represents the slope of the line tangent to the graph of F `(X)=X^3` at `X=0.8`.

5. Press [←][Y=] **EDIT** [▼] [+/−] [ENTER] to change the expression in **EQ** to `DQ(−0.8,H)`.

   Press [←][TABLE] and fill in values for H as before, to approximate $\lim\limits_{H\to 0} \frac{(-.8+H)^3-(-.8)^3}{H}$:

H	*DQ*(-.8,*H*)	*H*	*DQ*(-.8,*H*)
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
0.0001		-0.0001	

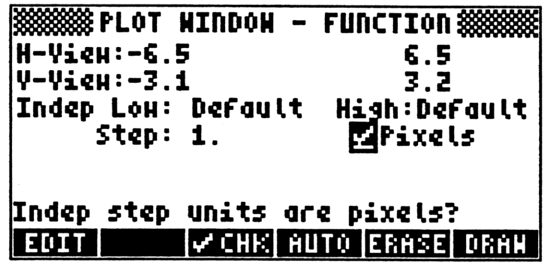
6. What do you get for the limit? ______________________ This value is the derivative of `X^3` at `X=−0.8` and represents the slope of the line tangent to the graph of `Y=X^3` at `X=−.8`.

Press [←][2D/3D], [▼] down to `INDEP`, then [X][ENTER] to reset the independent variable to `X`. Press [←][Y=] **DEL** **ADD** [ALPHA][F][←][()][X] [ENTER]. Press **ADD** again, then [ALPHA][←][D][1] [ALPHA][F][←][()][•][8][▶][X][←][()][X][−][•][8][→][▲][+][ALPHA][F][←][()][•][8][ENTER].

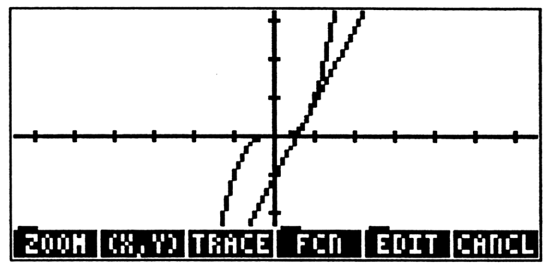
Your PLOT - FUNCTION screen should appear as shown here.

The expression d1F(.8) gives the slope of the line tangent to the graph of F at X=.8, i.e. $F'(.8)$. Thus the equation for Y2 is the Taylor form of the equation of the tangent line to the graph of Y=X^3 at X=0.8.

Press [←][WIN] and set your window up as shown here:



Then press [←][2D/3D] and check that the settings agree with those shown here.



Finally, press **ERASE** **DRAW** to graph F(X) and the tangent line. Your graph should look like this:

Press **TRACE** **(X,Y)**, and use [▶] to trace over to X=0.8. Then press any menu key ([F1] through [F6]) to recover the menu. Now press **FCN** **SLOPE**. Compare this value with your answer from above.

Press [CANCEL] to get off the graph screen, then [←][Y=]. Highlight Y2, and press **EDIT**. Press [▼] [▼] [▶] [+/−] [▲] [▶] [▼] [▶] [+/−] [▶] [▶] [▶] [▼] [+/−] [ENTER]. This should change all the .8's to −.8's and therefore make Y2=d1F(−.8)*(X+0.8)+F(−0.8). Press **DRAW** to graph F(X) and Y2 to see the tangent to the graph of Y=X^3 at X=−0.8. Trace over to X=−0.8, press any menu key to recover the menu, then **FCN** and choose SLOPE. Compare this value with your answer from above.

7. What do you notice about the two tangent lines? ____________________________

8. Try to offer an explanation for this. Can you generalize? ____________________________

Press [CANCEL] [←][Y=] and delete Y2. Press **ERASE** **DRAW** and see the graph of Y=X^3. Press **FCN** [NXT] **F'**. Trace along to X=.8, then to X=−.8. For any given X, the Y coordinate on the graph of F' tells you the slope of the line tangent to F at that X.

Part 2

Press [CANCEL] to get back to the stack. Not all functions behave as well as a simple polynomial like X^3. Try repeating this activity for F(X)=1+ABS(COS(π*X/2)), at X=1.

First, define F(X): [→]['] [ALPHA][F][←][()][X] [▶] [→][=] [1] [+] [←][ABS] [COS] [←][π][×][X][÷][2] [ENTER]. Press [←][Y=] [NXT] CLEAR OK [NXT] ADD, then [ALPHA][D][ALPHA][Q] [←][()][1][SPC][ALPHA][H] [ENTER], to put DQ(1,H) into EQ. Press [←][2D/3D], [▼] down to INDEP, and press [ALPHA][H] [ENTER] to change the independent variable back to H.

9. Press [←][TABLE] to go to the table. It is important that you evaluate both a left-hand and a right-hand limit for the difference quotient. (Recall the trick of taking -TAB and storing it to TAB, to get the values on the opposite side of 0 quickly.) Record your values in this table:

H	$DQ(1,H)$	H	$DQ(1,H)$
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
0.0001		-0.0001	

10. What is $\lim_{H \to 0^+} \dfrac{(1 + \text{ABS}(\cos(\pi(1+H)/2))) - (1 + \text{ABS}(\cos(\pi(1)/2)))}{H}$? ________________

11. What is $\lim_{H \to 0^-} \dfrac{(1 + \text{ABS}(\cos(\pi(1+H)/2))) - (1 + \text{ABS}(\cos(\pi(1)/2)))}{H}$? ________________

12. What is $\lim_{H \to 0} \dfrac{(1 + \text{ABS}(\cos(\pi(1+H)/2))) - (1 + \text{ABS}(\cos(\pi(1)/2)))}{H}$? ________________

13. You should be able to observe from the table values that the two-sided limit (and hence the derivative) does not exist for F(X)=1+ABS(COS(π*X/2)) at X=1. Can you observe this graphically, too?

    Press [←][2D/3D], arrow down to INDEP, and press [X] [ENTER] to change the independent variable back to X. Highlight the EQ field and press [→]['] [ALPHA][F][←][()][X] [ENTER] ERASE DRAW. Then press FCN [NXT] [F] to graph the derivative. How does it differ from the derivative of X^3?

    ____

    ____

    ____

Teacher Notes

Before doing this activity, students should be relatively familiar with defining functions, and looking at graphs and tables. They should understand the language of limits, including what is meant by one-sided limits. The activity is appropriate to use any time after the definition of derivative has been covered. It is particularly useful to demonstrate the difference between a function that is differentiable at a point, and a function that is not differentiable at a point.

Answers

1.

H	*DQ*(0.8,*H*)	*H*	*DQ*(0.8,*H*)
0.1	2.17	-0.1	1.69
0.01	1.9441	-0.01	1.8961
0.001	1.922401	-0.001	1.917601
0.0001	1.92024001	-0.0001	1.91976001

2. *About 1.92*

3. *About 1.92*

4. *Yes*

5.

H	*DQ*(-.8,*H*)	*H*	*DQ*(-.8,*H*)
0.1	1.69	-0.1	2.17
0.01	1.8961	-0.01	1.9441
0.001	1.917601	-0.001	1.922401
0.0001	1.91976001	-0.0001	1.192024

6. *About 1.92*

7. *The two tangent lines have equal slopes.*

8. *For an odd function, the slope at (x,f(x)) is the same as the slope at (-x,f(-x)). The symmetry in the graph of an odd function guarantees this.*

9.

H	*DQ*(1,*H*)	*H*	*DQ*(1,*H*)
0.1	1.564345	-0.1	-1.56434
0.01	1.570732	-0.01	-1.57073
0.001	1.570796	-0.001	-1.5708
0.0001	1.570796	-0.0001	-1.5708

10. *1.570796*

11. -1.5708

12. The limit does not exist, since the two one-sided limits are different.

13. Here, the derivative takes big jumps at every integer value of x.

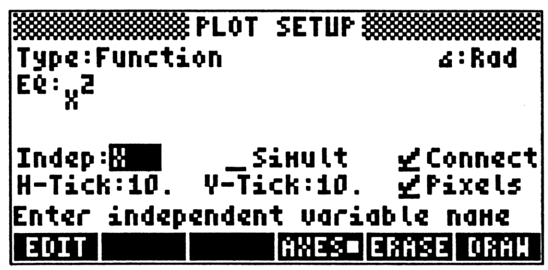
The Derivative and Approximations

No matter how "wavy and curvy" the graph of a function may look from a distance, when you look at it closely—very closely—it usually begins to resemble a straight line. This suggests that you might be able to approximate the function with a line in that very limited region. In this activity, you're going to see how zooming in to a graph can help you investigate the behaviors of such function approximations. You'll discover some startling results about the tangent line approximation to a curve near the point of tangency. You'll also be amazed to compare the behaviors of the errors resulting from two different approximations to the derivative.

Power Zooming

Zooming in on the graph of function—at a point where the function is differentiable—should produce a line whose slope is equal to the derivative of the function at the point, so long as you zoom with *equal scaling* horizontally and vertically. Otherwise you can't detect differentiability. You can see this when you compare the results of zooming with equal and unequal factors.

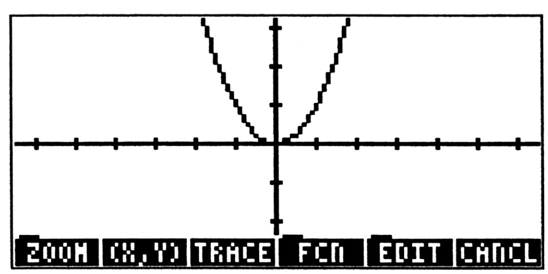
For example, zooming in on the graph of $f(x) = x^2$ at (0,0) with equal scaling factors should produce a horizontal line. Try it.

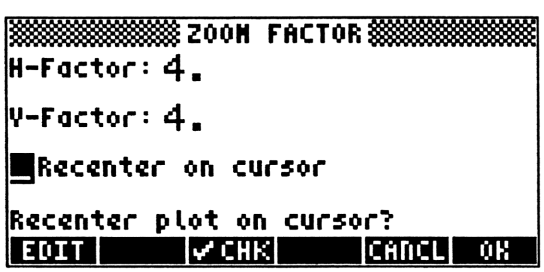


First, press [←] [2D/3D] [NXT] RESET [▼] OK [NXT] to reset the plot setup to default values. Then use [▼] to highlight the EQ field, and press [X] [y^x] [2] [ENTER]. Be sure your screen agrees with the one shown here:

Next, press [←] [WIN] [NXT] RESET [▼] OK [NXT] to reset the viewing window to defaults, as shown here.

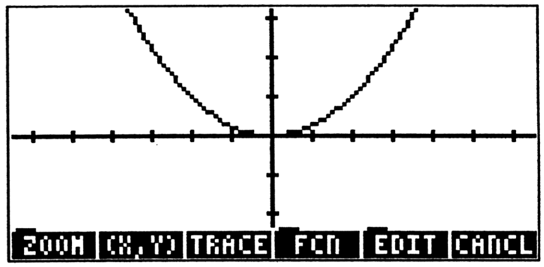
Then press ERASE DRAW, and you should see this graph:

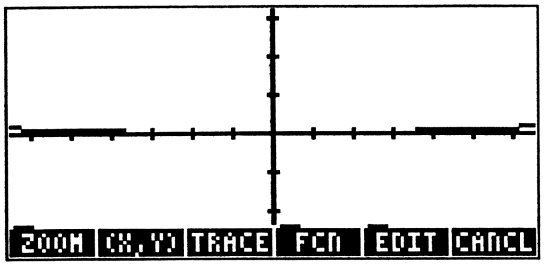




Press ZOOM ZFACT and set the options as shown here.

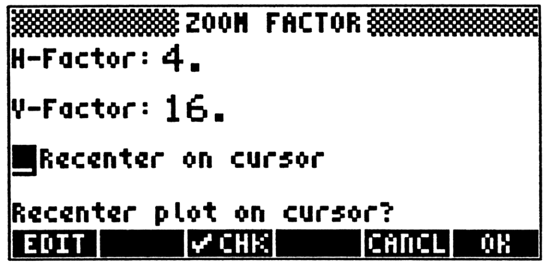
Press **OK** then **ZIN** to zoom in to the graph at (0,0).



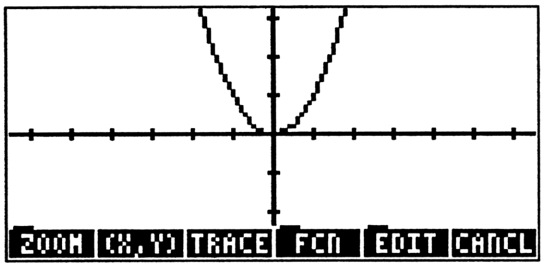


Press **ZOOM** **ZIN** three more times.

Notice how the graph flattens out, suggesting that the derivative of $f(x) = x^2$ at (0,0) is 0.



Now, press **ZOOM** **ZDFLT** **ZOOM** **ZFACT** and set the zoom factors to 4 and 16 for **H-Factor** and **V-Factor**, as shown:



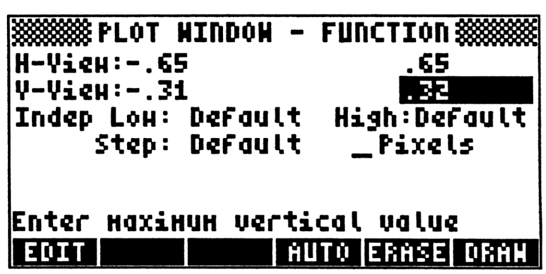
Press **OK** **ZIN** to zoom in.

If you repeatedly zoom in to this graph at (0,0) with these particular unequal zoom factors, its shape remains unchanged. Why? Because you are repeatedly multiplying the screen positions of your X-values by 4 and your Y-values by 16.

Consider a point on your graph, (X_A, Y_A), which is $(X_A, X_A^{\,2})$. After you zoom in, this point appears on the screen at $(X_B, Y_B) = (4X_A, 16X_A^{\,2}) = (4X_A, (4X_A)^2)$—still on the curve $Y = X^2$. By contrast, if the X- and Y- zoom factors had instead been, say, 4 and 8, your new point would have appeared at $(4X_A, 8X_A^{\,2}) = (4X_A, (4X_A)^2/2)$, which is not on $Y = X^2$; it's on $Y = X^2/2$, which is flatter.

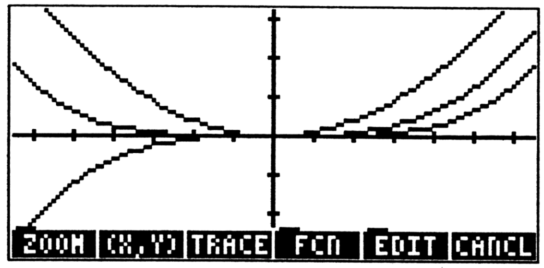
In general, if you zoom in by a factor of z horizontally and z^n vertically (call it a power zoom of degree n) on the graph of $f(x) = x^n$, the graph's shape will remain unchanged. If you power zoom in with degree n on a graph of $f(x) = x^m$, where $m > n$, the graph will become flatter; if $m < n$, the graph will become steeper.

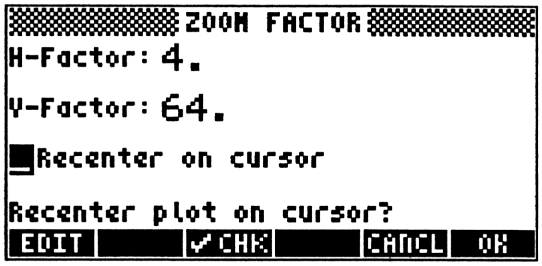
Try this. Press (CANCEL) to exit the graph screen, then (⬅)(Y=) **ADD** (X)(y^x)(3)(ENTER) **ADD** (X)(y^x)(4)(ENTER) to enter the functions shown here.



Then press (⬅)(WIN). Enter the values shown here for **H-View** and **V-View**.

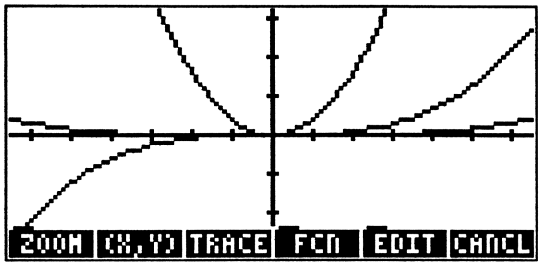
Then press **ERASE** **DRAW**.



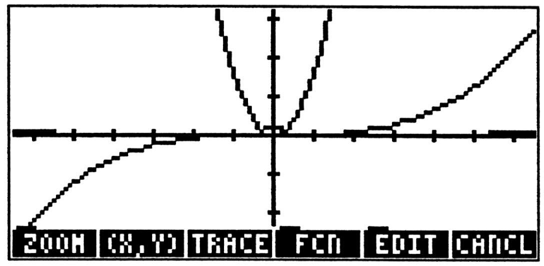


On the graph screen, press **ZOOM** **ZFACT** and enter the zoom factors shown here. Notice that the `V-Factor` is the cube of the `H-Factor`. In other words, $n = 3$.

Now press **ZOOM** **ZIN** once...

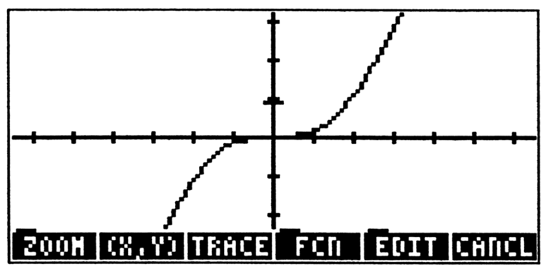


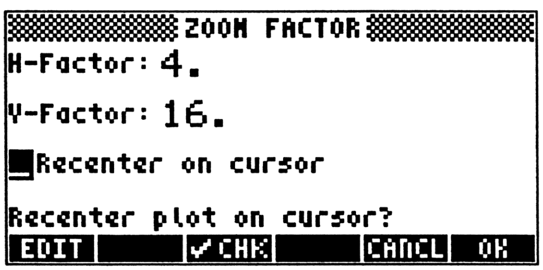
... and again: **ZOOM** **ZIN**.



The graph of $y = x^3$ (where $m = 3 = n$) stays unchanged; the graph of $y = x^2$ (where $m = 2 < n$) gets steeper; the graph of $y = x^4$ (where $m = 4 > n$) flattens out. This fact lets you measure how well two graphs (such as a function and its linear approximation) agree near a point. You graph the *difference* between the two functions, zoom in with various power zooms to observe how that difference graph behaves, and thereby discern the order of agreement of the original two graphs.

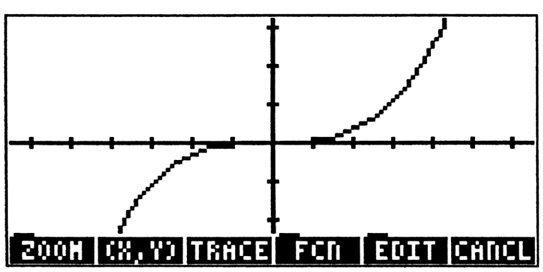
For example, to compare the graph of $y = \sin(x)$, with its tangent line, $y = x$, at $x = 0$, press [CANCEL] [←][Y=] [NXT] **CLEAR** **OK** [NXT] **ADD** [X] [−] [SIN][X] [ENTER] [←][WIN] [NXT] **RESET** [▼] **OK** [NXT] **ERASE** **DRAW**, to see this graph:

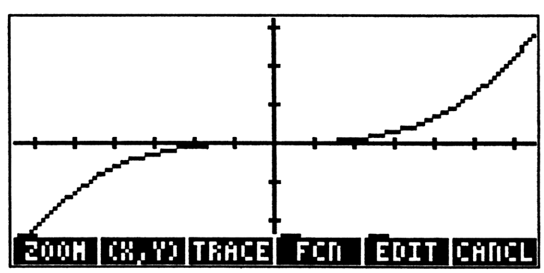




To learn more about this difference function, try a power zoom—start with a degree (n) of 2. Press **ZOOM** **ZFACT**, and enter the factors shown here (left).

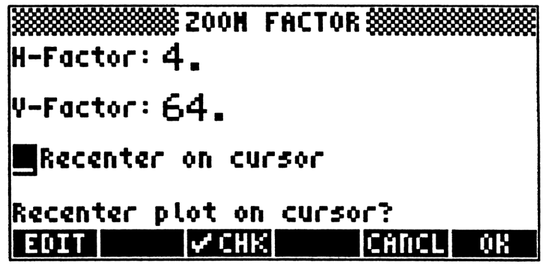
Press **OK** **ZIN** to see the graph change (below left). Press **ZOOM** **ZIN** to see it change again (below right).



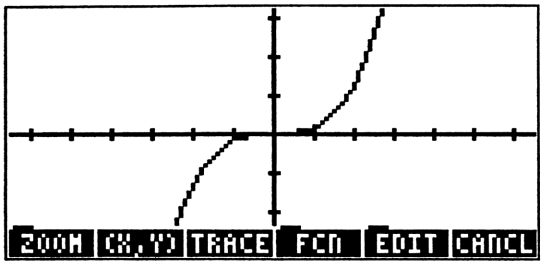


The graphs are getting flatter as you zoom, suggesting that you need to increase the degree, n, of your power zoom.

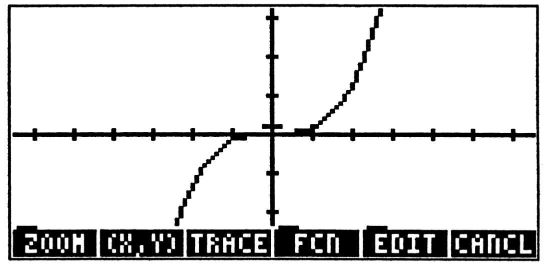
So press ZOOM ZDFLT to get back to the default view, then ZOOM ZFACT, and try degree of 3 instead, as shown:



Now OK ZIN to zoom....



Then zoom again: ZOOM ZIN....



You can see that the graph remains unchanged. You can therefore say that the order of agreement between $\sin(x)$ and x at $x = 0$ is 3.

1. If you're using one function to approximate another, you'd like a higher order of agreement, since higher degree power functions approach zero quickly, and stay near zero over a wider interval. Explain why power functions behave this way.

____

____

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What's so special about the tangent line?

You can use this same analytic tool—power zooming on a difference graph—to examine some Calculus concepts. First of all, why use the tangent line to approximate a function near a given point (the point of tangency)? Any other line intersecting the curve at that point could be used to approximate. What's so special about the tangent line? Try an example.

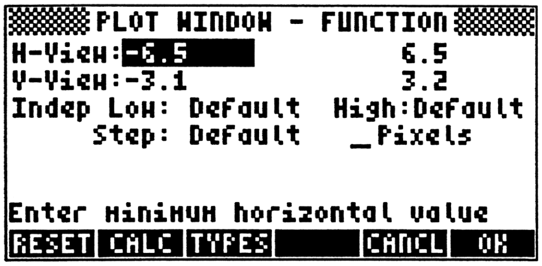
In Y1, define the difference (error) function between the function $y = e^x$ and its tangent line at $(0,1)$.

In Y2, define another, similar difference function, but with a slightly non-tangent line that intersects $y = e^x$ at $(0,1)$.

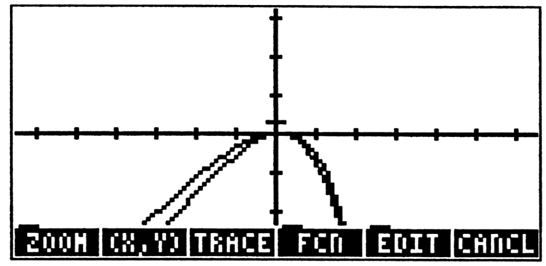
Press [CANCEL] [↰][Y=] DEL, then

ADD [X] [+] [1] [−] [↰][eˣ] [X] [ENTER]

ADD [1] [·] [2] [X] [+] [1] [−] [↰][eˣ] [X] [ENTER].



Next, press ⬅ WIN NXT RESET ▼ OK, to set the plot window as shown here.

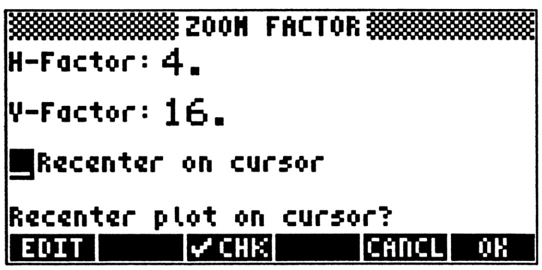


Now just press NXT ERASE DRAW, to see this:

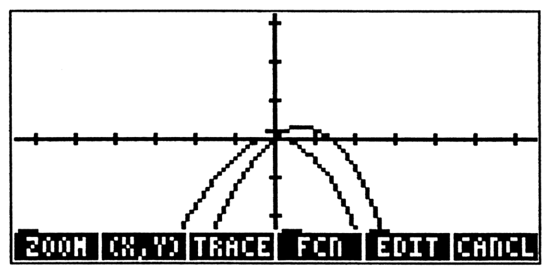
2. Why is the tangent line error negative?

____

____

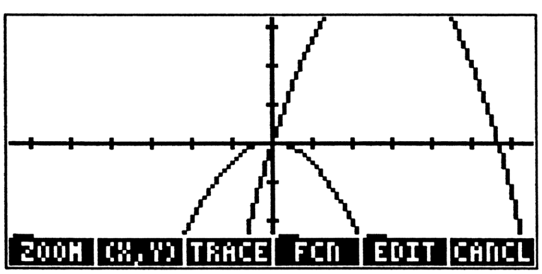


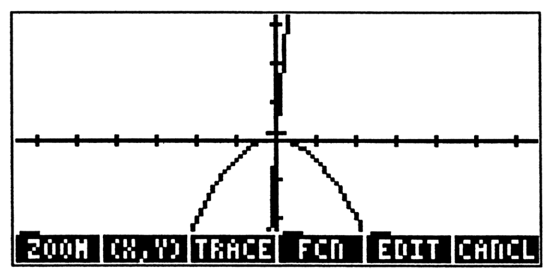
Press ZOOM ZFACT, and set H-Factor to 4 and V-Factor to 16 for a degree 2 power zoom.



Press OK, and back on the graph, press ZIN.

Zoom in a few more times, observing how the graphs change. The shape of the error of the tangent line remains unchanged; the shape of the error of the other line becomes steep and linear. Steep error is bad!





3. What is the practical significance of the difference in behavior (i.e. quadratic for the tangent line and linear for the "other" line) of the errors in these two linear approximations of e^x?

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Comparing the Difference Quotient to the Symmetric Difference Quotient

You can also use this same zooming technique to analyze the errors in using a one-sided difference quotient to approximate the derivative of a function at a point, as compared with the two-sided or symmetric difference quotient.

To create an example that does just that, you need to define some functions on the home screen. Press [CANCEL] as necessary to return to the stack.

First, define $F(X)$ = arctan(X): [↰][DEF] [↱]['] [ALPHA][F][↰][( )] [X] [▶] [↱][=] [↰][ATAN][X] [ENTER]

Next, define the one-sided different quotient, $DQ(X,H) = (F(X+H)-F(X))/H$:
[↰][DEF] [↱]['] [EQW] [ALPHA][D][ALPHA][Q] [↰][( )] [X] [ALPHA][H] [▶] [↱][=] [↰][( )] [ALPHA][F]
[↰][( )][X] [+] [ALPHA][H] [▶] [▶] [−] [ALPHA][F] [↰][( )][X] [▶] [▶] [▶] [▶] [÷] [ALPHA][H]
[↱][▲] [↱][COPY] [ENTER] [ENTER]. (This copying will save you keystrokes below.)

Now enter the symmetric difference quotient, $SDQ(X,H)$ = (F($X+H$)–F($X-H$))/($2H$):
[↰][DEF] [↱]['] [EQW] [↱][PASTE] [▼] [ALPHA][ALPHA][S][D][Q][ALPHA] [↰][( )] [X][SPC] [ALPHA][H]
[▶] [▶] [▶] [−] [ALPHA][H] [▶] [×] [2] [ENTER] [ENTER].

In order to measure the error in each of these approximations, you need to know the exact value of the derivative of arctan(X) at some value of X. The calculator knows the formula, which you'll define as another function, $D(X)$: [↰][DEF] [↱]['] [ALPHA][D] [↰][( )] [X] [▶] [↱][=] [EQW] [↱][∂] [X] [▶] [ALPHA][F][↰][( )][X]
[↱][▲] **EVAL** [ENTER] [ENTER].

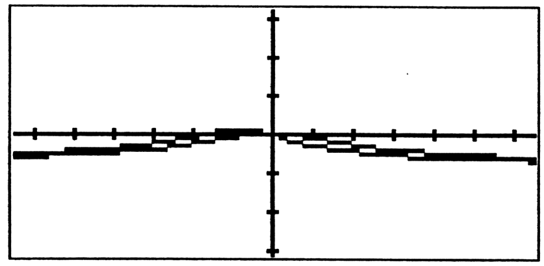
Now you're ready to define, graph, and look at a table of values for your two error functions. Before you do so, however, you'll need to press [↰][2D/3D] and change the `Type` to `Function`, if necessary. Also, change `INDEP` to `H` (by highlighting the `INDEP` field and pressing [↱]['][ALPHA][H]). After all, you'll be expressing the two errors as functions of H (the increment in X) in each of the two approximations.

Now enter the function $DQ(A,H)-D(A)$, the error in using the one-sided difference quotient to approximate the derivative of `ATAN(X)` at $X = A$: [↰][Y=] [NXT] **CLEAR** **OK** [NXT] **ADD** [ALPHA][D][ALPHA][Q]
[↰][( )] [ALPHA][A] [ALPHA][H] [▶] [−] [ALPHA][D] [↰][( )] [ALPHA][A] [ENTER].

Do likewise for the symmetric difference quotient error: **ADD** [ALPHA][ALPHA][S][D][Q][ALPHA] [↰][( )]
[ALPHA][A][SPC] [ALPHA][H] [▶] [−] [ALPHA][D] [↰][( )] [ALPHA][A] [ENTER].

The Y= screen should look like this:

Press ENTER to go back to the stack. You can choose any value for A, since the arctangent is differentiable for all X. Use $A = 0.5$. Press · 5 STO▶ ALPHA A ENTER.



Press ← WIN NXT RESET ▽ OK NXT ERASE DRAW. Your graphs should look like this.

4. Compare the errors in the two different derivative approximations in this default window.

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5. Why do both errors seem to approach zero as values of H approach zero?

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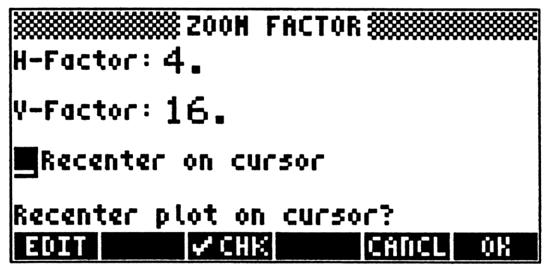
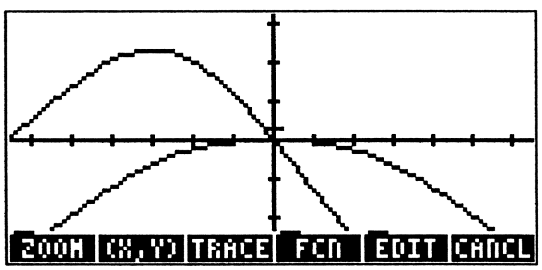
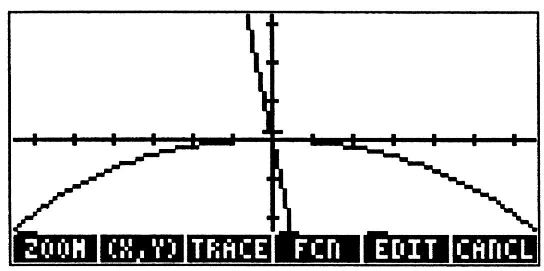


Press ZOOM ZFACT and set the factors as shown here. Note that this represents a power zoom of degree 2.



Press OK ZIN, and you'll get the graph shown here.

One more zoom in and you'll see this graph:



Keep in mind what you're looking at here. You're seeing the errors in using the difference quotient (DQ) and the symmetric difference quotient (SDQ) to approximate the derivative of $f(x) = \arctan(x)$ at $x = 0.5$. The errors are graphed here as functions of H, the increment in x (sometimes called Δx).

The results are amazing! The error in DQ appears to be linear—and the graph gets steeper as you perform a degree-2 power zoom. The error from using the SDQ appears to behave like a quadratic; its shape remains essentially unchanged as you do a degree-2 power zoom. (Again: steep error is bad, and higher-degree behavior in error functions is better than lower-degree behavior.)

6. Explain why the error from both approximations gets larger as H gets farther from 0 for this function.

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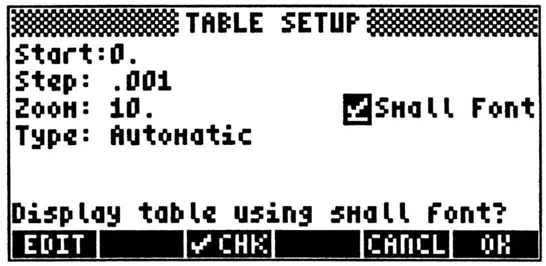
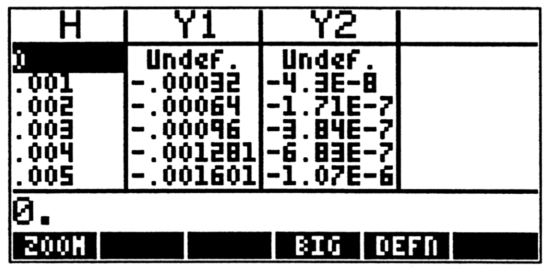
It's interesting to examine this same activity numerically by zooming in to the table of values near X=0.

Press [CANCEL] to get off the graph screen, then [←][TBLSET]. Set up the table as shown here:



Now press [←][TABLE]. It should look like this:



Notice that Y1, the DQ error, has a constant difference of about .00032, indicating linearity. For Y2, the SDQ error, the second differences are constant.

Teacher Notes

This activity is definitely best performed as a teacher-directed exercise, perhaps with a student performing the keystrokes. The syntax can be a bit daunting, but the mathematical rewards are ample and make the effort worthwhile. The notion of using zooming as an analytic tool is relatively new. It is worth introducing early to students, and revisiting as occasions arise. For example, when Taylor Polynomials are covered, it is appropriate to examine the behavior of the "remainder" by subtracting the polynomial from the function that generated it.

This activity was deliberately written up using generic functions; it will work for any function F *that is differentiable at* $x = A$. *In fact, after having worked through the activity once with* $F(X) = \arctan(X)$, *with* $DQ(X,H)$ *and* $SDQ(X,H)$ *defined in terms of* F, *it would be instructive to have students repeat the activity each with a different function,* F, *and/or point* $x = A$. *Just go back and redefine* F *and* $D(X)$, *the derivative of* F. *Everything else stays the same.*

In the final analysis of the table values, you might try demonstrating the first and second differences by generating a list of inputs using the `SEQ` *command, as follows:*

```
SEQ(X,X,.001,.010,.001) [STO▶] XLIST
DEFINE('EDQ(X,H)=DQ(X,H)-D(X)')
DEFINE('ESDQ(X,H)=SDQ(X,H)-D(X)')
DOLIST(XLIST,'EDQ') [STO▶] EDQLIST
DOLIST(XLIST,'ESDQ') [STO▶] ESDQLIST
```

Then `ΔLIST(EDQLIST)` *once to show the constant first differences (indicating linearity); and* `ΔLIST(ESDQLIST)`, *then* `ΔLIST(ANS(1))` *to show the constant second differences in* `SDQ`*'s errors.*

You might also be able to use `DOLIST(`*inputlist*`,'Y1')` *and* `DOLIST(`*inputlist*`,'Y2')` *to generate lists of outputs from the errors, then* `ΔLIST` *those to show the constant first and second differences, respectively.*

Acknowledgment

The idea for the tangent line error analysis was inspired by a talk Don Kreider delivered at the Technology in Calculus with Advanced Placement (TICAP) conference in Clemson, SC, in 1993.

Answers

1. *If* $-1 < x < 1$, *then multiplying x by itself results in a product smaller than x in absolute value. The more times a number between -1 and 1 is multiplied, the closer the product is to 0. Similarly, if the number is greater than 1 or less than -1, it gets bigger and bigger in absolute vale as it is multiplied by itself. These two observations combine to explain why higher degree power functions get to zero faster and stay there longer.*

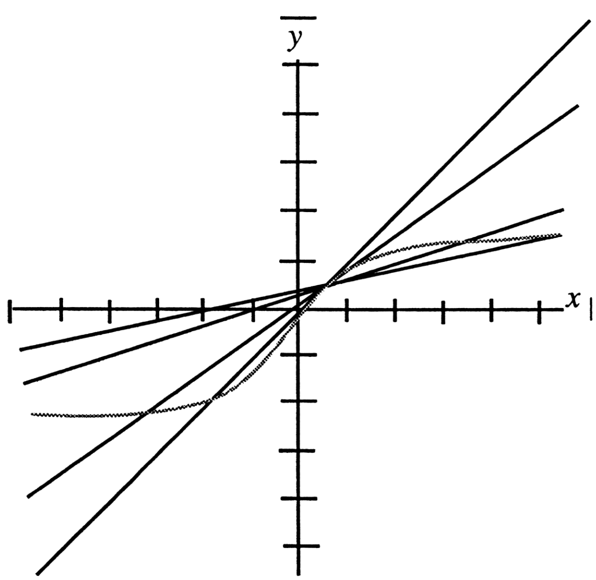
2. *The graph of* $y = e^x$ *is concave up for all x since its second derivative,* e^x, *is greater than 0 for all x. That means its slope is always increasing. The slope of the tangent line is constant, and equal to the slope of* $y = e^x$ *at* $x = 0$. *So the tangent line stays below* $y = e^x$, *making the error negative.*

3. *For values of x close to the point of tangency, the linearity of the "other" line's error means that as you move away from the point of the approximation, the error increases at a constant rate. By contrast, for the quadratic tangent line error, the error is flat at the point of tangency, and has a relatively small slope nearby. This makes those errors stay closer to zero.*

4. *In the default window, there is little difference between the two approximations. Both seem to be close to zero for* H *close to zero, and it's hard to see much difference in the shapes.*

5. *Since the derivative is defined to be the limit of the difference quotient as* H *approaches 0, it makes sense that the difference between the approximations and the exact value of the derivative will be closer to zero as* H *gets close to 0.*

6. *The graph below shows* $y = \arctan(x)$ *and the tangent line at* $x = 0.5$, *as well as several other lines that result from taking successive larger steps to the right of 0.5. Clearly, the slopes of those lines gets smaller and smaller, and farther and farther away from the slope of the tangent line, resulting in larger errors.*



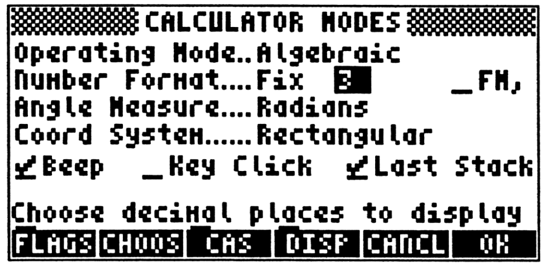
Fundamental Theorem Investigation

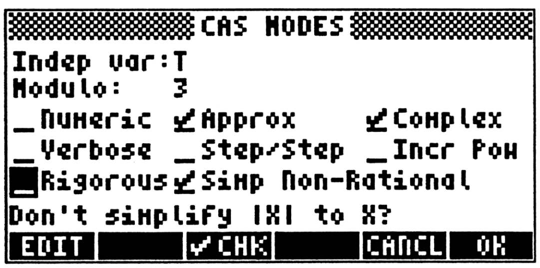
In this activity, you will work with a function defined by a definite integral, exploring connections between such a function and the integrand in its definition. This activity should give you graphical and numeric views of the First Fundamental Theorem of Calculus.

First, define $F(X) = .5 - 1.5\sin(X^2)$ and $A(X) = \int_{-2.5}^{x} F(T)dT$.

For $F(X)$, type [⬅][DEF] [➡]['][ALPHA][F] [⬅][( )] [X] [▶] [➡][=] [.][5] [−] [1][.][5] [SIN][X] [y^x] [2] [ENTER]. To enter $A(X)$, it's more convenient to use the Equation Writer: [⬅][DEF] [➡]['] [EQW] [ALPHA][A] [⬅][( )] [X] [▶] [➡][=] [➡][∫] [+/−] [2][.][5] [▶] [X] [▶] [ALPHA][F] [⬅][( )] [ALPHA][T] [▶] [ALPHA][T] [ENTER] [ENTER]

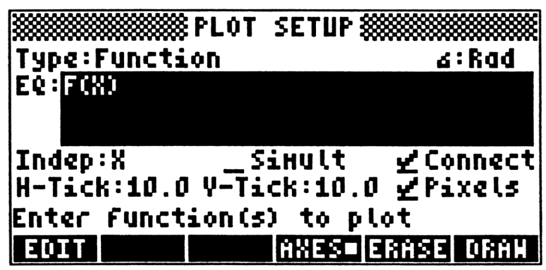
At the [MODE] screen, change the **Number Format** to **Fix 3**, so that three digits to the right of the decimal point will be displayed ([MODE] [▼] [CHOOS] [▼] [OK] [▶] [3] [OK]). Be sure that **ALG**ebraic Mode is set, too. The **CALCULATOR MODES** screen should appear as shown here.

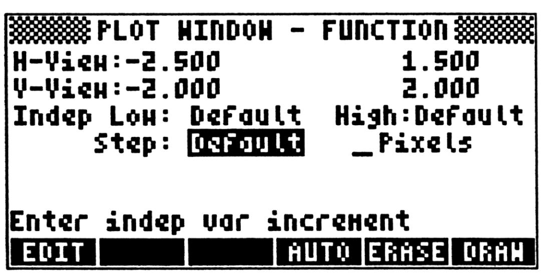




Next, press [CAS] and check that your **CAS MODES** agree with the settings shown here.

Press [OK] [OK], then [⬅][2D/3D] and check that the **PLOT SETUP** is configured as shown at right. (To put $F(X)$ into EQ, highlight **EQ** and press [➡]['] [ALPHA][F] [⬅][( )] [X] [ENTER].)





Now press [⬅][WIN] to adjust the **PLOT WINDOW - FUNCTION** screen. Set up your viewing window so that **X** is between **-2.5** and **1.5** and **Y** is between **-2** and **2**, as shown. Then press [ENTER] to get back to the stack and accept these settings.

Now generate a list of inputs for $A(X)$ using the SEQ command:

[⬅][PRG] [6] [ENTER] [2] [ENTER] [8] [ENTER] [X] [➡][,] [X] [➡][,] [+/−][2][.][5] [➡][,] [1][.][5] [➡][,] [.][1][2] [▶] [STO▶] [X] [ALPHA] [ALPHA] [L] [I] [S] [T] [ENTER].

To generate a list of outputs from $A(X)$, use the command DOLIST(XLIST,'A'), which applies the function A to each element of XLIST. Type [←][PRG] [6] [ENTER] [2] [ENTER] [ENTER] [X][ALPHA][ALPHA] [L][I][S][T][ALPHA] [→][,] [→][''] [ALPHA][A] [▶][▶] [STO▶] [ALPHA][ALPHA] [A][L][I][S][T] [ENTER].

The calculator may prompt you to change to Approximate mode and Complex mode in order to evaluate the integrals. Accept these changes. Then it will take a couple of minutes to evaluate the integrals. Note that $A(X)$ measures the accumulated signed area from the left edge of the graph screen to X.

Notice that each element of ALIST is the result of a definite integral. For example, the second element of XLIST, which is -2.380, produces this second element of ALIST:

$$A(-2.380) = \int_{-2.5}^{-2.380} .5\text{-}1.5\sin t^2 dt$$

1. Why is 0 the first component of ALIST? ____________________________

Graph the integrand function, $F(X)$, in the window. Use the graph to answer these questions:

2. a. Find the two negative X-intercepts on the graph of F in the window specified:

   ____________________________ and ____________________________.

   b. Find the X-coordinate of the positive X-intercept of $F(X)$. ____________________________

You'll need these later. While you're looking at the graph, take note of the sign of F for values of X a little larger than -2.5. You might want to refer to this in the following questions.

3. Why is the second component of ALIST greater than 0?

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4. Why is the third value of ALIST, which represents $A(-2.260) = \int_{-2.5}^{-2.260} .5\text{-}1.5\sin(t^2)dt$,

   greater than the second component of ALIST, which represents $A(-2.380)=\int_{-2.5}^{-2.380} .5\text{-}1.5\sin(t^2)dt$?

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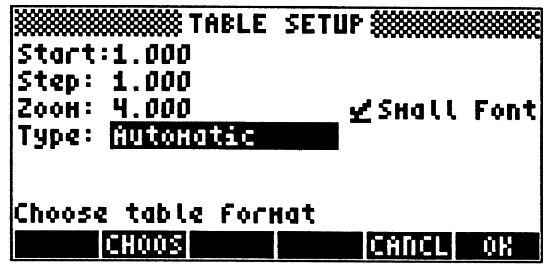
Press [←][2D/3D] and change the plot type to Parametric, by pressing CHOOS [▽][▽] OK.

To enter '(XLIST(T),ALIST(T))', at the EQ field press [→]['] [←][()] [ALPHA][ALPHA][X][L][I][S][T] [←][()] [T] [▶][→]['] [A][L][I][S][T][←][()] [T] [ENTER].

Change the independent variable to T. PLOT SETUP should look like this:

Press [←][WIN] and change the Independent Low to 1, High to 34, and Step to 1.



Press [←][TBLSET] and set Start to 1, Step to 1, and Type to Automatic.

Press [←][TABLE] and use the table to answer the following questions:

5. Looking at the values of ALIST, the outputs of the area accumulation function, for what approximate value of X does $A(X) = \int_{-2.5}^{x} .5\text{-}1.5\sin(t^2)dt$ reach its first local maximum?

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6. Explain why the value of $A(-.94) = \int_{-2.5}^{-.94} .5\text{-}1.5\sin(t^2)dt$ is smaller than $A(-1.18) = \int_{-2.5}^{-1.18} .5\text{-}1.5\sin(t^2)dt$. Keep in mind that $A(X)$ is measuring signed area.

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7. For what approximate value of X does $A(X) = \int_{-2.5}^{x} .5\text{-}1.5\sin(t^2)dt$ reach its first local minimum?

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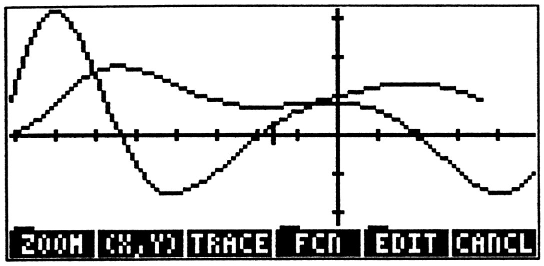

Without erasing the graph of F, overlay a plot of ALIST vs. XLIST on top of your graph of F by pressing DRAW from the 2D/3D window. (DO NOT PRESS ERASE!!)

8. Compare your answers to numbers 5 and 7 to your answers in number 2a. Carefully explain why they are similar.

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9. What happens with $A(X)$ at the point found in number 2b?

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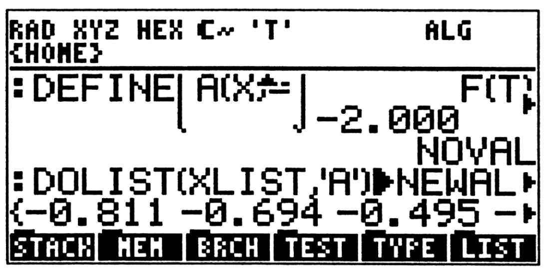
(Verify your answer numerically by looking at the table of values for $A(X)$.)

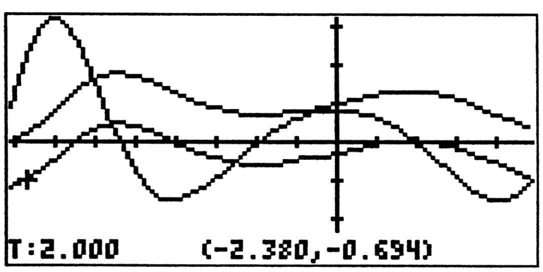
Redefine $A(X)$, changing the lower limit of integration from -2.5 to -2.

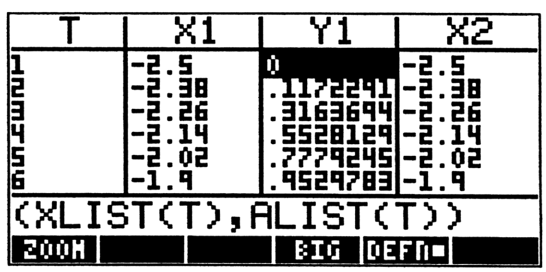
To do this, go back to the stack screen and press ▲ until the command to define $A(X)$ is highlighted. Now press ENTER, move over just to the right of -2.5, and press ⇐ ⇐ to erase the .5, then ENTER to redefine $A(X)$. Move up again to the DOLIST command, and edit it so that it appears as shown:



When you press ENTER, the new A, with -2 as the lower limit, is evaluated for all the inputs in XLIST the results stored in NEWALIST. (Again, it takes a couple of minutes to evaluate the 34 integrals!)

Press ← Y= EDIT → COPY ENTER ADD → PASTE ▼ ▶ EDIT ALPHA ALPHA N E W ENTER ENTER. Press DRAW, (do **not** ERASE), study the graphs.... Press CANCEL ← TABLE and study the table:





10. Scroll through the table and determine approximately where the values of this new function, A, achieve their first local maximum ______________________

11. Where is the first local minimum? ______________________

12. How do these answers compare with your answers from numbers 5 and 7? ______________________

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13. Why is the graph of NEWALIST below the graph of ALIST? (Remember, the only difference between the two is the lower limit of integration.)

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14. Explain why the graphs of ALIST and NEW ALIST have the same shape.

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15. (Challenge for the experts.) Find the locations of any points of inflection on ALIST and NEW ALIST. Explain what these locations have in common with features on the graph of F.

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Teacher Notes

This activity should shortly precede the Difference Quotient for the Area Function activity. It gives students experience working with functions defined by an integral, particularly with regard to interpreting signed area and the order of limits of integration. By itself, it strongly suggests the Fundamental Theorem, the connections between locations of extrema on the graph of the area function and sign changes on the graph of the integrand. Before attempting this activity, students should have experience with defining functions, using the definite integral function, drawing graphs and looking at tables of values, and using the FCN tools on the graph screen.

Answers

1. *The first element of* ALIST *is* $A(-2.5) = \int_{-2.5}^{-2.5} .5-1.5\sin(t^2)dt$, *which is 0.*

2 *a.* $X = -1.674$ *and* $X = -0.583$ *b.* $X = 0.583$

3. $A(-2.380) = \int_{-2.5}^{-2.380} .5-1.5\sin(t^2)dt > 0$ *since the integrand is positive on the interval* $[-2.5, -2.380]$

4. $A(-2.260) > A(-2.380)$, *since the integrand is positive from -2.380 to -2.260. You have added more "positive" area.*

5. *At about* $X = -1.66$.

6. $F(X)$, *the integrand, is negative for all X between -1.18 and -0.94. So you are adding "negative" area over this interval, making the values of* $A(X)$ *decrease.*

7. *At about* $X = -0.58$.

8. *The answer to 5,* $X = -1.66$, *tells where the values of A reach a local maximum—where you stop accumulating positive area (increasing the area function) and start accumulating negative area (decreasing the area function). This is where the integrand changes sign from positive to negative (an x-intercept of the graph of F). Likewise, the answer to 7,* $X = -0.58$ *tells where the values of A reach a local minimum—where you stop accumulating negative area and start accumulating positive area. This is where the integrand changes sign from negative to positive (another x-intercept of the graph of F).*

9. *At* $X = -0.583$, *A reaches another local max, since F changes sign from negative to positive there.*

10. *At about* $X = -1.66$.

11. *At about* $X = -0.58$.

12. *They are the same.*

13. *The graph of* NEWALIST *is below the graph of* ALIST *because the first entry in* NEWALIST *is a negative number (because you're integrating a positive function from a larger X to a smaller X). By contrast,* ALIST *starts out at 0.*

14. NEWALIST *and* ALIST *are both governed by the area under the graph of the integrand function, F, which would control the shape of any function that uses it as the integrand. Differing lower immigration limits merely shift the graph vertically.*

15. *Points of inflection occur on the graph of A when it undergoes an extreme rate of change. This happens when the integrand is farthest from the X - axis—where the largest little "chunk" of area is added. In other words, F is the derivative of A, so F' is the second derivative of A. And where does* $F' = A''$ *change sign? Wherever F changes from increasing to decreasing or vice-versa—at the extrema on the graph of F. These points occur at* $X = -2.171$, $X = -1.253$, $X = 0$, *and* $X = 1.253$.

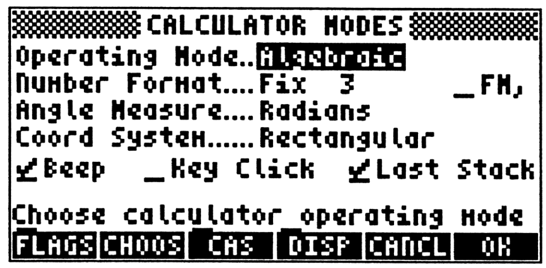
The Difference Quotient for the Area Function

In this activity, you'll work with the function $A(x) = \int_2^x \frac{1}{1+t^2}\,dt$ for $x = 0, 0.1, 0.2, \ldots, 3.0$ (some 31 values of x in all). Here is a broad outline of the procedure. Specific instructions follow.

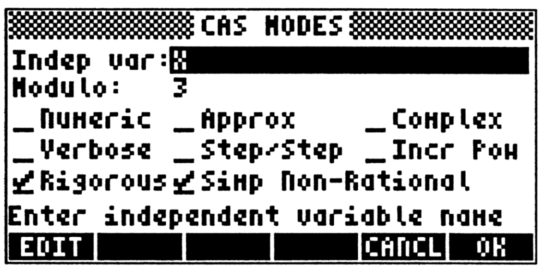
1. Make a table of inputs and outputs, $(x, A(x))$.
2. Plot the graph of $y = A(x)$ from your table.
3. "Eyeball" a sketch of the derivative of $A(x)$, $G(x) = A'(x)$, by estimating slope at several points.
4. Use your table from step 1 to form the difference quotient, for $x = 0, 0.1, 0.2, \ldots, 2.9$.
5. Plot $y = G(x)$ and compare to your graph from step 3.
6. Plot $f(x) = \dfrac{1}{1+x^2}$.

Then you're going to repeat these six steps, but replace 2 with -2 as the lower limit of integration in $A(x)$.

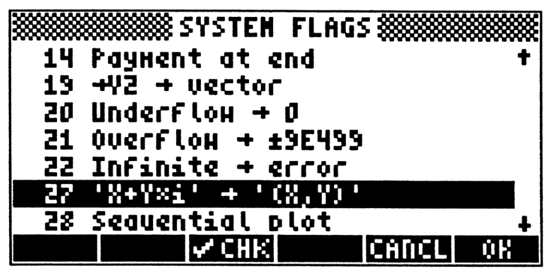
With this outline of tasks, now here are the details.



Press MODE and set the CALCULATOR MODES as shown here:



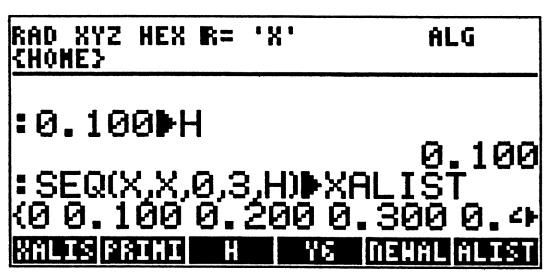
Then press CAS, and set the CAS MODES as shown at left.



Press OK FLAGS, arrow down to flag 27, and uncheck it by pressing CHK.

Press OK OK to get back to the stack, then 0 . 1 STO▶ ALPHA H ENTER, to store .1 in the variable H, which controls the increment in X.

Next, to generate the list of inputs and store it in XLIST, press ← PRG 6 ENTER 2 ENTER 8 ENTER X → , X → , 0 → , 3 → , ALPHA H ▶ STO▶ X ALPHA ALPHA L I S T ENTER.



Check that your screen looks like this:

Now define $F(X) = 1/(1+X^2)$ and $A(X) = \int_2^x F(T)dT$.

For $F(X)$, type [⇦][DEF] [⇨]['] [ALPHA][F] [⇦][()] [X] [▶] [⇨][=] [1] [÷] [⇦][()] [1] [+] [X][yˣ][2] [ENTER].

To enter $A(X)$, it's more convenient to use the Equation Writer: [⇦][DEF] [⇨]['] [EQW] [ALPHA][A] [⇦][()] [X] [▶] [⇨][=] [⇨][∫] [2] [▶] [X] [▶] [ALPHA][F] [⇦][()] [ALPHA][T] [▶] [ALPHA][T] [ENTER] [ENTER]

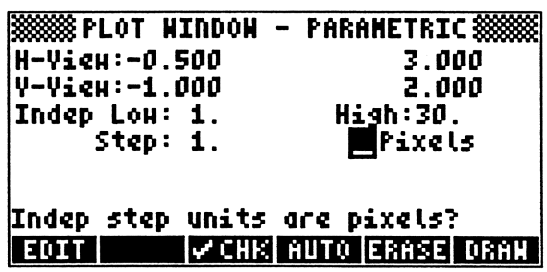
Store the list of area function values in ALIST via the command DOLIST(XLIST,'A') [STO▶] ALIST. ([⇦][PRG] [6][ENTER] [2][ENTER] [ENTER] [X][ALPHA][ALPHA][L][I][S][T][ALPHA] [⇨][,] [⇨]['] [ALPHA][A] [▶] [▶] [STO▶][ALPHA][ALPHA][A][L][I][S][T] [ENTER].) This command applies the function A to each component in XLIST, storing the results in another list, ALIST. (The calculator will ask you if you want to switch to approximate mode. Use OK to choose YES. Then be patient—the list processing takes a little time.)

1. List the first 10 values for XLIST and ALIST in this table:

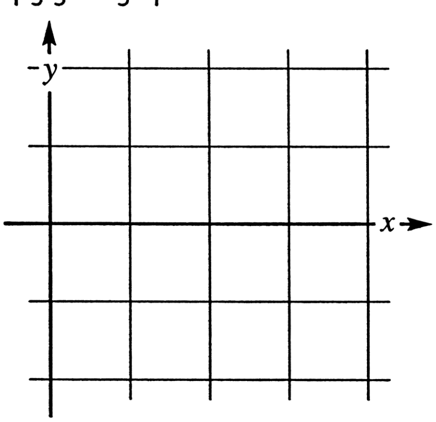
XLIST										
ALIST										

Have your teacher initial your work before continuing: ____________________

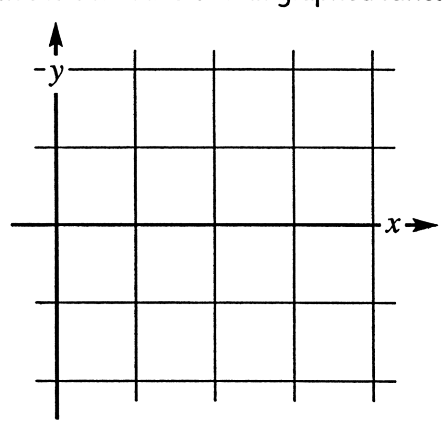
To graph the values in ALIST versus XLIST, use the parametric plot type. Set up [⇦][2D/3D] (Plot Setup) and [⇦][WIN] as shown below:



2. Copy your graph onto the axes below here.



3. Sketch the derivative of that graphed function here.



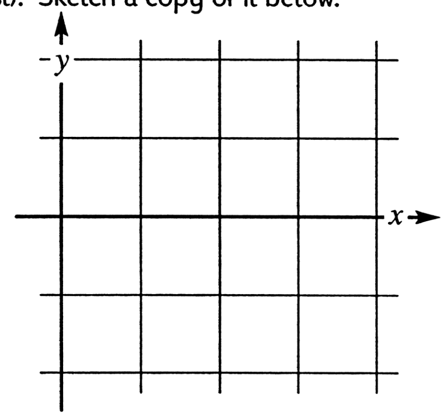
Now generate a list of the difference quotient values and store it in DQLIST, by using the commands ΔLIST(ALIST)/H STO▶ DQLIST. (At the stack screen, press ← MTH 3 ENTER ENTER ← () ALPHA ALPHA A L I S T ALPHA ▶ ÷ ALPHA H ▶ STO▶ ALPHA ALPHA D Q L I S T ENTER.)

4. Copy the first ten items from DQLIST into the table below. Note that these values in DQLIST represent an approximation to the derivative of the function whose outputs appear in ALIST. Have your teacher initial your work before continuing: ______________________

DQLIST										

To see a graph DQLIST versus XLIST, set up ← 2D/3D as shown. (At ← 2D/3D, highlight EQ, then EDIT ▼ ▶ EDIT ▶ ← ALPHA ALPHA D Q ENTER ENTER.)

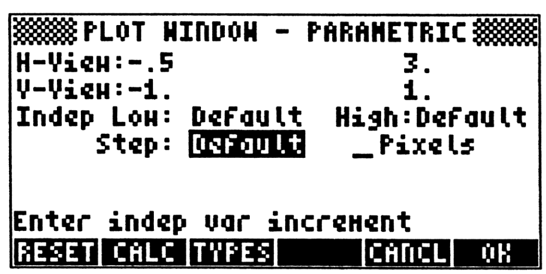
Then press DRAW to see the graph (and be careful NOT to press ERASE first). Sketch a copy of it below:



5. How does your sketch in step 3 compare with the plot above? ______________________

____

6. Now overlay the graph of $F(X)$ on this plot: Go to ← 2D/3D and change EQ to '(T,F(T))'. In the Plot Window screen (← WIN), change Indep Low, High, and Step back to defaults by pressing (NXT) RESET OK with the cursor in each field. Your Plot Setup and Plot Window screens should be:



7. Plot the graph (NXT DRAW), then (X,Y), TRACE and use the arrow keys to move the cursor. What do you notice about the graph of F? ______________________

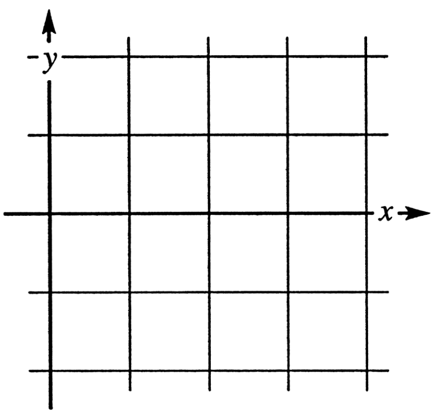
____

Now change the lower limit of integration from 2 to -2 in $A(X)$: Go back to the stack, press (▲) until you find the command defining $A(X)$, then **EDIT**. Move the cursor on top of the lower limit of integration, 2, and press (+/–) (ENTER). Then go up and highlight the DOLIST command to evaluate all 31 integrals and regenerate ALIST. Press (ENTER)(ENTER) to evaluate the integrals again and store them in ALIST (and it will take awhile). To graph them, put '(XLIST(T),ALIST(T))' into EQ again), change the window settings so T starts at 1, stops at 30, and steps by 1, as shown in the PLOT WINDOW – PARAMETRIC screen on page 56.

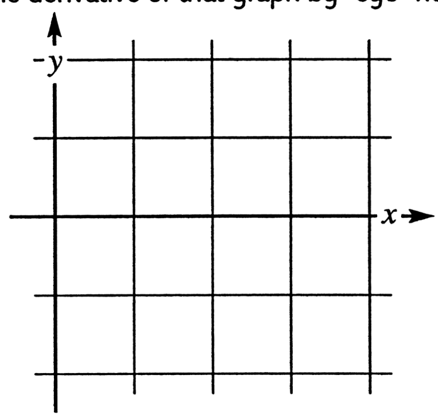
8. Fill in the first ten entries in the table for XLIST, ALIST with the lower limit of integration at -2.

XLIST										
ALIST										

9. Graph ALIST vs. XLIST here:



Sketch the derivative of that graph by "eye" here:



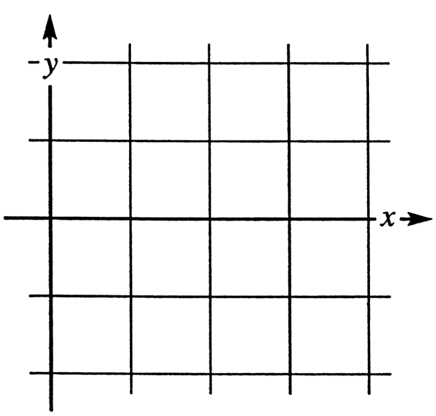
11. How do the graphs of 9 and 10 compare with the graphs of 2 and 3? ____________________

____

12. List the first ten values from DQLIST, the difference quotient for ALIST:

DQLIST										

13. Plot DQLIST versus XLIST:



How does this graph compare with the graph from step 5, when the lower limit was 2?

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Teacher Notes

This student lab activity comes from exercises at the end of section 6.4 in ***Dick and Patton,*** *Calculus of a Single Variable (ITP, 1994). The idea is to make a table of values for the area function, sketch a graph, approximate the graph of the derivative of the area function, then approximate that derivative numerically by forming difference quotients, making a scatter plot of those approximations. This scatter plot is then compared with the graph of the integrand of the original area function. Students must know how to graph parametric type expressions, and how to set up the calculator modes. They should also understand what the definite integral is, and how to evaluate it on the calculator.*

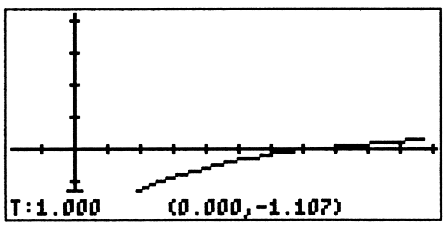
Answers

1.

XLIST	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ALIST	-1.107	-1.007	-0.91	-0.816	-0.727	-0.644	-0.567	-0.496	-0.432	-0.374

Students must have the teacher initial their work after finding XLIST *and* ALIST. *If they haven't obtained the correct values in* XLIST *and* ALIST, *the rest of the activity will be fruitless!*

2.

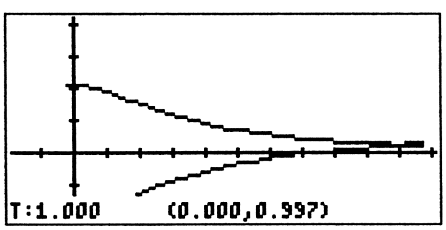


3. *The derivative sketch should be positive and decreasing.*

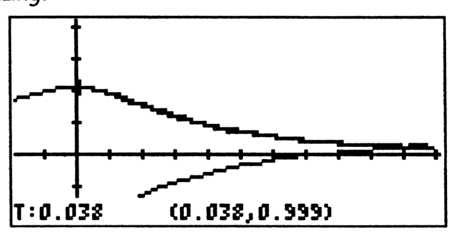
4.

DQLIST	0.999	0.977	0.941	0.89	0.831	0.768	0.703	0.64	0.581	0.526

5. *Here's the graph of* DQLIST *versus* XLIST. *It should be the same as the students' sketch in step 3.*



6. *And here is the graph of $F(X)$ laid on top of the scatter plot. Amazing!*



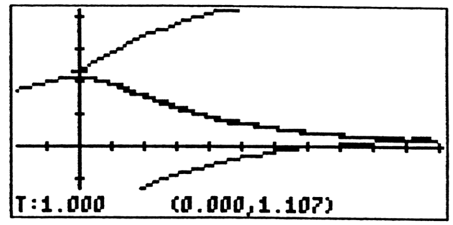
7. *The graph of F slams right through the scatter plot of the difference quotient for $A(X)$.*

8.

XLIST	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
ALIST	1.107	1.207	1.305	1.399	1.488	1.571	1.648	1.718	1.782	1.84

The only difference in the students' work for the new lower limit should be in ALIST *(step 1) and its graph (step 2). The new graph of* ALIST *vs.* XLIST *is a vertical translation of the old (by about 2.2).*

9. *This graph is the translated A(X), superimposed on the graphs of the original A(X), F(X), and* DQLIST.



10. *Same as 3.*

11. *The graph of the new* ALIST *is about 2.2 units above the old. The graph of its derivative is the same as before.*

12. *Same as 4.*

DQLIST	0.999	0.977	0.941	0.89	0.831	0.768	0.703	0.64	0.581	0.526

13. *Same as 5.*

To complete the effort to offer a truly multirepresentational approach to the FTC, here's some verbal work.

What exactly does the Fundamental Theorem of Calculus say? Students are remarkably inept at restating the result in their own words. Here's an attempt to state the FTC in alternative words to try to enhance or deepen students' understanding.

"For any continuous function, f, if $A(x) = \int_a^x f(t)dt$, *then* $A'(x) = f(x)$*."*

Now, try to state in words the meaning of the expressions on left and right hand sides of the conclusion equation. The left hand side says, "How fast is the function that measures the accumulation of a quantity changing?" The right hand side says, "Precisely as fast as the quantity is large."

You could also restate the theorem—and recast the interpretation—this way: "if $x(t) = \int_a^t v(u)du$, *then* $x'(t) = v(t)$,

where $x(t)$ *is measuring accumulated distance traveled by an object traveling with velocity* v*." The left hand side of the conclusion says, "How fast is the function that measures accumulated distance traveled changing?" The right hand side answers, "Precisely as fast as I am going!"*

Differential Equations, Euler's Method and Slope Fields

In this activity, you will examine the differential equation $\frac{dy}{dx} = \frac{2x}{1+y}$, with $y(0) = 1$.

First, you'll find an approximate solution to the differential equation numerically, using Euler's Method. Next, you'll find an exact solution, using analytic methods. Then you'll look at graphs of your Euler approximation, the exact solution, and the slope field generated by the differential equation. Finally, you'll take a look at the errors introduced by he Euler's Method approximation.

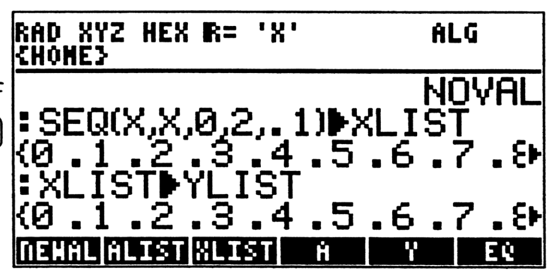
With Euler's method to approximate the solution to a differential equation, you generate a sequence of ordered pairs, (x_k, y_k), where $x_k = x_{k-1} + \Delta x$, and $y_k = y_{k-1} + m\Delta x$, where m is the value of dy/dx at (x_{k-1}, y_{k-1}). Essentially, you're just using short line segments, spliced together, to approximate your solution curve.

For example, here's one way to use Euler's Method to approximate $y(2)$ for the differential equation given above. Use $\Delta x = 0.1$.

First, define $F(X,Y)$ as the slope. On the home screen, with the calculator in default modes, type [↰][DEF] [↱][''] [ALPHA][F] [↰][()] [X] [↱][,] [ALPHA][Y] [▶] [↱][=] [2] [X] [÷] [↰][()] [1] [+] [ALPHA][Y] [ENTER].

Next, generate the list of X's, using the SEQ command: [↰][PRG] [6] [ENTER] [2] [ENTER] [8] [ENTER] [X] [↱][,] [X] [↱][,] [0] [↱][,] [2] [↱][,] [.] [1] [▶] [STO▶] [X] [ALPHA][ALPHA] [L] [I] [S] [T] [ENTER]

Now create a list to store the Y-values, by making a copy of XLIST: [VAR] ([NXT]) [XLIST] [STO▶] [ALPHA][ALPHA] [Y] [L] [I] [S] [T] [ENTER]. Your screen should look something like this:



Now you're ready to work on Euler's method. Enter the short program shown on the screen below. (Notes: Hold down [↱] while pressing [,] to get the semicolon, ; . You may need to use the [NXT] key on the VAR menu to get the menu items you need.)

[↱][« »] [ALPHA][N] [+] [1] [STO▶] [ALPHA][N] [↱][;] [↰][MATRICES] [ENTER] [1] [ENTER] [↱][''] [VAR] [YLIST] [▶] [↱][,] [ALPHA][N] [↱][,] [ALPHA][Y] [▶] [↱][;] [ALPHA][Y] [+] [ALPHA][F] [↰][()] [X] [↱][,] [ALPHA][Y] [▶] [×] [ALPHA][H] [STO▶] [ALPHA][Y] [↱][;] [X] [+] [ALPHA][H] [STO▶] [X] [↱][;] [↰][{}] [X] [↱][,] [ALPHA][Y] [ENTER]

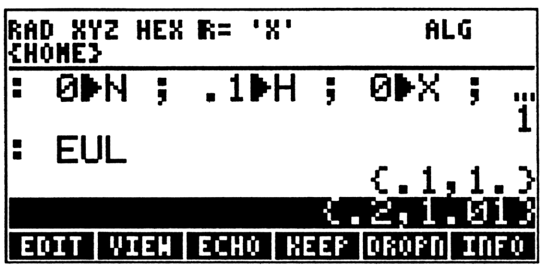
Store this program into a variable named EUL: [STO▶] [ALPHA][ALPHA] [E] [U] [L] [ENTER]

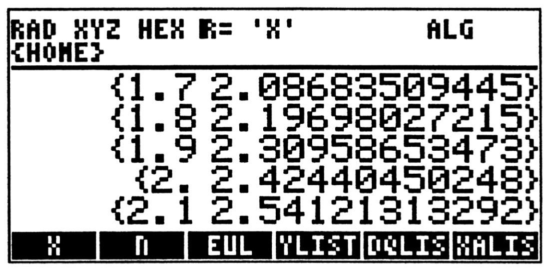
Here's a brief description of what each command in the program does:

`N+1▶N;`	Increment the counter, N, by one.
`PUT('YLIST',N,Y)`	Store the current value of Y into YLIST at position N.
`Y+F(X,Y)*H▶Y;`	(This is where the action is!) Compute the new value of Y using Euler's method, then store the result into Y.
`X+H▶X;`	Increment X by H, getting to the next point.
`{X,Y}`	Display the values of X and Y.

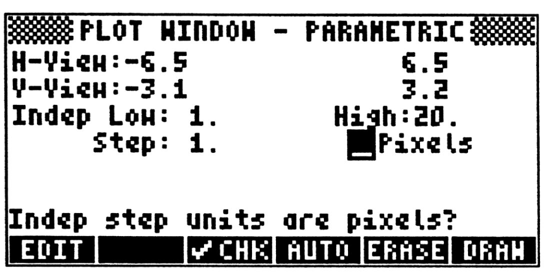
N is used to determine which term in the sequence of approximations you're on; H serves as Δx; and the initial condition is $(X,Y) = (0,1)$. Prepare to run the program by setting the initial values of the variables: [0] [STO▶] [ALPHA][N] [↱][;] [·][1] [STO▶] [ALPHA][H] [↱][;] [0] [STO▶][X] [↱][;] [1] [STO▶] [ALPHA][Y] [ENTER]

Press **EUL** [ENTER] to execute the program, then repeatedly press [ENTER] to execute it again and again. Stop when the first number displayed in the X list is 2.1. (Note that Y is "updated" in the program after it has been stored in YLIST, so in order to get the values up to $X = 2$, you need to display the next ordered pair Euler's Method would generate.) The first screen below (left) shows the first couple of steps; the second screen below (right) shows the last few steps.

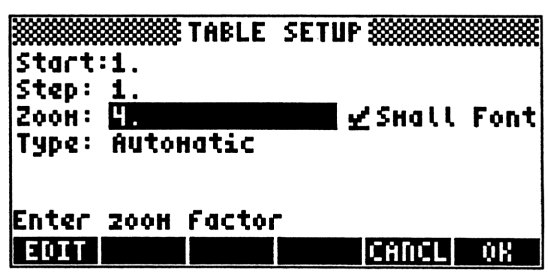




To look at the table of inputs and outputs, press [↰][2D/3D], and set up the screen as shown here. (You might also want to press [MODE], then **FLAGS**, and uncheck Flag 27, so that complex numbers will be displayed in (X,Y) format.)

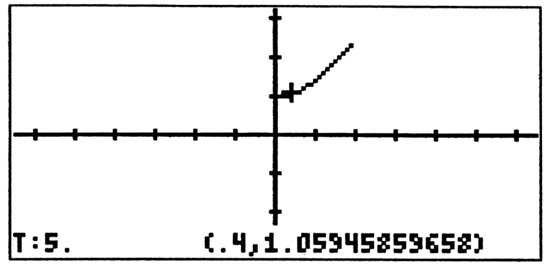


Press [↰][WIN] and set up as shown here (left).



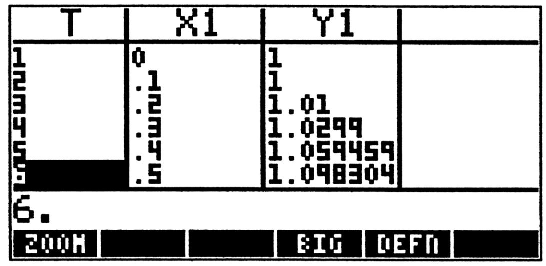
Press [↰][TBLSET] and set up as shown here (right):

Now press [←][2D/3D] once again, then **ERASE** **DRAW** to see the graph.



Notice that you can press **TRACE** **(X,Y)** to trace on the scatter plot.

Press [CANCEL] to leave the graph, then [←][TABLE] to look at the table:



1. Show the computations that give the values of Y1 at $X = 0.1$ and at $X = 0.2$.

____

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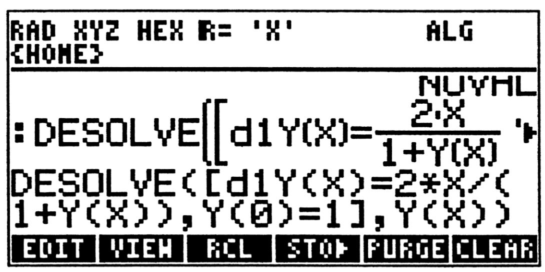
Solving the differential equation analytically

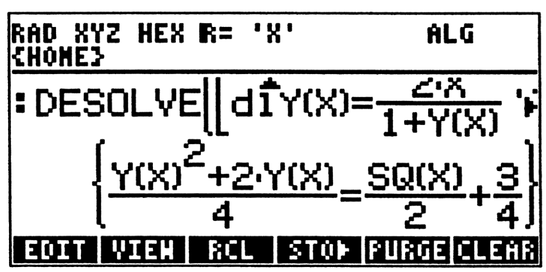
The HP 49G can solve this differential equation analytically. Go to the stack, purge the variable Y from memory: [TOOL] **PURGE** [→]['][ALPHA][Y][ENTER]. Also, set Exact mode on the [MODE] **CAS** screen.

Then [←][CALC] [3] [ENTER] [ENTER] [←][[ ]] [ALPHA] [←][D][1] [ALPHA][Y] [←][( )] [X] [▶] [→][=] [2] [×] [X] [÷] [←][( )] [1] [+] [ALPHA][Y] [←][( )] [X] [▶] [▶] [→][,] [ALPHA][Y] [←][( )] [0] [▶] [→][=] [1] [▶] [→][,] [ALPHA][Y] [←][( )] [X].

Your edit line should now look like this:

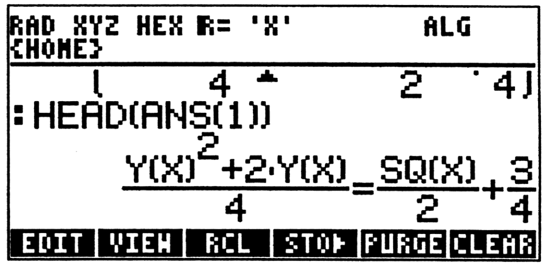


Now press [ENTER] to solve the differential equation using DESOLVE:



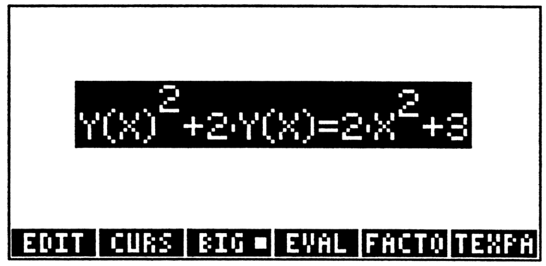
The idea is to graph that solution over the top of your Euler plot. To do so, you need to extract the solution equation from the list where DESOLVE left it.

Applying the command HEAD to the list ANS(1) will do this. (HEAD returns the first element of a given list.) Press ←PRG 6 ENTER ENTER 7 ENTER ←ANS ENTER.

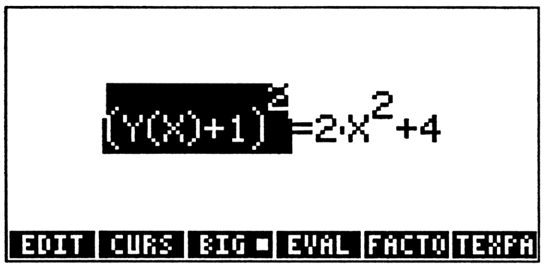


(Now, before you go any farther, go to MODE CAS, ▽ down to the Rigorous field and uncheck it. This switch will mean that simplifying the square root of Y(X)^2 will give Y(X), rather than ABS(Y(X)). Press OK OK.)

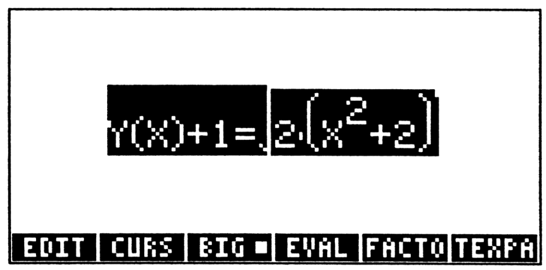
Press ▽ to bring your equation into the Equation Writer, where you can now set about isolating Y(X).



First, multiply both sides of the equation by 4: × 4 →△ EVAL EVAL.



Then complete the square in Y(X) by adding 1 to both sides: + 1 →△ EVAL EVAL. Press ▽ to highlight the left-hand side, then FACTO to factor it.



Press △ to highlight the entire expression, then √x EVAL to take the square root of both sides and simplify.

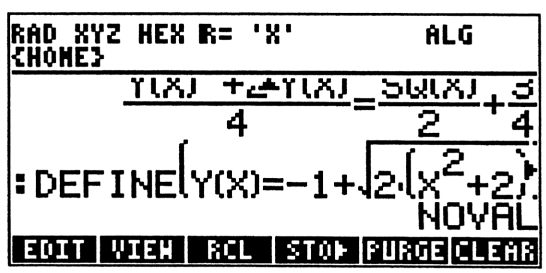
2. Why do you want only the positive square root of Y^2?

____

____

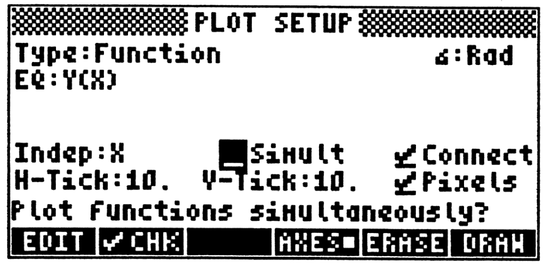
____

Finally, subtract 1 from both sides and simplify: − 1 →△ EVAL EVAL.

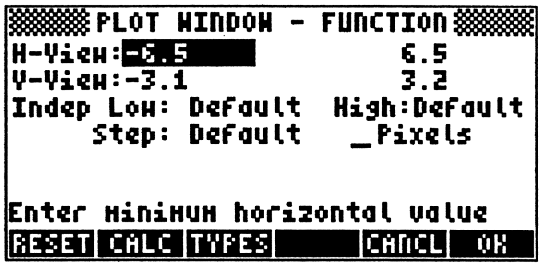


Press →COPY CANCEL to copy this result and return to the stack. Define Y(X): ←DEF →' →PASTE ENTER.

To graph this on top of your earlier scatter plot, you need to change the plot Type to Function, put Y(X) into EQ, and change the independent variable to X.

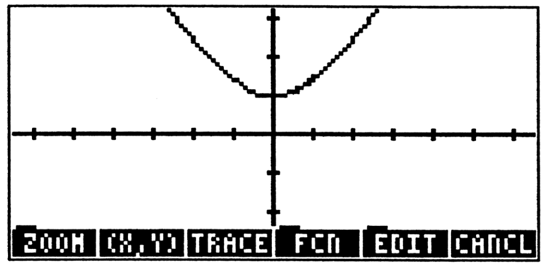


Press [←][2D/3D] and set up the screen as shown.



Press [←][WIN] and set up as shown. (A quick way is to press [NXT] RESET [▼] [ENTER].)

Press [NXT] DRAW ....



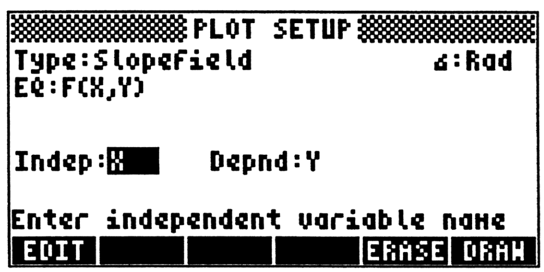
You should see the graph of the solution curve slam right through the Euler's method approximation points!

3. The solution curve appears to be an even function—symmetric over the y-axis. What about the slope relation, $F(X,Y) = -2X/(1+Y)$, assuming $Y > 0$? ______________________

____

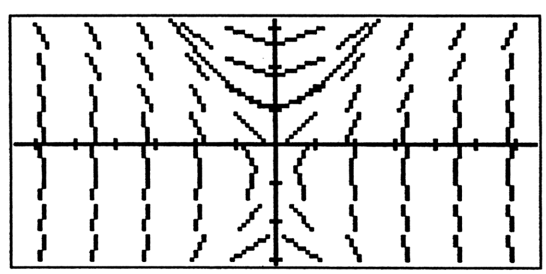
Overlaying a slope field

Press [CANCEL] to get off the graph screen, then [←][2D/3D] to go to the PLOT SETUP screen.



With the Type field highlighted, press [ALPHA][S] [ALPHA][S], which will change the plot type to Slopefield. Highlight the EQ field, and press [→]['] [ALPHA][F] [←][()] [X] [→][,] [ALPHA][Y] [ENTER].

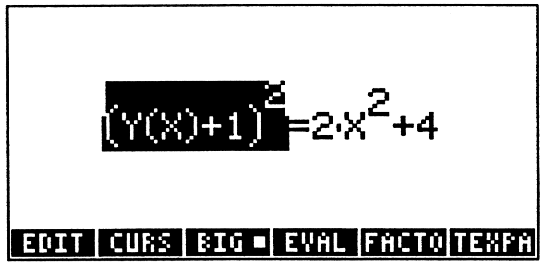
Then press [←][WIN], and set the window up as shown (left).



Then DRAW the graph (without erasing!) to superimpose the slopefield graph on top of the solution curve:

The fact that your solution curve "flows" smoothly through the slope field is confirming evidence that your analytic work is all correct. The slope field also suggests that the solution is asymptotic.

In fact, if you look closely at the analytic solution you obtained above (reproduced here), you can see that it is rather straight forward to rewrite that solution in standard form for the equation of a hyperbola.



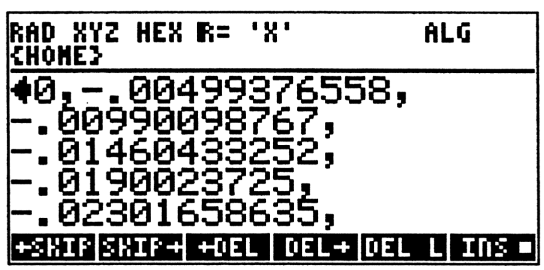
4. Show that, if you do, the equation is $\frac{(y+1)^2}{4} - \frac{x^2}{2} = 1$. Your solution consists of just the top branch of the hyperbola.

Extension: Looking at the Errors

Now the real fun starts! You're ready to look at the errors in the Euler's method approximation. All you need to do is subtract the exact values on the solution curve from the Euler approximations in `YLIST`. You'll store the result in `ERRLIST`. The command `YLIST-DOLIST(XLIST,'Y')▶ERRLIST` does it.

Back at the stack, press [VAR] so that you can use the menu key shortcuts to type the variable names. (Again, you may have to use [NXT] and/or [↰][PREV] to get to other pages of the menu.) Here goes: **YLIST** [−] [↰][PRG] [6] [ENTER] [2] [ENTER] [ENTER] **XLIST** [↱][,] [↱]['] [ALPHA][Y] [▶] [▶] [STO▶] [ALPHA] [ALPHA] [E][R][R][L][I][S][T] [ENTER] (You may have to accept the machine's suggested switch to approximate mode.)

After computing the errors, press [▼] and you can see them:



5. The first entry is virtually equal to zero. Why?

6. Why are all the other entries negative?

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7. What should happen to the errors as X-> infinity? Why?

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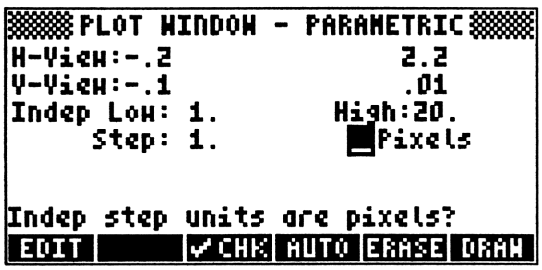
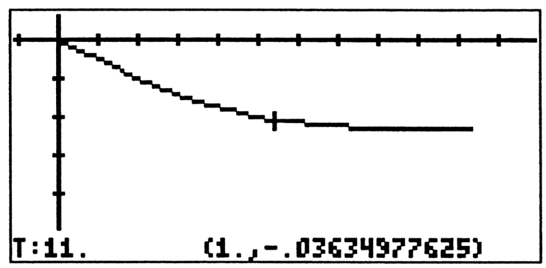
To graph the error list, press [CANCEL] [↰] [2D/3D] and change the **Type** back to **Parametric**, the Indep variable back to **T**, and put the expression shown into **EQ**. (Be sure to enclose the equation within "tick" marks: ' ')



Press [↰] [WIN] and set the window as shown (left).

Now **ERASE** and **DRAW** the graph:



8. Notice that the graph of the errors is always below the x-axis. Why?

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It also appears, if you look closely at the values in the error list, that the errors seem to start to tend back towards 0 near $x = 2$. Explain why the errors ought to get closer and closer to zero as x gets large.

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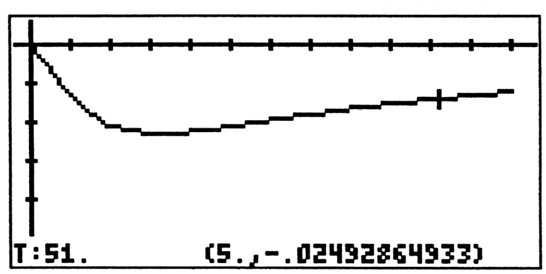
It would be an interesting extension to continue on up to $x = 6$ or so, and show that the errors do indeed approach 0. Here's a graph of the errors out to $x = 6$:



Teacher Notes

This is most appropriately done as a teacher-directed activity, with students following the teacher's work, but not actually typing keystrokes. Rather, the teacher should explain all the steps, and ask appropriate questions during the activity. Some questions have been suggested throughout the activity.

Answers to Suggested Questions

1. *The slope where* $X = 0$ *and* $Y = 1$ *is* $2(0)/(11) = 0$. *And* $Y(0.1) \approx 1 + 0(.1) = 1$.
   The slope where $X = 0.1$ *and* $Y = 1$ *is* $2(0.1)/(1+1) = 0.1$. *And* $Y(0.2) \approx 1 + 0.1(.1) = 1.01$.

2. *Your solution contains the point* $(0,1)$, *which is the vertex of the top branch of the hyperbola, where* $Y > 0$.

3. *Assuming* $Y > 0$, *the slope* $F(-X,Y) = 2(-X)/(1+Y) = -2X/(1+Y) = -F(X,Y)$. *Thus the slope function is symmetric with respect to the origin.*

4. $(Y+1)^2 = 2X^2 + 4$, *so* $(Y+1)^2 - 2X^2 = 4$ *and* $(Y+1)^2/4 - 2X^2/4 = 1$. *Thus* $(Y+1)^2/4 - X^2/2 = 1$.

5. *The initial condition,* $X = 0$ *and* $Y = 1$, *is the first point in the sequence of Euler approximations. The solution curve also goes through the point* $(0,1)$. *The difference is not exactly 0, since the exact solution involves irrational constants.*

6. *The Euler approximations are all too small, since the hyperbola is always concave up. The slope on the curve is always increasing, while along the short line segments used by Euler's method, the slope is constantly equal to the slope on the curve at the left endpoint of each interval. This makes the tangent lines lie under the curve.*

7. *As X approaches infinity, the graph of the hyperbolic solution curve approaches a linear asymptote. Consequently, short tangent line segments become closer and closer to the actual solution curve. This causes the errors to approach 0!*

8. *The Y-coordinate on the graph of the errors tells you the difference between the Euler approximation and the exact value on solution curve. Since the Euler approximation is less than the exact value, this difference is negative.*

Introduction to Histograms

Rolling the Dice

A histogram is a useful graph when you're analyzing one-variable data. On the horizontal axis, it displays all possible values of the single variable; on the vertical axis, it shows the number of times (called the frequency) that the returned value is in a particular interval.

The purpose of this activity is to show how you can easily construct a histogram on the HP 49G and to give you experience in using histograms to convey information. In this case, the variable will be the sum of the roll of two dice. You'll gather data over 50 rolls, and construct a histogram from this data.

Collecting data

Turn on the calculator, press [⇦][2D/3D]. If the plot type is not set to histogram, highlight the **Type** field, press **CHOOS** and select **Histogram**. Now highlight the **ΣDAT** field and press **EDIT**. This will shift the calculator into the Matrix Writer. You're going to record your data in the first column of a matrix screen, so for convenience, make sure the **GO↓** is selected. You're now ready to start the experiment.

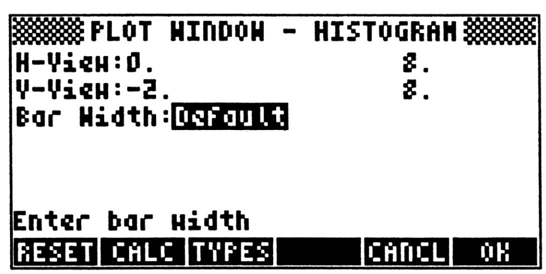
Roll two dice, type the sum into the calculator and press [ENTER]. You should see that sum in row 1, column 1 of the Matrix Writer screen; and row 2, column 1 should now be highlighted. Continue rolling and entering until you have filled up 50 entries in column 1 with the sums of each roll. When all 50 rolls are recorded press [ENTER] again, to go back to the plot setup screen.

1. What are the largest and smallest possible values for this data? ______________________________

   What are the largest and smallest values that you have recorded? ____________________________

Creating a histogram

First go to the **PLOT WINDOW** screen—press [⇦][WIN]. This is where you input the horizontal and vertical view intervals and the bar width. Begin with the settings shown here. (The default bar width is one unit, which will suit your purpose.) Press [ENTER] to accept all the settings and exit the screen.



Next, press [⇦][2D/3D] again to see the **PLOT SETUP** screen. Adjust the settings to those shown here. Be sure that **AXES■** appears in the menu line. (If not, press **AXES** to change it.)

Now press **ERASE** **DRAW** to generate the plot....

The resulting histogram doesn't tell the whole story, does it? What might be wrong?

For starters, it looks as if the plot window is not appropriate for the data—look at the horizontal axis. Press (CANCEL) twice, then (↰)(WIN). Change the **H-View** to show from **0** to **13**, so that you'll see all possible dice sum values represented on the horizontal axis. Press **ERASE** **DRAW** to generate a new histogram....

The result looks better, but some of the bars may extend beyond the top of the screen, depending on your data. (This will probably happen if a particular dice sum appeared more than 8 times in your 50 trials.) If so, you'll need to adjust the vertical view—increase the **V-View**: Go back to the plot window screen (by pressing (CANCEL)), and change the **V-View** to show from **0** to **10**. Press **ERASE** **DRAW** to generate a new histogram.... If any of the bars still reach the very top of the screen, try **0** to **11** for V-View. Keep increasing until there is only a small amount of blank space above the tallest bar.

You should now have an excellent looking histogram of the data.

Analyzing the data

Notice that you can move the crosshairs around the screen using the four arrow keys. Also, notice that you can see the coordinates of the crosshairs by pressing **X,Y**.

2. Position the crosshairs at the top, center of the first bar of the histogram. What are the coordinates of the crosshairs at this position? ______________________________

3. What dice sum is represented by this bar? ______________________________

   What is the frequency of this roll? ______________________________

4. Continue positioning the crosshairs at the tops of the bars, and write the frequencies here, labeling each with the dice sum (variable value) it represents:

   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________
   ________ ________

5. The variable value that occurs most often is called the mode. What is the mode in this experiment and what is its frequency? ______________________________

6. Most of the time, the mode in this experiment will turn out to be 7. (You can confirm this by looking at the modes from other groups in the class.) That is, when rolling two dice, the most likely sum of the dice will be 7. Explain why this is so.

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The last plot window value you'll change is the Bar Width value. Go to the PLOT WINDOW screen (⬅ WIN) and change this from Default (which is 1) to 2. Replot the histogram. (You may have to change the V-View value again.)

7. Write a short paragraph explaining how this change affected the histogram.

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8. What dice roll sums are represented in the first bar of the histogram?

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Teacher Notes

This activity may be the students first experience with the histogram plot type. A fair amount of time is spent in the plot window screen in the hopes that students will understand what each of the settings mean. Students should be encouraged to experiment with different settings and try to predict what effect a changed setting will have on the plot. The goal is to set up a plot that contains all of the data without a lot of wasted space.

Materials needed

Two dice for each group; at least one HP 49G for each group.

Terms introduced

histogram, frequency, mode.

Calculator functions introduced

Histogram plot type, histogram plot window, generating a histogram, moving crosshairs on a histogram.

Solutions to Selected Exercises

1. *Dice sums must be between 2 and 12 inclusive. The largest and smallest dice sums observed may vary.*

3. *Answers may vary but they should be related to answer 2. Keep in mind that the intervals are half open. In this case, the first interval is [0,1), the next is [1,2), and so on. Look for the correct identification of dice roll sum and frequency. Also look to see that each of these are real numbers.*

5. *Keep in mind that the mode is the value of the dice sum that occurs most often, not its frequency.*

6. *The explanation will vary with the background of the student. Listing all possible combinations of 2 dice rolls works just fine.*

7. *The dice sums 2 and 3 are represented in the first interval. The bars associated with some of the more frequent intervals are likely to "go through the roof."*

Extensions

Use the data to discuss the probability of rolling a particular dice sum. Compare that with the probability for that dice sum as computed by listing all possible dice sums. Are these different? Why?

Use the binomial probability to discuss the probability of rolling a particular dice sum a particular number of times. Compare this with the individual group results.

Using Histograms, Frequencies and Single-Variable Statistics

Mrs. Green's Algebra Classes

In this activity, you will gain experience in setting up histograms on your own. You will also be introduced to bin sorting and some of the more common single-variable descriptive statistics.

Mrs. Green teaches two freshman algebra courses, each with 21 students. The results of her most recent test, out of a possible 100 points, are as follows:

Class 1	Class 2
94	71
92	79
58	80
82	75
99	74
51	84
93	76
96	66
75	77
74	62
64	75
79	96
65	68
53	75
87	73
62	86
54	67
62	80
95	68
63	83
83	66

Mrs. Green uses this grading scale (where s is a score):

$90 \le s \le 100$ = A
$80 \le s < 90$ = B
$70 \le s < 80$ = C
$60 \le s < 70$ = D
$s < 60$ = F

Enter the test score data into a 2-column matrix, with Class 1 in column 1 and Class 2 in column 2.

Then create a histogram that displays the scores from Class 1. Make sure that the **Col̲umn** field is set to **1** in the **PLOT SETUP** screen (⬅2D/3D). In the **PLOT WINDOW** screen (⬅WIN), use a **Bar Width** that will put all of the scores earning a particular letter grade together, and set the window dimensions appropriately. This may take a bit of thought. (Hint: Do not start **H-View** at **0**.)

1. Write down the Plot Window settings that you used.

   H-View: ______ ______

   V-View: ______ ______

   Bar Width: ______ ______

2. Using the histogram you just generated, determine the most common letter grade assigned for Class 1 (i.e. the letter grade mode for Class 1). ____________________________________

Now make a histogram of the Class 2 data. (Press CANCEL twice to leave the plotting environment, then press ⬅2D/3D and set the **Column** field to **2**. Press **ERASE** then **DRAW**. You may find it necessary to change the window dimensions to see the tops of all of the bars.)

3. What is the letter grade mode for Class 2? ____________________________________

Sorting into bins

It is also possible to numerically "see" the frequencies of each letter grade in each of the classes. Press ➡STAT. This brings up the statistics sub menu. Select **Frequencies...** and press ENTER to go to the **FREQUENCIES** screen. On this screen, the same ΣDAT variable is used for data input, and the **Column** field has the same meaning as in the **PLOT WINDOW** screen, too. But the other settings work a little differently:

> **X-Min** determines the smallest number considered for sorting into frequency bins.
> **Bin Count** determines the number of bins created.
> **Bin Width** determines the interval width for each of the bins.

Example: With **X-Min** set to **2**, **Bin Count** set to **3**, and **Bin Width** set to **5**, the Frequency command would sort your data into 3 bins with intervals [2,7), [7,12), and [12,17). Any value smaller than 2 would be considered a lower outlier; any value 17 or larger would be considered an upper outlier. (Here, outliers are simply values outside of the entire sorting interval.)

The Frequency command returns two vectors as the results of its sorting: a column vector of the frequencies (one frequency for each bin) and a two-element row vector containing the numbers of lower outliers upper outliers.

4. Fill in the Frequency screen with settings that you think will sort Class 1 into the appropriate grade bins for Mrs. Green. Record what you think these settings should be below.

   Col: _____

   X-Min: _____

   Bin Count: _____

   Bin Width: _____

Press OK to see the result. (You may have to ▲ up to highlight and view the column vector to answer the following question.)

5. How many students from Class 1 will receive a letter grade of B on this test? _________

Now repeat the process to examine the grade frequencies for Class 2.

6. How many students from Class 2 will receive a letter grade of A on this test? _________

7. Which class do you think did better on the test? Why?

   ____

   ____

Computing statistics

It's easy to compute many common single variable statistics with the HP 49G. Press [→][STAT] to bring up the statistics sub menu. Select Single-var... and press [ENTER]. You will see a screen that allows you to pick a column of ΣDAT to analyze, choose sample or population statistics, and choose from among 6 different statistics computed for the ΣDAT column.

In this case, you want to examine Class 1, so set the Column field to 1. This data represents everyone in class—the entire population—so highlight the Type blank, press CHOOS, and select Population. Finally, check all six of the statistics blanks. Press OK.... The six statistics you requested will then be computed and displayed on the stack. You are probably familiar with the *mean*, and you can probably guess what the *maximum* and *minimum* are, but what about the other statistics?

The *total* is simply the sum of all of the entries in the column. In this case, it is the sum of all of the test grades in Class 1.

The *variance* and *standard deviation* indicate the amount that the data is spread out from the mean. (Note that the standard deviation is the square root of the variance.) The standard deviation will be used much more in later activities.

8. Record the six statistics computed for Class 1.

Mean	___________	Total	___________
Std Dev	___________	Maximum	___________
Variance	___________	Minimum	___________

Repeat the statistics computation process for Class 2.

9. Record the six statistics computed for Class 2.

Mean	___________	Total	___________
Std Dev	___________	Maximum	___________
Variance	___________	Minimum	___________

10. Other than adding up all of the individual scores, how else could you compute the total points scored by a class? ____

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11. Write a short paragraph comparing the scores of Class 1 as a whole to Class 2 as a whole.

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Teacher Notes

The data in this lab has been conjured, but with a purpose. The test scores from each of Mrs. Green's two algebra classes are quite different, but each class has the same mean and the same median. The point here is that there is more to describing a distribution that just conveying the central tendencies. Students are asked to set up a histogram using a bar width that will sort the scores into letter grades. Very little calculator direction is offered; in fact, they must tell you what settings they used. A bit of trial-and-error here is expected. The frequency screen is introduced so that students can numerically examine the grade distributions. Students then compute some single variable descriptive statistics.

Materials needed

At least one HP 49G per group.

Terms introduced

Bin sorting, population, mean, maximum, minimum, total, variance, standard deviation.

Calculator functions introduced

Frequencies, single-variable statistics.

Solutions to Selected Exercises

1. `H-View: 40 110` `V-View: -2 7` `Bar Width: 10`
   This gives an aesthetically pleasing view of the Class 1 histogram. (Some variation on these settings is to be expected.) Notice that `V-View` *starts at* `-2` *so that you can still see the bottom of the plot with the menu in place. But note, too, that the upper* `V-View` *setting is too small for the Class 2 histogram.*

2. *The most common letter grade (i.e. the letter grade mode) in Class 1 is an A. It has a frequency of 6.*

3. *The letter grade mode in Class 2 is a C. It has a frequency of 9.*

4. `Col: 1` `X-Min: 50` `Bin Count: 5` `Bin Width: 10`

5. *Three students will receive B's on this test.*

6. *One student will receive an A on this test.*

7. *This is an open-ended question meant to initiate some student reflection on the data.*

8. *For Class 1 (to 3 decimal places)* `Mean: 75.286` `Total: 1581`
   `Std Dev: 15.670` `Maximum: 99`
   `Variance: 245.537` `Minimum: 51`

9. *For Class 2 (to 3 decimal places)* `Mean: 75.286` `Total: 1581`
   `Std Dev: 7.839` `Maximum: 96`
   `Variance: 61.442` `Minimum: 62`

10. *The total can be computed by multiplying the mean by the number of scores.*

11. *This is an open-ended question. Student responses should include a discussion about the spread of the test scores.*

Extensions

Ask "Which class do you think did better on the test and why?"

Have students compute the median (75). Discuss how the scores from two different classes could have the same mean and the same median.

Create box and whisker plots for the data from each class.

Discuss variance and standard deviation.

Scatterplots and Linear Fitting Data

Predicting Heights from Shoe Sizes

Do taller people tend to have bigger feet? If so, is there a way to predict a person's height from the size of their footprint? You'll try to answer these two questions in this activity. Along the way, you'll be introduced to a kind of graph that's very useful for analyzing data of two variables—called a scatterplot—and to a method of fitting an equation to this two variable data.

Collecting the data

You must first gather the shoe sizes (corrected for male or female) and the heights (in inches) of everyone in the class. Once that is accomplished, use the Matrix Writer to enter the data into a matrix on the calculator with shoe sizes in the first column and heights in the second column. Save this matrix in the ΣDATA variable.

Creating a scatterplot

Press (⬑)(2D/3D) to go to the **PLOT SETUP** screen, highlight the **Type** field, press **CHOOS** and select **Scatter**. The contents of the ΣDATA field should show the shoe sizes and heights you have already entered.

You want shoe size to be the independent variable (graphed on the horizontal axis); height will be the dependent variable (graphed on the vertical axis). You define the independent and dependent variables in the Cols field. The first number identifies the column to be plotted along the horizontal axis—set this to **1**. Set the second number to **2**, to plot the second column of ΣDAT along the vertical axis. Make sure that **H-Tick** and **V-Tick** are both set to 10 pixels, then press (ENTER) to save these settings.

The last thing to do before you generate a scatterplot is to visit the **PLOT WINDOW** screen ((⬑)(WIN)) and make any needed changes. You must make good decisions for the values of **H-View** and **V-View** based on the data. (Remember, shoe size will be on the plotted horizontal axis and height will be plotted on the vertical axis.) The goal is to make a plot that contains all of the data, without a lot of wasted space on the edges.

Press **ERASE** **DRAW** to generate the scatterplot.... Each of the points shows a person's data as the ordered pair (shoe size, height). You may have to return to the **PLOT WINDOW** screen (by pressing (CANCEL)) and adjust the settings to get the viewing window just right.

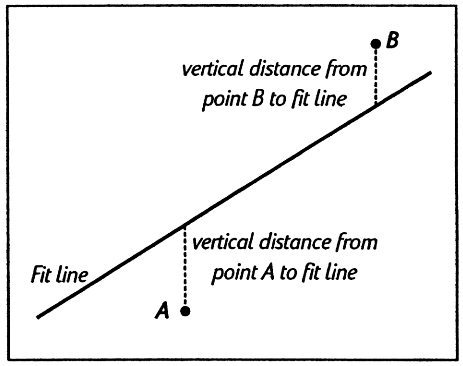
Can you spot any trends in this data? Circle the word in each bracket that best describes the plot.

1. The points seem to go from { lower left , upper left } to { lower right , upper right }.

2. As shoe size increases, height { decreases , increases }.

Fitting a line to the data

One of the most useful things you can do with a scatterplot is to find the line (or some other curve) that represents the data. Computing this line by hand, called "the least squares fit line" or simply "the line of best fit," is a bit complicated, but the idea is simple.



The figure at right shows two data points and their vertical distances to a line. The line of best fit is the line that makes the sum of the distances of all of the data points to the line as small as possible.

The HP 49G makes it easy to compute and draw this line of best fit.

First, you must be sure that the calculator is set up to draw a line and not some other curve. Press CANCEL twice to get back to the stack, then ⮣ STAT, and select `Fit data...` from the list to get the `FIT DATA` screen. You should see ΣDAT (whose values should look familiar), the `X-Col` and `Y-Col` fields (which should be set to `1` and `2`, respectively), and the `Model` field. This field should be set to `Linear Fit` in order to fit a line to the data. (If not, **CHOOS** `Linear Fit` from the list.)

Press **OK** to return to the stack, where you should see three levels of information. On level 3, you will see the expression of the line of best fit for the data. On level 2, you will see the correlation coefficient, a measure that you will look at later. On level 1, you will see the covariance between the two variables.

Press ▲ twice to highlight the best-fit expression, then press **VIEW** to see it. You may have to press ◀ several times to see the whole thing.

3. Write down the equation of the best-fit line.

____

To see the best-fit line superimposed on the scatterplot of the data, press ⮢ GRAPH, then press **STATL**. (And notice that the features available on any function plot—such as tracing or finding roots—are now available with respect to the best fit line.)

4. If a boy's data point lies below the line of best fit, would you say that this boy is "too tall for his feet," or "his feet are too big for his height"—based on the class data? Explain.

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The equation of the best-fit line is the linear equation that most closely links shoe size to the height of a person in the class, for the data given.

In fact, you can use this equation to predict the height of any person based on their shoe size. If you substitute a person's shoe size into X in the linear fit equation, the resulting value will be the prediction of that person's height based on the class data.

5. Predict your own height using the best fit linear equation. How close is this prediction to your actual height?

____

____

6. Predict the height of a classmate using the best fit linear equation. Use someone with a different shoe size. How close is this prediction to his/her actual height?

____

____

7. Predict the height of your teacher using the best fit linear equation. How close is this prediction to their actual height?

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8. Michael Jordan wears a men's size 16 shoe. Predict his height using the best fit linear equation.

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9) What data could you collect to make the prediction of Michael Jordan's height, based on his shoe size, more accurate?

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Teacher Notes

Important note: Shoe sizes must be "corrected" in a mixed gender class. If you use women's size as the standard, then add 2 to each male's shoe size (including Michael Jordan's); if you use men's size as the standard, then subtract 2 from each female's shoe size. (You may want to create a data table on an overhead transparency that students fill in once their heights have been measured.)

This activity is truly an experiment—it turns out different for each class. This may be students' first experience with two-variable data. They're asked to generate a scatterplot, but direction is given mainly in areas where the scatterplot procedure is different than previous plots. Little direction is given for the plot setup screen. Students have to look at the data and make good decisions.

Materials needed

One yardstick (or meterstick with inches) per group, and at least one HP 49G per group.

Terms introduced

scatterplot, independent variable, dependent variable, least squares fit line.

Calculator functions introduced

Scatterplot, fit data.

Solutions to Selected Exercises

1. *The points will tend from lower left to upper right.*
2. *As shoe size increases, height tends to increase.*
4. *His feet are too big for his height. For his shoe size, the he is shorter than is typical for the class (judging by the line of best fit).*
8. *Michael Jordan is 6'6" (78 inches) tall.*
9. *Use all adults as a sample population for drawing the best fit line. Use only tall, male adults for drawing the best fit line. Use only professional basketball players for drawing the best fit line.*

Extensions

Discuss ways in which the accuracy of this experiment could be improved. (Instead of using shoe sizes, you could measure the length of peoples feet. You could also include a larger population for the scatterplot.)

Interesting angle: Crime scene investigators get a rough idea of the height of a criminal by examining footprints left at the scene.

Ask students to try to think of other pairs of quantities that might have a linear relationship. Encourage them to formulate testable questions, construct surveys, collect data, and analyze the results.

Linear and Power Fitting Data

Speed and Stopping Distance

A car's stopping distance depends on several things, but by far the most important controllable factor is the speed, for two reasons. First, brakes must work much harder to stop a faster car. Second, a faster car will travel farther during the driver's reaction time. A car traveling at 70 mph travels 102 feet in one second.

The data

This data table shows the stopping distances of a typical car driven by a driver with a typical reaction time on a dry road.

Speed [MPH]	Stopping Distance [ft]
10	14
20	38
30	71
40	114
50	166
60	227
70	298
80	378
90	467
100	566

Enter this data into a matrix on the calculator with speed in the first column and stopping distance in the second column. Save this matrix in the ΣDATA variable.

Creating a scatterplot

Press [←][2D/3D], set the plot **Type** to **Scatter**, the first **Col** number to **1** and the second **Col** number to **2**. Be sure **H-Tick** and **V-Tick** are both set to 10 pixels, then press [ENTER] to save these settings. Next, press [←][WIN] and set up a plotting window appropriate for the data. Then generate the scatterplot by pressing [ERASE] [DRAW].... You should see all of the data points, without a lot of wasted space around the sides of the display. If necessary, press [CANCEL] to get back to the **PLOT WINDOW** screen and readjust the settings.

Modeling the data

First, investigate a linear fitting model of this data. Press [→][TBLSET], and select **Fit data...**. You should see ΣDAT (which should look familiar), **X-Col** and **Y-Col** (which should be set to **1** and **2**, respectively), and the **Model** field. This field should be set to **Linear Fit** so you can fit a line to the data. (If it's not, highlight the **Model** field, press [CHOOS] and select **Linear Fit** from the list.) Press [OK] to return to the stack.

1. Write down the equation of the line that best fits the data. You may need to use an arrow key and **VIEW** to see the entire expression. ______________________________

2. What is the correlation coefficient for this model of the data? ____________________

3. What stopping distance does this line predict for a speed of 100 MPH? ____________________

Press (→)(TBLSET) and choose `Fit data...` from the list. Then press **PRED** to get the `PREDICT VALUES` screen. Make sure that the `Column` settings are correct and that the model is set to `Linear Fit`. Highlight the `X` field and enter `100`, then highlight the `Y` field and press **PRED**.

4. What's the meaning of the number in the `Y` field, in terms of speed and/or stopping distance?

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Now enter `120` in the `Y` field, highlight the `X` field and press **PRED**.

5. What's the meaning of the number in the `X` field, in terms of speed and/or stopping distance?

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When you compare your predictions above with the actual data in the table, the linear model seems to leave a lot to be desired. Press (←)(GRAPH), then **STATL**, to show the scatterplot with the line of best fit superimposed. Can you see the shortcomings of the linear model?

Press (CANCEL) twice to go back to the stack. To fit a different model to the data, press (→)(TBLSET), select `Fit data...` from the list, and change the `Model` to `Power Fit`. Press **OK** to see these results.

6. What is the power equation that best fits the data?

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7. Record the correlation coefficient associated with this model of the data. ____________________

Press (←)(GRAPH), then **STATL**, to show the scatterplot, the power equation of best fit and the line of best fit.

8. What stopping distance does the power equation predict for a speed of 100 MPH? ______________

9. In this model, what's the maximum speed you could go and still stop within 120 feet? ____________

10. If you're driving down a city street at 25 MPH, and a small child runs across the street 57 feet in front of you, by how many inches will you miss the child, based on the power fit model? ______________

11. What speed doubles the 55 MPH stopping distance in this power fit model? ________________

Teacher Notes

This data was derived from the 60-to-0 braking distance of a 1995 Ford Taurus SHO. The typical accelerator-to-brake reaction time was assumed to be 0.7 seconds. All of the other data points were computed using a little physics (see Giancoli, 1991, p. 128, or the work chapter in any other physics text).

Materials needed

At least one HP 49G calculator per group.

Terms introduced

Fitting non-linear data.

Calculator functions introduced

Power fit model, predict values.

Solutions to Selected Exercises

1. *Rounding to three decimal places:* $y = 6.133x - 103.4$. *The calculator will display only the right side of this equation (and in either descending or ascending degree, depending on your CAS setting for* Incr Pow*).*
2. *To three decimal places: 0.982*
3. *Using the result from question 1:* $6.133(100) - 103.4 = 509.9$
4. *This number, which matches the result from question 3, gives the stopping distance (in feet) associated with a speed of 100 MPH, using the linear model.*
5. *This number (approximately 36.428) represents the speed in MPH associated with a stopping distance of 120 feet, using the linear model.*
6. *Rounding to 3 decimal places (which can introduce a lot of error):* $y = 0.307x^{1.619}$
7. *To three decimal places: 0.999.*
8. *Using the* PREDICT VALUES *screen: 531.383 feet.*
9. *Using the* PREDICT VALUES *screen: 39.890 MPH.*
10. *About 3.75 inches.*
11. *About 84.399 MPH.*

Extensions

Measure out 264 feet to illustrate the stopping distance predicted for 65 MPH. (And remember: This model assumes dry road and good reaction times!)

Discuss the correlation coefficient associated with each of the models.

The calculator also has a `Best Fit` *option, which will pick the best model among linear, power, exponential or logarithmic. Is the power model the best one for this data?*

Ask students to try to think of other pairs of quantities that might be related. Encourage them to formulate testable questions, construct surveys, collect data and analyze the results.

Introduction to Confidence Intervals Around a Mean

SAT-M Scores and Speeding Cars

The Scholastic Aptitude Test for Mathematics (SAT-M) is widely used to measure readiness for college math. The test is developed and tried out on a "standardization group" to adjust the scoring so that the group mean is 500 and the standard deviation is 100. The test-taking population (all students who take the test in a given year) generally has a lower mean score. In 1991, the population mean for the SAT-M was $\mu = 474$.

The reasoning behind confidence intervals

Suppose that you took a random sample of 500 students who took the SAT-M in 1991. Are you guaranteed that the mean for this sample will be $\bar{x} = 474$? Of course not. You can expect some variability in sample means from sample to sample. If you took several random samples of the population you might expect some of the sample means to be smaller than 474, and some to be larger. Recall from the central limit theorem that sample means of a particular size are distributed normally, regardless of the population distribution. Furthermore, if the population has a mean of μ and standard deviation of σ, then the sample mean distribution will be $N(\mu, \sigma/\sqrt{n})$, where n is the size of each of the samples.

Imagine that you took repeated random samples from the 1991 population of SAT-M test takers and computed the mean for each sample. Your variable would then be $\bar{x}$. You should expect $\bar{x}$ to be normally distributed with a mean of 474 (the population mean). Now assume that the population standard deviation is 100, then the standard deviation for your distribution of sample means would be: $\sigma_{\bar{x}} = \dfrac{100}{\sqrt{500}} \approx 4.5$

1. You may be used to viewing $\bar{x}$ as a statistic. Now it is being referred to as a variable. Write a short paragraph explaining how you are viewing $\bar{x}$ differently now.

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Recall that in a normal distribution, about 95% of the variable values will lie within two standard deviations of the distribution mean. In our example, we should expect that 95% of the means of our random samples should be within $2(4.5) = 9$ points of $\mu = 474$. To say that $\bar{x}$ lies within 9 points μ is the same as saying that μ lies within 9 points of $\bar{x}$. Thus, for 95% of the sample means, μ will be between $\bar{x} - 9$ and $\bar{x} + 9$. The interval of numbers between $\bar{x} \pm 9$ is called a 95% confidence interval for μ.

2. A random sample of 500 students from the population of all 1991 SAT-M test takers has a mean score of 480. Compute, by hand, the 95% confidence interval for μ based on this sample. ____________

Using the HP 49G calculate confidence intervals

Turn on the calculator and press [→][STAT]. Select the `Conf. Interval` option. In that choose box, select `Z-INT: 1μ.`, which will take you to an input screen. To recompute problem 2, use these values:

$\bar{x}$: 480	n: 500
σ: 100	C: 0.95

Press **OK**. When the computation is finished, you will see a screen that contains two z-scores that correspond to the endpoints of the interval that contains 95% of the area under the standard normal curve. These are sometimes called the critical z-scores. You will also see numbers labeled as μ min and μ max. These correspond to the endpoints of the confidence interval. Now press **GRAPH**. This shows the standard normal curve with the critical z-scores identified. Below the curve is a scale showing the sample mean lined up with the standardized mean of 0, and μ min and μ max lined up with the critical z-scores. Press **TEXT** to return to the previous display or press **OK** to return to the stack.

3. Did the calculator return results identical to the results of your hand calculations in problem 2? Why or why not? ____

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In practice, the population mean μ, is usually not known. Statisticians use confidence intervals based upon a sample mean to estimate the population mean.

4. Suppose you are interested in the 1991 SAT-M scores for the population of students that come from California. A random sample of 300 California students that took the SAT-M is examined and a sample mean of 471 is computed. Assume that the population standard deviation is still 100. What is the 95% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on this sample? ______________________________

5. What is the 99% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on the sample from problem 4? ______________________________

6. What is the 90% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on the sample from problem 4? ______________________________

7. Write a paragraph describing the effects of the confidence level on the size of the confidence interval. Why does it work this way? ______________________________

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This is a good time to mention some cautions about confidence intervals:

A 95% confidence interval for μ does not say that the probability is 0.95 that μ falls within the interval. It merely says that the interval was computed in a way that gives correct results in 95% of all of the samples. No randomness remains after a particular sample is selected; the population mean either falls within the confidence interval computed upon that sample or it doesn't.

Note that samples must be selected randomly from the population. The method of computing confidence intervals relies on the central limit theorem, and the central limit theorem applies only to random samples from a population.

So far, the method of computing confidence intervals relies on knowing the population standard deviation, σ. This is rarely true in practice. In the next section, you will see that there is a way around this limitation.

Confidence intervals in practice

In the method outlined above, deciding on a 95% confidence level determined two z-scores between which lay 95% of the area under the standardized normal curve (namely $z \approx \pm 1.96$). This is graphically illustrated by the HP 49G.

Knowing $\bar{x}$ (the sample mean), n (the sample size), and σ (the population standard deviation), you can algebraically find the value of μ for each z-score, using the relationship $z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$.

These calculated values of μ could give the confidence interval for the population mean, although it is rare that you need an estimate for the population mean. However, you do know σ, the population standard deviation, and it seems natural to substitute s, the sample standard deviation, for σ in the above formula.

Doing so does not give a z-score, however. It gives a new statistic called a t-score: $t = \frac{\bar{x} - \mu}{s / \sqrt{n}}$

Unlike the z statistic, the t statistic does not have a normal distribution. It has a distribution called the t distribution. Furthermore, there is a different t distribution for every sample size n. Statisticians classify t distributions in terms of *degrees of freedom*. The degrees of freedom of a t distribution is equal to $n - 1$.

For example, suppose you want to use the mean and standard deviation from a sample of 500 to compute a 95% confidence interval for the population mean. You must find the two t-scores between which lies 95% of the area under the curve of the t distribution with 499 degrees of freedom. Fortunately, the HP 49G does most of this for you....

Revisiting problem 4: Suppose that a random sample of 300 California students who took the SAT-M is examined and a sample mean of 471 is computed. Instead of assuming a population standard deviation of 100, you now compute the standard deviation of the sample to be 92. What is the 95% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on this sample?

Press [→][5], select the `Conf. Interval` option, and choose `T-INT: 1μ.`, which will take you to an input screen. Enter these field values: `x̄: 471` `n: 300` `sx: 92` `C: 0.95`

Press **OK**. When the computation is finished, you will see a screen that contains two *t*-scores that correspond to the endpoints of the interval that contains 95% of the area under the curve of the *t* distribution with 299 degrees of freedom. These are sometimes called the critical *t*-scores. You will also see numbers labeled as μ min and μ max. These correspond to the endpoints of the confidence interval.

Now press **GRAPH**. This shows the appropriate *t* distribution curve with the critical *t*-scores identified. Below the curve is a scale showing the sample mean lined up with the *t* distribution mean of 0, and μ min and μ max lined up with the critical *t*-scores. Notice the *t* distribution looks very similar to the normal distribution. In fact, the *t* distribution resembles the normal distribution more closely as the number of degrees of freedom increases. Press **TEXT** to return to the previous display or press **OK** to return to the stack.

The speeds of 25 cars traveling past a point on a country road are measured on a sunny day. These speed (in MPH) are given here:

56	49	44	61	67	43	
51	53	57	60	59	47	
55	58	67	43	66	62	
64	51	61	61	48	56	62

8. Find the 95% confidence interval for μ based on this sample. ______________________________

9. Find the 99% confidence interval for μ based on this sample. ______________________________

10. Find the 90% confidence interval for μ based on this sample. ______________________________

11. Does the confidence level have the same general effect on the size of the confidence interval as in question 7? ____

12. Are these estimated values for μ good indications of the mean speed of all of the cars that have ever passed this point in the road? Write a paragraph explaining why or why not.

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Teacher Notes

This activity may represent students' first experience with inferential statistics. It represents a substantial leap forward from the last activity because it assumes several statistical concepts, including: population and sample means and standard deviations; experience with finding areas under the standard normal distribution; and a conceptual awareness of the central limit theorem. The reasoning behind the process of finding confidence intervals around a population mean is discussed at length before the HP 49G is used. Confidence intervals around a population mean for an unknown population standard deviation are also discussed, necessitating the introduction of the t distribution.

Materials needed

At least one HP 49G per group.

Terms introduced

Central limit theorem, confidence level, confidence interval around a mean, critical z-scores, t-score, t distribution, degrees of freedom, critical t-score.

Calculator functions introduced

Confidence intervals: Z-INT: 1 μ, T-INT: 1 μ, confidence interval graphs

Solutions to Selected Exercises

1. *You are now analyzing a set of data that represents many sample means.*

2. *(471, 489)*

3. *The area under the normal curve between two standard deviations below the mean to two standard deviations above the mean is only approximately 95%. Look at the critical t-scores returned by the calculator. Round-off error in computing σx in question 2 also contributes to differences in the two answers.*

4. *To three decimal places: (459.684, 482.316)*

5. *To three decimal places: (456.128, 485.871)*

6. *To three decimal places: (461.503, 480.497)*

Note: the sample mean and standard deviation has to be computed before these confidence intervals can be computed.

8. *To three decimal places: (52.980, 59.100)*

9. *To three decimal places: (51.893, 60.187)*

10. *To three decimal places: (53.503, 58.577)*

12. *No, this sample is biased toward good road conditions.*

Extensions

Ask students to think of a survey question to ask students in the school upon which a mean confidence interval could be computed.

Determine a random sample, conduct the survey, collect data, and analyze the results.

Introduction to Significance Testing

More SAT-M Scores

Statistical inference has two major goals. One is the estimation of population statistics by computing confidence intervals based upon sample statistics. The other goal is the assessment of statements about a population. Such statements can be examined through tests of significance, a process illustrated here.

The reasoning behind tests of significance

While discussing SAT-M scores, a teacher states that these scores overestimate the ability of typical high school seniors, because the test is taken by predominantly college bound students. The teacher goes on to claim that if all students took the SAT-M, the mean would be no more than 450, not the national mean of 471. You disagree. You think that the mean would be bigger than 450. How can your claim be tested?

One way to test the claim is to test every single high school senior in the nation and compute the mean. Of course, this would cost millions of dollars and require the cooperation of millions of people. Fortunately, there is a way to test this claim based on a random sample—at a certain level of statistical significance (more on that idea later).

Suppose that a random sample of 300 high school seniors is selected nationwide and given the SAT-M (if they did not already take it). Suppose that the mean for this sample is 461. Does this information support your claim on its own? Not really. It is possible that a different random sample might yield a mean of 443, due to natural variability. So how do you know if the sample mean of 461 really supports the claim that if all students took the SAT-M, the mean would be greater than 450?

The first step, is to state the null hypothesis (abbreviated H_0)—the statement that significance testing will allow you to accept or reject. H_0 is the claim that you're trying to find evidence *against*—usually a statement of no effect, or no difference. In this case, it is: "The mean SAT-M score of all U.S. high school seniors is equal to 450." Symbolically, you would write H_0: $\mu = 450$. Notice that the null hypothesis is stated relative to a population variable. You already know that the mean of your random *sample* is 461, not 450. You're trying to determine if this is enough evidence to conclude that the *population* mean is not 450.

The next step is to state the opposing, or alternative, hypothesis (abbreviated H_a)—the statement you're seeking evidence for. In this case, H_a is: "The mean SAT-M score of all high school seniors is greater than 450." Symbolically, you would write H_a: $\mu > 450$.

Next, you specify the significance level of the test (denoted as α)—the "confidence" requirement you wish to put on the data. For example, if you choose a significance level of $\alpha = 0.05$, you are requiring that the data provide evidence against H_0 that is so strong that the test would give wrong results (this is, cause you to reject a true H_0) no more than 5% of the time (or 1 time in 20). Common significance levels are 0.10, 0.05 and 0.01.

Finally, suppose you know that the SAT-M scores are normally distributed with a population standard deviation of 100.

You're now ready to apply a test of significance. To summarize: H_0: $\mu = 450$ H_a: $\mu > 450$
$\sigma = 100$ $\bar{x} = 461$

Now, as you recall, a z-score is a measure of how far the sample mean is from a claimed population mean.

So you can compute a z-score based on the above information: $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{461\text{-}450}{100/\sqrt{300}} \approx 1.905$

Is this difference extreme enough to reject the null hypothesis? You may recall (or can verify from a table) that about 95% of the area under the standard normal curve lies below $z = 1.645$. That means that about 5% of the area under the standard normal curve lies above $z = 1.645$. Thus $z = 1.645$ is called the critical z-score for a significance level of $\alpha = 0.05$. The z-score you computed based on your data is 1.905, which is greater than the critical z-score of 1.645. Therefore, there is enough evidence to reject H_0. We can say, at a 0.05 level of significance, that the mean SAT-M score of all high school seniors is more than 450.

Using the HP 49G to test significance

The purpose of the previous section was to provide the reasoning behind the process of significance testing. In reality, most statisticians perform these tests using a computer or calculator. Try redoing the previous problem using the HP 49G. This will illustrate the steps involved in using the significance testing features, and the graphical display should clarify the ideas behind that testing.

Turn on the calculator and press (→)(STAT). Select the `Hypoth. Tests...` option and get the `Hypothesis Tests` choose box. (Significance tests are sometimes called hypothesis tests.) Select `Z-Test: 1μ..`, which will take you to an input screen. Then, to recompute the above problem, just enter these values into their respective fields: `μ: 450` `x̄: 461` `σ: 100` `n: 300` `α: 0.05`

Press **OK**. You'll then see the `Alternative Hypothesis` choose box. Select `μ>450` and press **OK**. After some calculation, the calculator will return the z-score associated with $\bar{x} = 461$ (this should be familiar), a statistic known as the P-value, the critical z-score (also explained above), and a statistic called the critical $\bar{x}$. Notice that the very top of this window tells you to reject $\mu = 450$, the null hypothesis, H_0. This is that same result you got by hand.

1. Record the values of z, P, critical z, and critical $\bar{x}$ for this test.

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The P-value is the probability, assuming H_0 is true, that the test will indicate that you should reject H_0. The smaller the P-value, the stronger the evidence against H_0. In fact, you reject H_0 precisely when $P < \alpha$. Another way to decide whether to reject H_0 is by looking at the critical $\bar{x}$. The critical $\bar{x}$ is the value of the sample mean that is on the boarderline in terms of providing evidence to reject or not reject H_0. In this case, your measured sample mean is more extreme (farther from $\mu = 450$) than the critical $\bar{x}$, so you reject H_0.

Now press GRAPH. You'll see the standard normal curve. Notice the value of R shown on the curve with the arrow pointing to the right. This shows the reject region for this test.

Under the standard normal curve is a different scale, upon which the SAT-M score is plotted. Notice that the null hypothesized mean score aligns with the mean z-score. Also see that the critical $\bar{x}$ aligns with the value of R. Finally, look at your sample $\bar{x}$. It is in the reject region on the standard normal curve, which is the same as saying that it is more extreme than the critical $\bar{x}$.

2. Perform a new test, this time with a significance level of $\alpha = 0.01$. What is the result of this test (reject or accept H_0)?

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3. What are the values of z, P, critical z, and critical $\bar{x}$?

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4. Did you really need to perform this test over to decide whether to reject H_0, or did the results of the previous test provide enough information? Explain.

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The above test is called a one-tailed test, because the reject area is contained in just one tail of the standard normal distribution—the upper tail, since the alternative hypothesis was H_a: $\mu > 450$. Similarly, if the alternative hypothesis had been H_a: $\mu < 450$, that would also have been a one-tailed test, since the reject region would be contained in the lower tail of the distribution. However, if the alternative hypothesis had been H_a: $\mu \neq 450$, you would have had a two-tailed test. In such a test, you don't make a claim that the mean of the population is specifically greater than, or less than, 450. Rather, you reject the null hypothesis if the sample mean were different enough from 450 in either direction.

5. Use the HP 49G to test H_0: $\mu = 450$, H_a: $\mu \neq 450$, at the 0.05 level of significance based on the same sample. What is the result of this test (reject or accept H_0)? Don't forget to look at the graph for this test. Do you see why it is called a two-tailed test? ______________________

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   ____

6. Record the upper and lower critical $\bar{x}$ values, and the critical z-scores.

   ____

7. Use the HP 49G to compute the 0.95 level confidence interval for μ based on this sample. Write down this interval along with the critical z-scores. ______________________________

____

8) Use the HP 49G to test H_0: $\mu = 450$, H_a: $\mu \neq 450$, at the 0.01 level of significance based on the same sample. Record the upper and lower critical $\bar{x}$ values, and the critical z-scores.

____

9) Use the HP 49G to compute the 0.99 level confidence interval for μ based on this sample. Write down this interval along with the critical z-scores. ______________________________

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10) Write a short paragraph about the connection between a two-tailed test of significance based on μ, and a confidence interval for μ. ______________________________

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Teacher Notes

This activity describes in some detail the ideas behind significance testing of claims about a mean. After a discussion based on an example, the HP 49G is used to recompute the same example. Students are also asked to compute confidence intervals for a population mean. This is done to reinforce the previous activity, and to make the connection between confidence intervals and two-tailed tests of significance. A known population standard deviation is assumed throughout this activity so that the standard normal distribution can be used. This assumption is dropped in the next activity.

Materials needed

At least one HP 49G per group.

Terms introduced

Test of significance, null hypothesis, alternate hypothesis, significance level, P-value, critical $\bar{x}$*, one-tailed test, two-tailed test.*

Calculator functions introduced

Test of hypothesis: Z-Test: 1 μ

Solutions to Selected Exercises

All numerical answers rounded to three decimal places

1. $z = 1.905, P = 0.028$, *critical* $z = 1.645$, *critical* $\bar{x} = 459.497$.

2. *This test advises to accept* H_0.

4. *No,* z *and* P *do not depend on* α*. In this case, reject* H_0 *when* $P = 0.028 < \alpha$*. Because* $\alpha = 0.01$*, you accept* H_0.

5. *This test advises to accept* H_0*. Looking at the graph, the two reject regions (one at each tail) are visible.*

6. *Critical* $\bar{x} = \{449.6, 472.3\}$*, critical* $z = \{-1.960, 1.960\}$*. Also note that the P-value for this two-tailed test is twice the P-value for the one-tailed test in question 1.*

7. *The 95% confidence interval for* μ *is* $(449.684, 472.316)$.

8. *Critical* $\bar{x} = \{446.1, 475.8\}$*, critical* $z = \{-2.576, 2.576\}$.

9. *The 99% confidence interval for* μ *is* $(446.128, 475.871)$.

10. *A two-sided significance test at level* α *rejects a hypothesis* $H_0\!: \mu = \mu_0$ *exactly when the value of* μ_0 *falls outside a level–*α *confidence interval for* μ.

Extensions

Discuss Type I and Type II errors.

Discuss the role of sample size on tests of significance and confidence intervals.

Practical Significance Testing

Speeders and Lightbulbs

In the previous activity, the significance testing procedure used the standard normal distribution to compute a z-score based upon a random sample, then compare it to a critical z-score based upon α, the desired level of significance. This use of the standard normal distribution depends upon knowing σ, the population standard deviation. In practice, it is quite rare to know σ. You were in a similar situation when you considered confidence interval. In that case, you used s, the sample standard deviation in place of the population standard deviation, which necessitated using a t-distribution in place of the standard normal distribution. The same situation exists for significance testing.

Significance testing with an unknown population standard deviation

The speeds of 25 cars traveling past a point on a country road is measured on a random sunny day. These speed are given below. (This is the same sample used in the activity on confidence intervals.)

56	49	44	61	67	43	51	53	57	60	59	47	55
58	67	43	66	62	64	51	61	61	48	56	62	

1. Find the mean and standard deviation for this sample.

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Residents along the road claim that the average car goes faster than the posted 55 MPH speed limit. The mean computed above does not really provide convincing evidence that the claim is true, so you decide to apply a test of significance at the $\alpha = 0.10$ level.

2. What should the null hypothesis be for this test? Be sure to write it using the standard symbolism.

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3. What should the alternative hypothesis be for this test? Be sure to write it with standard symbolism.

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4. Will this be a one-tailed or two-tailed test?

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Turn on the HP 49G, press [→][STAT]. Select the **Hypoth. Tests...** option. From the **Hypothesis Tests** choose box, select **T-Test: 1μ**, which will take you to an input screen. Enter the proper values in each of the fields (and note the explanation that appears on the bottom of the screen when you highlight each field). When you're finished, press **OK**.

5. Would you accept or reject the null hypothesis? ______________________________

6. Does this support the residents' claims? Why or why not? ______________________________

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Remember to look at the graph of this situation.

Testing light bulbs

You are the quality control manager for the Bright Idea Lightbulb Company. Printed on the outside package of each lightbulb is the claim "Average life 1000 hours." You decide to test the claim by taking a random sample of 10 bulbs off the production line and measuring their life. The results (in hours) are:

989 1012 997 997 999 992 994 996 998 996

7. Find the mean and standard deviation for this sample. ______________________________

You decide to perform a test of significance at the $\alpha = 0.05$ level based on this sample.

8. What should the null hypothesis be for this test? Be sure to write it using the standard symbolism.

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9. The packaging will be changed if the population mean life turns out to be significantly different from 1000 hours (either smaller or larger). What should the alternative hypothesis be for this test?

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10. Will this be a one-tailed or two-tailed test? ______________________________

11. Do you accept or reject the null hypothesis? ______________________________

12. Do you have to change the packaging? ______________________________

13. The company president wants a report of the significance test—write it below. Be sure to include information about sample size, test method, the values of the sample statistics, and all of the values (and their meanings) that the calculator returned. ______________________________

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14. What is the 95% confidence interval for the mean bulb life of the population (in hours)? __________

Teacher Notes

In this activity, tests of significance about a population mean are applied in situations where the population standard deviation is not known. This is far more realistic than the previous activity. Due to the work already done with confidence intervals, the t statistic is used without much fanfare. Both situations in this activity involve a claim about a population mean that appears to be supported by a sample mean. These sample mean differences do not turn out to be statistically significant.

Materials Needed

At least one HP 49G per group.

Calculator functions introduced

Test of hypothesis: Z-Test: 1μ

Solutions to Selected Exercises

All numerical answers rounded to three decimal places

1. $\bar{x} = 56.04$ MPH, $s = 7.413$.
2. H_0: $\mu = 55$.
3. H_a: $\mu > 55$.
4. This is a one-tailed test.
5. Accept H_0 ($P = 0.245$).
6. This test does not support the residents claims because the significance test advises the acceptance of H_0: $\mu = 55$. In other words, even though the sample mean speed was over 55 MPH, the difference is not statistically significant.
7. $\bar{x} = 997$ hours, $s = 6.055$.
8. H_0: $\mu = 1000$
9. H_a: $\mu \neq 1000$
10. This is a two-tailed test.
11. Accept H_0 ($P = 0.152$).
12. The packaging can stay; the significance test advises the acceptance of H_0: $\mu = 1000$. The sample mean life was less than 1000 hours (9 bulbs out of the 10 samples had a life less than 1000 hours), but the difference is not statistically significant.

14. *The 99% confidence interval for μ is (992.668, 1001.331). Be careful to use the* `T-Int: 1μ` *confidence interval option. Notice that the sample mean lies within this interval.*

Extensions

Discuss the role of the significance level. For what kinds of tests would one require a high level of significance? Medical trials and aviation testing comes to mind....

Because of variability associated with human behavior, many claims regarding psychological and education treatment are held to a much lower level of significance (0.10 or even 0.20). Discuss this idea.

Ask students to try to think of claims that they would like to test. Encourage them to formulate testable questions, construct surveys, collect data, and analyze the results. Be careful that student generated claims do not involve the comparison of two sample means.

Comparing Two Means

State SAT-M Scores and Lightbulb Improvements

So far, all of the significance tests you have performed have been on a single mean. But suppose you want to compare the means of two different populations? Significance tests exist for this situation, too. One test, based on the standard normal distribution, requires that both population standard deviations be known. The other test, based on a t-distribution, does not require that. In either case, though, the tests each have these two requirements: ***(i)*** Each of the samples must be from distinct populations; ***(ii)*** The quantity being measured in one group must be independent of the quantity being measured in the other group.

Comparing two means with known population standard deviations

Suppose you want to compare the SAT-M scores from high school seniors in Illinois (population #1), to the SAT-M scores from high school seniors in California (population #2). This is possible, because the situation satisfies the above two criteria: Each population is distinct (since they have no common members), and the results are independent, since the scores from one state do not depend on the scores from another state.

Here are the results of random samples of seniors' test scores in each state:

	IL	**CA**
Mean Score	482	473
Sample Size	150	120

Suppose that each population standard deviation is 100 (although they don't have to be the same to apply the test), and that you want a significance level of $\alpha = 0.20$. Keep in mind that you're interested only in whether the population means are different; you are not testing whether a particular one is larger.

1. What should the null hypothesis be for this test? Be sure to write it in the standard symbolism.

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2. What should the alternative hypothesis be for this test? Be sure to write it in the standard symbolism.

   ____

3. Will this be a one-tailed or two-tailed test? ______________________________

Turn on the HP 49G and press [→][STAT]. Select the `Hypoth. Tests...` option. From the `Hypothesis Tests` choose box, select `Z-Test:μ1-μ2`, which will take you to an input screen. Enter the proper values in each of the blanks (noting the explanations that appear on the bottom of the screen when you highlight a input blank). When you finish, press OK.... You should see that the results of the test are to accept the null hypothesis. In this case, you'd say that the difference in mean score on the SAT-M of the high school seniors from Illinois compared to that from California is not statistically significant at the $\alpha = 0.20$ level.

Comparing two means with unknown population standard deviations

The engineering department from the Bright Idea Lightbulb Company has devised a new manufacturing process that they think will increase the life of the light bulbs, while costing no more than the old process. As quality control manager, you select a random sample of 10 bulbs from each manufacturing process. Each bulb's life is measured in hours. Here are the results:

Sample 1 (old process)	990	1010	997	996	1001	992	995	996	994	996
Sample 2 (new process)	1021	1012	1015	1025	1019	1017	1022	1019	1017	1018

4. Compute the mean and standard deviation for each sample. ____________________

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The mean of sample 2 surely looks higher than that of sample 1, but you decide to test to see if the life of the bulbs from the new process is significantly higher than those from the old process at the $\alpha = 0.05$ level. (Remember: A significance level of 0.05 means that you are requiring evidence against H_0 that is so strong that the test would give wrong results no more than 5% of the time—1 time in 20—if H_0 is in fact true.

5. What should the null hypothesis be for this test? Be sure to write it using the standard symbolism.

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6. What should the alternative hypothesis be for this test? Be sure to write it in standard symbolism.

____

7. Will this be a one-tailed or two-tailed test? ____________________

Perform the test using the HP 49G the same way as before, except this time choose the **T-Test: $\mu_1 - \mu_2$** option.

8. Does the new manufacturing process result in significantly ($\alpha = 0.05$) longer-lasting bulbs?

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9. The president of the company wants a report on the new manufacturing process. Write this report below. Be sure to include information about sample size, test method, the values of the sample statistics, and all of the values (and their meanings) that the calculator returned. ____________________

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Teacher Notes

The purpose of this activity is to show how a test of significance can be applied to comparing two means. As in previous activities, when both population standard deviations are known, the standard normal distribution can be used. When population standard deviation are unknown, the t distribution must be used.

Materials Needed

At least one HP 49G per group.

Calculator Functions Introduced

Significance tests: Z-Test: $\mu_1 - \mu_2$, *T-Test:* $\mu_1 - \mu_2$

Solutions to Selected Exercises

1. H_0: $\mu_1 = \mu_2$
2. H_a: $\mu_1 \neq \mu_2$
3. *This is a two-tailed test.*
4. *For sample 1:* $\bar{x} = 996.7,\ s = 5.519$
   For sample 2: $\bar{x} = 1018.5,\ s = 3.659$
5. H_0: $\mu_1 = \mu_2$
6. H_a: $\mu_1 < \mu_2$
7. *This is a one-tailed test. Note that you are not doing a pooled analysis. (See Moore & McCabe, 1993, p. 542.)*
8. *Yes, the test advises to reject* H_0.

Extensions

A common mistake is to use the significance test for the comparison of two means to analyze a pre-test, post-test situation. For instance, suppose different, but parallel, SAT tests are given to the same class. One is given at the beginning of an SAT preparation class, the other at the end. The means from these two tests cannot be compared using the type of test in this activity, because the situation does not fulfill the two requirements stated in the activity. Instead, a matched pairs analysis must be used. Discuss this common mistake and introduce the technique of matched pairs (see Moore & McCabe, 1993, p. 506).

Ask students to try to think of claims comparing two means that they would like to test. For instance, do seniors make more per hour in part time jobs than juniors? Encourage them to formulate testable questions, construct surveys, collect data, and analyze the results. Be careful that student-generated claims fulfill the two requirements needed for this type of test.

There are tests of significance that apply to statistics other than the mean. The HP 49G can also perform significance tests for proportions (see Moore & McCabe, 1993, p. 574).