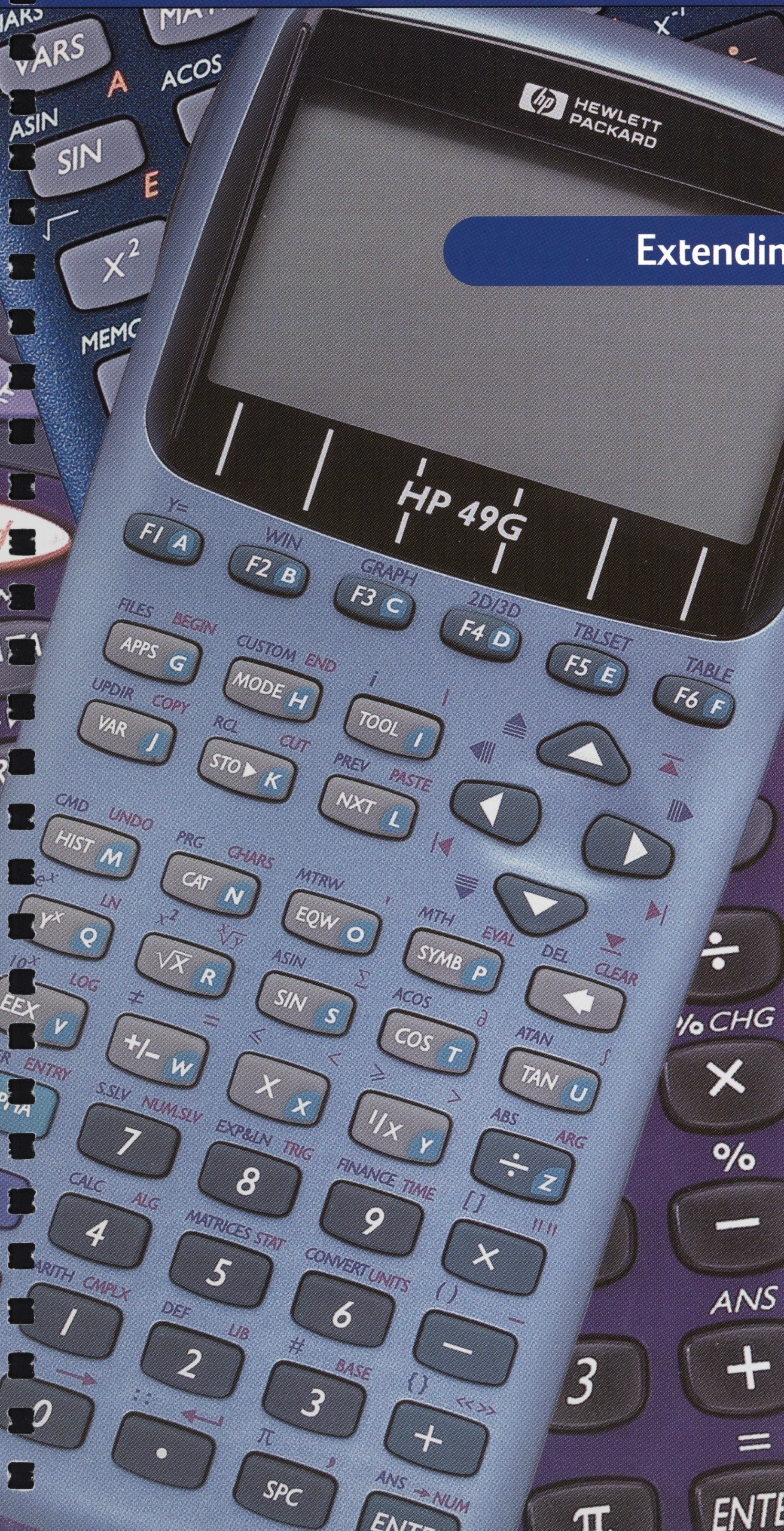
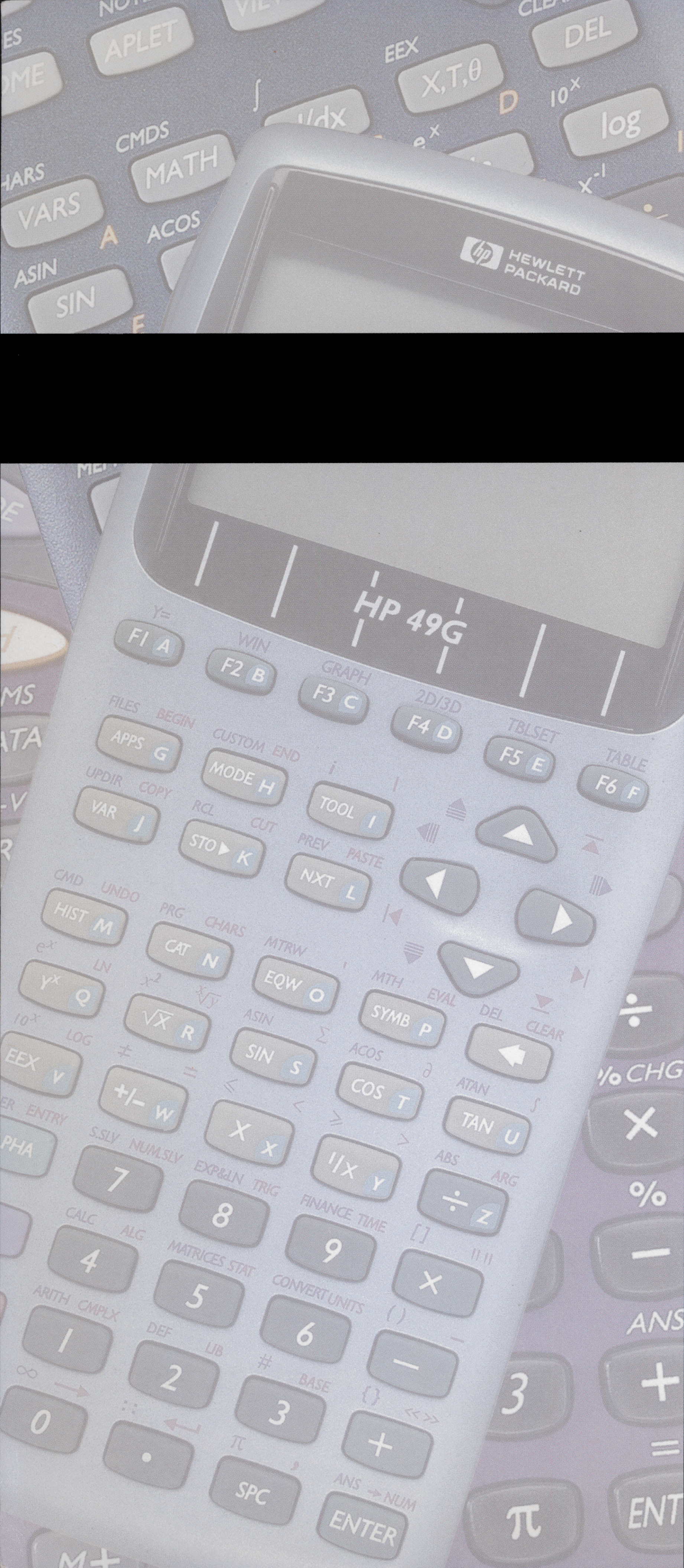
Extending Limits in Mathematics



Classroom

math activities

► with the HP 49G

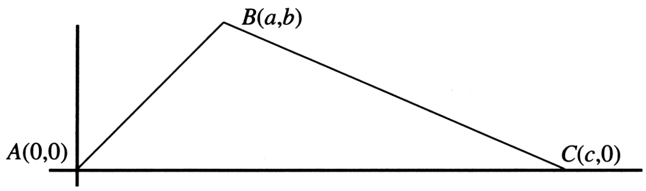


HEWLETT PACKARD

Extending Limits in Mathematics
Classroom Math Activities
with the HP 49G

A Coordinate Geometry Proof

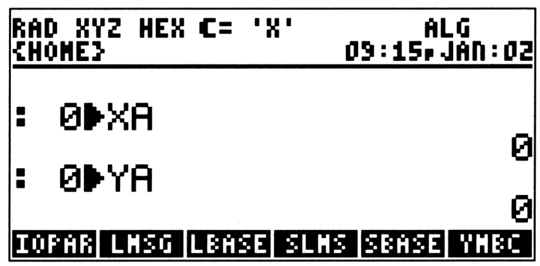
Did you realize that you can use your HP 49G to help prove theorems in coordinate geometry? You can! For example: Prove that in any triangle, the segment connecting the midpoints of any two sides is parallel to the third side and had a length equal to $^1/_2$ the third side.



In other words, using the above triangle, show that the segment connecting the midpoints of AB and BC is parallel to AC and has a length equal to $^1/_2$ AC.

The first step is to store each of the coordinate values of the labeled diagram into its own variable.

To store the *x*- and *y*-coordinates of point *A* into the variables XA and YA, do this: [0] [STO▶] [X][ALPHA][A] [ENTER]; and [0] [STO▶] [ALPHA][Y][ALPHA][A] [ENTER].

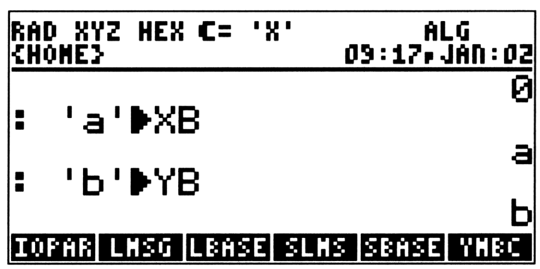


Notice how your inputs echo on the left side, and the results appear on the right.

(Also, notice that if you press [VAR], you will see your variables appear on that menu as you store them.)

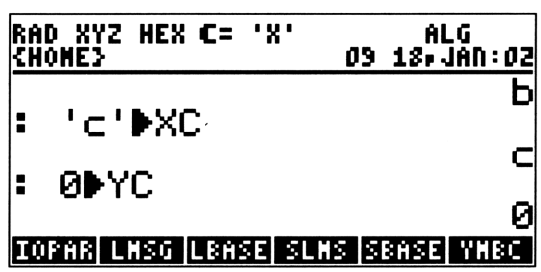
Similarly, store the coordinate values of point *B* (*a* and *b*) into the variables XB and YB, as follows:

[↱]['] [ALPHA][↰][A] [▶] [STO▶][X][ALPHA][B] [ENTER]; and [↱]['] [ALPHA][↰][B] [▶] [STO▶][ALPHA][Y][ALPHA][B] [ENTER].



Finally, store the coordinate values of point *C* (*c* and 0) into the variables XC and YC:

[↱]['] [ALPHA][↰][C] [▶] [STO▶][X] [ALPHA][C] [ENTER]; and [0] [STO▶] [ALPHA][Y] [ALPHA][C] [ENTER].

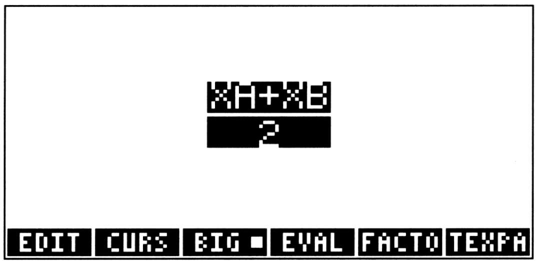


Notice in your variable (VAR) menu, as the variable list updates, the newer items appear on the left, while older ones move to the right. To see items other than the most recently stored six, simply press NXT—repeatedly, if necessary, until you see all items and return to the menu's first "page."

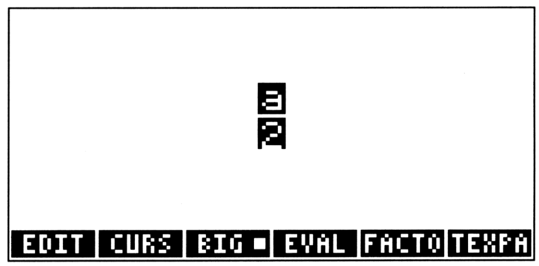
Now that you have all the individual coordinates stored, you can calculate the coordinates of the midpoint of side *AB* of the triangle. Start with the x-coordinate:

Press EQW to go into the Equation Writer, then type X ALPHA A + X ALPHA B ▶.

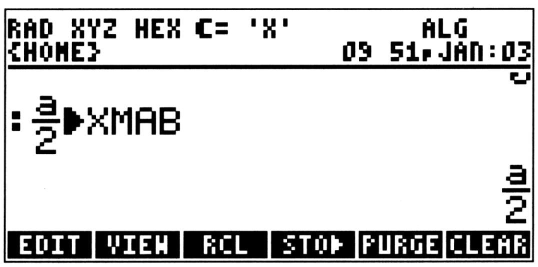
This should produce and highlight this expression:



Now complete the midpoint formula, by typing ÷ 2 ▶.



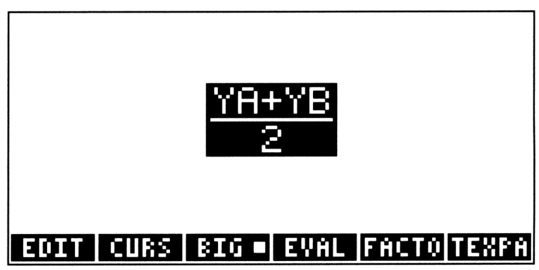
Use ↱ EVAL to perform and simplify the calculation:



Save this result in its own variable name, XMAB ("the X-coordinate of the Midpoint of segment AB"):

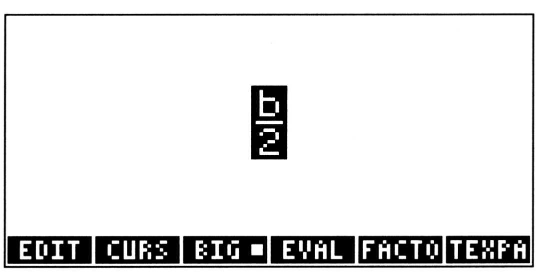
ENTER STO▶ X ALPHA ALPHA M A B ENTER.

Now, go back to the Equation Writer to find the y-coordinate of the midpoint similarly:



EQW ALPHA Y ALPHA A + ALPHA Y ALPHA B ▶ ÷ 2 ▶...

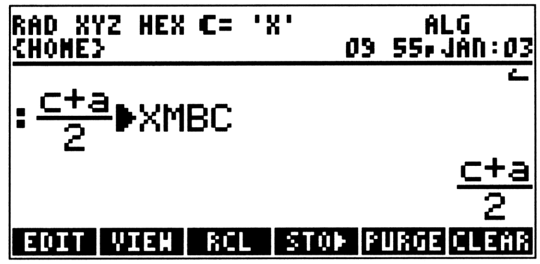
↱ EVAL...



...ENTER STO▶ ALPHA ALPHA Y M A B ENTER.

That takes care of the coordinates of the midpoint of *AB*. The next step is to do likewise for the coordinates of the Midpoint of segment BC of the triangle.

[EQW] [X][ALPHA][B] [+] [X][ALPHA][C] [▶] [÷] [2] [▶]
[↱][EVAL] [ENTER] [STO▶][X][ALPHA][ALPHA][M][B][C] [ENTER].



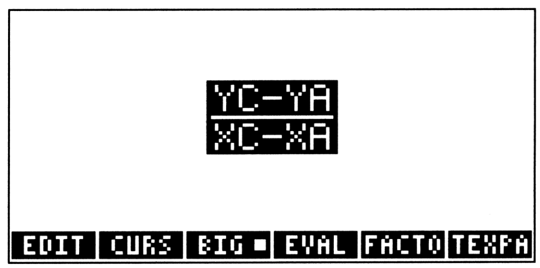
(Again, notice the variables appearing on the VAR menu as they are created.)

[EQW] [ALPHA][Y][ALPHA][B] [+] [ALPHA][Y][ALPHA][C]
[▶] [÷] [2] [▶] [↱][EVAL] [ENTER]
[STO▶][ALPHA][ALPHA][Y][M][B][C] [ENTER].

Now that the values are all stored, you can prove the property asserted by the theorem. For example, to demonstrate that the segment connecting the midpoints is parallel to the base, find and compare the slopes.

Start with the slope of the base:

[EQW] [ALPHA][Y][ALPHA][C] [−]
[ALPHA][Y] [ALPHA][A] [▶] [▶] [÷] [X][ALPHA][C]
[−] [X][ALPHA][A] [▶] [▶] [▶].

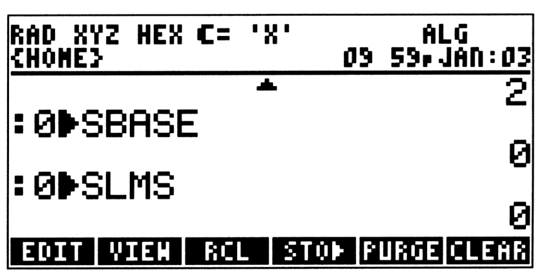


Fnd the result and store it in an aptly named variable:

[↱][EVAL] [ENTER] [STO▶][ALPHA][ALPHA][S][B][A][S][E]
[ENTER].

Now calculate and store the slope of the line connecting the midpoints:

[EQW] [ALPHA][ALPHA][Y][M][B][C][ALPHA] [−]
[ALPHA][ALPHA][Y][M][A][B][ALPHA] [▶] [▶]
[÷] [ALPHA][ALPHA][X][M][B][C][ALPHA]
[−] [ALPHA][ALPHA][X][M][A][B][ALPHA] [▶] [▶] [▶]
[↱][EVAL] [ENTER] [STO▶][ALPHA][ALPHA][S][L][M][S] [ENTER].

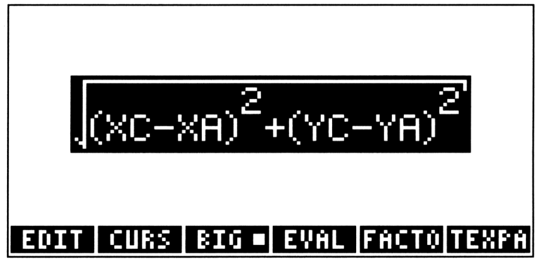


Property confirmed: The two slopes are equal, so those two segments are parallel!

What about the other assertion of the theorem—the relationship of the lengths of these two segments?

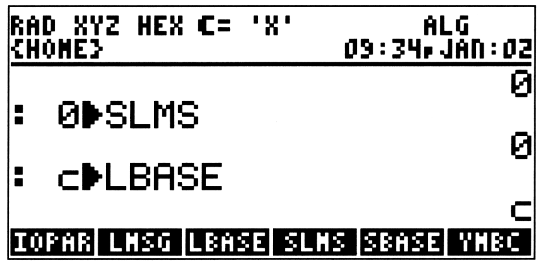
Go to the Equation Writer and use the general formula for the distance between two points, plugging in the variables you created.

Try the Base Length first: $\sqrt{(XC - XA)^2 + (YC - YA)^2}$



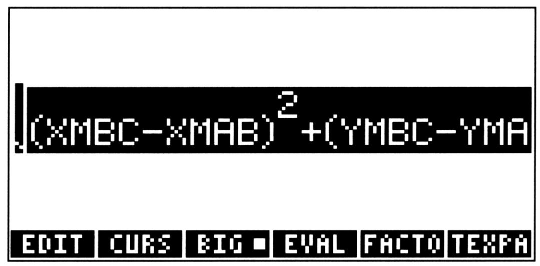
[EQW] [√x] [X] [ALPHA] [C] [−] [X] [ALPHA] [A] [▶] [▶] [y^x] [2] [▶] [+] [ALPHA] [Y] [ALPHA] [C] [−] [ALPHA] [Y] [ALPHA] [A] [▶] [▶] [y^x] [2] [▶] [▶] [▶].

Now calculate and store:



[↱] [EVAL] [ENTER] [STO▶] [ALPHA] [ALPHA] [L] [B] [A] [S] [E] [ENTER].

Do likewise for the segment connecting the midpoints:



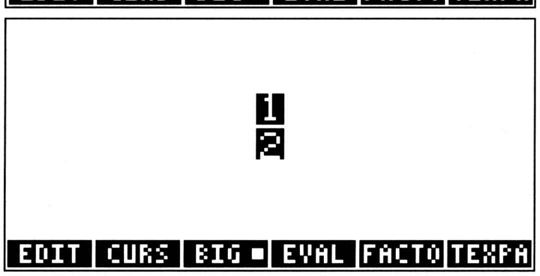
[EQW] [√x] [ALPHA] [ALPHA] [X] [M] [B] [C] [ALPHA] [−] [ALPHA] [ALPHA] [X] [M] [A] [B] [ALPHA] [▶] [▶] [y^x] [2] [▶] [+] [ALPHA] [ALPHA] [Y] [M] [B] [C] [ALPHA] [−] [ALPHA] [ALPHA] [Y] [M] [A] [B] [ALPHA] [▶] [▶] [y^x] [2] [▶] [▶] [▶].

Calculate and store:

[↱] [EVAL] [ENTER] [STO▶] [ALPHA] [ALPHA] [L] [M] [S] [G] [ENTER].

To confirm the relationship of the lengths of the two segments, just set up their ratio:

[EQW] [ALPHA] [ALPHA] [L] [M] [S] [G] [ALPHA] [÷] [ALPHA] [ALPHA] [L] [B] [A] [S] [E] [ALPHA] [▶]

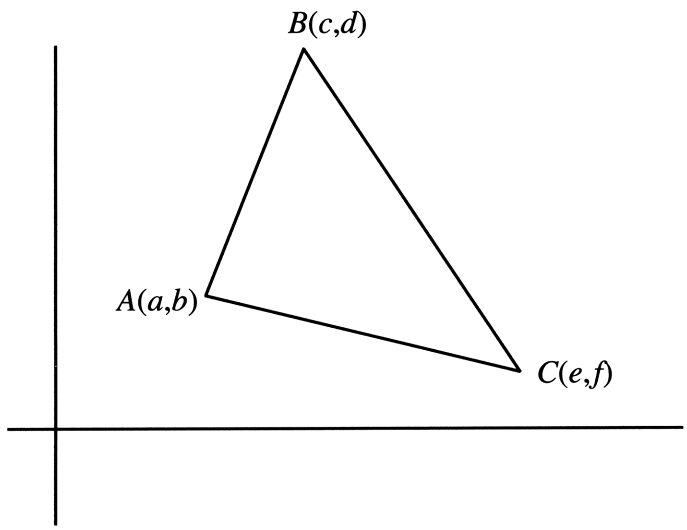


Now compute it: [↱] [EVAL]

Property confirmed: The segment connecting the midpoints of two sides of a triangle is $^1/_2$ the length of the base!

Follow-Up Activity

The proof you just completed was made simpler by a convenient choice of a triangle whose base sits on the x-axis. But of course, the proof should still hold for a triangle of any orientation, like this:



Use the above general triangle and your HP 49G to re-confirm the proof.

1. What changes might you expect in your calculations?

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2. How might you have to adjust to compare results this time?

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Teacher Notes

With its ability to work with variables and the simplicity of the Equation Writer, the HP 49G affords a great opportunity to connect the properties studied in a Geometry course with higher-level coordinate algebra. This activity walks through a coordinate proof and then provides an extension on the same topic.

While the points raised are fairly simple, the proof shows the value of using ratios to relate things instead of the usual "just show that visually the things are the same, therefore equal."

Skills Used:	*Properties of segments connecting the midpoints of a triangle* *Slope of a line segment* *Distance formula* *Midpoint formula*
Skills Introduced:	*Storing and Manipulating Variables on the HP49G* *Relating Lengths and Slopes of segments.*

Frequently in doing proofs on the calculator like this, more errors arise from labeling than from actual bad computation. Note the careful choice of meaningful variable names—and the use of single-character, lowercase names only for the coordinate values. (Note, too, that while you may choose lengthy names to clarify their contents, names that are too long won't entirely fit in a menu item on the VAR menu, thus compromising their utility as mnemonics.)

The use of the general distance formula is not necessary to find the lengths in the proof, of course. Since you're dealing with horizontal lines, you'd get the same result simply by subtracting the x-coordinates. This is worth some discussion if a student mentions it; if not, it's worth pointing out. The general calculation is definitely necessary in the follow-up activity, however. In fact, that gets rather messy, and the two distances don't necessarily appear to be all comparable upon mere inspection. The power of the machine shows the ratios to be either $^1/_2$ or 2 (depending upon the way they're set up).

As set up, the proof uses horizontal lines, so the slope values of zero should come as no surprise. Hopefully, at least one student will jump on that point quickly. In the follow-up, the non-zero slopes are still fairly simple and can, with some prodding, be seen intuitively to be equal by the better students. Better for everyone, though, is to set up those slopes in a ratio for comparison. The result should be 1. (If the results come out to be −1, make sure the discussion goes to the fact that with the slope being a difference quotient, the order of the variables is important.)

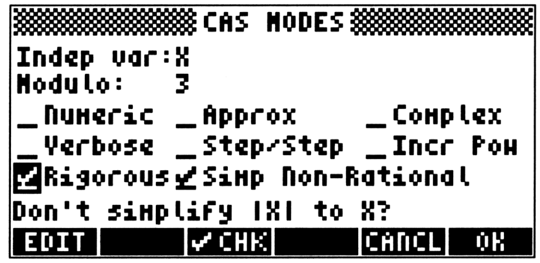
Another good discussion opportunity arises when the calculator produces an absolute value when calculating the distance results.

Three Approaches to the Derivative: Graphical, Numeric and Analytical

What does a derivative "look like?" How does it behave? Here's an exercise that gives you three different ways to think about a derivative.

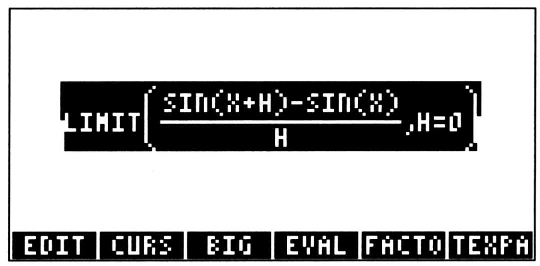
As you know, for any function, $g(x)$, its derivative, $g'(x)$, is defined as $\lim\limits_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$ for all x, if the limit exists. For the function $g(x) = \sin(x)$, this definition becomes $g'(x) = \lim\limits_{h \to 0} \dfrac{\sin(x+h) - \sin(x)}{h}$

First purge the variables X and H. Then to evaluate this limit, start by entering it into the HP 49G's Equation Writer and using the symbolic algebra tools.



First, set the modes correctly. Press MODE CAS, and make sure your screen appears as shown here:

Press OK OK to accept the mode settings, then go to the equation writer, EQW. Enter the difference quotient first: SIN X + ALPHA H ↱ ▲ − SIN X ↱ ▲ ÷ ALPHA H. Now highlight the entire expression, ↱ ▲, then apply the limit command to the highlighted expression: ↰ CALC 2 ENTER 2 ENTER ALPHA H ↱ = 0 ↱ ▲.



If BIG■ appears in the menu (i.e. if the "big" font is currently selected) change it to BIG (press it once) to select the smaller font, so that you can see the entire expression.

The calculator is fully capable of evaluating this limit—press EVAL.... Voila!

How is this done? Magic? No—people have been evaluating limits long before computer algebra systems. They simply expanded the quotient expression and looked at its behavior as H became very small....

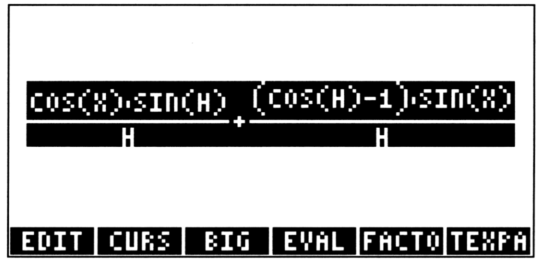
Press ↱ UNDO to recover the limit expression. Now use a trigonometric identity to expand `SIN(X+H)`. Press ▼ ▼ ▼, which will highlight `SIN(X+H)`. Then press TEXPA, which will apply the trig identity `SIN(A+B)=SIN(A)*COS(B)+COS(A)*SIN(B)`. Press ▲ ▲ to highlight the entire quotient. Now split the fraction into two terms by applying `PARTFRAC` (↰ ARITH 2 ENTER 1 4 ENTER).

Now you have the following limit expression, equivalent to your original:

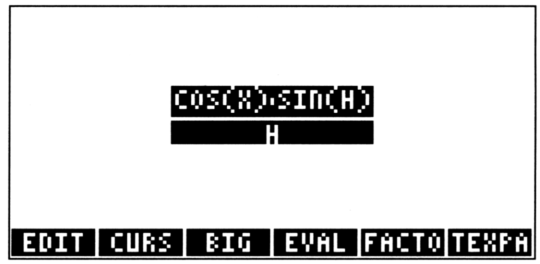
$$\lim_{H \to 0} \frac{COS(X)SIN(H)}{H} + \frac{(COS(H) - 1)SIN(X)}{H}$$

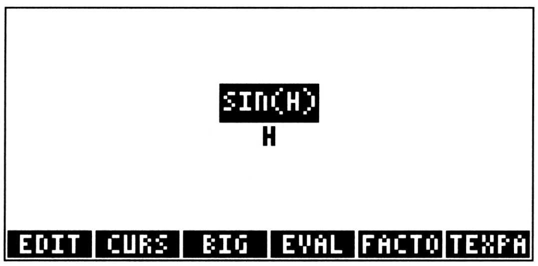
To find the limit of a sum, notice that you can find the sum of the limits (of each addend). Also, note that in the first term, COS(X) does not depend on H, so it can be factored through the limit. Similarly, in the second term, SIN(X) can be factored through. All you really need to do is find how SIN(H)/H and (COS(H)−1)/H behave for H near 0. To do that, use the calculator's table and graphing features.

First, press [→][COPY] to make a copy of the expression in the limit, as shown here.



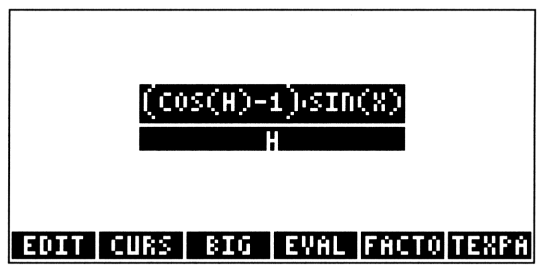
Now press [CANCEL] [←][2D/3D] and set **Type** to **Function**. Then [←][Y=] [NXT] **CLEAR** **OK** [NXT] **ADD** [◄][→][CLEAR]. In the Equation Writer, [→][PASTE] the expression you copied earlier. Now delete the second addend, [▼] [►] [←][DEL]. Your display should now appear like this:

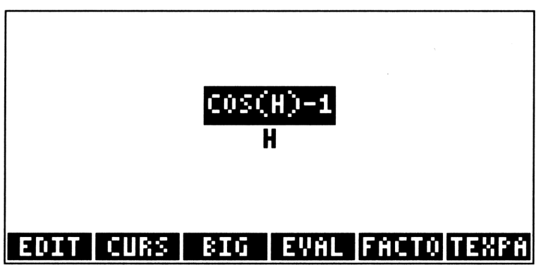




To delete the COS(X) factor, press [▼][▼] [←][DEL] [←][DEL]. Your display should appear as shown here.

Press [ENTER] to put that expression into Y1. Then press **ADD** [◄] [→][CLEAR] [→][PASTE] to use the original expression, and [▼] [←][DEL]. Your display should look like this:

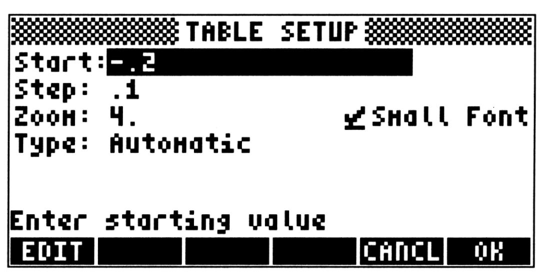




Press [▼][▼] [►] [←][DEL] [←][DEL]. Your display should appear as shown here (left).

Press [ENTER] to put that expression into Y2.

Now press [←][2D/3D] and set up your plot as shown below, left. (Be sure to change the independent variable to H.) Then press [←][TBLSET] and set up the as shown below, right.



Press (←)(TABLE) DEFN. Press ▽▽ ▷ to position the cursor on the row with H=0 and in the column with SIN(H)/H. Zoom in to the table twice, by pressing ZOOM OK ZOOM OK, and you'll see outputs for SIN(H)/H and (1-COS(H))/H for values of H near 0.

1. What does it appear $\lim_{H \to 0} \frac{SIN(H)}{H}$ is? ______________________

2. What does it appear $\lim_{H \to 0} \frac{COS(H) - 1}{H}$ is ? ______________________

Go back to your separated, factored limit expression:

$$\lim_{H \to 0} \left(\frac{SIN(X + H) - SIN(X)}{H} \right) = COS(X) \lim_{H \to 0} \frac{SIN(H)}{H} + SIN(X) \lim_{H \to 0} \left(\frac{COS(H) - 1}{H} \right)$$

Replace each limit expression on the right hand side of this equation with your answers from above.

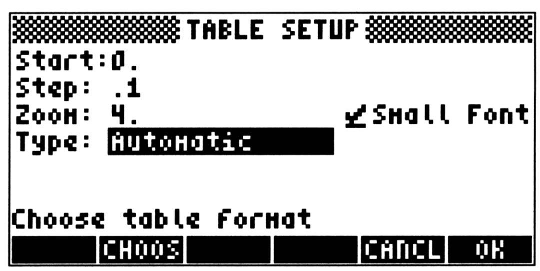
3. What is $\lim_{H \to 0} \left(\frac{SIN(X + H) - SIN(X)}{H} \right)$? ______________________

4. If $g(x) = \sin(x)$, what is its derivative, $g'(x)$? ______________________

Take a graphical and numerical look. Press (←)(2D/3D), highlight the INDEP field, and change it to X ((X) (ENTER)). Press (←)(Y=)(NXT) CLEAR OK (NXT) ADD. Enter (SIN(X+.001)-SIN(X))/.001 for Y1 . (This is the difference quotient for SIN(X), with a small value, .001, substituted for H.) Enter the expression COS(X) for Y2. Press ERASE DRAW to see the graphs.

5. Trace along the two graphs. What do you notice? ______________________

6. What does this say about the derivative of $g(x) = \sin(x)$? ______________________



Press (←)(TBLSET), and set up the table as shown here:

Press (←)(TABLE) and look at the resulting table of values.

7. What do you notice about the values for the difference quotient and COS(X)?

____

8. What does this tell you about the derivative of $\sin(x)$? ______________________

9. How could you change the definition of Y1 to make the values of Y1 and Y2 even closer?

____

Teacher Notes

This activity introduces the idea of the derivative function, and introduces connections between the graph of a function and its derivative. Students should know the definition of derivative at a point. They should also be familiar with defining functions and looking at graphs and tables before doing this activity. Most keystrokes are shown, but some familiarity with graphs and tables is assumed.

As a further exploration, students can investigate the difference in behavior between the difference quotient definition of derivative, covered here and the symmetric difference quotient, which states $g'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x - h)}{2h}$

They can do this by looking at the errors in the two approximations—subtracting the exact value of the derivative from the difference quotient (or symmetric difference quotient) approximation.

Answers

1. *1*

2. *0*

3. *cos(x)*

4. *cos(x)*

5. *The two graphs are virtually indistinguishable in the default viewing window.*

6. *This gives graphical evidence that the derivative of sin(x) is cos(x).*

7. *The outputs from the difference quotient are very close, but not exactly equal, to the outputs from cos(x).*

8. *The answer to number 7 gives numeric support to the conclusion that the derivative of sin(x) is cos(x).*

Introduction to Confidence Intervals Around a Mean

SAT-M Scores and Speeding Cars

The Scholastic Aptitude Test for Mathematics (SAT-M) is widely used to measure readiness for college math. The test is developed and tried out on a "standardization group" to adjust the scoring so that the group mean is 500 and the standard deviation is 100. The test-taking population (all students who take the test in a given year) generally has a lower mean score. In 1991, the population mean for the SAT-M was $\mu = 474$.

The reasoning behind confidence intervals

Suppose that you took a random sample of 500 students who took the SAT-M in 1991. Are you guaranteed that the mean for this sample will be $\bar{x} = 474$? Of course not. You can expect some variability in sample means from sample to sample. If you took several random samples of the population you might expect some of the sample means to be smaller than 474, and some to be larger. Recall from the central limit theorem that sample means of a particular size are distributed normally, regardless of the population distribution. Furthermore, if the population has a mean of μ and standard deviation of σ, then the sample mean distribution will be $N(\mu,\sigma/\sqrt{n})$, where n is the size of each of the samples.

Imagine that you took repeated random samples from the 1991 population of SAT-M test takers and computed the mean for each sample. Your variable would then be $\bar{x}$. You should expect $\bar{x}$ to be normally distributed with a mean of 474 (the population mean). Now assume that the population standard deviation is 100, then the standard deviation for your distribution of sample means would be:

1. You may be used to viewing $\bar{x}$ as a statistic. Now it is being referred to as a variable. Write a short paragraph explaining how you are viewing $\bar{x}$ differently now.

   ____

   ____

   ____

Recall that in a normal distribution, about 95% of the variable values will lie within two standard deviations of the distribution mean. In our example, we should expect that 95% of the means of our random samples should be within $2(4.5) = 9$ points of $\mu = 474$. To say that $\bar{x}$ lies within 9 points μ is the same as saying that μ lies within 9 points of $\bar{x}$. Thus, for 95% of the sample means, μ will be between $\bar{x} - 9$ and $\bar{x} + 9$. The interval of numbers between $\bar{x} \pm 9$ is called a 95% confidence interval for μ.

2. A random sample of 500 students from the population of all 1991 SAT-M test takers has a mean score of 480. Compute, by hand, the 95% confidence interval for μ based on this sample. ____________

Using the HP 49G calculate confidence intervals

Turn on the calculator and press (→)(STAT). Select the `Conf. Interval` option. In that choose box, select `Z-INT: 1μ.`, which will take you to an input screen. To recompute problem 2, use these values:

```
x̄: 480        n: 500
σ: 100        C: 0.95
```

Press `OK`. When the computation is finished, you will see a screen that contains two z-scores that correspond to the endpoints of the interval that contains 95% of the area under the standard normal curve. These are sometimes called the critical z-scores. You will also see numbers labeled as μ min and μ max. These correspond to the endpoints of the confidence interval. Now press `GRAPH`. This shows the standard normal curve with the critical z-scores identified. Below the curve is a scale showing the sample mean lined up with the standardized mean of 0, and μ min and μ max lined up with the critical z-scores. Press `TEXT` to return to the previous display or press `OK` to return to the stack.

3. Did the calculator return results identical to the results of your hand calculations in problem 2? Why or why not? ______________________________

   ____

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In practice, the population mean μ, is usually not known. Statisticians use confidence intervals based upon a sample mean to estimate the population mean.

4. Suppose you are interested in the 1991 SAT-M scores for the population of students that come from California. A random sample of 300 California students that took the SAT-M is examined and a sample mean of 471 is computed. Assume that the population standard deviation is still 100. What is the 95% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on this sample? ______________________________

5. What is the 99% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on the sample from problem 4? ______________________________

6. What is the 90% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on the sample from problem 4? ______________________________

7. Write a paragraph describing the effects of the confidence level on the size of the confidence interval. Why does it work this way? ______________________________

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This is a good time to mention some cautions about confidence intervals:

A 95% confidence interval for μ does not say that the probability is 0.95 that μ falls within the interval. It merely says that the interval was computed in a way that gives correct results in 95% of all of the samples. No randomness remains after a particular sample is selected; the population mean either falls within the confidence interval computed upon that sample or it doesn't.

Note that samples must be selected randomly from the population. The method of computing confidence intervals relies on the central limit theorem, and the central limit theorem applies only to random samples from a population.

So far, the method of computing confidence intervals relies on knowing the population standard deviation, σ. This is rarely true in practice. In the next section, you will see that there is a way around this limitation.

Confidence intervals in practice

In the method outlined above, deciding on a 95% confidence level determined two z-scores between which lay 95% of the area under the standardized normal curve (namely $z \approx \pm 1.96$). This is graphically illustrated by the HP 49G.

Knowing $\bar{x}$ (the sample mean), n (the sample size), and σ (the population standard deviation), you can algebraically find the value of μ for each z-score, using the relationship $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$.

These calculated values of μ could give the confidence interval for the population mean, although it is rare that you need an estimate for the population mean. However, you do know σ, the population standard deviation, and it seems natural to substitute s, the sample standard deviation, for σ in the above formula.

Doing so does not give a z-score, however. It gives a new statistic called a t-score: $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$

Unlike the z statistic, the t statistic does not have a normal distribution. It has a distribution called the t distribution. Furthermore, there is a different t distribution for every sample size n. Statisticians classify t distributions in terms of *degrees of freedom*. The degrees of freedom of a t distribution is equal to $n - 1$.

For example, suppose you want to use the mean and standard deviation from a sample of 500 to compute a 95% confidence interval for the population mean. You must find the two t-scores between which lies 95% of the area under the curve of the t distribution with 499 degrees of freedom. Fortunately, the HP 49G does most of this for you....

Revisiting problem 4: Suppose that a random sample of 300 California students who took the SAT-M is examined and a sample mean of 471 is computed. Instead of assuming a population standard deviation of 100, you now compute the standard deviation of the sample to be 92. What is the 95% confidence interval for μ, the mean of the population of California SAT-M test takers in 1991, based on this sample?

Press [→][5], select the Conf. Interval option, and choose T-INT: 1μ., which will take you to an input screen. Enter these field values: $\bar{x}$: 471 n: 300 sx: 92 C: 0.95

Press OK. When the computation is finished, you will see a screen that contains two *t*-scores that correspond to the endpoints of the interval that contains 95% of the area under the curve of the *t* distribution with 299 degrees of freedom. These are sometimes called the critical *t*-scores. You will also see numbers labeled as μ min and μ max. These correspond to the endpoints of the confidence interval.

Now press GRAPH. This shows the appropriate *t* distribution curve with the critical *t*-scores identified. Below the curve is a scale showing the sample mean lined up with the *t* distribution mean of 0, and μ min and μ max lined up with the critical *t*-scores. Notice the *t* distribution looks very similar to the normal distribution. In fact, the *t* distribution resembles the normal distribution more closely as the number of degrees of freedom increases. Press TEXT to return to the previous display or press OK to return to the stack.

The speeds of 25 cars traveling past a point on a country road are measured on a sunny day. These speed (in MPH) are given here:

56	49	44	61	67	43	
51	53	57	60	59	47	
55	58	67	43	66	62	
64	51	61	61	48	56	62

8. Find the 95% confidence interval for μ based on this sample. ______________________________

9. Find the 99% confidence interval for μ based on this sample. ______________________________

10. Find the 90% confidence interval for μ based on this sample. ______________________________

11. Does the confidence level have the same general effect on the size of the confidence interval as in question 7? ____

12. Are these estimated values for μ good indications of the mean speed of all of the cars that have ever passed this point in the road? Write a paragraph explaining why or why not.

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Teacher Notes

This activity may represent students' first experience with inferential statistics. It represents a substantial leap forward from the last activity because it assumes several statistical concepts, including: population and sample means and standard deviations; experience with finding areas under the standard normal distribution; and a conceptual awareness of the central limit theorem. The reasoning behind the process of finding confidence intervals around a population mean is discussed at length before the HP 49G is used. Confidence intervals around a population mean for an unknown population standard deviation are also discussed, necessitating the introduction of the t distribution.

Materials needed: *At least one HP 49G per group.*

Terms introduced: *Central limit theorem, confidence level, confidence interval around a mean, critical z-scores, t-score, t distribution, degrees of freedom, critical t-score.*

Calculator functions introduced: *Confidence intervals: Z-INT: 1 μ, T-INT: 1 μ, confidence interval graphs*

Solutions to Selected Exercises

1. *You are now analyzing a set of data that represents many sample means.*
2. *(471, 489)*
3. *The area under the normal curve between two standard deviations below the mean to two standard deviations above the mean is only approximately 95%. Look at the critical t-scores returned by the calculator. Round-off error in computing σx in question 2 also contributes to differences in the two answers.*
4. *To three decimal places: (459.684, 482.316)*
5. *To three decimal places: (456.128, 485.871)*
6. *To three decimal places: (461.503, 480.497)*
8. *To three decimal places: (52.980, 59.100)*
9. *To three decimal places: (51.893, 60.187)*
10. *To three decimal places: (53.503, 58.577)*

Note: the sample mean and standard deviation must be computed before these confidence intervals can be computed.

12. *No, this sample is biased toward good road conditions.*

Extensions

Ask students to think of a survey question to ask students in the school upon which a mean confidence interval could be computed.

Determine a random sample, conduct the survey, collect data, and analyze the results.

Introduction to Significance Testing

More SAT-M Scores

Statistical inference has two major goals. One is the estimation of population statistics by computing confidence intervals based upon sample statistics. The other goal is the assessment of statements about a population. Such statements can be examined through tests of significance, a process illustrated here.

The reasoning behind tests of significance

While discussing SAT-M scores, a teacher states that these scores overestimate the ability of typical high school seniors, because the test is taken by predominantly college bound students. The teacher goes on to claim that if all students took the SAT-M, the mean would be no more than 450, not the national mean of 471. You disagree. You think that the mean would be bigger than 450. How can your claim be tested?

One way to test the claim is to test every single high school senior in the nation and compute the mean. Of course, this would cost millions of dollars and require the cooperation of millions of people. Fortunately, there is a way to test this claim based on a random sample—at a certain level of statistical significance (more on that idea later).

Suppose that a random sample of 300 high school seniors is selected nationwide and given the SAT-M (if they did not already take it). Suppose that the mean for this sample is 461. Does this information support your claim on its own? Not really. It is possible that a different random sample might yield a mean of 443, due to natural variability. So how do you know if the sample mean of 461 really supports the claim that if all students took the SAT-M, the mean would be greater than 450?

The first step, is to state the null hypothesis (abbreviated H_0)—the statement that significance testing will allow you to accept or reject. H_0 is the claim that you're trying to find evidence *against*—usually a statement of no effect, or no difference. In this case, it is: "The mean SAT-M score of all U.S. high school seniors is equal to 450." Symbolically, you would write H_0: $\mu = 450$. Notice that the null hypothesis is stated relative to a population variable. You already know that the mean of your random *sample* is 461, not 450. You're trying to determine if this is enough evidence to conclude that the *population* mean is not 450.

The next step is to state the opposing, or alternative, hypothesis (abbreviated H_a)—the statement you're seeking evidence for. In this case, H_a is: "The mean SAT-M score of all high school seniors is greater than 450." Symbolically, you would write H_a: $\mu > 450$.

Next, you specify the significance level of the test (denoted as α)—the "confidence" requirement you wish to put on the data. For example, if you choose a significance level of $\alpha = 0.05$, you are requiring that the data provide evidence against H_0 that is so strong that the test would give wrong results (this is, cause you to reject a true H_0) no more than 5% of the time (or 1 time in 20). Common significance levels are 0.10, 0.05 and 0.01.

Finally, suppose you know that the SAT-M scores are normally distributed with a population standard deviation of 100.

You're now ready to apply a test of significance. To summarize: H_0: $\mu = 450$ H_a: $\mu > 450$
$\sigma = 100$ $\bar{x} = 461$

Now, as you recall, a z-score is a measure of how far the sample mean is from a claimed population mean.

So you can compute a z-score based on the above information: $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{461-450}{100/\sqrt{300}} \approx 1.905$

Is this difference extreme enough to reject the null hypothesis? You may recall (or can verify from a table) that about 95% of the area under the standard normal curve lies below $z = 1.645$. That means that about 5% of the area under the standard normal curve lies above $z = 1.645$. Thus $z = 1.645$ is called the critical z-score for a significance level of $\alpha = 0.05$. The z-score you computed based on your data is 1.905, which is greater than the critical z-score of 1.645. Therefore, there is enough evidence to reject H_0. We can say, at a 0.05 level of significance, that the mean SAT-M score of all high school seniors is more than 450.

Using the HP 49G to test significance

The purpose of the previous section was to provide the reasoning behind the process of significance testing. In reality, most statisticians perform these tests using a computer or calculator. Try redoing the previous problem using the HP 49G. This will illustrate the steps involved in using the significance testing features, and the graphical display should clarify the ideas behind that testing.

Turn on the calculator and press [→][STAT]. Select the `Hypoth. Tests...` option and get the `Hypothesis Tests` choose box. (Significance tests are sometimes called hypothesis tests.) Select `Z-Test: 1μ.`, which will take you to an input screen. Then, to recompute the above problem, just enter these values into their respective fields: `μ: 450` `x̄: 461` `σ: 100` `n: 300` `α: 0.05`

Press `OK`. You'll then see the `Alternative Hypothesis` choose box. Select `μ>450` and press `OK`. After some calculation, the calculator will return the z-score associated with $\bar{x} = 461$ (this should be familiar), a statistic known as the P-value, the critical z-score (also explained above), and a statistic called the critical $\bar{x}$. Notice that the very top of this window tells you to reject $\mu = 450$, the null hypothesis, H_0. This is that same result you got by hand.

1. Record the values of z, P, critical z, and critical $\bar{x}$ for this test.

___

The P-value is the probability, assuming H_0 is true, that the test will indicate that you should reject H_0. The smaller the P-value, the stronger the evidence against H_0. In fact, you reject H_0 precisely when $P < \alpha$. Another way to decide whether to reject H_0 is by looking at the critical $\bar{x}$. The critical $\bar{x}$ is the value of the sample mean that is on the boarderline in terms of providing evidence to reject or not reject H_0. In this case, your measured sample mean is more extreme (farther from $\mu = 450$) than the critical $\bar{x}$, so you reject H_0.

Now press GRAPH. You'll see the standard normal curve. Notice the value of R shown on the curve with the arrow pointing to the right. This shows the reject region for this test.

Under the standard normal curve is a different scale, upon which the SAT-M score is plotted. Notice that the null hypothesized mean score aligns with the mean z-score. Also see that the critical $\bar{x}$ aligns with the value of R. Finally, look at your sample $\bar{x}$. It is in the reject region on the standard normal curve, which is the same as saying that it is more extreme than the critical $\bar{x}$.

2. Perform a new test, this time with a significance level of $\alpha = 0.01$. What is the result of this test (reject or accept H_0)?

3. What are the values of z, P, critical z, and critical $\bar{x}$?

4. Did you really need to perform this test over to decide whether to reject H_0, or did the results of the previous test provide enough information? Explain.

The above test is called a one-tailed test, because the reject area is contained in just one tail of the standard normal distribution—the upper tail, since the alternative hypothesis was H_a: $\mu > 450$. Similarly, if the alternative hypothesis had been H_a: $\mu < 450$, that would also have been a one-tailed test, since the reject region would be contained in the lower tail of the distribution. However, if the alternative hypothesis had been H_a: $\mu \neq 450$, you would have had a two-tailed test. In such a test, you don't make a claim that the mean of the population is specifically greater than, or less than, 450. Rather, you reject the null hypothesis if the sample mean were different enough from 450 in either direction.

5. Use the HP 49G to test H_0: $\mu = 450$, H_a: $\mu \neq 450$, at the 0.05 level of significance based on the same sample. What is the result of this test (reject or accept H_0)? Don't forget to look at the graph for this test. Do you see why it is called a two-tailed test?

6. Record the upper and lower critical $\bar{x}$ values, and the critical z-scores.

7. Use the HP 49G to compute the 0.95 level confidence interval for μ based on this sample. Write down this interval along with the critical z-scores. ______________________________

____

8) Use the HP 49G to test H_0: $\mu = 450$, H_a: $\mu \neq 450$, at the 0.01 level of significance based on the same sample. Record the upper and lower critical $\overline{x}$ values, and the critical z-scores.

____

9) Use the HP 49G to compute the 0.99 level confidence interval for μ based on this sample. Write down this interval along with the critical z-scores. ______________________________

____

10) Write a short paragraph about the connection between a two-tailed test of significance based on μ, and a confidence interval for μ. ______________________________

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Teacher Notes

This activity describes in some detail the ideas behind significance testing of claims about a mean. After a discussion based on an example, the HP 49G is used to recompute the same example. Students are also asked to compute confidence intervals for a population mean. This is done to reinforce the previous activity, and to make the connection between confidence intervals and two-tailed tests of significance. A known population standard deviation is assumed throughout this activity so that the standard normal distribution can be used. This assumption is dropped in the next activity.

Materials needed: *At least one HP 49G per group.*

Terms introduced: *Test of significance, null hypothesis, alternate hypothesis, significance level, P-value, critical $\bar{x}$, one-tailed test, two-tailed test.*

Calculator functions introduced: *Test of hypothesis: Z-Test: 1 μ*

Solutions to Selected Exercises

1. $z = 1.905$, $P = 0.028$, critical $z = 1.645$, critical $\bar{x} = 459.497$. (All numerical answers here are rounded to 3 decimal places.)

2. This test advises to accept H_0.

4. No, z and P do not depend on α. In this case, reject H_0 when $P = 0.028 < \alpha$. Because $\alpha = 0.01$, you accept H_0.

5. This test advises to accept H_0. Looking at the graph, the two reject regions (one at each tail) are visible.

6. Critical $\bar{x} = \{449.6, 472.3\}$, critical $z = \{-1.960, 1.960\}$. Also note that the P-value for this two-tailed test is twice the P-value for the one-tailed test in question 1.

7. The 95% confidence interval for μ is $(449.684, 472.316)$.

8. Critical $\bar{x} = \{446.1, 475.8\}$, critical $z = \{-2.576, 2.576\}$.

9. The 99% confidence interval for μ is $(446.128, 475.871)$.

10. A two-sided significance test at level α rejects a hypothesis H_0: $\mu = \mu_0$ exactly when the value of μ_0 falls outside a level–α confidence interval for μ.

Extensions

Discuss Type I and Type II errors.

Discuss the role of sample size on tests of significance and confidence intervals.

Practical Significance Testing

Speeders and Lightbulbs

In the previous activity, the significance testing procedure used the standard normal distribution to compute a z-score based upon a random sample, then compare it to a critical z-score based upon α, the desired level of significance. This use of the standard normal distribution depends upon knowing σ, the population standard deviation. In practice, it is quite rare to know σ. You were in a similar situation when you considered confidence interval. In that case, you used s, the sample standard deviation in place of the population standard deviation, which necessitated using a t-distribution in place of the standard normal distribution. The same situation exists for significance testing.

Significance testing with an unknown population standard deviation

The speeds of 25 cars traveling past a point on a country road is measured on a random sunny day. These speed are given below. (This is the same sample used in the activity on confidence intervals.)

56	49	44	61	67	43	51	53	57	60	59	47	55
58	67	43	66	62	64	51	61	61	48	56	62	

1. Find the mean and standard deviation for this sample.

____

Residents along the road claim that the average car goes faster than the posted 55 MPH speed limit. The mean computed above does not really provide convincing evidence that the claim is true, so you decide to apply a test of significance at the $\alpha = 0.10$ level.

2. What should the null hypothesis be for this test? Be sure to write it using the standard symbolism.

____

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3. What should the alternative hypothesis be for this test? Be sure to write it with standard symbolism.

____

____

4. Will this be a one-tailed or two-tailed test?

____

Turn on the HP 49G, press [→][STAT]. Select the `Hypoth. Tests...` option. From the `Hypothesis Tests` choose box, select `T-Test: 1μ`, which will take you to an input screen. Enter the proper values in each of the fields (and note the explanation that appears on the bottom of the screen when you highlight each field). When you're finished, press [OK].

5. Would you accept or reject the null hypothesis? ______________________

6. Does this support the residents' claims? Why or why not? ______________________

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Remember to look at the graph of this situation.

Testing light bulbs

You are the quality control manager for the Bright Idea Lightbulb Company. Printed on the outside package of each lightbulb is the claim "Average life 1000 hours." You decide to test the claim by taking a random sample of 10 bulbs off the production line and measuring their life. The results (in hours) are:

989 1012 997 997 999 992 994 996 998 996

7. Find the mean and standard deviation for this sample. ______________________

You decide to perform a test of significance at the $\alpha = 0.05$ level based on this sample.

8. What should the null hypothesis be for this test? Be sure to write it using the standard symbolism.

____

9. The packaging will be changed if the population mean life turns out to be significantly different from 1000 hours (either smaller or larger). What should the alternative hypothesis be for this test?

____

10. Will this be a one-tailed or two-tailed test? ______________________

11. Do you accept or reject the null hypothesis? ______________________

12. Do you have to change the packaging? ______________________

13. The company president wants a report of the significance test—write it below. Be sure to include information about sample size, test method, the values of the sample statistics, and all of the values (and their meanings) that the calculator returned. ______________________

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14. What is the 95% confidence interval for the mean bulb life of the population (in hours)? __________

Teacher Notes

In this activity, tests of significance about a population mean are applied in situations where the population standard deviation is not known. This is far more realistic than the previous activity. Due to the work already done with confidence intervals, the t statistic is used without much fanfare. Both situations in this activity involve a claim about a population mean that appears to be supported by a sample mean. These sample mean differences do not turn out to be statistically significant.

Materials Needed

At least one HP 49G per group.

Calculator functions introduced

Test of hypothesis: Z-Test: 1μ

Solutions to Selected Exercises

All numerical answers rounded to three decimal places

1. $\bar{x}$ = 56.04 MPH, s = 7.413.
2. H_0: $\mu = 55$.
3. H_a: $\mu > 55$.
4. This is a one-tailed test.
5. Accept H_0 (P = 0.245).
6. This test does not support the residents claims because the significance test advises the acceptance of H_0: $\mu = 55$.
   In other words, even though the sample mean speed was over 55 MPH, the difference is not statistically significant.
7. $\bar{x}$ = 997 hours, s = 6.055.
8. H_0: $\mu = 1000$
9. H_a: $\mu \neq 1000$
10. This is a two-tailed test.
11. Accept H_0 (P = 0.152).
12. The packaging can stay; the significance test advises the acceptance of H_0: $\mu = 1000$. The sample mean life was less than

1000 hours (9 bulbs out of the 10 samples had a life less than 1000 hours), but the difference is not statistically significant.

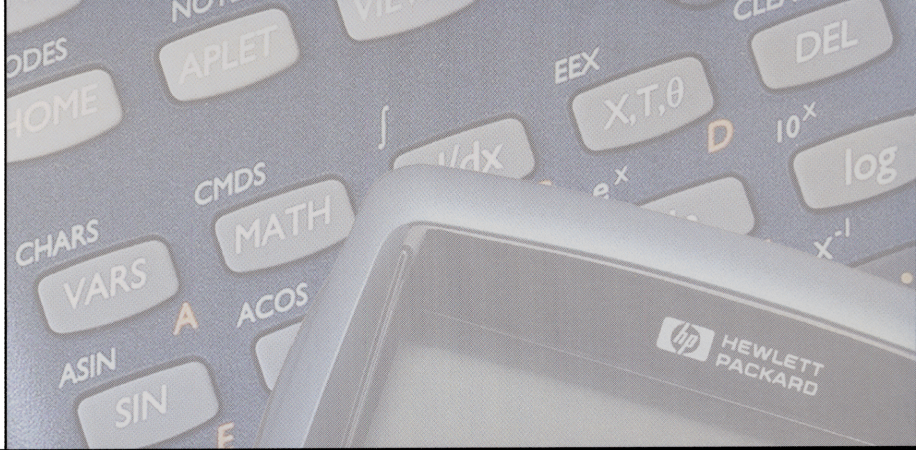
14. *The 99% confidence interval for μ is (992.668, 1001.331). Be careful to use the* `T-Int: 1μ` *confidence interval option. Notice that the sample mean lies within this interval.*

Extensions

Discuss the role of the significance level. For what kinds of tests would one require a high level of significance? Medical trials and aviation testing comes to mind....

Because of variability associated with human behavior, many claims regarding psychological and education treatment are held to a much lower level of significance (0.10 or even 0.20). Discuss this idea.

Ask students to try to think of claims that they would like to test. Encourage them to formulate testable questions, construct surveys, collect data, and analyze the results. Be careful that student generated claims do not involve the comparison of two sample means.



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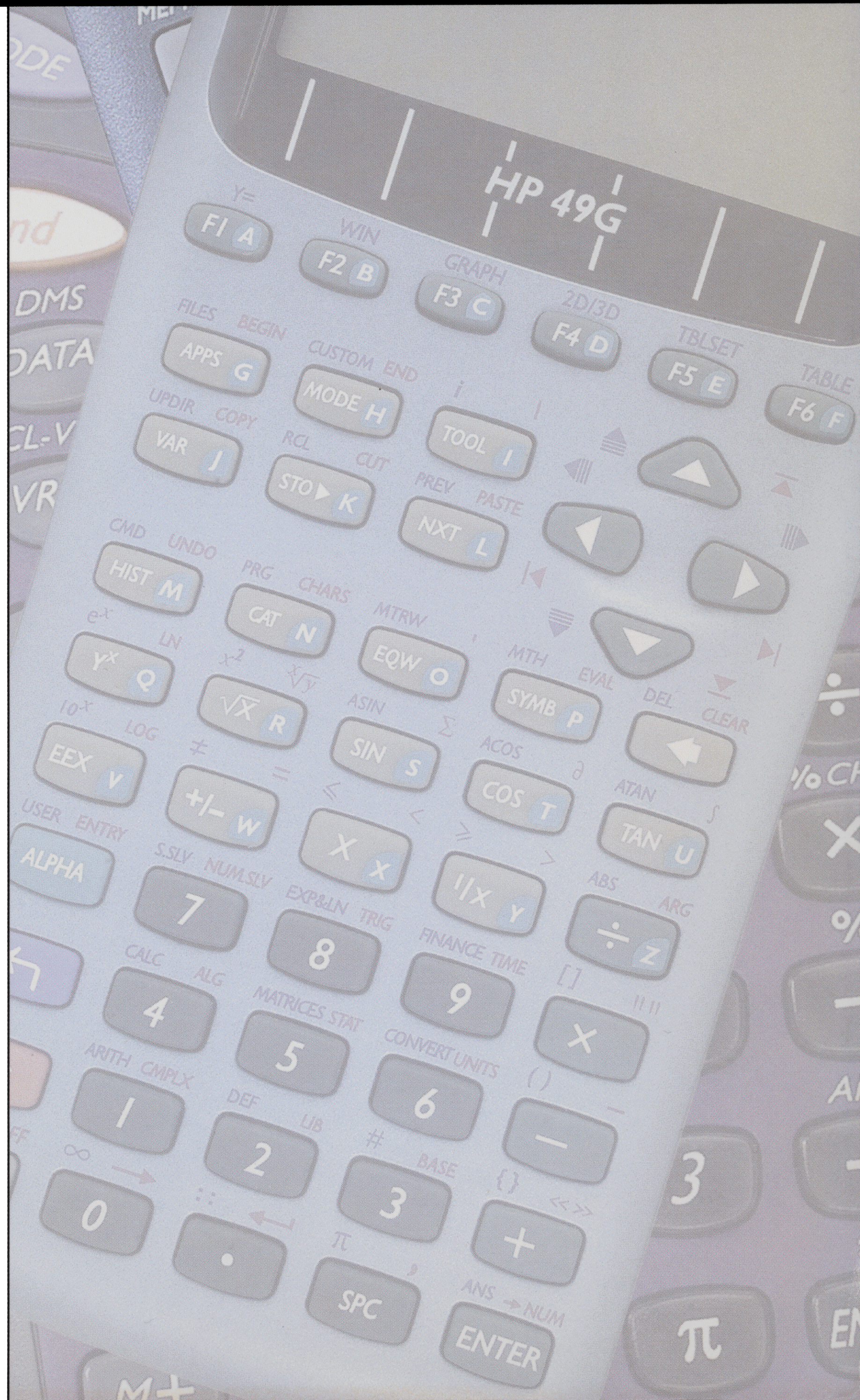

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