an introduction to

Cashflow Analysis

TIME VALUE OF MONEY CONCEPTS
IN THE VALUATION OF ANNUITIES,
STOCKS, BONDS, MORTGAGES,
INCOME REAL ESTATE, LEASES
TRUST DEED NOTES

ROBERT J. DONOHUE CCIM

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INTRODUCTION TO CASHFLOW ANALYSIS

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IN LOVING MEMORY OF MY PARENTS,
JOHN AND ILA DONOHUE

An Introduction to Sixth Edition

All investment is a matter of exchanging something of value today for what we hope may become something of greater value tomorrow. When we express these exchanges in terms of money, we create *cashflow* situations. All investments involve cashflows, and the value of every investment is the Present Value of all the future cashflows which the investor can hope to accrue over the entire holding period, discounted at an acceptable rate.

This text attempts to uncover and illustrate, for those whose primary professional concern has *not* been finance, the basic financial concepts and principles involved in investment problems and decisions. Examples are drawn from a wide variety of financial investments: stocks, bonds, real estate, promissory notes, leases, mortgages, all set in the practical circumstances of everyday business.

The text makes ample use of the financial calculator, but assumes the reader initially has only the most basic skills in using the machine. It uses the machine as a convenience-tool rather than as an end in itself. It is a text aimed at *understanding and employing financial concepts* and not another book on how to operate a financial calculator. This seems to me to be an important goal since the skills required to solve financial cashflow problems depend more on a basic understanding of the underlying concepts and creative thinking than on the ability to push keys by rote.

In this revision of the text, we have added a chapter pertaining to investment *risk*. We have also added a glossary containing definitions, and sometimes a short explanation of a financial term or related concept. We have revised the Appendix which now contains the mathematical derivation of the most frequently used cashflow formulas. It also includes an amplification of the Inflation-Adjusted Rate which is sometimes troublesome to some readers.

Although there are many excellent hand-held financial calculators, it is impractical to write a text such as this for every different machine. Examples in the text are rendered in terms of Hewlett-Packard's 12C model. If you have a calculator other than the HP–12C, you should take a few minutes to learn how its financial registers can be accessed. After that, following the methodology here is not at all difficult since all financial calculators use the same symbols.

The text also presents solutions using a computer spreadsheet. Examples are given in terms of MS-Excel,® but a spreadsheet from Works® or Lotus 1-2-3® will do as well since the differences among these programs are very small.

The reader may notice some small differences in the values calculated by the hand-held calculator vs. those calculated by the computer. But the majority of differences found will be the result of 'lifting' (reading) calculated values from one register and manually reinserting them into another. In almost all cases, we transpose values from one register to another thereby preserving the accuracy of the calculation to the ninth decimal place. This accounts for some small differences.

The key to mastery of these concepts is in the solution of business problems. Therefore the text makes ample use of problems and their solutions, step by step, to help the reader extract the financial principle involved. I have also included a chapter with over 60 word problems and their solutions because the process of mastering these techniques is a process of abstracting principles from specific, relevant examples. Financial principles become evident only through the working of many problems.

In depicting the calculator keystrokes we have sometimes omitted drawing a box around the key. In instances where the omission was deemed likely to be confusing, or where the key is used for the first time, we have boxed the key to help identify it. Also, we have used a shorthand to identify key locations on the HP–12C: a location (1,3), for example, denotes the key located on the first (top) row, third key (from the left). $\boxed{\text{PV}}$

Lastly, I wish to thank the students at the University of California, Irvine, for their excellent suggestions and for their invaluable assistance in ferreting out errors, the inevitable typos, and miscalculations. I am also pleased and thankful for the comments and support of many teachers in various universities and colleges throughout the country where this book now serves as the text for *Cashflow Analysis* courses in financial planning programs.

I also remain grateful to Dr. Charles J. Cuny of the University's Graduate School of Business who was especially helpful in developing the section on Bond Duration.

I hope *An Introduction to Cashflow Analysis* will be a significant, rewarding and stimulating learning experience for you.

Robert J. Donohue CCIM

Irvine, California
2002

Table of Contents

Sixth Edition

Chapter Six: Valuing Cashflows from Bond

Definition of bonds and debentures • nomenclature of bonds • coupon rate • reversionary-maturity value • callable bonds • calculating bond yields (YTM) • determining bond price • accrued interest • call dates • spot rate for money • yield curve • treasury bonds • calculating bond Duration • Modified Duration • Convexity • adjusting for Convexity • Duration and bond portfolio • zero-coupon bonds • U.S. Inflation-Adjusted bonds • safe rate for money

Chapter Seven: Evaluating Risk

Risk as component of discount rate • historical and probable deviations • variance • standard deviation • probability of returns • meaning of standard deviation • correlation coefficient • co-variance • Tchebysheff's theorem • Empirical Rule • Semi-variance • Semi-standard deviation • Sharpe Index • market /systematic risk • unique /unsystematic risk • portfolio returns • Market Portfolio Theory • Beta • Alpha • R-Squared • Capital Asset Pricing Model • Arbitrage Pricing Theory • Monte Carlo Simulation • elements of bond risk • bond rating services

Chapter Eight: Understanding Promissory Notes and Mortgages

Trust deeds and mortgages • amortizing loans • constructing amortization schedule • determining loan payments and remaining balances • balloon payments • adjustable-rate mortgage (ARM) • buydown mortgages • graduate payment mortgage • All-Inclusive Deed of Trust (AITD/wraparound) • reverse mortgages • loan qualification • loan ratios • debt coverage/service ratio • annual percentage rates (APR) • loan constants

Chapter Nine: Valuing Cashflows from Real Estate

Quantifying and valuation of income • vacancy and credit losses • operating expenses • income statement • cashflow after tax • building a pro-forma • depreciation • amortization of loan points • reversion value • capital gains tax • depreciation recapture • net proceeds after-tax • selecting a discount rate • calculating IRR • band of investments

Chapter Ten: The Role of Leverage

Neutral, positive and negative leverage • stocks and bonds • leverage and appreciation effect on ROE • maximum leverage on real estate • effect on IRR

Chapter Eleven: Problems, Problems, Problems

A collection of cashflow problems with supplied answers

Appendix

Derivation of PV-FV cashflow formulas • annuity formulas • additional information on Inflation-adjusted Rate

Glossary

A collection of frequently used financial terms and concepts

Index

Chapter 1 Present Value & Future Value of Cashflows

The Time Value of Money is an important financial concept, since many investments in business (and in our personal lives, as well) involve the payment or receipt of money over a period of time. And, as most experienced business people know, financial value is a combination of cash paid today, and the terms for the balance of the payment tomorrow.

Most often affected are our major business investments: the purchase or sale of a major income property; a direct investment in a business venture; an investment in a partnership which, in turn, invests in a business or real estate venture; the lease of both personal and real property; business and personal loans; investments in stocks and bonds; and a host of transactions which create cashflows over time.

We are concerned with our ability to measure and compare the value of cash to be paid tomorrow with an equivalent amount paid today, and with tomorrow's value of an amount invested today. These concepts involve Future Value cashflows and Present Value cashflows. But first, let's define our terms and establish a few basic concepts.

What is a Cashflow?

Cashflows are not profits, although some use the terms interchangeably. A Cashflow, in the context in which we will be working, simply means the flow of money from one set of pockets into another. The party providing the cash experiences a negative out-of-pocket cashflow; the party receiving the cash experiences a positive into-the-pocket cashflow. Every financial transaction involves both a positive cashflow and a negative cashflow.

In a more technical (and more correct) sense, *cashflow* may refer to cashflow from operations, or from interest and dividends received, or from interest paid. For our purposes we are interested in net earnings from operations before deductions for interest, taxes, depreciation and amortization (EBITDA).

Present Value and Future Value

Both these terms, Present Value and Future Value, rely for their meaning on the fact that money has a *time-value*. By *time-value*, we mean that money can be worth more or worth less than its apparent face or *nominal* value, depending on when it is to be paid out or when it is to be received.

For example, a tax which can be deferred has a lower Present Value than its nominal amount because a smaller sum can be invested today to grow to the amount necessary to pay the full tax in the future. Cashflow analysis concerns itself with the Present and Future Values of money paid or received over time, often measuring these values against some pre-selected investment yardstick.

Nominal vs. Real Profits and Losses, Constant Dollars

There is a distinction to be made between nominal and real profits. If we invest today's $1 in an investment which will add to our wealth, the $1 will grow into some Future Value. If the $1 is invested in such a way that it just keeps pace with inflation, we will have preserved the **constant** or **purchasing** value of the **dollar**. We will have a *nominal* profit, but in actuality the future dollar received will have no more buying power than the original dollar invested.

If we invest $1 in such a way that the amount realized in the future will buy more than $1's worth of goods or services, we will have made a **real** profit.

If our investment of $1 today increases in value, but does not keep pace with inflation, we will have suffered a real **loss** in purchasing power, even though we may have a nominal profit. The concept of nominal and real profits and losses is important to us all, but especially important to those whose responsibility it is to plan for real profits, or to plan to meet future expenses with constant dollars.

The Sign Convention

Since every financial transaction involves at least one positive and one negative cashflow, the financial description of the cashflows must always include at least one positive and one negative sign. Whenever you enter a cashflow problem involving only a PV and a FV, the PV and FV values must be opposite in sign. This rule is so inviolable that if you enter a cashflow problem into a financial calculator or computer without designating a giver (–) and a receiver (+), the calculator will display an "Error 5" message while the computer spreadsheet delivers a #NUM error sign.. You will know that you did not observe the proper sign convention, which recognizes that in every cashflow sequence someone gives (–) and someone gets (+) the cash..

Cashflows may be either positive or negative, depending on the vantage point from which you view the transaction. If you are a banker, the loan of $100,000 would be a negative (out-of-pocket) cashflow event for your bank. Later, re-payments to your bank would be positive cashflow events. If you are the borrower, the initial receipt of $100,000 would be a positive cashflow; later, the same re-payments would be negative (out-of-pocket) cashflows.

Importance of Maintaining Single Vantage Point

In analyzing cashflows, you must discipline yourself to adopt a single vantage point from which to describe and analyze the cashflow transaction – and then stick to it. You must either be the lender or the borrower; the lessor or the lessee; the mortgagor or the mortgagee. Viewing a loan transaction from the lender's point of view one moment, and then from the borrower's point of view the next, is a sure-fire way to introduce fatal signing errors into the structure of your cashflow analysis. Any analysis based on this error will itself be incorrect. Be consistent. Be either the lender or the borrower; either the lessor or the lessee; either the mortgagor or the mortgagee. In sticking to one point of view, you will avoid signing errors. Or, to put it another way, you will avoid crediting (+) cash to a party when you should be debiting (–) the same party.[1]

Failure to follow this rule is one of the most common errors in cashflows analysis.

Depicting Cashflows

It helps a great deal if you can "see" the cashflow situation with which you are working. A very useful graphic device to accomplish this is a **T–Bar,** which will enable you to portray the negative and positive movements of cash *and their timing* through the entire transaction. If you learn to use this device to illustrate the flow of the cash you are attempting to evaluate, it is difficult to imagine any cashflow problem which will not yield to a proper analysis.

The vertical T-Bar graphic used here has advantages when compared to the horizontal graphic devices pictured in the handbook of most financial calculators because the T-Bar can be expanded to the right to add and to the left to subtract cashflows, whereas the horizontal graphic cannot be so easily modified. The ability to expand the T-Bar laterally means that other cashflows which are concurrent in time can be added to, or subtracted from, the principal cashflow. This ability to add and subtract T-Bars is a great aid in visualizing, simplifying and condensing the overall transaction.

Fortunately the T–Bar which we will use contains a "place setting" for all financial elements which correspond to the keys of the (horizontal) financial register of your HP–12C. These are the keys which extend from positions (1,1) to (1,5) on the keyboard.[2] They are also keys which are common to <u>all</u> financial calculators, and they correspond to some of the symbols used by computer spreadsheets[3] in financial formulas:

n	i	PV	PMT	FV

1 Wherever possible view the transaction from the point of view of the investor.

2 (1,1) indicates the first key in row 1: (#key, #row)

3 Computer spreadsheets use **nper** instead of **n** to denote time periods, and **rate** to denote **i.**

Therefore if you can depict the problem graphically with the aid of the T–Bar, you can transfer the elements of the graphic to the calculator for a rapid solution.

Consider, for example, a T–Bar constructed to represent a loan by a banker for $1,000 to be re-paid monthly over a five-month period, including interest at an annual rate of 10%. This loan will be depicted from the point of view of the banker-lender who advances the cash.

THE BASIC T–BAR WILL LOOK LIKE THIS

	n	(PV)
	1	PMT
	2	PMT
i =	3	PMT
.10/12	4	PMT
	5	PMT + FV

The amount of –$1,000 will be entered in the T–Bar in the position occupied by **PV** (**P**resent **V**alue), at the top of the T–Bar. It is entered as a negative number because it represents a negative (out of-pocket) cashflow for the banker-lender. Payments made by the borrower are represented as positive numbers because these are positive (into the pocket) cashflows to the same banker-lender from whose vantage point we are describing the transaction.

Next, the number of periods is entered under the letter **n**(umber). Here we shall enter the numbers 1,2,3,4 and 5 representing the 5 separate periods over which a cashflow will be received as a periodic payment. A "period" can be any measure of time as you wish to define it: it may be a day, month, year, quarter, half-year – or any other period of time you choose. In this example it represents monthly periods.

The position indicated by **FV** (**F**uture **V**alue) will be represented by a zero (0), because the loan is meant to be fully repaid at the end of the fifth month (period) and (0) will remain as the future balance. The position **FV** in loan situations always represents the *remaining balance* of the loan. In other situations it represents a value present at the end of a sequence of cashflows. For example, if you rent equipment to others, FV would be the *reversionary value* , or worth, of your property at the end of the lease. If you are leasing an automobile, dealers call the Future Value of the auto at the end of the lease the *residual value.*

The interest rate **i** is annotated to the left of the T–Bar as a mnemonic to remind us that the interest rate is 10%. We have divided the interest rate by 12 to convert it to a monthly rate. Take note that in order to enter a 10% interest rate into $\boxed{i}$ you should *not* enter into the calculator the decimal 0.10. Any number entered into the key $\boxed{i}$ is automatically divided by 100. Therefore to enter an interest rate of 10%, you should simply key-in the number 10, and press the $\boxed{i}$ button (1,2). The calculator expects a number entered into $\boxed{n}$ to be a *percent,* not a decimal.

Be Time Consistent With The Variables

Another very common error made in dealing with cashflows is that the interest rate **i** per period does not conform with the period, **n**, for which payments (**PMTs**) are to be made or received. If payments are to be considered monthly payments, the interest rate in **i** should be a monthly interest rate (e.g. 0.10 ÷ 12); if the payments are to be considered quarterly payments, the number in **i** should represent the quarterly interest rate (e.g. 0.10 ÷ 4); if the payments are to be annual payments, the number in **i** should represent the annual interest rate (e.g. 0.10 ÷ 1).[4] The interest rate **i** should always be time-consistent with the period **n** and with the time period for the **PMT.**

The last position is the **PMT** position. If we solve this particular problem for the amount of payment necessary to repay the loan, the solution will be transferred to the **PMT** position to represent the completed cashflow event.

Our T–Bar will now look like this:

		-1,000.00	**(PV)**
EOM [5]	1	205.03	**(PMT)**
	2	205.03	
i = 10%/12	3	205.03	
	4	205.03	
	5	205.03 + 0	**(FV)**

Let's now transfer the elements of the T-Bar to the financial registers of the calculator:

n	i	PV	PMT	FV
5	0.83	-1,000.00	?	0

Here are the keystrokes and what should appear in the display window after each entry:

Key-In	Display Shows
5 [n]	**5.00** [6]
10 [g] [i]	**0.83**
1000 [CHS] [PV]	**-1,000.00**
0 [FV]	**0.00**
solving ....(push) [PMT]	**205.03**

4 The value (0.10 ÷12) would be entered as 10 ÷12. The calculator will divide the result by 100.

5 **E**nd **of M**onth. **EOY** = **E**nd of **Y**ear; **EOQ** = **E**nd of **Q**uarter, etc.

6 If your calculator does not show two decimal places, set it to two places by pressing [*f*] **2**. Be aware, however, that although only 2 decimal places are shown, the calculation uses up to 9 decimal places in it computations.

> **TIP** The second entry, 10 [g] , is a convenient way of dividing the interest rate by 12. Whenever you precede the entry of a number into [i] by first pressing the blue key [g] **(4,3)** the number entered into [i] is automatically divided by 12. If you need to express the interest rate for any period other than monthly, do it using the keypad; e.g. if **n** were a quarterly period, then you would handle the interest rate as follows: (10 [Enter] 4 [÷]) delivers 2.5 which is then directly entered into [i] . The [CHS] key (1,6) **CH**anges the **S**ign to the opposite sign. In this case it changes a positive 1,000 to a negative –1,000.

Suppose, on the other hand, that you elect to depict the cashflow from the viewpoint of the borrower. Everything is the same except that the signs are reversed:

		+1,000.00	**(PV)**
EOM	1	-205.03	**(PMT)**
	2	-205.03	
i = 10%/12	3	-205.03	
	4	-205.03	
	5	-205.03 + 0	**(FV)**

If you enter these data into the calculator, the result will be:

Key In	**Display Shows**
5 [n]	**5.00**
10 [g] [i]	**0.83**
1000 [PV]	**1,000.00**
0 [FV]	**0.00**
solving .... [PMT]	**–205.03**

or,

n	**i**	**PV**	**PMT**	**FV**
5	0.83	-1,000.00	?	0
		Solving ······▶	**205.03..**	

It's a good idea to approach every investment situation from the vantage point of the investor. Therefore every investment situation will begin with a negative cashflow, the amount of the initial investment.

Concept of Present Value, Future Value

The Present Value, **PV**, of one dollar held today is one dollar because it can buy one dollar's worth of goods or services. The **FV** of one dollar invested today, however, cannot be determined until you decide how many periods, **n**, will elapse before you receive it, and the rate of growth, **i**, which the cash will earn for the number of periods over which it is invested.

If we are trying to determine the future value of a present value dollar, held for **n** periods and earning interest at the rate **i** per period, then **i** is called an ***interest*** rate.[7] If, on the other hand, we want to determine the present value today of a sum to be received **n** periods in the future, **i** is called a ***discount*** rate. In other words, **i** can represent either an *interest* rate or a *discount* rate, depending on the direction in time we want to go. Going forward in time, **i** is an *interest* rate; but going backward in time, **i** is a *discount* rate.

An interest rate and a discount rate are different sides of the very same coin.

That's why these calculations are often referred to as calculations involving the "present worth of money" or the "time value of money." These phrases suggest, as we have already noted, that cash has a value today, and a different value tomorrow. The value of money is *time*-dependent.

The Underlying Basic

There is a mathematical relationship between the Future Value (**FV**) of a dollar and its Present Value (**PV**), a relationship which is very important to understand. It is described by this simple expression:

$$\mathbf{PV} = \frac{\mathrm{FV}}{(1+\mathrm{i})^{\mathrm{n}}}$$

where,

PV	is the value of an amount of cash today – its Present Value
FV	is the Future Value of the cash to be received in the future
i	is the interest (or discount) rate to be applied
n	is the number of periods (time) between the PV and the FV

Mathematically, the relationship tells us that the **PV** of a single sum to be received in the future (**FV**), is equal to the **FV** divided by the expression (1 + **i**) raised to the power **n**. Before going on, let's refresh our memories about what it means to *raise a number to a power.*

When we raise a number to a *power* **n**, we simply multiply a number times itself for as many times as is indicated by the number **n**. The number **n** is called the *exponent*, and is written x^n.

7 In some cases the letter **r** is used to indicate rate.

Therefore:

$$
\begin{aligned}
10^{1} &= 10 \\
10^{2} &= 10 \times 10 = 100 \\
(1+.10)^{4} &= 1.10 \times 1.10 \times 1.10 \times 1.10 = 1.46 \\
3^{6} &= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729 \\
10^{-1} &= 1/10 \\
10^{-2} &= 1/100 \\
10^{1/5} &= 10^{0.20} = 1.5849 \text{ (the } 5^{th} \textit{ root} \text{ of 10)} \\
10^{0} &= 1
\end{aligned}
$$

Any number or expression raised to the zero power equals one.

Knowing these simple facts enables you to convert the future receipt of a single sum of money into a **P**resent **V**alue (today), provided you specify a *discount* rate (since we are moving backward in time), and further specify the number of periods, **n** , between **PV** and **FV.**
Solve this problem, please:

> You agree to take part of your consultation fee in the form of a note which calls for you to receive $5,000 at the end of 3 years. The notes carries no interest rate and no intermediate payments.
> What is the present value (or present worth) of this note?

It is easy to see that:

PV = ?
FV = 5,000
n = 3

But what is **i** ?

You determine that if you had a comparable sum in your hand today, you could invest it (in another investment of similar risk[8]) at an annual interest rate of 10%. Therefore, you assign **i** a value of 10% or 0.10. Now you can substitute into the formula above all the values you have at hand:

$$\textbf{PV} = \frac{FV}{(1+i)^{n}} = \frac{5{,}000}{(1+.10)^{3}}$$

$$\textbf{PV} = \frac{5{,}000}{(1+.10)(1+.10)(1+.10)} = \frac{5{,}000}{1.33} = \textbf{3,756.57}$$

Therefore your promissory note has a Present Value, under the terms and conditions stipulated, of not $5,000, but $3,756.57 today. To someone else who has a different opportunity cost, this note will have a different Present Value. Therefore the (present) value of this note is, in the last analysis, fairly *subjective* because it depends on the rate at which the future PMTs are discounted by the person doing the discounting. *This is a fact with all investments.*

8 The rate which could be earned on the next best investment of similar risk is known as the "Opportunity Cost" of money. In this case you estimate that if you were paid your fee at closing you could invest it to earn 10% per year. Therefore your Opportunity Cost is 10%.

Using the Calculator's Financial Register

Now that you understand how PVs are determined, you could process one with a discount-store calculator. But financial calculators, such as the HP–12C,[9] make the entire exercise very easy. Simply enter the data into the appropriate financial registers:

n	i	PV	PMT	FV
3	10.00	?	0	5,000.00
	Solving	▶ **–3,756.57**		

The keystrokes are:

	Key-In	**Display Shows**
	3 $\boxed{n}$	**3.00**
	10 $\boxed{i}$	**10.00**
	0 $\boxed{PMT}$	**0.00**
	5000 $\boxed{FV}$	**5,000.00**
solving	$\boxed{PV}$	**– 3,756.57**

[The fact that PV is shown as a negative number is a convention of the calculator, and does not necessarily indicate a negative number.]

Determining the Future Value of a Single Sum

It is frequently required to estimate the value to which a single sum invested today (a PV) will grow if invested at **i** rate of interest for **n** periods. For example:

> Smith invests $300,000 in a parcel of vacant land in the "path of progress." He estimates that land values will grow at an annually compounded rate of 8% over the next 5 years.
>
> What is the projected value (FV) of Smith's land at the end of 5 years?

In order to answer this question, we need only to rearrange the PV-FV formula:

$$\mathbf{PV} = \frac{FV}{(1+i)^n}$$

$\mathbf{FV} = PV * (1+ i)^n$ (*This is the formula for compound interest*)

Therefore – $\mathbf{FV} = 300{,}000 \times (1 + .08)^5$

$\mathbf{FV} = 300{,}000 \times 1.08 \times 1.08 \times 1.08 \times 1.08 \times 1.08$

$\mathbf{FV} = 440{,}798.42$

9 All financial calculators use this sequence.

Using the calculator, the problem is easily solved:

n	i	PV	PMT	FV
5	8.00	-300,000	0	?
			Solving…	**440,798.42**

The key strokes are:

Key In	Display Shows
5 [n]	**5.00**
8 [i]	**8.00**
300000 [CHS] [PV]	**–300,000.00**
0 [PMT]	**0.00**

solving,

Key In	Display Shows
[FV]	**440,798.42**

When i Is a Negative Number

Suppose Smith were absolutely wrong in his projection, and the value of the land *declined* at the rate of 8% per year for 5 years.

What would Smith's lot then be worth?

FV $= 300{,}000 \times (1 - .08)^5$
FV $= 300{,}000 \times 0.92 \times 0.92 \times 0.92 \times 0.92 \times 0.92$
FV $= 197{,}724.46$

On the calculator –

n	i	PV	PMT	FV
5	**–8.00**	-300,000	0	?
			Solving	**197,724.46**

The keystrokes are virtually identical except that the interest rate, **i** , is entered as a negative number:

Key In	Display Shows
5 [n]	**5.00**
8 [CHS] [i]	**– 8.00**
300000 [CHS] [PV]	**–300,000.00**
0 [PMT]	**0.00**
solving .... [FV]	**197,724.46**

Try another:

> You deposit $10,000 in a bank which compounds interest at the end of every month at the annual rate of 4.5% per year.
>
> What will be your bank balance at the end of 5 years?

The thorn in this problem is that the interest rate is expressed annually but applied every month. Therefore **i** per period is not 4.5% but (4.5% ÷ 12).[10] Since the annual interest rate is to be expressed and applied monthly, you must also convert the periods **n** to months.

In 5 years there are 12 x 5 = 60 months. Therefore **n** = 60.

Now solve the problem using your non-financial calculator.

$$\mathbf{FV} = \mathbf{PV \times (1+i)^n}$$
$$\mathbf{FV} = 10{,}000 \times (1 + (0.045 \div 12))^{60} \qquad \text{(see footnote[11])}$$
$$\mathbf{FV} = 10{,}000 \times (1 + .00375)^{60}$$
$$\mathbf{FV} = 10{,}000 \times 1.25180$$
$$\mathbf{FV} = 12{,}517.96$$

---

10 This will result in compounding of the rate.

11 There is an alternate way to calculate the value of $(1 + (0.045 \div 12))^{60}$. Follow these steps:

Key-In	Display
f 6	0.000000
.045 Enter	0.045000
12 ÷	0.003750
1 +	1.003750
60 [y^x] (2,1)	1.251796
10000 [x]	12,517.9582
f 2	12,517.96

It's much easier using a financial calculator:

n	i	PV	PMT	FV
60	4.5/12	-10,000	0	?
			Solving ▶	**12,517.96**

The keystrokes are:

Key In	**Display Shows**	
5 [g] [n]	**60.00**	
4.5 [g] [i]	**0.38**	**(0.375, actually [12])**
10000 [CHS] [PV]	**–10,000.00**	
0 [PMT]	**0.00**	
solving .... [FV]	**12,517.96**	

PV and FV Derived from Periodic Payments

So far we have determined PVs and FVs without any regard to PMTs. But the financial world is simply not that simple. Most financial cashflows involve payments made or received periodically. To complicate matters slightly, these periodic PMTs can be either negative cashflows or positive cashflows. The difference between a FV and a PMT is that a FV occurs only once, while PMTs can occur *at the beginning* or *at the end* of each period for as many times as may be indicated by **n**. Any value you enter into FV, however, <u>is always regarded by the calculator as occurring at the end of period **n**, and occurs only once.</u>

This is as good a time as any to tell you that this "horizontal" financial register –

n	**i**	**PV**	**PMT**	**FV**

can be used in problems *only* where there are no PMTs or where the PMTs are equal. By "equal" we mean equal in amount *and* equal in sign (+,–). If the payments are of different amounts, or if, though equal in amount, they change signs during the cashflow series, this register cannot be used. There is a separate register which will solve these problems, and we will get to that shortly.[13]

---

12 Press [f] 3 to view the number with the decimal point set to three places

13 If the cashflow series has a period in which no payment is received or made, this period must be represented by a zero. Cashflows containing zero payments combined with either positive or negative payments are **uneven** cashflows. The "horizontal" registers cannot be used.

The value of the number entered into the PMT register is a payment either made or received *in each period*, **n**. For example, suppose you expect to receive $100 per month for the next 2 months. If you enter the number 100 into PMT and the number 2 into **n**, you are informing the calculator that the cashflow will be positive (into your pocket), will be equal to 100, and will occur 2 times.

n	i	PV	PMT	FV
2			100	0

Now let us inform the calculator that the rate **i** is 10% annually. You may not enter 10 directly into **i** because 10% is the annual rate and you have already defined that **PMT**[14] is a monthly cashflow. Therefore you must convert the annual rate into a monthly rate.

Key In	Display Shows
10 [g] [i]	0.83

The registers now contain:

n	i	PV	PMT	FV
2	0.83		100	0

You must now decide whether to move forward in time to solve for FV, or move backward in time to solve for PV. If we move forward in time to solve for the Future Value of these payments, we ask the calculator to determine the interest on the payments received, and to add the interest to the accrued payment amounts, and to display the results in the FV register.

Payments at the Beginning or End of Period

But before we can go on, we need to discuss *when* these PMTs are to be received. We know that they are to be received at monthly intervals, but the question is: "At the beginning or at the end of the month?" It makes a financial difference.

The difference between PMTs received from a lease and those from a mortgage, for example, will serve to illustrate this important point. Lease PMTs are customarily made at the beginning of the month (**BEG**). Therefore if you were to receive two lease payments of $100 per month, you could earn interest on the first payment of $100 for two months; you could also earn interest on the second month's PMT of $100 for one month.

If these PMTs flowed from a mortgage, however, the payment of $100 would be made at the end of the month (**END**). In this case you could earn interest on the first payment for only one month, and you would earn no interest on the second payment of $100 because it occurs at the end of the second month, which is also the end of the cashflow period under consideration. Therefore you

14 Whenever a PMT is involved, the timing of the PMT defines the timing for the values in **i** and **n**.

need to inform the calculator when the PMTs will occur: at the **BEG**inning or at the **END** of the month.

On the lower face of keys (1,7) and (1,8) you will see – printed in *blue* – the letters **BEG** on (1,7) and **END** on (1,8). You must access these functions by first pressing the blue key, $\boxed{g}$ (4,3) and then pressing either **BEG** or **END**. Depending on which you choose, you inform the calculator that the PMT will occur either at the beginning or end of the period **n**. When **BEG** is selected the word "**BEGIN**" will appear in the display window. When **END** is selected, the word "**BEGIN**" will not appear (the same position in the window will be blank).

Let's assume that in our example the PMT of 100 will flow from a lease and will occur at the beginning of the period. We need, then, to set the calculator to "**BEGIN**." Once that is done we can solve the problem.[15] Here are the keystrokes:

Key In	Display Shows
$\boxed{g}$ $\boxed{\text{BEG}}$ (1,7)	**0.00**
2 $\boxed{n}$	**2.00**
10 $\boxed{g}$ $\boxed{i}$	**0.83**
100 $\boxed{\text{PMT}}$	**100.00**
solving …….. $\boxed{\text{FV}}$	**–202.51**

Calculator window now reads "BEGIN" indicating that all PMTs will be treated as having been received at the beginning of each **period.**

BOP (Beginning of Period)

n	i	PV	PMT	FV
2	0.83…	0	100	?
			Solving	**–202.51**

Now consider that the PMTs flow from a mortgage or bond you hold. Since mortgage and bond payments are made *in arrears* (at the end of the period), you need to inform the calculator to treat the PMTs as end-of-the-period PMTs. There is no need to key in all the values again. Simply change to **END** (**g, End** on key 1,8) and re-solve by pressing FV:

n	i	PV	PMT	FV
2	0.83…	0	100	0
			Solving	**–200.83**

15 On computer spreadsheets this choice is available in relevant financial function commands by the insertion of either 0 or 1 at the "**TYPE**" position in the formula; e.g. =PMT(rate, nper, PV, FV,**TYPE**). When TYPE = 1, PMTS are BOP. When TYPE = 0, PMTs are EOP. Default = 0

Combining PVs & FVs with Positive & Negative PMTs

A savings account is an excellent example of a combination of an initial PV and subsequent PMTs.

> Suppose that you deposit (a deposit is a negative cashflow to you) $10,000 in a savings account which pays interest **i** at the rate of 5% per year. You also plan to deposit an additional $2,309.75 to this account at the end of the year.
> Interest is credited to your account annually.
> What would be your account's balance be at the end of year 1 ?

n	i	PV	PMT	FV
1	5	–10,000	–2,309.75	?
			Solving..	**12,809.75**

Note that PMT is also a negative number because it represents an out-of-pocket cashflow to you.

At the end of the first year ('period') your balance would consist of a recent PMT of $2,309.75 plus the original $10,000 deposit plus the interest earned on the original $10,000 deposit for one year ($500). This explains the total ($10,000 + 500 + 2309.75 = $12,809.75). You earned no interest on the $2,309.75 you deposited because it was deposited at the end of the last period.

Suppose now, that after entering the original sum of $10,000 in your account, you *withdraw* the amount of $2,309.75 at the end of the year. What would be your balance after the first year's withdrawal?
Let's change the sign of the PMT to reflect an into-your- pocket cash flow:

n	i	PV	PMT	FV
1	5	–10,000	+2,309.75	?
			Solving.	**8,190.25**

During the year, your original deposit earned interest at the rate of 5% annually and therefore the total in the account was $10,500. But after your year-end withdrawal of $2,309.75, your balance fell to $8,190.25 ($10,500 - 2,309.75).

What would be your balance at the end of 5 years?

n	i	PV	PMT	FV
5	5	–10,000	+2,309.75	?
			Solving.	**0.01**

(Did you make yourself a fully-amortizing loan payable $2,309.75 annually for 5 years, including interest at the rate of 5% per annum? Think about this.)

Discounting Amounts To Be Received in the Future

So far we have dealt with converting PVs, either alone or combined with positive and negative PMTs, into FVs. An equally important skill involves converting PMTs into Present Value amounts, single sums (FVs) into Present Values, and the combination of FVs and PMTs into PVs. Let's take simple future PMTs first.

> Suppose that you grant your neighbor, Jones, a 5-year easement over your property in return for his promissory note which specifies monthly payments of $125 for the next five years. You visit three banks to determine how much you can borrow against this note. You are well known to each banker and have the financial ability to guarantee the payments on the note.
> **Banker A** tells you that he will lend 50% of the value of the (note) PMTs, discounted at 10% annually.
> **Banker B** tells you that he will lend 55% of the value of the (note) PMTs, discounted at 11% annually.
> **Banker C** tells you that he will lend 60% of the value of the (note) PMTs, discounted at 12% annually.
> Which is the best offer?

There is a priority involved in solving this problem: first, you must determine the PV of the note in the hands of each banker; then you must determine what percentage of the PV he will lend. Let's quickly determine the PV of this note to each banker:

Banker A:

n	i	PV	PMT	FV
60	**10**/12	**?**	125.00	0
	Solving ▶	**–5,883.17**		

Banker B:

n	i	PV	PMT	FV
60	**11**/12	**?**	125.00	0
	Solving ▶	**–5,749.13**		

Banker C:

n	i	PV	PMT	FV
60	**12**/12	**?**	125.00	0
	Solving.. ▶	**–5,619.38**		

Therefore **Banker A** estimates the PV of this note at $5,883.17; **Banker B** values it at $5,749.13, while **Banker C** holds for a value of $5,619.38.

Notice that as the discount rate increases, the PV of the future PMTs decreases. You should make permanent mental storage of the fact that when discounting cashflows, the higher the discount rate applied, the lower the Present Value of the cashflow.[16]

To determine the amount each banker will lend on this note we need only multiply the PV value times the percentage listed above:

Banker	PV	% of Value	Loan Amount
A	5,883.17	50	2,941.59
B	5,749.13	55	3,162.02
C	5,619.38	60	3,371.63

Obviously, **Banker C** will advance a greater amount on the collateral of your note even though he discounts the value of the note at a higher (12%) rate.

The problem above involves the discounting of future PMTs only. No FV was involved. Frequently, however, cashflows to be received in the future come in the form of periodic PMTs together with a lump sum at the end of the period. Many Promissory Notes are a good example of periodic PMTs combined with a FV (the loan payoff).

Discounting a Promissory (Trust Deed) Note

"Discounting" a promissory note to determine its *market value* is simply the process of determining the Present Value of future cashflows, having selected a desired yield or discount rate. Consider, for example, the following common situation:

> A client has negotiated the sale of her residence. The price is satisfactory but the buyer's offer requires the client/seller to carry back a promissory note in the amount of $25,000, payable monthly, *interest only*, at the rate of 10% per year, with the remaining balance and interest all due and payable three years "from date" (the date of the note). Your client/seller agrees to accept the price only if you can liquidate the note at the close of escrow. A physician whom you know invests in discounted promissory notes for his 401-K account. He agrees to buy the note provided it provides him a 12% annual yield.
>
> At what price must the note be sold into the physician's retirement account, and what will be the percent discount of the note from its face value?

16 Bonds behave this way also. As the interest rates rise, bond values fall. See-saw Majorie Daw: a Discount rates sits on one end of the teeterboard and the Present Value sits on the other.

Let's enter the cashflow represented by the note into the T-Bar:

			–25,000 **(PV)**
	EOM	1	208.33... **(PMT)**
		2	208.33
i = 10%/12		:	:
		36	208.33 + 25,000 **(FV)**

Note that the principal payoff (FV) occurs at the **E**nd **O**f **M**onth 36 and not in the following month. It is a common mistake to set up a problem such as this indicating 36 monthly payments and then to place the payoff (FV) one period later. This mistake converts the term of the note, **n**, to 37 months, not 36 months as it should be.

The T-Bar now depicts the way this note will perform. But the problem to be solved is to determine the Present Value of this cash flow discounted at 12%, the minimum rate the investor requires. In other words:

		? **(PV)**
EOM	1	208.33 **(PMT)**
	2	208.33
	:	:
$\boxed{\textbf{i = 12\%/12}}$	:	:
	36	208.33 + 25,000 **(FV)**

We can now transfer these values to the calculator and solve for PV.

n	i	PV	PMT	FV
36	12/12	?	208.33...	25,000
	Solving	▶ **23,745.52**		

Therefore the physician can buy this note for $23,745.22 and earn exactly 12% per annum on his invested cash for 3 years. His PMTs will be $208.33 per month for 35 months with a final payment of $208.33 plus the remaining balance of $25,000.[17] The keystrokes for this problem are as follows:

Key In	Display Shows
36 [n]	**36.00**
12 [g] [i]	**1.00**
208.33 [PMT]	**208.33**
25000 [FV]	**25,000.00**
solving, ... [PV]	**-23,745.22**

17 Notes which provide for interest-only payments are know as *straight notes.*

> Suppose, for a moment, that the physician requires not a 12% return on his cash invested, but a 15% return.
>
> What would the PV of this note become?

Again, it is not necessary to re-enter all the values. Simply "write over" the discount rate **i** as follows:

	Key In	**Display Shows**
	15 [g] [i]	1.25
solving, ...	[**PV**]	–21,994.98[18]

n	**i**	**PV**	**PMT**	**FV**
36	**15**/12	**?**	208.33	25,000
	Solving ········▶	**–21,994.98**		

As you can see, the higher the discount rate, the lower the PV of the cashflow.

The percentage of discount from face value is also easily expressed.
In the first case:

Key In	**Display Shows**
25000 [**Enter**]	25,000.00
23,745.22 [Δ%] (2,4)	–5.02

Therefore the seller would be required to discount the $25,000 note 5.02% to deliver a 12% yield to the physician. In the second case:

Key In	**Display Shows**
25000 [Enter]	25,000.00
21,994.98 [Δ%]	–12.02

If a 15% yield were required, the note would be discounted 12.02% from face value.

The first two sets of examples and problems involved the discounting of PMTs only, and then the discounting of PMTs combined with a FV to be received at the end of the periods. If you are required to determine the PV of a single future sum, FV, you have the easiest problem of all.

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18 The PMT is 1/12th of (25,000 *10%) which results in 208.333333.... but this would be collected as $208.33.

For example, here's a dramatization of the PV of a FV, discounted at a certain rate:

> Suppose your home has a current market value of $250,000.
> Would you sell it for $50,000?
> "Of course not," you say.
> But what if you were offered $1,000,000 for your property?
> Most individuals would jump at the chance.
> But you haven't yet been told *when* you would receive the cash.
>
> If you were told that the cash would be paid in 50 years (a bit hyperbolic, but play along), would you still accept the offer?

The situation really poses the financial question: "What is the PV (today) of $1,000,000 to be received 50 years in the future?

In order to answer the question, you must again choose a discount rate **i,** which will enable you to move the value of money backward in time. Let's assume that if you had a large sum of money in hand, you could safely invest the cash to earn 8% per year, net. This, then, is the *opportunity cost* of money to you. It is also the discount rate you would use to value the offer, and to convert the promise of a future receipt of cash into its equivalent cash value today:

n	i	PV	PMT	FV
50	8.00	?	0	1,000,000
	Solving ▶	**–21,321.23**		

Now that you know you would not accept the $1,000,000 offer because it has an equivalent Present Value of only $21,321.23, what is the longest period of time you could wait to receive the money without losing the present value of your $250,000 home?

We can solve this problem by entering –250,000 into the PV register and solving for **n,** the number of periods required for $250,000 to grow to $1,000,000 if the PV grows at the rate of 8% annually. But before we proceed…..

Uncovering an Idiosyncrasy of the HP–12C

We can also restate this problem to ask "Over how many periods must we discount the receipt of $1,000,000 to create a Present Value of $250,000, if we discount at the rate of 8% per period?"

n	i	PV	PMT	FV
?	8.00	**–250,000**	0	1,000,000
19 ◀	**Solving**			

Caveat

When solving for **n**, the HP–12C calculator[19] rounds up the answer and delivers the value of **n** only in whole numbers (integers). In this example, the answer for **n** *appears* to be 19 years. But the calculator has rounded the answer up to the next highest integer.[20] If you re-solve this problem for FV using **n** = 19 you will find a different value for FV:

n	i	PV	PMT	FV
19	8.00	**–250,000**	0	?
			Solving ▶	**$1,078,925.27**

The fact that the FV exceeds $1,000,000 means that **n** should be slightly *less* than 19.00 years. Unfortunately, there is no way, *using the financial registers* of this particular calculator,[21] to determine exactly what **n** should be. This problem surfaces again in dealing with amortizing promissory notes, and we will amplify on its significance in a later chapter.

Determining a Yield (i) From Even Payments

> Suppose the physician in the previous example desires to know the rate of return he could achieve if he could buy the seller's carry-back note of $25,000 at a 20% discount from its face value.
>
> What would his return be?

Remember that the note is scheduled to pay interest-only PMTs of $208.33 each month for 35 months and then a final payoff of the last interest PMT and the FV (balance) of $25,000. This variation is a problem in determining the yield, or discount rate, on the investment of $20,000. A T-Bar constructed to depict this cashflow would look like this:

		-20,000 (**PV**)
EOM	1	208.33 (**PMT**)
	2	208.33
	⋮	⋮
i = ?/12		
	36	208.33 + 25,000 (**FV**)

19 This is not a problem with other HP machines such as the 17B and 19B.

20 **n** rounds UP to the next higher integer whenever the decimal portion of the answer exceeds .005

21 There is a way to do this on the HP 12-C using the natural logarithm LN (2,3) function: $(1.08)^n$ = FV/PV, or, 1,000,000/250,000 = 4. $(1.08)^n = 4$. Therefore n = LN(4) ÷ LN(1.08) = 18.012937 (years). (Use key (2,3) to determine values of LN.) Substituting 18.012937 in n will deliver FV = 1,000,000.

Transferring these values to the calculator:

n	i	PV	PMT	FV
36	?	–20,000	208.33	25,000
Solving ▶	**1.564...**			

The answer obtained, 1.564%, is the interest rate **per period**. Because we expressed both the period **n** and the **PMT** in months, however, we need to multiply this monthly answer by 12 to express the annual rate of **18.77%**.

Distinguishing Between PMTs and FV

If you are dealing with only one PMT to be received at the end of the period, or one FV at the end of the period, it doesn't matter whether you treat the sum as a PMT or as a FV.
For example:

> Determine the PV of a sum, $5,000, to be received one year in the future and discounted at 10%.

As an EOP PMT:

n	i	PV	PMT	FV
1	10.00	**?**	**5,000**	0
	Solving ▶	**–4,545.45**		

As a FV:

n	i	PV	PMT	FV
1	10.00	**?**		**5,000**
	Solving ▶	**–4,545.45**		

If **n** is something other than 1, however, it does make a difference. Here's why...

If **n** = 2, and $5,000 is entered as a PMT, the calculator considers the sum of $5,000 to be received twice, once at the end of period 1 and again at the end of period 2. If **n** = 2, and the $5,000 is entered as a FV, the calculator considers that the sum of $5,000 is to be received only once, at the end of the second period.

If **n** = 2, and $5,000 is entered both as a PMT and as a FV, the calculator considers that the PMT of $5,000 is to be received twice and the FV to be received only once, at the end of period 2.

If you instruct the calculator to consider the receipt of the PMT as occurring at the beginning of each period, such instruction will not alter the fact that the calculator *always treats the receipt of a FV to be a one-time event always occurring at the end of period **n**.*

Chapter Summary

In this chapter, you have learned how to determine the Future and Present Values of cashflows involving single sums and *even* payments. A number of points are worth re-emphasizing:

1. Cashflows represent financial transactions. There is a flow of money from one pocket across to another.
2. Technically, a cashflow is a company's net operating profit after taxes, but with all non-cash items added back in: depreciation and amortization deductions are non-cash items.
3. In analyzing a cashflow transaction, it is important to adopt one point of view regarding the flow of cash. In most cases, it is helpful to adopt the position of the investor, or the person who advances the cash into an investment.
4. The signs of a cashflow are important when payments (PMTs) are involved. A positive PMT usually will indicate the receipt of cash, while a negative cashflow will represent additionally invested cash.
5. Every transaction must have a positive and negative cashflow. If PMTs are not involved, entering a FV as a positive number will deliver a negative PV. This is simply a convention of the calculator, and does not represent a loss. If a cashflow is entered in PV as a negative number, the FV will always be positive.
6. The application of *interest* rates permits us to express the future worth (FV) of a single sum, or a series of periodic, equal payments, or the combination of a single sum together with periodic payments . *Interest rates* move money forward in time.
7. The application of *discount* rates permits us to express a single sum (a Future Value), or a series of equal payments (PMTs), or a combination of the two, as a Present Value (PV). **Discount rates** move both kinds of money values **backward** in time.
8. The mathematical relationship between a single sum Present Value of money and a single sum Future Value of money is expressed by the equation:

$$\mathbf{PV} = \frac{FV}{(1+i)^n}$$

9. When equal PMTs are involved, this formula is opened up to show:

$$\mathbf{PV} = \frac{\mathbf{PMT}}{(\mathbf{1+i})^1} + \frac{\mathbf{PMT}}{(\mathbf{1+i})^2} + \frac{\mathbf{PMT}}{(\mathbf{1+i})^3} \cdots\cdots \frac{\mathbf{FV_n}}{(\mathbf{1+i})^n}$$

Chapter 2 Uneven Cashflows

In Chapter 1, we worked with cashflows which either did not contain PMTs or, if they did, the PMTs were equal, both in amount and sign.. These situations occur with fixed-rate mortgages, payments under most promissory notes, the conversion of a single sum annuity to a Present Value, or the extension of a series of equal annuity PMTs into a Future Value or Present Value.

But many financial situations involve payments which change from period to period, either in amount or from a positive to a negative sign. Most income-producing investments involve uneven cashflows, and some even involve negative cashflows.

Whenever you need to deal with uneven **PMT**s, the "horizontal" financial register which has been used,

n	i	PV	PMT	FV

is no longer applicable. You need to learn a different method. Consider a situation in which you are scheduled to receive the following cashflows:

		?
End of Period	1	100
	2	200
i = .10	3	300

What is the **PV** of this series of **PMT**s?

The alternate financial register we must use in dealing with uneven cashflows is composed of keys (1,3), (1,4) and (1,5); specifically the functions printed *in blue* on the lower facet of these keys.[1]

1 Users of HP 17 and 19 models can access the uneven cashflow registers via the TVM /CFLO registers

These functions are, respectively:

n	i	PV	PMT	FV
		CFo	**CFj**	**Nj**

You already know that in order to access these functions *in blue* you must first press the blue key [g] (4,3). You can consider that **CFo** stands for Cash**F**low**o**riginal, or that certain amount of cash invested at the very beginning of an investment. When it represents the initially invested cash, it will be entered as a **negative** number.

CFj stands for the *separate and different*, individual PMTs, which you will enter per period [n]. **Nj** stands for the number of times a particular cashflow, **CFj,** will occur. We will continue to use the [i] register as before.

The first rule to observe is that when you begin a *new* cashflow series involving uneven payments, you should set **Cfo** to zero.[2] By doing so you automatically reset the **n** register to zero and alert the calculator to get ready to *count* the number of *different* **CFj**s you are about to enter. You should do this even though (as in the problem above) there is no **CFo** - no original investment. Once done, the calculator will automatically keep track of the number of *different* cashflows entered, increasing **n** by **1** each time you enter a *different* cashflow into **CFj**. Let's try it:

Key In	Display Shows	Comment
[*f*] [CLX]	**0.00**	**clears all memory (except PRGM)**
0 [g] [CFo]	**0.00**	**sets the n register to 0 and installs 0 in memory cell 0**
100 [g] [CFj]	**100.00**	**installs 100 in memory cell 1**
200 [g] [CFj]	**200.00**	**installs 200 in memory cell 2**
300 [g] [CFj]	**300.00**	**installs 300 in memory cell 3**

Now that you have these uneven cashflows entered into the memory registers of the calculator, you need to inform the calculator of the *discount rate* you wish to use to convert these future values into a single Present Value. The T–Bar indicates that the discount rate [i] will be 10%.

Continue by keying–in:

10 [i]	**10.00**	**enters 10% into register [i]**

2 Whenever appropriate. Some cashflows specify an up-front investment payment. This payment should be entered into CFo as a negative.

You will notice that the top (gold) facet of key (1,3) is marked **NPV**. This stands for **N**et **P**resent **V**alue. It will also deliver the **P**resent **V**alue if no initial value has been entered into **CFo**. This is the case in this example.

Therefore go ahead and solve for (**N**)**PV**:
solving,

f NPV (1,3)	**481.59**

Therefore the Present Value of this series of future cash payments of \$100, \$200 and \$300, discounted at 10% per period, is \$481.59.

Just how did the calculator arrive at this answer? – Just the way we demonstrated in the first chapter when we considered the relationship:

$$\mathbf{PV} = \frac{FV}{(1+i)^n}$$

In this case,

$$\mathbf{PV} = \frac{100}{(1+.10)^1} + \frac{200}{(1+.10)^2} + \frac{300}{(1+.10)^3}$$

$$\mathbf{PV} = 90.91 + 165.29 + 225.39$$

$$\mathbf{PV} = 481.59$$

Verifying Accuracy of Entries

You can check for the accuracy of your entries by recalling the various storage registers to verify what they contain. Recall the value in the **n** register:

Key In	Display	Shows Comment
RCL (4,5) n	**3.00**	**Shows number of *different* CFjs**

Recall the values from the various registers in which the calculator stored your **CFo** and **CFj**s. The memory cells used are in the numerical keypad, 0 through 3.

RCL 0	0.00	Shows 0.00 stored in memory cell 0
RCL 1	100.00	Shows 100.00 stored in memory 1
RCL 2	200.00	Shows 200.00 stored in memory 2
RCL 3	300.00	Shows 300.00 stored in memory 3

Now that you know *where* these values are stored, you can easily change them, if you need to, by "writing over" the stored value and entering the new value directly into the memory cell/register. For example, suppose that the value 200 which you entered into memory cell 2 should have been 500. How can you change this single entry without having to re-enter all the values?

Key In	Display Shows	Comment
500 [STO] (4,4) **2**	500.00	Replaces the value of 200 in memory cell 2 with 500 [3]

solving,

[f] [NPV]	729.53	New Present Value

Adding Cashflows to a Series

If you wish to continue to add cashflows to this series, simply continue to enter the new ones from where you left off. (Don't enter 0 into **CFo** because this would reset the "counter," **n,** equal to 0). Let's add two additional cashflows: 600 and 750.

Key In	Display Shows	Comment
600 [g] [CFj]	**600.00**	**Stores 600 in memory cell 4**
750 [g] [CFj]	**750.00**	**Stores 750 in memory cell 5**

solving,

[f] [NPV]	**1605.03**	**New PV**

Using [Nj] To Indicate Repeating Cashflows

Cashflows frequently repeat themselves for a number of periods before changing. Consider a situation in which a cashflow of $1,000 is received for 6 periods; then the cashflow increases to $1,200 for 3 periods and finally rises to $1,500 for the last 2 periods.

What is this cashflow's presently worth (what is its **PV?**) to an investor who requires a 12% return per period on his money?

End of Period

i =12%

	(?)	
1	1000	
2	1000	
3	1000	6 times
4	1000	
5	1000	
6	1000	

3 The (4,4) indicates the key position, not the entry.

7	1200	
8	1200 ←	3 times
9	1200	
10	1500	
11	1500 ←	2 times

Key In	Display Shows	Comment
[*f*] [CLX]	**0.00**	**Clears memory**
0 [g] [CFo]	**0.00**	**sets the n register to 0 and installs 0 in memory cell 0**
1000 [g] [CFj]	**1,000.00**	**Stores 1000 in memory 1**
6 [g] [Nj]	**6.00**	**Informs the calculator that this cashflow occurs 6 times**
1200 [g] [CFj]	**1,200.00**	**Stores the second *different* cashflow in memory cell 2**
3 [g] [Nj]	**3.00**	**Informs the calculator that this cashflow occurs 3 times**
1500 [g] [CFj]	**1,500.00**	**Stores the third *different* cashflow in memory cell 3**
2 [g] [Nj]	**2.00**	**Informs the calculator that this cashflow occurs 2 times**

Now inform the calculator of the discount rate you wish to use.

12 [i]	**12.00**	**Enters 12% into the i register**

Before solving for the answer, check to see how many *different* cashflows the calculator has stored. You have entered three *different* cashflows: one occurring 6 times, a second occurring 3 times, and the third occurring 2 times.

[RCL] [n]	**3.00**	**Shows that 3 different** cashflows have been entered.

Also, check to see that the value of each of the 3 different cashflows stored in memory is correct:

[RCL] **1**	**1000.00**	**Value stored in memory cell 1**
[RCL] **2**	**1200.00**	**Value stored in memory cell 2**
[RCL] **3**	**1500.00**	**Value stored in memory cell 3**

solving,

[f] [NPV]	**6,485.79**	**Present Value of 11 sequential cashflows as specified.**

Some Limitations on the Number of Cashflows

The HP-12C calculator will accept 10 cashflows and store each in Registers 0 through 9, and *up to* 10 more in additional registers **.0** through **.9**.[4] Each of these cashflows may be repeated a maximum of 99 times. Therefore. if you had a certain cashflow, 180, which occurred 100 times, you could not enter 100 into **Nj** since it exceeds the calculator's limit of 99 repetitions per memory cell (register). You could, however, enter this cashflow as follows

180 [g] [CFj]	**180.00**
98 [g] [Nj]	**98.00**
180 [g] [CFj]	**180.00**
2 [g] [Nj]	**2.00**

Any other combination of numbers (**Nj**s) totaling **100** would do as well. But if you attempt to enter any one cashflow 100 times or more, the calculator will display an **Error Code (6)** message.

If, however, the cashflow series consists of more than 20 *different* cashflows, then the longer formulas (see Appendix) or a computer spreadsheet can be used.

Adjusting for Incorrect Entries

Now and then you will make errors in entering cashflows. If the cashflow contains only a few entries, it's not terribly inconvenient to clear the memories and re-enter the data. But if there is a long series of **CFj**s, re-entering all the data from scratch can be frustrating. Here's how to correct for data entry errors.

Wrong Amounts

If you have entered an incorrect amount in the series, simply key-in the correct amount, press [STO] (4,4) and deposit the corrected amount in the cell (R_X) in which it belongs.
For example, enter the following cashflow series in your calculator:

4 The 12–C handbook defines these 20 storage registers as $\mathbf{R_0}$ through $\mathbf{R_9}$ and $\mathbf{R._0}$ through $\mathbf{R._9}$ Note the decimal points in the second set . You can examine the contents of any register by using the recall **RCL** key (4,5) followed by number of the cell you wish to see. There are times when some of these memory cells are used for other functions. In those cases, fewer than 20 storage cells may be available.

		(PV?)
End of Period	1	100
	2	**400**
i = 10%	3	500

The second entry, **400**, should have been **300**. To make the correction, follow these steps:

Key In	**Display Shows**	**Comment**
300 [STO] 2	**300.00**	**Over-rides the value in R_2**
		Changing it from 400 to 300

Re-solving,

[*f*] [**NPV**]

If you make this correction properly, the PV of the series will change from 797.15 to 714.50.

Wrong Njs

Occasionally the number of times a unique cashflow occurs is mis-entered in Nj. The wrong Nj entry can be replaced without disturbing the remainder of your entries.

For example, suppose that you entered the following set of values:

		(PV?)	
End of Period	1	100	1
	2	**300**	
i = .10	3	**300**	**2**
	4	**300**	
	5	**300**	
	6	500	3
	7	700	4
	8	900	5

The error consists in the fact that the 2nd, separate cashflow of 300 occurs 3 times not 4 times.

The T-Bar should have been:

		(?) PV	
End of Period	1	100	1
	2	**300**	
i = .10	3	**300**	2
	4	**300**	
	5	500	3
	6	700	4
	7	900	5

Here are the keystrokes to correct this error:

	Key In	Display Shows	Comment
	2 [n]	2.00	**Informs the calculator that a change is to be made affecting the second CFj entry**
	3 [g] [Nj]	3.00	**Changes Nj from 4 to 3**
	5 [n]	5.00	**Restores n to 5, the correct number of <u>separate</u> CFj entries**
Re-solving,	[*f*] [NPV]	?	

If you have made these adjustments correctly, the NPV will change from 2,016.72 to 1,936.58.

The difference is a loss of the Present Value of one entry of 300, which, however, is offset by the fact that the 3rd, 4th and 5th cashflows each advance one period in time, and gain in Present Value.

The value stored in [Nj] may be viewed by inserting into [n] the number of the cashflow you wish to examine. Then [RCL] [g] [Nj] to view the value stored for that cashflow.

The most common error made in changing [Nj] entries is to forget to restore the proper [n] value following the corrections. Note that [n] , in this case, should be = 5, both before and after the correction. Recall [n] to verify this. If it is not 5, make the necessary change.

Uneven Payments Made at the Beginning of the Period

When we dealt with even cashflows using the "horizontal" registers,[5] we could alter the timing of the PMTs by use of the blue key, **g,** followed by keys **BEG** ((1,7) and **End** (1,8). When we deal with uneven cashflows, however, these keys have absolutely <u>no</u> effect on the timing of the payments. All PMTs entered into the "uneven cashflow" registers are regarded as occurring at the end of the period. Therefore we must have another way to structure the cashflow series when the first PMT occurs at the very beginning of the series, and therefore ought not be discounted. There are two methods we can use to accomplish this.

First, we can simply remove the first cashflow from the series and treat (discount) all remaining cashflows as though they occur at the end of the period. After the PV of this series is determined, we can add back the value of the cashflow we removed. The result will be the correct **PV** of the series. Alternately, and more conveniently, we can store the first cashflow in register 0 as a **positive** number, and enter all the remaining cashflows in the normal manner. When the calculator

5 Keys (1,1) through (5,5)

has finished determining the PV of the series, it will "look" into the 0 register and add whatever value is there to the total it has computed. The result will be the correct **PV** of the series. The calculator will not discount any value in memory Register 0.

Valuation of Leases

The ability to discount uneven cashflows is a requirement for the valuation of leases since almost all leases furnish uneven **PMTs** in the form of rents which change from period to period. Since leases can be bought and sold, and also used as collateral for loans, valuing a lease is a pre-condition of many financial transactions. Consider this situation:

> You have just completed the leasing of your 24,000 S.F. industrial building to a credit tenant[6] for 7 years.
> The tenant has contracted to pay rent as follows:
> First three years ....$12,000 per month
> Next two years ....$13,000 per month
> Last two years ....$14,000 per month
> As owner of the building, you are pressed for cash and would like to sell the right to collect these lease PMTs to an investor.
> If the investor requires a 15% annual return, what is the likely (present) value of the lease to this investor?

		PV?	
End of Period	**2**	12,000	
	"	"	35months
i = 15%/12	36	12,000	
	37	13,000	
	"	"	24 months
	60	13,000	
	61	14,000	
	62		
	"	"	24 months
	84	14,000	

This is a problem which requires you to discount the future receipt of the monthly rent at a nominal rate of 15% per year. Note that rent is paid monthly, while the required rate of return in the problem is expressed annually.

6 A credit tenant is one whose assets are sufficient to act as security for the entire amount due under the lease, even if the tenant chooses to vacate.

The keystrokes would be:

	Key In	Display Shows	Comment
	12,000 g CFo	**12,000.00**	**Stores the first lease PMT as a positive in register 0**
	12,000 g CFj	**12,000.00**	**Enters 12,000 as a PMT**
	***35* g Nj**	**35.00**	**Enters 35 PMTs in register 1**
	13,000 g CFj	**13,000.00**	**The succeeding PMTs**
	24 g Nj	**24.00**	
	14,000 g CFj	**14000.00**	
	24 g Nj	**24.00**	
	15 g i	**1.25**	**The *monthly* discount rate**
solving	**f NPV**	**662,811.59**	**The PV of the lease**

If the investor pays exactly $662,811.59, and if the tenant pays the rent as scheduled, the investor will realize a 15% rate of return (yield) on his money over the term of the lease. If he pays *more* than $662,811.59, he will earn *less* than 15%, since a higher Present Value requires a lower discount rate. If he pays *less* than $662,811.59, he will have a *higher* yield since a lower Present Value requires the application of a higher discount rate.

The discount rate *is* the investor's yield.

The Concept of *Net* Present Value (NPV)

Most investments involve up-front cash. These out-of-pocket, negative cashflows can be represented by entering the amount of the original investment in **CFo** as a negative number. The calculator stores **CFo** in register R_0 (in memory cell 0 on the numeric keypad).

When you press ***f*, NPV (1,3)**, after having entered a series of cashflows, the calculator responds by determining the Present Value of each individual cashflow, discounting each one back to a **PV** using the discount rate you entered in i . Then it adds the resulting present values together, including the value in register R_0. When the value in register R_0 (**CFo**) is 0, the result is not different from the total of the PV for all future cashflows and the result delivered is the **Present Value** of the cashflow. But when **CFo** is a negative number, indicating an out-of-pocket initial cashflow, the calculator adds the positive and negative results and displays the **Net Present Value.** Therefore the summation of all the **PVs + (– CFo) =** N**et P**resent **V**alue**.**

The NPV of a series is nothing more than the PV of a series minus the amount of the original investment.

For example,

> Let's assume the buyer of the lease on the industrial building in the preceding example negotiated to acquire the lease for $580,000.
>
> What would be the **NPV** of this lease in the hands of this investor?

Hopefully, your calculator still retains the values from this recent problem. (If not, please re-enter them.) If so, you need only add the **CFo** value, –$580,000, to the first PMT, +$12,000, and place the total (–$568,000) in register R_0 in order to structure a Net Present Value calculation.[7] Here are the keystrokes:

Key In	Display Shows	Comments
568,000 [CHS] [STO] 0	**-568,000.00**	**Changes the sign, stores –568,000 in R_0, the cell in which the CFo is stored. Using STO instead of [g] [CFo] avoids resetting n to zero.**
[f] [NPV]	**82,811.59**	**The NPV, or the sum of the present value of $662,811.59 & the initial investment of –$580,000.**

Usefulness of the NPV Method

The NPV tells us whether or not the **Present Value** of the future cashflows is equal to, less than, or greater than the amount of cash initially invested. Many (if not most) finance professionals prefer the **NPV** method of ranking an investment over any other method. In theory, any number of alternative investments can be reduced to a **NPV** number and then prioritized or rank-ordered. Investments which have a negative **NPV** would be eliminated from consideration because the financially equivalent present worth of the future returns would be less than the cash required to obtain them in the first place.

Of those alternatives which result in a positive **NPV,** *and which carry comparable risks,* the investment which would deliver the highest **NPV** *may* be the logical choice. If the investments carry dissimilar degrees of risk, ranking them according to their **NPV**s would be inappropriate *unless* the **NPV** of each investment were determined using a risk-adjusted discount rate.

7 The investor would pay $580,000 but immediately receive the first month's rent of $12,000.

Discount Rate Varies with Risk

We have said little about the **risk** associated with the probability of receiving future cash flows. The difference in the degree of risk associated with each financial alternative should be reflected in the selected discount rate. A higher risk calls for a higher discount rate, and, as you already know from the schoolyard seesaw analogy, a higher discount rate results in a lower **P**resent **V**alue. Therefore a riskier investment usually has a lower investment value than a less risky one because the selection of a higher discount rate results in a lower **P**resent **V**alue.

We will expand on the calculation of risk and the selection of a discount rate in Chapter 7, but it should be noted that the use of the NPV method to evaluate an investment opportunity requires the investor to *furnish a discount rate* commensurate with perceived risk — whereas the IRR *forces* a discount rate without any regard to risk.

Return on Investment (ROI) - a Profitability Index

A variation of the **NPV** method is to determine the Present Value of the future returns and divide this number by the amount of the total [8] original investment, the **CFo**. In the example above, the Return on Investment (**ROI**) for the investor would be:

$$\frac{\sum \mathbf{PVs}}{\mathbf{CFo}} = \frac{\mathbf{662{,}811.59}}{\mathbf{580{,}000}} = \mathbf{1.14}$$

This indicates that the investor may recover 100% of his capital and an additional profit of 14%

The Internal Rate of Return (IRR)

The *single* discount rate which will produce a **N**et **P**resent **V**alue = 0 is called the **I**nternal **R**ate of **R**eturn (IRR). Because the IRR is an important measure of investment performance, we will defer any detailed discussion of it to Chapter 4. But before we leave this section, let's address a few anomalies.

Negative Cashflows

Other than the initial cashflow, **CFo**, all of our examples have used positive cashflows occurring in later periods. But there are many circumstances in which negative cashflows occur in financial investments. When you discount a negative future cashflow (or **PMT**) the result is a negative Present Value:

$$\frac{-\mathbf{FV}}{(1+\mathbf{i})^{\mathbf{n}}} = -\mathbf{PV}$$

8 Equity + Debt

Contrary to popular opinion, negative cashflows, in and of themselves, are not necessarily indicative of a poor investment. Consider a situation in which a $1,000 investment is scheduled to return the following cashflows:

	CFo	(1,000)
End of Period	1	100
	2	200
i = 10%	3	1200

If the initial investment of $1,000 is made in one lump sum at the beginning of the investment period, the **NPV** of this investment is:

	1000 [CHS] [g] [CFo]	**-1,000.00**
	100 [g] [CFj]	**100.00**
	200 [g] [CFj]	**200.00**
	1200 [g] [CFj]	**1,200.00**
	10 [i]	**10.00**
solving,	**[f] [NPV]**	**157.78**

[You can also *simultaneously* solve for the IRR:

[f] [IRR] (1,5)	**16.16%]**

But if the initial $1,000 could be paid in two annual (EOP) installments of $500, note the effect on the **NPV:**

	NPV=	CFo **0** + ∑PV
End of Period	1	100 – 500 = –400
	2	200 – 500 = –300
i = .10	3	1200

Translating,

	Key In	Display Shows
	0 g CFo	**0.00**
	400 CHS g CFj	**-400.00**
	300 CHS g CFj	**-300.00**
	1200 g CFj	**1,200.00**
	10 i	**10.00**
solving,	**f NPV**	**290.01**[9]

---

9 Partnership contributions often involve this type of staged investment

The effect on the IRR is also dramatic:

[f] [IRR] **39.72**

Many limited partnership interests have been sold using “staged” investor contributions. These initial payments are negative (out-of-pocket) payments which may occur simultaneously with return payouts (if any) from operations. The net cashflow position can be depicted by adding the T-bars representing both cashflows :

	Staged Investment	Operating Returns	Net Cashflow
	0	0	0
EOP 1	(1,500)	590	(910)
EOP 2	(1,500)	640	(860)

Although staged initial investments may improve the investor’s yield, the General Partner will charge interest on the deferred amounts, thereby reducing the net payback. These interest payments should be included as negative cashflows in the periods in which they are payable.

Calculating PV and NPV on the Computer's Spreadsheet

Spreadsheets handle the PV and NPV of cashflows in a manner quite similar to the HP-12C.

The computer function [=**PV(*rate, nper, pmt, fv, type*)**] accepts only identical payments (PMTs), and is therefore equivalent to the "horizontal" registers[10] of the HP–12C. The ***type*** position in the formula provides an opportunity to solve for PMTs received at the end (EOP) or at the beginning (BOP) of the period. Replace ***type*** with a 1 if PMTs are received BOP; enter 0 if PMTs are received at EOP. The default (do nothing) value = 0.

> **Tip**: To call up the formula for any Excel function, type in the function desired and press Ctrl Shift A;
> e.g. = **IRR** Ctrl Shift A will deliver
> **=IRR(values, guess)**
> This shortcut may not work on laptops which use a condensed version of Excel.

If PMTs are *uneven*, the function[11] [**=NPV(*rate, CF_1, CF_2, CF_2, ... CF_n,*)**] must be used. But this function, as is true with the HP-12C, ***returns only the Present Value of the series, not the NPV !***

Therefore if the **Net** Present Value of the uneven cashflow is desired, the original investment value (**CF_0**) must be subtracted from the result, just as we have done above. Do not make **CF_0** the

10 Keys 1,1 through 1,5.
11 Analogous to the "vertical" registers.

first cashflow in the spreadsheet's menu formula; doing so will effectively push all succeeding cashflows back one period in time and distort (lessen) the value of the PV.
If the uneven cashflows are received at the beginning of each period (BOP), then the first cashflow must also be extracted from the series and added to the result, since there is no ***type*** position in the NPV formula. Excel's function for an uneven cashflow when the PMTs are received BOP would be:

[= NPV(rate, CF_2.....CF_n) – Initial Investment + CF_1]

Varying the Discount Rate

The selection of a *single* discount rate implies that the risk of receiving future payments from an investment will be equal over all time periods. In most circumstances this will not be so. As we attempt to anticipate farther and farther into the future, the margin for error usually increases, as does the risk of not receiving our return cashflows. Raising the exponent **n** on the expression $(1+i)^n$ compensates for the time delay in receiving a future cashflow, but it does not reflect the risk of receiving $cashflow_n$. That risk must be built into the selection of the discount rate, **i.**

It's worth noting, however, that the discount rate need not always increase. For some new ventures the earliest cashflows are the riskiest while a concept or product is tested. Once a project has proven-out, it may be entirely appropriate to lower the discount rate, even though **n** continues to increase to reflect the time value of money.

For example, it is quite common to value the worth of established small business ventures by *varying* the discount rate applied to the future cashflows from the operation of the business. Unfortunately there is no convenient way[12] to instruct the calculator to vary the discount rate to be applied to future cashflows. If different discount rates are to be used, the **PV** of each future cashflow must be determined using its particular discount rate. Then the total of all the resulting individual **PV**s, the can be determined by simple addition. Consider this situation:

> Smith is considering the purchase of his first small business, a well-located automatic laundry in a newly developing section of town. The business has been in operation for just two years and is doing quite well. An examination of the operating records, together with estimates of increased usage by a growing local population, lead Smith to believe that the business could produce significant gains. In particular, a new apartment house is to be constructed only a short distance from this location within the next year and will be in operation in two years. If this happens, Smith estimates the following amounts of net cash from operations:

Year 1	$35,000	Year 4	$65,000
Year 2	$38,500	Year 5	$70,000
Year 3	$60,000		

12 The HP-12C can be programmed to perform this kind of calculation, but programming a hand-held calculator is cumbersome in comparison to solving this kind of problem on a computer's spreadsheet

> Smith is comfortable with discounting the first year's income @ 25%, and the second year's income @ 30%. But his confidence for the outyears is shakier, and he feels that he must have a 50% return to justify the risk that the apartment house may never be built.
>
> What price should Smith pay for this business?

This is the kind of problem which cannot be solved on a financial calculator in one operation because you cannot apply different discount rates to an uneven cashflow. We can, however, solve the problem by breaking it down into its component parts, determining the PV for each part, and then adding the resultant PVs together:

Let's construct three T-Bars:

EOY		
1	35,000	

i = .25

1	**0** ←
2	38,500

i = .30

1	**0** ←
2	**0** ←
3	60,000
4	65,000
5	70,000

i = .50

Note that zeros have been inserted in the second and third T-Bars to maintain the time relationships among the payments. If these three T-Bars were added together laterally, they would replicate the original cashflow series.

By applying the methods we have already described, the following PVs are calculated:

PV Year 1 @ 25%	28,000
PV Year 2 @ 30%	22,781
PV Yrs. 3,4,5 @ 50%	39,835
Σ PVs	**$90,616**

The value of the business is $90,616 *because this is the sum of all the future cashflows, which the owner can expect to receive , discounted at acceptable rates over his intended holding period.*

Smith's valuation problem could also be presented as:

$$\mathbf{PV} = \frac{\$35{,}000}{(1+0.25)^1} + \frac{\$38{,}500}{(1+0.30)^2} + \frac{\$60{,}000}{(1+0.50)^3} + \frac{\$65{,}000}{(1+0.50)^4} + \frac{\$70{,}000}{(1+0.50)^5} = \mathbf{\$90{,}616}$$

If it could be assumed that the business will continue to produce income beyond the 5^{th} year, and that this income would remain at $70,000 annually, the PV of that future income, continuing indefinitely (in perpetuity), would be \$70,000 / 0.50, or \$140,000. But this Present Value would be the value at the end of the 5th year. Its Present Value today would be:

$$\mathbf{PV} = \frac{\$140{,}000}{(1+0.50)^5} = \$18{,}436$$

This low value reflects a cashflow discounted at a high rate and to be received so far in the future. If Mr. Smith were forced to do so as the result of negotiations, he would include the PV of the reversion value to his 5-year calculations. His total price would be \$90,616 + 18,436 = \$109,052

Using Excel To Calculate PVs Using Varying Discount Rates

The easiest method of determining PVs (and FVs) for uneven cashflows with varying discount rates is to construct a spreadsheet, such as Excel. The formula is the basic formula:

$(PV) = \frac{PMT}{(1+i)^n}$ and is inserted in Column D, below. (The formula appears in Column E).

	A	B	C	D	E
1	**Period**	**CFlow**	**Disc.Rate**	**PV**	**Formula in D**
2	**1**	35,000	25%	28,000.00	=B2/(1+C2)^A2
3	**2**	38,500	30%	22,781.07	=B3/(1+C3)^A3
4	**3**	60,000	50%	17,777.78	=B4/(1+C4)^A4
5	**4**	65,000	50%	12,839.51	=B5/(1+C5)^A5
6	**5**	70,000	50%	9,218.10	=B6/(1+C6)^A6
7			**Total PV**	**90,616.46**	=Sum(D2:D6)

If these cashflows had occurred at the beginning of the period, the value of the exponent would be decreased by 1: i.e. [=B2/(1+C2)^(A2-**1**)]. Since the value of (A2–1) would then be equal to zero, the first cashflow would not be discounted at all.

A fair question at this point might be: "Given these cashflows, and given the Present Value which you have calculated, what would be Smith's overall return (yield) on his investment over the depicted holding period?" This question asks for the Internal rate of Return (IRR) on the investment.
That return is:

Initial Investment	**– 90,616.46**
1	35,000
2	38,500
3	60,000
4	65,000
5	70,000
IRR =	**43.3%**

Note that although the Present Value of this series of cashflows was arrived at using different discount rates per period, we can still express the *overall* yield by use of the IRR.

This example, though simple, is very important since it is the fundamental method of valuing <u>any</u> investment.

The value of *any* investment is equal to the value of all the future cashflows, including any reversion amount, which will accrue to the investor, discounted at an acceptable rate.

Chapter Summary

1. Whenever PMTs are unequal in amount *or* sign, the determination of the Present Value of a cashflow must be made by using the CFo, CFj and Nj registers, or by a spreadsheet.
2. The determination of the Present Value of an uneven cashflow (EOP) follows the same algebraic formula used in the determination of the PV of even cashflows:

$$\mathbf{PV} = \frac{C_1}{(1+i)^1} + \frac{C_2}{(1+i)^2} + \frac{C_3}{(1+i)^3} + \frac{C_4}{(1+i)^4} + \frac{C_5}{(1+i)^5} \quad \ldots\ldots\ldots\ldots$$

3. The HP-12C can never accommodate more than 20 *separate, distinct,* cashflow entries, and at times less than 20. Each separate cashflow, however, may be repeated up to 99 times *per storage register*.
4. There is no button to adjust uneven cashflows for **BEGIN** or **END** of period. When the first cashflow occurs at the beginning of the period it should be removed from the cashflow stream, the PV of the remaining cashflows determined, and then the original cashflow added back. When a cashflow occurs at the beginning of a series, it is not subject to a discount.
5. Discounting all future cashflows and totaling their numbers results in the Present Value of an investment. If the original cash invested is then subtracted, the result is the **Net Present Value** of an Investment. If no initial cash were invested, the result would simply be the **Present Value** of the investment.
6. A positive NPV indicates hat the investor will receive a yield at least equal to the discount rate used to determine the NPV. A negative NPV indicates that the Present Value of all future cashflows, discounted at the rate used, will be less than the amount the investor paid to secure these cashflows. The investor will not reach his targeted rate of return.
7. That single discount rate which, when applied to all future cashflows, results in a total PV exactly equal to the original investment is called the **Internal Rate of Return**.
8. Risk is the likelihood that a beneficial future event (reward) may never happen, or that an adverse event will happen. High risk situations indicate the use of high discount rates. <u>Risk runs with reward</u>.

9. Discounting a Future Value (or PMT) which is negative results in a negative Present Value.
10. In those instances when different discount rates are to be applied to a cashflow series, the separate PVs must be determined for each discount rate. The resulting PVs can then be added together to arrive at the value of the investment. Once the PV is known, the overall yield can be expressed in the form of the Internal Rate of Return by using the PV as the initial investment and using the <u>original</u> cashflows in the computation.

Chapter 3 Annuities & Rates

Financial texts refer to a regularly occurring sum of cash, either paid or received, as an "annuity." "Annuity" once meant a sum paid on an annual schedule,[1] but the term is now more loosely applied to any sum paid on a *regular* basis.

When a regular PMT is received over a limited time, we can say that it is a *finite* annuity. If the PMT is to be received forever, as it might be in the case of an endowment, it is known as a *perpetual* annuity.

Both finite and perpetual annuities can provide payments either at the beginning or end of the PMT period. Annuities which provide PMTs at the beginning of the period are known as *Annuities Due*; those whose PMTs are furnished at the end of the period are *Ordinary Annuities.* The PMT may be *fixed or variable*[2] resulting in *Fixed/Variable, Ordinary Annuities or Annuities Due.*

The subject of annuities is a very important one for the serious investor, for the corporate financial office, the banker, the financial planner and for almost all other financial professionals because so much of their practices revolve around the purchase, sale or valuation of *Annuities.*

For example, a banker who makes a loan may be considered to be buying a finite annuity from the borrower: the banker pays an up-front lump sum for the right to receive a stipulated PMT over a pre-specified number of periods. The borrower is selling an annuity: the obligation to make the PMTs in return for the receipt of a lump sum today. Both lenders and borrowers are dealing in *annuities.*

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1 L. Annus, year.

2 A "variable annuity" as used here simply means a PMT which changes from time to time. It does not mean an investment contract sold by insurance companies.

Perpetual Annuities

Consider the following situation:

> An alumnus wishes to endow his alma mater in order to provide annual operating funds of $100,000 per year *forever.* He determines that the invested principal (corpus) of the endowment could earn 8% each year
>
> What must be the amount of his contribution if payments will begin one year from the effective date of the endowment.?

This situation represents a minor problem for us in that the PMT of $100,000 is expected to be received *"forever."* Therefore **n** would have to be set to infinity – a challenge to which neither the calculator nor the computer's spreadsheet is quite equal.

You can see that the amount of the endowment must be equal to the sum of the PVs of all future PMTs, discounted to infinity, at 8% per year, or,

$$\mathbf{PV} = \frac{100{,}000}{(1+.08)^1} + \frac{100{,}000}{(1+.08)^2} + \frac{100{,}000}{(1+.08)^3} + \frac{100{,}000}{(1+.08)^4} \ldots\ldots\ldots\infty$$

There is a way to calculate the PV of an infinite series of ***equal*** PMTs. This calculation reduces to a simple but precise formula[3] for the sum of the present values of an infinite series:

$$\mathbf{PV} = \frac{C}{i} \quad \text{(where C is the } \textit{equal} \text{ PMT)}$$

Therefore the alumnus must set aside: $\mathbf{PV} = \dfrac{\$100{,}000}{.08} = \$1{,}250{,}000$

This kind of valuation of an infinite series of future PMTs is important because it forms the basis for the *Capitalization Method* of valuing a level stream of income from real estate and sometimes from stocks. We will discuss the "cap method" of establishing the value of income-producing property by "capitalizing" its net operating income, and the valuation of stocks by "capitalizing" dividends, in later chapters.

When the perpetual PMT is scheduled to be paid at the beginning of the period (BOP), one additional payment of **C** is required, and the formula becomes:

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{i}} + \mathbf{C},$$

or alternately,

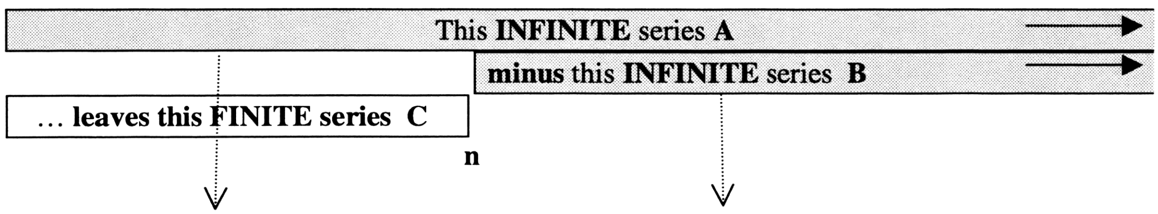
$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{i}} * (1+i)$$

3 The mathematical derivation can be found in the Appendix.

Present Value of Annuities for a Definite Period

A more common need in building financial plans is the need to capitalize a trust or an account to meet future cashflow requirements of an education fund, a retirement plan or any special-needs account. These kinds of accounts can be funded in one lump sum but are more commonly provided for by periodic payments which will grow to meet the future need.

The Present Value of a series of equal cashflows for a finite period, **n**, is really the difference between the Present Values of two infinite series of cashflows, one of which begins today and the other at the end of period **n:**



Or, $\quad \mathbf{PV_A} = \dfrac{\mathbf{C}}{\mathbf{i}} \quad$ **minus** $\quad \mathbf{PV_B} = \dfrac{\mathbf{C}}{\mathbf{i}}$

But $\mathbf{PV_B}$ occurs **n** periods in the future. Therefore we must discount PV_B by $(1+i)^n$ to be able to express its value today. The Present Value of a finite series of *equal* cashflows **n** long, discounted at rate i becomes:

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{i}} - \frac{\mathbf{C}}{\mathbf{i} * (\mathbf{1}+\mathbf{i})^{\mathbf{n}}} \quad \textbf{or,}$$

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{i}}\left(\mathbf{1} - \frac{\mathbf{1}}{(\mathbf{1}+\mathbf{i})^{\mathbf{n}}}\right)$$

Fortunately, the determination of the Present Value of a finite series of *equal* cashflows is a problem easily handled by either the calculator or a computer's spreadsheet. Whenever a Future Value is also involved, its Present Value can be added to (or subtracted from) the Present Value of the cashflows:

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{i}}\left(\mathbf{1} - \frac{\mathbf{1}}{(\mathbf{1}+\mathbf{i})^{\mathbf{n}}}\right) \pm \frac{\mathbf{FV}}{(\mathbf{1}+\mathbf{i})^{\mathbf{n}}}$$

Notice that in this case the Future Value is discounted by the expression $(1+i)^{\mathbf{n}}$, where **n** is also the timing of last cashflow in a series **n** long. This indicates that the last cashflow occurs simultaneously with the last PMT (C). When the last cashflow occurs one period after the last PMT, as it does in equipment and auto leasing, the FV must be discounted at $(1+i)^{\mathbf{n+1}}$.

Perpetual Annuities Adjusted for Growth or Inflation

It is often necessary to determine the Present Value of an annuity which grows by a constant factor each period. An increase to offset inflation could be one kind of growth factor adjustment. Here's an example:

> The alumnus who wishes to endow his alma mater realizes that a constant sum of $100,000 each year will be subject to inflation, so that each dollar in later years will lose purchasing power. In order to allow for inflation, he must set aside a sufficiently large sum today to allow for an estimated 4% annual inflation rate.
> With what amount must he now capitalize the fund?

Once again, the future annual amounts will have a Present Value of...

$$\mathbf{PV} = \frac{100{,}000}{(1+.08)^1} + \frac{100{,}000(1+\mathbf{g})^1}{(1+.08)^2} + \frac{100{,}000(1+\mathbf{g})^2}{(1+.08)^3} + \frac{100{,}000(1+\mathbf{g})^3}{(1+.08)^4} \;.....\infty$$

where **g** = inflation (or growth) factor.[4] Note that the first annual payment of an ordinary annuity is *not* adjusted for inflation since it is stipulated. <u>This is an important observation.</u>

The formula for the PV of an infinite series of *equal* payments (**C**), payable at the end of the period, invested at rate **i** and growing at a constant rate **g** per period simplifies to:

$$\mathbf{PV} = \frac{\mathbf{C}}{(\mathbf{i}-\mathbf{g})}$$

It almost goes without saying that **g** must always be less than **i**.

Therefore, if the alumnus anticipates future inflation to be 4% per year, he must contribute:

$$\mathbf{PV} = \frac{\$100000}{(.08-.04)} = \$2{,}500{,}000$$

Notice that the first cashflow in an Ordinary Annuity is not incremented by **g**: it is *stipulated.* Notice also that the required amount of the endowment decreases with any combination of increasing rates of return **i** and decreasing rates of inflation rate **g**.[5] If the endowment could be managed for a greater annual return (e.g., **i** = 0.10), in a 3% inflationary environment (**g** = .03), the amount of the required endowment could be reduced substantially:

$$\mathbf{PV} = \frac{\$100000}{(.10-.03)} = \$1{,}428{,}571.$$

---

4 Inflation is but one kind of growth factor.

5 The result is valid only as long as **i** is greater than **g**. The value of **g**, however, may be negative.

In dealing with annuities both the rate of return on invested cash and the prospective inflation rate are important factors.

PV of Finite Annuities Subject to a Growth Factor

> If we continue with the endowment situation, we can postulate that our flagging alumnus has decided to limit his generosity to 20 years. In order to provide $100,000 in *constant dollars*[6] annually, we can ask what amount must now be set aside to offset inflation of 4% per year over the next 20 years? Cash in the fund will still earn 8% per year, and the first payment will still be made at the end of the first year (an Ordinary Annuity).

We need a formula for the Present Value of a **finite** cashflow series, growing at rate **g** per period and which returns a yield of **i** per period. Payments are limited to 20 years, **n**. The first payment begins at the end of the first year in which the fund is established (an Ordinary Annuity). The first payment, though occurring at the end of the first period, is unadjusted for inflation because it is stipulated.

$$\mathbf{PV} = \frac{C}{i - g} * \left[1 - \left[\frac{(1+g)}{(1+i)}\right]^n\right]$$

$$\mathbf{PV} = \frac{C}{.08 - .04} * \left[1 - \left[\frac{(1+.04)}{(1+.08)}\right]^{20}\right]$$

$$\mathbf{PV} = \frac{\$100{,}000}{.04} * (1 - 0.47010)$$

$$\mathbf{PV} = \$2{,}500{,}000 * 0.52990 = \mathbf{\$1{,}324{,}746.14}$$

Although this formula seems to be a bit complicated, it is quite useful when the number of periods increases beyond 20, since the HP-12C is limited to a maximum of 20 *different* cashflows.[7]

The Annuity Due

If the first payment is to be disbursed at the time the fund is established (creating an Annuity Due), then we seek the formula for a finite series of cashflows, growing at rate **g** per year, and discounted at rate **i** per year, and whose payments occur at the **BOP**.

---

6 Dollars which are adjusted for inflation to maintain their purchasing power.

7 Even though each of these cashflows may be repeated up to 99 times per storage register.

This formula is:

$$\textbf{PV} = \frac{C\,(\mathbf{1+i})}{(i - g)} \;*\; \left[1 - \left[\frac{(1+g)}{(1+i)}\right]^{n}\right]$$

$$\textbf{PV} = \frac{\$100{,}000\,(1+.08)}{(.08 - .04)} \;*\; \left[1 - \left[\frac{(1+.04)}{(1+.08)}\right]^{20}\right]$$

$$\textbf{PV} = \frac{\mathbf{\$108{,}000}}{\mathbf{.04}} * (\mathbf{1 - .470102}) = \$1{,}430{,}725.83$$

Note that dividing the formula for an Annuity Due by the formula for its Ordinary Annuity counterpart results in (1+ i). This is a handy relationship to remember:

PV of Ordinary Annuity * (1+i) = PV of Annuity Due

When the PMTs are to be made on a monthly basis both **i** and **g** must be expressed monthly. Therefore (1+.08) becomes $(1+\frac{.08}{12}) = \frac{12.08}{12}$. The expression (.08 – .04) becomes $\frac{.04}{12}$.

Note that this is mathematically quite different from $\frac{(1.08)}{12}$, a value often incorrectly applied to this formula; $(1+\frac{.08}{12})$ is a value greater than 1, while $\frac{(1.08)}{12}$ is a value less than one.

Handling Constant Growth Cashflows Which Exceed Calculator Memory

There are many instances when you will be called upon to determine the Present Value of a sum which grows at a constant rate over very long periods of time. For example:

> Determine the present value of an Annuity of $1,000 which grows at a constant rate of 4.5% per annum over 25 years. Payments are received at the EOP, and the applicable discount rate is 8% p.a..

This situation presents two problems: first, to determine each of the 24 succeeding Annual cashflows, increased at the rate of 4.5% p.a.; then to enter these 24 different cashflows into a calculator which can accept only 20 different entries. The problem may also be solved on a spreadsheet, but requires set-up time.

The solution is much more easily handled by applying the formula for the **PV** of a finite, O*rdinary* Annuity, growing at a *constant* rate **g,** for **n** periods, **PMT**s at EOP:

$$\mathbf{PV} = \frac{\mathbf{C}}{\mathbf{(i - g)}} * \left[1 - \left[\frac{(1+g)}{(1+i)}\right]^n\right]$$

$$\mathbf{PV} = \frac{\$1{,}000}{(.08 - .045)} * \left[1 - \left[\frac{(1+.045)}{(1+.08)}\right]^{25}\right]$$ see footnote[8]

PV = \$28,571.43 * 0.56115 = **\$16,032.94...**

Adjusting Interest Rates To Compensate for Inflation

There is a second convenient method to adjust cashflows for inflation, and it holds true in most cases. Consider the following rationale: If \$1 is invested to yield an 8% return in one year, the FV of the investment will be:

FV = PV (1+.08) = \$1* (1.08) = \$1.08

If, however, inflation has advanced during this period at a 4% annual rate, then in order to achieve a true (constant dollar) 8% yield, the investor must receive a return which includes an allowance for inflation:

FV = PV *(1+.08) * **(1+.04)** = 1.12320. Rate = 1.12320 – **1** * 100 = 12.32%

If no allowance is made for inflation, the apparent (nominal) rate of return must be discounted by the inflation rate[9] in order to express the result in constant dollars.

$$FV = PV * \frac{(1+\text{ nominal rate})}{(1+\text{ inflation rate})} = \frac{1.08}{1.04} = PV * 1.\underline{03846}$$

The *inflation-adjusted rate* will be: 1.03846 – 1 = .03846 * 100 = 3.846%.[10]

This **Inflation-Adjusted Rate** (IAR) is handy whenever we want to express financial answers in terms of *constant dollars*, i.e. dollars adjusted for the effect of inflation.
For example:

> What sum most be invested today to provide for *constant dollar annuities* of \$100,000 for 20 years if cash in the investment earns 8% p.a. and inflation averages 4% p.a.? Payments are **BOP**.

8 Divide 1.045 by 1.08, press Enter, key in 25 and press key (2,1) to raise the result to the 25th power.
9 Subtracting the inflation rate from the nominal rate is imprecise.
10 Remember that the HP-12C accepts only percentages in the $\boxed{i}$ key.

This problem proposes that an annuity of $100,000, payable at the beginning of the first year, be increased by 4% each *subsequent* year. It asks for the PV of this inflation-adjusted cashflow series when discounted at the rate of 8% per year. We could also restate this problem to ask :

> What lump sum must be invested today at 8% p.a. to furnish an annuity of $100,000 for 20 years, which annuity increases each subsequent year to keep pace with inflation of 4%?
> Payments are **BOP**.

This problem can be conveniently handled by use of the inflation-adjusted rate of 3.846% which was calculated above:

BOP

n	i	PV	PMT	FV
20	3.8461…	?	100,000	0
	Solving…	**-1,430,725.84**		

Note that the PV of this cashflow series would be much lower if each subsequent PMT were *not* adjusted for inflation:

BOP

n	i	PV	PMT	FV
20	8	?	100,000	0
	Solving…	**-1,060,359.92**		

The **IAR** may also be used to solve this type problem:

> What annual annuity of constant purchasing power, *to be paid at the beginning of the year*, could be furnished for 20 years by an endowment of $1,430,725.84, if funds earned 8% per year and inflation averaged 4% per year.?

BOP

n	i	PV	PMT	FV
20	3.846…	**–1,430,725.84**	?	0
		Solving…	**100,000**	

Errors When *Ordinary* Annuities Are Involved

When the financial situation involves *Ordinary* Annuities (PMTs EOP), the use of the IAR will introduce a small but potentially significant error when solving for either the PV or the PMT.

This error is caused by the fact that the IAR inflates every PMT by (1+g). This is not a problem in the case of an Annuity Due (PMTs BOP) since the first PMT is always set aside by both the calculator *and* the computer and is neither increased nor discounted by the inflation rate. But when dealing with an Ordinary Annuity, the first PMT is *stipulated* and occurs at the end of the first period. The use of the IAR inflates this stipulated PMT by (1+g) before discounting it, and thereby introduces an error into the answer equal to (1+g)

Therefore the use of the IAR in determining the PV of a series composed of *Ordinary Annuity* PMTs (EOP) results in a value which is (1+g) *too high.* To correct for this error, solve as for EOP, then *divide* the answer by (1+g). (Remember to express **g** as the **g/period**.)

> For additional information and examples in the use of the Inflation-Adjusted Rate, please consult the Appendix.

When attempting to determine the regular amount (PMT) which can be supported by an *Ordinary Annuity* of a given value, the use of the IAR will result in an annuity (PMT) which is *too low.* To correct for this error, solve for the PMT as for an EOP, then *multiply* the answer by (1+g).

Nominal vs. Effective Interest Rates

The difference between *nominal* rates of interest and *effective* rates of interest is created by the effect of compounding. For example, if $1 is invested for 1 year at a 10% nominal rate of interest applied annually, the calculation of the FV is straightforward:

$$\mathbf{FV} = \text{PV}\ (1+i\ )^n$$
$$\mathbf{FV} = \$1*(1+.10)^1\ = \$1.10$$

The *effective* interest rate **i** = $\mathbf{FV} - 1, = \ \$1.10 - \$1, \ = 0.10\ = 10\%$

But when the stated (nominal) interest rate is compounded over a number of periods, the result is different. Suppose the annual interest rate of 10% is broken up into a monthly rate and compounded 12 times during the year. Then,

$$\mathbf{FV} = \text{PV} * (1+\frac{.10}{12})^{12}\ = 1.1047 - 1 = 10.\mathbf{47}\%$$, which is the ***effective* rate**

As the compounding period becomes more frequent, the effective rate of interest increases. If the annual 10% rate is divided into daily rates[11] and compounded daily then,

$$\text{FV} = \text{PV} * (1+\frac{.10}{360})^{360}\ = 1.\mathbf{10516}$$

[11] Using a banker's year = 360 days.

The *effective* rate has risen to 10.516%. This trend toward higher and higher *effective* rates continues as **n** approaches infinity.

To determine an *effective* rate, simply determine the Future Value of 1 (one), compounded over **n** periods at nominal rate **i** per period. For example,

> What is the effective rate when a 9% interest annual rate is compounded **quarterly**?

n	i	PV	PMT	FV
4	9/4	**-1**	-	?
			Solving...	**1.09308...**

If the effective rate is given and the nominal rate is sought, reverse the process and put (1 + effective rate) into FV; put (–1) into PV, and then solve for **i.**

n	i	PV	PMT	FV
4	?	**-1**	-	1.093083319...
Solving...	**2.25**			

The result, 2.25%, is the rate *per quarter*. Multiplying by 4 delivers the annual rate, 9.0%.

Continuously Compounded Rates

If we begin with an interest rate equal to 100% payable over **n** periods, the interest rate, **i** , per period **n** will be $\frac{100\%}{n}$ and the Future Value of 1 compounded for **n** periods becomes:

$$\mathbf{FV} = 1 * (1 + \left(\frac{\mathbf{100\%}}{\mathbf{n}}\right)^{\mathbf{n}})$$

As **n** grows, the value of this expression converges to a limit:

$$\mathbf{FV} = 1 * (1+\frac{1}{100})^{100} = 2.7048$$

$$\mathbf{FV} = 1 * (1+\frac{1}{1000})^{1000} = 2.71692$$

$$\mathbf{FV} = 1 * (1+\frac{1}{1000000000})^{1000000000} = 2.71828$$

This "magical" number, **2.71828**, is known as ***epsilon***, represented by the symbol **e**. Epsilon[12] represents the absolute limit to which **1** can grow when the rate of growth (100%) is divided into infinitely small parts and compounded over an equally infinite number of periods.[13]

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[12] Epsilon, like pi (π), is an irrational number which cannot otherwise be expressed exactly as the division of two integers and whose digits do not form a repeating pattern. Epsilon forms the base for Napier's natural logarithms, a system which is available on the HP–12C. See keys(2,2 and 2,3)

The result is:

$$\mathbf{FV = PV*(e)}$$

When the rate **i** is other than 1, and **n** is other 1, the formula becomes:

$$\mathbf{FV = PV*(e)^{(i * n)}}$$

Where **i** is the rate **per period** and **n** is the *corresponding* number of periods over which **i** applies.

For example, $1 *continuously* compounded over 2 years at an annual interest rate **i** of 10% would have a maximum future value of:

$$\mathbf{FV = \$1*(e)^{(i * n)}}$$

$$\mathbf{FV = \$1*(e)^{(0.10*2)}}$$

$$\mathbf{FV = \$1.22140276...}$$

Banks often advertise daily compounding.[14] The results obtained by compounding for 365 days per year over a two-year period are, for all practical purposes, virtually equal to that which would be produced by using continuous compounding:

n	i	PV	PMT	FV
365*2	10/365	**-1**	0	?
			Solving…	**1.221369..**

Making Practical Use of Continuous Compounding

We have seen that the present value of a cashflow can be calculated when the cash is available either at the beginning or at the end of a period. But this fact doesn't help the CFO[15] who must provide capital for equipment to be purchased over the course of the coming fiscal year.

If he sets aside all the cash needed at the beginning of the year, he may overestimate the amount required, since cash not used for early purchases can earn interest until needed. If he assumes that all the required cash will be needed at the end of the year, he may fall short, since any withdrawals made during the year will deprive him of interest to be earned on his invested capital.

This problem can be mitigated, though not entirely overcome, by budgeting the cash required *as though* it were to be supplied continuously. This is where epsilon proves useful.

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[13] This rate of growth is sometimes referred to as the "organic rate of growth:" one cell gives rise to two, two give rise to four, four become eight etc.. A 100% growth factor.

[14] Banks are prohibited from using continuous compounding.

[15] Chief Financial Officer

The value $\mathbf{(e)}^{(i * n)}$ can be inserted in place of $\mathbf{(1+i)^n}$ or $\mathbf{(1+g)^n}$ in any of the formulas we have discussed, and is particularly useful when we wish to express the Present Value of PMTs distributed evenly (continuously) over time. While the PMTs cannot actually be made continuously, the use of ***epsilon*** is very handy in compiling capital budgets where PV estimates based on PMTs required at the beginning of the year or at the end of the year do not reflect the actual timing of the required outflows.

For example:

> The alumnus who decided to endow his alma mater with an Annual Annuity of $100,000 now decides that the cash should be made continuously available over a 20-year period.
> With what amount must he now capitalize a trust which can earn 8% per year.?

Recall the basic formula for an annuity for a finite time:

$$\mathbf{PV} = \frac{C}{i} * \left(1 - \frac{1}{(1+i)^n}\right)$$

We can insert $\mathbf{(e)^{i*n}}$ into the formula in place of $\mathbf{(1+i)^n}$

$$\mathbf{PV} = \frac{C}{i} * \left(1 - \frac{1}{(e)^{i*n}}\right)$$

$$\mathbf{PV} = \frac{\mathbf{\$100{,}000}}{\mathbf{0.08}} * \left(\mathbf{1} - \frac{\mathbf{1}}{\mathbf{(e)^{0.08*20}}}\right) = \frac{\$100{,}000}{0.08} * \left(1 - \frac{1}{(e)^{1.6}}\right)$$

$$\mathbf{PV} = \$1{,}250{,}000 * 0.798103 = \mathbf{\$997{,}629.35}$$

This result, $997,629, is slightly *more* than would be expected if all PMTs were assumed to be made at the end of the period ($981,815), and slightly *less* than the amount which would be required if all PMTs were to made at the beginning of the period ($1,060,360). It is also different from the average of the two ($1,021,087).

Continuous compounding or discounting of cashflows to be paid or to be received over a definite time period is a very useful tool in capital budgeting when the timing of the outflows is uncertain.

Inflation Rates and Jellybeans

Almost all financial professionals and jellybean fanciers have a keen interest in the rate of inflation since inflation reduces the purchasing power of the dollar. One dollar at the beginning of the 21st century buys as many jellybeans as did only **3** cents at the beginning of the 20th century.

On another level, the cost of replacing machinery purchased 10 years ago will be much higher than its previous acquisition price. The amount of money required for a college education or for a reasonably comfortable retirement will also be much greater for you than for your parents.

In our discussion of Present Values we have provided two formulas[16] which express the Present Value of a fixed payment, C, which increases at a constant rate:

For Perpetual Ordinary Annuities: $PV = \frac{C}{i-g}$ and,

For Finite Ordinary Annuities: $PV = \frac{C}{i-g} * \left[1 - \left(\frac{1+g}{i+i}\right)^n\right]$

The **g** in both these formulas pertains to a constant rate of growth in the PMT, C, and indicates that C will grow at rate **g** per period, for the number of periods represented by **n**.

When we need to express the Future Value of either a Present Value or a series of fixed PMTs, or a combination of the two, in terms of today's dollars, we can also employ the Inflation-Adjusted Rate. For example, consider Ann Brown's situation:

> Ann is planning on retiring 20 years from today.
> She estimates that she could live comfortably for 25 years on a monthly income of $4,000 *as measured in today's dollars*, provided this amount is increased each month to offset inflation. Upon retirement, she is willing to invade the principal of her account to make the plan work. She can invest her savings today and during retirement to realize a 9% annual return in an inflationary environment anticipated to average 3.0% per year.
> Ann has $150,000 with which to open her retirement account.
> How much more must she invest at the beginning of each month, *starting today*, in order to achieve her goal?

First, we need to determine the equivalent monthly amount, adjusted for inflation, which Ann must have at the time of her retirement. We can do this by inflating today's $4,000 amount by the anticipated inflation rate of 3% per year (compounded monthly) for 240 months.

n	**i**	**PV**	**PMT**	**FV**
240	0.25	-4000	0	?
			Solving...	**7,283.02**

This amount, $7,283.02, will be her first monthly withdrawal at the time of retirement and will be equivalent in purchasing power to a $4,000 sum today.
Next, we need to calculate the required *capital value* of her account <u>at the time of her retirement.</u> This capital amount must reflect the yield of 9% (compounded monthly) on invested cash, and must also allow for monthly increases in her future monthly retirement checks at the rate of 3%/12 to offset inflation. We need to use the IAR.

---

[16] And two additional formulas for the BOP situations.

BOP

n	i	PV	PMT	FV
300	0.49875..	?	7,283.02	**0**
	Solving	**–1,137,626.90**		

In order to accumulate this amount by the time she retires, Ann can contribute *any* combination of PV and PMTs which, when invested at 9% (compounded monthly), will accrue $1,137,626.90 in 20 years (240 months). Since she has $150,000 to start today, her additional beginning-of–the-month contribution must be:

BOP

n	i	PV	PMT	FV
240	**9/12**	–150,000	?	**–1,137,626.90**
		Solving…	**-351.11**	

We have not used the IAR in this latter calculation because we have already compensated for inflation in the determination of the FV, $1,137,626.90. We did this by adjusting the PMT to reflect inflation. This value is the PV equivalent of the inflation-adjusted monthly payments she will receive over her 300 months of retirement.

If it were possible, Ann could simply contribute $189,315.73 today, with no monthly contributions, and still achieve her FV goal. (Keep this number in mind; we will need it shortly.)

How can we be sure that this plan will work? One way is to draw an amortization table in which the monthly payment of $7,283.02 increases at the rate of 3%/12 per month to offset inflation, while the balance of the account earns the nominal 9%/12 rate per period. Withdrawals are made at the beginning of each month.

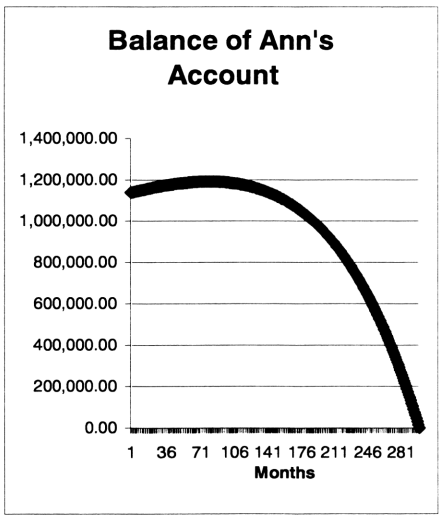
Payment	Starting Bal.	Withdraw	Balance	Interest	Ending Bal.
1	1,137,626.**87**	7,283.02	1,130,343.85	8,477.58	1,138,821.43
2	1,138,821.43	7,301.23	1,131,520.20	8,486.40	1,140,006.60
3	1,140,006.60	7,319.48	1,132,687.12	8,495.15	1,141,182.28
4	1,141,182.28	7,337.78	1,133,844.50	8,503.83	1,142,348.33
5	1,142,348.33	7,356.12	1,134,992.21	8,512.44	1,143,504.65
▼					
298	45,639.08	15,288.78	30,350.31	227.63	30,577.93
299	30,577.93	15,327.00	15,250.93	114.38	15,365.32
300	15,365.32	15,365.32	(0.00)	(0.00)	0.00[17]

[17] The value $1,137,626.**87** was generated by Excel and is slightly more accurate than the HP–12C

A second proof is available: we can discount the required FV of $1,137,626.90 by the compounded inflation rate $(1+.03/12)^{240}$ to produce the equivalent PV today. This PV is $624,810.53. If we had this sum available today, what monthly retirement amount would it support, adjusted for inflation, given a 9% yield and a 3% inflation factor?

BOP

n	i	PV	PMT	FV
300	0.49875	–624,810.53	?	**0**
		Solving...	**4,000**	



The progress of Ann's account *after* retirement is depicted in the graph at the left.

Her balance initially increases, but as the monthly withdrawals increase, adjusted for inflation, the balance of her account reverses and begins to decline. This will occur in the 80th month of the schedule.

This type account cannot be amortized on the HP–12C *which can amortize only Ordinary Annuities.* The **BEG** function has no effect. Therefore the amortization of an Annuity Due must be done on a spreadsheet in a manner similar to what is shown in the table above.

The Graduated Payment Annuity

One shortcoming of many annuity plans is that they call for level payments in each of the payment periods. For example, Ann may feel it difficult just now to contribute $351.11 each month to her retirement plan.

> *"Isn't there a way I can start with a smaller amount this year and gradually increase it in later years?"* she asks.

Fortunately, there is a way.

In your discussion with Ann, she commits to a 5% increase in her contributions each year over the first 10 years of the plan *provided* she can start with a lower initial monthly PMT. Beginning with the 11th year, she would prefer no increases since she foresees that she will be at the peak of her earning curve.

You will recall that we calculated that Ann could achieve her required FV of $1,137,626.90 by any combination of a current (PV) amount plus monthly contributions. We calculated that if she were to make no monthly contributions, she would need to contribute $189,315.73 *today*. Since she is still willing to advance $150,000 today, she would be short the PV difference, $39,315.73.

It is this PV amount which we need to supplement by means of regular payments which increase 5% per year over the first ten years of her plan, and then level PMTs thereafter. Let's assume that we start with $1.00 in PMTs.

Year	**Amount Per Month**	**No. of PMTs**
1	1	**11**
2	1.05	12
3	1.1025	12
4	1.15763	12
5	1.21551	12
6	1.27628	12
7	1.3401	12
8	1.4071	12
9	1.47746	12
10	1.55133	12
11 - 20	1.55133	99
	1.55133	21
	Total	239

These PMTs increase each year by 5% and total only 239 in number because we have removed the first PMT, which is not to be discounted in a BOP series. We have also broken up the **11-20** year interval since the HP–12C registers are limited to 99 repetitions per register.

With these values entered in the uneven registers, and with the first PMT of $1.00 in register CFo, find the PV of the series when discounted at 9%/12 per period. The result is $146.84564, the PV of the series.

Therefore we see that this series of PMTs, starting with $1.00 and incremented as shown, will support a PV of $146.84564.

Fortunately there is a proportion between the PV of the schedule we have created, beginning with $1.00, and the total we need to support $39,315.73. This proportion is the ratio of the first PMT to the total PV in each of the two sets:

$$\frac{\text{PMT}}{\text{PV}} = \frac{\$1.00}{146.84564} = \frac{\mathbf{X}}{39{,}315.73}$$

Solving for the first PMT, **X** = $267.74 (rounded) The entire schedule follows:

Year	**PMT Per Month**	**No of Times**
1	267.735...	12
2	281.12	12
3	295.18	12
4	309.94	12
5	325.43	12
6	341.71	12
7	**358.79**	12
8	**376.73**	12
9	**395.57**	12
10	**415.34**	12
11 - 20	**415.34**	120
	Total Pmts.	240

If we discount the PMT schedule in the table at the left by 9%/12, its Present Value is found to be $39,315.68 (The table shows rounded up values for the PMT but we have not rounded them up on the calculator.)

Notice, also, that the PMTs required under the graduated payment plan beginning in the 7th year exceed that which would have been required using a level PMT ($351.11). This is true of many graduated annuities (mortgages included) in order to preserve the same Present Value. But these increased PMTs occur well along in Ann's earning curve, hopefully at a time when she can more comfortably meet the larger PMTs.

Meanwhile, the reduction in the required initial PMT, in this case from $351.11 to $ 267.74 (–24%), may make the financial plan not only feasible for Ann but even financially comfortable.

This plan for a Graduated Payment Annuity can be configured in any format desired: monthly or annual increases in PMTs can be raised, lowered (or even temporarily suspended) for any number of periods. This method provides for unlimited flexibility in achieving a future investment goal.

Be aware, however, that although a higher incremental rate will lower the initial payments, a higher incremental rate will also require higher payments later in the schedule to preserve the PV of the annuity. If the rate is too high, it could jeopardize the plan by presenting payments in the later years which could be difficult to meet.

The rate used in the Graduate Payment Annuity may either be the nominal rate or the Inflation-Adjusted Rate.

Remember to begin the PMT series with $1.00. Inflate the periodic payments by the selected incremental (growth) rate. Then discount the entire series by the projected investment rate. The resulting PV should then be made proportional to the PV to be raised. Always deal with PVs, never with PMTs or FVs.

Future Inflation Rates ?

In constructing financial or investment plans, some feeling for future inflation rates is required. There was a time in our recent economic history when the rate of inflation changed very little. In the period following WW II until the early 1960s,[18] inflation remained below 1% per year. Since then, however, the annual rate of inflation has varied greatly, reaching a peak of 13.5% in 1981.

There are economic periods during which inflation is caused by the *push–pull* of a demand for goods and services exceeding supply. At other times inflation is caused by *cost-push* factors such as when OPEC[19] raises the price of oil.

If past equals prologue the investor/planner may obtain some help and perspective from historical data which are available on a monthly schedule from the U.S. Department of Labor's web site: "http://www.bls.gov/data/".

The Department publishes two indices which are intended to measure the change in the cost of goods and services. The first index is the *Consumer-Price Index –All Urban Consumers* (CPI-U); the second is *Urban Wage Earners and Clerical Workers* (CPI-W).

The CPI-U samples about 87% of the population of the United States and is based on the expenditures of *all* families living in urban areas. The CPI-W is a sub-set of the CPI-U and is based on the expenditures of all families in urban areas who meet the additional requirement that

[18] The beginning of President Lyndon Johnson's "Great Society" which called for guns *and* butter.

[19] Organization of Petroleum Exporting Countries, a cartel of oil-producers.

more than one-half of the family's income must be earned from clerical or hourly-wage occupations. The CPI-W represents about 32% of the total U.S. population.

The CPI-W is widely used in the adjustment of labor contracts, over 60% of which carry COLA clauses,[20] and in annual changes in Social Security benefits. The CPI-U is more commonly used to adjust escalation clauses in business contracts, such as commercial leases.

The *Core Rate of Inflation* is similar to the CPI but excludes the costs of energy and food, which tend to be volatile. But since these are important factors in many budgets, critics contend that their elimination does not truly represent "the cost of living." CPI indices are available on a regional as well as on a national basis.

How to Compute Rate Changes in the CPI

To compute the percent change in the CPI simply divide the new index number by the older index number and subtract 1. For example the national CPI-U ending December 1995 was 153.5.[21] The ending average for December 1999 (a 4-year interval) was 168.3. Therefore the change in the rate is $\frac{168.3}{153.5} = 1.0964 - 1 = 0.0964 \times 100 = 9.64\%$. This is the amount by which the CPI increased *over* 4 years.

The average rate of increase *per year*, however, was:

n	i	PV	PMT	FV
4	?	–153.5	0	168.3
Solving...	**2.33%**			

Many economists consider the following values as indicating the degree of inflation:

Range of Price Changes	Condition Described as
0.0-2.0%	Price Stability
2.1–3.0%	Low Inflation
3.1–5.0%	Moderate Inflation
5.1–10.0 %	High Inflation
10.1–20.0%	Price Instability

During Allan Greenspan's reign as Chairman of the Federal Reserve Board, the targeted core rate of inflation was between 2.00–2.5%

20 Cost Of Living Adjustments

21 Based on the 3-year average for the period 1982-1984 = 100. Revisions are made about every 10 years.

Chapter Summary

1. An annuity is a sum of cash paid or received over a regular schedule.
2. An annuity may deliver payments forever or for a limited (finite) period of time. Those annuities whose payments are made at the beginning of the period are called Annuities Due; those whose payments are made at the end of the payment period are Ordinary Annuities.
3. A finite annuity is the difference between two perpetual annuities: one whose PMTs begin now, and one whose PMTs begin in the future at the end of period **n**.
4. Th amount of the payment under both perpetual and finite annuities can be easily adjusted to reflect a constant rate of growth. When the adjustment is made using the so-called Inflation-Adjusted Rate (IAR), the Present Value of an Ordinary Annuity will be overstated by (1+g), where **g** is the rate of growth in the PMT per period. When the IAR is used to determine the PMT of an Ordinary Annuity, the result will be understated by (1+g). This problem does not occur with an Annuity Due. Use of the "long formulas" avoids these errors entirely.
5. The "long formulas" which have been presented for both Ordinary and Annuities Due are also useful in dealing with growth annuities whose number of different payments extends beyond the maximum capacity of the calculator (20 *different* PMTs).
6. *Effective* interest rates are created as the result of compounding in which interest is earned on interest which has accrued. More frequent compounding results in higher FVs. As the compounding periods increase per year, the effective rate converges to a limit. When a sum doubles in value each period, an infinite number of compounding periods results in the constant *epsilon*, 2.71828. Continuous compounding using epsilon simulates the payout (or receipt) of a stream of PMTs over an infinite number of compounding periods and is useful in capital budgeting.
7. Inflation results from both the push-pull of a demand for goods and services in excess of supply, and from the cost-push effect of a rise in commodity prices. All financial plans must be designed to compensate for the loss of purchasing power as the result of inflation.
8. Graduated Payment Annuities can be designed to create investment programs featuring gradually increasing planned contributions. These plans may make certain financial plans feasible by reducing the contributions required to begin and maintain the plan.
9. The U.S. Department of Labor provides monthly data on the cost of living for groups of families. These data often serve as guideposts for the increase in labor and business contracts and can be accessed via the world-wide-web.

Chapter 4 The Internal Rate of Return

The IRR is undoubtedly the most arcane of all the indices of financial performance. Few investors outside the institutional investment community use the IRR. The average investor has only a muddy understanding of it, if at all, and is typically not motivated to study its usefulness. Yet, properly understood and used, it is a valuable yardstick of overall investment performance because it measures the rate of return on the investment over the entire holding period.

We have already defined the IRR in Chapter 2 in the section dealing with uneven cashflows. The IRR is a discount rate, but a very special and unique discount rate. The IRR is:

...that certain, single discount rate, which - when applied to each future cashflow - results in Present Values the total of which is exactly equal to the initial investment.[1]

IRR Under Different Aliases

Bankers have a special name for their IRR: they call it "yield." The true interest rate charged to borrowers (which, under federal Regulation Z, a lender must disclose to a borrower) is more commonly known as the **APR**, the "Annual Percentage Rate."

When next you enter a bank lobby and see a stanchion sign advertising a Certificate of Deposit rate as "4% (4.38% APR)," you will recognize that the stated, or **nominal**, rate of interest paid by that depository is 4%, but that as a result of compounding, the **yield**, or the **effective** rate of interest, to the depositor is 4.38%. IRR, Yield and APR are all one and the same measuring stick under different aliases. A banker's "yield" = automobile dealer's "APR" on auto leases = investor's "IRR" on an apartment house investment = bond dealer's Yield to Maturity (YTM).

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1 If PV = CFo, and CFo is made negative, then PV + (–CFo) = 0.

IRR - a Calculator or Computer Supplied Discount Rate

By this time you should be comfortable in your ability to discount a series of cashflows by a given discount rate. Oftentimes an investment problem does not supply an initial cash investment (CFo); it's up the analyst to determine an appropriate discount rate and apply it to future cashflows to determine the Present Value of the cashflow.

At other times, however, the cash initially required for the investment is specified, and the subsequent cashflows are estimated based on economic and financial assumptions and projections of future performance by means of a *pro-forma*. The problem then becomes to determine the IRR on the investment given these particular cashflows. Given these data, the calculator – not the investor – supplies the discount rate, or yield.

Let's take a very simple situation to demonstrate the IRR.

In Chapter 2, you worked through an example of valuing a small business opportunity (the automatic laundry) using different discount rates for different periods in the cashflow[2].

		(125)
End of Period	1	50
	2	60
	3	70

Let's examine this kind of structure again (but using smaller and simpler numbers for convenience).

Let's assume that the receipt of 50 at the end of period 1 may be quite secure and we would be justified in using a 10% discount rate against the first PMT. But because the receipt of the second PMT, 60, is less likely, a discount rate of 15% is more appropriate. Lastly, assume that the receipt of the third PMT, 70, is even less likely, and a discount rate of 20% is used.

The calculator cannot discount all three PMTs in one operation, using different discount rates. But we can determine the Present Value for each separate PMT, using its appropriate discount rate, and then add the sum of the PVs together to determine Σ PV. Let's do that.

$$(1) \qquad \mathbf{PV_1} = \frac{50}{(1+.10)^1} = 45.45$$

$$(2) \qquad \mathbf{PV_2} = \frac{60}{(1+.15)^2} = 45.37$$

$$(3) \qquad \mathbf{PV_3} = \frac{70}{(1+.20)^3} = \underline{40.51}$$

$$\mathbf{\Sigma\, PV} = 131.33$$

2 In fact, this is the method commonly used to evaluate the worth of small businesses. Their future net operating income is estimated and then discounted, typically at increasing annual rates, over a period of future years. The PV of the future annual NOIs becomes the asking price of the business.

But if it is given that the CFo of this investment is 125, the task becomes to determine that single discount rate which will cause the ΣPV of PV_1, PV_2, and PV_1, to total 125, not 131.33. We begin by entering the cashflows as we have previously:

Key In	Display Shows	Comments
f CLX	0.00	Clears registers
125 **CHS g CFo**	-125.00	Stores CFo in R_0; sets n = 0
50 **g CFj**	50.00	Stores value in R_1
60 **g CFj**	60.00	Stores value in R_2
70 **g CFj**	70.00	Stores value in R_3
solving,		
f IRR (1,5)	**19.441**	The **IRR** of this cashflow

Iteration

You probably noticed that the calculator flashed "..running.." and that it took some time to find the answer. The reason for this is that there is no algorithm built into a calculator (or computer) chip which will lead directly to the IRR. Instead, the calculator proposes a discount rate to itself and "tests and fits"[3] to see whether this discount rate will result in an ΣPV which, when added to the –CFo,[4] will exactly total zero. If the result is less than zero (negative), the calculator chose too high a discount rate. If the addition of PV and –CFo results in a value greater than zero (positive), the discount rate chosen was too low.

In either case, the calculator continues to "test and fit" a discount rate until it arrives at that single rate which will produce a PV which, when added to the negative value of CFo will net precisely to zero. That rate is the Internal Rate of Return for this cashflow series. You can readily see why the IRR is alternately defined as *that single discount rate which will cause the Net Present Value of a cashflow to equal zero.*

Your calculator now contains the IRR which is stored in register **i**. Leave it there for the following computations. Now, using the IRR stored in **i** as the discount rate,[5] determine the PV for each of the cashflows and total them.

$$(1) \quad \mathbf{PV_1} = \frac{50}{(1+.19441)^1} = 41.86$$

$$(2) \quad \mathbf{PV_2} = \frac{60}{(1+.19441)^2} = 42.06$$

$$(3) \quad \mathbf{PV_3} = \frac{70}{(1+.19441)^3} = \underline{41.08}$$

$$\Sigma\ \mathbf{PV} = \mathbf{125.00} \quad \text{(the CFo)}$$

3 This repetitive process is called Iteration.

4 The addition results in zero because the CFo is entered as a negative number.

5 Retrieve the IRR from i, add one to it, store it in register 9 and use it as the denominator of the fraction.

This proves that the single discount rate which we computed, 19.441%, when applied to each future cash flow, will produce three PVs, the sum of which is exactly equal to our original investment of $125.00. That single discount rate *is* the IRR.

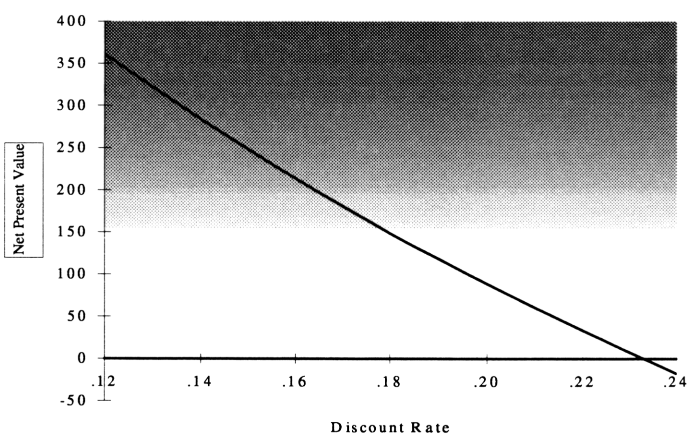
The IRR by Construction

Another way we might determine the IRR is by simple *construction*. You may have once solved problems in geometry by constructing physical models of the problem and then measuring to find the answer.

Examine the following cashflow, then plot 7 Net Present Values using as discounts 12%, 14%, 16%, 18%, 20% , 22% and 24%.[6] (Plot the NPVs against the discount rate.)

		-1,000
End of Period	1	200
	2	300
	3	400
	4	500
	5	600

You should have obtained the **Net Present Values** shown in the table below for each discount rate.
If you plot these NPVs against the Discount Rate, you can approximate where the resulting curve will cross the *x* (horizontal) axis. This point (where NPV=0) represents the IRR.



Discount Rate %	Net Present Value $
12	361
14	284
16	213
18	149
20	89
22	33
24	-18

As you can see, the curve crosses the horizontal axis, representing NPV =0, about half way between 22% and 24%. The actual IRR is 23.3%

Therefore if you can construct a few points which represent a number of NPVs which decline and eventually turn from positive to negative (cross the x axis), you should be able to sketch in a curve and make a very useable estimate of the IRR.

6 Enter the cashflows only once; then change the discount rate in **i** to obtain the NPV.

Purists may object to this approximation of the IRR as being too imprecise. But simply because the computer or calculator can determine an IRR to the nth decimal point doesn't mean that the IRR value is indeed that precise. After all, it is based on estimates of future cashflows which are also subject to error.

Nevertheless, there are those who are wont to quote the IRR to the nth decimal place. But unless the calculation has been made on *historical* data, this kind of financial preciosity is, at best, ludicrous and at its worst, potentially misleading. A CEO who requires a 25% yield on a project would most probably not reject a project whose IRR is 24.5%. He or she is more likely to accept the project and work to find a way to improve the return.

The IRR By Interpolation

NPV	Discount Rate
+33	22%
0	23.3%
-18	24%

There is still another way to arrive at this special number. Notice in the diagram to the left, that the Net Present Values (left side) cross the zero NPV line from positive to negative between the NPVs points of +33 and –18. These NPVs correspond to discount rates (right side) of 22% and 24% respectively.

The absolute distance between 33 and –18 is 51. A Net Present Value of zero, therefore, would lie 33/51 along a line proceeding from 22 to 24, an absolute distance of 2. But 33/51 of 2 = 1.30. Therefore the IRR lies at a point which is 1.30 along the line from 22% to 24%, or 22.0% + 1.30% = 23.3%.

What Does The IRR Tell Us ?

These graphic examples of the IRR are meant to emphasize that the IRR is a particular, ***unique*** discount rate: It is that *single* discount rate which will render the NPV of a cashflow equal to 0. Since the discount rate is also a yield, the IRR is the yield which can be expected *if* the investment is held for the entire length of time indicated by **n** and *if* each period of the investment delivers the forecasted cashflow. If one period is added or lost, or if any single cashflow is changed, you can see that the IRR must also change.

What the IRR really tells us is the periodic rate at which we are accumulating wealth ***on the cash remaining in the investment***, measured over the entire life of the investment. In most cases, the cash which we have remaining in an investment, at any one moment, is our net equity in the investment. Therefore one use of the IRR may be to help us choose, from investment opportunities of similar risk, that investment most likely to render us the greatest yield or return on invested equity (ROE).

In addition to helping make an initial investment decision, the IRR can also help answer questions which arise as the result of changes made in our investment strategy. For example:

1. What effect on future yield can I anticipate as the result of refinancing?
2. When should I sell the investment?
3. Is the indicated IRR return commensurate with the discernible risk in a particular investment which is being offered to me?

Limitations to the IRR

The IRR is a valuable tool, but there are some limitations.

The investment which produces the highest IRR may not always be the best investment, either from the point of risk,[7] or from a financial point of view.

Consider this cashflow:

		-1,000
End of Period	1	0
	2	0
$\boxed{i = ?}$	3	2,500

Key In	Display Shows	Comments
f CLX	0.00	Clears all memory
1000 **CHS g CFo**	-1,000.00	Stores value in R_0
0 **g CFj**	0.00	Stores 0 in R_1
2 **g Nj**	2.00	0 PMT in R_1 occurs twice
2500 **g CFj**	2,500.00	Last PMT into R_2
solve,		
f IRR	**35.72**	**IRR** for this series

This percentage return would be considered excellent by most investors. But the fact that the entire cash return occurs at the end of the investment period suggests there may be significant risk associated with this investment.[8] The old adage "..a bird in hand is worth two in the bush..." was probably coined following someone's sobering experience in which a rich investment pay-off, promised far in the future, never materialized.

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7 The IRR is a number *derived* from other numbers at hand. It is not a discount rate *selected* as the result of careful risk assessment. Therefore the IRR neither measures nor accurately reflects risk.

8 The investor must wait three periods to finds out if the investment will pay off.

The second caution to be exercised in selecting an investment based on the IRR alone is that the discount rate, or percentage yield, ought not be the sole objective of investing. The objective is maximum dollars gained over the distance run, not percentage rates achieved.[9]

To illustrate, let's propose two investment opportunities available to an investor with $50,000. The first requires an initial investment of $45,000 and returns the following cashflow:

		–45,000
End of Period	1	10,000
	2	20,000
	3	30,000

The IRR of this investment is 13.3% and will return a raw total of $60,000 over 3 periods. But the investor will have $5,000 extra which he has not yet invested.

The second investment consumes the investor's entire $50,000 and generates the following cashflow:

		–50,000
End of Period	1	8,000
	2	15,000
	3	45,000

The IRR for this investment is less, 13.0%. But it will return a raw total of $68,000, $8,000 more than the first investment.

Unless the investor, after making the first investment of $45,000, can invest the remaining $5,000 in an investment which can earn 16.96%,[10] he may favor the second investment even though it may be somewhat more risky and carries a lower IRR. The second investment simply returns more (quantity) dollars over the same holding period. "Percentages" are not the sole selection standard of a "good" investment. Nor do they pay the rent.

Multiple IRRs for the Same Cashflow

A popular pastime for financial academes is to point out that more than one IRR may exist for the same cashflow.[11]

The following example is illustrative only. There are very few instances in the real world when multiple IRR solutions become a business problem.[12]

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9 This is precisely why many wealthy investors tend to avoid high-risk/high yield investments. Smaller percentage returns on large capital investments return very satisfactory dollar results.

10 $5,000 invested for three years requires a 16.96% annual return rate to yield $8,000.

11 Following Descartes "rule of signs," there are as many solutions to a polynomial as there are changes in sign. Do not count the initial investment (PV).

12 One great financial skill worth cultivating is the ability to step back from a situation and ask "Does this result make sense?"

Consider this cashflow:

		– 10,000
End of Period	1	60,000
	2	(110,000)
	3	60,000

$$\textbf{CFo} = \frac{PMT_1}{(1+i)^1} + \frac{PMT_2}{(1+i)^2} + \frac{PMT_3}{(1+i)^3}$$

$$10{,}000 = \frac{60000}{(1+i)^1} + \frac{(110000)}{(1+i)^2} + \frac{60000}{(1+i)^3}$$

There are three possible solutions for i : 0%, 100% and 200%.

1) At 0%

$$10{,}000 = \frac{60{,}000}{(1+0)^1} + \frac{(110{,}000)}{(1+0)^2} + \frac{60{,}000}{(1+0)^3}$$

$$= 60{,}000 - 110{,}000 + 60{,}000$$
$$= 10{,}000$$

2) At 100%

$$10{,}000 = \frac{60{,}000}{(1+1)^1} + \frac{(110{,}000)}{(1+1)^2} + \frac{60{,}000}{(1+1)^3}$$

$$= \frac{60{,}000}{2} + \frac{(110{,}000)}{4} + \frac{60{,}000}{8}$$

$$= 30{,}000 - 27{,}500 + 7{,}500$$
$$= 10{,}000$$

3) At 200%

$$10{,}000 = \frac{60{,}000}{(1+2)^1} + \frac{(110{,}000)}{(1+2)^2} + \frac{60{,}000}{(1+2)^3}$$

$$= \frac{60{,}000}{3} + \frac{(110{,}000)}{9} + \frac{60{,}000}{27}$$

$$= 20{,}000 - 12{,}222 + 2{,}222$$

$$= 10{,}000$$

Reinvestment Rate of Payments – Does It Matter?

Another frequently asserted limitation of the IRR is that the attainment of the Internal Rate of Return depends upon the rate at which the PMTs *flowing from* the investment are reinvested. Some contend that unless the PMTs derived from the investment can be reinvested at a rate equal to or higher than the IRR, the IRR is illusory.

Consider this T-Bar and tabulation of a cashflow showing the progress of an initial investment of $479.59. The PMTs (100, 125 ...etc.) are cashflows produced by, *and removed from*, the investment as they occur. The IRR of this cashflow is 15%.

		–479.59)
End of Period	1	100
	2	125
IRR = 15%	3	150
	4	175
	5	200

You will recall that a discount rate and an interest rate are opposite sides of the same coin. The IRR is a discount rate. Therefore the IRR can also be used as a yield, or interest rate.

The table below depicts the progress of the investment as the cash, *remaining in the investment* after withdrawal of the PMTs, earns interest at the IRR rate:

Beginning Balance	**Interest Rate**	**Principal Plus Interest**	**Withdrawal (PMT)**	**Remaining Balance**
479.59	0.15	551.53	**-100**	451.53
451.53	0.15	519.26	**-125**	394.26
394.26	0.15	453.40	**-150**	303.40
303.40	0.15	348.91	**-175**	173.91
173.91	0.15	200.00	**-200**	**0.00**

The table shows that the net balance of the investment will be zero at the end of the holding period *after having returned the investor a steady 15% on cash remaining in the investment..* During the period of the investment, the *remaining balances* always earn a return at the IRR rate. The rate at which the PMTs could be reinvested *does not* affect the IRR of this cashflow. Therefore the assertion that the IRR depends upon the reinvestment rate of the PMTs is entirely false.

Aside from these limitations, most professionals consider that the **N**et **P**resent **V**alue of an investment to be a better measuring stick because: 1) it deals with cash returns and not percentages; 2) it forces the investor to address the issue of risk by requiring the active selection of a discount rate rather than forcing one mathematically; 3) it offers the opportunity to assign different discount rates for different periods of the investment.

Geometric Mean vs. IRR Rate of Return

The question often arises as to the proper method of measuring the return from an investment. Should the annual returns over a number of years be added together and averaged, or should they be multiplied together, or should some other method be used? The difference in these rates is best explained by an example.

An investor realized the following gains from five separate investments: 10%, 12%, !4%, -7% and 9%. We can say that the *average* gain from all his investments was (10% + 12% + 14% + (-7%) + 9%)/5 = 7.60% per year. We can substitute 7.60% for each of the separate gains and the total would be the same (38%), as would the average, 7.60%[13]

If, however, one investor realized these same successive rates of return from a *single* investment over a 5-year period, the average return would be $[(1.10)(1.12)(1.14)(1-.07)(1.09)]^{1/5}$ = $1.42372^{0.20}$ = 1.07321 −1 ∗ 100 = 7.321% gain per year. In this case, we can substitute 1.07321 for each of the annual rates of return, and have the same result 1.42372.[14]

The first example illustrates the *arithmetic* mean, 7.60%, while the second example illustrates the *geometric* mean, 7.32%. The arithmetic mean involves the *addition* of factors, while the geometric mean involves the *multiplication* of factors.

If the cashflows from an investment, $x.....x_n$, are equal, then the arithmetic mean and the geometric mean are also equal. In all other cases, the relationship between the arithmetic mean and the geometric mean becomes:

$$\textbf{Log}\ (\text{GMean}) = \left[\sum_{i=1}^{n} \textbf{log}\ x_i \right] / \ n$$

If the same annual cash returns were produced by, and withdrawn from, the investment, the Internal Rate of Return would be 7.92%. This rate of return is different from both the arithmetic mean and from the geometric mean rates of return.

The geometric mean assumes that the proceeds from the previous period *remain* in the investment and are compounded forward, while the IRR assumes that the periodic proceeds are *withdrawn* from the investment. The IRR, therefore, measures the periodic rate of return on cash remaining in the investment as measured over the entire holding period. The geometric mean measures the yield from the re-investment of the dividends (proceeds), as one might do with a mutual fund held in a tax-deferred account.

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13 This a fairly meaningless number since we don't know the dollars involved in each separate investment. We would be better informed with the weighted return ($ * %) for each investment.

14 We used the expression (1-.07), which equates to 0.93, because the calculation of the Geometric Mean does not allow for negative numbers

The IRR and the Problem of Negative Cashflows

The IRR calculation handles negative cashflows which may occur in the investment series in exactly the same way it does positive payments: it discounts them and produces a negative PV. These negative PVs are added to the positive PVs to produce Σ PV.

Consider the following cashflow:

		–50,000
End of Year	1	10,000
	2	–5,000
	3	85,000

The IRR of this investment is 23.45% The investor in this situation is being called upon to supply an additional $5,000 at the end of year 2. The assumption implicit in this financial structure is that the investor can invest a certain sum at the beginning of the investment (an additional CFo) at the same internal return rate of 23.45% for 2 years in order to accumulate %5,000 in cash to cover this future negative PMT.

It's easier this to see this in the following format:

$$PV \ \frac{10000}{(1+.2345)^1} + \frac{\mathbf{-5000}}{(1+.2345)^2} + \frac{85000}{(1+.2345)^3} = 8{,}100.47\ \mathbf{-3{,}280.88} + 45{,}180.41 = 50{,}000$$

How many investments abound in which you can invest so small a sum, $3,280.88, for 2 years and realize a return of 23.45% each year?

The Modified Internal Rate of Return: MIRR/FMRR

This distortion in the IRR when significant negative cashflows occur is the reason for Modifying the IRR to produce the MIRR: the Modified Internal Rate of Return – or, as some would have it, the FMRR (Financial Management Rate of Return).

For those who wish to use the IRR method for cashflows containing significant negative PMTs, the MIRR method compensates for the fact that the IRR discounts negative sums at the same rate as for positive sums. The financial reality is that it is extremely difficult to invest small initial sums (PVs) at a large interest rate. Let's reconsider the previous example:

The IRR of this investment is 23.45%. This investment requires two negative cashflows from the investor: one at the very beginning in the amount of –$50,000 (the CFo), and a second, –$5,000, at the end of the second period. The Present Value of the second negative PMT, discounted at the IRR of 23.45% for two periods is –$3,280.88.

		– 50,000
End of Year	1	10,000
	2	– 5,000
	3	85,000

Since the probability of being able to invest so small an amount, $3,280.86, at so large a rate of return, 23.45%, for so short a period is usually quite low, we need to adjust this cashflow to reflect a more realistic estimate of return.

One method of covering this future cash requirement would be to set aside an additional initial amount (together with the original $50,000) which, invested at a conservative or ***safe rate***[15] of interest, say 5%, would grow to $5,000 at the end of the second period. The availability of this cash would offset the negative cashflow at the end of the second period.

This required amount would be:

n	i	PV	PMT	FV
2	5	?	0	5,000
	Solving...	**-4,535.15**		

The adjusted T-Bar would look like this:

		–50,000 – 4,535.15 = –54,535.15
End of Period	1	10,000
	2	0
	3	85,000

The MIRR of this cashflow is 22.39%, a reduction from the original IRR of 23.45%. (Solve for the MIRR exactly as you would for the IRR.)

A second approach would be to reinvest the $10,000 which will be available at the end of period 1 in the same conservative investment of low risk in order to have cash available to cover the second negative cashflow. If we could invest $10,000 for one period @ 5% interest, the revised T-Bar would show:

		–50,000
End of Period	1	10,000
	2	–5,000 + 10,500 = 5,500
	3	85,000

or,

		–50,000
End of Period	1	0
	2	5,500
	3	85,000

Now if we solve for the MIRR of this cashflow, we obtain 22.42%, a slightly higher yield than the previous solution.

The third and most efficient method recognizes that it is unnecessary to use *all* the $10,000, since only a portion of it, invested for one period @ 5%, would yield just enough to cover the negative cashflow at the end of the second period. How much, then, of the $10,000 needs to be carried forward for one period in an investment earning 5% to yield $5,000?

15 An interest rate derived from an investment of minimal risk; e.g. T-Bills.

n	**i**	**PV**	**PMT**	**FV**
1	5	**?**	0	**5,000**
	Solving...	**-4,761.90..**		

The T–Bar now becomes,

		(50,000)				(50,000)
End of Period	1	10,000 – 4,762 ¬		=	1	5,238
	2	-5,000	+ 5,000	=	2	0
	3	85,000		=	3	85,000

The MIRR of this series is 22.94%, the best yield of all three methods.

From a financial standpoint, large negative PMTs occurring well down in the cashflow sequence are best managed by planning to reinvest a *sufficient* portion of previous cashflows at a conservative, or *safe*, rate to cover the future negative PMT. This would be cash-sparing, since it would require no additional up-front investment. If the negative PMTs occur early in the cashflow series, however, there may be no choice but to provide for them by an additional CFo invested separately in a low-risk investment at a safe rate of interest. In any case, the MIRR will *always* be less than the IRR.

Providing most efficiently for the future negative cashflows which may be encountered in an otherwise attractive investment is a hallmark of good cashflow management and will improve overall yield.

Note: The HP–12C handbook suggests 1) discounting, at the safe rate, all negative cashflows back to Present Value and 2) compounding all positive cashflows forward at the "re-investment rate" to produce a Future Value, then 3) to solve for the IRR using the PV and FV so determined. This method is repeated in various spreadsheet programs, such as Excel, as a way to eliminate negative cashflows in a series.

This method will *always* result in a lower investment yield than the third method described above because amounts added to CFo have a pronounced negative effect on the IRR. The HP method is also one source of the commonly heard "the IRR depends on the re-investment rate." If one *chooses* the re-investment rate, or uses an existing "hurdle rate, "then the "IRR" is *forced.*, not calculated. Then why not simply use the NPV method?

Chapter Summary

1. The Internal Rate of Return is that *single* discount rate which will convert all future payments to present values whose sum will be exactly equal to the cash originally invested.
2. The IRR is not an average of periodic or "spot" rates of return, but rather a single discount rate expressed on a per-period basis which may be realized if the investment delivers the projected cashflows and is held for all the planned number of periods.
3. The calculations necessary to determine the IRR do not involve risk assessment, although a high IRR may suggest risk. Therefore the investor or financial analyst may wish also to compute the Net Present Value of the investment, a process which requires a discount rate to be selected with due consideration given to risk.
4. There are certain technical limitations to the IRR, such as the possibility of multiple solutions. Though real, the possibility is considered remote when dealing with real-world investments. Nevertheless, the investor/analyst should be alert when dealing with cashflows whose PMTs change sign (+ –) frequently. There will potentially be as many separate IRR solutions as there are changes in sign.
5. While the calculation of the IRR is mathematically precise, the estimates of future PMTs upon which the IRR is based may be moderate, conservative or highly optimistic. Therefore expressing an IRR to the second decimal point may be grossly misleading by implying a degree of precision to the whole exercise which is inconsistent with the precision of underlying assumptions.
6. The Modified Internal Rate of Return, MIRR (or Financial Management Rate of Return - FMRR), overcomes some of the problems posed by negative PMTs which occur in a cashflow series. In order to maintain the highest possible rate of return, *sufficient* cash flowing from a previous period or periods should be reinvested at a conservative rate to cover future negative cashflows.
7. Only when there is no previous cashflow available to fund a future negative cashflow should additional cash be added to the Present Value (initial investment). Doing so has a disproportionately negative effect on the IRR.
8. The calculated IRR is not at all dependent upon the rate at which outflowing PMTs can be reinvested in a subsequent investment. But prudent portfolio management will take into consideration the options available for the reinvestment of funds in managing the entire amount available for investment.
9. The IRR is distinct from the geometric mean rate of return from an investment. The geometric mean is the multiple of all the previous rates of return multiplied together, and assumes that the proceeds from the investment will remain in the investment. The IRR is a measure of the rate of return remaining in an investment and assumes that the periodic cashflows will be removed from the investment.

Chapter 5 Cashflows from Stocks

Because stocks and bonds comprise an important part of a well-balanced investment portfolio, it is essential to understand how these financial instruments return cashflow to the investor and how a rate of return on this cashflow can be measured.

An analysis of the cashflows produced by stocks and bonds, however, is not the equivalent of an analysis of the economic fundamentals which may produce these cashflows. That's a task for the stock or bond analyst. But given economic projections relevant to an issue, and the cashflows they imply, it's our task to measure and rate the returns which can be anticipated from ownership and to estimate a market value.

Cashflows from Stocks

Many investors buy stocks on their past record. But the present value of any financial investment is not measured by what has already happened, but rather by what is likely to happen in the future. Most of these same investors would not buy a 20–year old racehorse because of the number of races he has already won; they would probably be more interested to know how many races he *can* win in the future.

Stocks, too, depend for their present market value on their future cashflows. These future cashflows can be separated into two neat piles: first, the dividends which a stock may pay over the contemplated holding period, and second, the amount of gain – or loss – it will produce when sold at the end of the holding period.

The dividend which a stock pays can be regarded as a periodic **PMT.** For the most part, companies which pay dividends increase and decrease their dividend amounts depending on current earnings and the firm's forecasted requirement for cash to fund future projects. Most companies prefer to hold the dividend, or its annual increase, fairly stable, since a decline in dividends tends to result in a prompt loss of stock value.

The gain (or loss) involved in the valuation process includes the amount of appreciation (or loss) in value which the investor will experience when he sells the stock. This final cashflow is exactly equivalent to our concept of the reversionary or *Future Value*.

But in order to maintain touch with the practitioners, we will use $\mathbf{DVD_n}$ to represent the periodic $\mathbf{PMT_n}$, $\mathbf{P_o}$ to represent the present value (**PV**), and $\mathbf{P_n}$ to represent the final price (**FV**) when the stock is sold in $\mathbf{year_n}$.

The Market Capitalization Rate

If a stock which pays dividends[1] were to be held for one year, the return rate, **r**, would be determined as :

$$\mathbf{r} = \frac{DVD + (Pn - Po)}{Po}$$

Note that the price P_0 can be expressed as a function of the rate **r:**

$$Po = \frac{DVD + Pn}{(1+r)}$$

This rate, **r**, is the **market capitalization rate**[2], the rate at which the market expresses the present value of future income. A stock's price is a function of its market capitalization rate:

Market sectors tend to carry market-sector capitalization rates, while individual stocks within that sector carry capitalization rates generally within the range for the sector. The capitalization rate reflects a safe rate of return on cash invested plus a *premium* commensurate with perceived risk for the particular investment.

Methods of Measuring Value

Stocks can be valued by one of three basic methods, each related to the dividend circumstances of the individual stock in relation to growth. These circumstances include:

1. Stocks which pay high dividends but offer little or no growth potential
2. Stocks which blend dividends with some growth prospects
3. Stocks which pay no dividends but represent high growth potential.

Valuation by Capitalization of Dividend

The first and most historical approach to valuation is the capitalization of future dividends. This method is based on the valuation of a *perpetual ordinary annuity*, described in some detail in Chapter 3. Implicit in this method are two critical assumptions:

1. Dividends will continue in perpetuity (i.e., *forever*)
2. Dividends will neither increase nor decrease over the holding period

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1 Dividends are paid quarterly, but for convenience we will consider the dividend to be an annual PMT.

2 Not to be confused with total market capitalization which equals stock value times the total number of shares outstanding.

Since the Present Value of a perpetual annuity, C, is determined by $PV = \frac{C}{i}$, a stock whose dividend is \$2.50 per year, capitalized at 15%, would indicate a market value of

$$PV = \frac{\$2.50}{0.15} \text{ or } \$16.67.$$

The reciprocal[3] of 0.15 is 6.67 Therefore we can also express the value of the stock as a ***multiple*** of its dividend.

> **The expression** $\frac{\$2.50}{0.15}$ is the same as $\$2.50 * \frac{1}{0.15}$, or \$2.50 * 6.67 = \$16.67

The underlying rationale of this method regards the amount of earnings/share (EPS) as irrelevant to the investor who is most interested in the cash to be received over the span of his holding period, which, in the case of a perpetuity, is forever. Since this holding period is infinite, the reversion value of the stock becomes insignificant and can be disregarded.[4]

If the dividends and the price of the stock are estimated to increase 10% per year, then a stock which is capitalized @ 15% will have a market value which follows the rules for a perpetual Ordinary Annuity whose DVD (PMT) increases at rate **g** :

$$PV = \frac{DVD}{r - g} \quad \text{or,} \quad PV = \frac{\$2.50}{0.15 - 0.10} = \$50.00$$

While the valuation of a perpetuity subject to a constant growth rate may be proven mathematically, it can also be shown to be true by examining a holding for 100 'Stock Years,' which, for all practical purposes, can be equated to a perpetuity.[5]

The table on page 5-4 shows the respective contributions of the dividends and the final price to *total Present Value* over a holding period of 100 years for a stock paying \$2.50 per year in dividends, increasing at the rate of 10% per year. The reversion value of the stock (\$50) is also increasing 10% per year.

Note that the sum of the two present values (PV of cumulative dividends + PV of reversion value) always totals \$50.00. The future sales price of \$689,030.62 (a single cashflow event occurring in Year 100) contributes only 59¢ to today's value when discounted at 15% per year.

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3 The reciprocal of a number = 1/number

4 The PV of a \$100 to be received in 100 years, discounted @ 8% per year is only 4.5 cents.

5 Core States Financial, founded in 1781 as Philadelphia National, has paid dividends continuously since 1844. It is one of a group of 200 U.S. firms that are at least 100 years old.

	Future Values Growing @ 10%/yr.			Present Values Discounted @ 15%/...		
Year	**Dividend**	**Reversion Value**	**PV of Dividend**	***Cumulative* PV of Dividend**	**PV of Reversion Value**	**Total PV**
0		$50.00	$0.00	**$0.00**	**$50.00**	**$50.00**
1	$2.50	$55.00	$2.17	**$2.17**	**$47.83**	**$50.00**
2	$2.75	$60.50	$2.08	**$4.25**	**$45.75**	**$50.00**
3	$3.03	$66.55	$1.99	**$6.24**	**$43.76**	**$50.00**
4	$3.33	$73.21	$1.90	**$8.14**	**$41.86**	**$50.00**
10	$6.48	$129.69	$1.60	**$17.94**	**$32.06**	**$50.00**
50	$293.48	$5,869.54	$0.27	**$44.58**	**$5.42**	**$50.00**
100	$34,451.53	$689,030.62	$0.03	**$49.41**	**$0.59**	**$50.00**

Therefore, the value of a stock can be expressed as the Present Value of an infinite stream of dividends, discounted at an acceptable rate of return.

When a stock is held for a number of periods and then sold, this infinite income stream is shared by two investors, the first of whom enjoys the PV of an infinite annuity less the PV of the second infinite annuity in the hands of the next owner. A review of Chapter 3, p. 3, may be helpful in capturing this concept which equates the PV of a stock paying dividends to the PV of a finite annuity.

It is not reasonable to assume that a stock's earnings will continue to grow at a high rate forever. But we will address this matter later. The rate of growth used in this example is for example purposes only.

Limitation of Capitalization of Dividends Valuation Method

The U.S. stock market of the 1990's behaved like a company which could do without paying dividends. While financial pundits point to an historical average annual growth rate of approximately 10% in the Dow Jones Industrial Average (DJIA), this rate assumes the reinvestment of *pre-tax* dividends. If this rate of growth were a valid number, the DJIA which stood at **157 in 1925** would stand at over **219,656 in 2002**. Instead, the DJIA in 2001 reached 11,000.

Therefore the average growth rate of the market in 76 years has been closer to 5.8% per year. Over this time, the dividend yield rate of the S&P 500 Index has averaged 4.5%, which accounts for the claimed 10% annual return over that period. Rather than increase its dividend, the U.S.

stock market in the 90's decreased the dividend payout in order to fund new investment. By 2002, the dividend yield on the S&P 500 Index had fallen below 2% - a level not seen since 1926.

This trend away from the payment of a dividend to other corporate maneuvers which increase stock price is reinforced by current tax law. Dividends are taxed as ordinary income while gains from price appreciation pay the much lower long-term capital gains rate. As a result, many firms actively pursue policies which enhance price appreciation at the expense of dividend distribution.

Retained earning not used to fund future growth may also be used to buy back the company's stock, thereby decreasing the number of shares outstanding and increasing the apparent earnings-per-share value. [6] Therefore the investor still receives the total of (DVD_n + Pn-Po), but with greater emphasis on (Pn–Po). Unfortunately, he must sell his holding to receive (Pn–Po).

Components of the Market Capitalization Rate

We have shown that the Present Value of a perpetual annuity subject to a growth factor is :

$$\mathbf{P_o} = \frac{\text{DVD}}{(r - g)}$$

Then it is also true that,

$$\mathbf{r} = \frac{\text{DVD}}{P_o} + g$$

or, to express it differently —

Market Capitalization Rate, r = Dividend Yield Rate[7] + Growth Rate.

The growth component, **g**, of the total return, **r**, is a product of a company's **Re-investment Ratio** times its anticipated rate of **Return on Equity (ROE)** or,

g (growth rate[8]) = Re-investment Ratio * ROE

Let's take a closer look at these two important elements of the earnings growth rate, the Re-investment Ratio and the Return-on-Equity.

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6 Many corporate executives are rewarded with stock options which provides an additional incentive.

7 The dividend yield ratio is equal to the dividend divided by the price. The dividend ratio is DIV/EPS.

8 In this case, **g** pertains to the growth of earnings.

The Re-investment Ratio

The portion of current net earnings which a company earmarks for distribution as dividends represents its **Payout Ratio,** or **Dividend Ratio,** which is = $\frac{\textbf{DVD}}{\textbf{EPS}}$. Don't confuse a stock's Dividend **Ratio** with its Dividend **Yield.** The Dividend Yield = $\frac{\textbf{DVD}}{\textbf{Price}}$.

That portion of retained earnings devoted to funding projects for future growth[9] is its **Re-investment Ratio** = **1** – $\frac{\textbf{DVD}}{\textbf{EPS}}$. The amount of *new* earnings which will flow from the re-investment of current earnings will depend upon the rate, or yield, at which these re-invested earnings earn new cash. This rate may be equal to, or above, but hopefully not below, the company's current rate of **Return on Equity.**

The Return on Equity (ROE)

Warren Buffet's dictum to investors is "Focus on return on equity, not earnings per share." The rate of Return on Equity is simply the rate at which net earnings are produced in relation to the total net value of assets at the disposal of management. For this reason, ROE is considered by many as the optimum yardstick of management's effectiveness in producing a profit using assets under its control.

There is no guaranty that reinvested retained earnings will yield a targeted rate of return. Undoubtedly management will carefully weigh each new potential investment opportunity to develop an estimate of yield (**IRR**) on the new cash to be invested.

Some companies evaluate these new opportunities by posting a required minimum Rate of Return on Equity invested, or "hurdle rate," and then calculate the Net Present Value of the prospective project using the hurdle rate as a discount rate. The firm's cost of funds is also commonly used as the "hurdle rate" on the theory that the company should be able to show a profit when the net yield from the project exceeds the cost of borrowed funds. Those projects with a negative NPV are typically rejected.

The average investor, however, is usually not privy to intramural details about new growth opportunities and anticipated future yields and thus is often forced to assume that the racehorse will run in the future as it has in the recent past.[10] He will attempt to measure the ROE which management has produced *recently*. To do that he must collect relevant information about costs,

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9 On the assumption that all retained earnings are re-invested. But this is frequently not the case.

10 In contrast to the professional investor who actively investigates prospects for future earnings.

profits, earnings and the number of shares currently outstanding.[11] He will use these data to estimate a current **ROE**[12] and hope that this rate will apply in the future.

One estimate of **ROE** might be:

$$\textbf{ROE} = \frac{\text{Earnings per share}}{\textbf{Book Equity per share}}$$

where, $$\textbf{Book Equity per share} = \frac{\text{Shareholder investment + retained earnings}}{\text{Total number of shares outstanding}}$$

Book Equity, however, may be very misleading for companies that have been long-established or for companies which have acquired other firms where "goodwill"[13] figured significantly in the acquisition price. As a result, the book value of the company may be significantly overstated.

The phrase "Shareholders Investment + Retained Earnings" may also include the acquisition of many hard assets which have long ago been fully depreciated. They are carried on the "books" at a value perhaps much lower than their current market value. Real estate owned (REO) is a prime example of such an asset. In these cases book value may be significantly understated.

After a number of years and a number of transactions, the term "book value," or "book equity" may have only historical significance unless a reassessment of the value of total assets and total liabilities is undertaken. In that case Book Equity may be Net Assets/Total Shares Outstanding.[14]

Value as a Combination of Dividends *plus* Growth

This second method of valuation begins by postulating that if a company had no growth opportunities, it would undoubtedly distribute all its current net earnings as dividends. This method first capitalizes total earnings as though they were wholly distributed dividends, then adds to this value the capitalized **Net** Present Value of new earnings.
Let's construct an example to illustrate this method:

> An industry analyst forecasts that the stock of ABC, currently selling for $50 per share, will earn $6.25 per share in the coming year, and that the price will advance to $55. The dividend is also expected to be increased to $2.50, in line with the company's current policy of distributing 40% of earnings to shareholders.

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11 Historic rates of return on equity are available from many financial service companies.

12 In his approach to Value Investing, Warren Buffet starts with "owners' equity" which he defines as "net owner earnings," i.e. reported earnings *plus* depreciation and amortization minus new capital investment. This amount is known as "free cashflow." His business valuations are based on this number.

13 The value by which the acquisition price exceeds the market value of the assets acquired.

14 On a fully diluted basis, meaning that issued option shares should be included in the total.

From this information we can determine ABC's *prospective* market capitalization rate:

$$\mathbf{r} = \frac{DVD_1 + (P_n - P_o)}{P_o} = \frac{2..50 + (55 - 50)}{50} = 0.15\text{, or } 15\%$$

Since ABC is earning \$6.25 its market value in a no-growth situation in which it would distribute all earnings as dividends would be $\frac{\$6.25}{0.15} = \$41.67$.

But ABC's current price is \$50. Therefore the market must attribute a present value of \$8.33 (\$50.00 – \$41.67) to future growth opportunities. How does it arrive at this extra value?

Since ABC will retain 60% of earnings for use in new projects and re-invest them at the current ROE, its return on these projects is expected to equal:

$$\begin{aligned}\text{New Earnings} &= \text{EPS} * \text{Re-investment Ratio} * \text{ROE}\\ &= \$6.25 * 60\% * \text{ROE}\end{aligned}$$

The historical ROE for this firm is 16.67%. Therefore ,

$$\begin{aligned}\text{New Earnings} &= \$6.25 * 60\% * 16.67\%\\ &= \$0.625\end{aligned}$$

The market will capitalize these new earnings @ 15%: $\frac{\$0.625}{0.15} = \$4.16667$.

But it would be inappropriate to accept the Present Value of new earnings without taking into account the amount of the initial investment made to produce them. The **Net** Present Value of these earnings, therefore, will be the present value of the new earnings less the initial investment, or \$4.16667 – \$3.75 = \$0.41667.

Since this Net Present Value will have a 10% growth expectancy, the market will assign a capitalized value to these earnings according to the formula for a perpetuity growing at rate **g:**

$$PV = \frac{\$0.41667}{(0.15 - 0.10)} = \$8.33$$

$$P_0 = \frac{\$6.25}{0.15} + \$8.33 = \$50.00$$

Therefore, a company now paying some dividends, but with growth opportunities, may have its stock valued as *–the capitalized value of fully distributed earnings*, **plus** *the capitalized value of **net** new earnings from new investments.*

This valuation is represented by the following formula:

$$\mathbf{P_o} = \frac{\text{EPS}}{\text{r}} + \left[\frac{\dfrac{[\text{EPS} * \text{RR} * \text{ROE}\ ]}{\text{r}} - \text{Initial Investment}}{(\text{r} - \text{g})} \right]$$

where, P_o = current stock value
EPS = earnings per share
RR = Reinvestment Rate
r = market capitalization rate
g = anticipate growth rate of earnings

This is fairly easy to understand when the company under consideration has but one product line or one service. When there are many product lines or many divisions, the ROE will be the weighted return on cash invested in each product line or division.

Importance of the IRR for New Projects

Companies whose stock value depends heavily on growth are especially sensitive to the yield (IRR) to be delivered by new projects and ventures. Consider the effect on ABC's stock price when a major new project is accepted which carries an IRR of 15%, which is only equal to the current market capitalization rate of 15%, and below current ROE of 16.67%:

Growth Rate = Reinvestment Rate * IRR
g = 60% * 15%
g = 9.0%

In terms of dollars,

New Earnings = EPS * Re-investment Rate * IRR
= $6.25 * 60% * 15%
= $0.5625

The market will capitalize these new earnings @ 15%: $\frac{\$0.5625}{0.15} = \$3.75$

The **NPV** of this additional earnings of $3.75 is zero (– $3.75 + 3.75 = 0). The value of the stock will become:

$$P_0 = \frac{\$6.25}{0.15} + \mathbf{\$0.00} = \$41.67$$

This example helps illustrate the sensitivity of a growth stock's price to changes in overall ROE. For this reason, company leadership is always alert to new projects and ventures which have prospective yields (IRRs) at least equal to the current market capitalization rate and, hopefully, higher. In the case of ABC, a new growth venture with an IRR of 20% would have a substantial, positive impact on stock value:

Growth rate = Re-investment Rate * IRR
g = 60% * 20%
g = 12%

New Earnings = EPS * Re-investment Rate * ROE
= \$6.25 * 60% * 20%
= \$0..75

The market will capitalize these new earnings @ 15%: $\frac{\$0.75}{0.15}$ = \$5.00, resulting in a Net Present Value of \$1.25 (\$5.00–3.75). Assuming continuing earnings from this venture and a growth rate of 10.0%, the new earnings will contribute \$15.00 to stock price: $(\frac{\$0.75}{(0.15-0.10)})$ =\$15.00

$$P_0 = \frac{\$6.25}{0.15} + \$15.00 = \$56.67$$

When a growth company which retains all earnings, and whose **ROE** is above its market capitalization rate, accepts new projects whose IRR is less than current ROE, it sacrifices the value premium which its stock currently enjoys.

When a growth company which retains all earnings accepts a new project whose **IRR** is less than its current market capitalization rate, it depresses the value of the stock below that which could be achieved by a distribution of all its earnings as dividends.

When a growth company which retains all earnings successfully pursues new projects whose IRR is in excess of its current market capitalization rate *and* current ROE, it adds substantially to stock value.

Growth Without Dividends

Companies with excellent future growth potential and opportunity generally retain a large share of earnings, and very often all earnings, since reinvestment of these earnings delivers new earnings at high yield rates. Investors remain happy with this arrangement as long as the rate of price appreciation in their shares is greater than what they could obtain by re-investing dividends elsewhere.

In those cases in which no dividends are paid, the value of the stock is delivered as a function of Earnings Per Share alone. – using the assumption that if there were no remaining opportunities for reinvestment the firm would eventually distribute all current net earnings as dividends.

$$P_0 = \frac{EPS}{(r-g)}$$

Using this method, which also assumes that earnings (EPS) will continue forever, the value of **g**, the growth rate, *is also assumed to continue forever.*

Of course it will not. Either the market will become saturated, or competition will erode high margins, or technology will change. Even without these adverse factors, as the company grows its total assets grow, and the increase in net income necessary to maintain the same *percentage* of ROE becomes more difficult to achieve. So the natural life cycle of most companies is much like our own: a vigorous youth, a stable maturity, a gradual decline.

Consequently, it is common to project high growth rates for only a limited number of years, then to decrease substantially the rate of growth for later years. An alternate approach, as suggested by Franco Modigliani, is to take initially only a portion of the expected growth rate in order to "average out" the overall rate over time.

As the growth rate, **g**, declines, the denominator of this fraction increases, and the stock's value decreases. Since the price of the stock is not defended by distributed earnings in the form of dividends, the effect of a small decrease in growth has an direct and very pronounced effect on stock price. Consider a decline in growth from 10% to 8%.

$$P_0 = \frac{EPS}{(0.15-0.10)} = EPS * \mathbf{20} \qquad P_0 = \frac{EPS}{(0.15-0.08)} = EPS * \mathbf{14.28}$$

A 20% reduction in a growth rate of 10% (from 10% to 8%), in a market which capitalizes earnings at 15%, will result in a 28.6% decline (from a multiple of 20 to 14.28) in stock value.

One can see why high growth companies which distribute no dividends are also regarded by the market as higher risk investment situations commanding higher capitalization rates, and why their stock price is so sensitive to changes in prospects for EPS..

The Price-Earnings Ratio and Growth Rate

It is often more convenient to express a stock's price as a function of its earnings (EPS).

We already know that the market capitalization rate of a pure growth stock: **(r-g)** = $\frac{EPS}{Price(P_0)}$.

This ratio always results in a fraction. For example, if EPS is \$5.00 and Price is \$100.00, **(r-g)** = 0.05. We can obtain the same result using the reciprocal of 0.05 which is 20 (1/.05). The reciprocal is the stock's ***price/earnings multiple***, or **P/E ratio:**

$$EPS * Multiple = Price$$
$$\$5.00 * 20 = \$100$$

Therefore the Price-Earnings multiple is the reciprocal of the denominator of our value of **(r-g).** Oftentimes a growth stock's multiple is mistaken for the reciprocal of its capitalization rate alone, when it is actually the reciprocal of **(r-g)**.

For example, in 1998 Microsoft, a high growth rate company which retains all earnings, enjoyed a market price of approximately \$150 on earnings of approximately \$2.96, resulting in a

Price/Earnings multiple of 51. Since Microsoft paid no dividends, its high capitalization rate reflected its dependency on future growth: $\$150 = \frac{\$2.96}{(r - g)}$.

The 5-year rate of growth (1993-1998) in earnings (**g**) for this firm was 27% per year. Therefore,

$$\$150 = \frac{\$2.96}{(r-g)} \text{ and } r\text{-}g = \frac{\$2.96}{\$150}$$
$$r = 0.01973 + g \text{ and } r = 0.01973 + 0.27$$
$$r = 0.28973 = 29\%$$

Therefore the market capitalization rate for Microsoft in 1998 was 29%. Its Price/Earnings multiple, 51, in the absence of any dividend, was buoyed up by its astonishing growth rate.[15]

It is worth noting that real estate investments use the reciprocal of the Price/Earnings ratio, the Earnings/Price ratio, or, in real estate terms, the Net Operating Income/Fair Market Value. This fraction is known as the *overall capitalization rate*.[16] While it is tempting to compare the P/E ratio of stocks to the E/P ratio (the "cap rate") of real estate, the "earnings" used in the real estate formula is really a pre-tax cashflow number (EBITDA[17]). Therefore the two 'capitalization' rates are not directly comparable.

PEG Ratios

Investors are generally quite interested in a firm's **g**, its growth rate. This is especially true in the case of younger and smaller firms which often pay no dividends, since appreciation in stock value is the investor's only reward. One measure applied to this situation is the PEG Ratio, which is defined as the Price-Earning Ratio divided by the anticipated annual growth rate.

The basis of this measurement is that if $\frac{P/E}{g} = 1$ then P/E = 1*g, and P = E*1*g, where **g** is the future annual growth rate expressed *as a number and not a percentage..*

If earnings increase, then the Price ought to increase as well in order to maintain the same relationship. If, however, the Price does not increase proportionate to Earnings, then the PEG ratio will fall below 1, perhaps indicating a market undervaluation. If the PEG ratio increases to a value greater than 1, then the Price is increasing at a rate proportionately greater than Earnings, suggesting the stock may be overvalued.

The PEG Ratio is applied most often to smaller companies which pay no dividends. As a company increases in size, a lower growth rate measured against a relatively fixed number of shares outstanding may still produce significant improvement in net income. In that case, the PEG ratio may be quite misleading.

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[15] In 2002 the SEC sanctioned Microsoft for the company's practice of deferring the recognition of profit. Over the years, Microsoft has always exceeded its profit forecast by exactly $0.01 per share.

[16] "Overall" because it includes capital sourced from debt.

[17] Earnings Before Interest, Taxes, Depreciation and Amortization deductions

Though a convenient and easy to use ratio, the PEG Ratio depends on a forward assessment of future earnings, and is therefore prone to forecasting errors. It is probably best used as an initial screen.

The Capitalization Rate, r, and the Safe Rate

Capitalization rates do not contain an allowance for inflation, but – in the case of stocks – are composed of a *safe interest rate* and an allowance, or *premium*, for risk. Therefore the value of **r** is the sum of the safe interest rate and a rate which reflects risk.

The *safe rate* of return on cash in the United States has remained remarkably constant over the last 20 years, averaging 3.50% – 4.00%. This rate is reasonably independent of the inflation rate. When inflation rates are high, nominal interest rates are also high but the difference between the nominal rate and the inflation rate generally falls within the historical range of the safe rate.

For example, a nominal interest rate on the 10-year Treasury bond in 2000 was 6.5% when the inflation rate for the year current was 2.5%, yielding a safe rate of 6.5 – 2.5% = 4.0%.

A valuable source for the determination of the safe rate is the U.S. Treasury's Inflation-Indexed Bonds (TIPS)[18] These instruments have upward adjustments made to the periodic payments and principal at maturity in order to remove the effect of inflation on bond yield. As such, they are excellent indicators of the current safe rate. At the same time the 10-year bond was at 6.5%, the 10-year Inflation-Adjusted (Indexed) Treasury Bond was 4.0%, indicating an inflation rate of approximately 2.5%. (See Chapter 6 for additional information.)

The safe rate is applicable to all investments, but an additional premium reflecting the stock's particular risk must be added in order to arrive at the appropriate capitalization rate, **r**.

Stock Value When Growth is Limited or Nearly Absent

Those companies whose growth prospects are very limited are confined to distributing sufficient earnings in the form of dividends to support the stock's current price.

Utility companies which usually have high payout ratios, for example, are in a near-constant process of applying for rate increases to maintain constant-dollar dividends and maintain shareholder value. Because of the time delay in obtaining such increases, utilities tend to lose market favor during times of significant inflation because they carry lagging market yields. During times of deflation, they tend to regain investors' favor because their yields are high in relation to declining interest rates.

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18 Find them on the bond page at the bottom of the column listing price quotes for U.S. Treasury securities. TIPS = Treasury Inflation-Protected Securities.

Cashflows from REITs

REITs (real estate investment trusts) represent a rather unusual situation since they must, by law, distribute 90% of their earnings to investors each year. This leaves little in retained earnings for reinvestment in new projects. Growth must be provided for by borrowed funds, by selling a property asset, or by issuing more stock.

While additional stock offerings raise new capital, they also dilute the per share earnings of the outstanding stock. Unless management can invest the proceeds to restore EPS on the increased number of shares, the REIT will experience a diminished ROE. Enough properties in the REIT mix operating at a diminished IRR will depress overall ROE, leading to lower stock prices.

Since the cashflows that are used to value the REIT stock are assumed to be *continuing* cashflows, proceeds from the sale of an asset need to be eliminated from inclusion in Net Operating Income. The remaining cashflows, Funds From Operations (**FFO**), should be used. Some REIT analysts add back non-cash items such as accounting depreciation and amortization to FFO funds, but then deduct actual cash estimates necessary to maintain current properties. These analysts measure **AFFO**, Adjusted Funds From Operations.

In the absence of sufficient new investment opportunities which will equal or better a REIT's current ROE, many REITs resort to increasing leverage when buying new properties and re-financing existing properties, especially when interest rates are lower than cap rates. Many REITs which were once leveraged 35% are now leveraged 50-60%.

Free Cashflow Per Share

The term "Free Cash Flow" signifies the amount of net after-tax earnings, after adding back depreciation and amortization deductions (non-cash items), which are not retained, reinvested or otherwise consumed in the business. Therefore Free Cash Flow refers to potentially distributable earnings per share.[19] The value of a stock can also be expressed as a function of its Free Cash Flow, which - to some extent- expresses value devoid of the aberration in earnings caused by deducting non-cash expenses. This index is often used in merger and acquisition (M&A) situations.

[19] This is a useful distinction since not all retained earnings may be used to fund growth opportunities.

Chapter Summary

While it may be somewhat satisfying to understand the basic financial theory underlying cashflow valuation of stocks, the devil is in the details. Here are some things to ponder.

1. Stocks, like all other investments, derive their Present Value, or worth, from the value of all future earnings, discounted at a yield (rate) acceptable to the holder of the stock.
2. The Market Capitalization Rate of a stock is a combination of its dividend yield and growth rate.
3. The growth rate is the product of that share of earnings applied to new business opportunities, the Re-investment Rate, multiplied by the IRR, or Return on Equity, for the proposed new venture(s).
4. Stocks which pay dividends are often valued by treating the dividend as a perpetual annuity, capitalizing the total annual dividend by a rate acceptable to the investor. This method is most appropriate when nearly all earnings are distributed.
5. Stocks may also be valued by capitalizing current earnings and adding the capitalized Net Present Value of new earnings. This method is more suitable to companies with significant growth prospects funded by retained earnings..
6. Stocks which pay little or no dividends rely entirely on gain from price appreciation for total return.
7. As companies grow in size and accumulated earnings, the difficulty of investing accumulated cash at a high rate of return begins to slow the rate of Return on Equity.
8. To bolster stock prices, these companies often buy back their own stock, and later initiate or increase dividend payout.
9. Stocks may also be valued by capitalizing Free Cashflow: after-tax net earnings, *plus* depreciation and amortization, *less* earnings retained for new investment.
10. Dividends contribute significantly to overall return rates and tend to moderate variations in a stock's growth rate.
11. Current earnings which are capitalized at the market capitalization rate are not guaranteed to infinity. Core businesses are also subject to risk, to change and to competition. Increased industry competition increases risk perception and tends to raise market capitalization rates and lower stock prices.
12. The market changes its perception of risk associated with a company's prospects for growth. High yields, especially in industries with low thresholds of entry, invariably invite competition which reduces market share, lowers prices, lowers profits, lowers earnings and eventually attenuates shareholder value.
13. A company may set aside a significant portion of its earnings for a "rainy day." This is especially true in cyclic industries such as autos and housing which frequently adopt a defensive posture as a means of self-preservation. These retained earnings, which are not invested at the company's market capitalization rate, increase net worth but tend to reduce the rate of return on equity.
14. High growth rates in new industries are difficult to sustain, not only because of the increased competition which they attract, but also because of rapidly changing technology. The growth factor, **g**, becomes harder and harder to accomplish.

Chapter 6 Cashflows from Bonds

Bonds are essentially different from equities (stocks) in that they represent no ownership interest in the company; they are, in fact, I.O.U.s made by the issuing entity and secured by assets of one sort or another.[1]

Bonds take on a nomenclature related to the issuing entity: those issued by a city, county or state are known as “municipal” bonds; those issued by corporations are “corporate” bonds, while those issued either by the federal government, or an agency of the federal government, are “Treasury” or “Agency” bonds.

There are many other sub-classifications of bonds which have very significant import for investors and financial planners. Those interested in this *anything-but-dry* subject of *bonds*[2] are encouraged to investigate and read further. For our part, we want to look carefully at how the value of bonds is determined in anticipation of the *cashflows* they promise to the investor.

Bond Value Predicated on Anticipated Cashflows

The value of *all* financial investments depends on the *Present Value* of cashflows which the owner can reasonably anticipate during his period of ownership.[3] Bonds are no exception. Bonds are somewhat unique, however, in that they almost always combine a series of payments, PMTs, with a reversionary value, FV. Bonds are often (and accurately) described as mortgages, but unlike mortgages which typically amortize to zero value,[4] bonds do not amortize to zero value:

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1 “Bonds” are debt instruments secured by a pledge of assets; “Debentures” are unsecured debt instruments.

2 During the go-go years of the late 90s, the idea of investing in *bonds* became somewhat déclassé and are often now referred to as 'fixed income securities.'

3 A recurrent theme.

4 Or at least are *scheduled* to do so.

they customarily have a reversionary value at maturity of $1,000 and therefore behave financially like interest-only $1,000 promissory notes.

In the case of bonds, the PMTs are the *interest* payments to be received during the holding period, while the Future Value (FV) is the amount of cash which the holder of the bond will receive when the bond is redeemed.

Timing and Reversionary Value of Bonds

Most bonds issued in the United States are designed to pay interest semi-annually, in arrears, and are therefore forms of an *ordinary finite annuity*. The last PMT which the bond holder receives at *maturity* is a combination of the redemption value of the bond *plus* the last interest payment.

In Chapter 2, we stressed the need for caution in characterizing the timing of the cashflows. You will recall that many reversionary sums (FVs) are due at the end of the holding period along with the last cashflow from operations, or – in the case of mortgages – with the last payment due in the schedule. Bonds fit this note of caution since the redemption value (FV) will always be made concurrently with the last interest payment. Do not schedule the last interest payment (PMT) and then schedule the redemption value (FV) one period later. Doing so will distort the timing of your calculations and will alter the true yield-to-maturity.

When Bonds Are "Called"

We are hedging the point a bit when we refer to the last payment as the *reversionary* value rather than simply referring to the usual maturity value of $1,000. This is because many issuers reserve the right to "call," or *redeem* the bond before its scheduled maturity date. Issuers insert call dates in bonds as an option to be exercised if interest rates decline. Should interest rates decline sufficiently, the issuer can redeem the outstanding higher-interest-rate bonds and replace them with securities paying a lower rate. In order to maintain the initial marketability of these bonds for investors seeking higher rates, the issuer pays a premium, or "call" price. In these cases, the "call" price is always greater than the par value of the bond.

Bond "Coupon" Rate and Maturity Date

The *coupon* [5] *rate* of the bond is the rate of interest stated in the bond document at the time of issuance. This is the annual interest rate which will be paid *over the life of the bond*. Therefore a bond issued with a 7% "coupon" will pay:

Coupon Rate x Face Value = Annual Interest Payment
0.07 x $1,000 = $70.00

5 Bond documents once had detachable coupons which were intended to be "clipped" and remitted to the bond manager for periodic payment. No longer used.

The PMT, $70 in this case, will usually not change.[6] Almost all bonds issued in the United States are designed to pay interest *semi-annually, in arrears.* Therefore the actual amount received by the bond holder in the example above will be $35.00 every 6 months. Therefore all compounding and discounting with regard to bonds is done on a twice-a-year basis.

The *maturity date* stated in the bond document is the date on which the issuer will fund the redemption value (FV) of the bond. U.S. Treasury and federal agency bonds tend to have the longest maturity dates (30-40 years) although some private (corporate) bonds have been issued with maturity dates as long as 100 years.[7] These super-long-term bonds, as the reader can readily appreciate by now, behave like perpetual finite annuities ($PV = \frac{PMT}{i}$).

Bond Terminology

In order to work through a consistent example, let's choose two United States Treasury obligations listed as follows:

	Rate	**Mo/Yr.**	**Bid**	**Ask**	**Ask/Yld**
#1)	3 1/2	Nov 09	98:13	99:13	3.71
#2)	5 1/2	Nov 09**n**	99:27	99:29	5.53

The **Rate** indicates the "coupon rate" (PMT) for the bond. Therefore bond #1 pays 3.5% of $1,000 annually, or $17.50, every six months. Bond #2 pays $27.50 every six months.

The **Mo/Yr.** indicates that both instruments will mature in November 2009. The small **n** indicates that the second "bond" is really a "Note." The fact that both instruments mature at the same time but carry different (original) interest rates indicates that they were issued at different times, bond #1 when rates were slightly lower, and bond #2 when rates were somewhat higher.

The **Bid** is the price offered for these instruments near the end of the trading day.[8] Treasury bonds, use "**ticks**" to represent a fraction of 1%: one "tick" equals 1/32 of 1%, or 31.25 cents Thirty-two ticks equal 1%.

In our pricing example of a Treasury bond, the colon between 98 and 13 indicates that the bond is priced at 98 **percent** of $1000 plus 13 "ticks," or 13/32nds of one percentage point. This price,

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6 Some bonds are issued with "step ups" in their coupon rate, while others have interest rates which "float, " i.e. are tied to an index, such as the U.S. Treasury bills.

7 In 1995, Columbia/HCA Healthcare issued $200 million of debt (bonds) due in 100 years. Other recent issuers of century debt are: Disney (1993), Coca-Cola (1994) and ABN Amro (1994). Bonds with very long maturity dates may be subject to re-classification as equities by the government, thus eliminating the deductibility of interest payments made to bond holders.

8 Bond prices quoted in the newspaper are usually not the last price of the day, but rather the price quoted at the time the news service gathered the quote.

98:13, means 98+13/32 %, or 98.40625 % of $1,000, or $984.0625.[9] Both Treasury notes and bonds are quoted using "ticks," but "bills" are quoted using hundredths of a point. Corporate bonds are quoted in fractions: e.g. 98 1/2 = 98.5% of $1,000, or $985.00.

The **Ask** is the price last asked by a potential seller of the instrument on this trading day. Yield-To-Maturity rates are calculated on **Ask** prices. All Treasury <u>bonds</u> are quoted as a percent of the current **Ask** quote. The **Ask/Yld** is the yield (IRR) which the buyer of the instrument would realize if he paid the Ask price and held the note to maturity (redemption) in November 2009.

Calculating Bond Yields

We can easily construct two T-Bars to represent the cashflows from these instruments from a specified date to maturity. For the sake of convenience, let's assume that both instruments mature on the very same day, three years from Nov. 1, 2006.

<u>Bond #1</u>

	(994.0625)[10]
	1 17.50
	2 17.50
$\boxed{r = ?}$	3 17.50
	4 17.50
	5 17.50
	6 17.50 + 1,000

The <u>Net</u> Present Value of these cashflows can also be depicted as:

$$\text{NPV} = -\underline{994.0625} + \frac{17.50}{(1+r)^1} + \frac{17.50}{(1+r)^2} + \frac{17.50}{(1+r)^3} + \frac{17.50}{(1+r)^4} + \frac{17.50}{(1+r)^5} + \frac{(17.50+1{,}000)}{(1+r)^6}$$

We are looking here for the value of **r** which will render the NPV = 0. We are looking for the bond's IRR, which, in bond parlance, is its Yield-to-Maturity.

Yield (r) = 1.85548% per six months, or 3.71097 % per year. (To determine the necessary keystrokes, consult Chapter 2 dealing with uneven cashflows.)

9 To convert the bond quote to price, simply multiply by ten: divide by 100 to convert a percentage to a decimal and multiply by $1,000 to represent face value.

10 Price converted from 99 + 13 ticks, {13/32/100 * 1,000} Convert ticks to a decimal, then divide by 100 to convert to a percent. Multiply by 1000 to find cash value of the 13 ticks. (Or, simply 13/32*10)

Bond #2

	(999.0625)
1	27.50
2	27.50
3	27.50
4	27.50
5	27.50
6	27.50 + 1,000

$\boxed{r = ?}$

Yield (r) = 2.76717% per six months,
or 5.53435% per year.

Using the Calculator to Determine Yield-To-Maturity

The HP 12-C provides a convenient on-board program to determine Yield-to-Maturity (**YTM**), or, given a desired YTM, the Price. This program is particularly useful when the settlement date does not fall on an anniversary of the issue date.

Let's first consider its facility in calculating YTM.

> A bond bearing a coupon of 7.75% matures on Nov. 10, 2005. Its current Ask price is 96.5.
> What is this bond's current **YTM** if purchased on Dec. 1, 1999 at the Ask price?

You already know that bond prices are quoted as a percent of $1,000. But the HP has been programmed to allow for this, so you need make no conversion from percent to dollars.[11]

Key In	Display Shows	Comments
96.5 [PV]	**96.50**	**Enters today's asking price**
7.75 [PMT]	**7.75**	**Enters coupon rate**
12.011999 [Enter]	**12.01**	**The settlement (purchase) date**[12]
solving		
11.102005 [f] [YTM]	**8.51**	**The Yield to Maturity (IRR)**

12 Both HP-17B and 19B calculators allow the selection of a maturity amount other than $1,000. Excel's bond functions also provide this flexibility.

12 Notice that the format for the date is M.DY (2,8). When this format is used, no date notation appears in the calculator's window. When the format D.MY is used (2,7), a notation of the format does appear in the window. The entire date format may be viewed by setting the decimal place to [f] 6.

Observe the date format (**M.DY**) carefully . Note that although the first of the month was used in the settlement date, the formatted date still reserved two places for the day, i.e. 12.**01**1999. The other date format, **D.MY**, key (2,7) could also have been used to format both dates with identical results. These date formats must actually be entered as either **m.ddyyyy** or **d.mmyyyy.** For practical reasons the formats are abbreviated to fit on the facade of the calculator key.

Determining Bond Price, Given YTM

When you are attempting to determine the current value of a bond the procedure is equally simple:

> A bond bearing a coupon of 7.75% matures on Nov. 10, 2005. The settlement date is Dec.1, 1999.
>
> What is this bond's current value to an investor who requires an 8.51% return rate?

Key In	Display Shows	Comments
7.75 [PMT]	**7.75**	**Enters coupon rate (as a % of $1,000)**
8.51 [i]	**8.51**	**Enters desired YTM**
12.011999 [Enter]	**12.01**	**The settlement (purchase) date[13]**
11.102005 [f] [Price]	**96.5037**	**The price as a % of $1,000**
Now press [x ≈y] **(3,4)**	**0.44712**	**The amount (as a % of $1,000) of accrued interest due the prior owner (seller) of the bond.**

Calculating Accrued Interest

In those cases in which the settlement date does not fall on a dividend date, the purchaser will need to reimburse the seller for the amount of interest which the seller has accrued up to the settlement date.

In this case, the calculator assumes that the maturity date of the bond is on Nov. 10, 2005. It then counts back in six month intervals to the purchase (settlement) date, Dec. 1, 1999, assuming that each six month interval will mark a dividend date. Therefore it marks Nov. 10, 1999 as a dividend date. The purchaser will receive the next full dividend of $77.50/2 on May 10, 2000 but will not have owned the bond for 100% of the period covered by the dividend payment. *The accrued interest*

---

13 Notice that the format for the date is M.DY (2,8). When this format is used, no date notation appears in the window. When the format D.MY is used (2,7), a notation of the format appears in the window. The entire date may be viewed by setting the decimal place to [f] 6.

is the interest earned by the prior owner from Nov.10, 1999 to Dec. 1, 1999, the settlement date. The interest for this period will be $4.47 due the previous owner. Therefore the accrued interest due the previous owner is added to the indicated price::

Continuing with the keystrokes to add the accrued interest to the cost of the bond:

$\boxed{+}$	**96.95089**	**Indicating a total cost of $969.51, including accrued interest.**

When Call Dates Are Involved

When a bond contains a call provision (the issuer's right to redeem the bond before its maturity date), the YTM of the bond is determined in one of two ways: if the coupon rate is *above* current market rates, the YTM is calculated to the call date on the assumption that the bond will be redeemed. When the coupon rate is *below* market rates, the YTM is calculated to the maturity date on the assumption that it will not be called in.

Call prices are always above par ($1,000), but the HP-12C's built-in program assumes that the FV of all bonds at maturity will always be $1,000. There is no way to override this maturity price to enter a value greater than par, such as a call price.

For example:

> A bond which carries a 7.75% coupon rate and a maturity date of Dec.1, 2005 also specifies a call date on Dec. 1, 2002 at 105. What will be the YTM for an investor who purchases the bond on Dec. 1, 1999 @ 96.5, if the bond is called?

Determining the YTM of this bond to its call date is made difficult because there is no way to inform the HP-12C calculator that the call price is 105.[14] But we can use a device to *approximate* the price.

We know that the investor is to receive 6 future semi-annual interest payments, including the last payment. Each of these will be equal to $38.75 ($77.50/2). Instead of delivering the $50 call premium at the end of the cashflow, we can adjust the remaining periodic interest payments to include an amount which would be the future *financial equivalent* of $50 over the next 6 payments, using the coupon rate of 7.75% as the interest rate.
What would these PMTs need to be?

n	i	PV	PMT	FV
6	7.75÷2...	**0**	?	50
		Solving...	**-7.561282**	

[14] The HP-12C can be programmed to determine both the yield and price of bonds specifying a redemption value other than $1,000. An example of this program is given in the HP-12C owner's handbook. These solutions, however, are more easily accomplished in Excel®, which has a specific function, YIELD, which allows for call prices. Excel's *Analysis Tool Pack* must be installed.

Therefore the receipt of an extra $7.561282 every six months has a Future Value of $50, the amount of the call premium. By including this payment with the regularly scheduled interest payment, we can provide the financial equivalent of $50 on the call date.

In other words, the receipt of $46.31182 ($38.75 + 7.561282 = $46.31182) for the remaining six payments has the same Future Value as the receipt of $38.75 for six payments together with a final premium payment of $50.00.

We can now employ the HP-12C to determine the YTM by use of a "new" annual coupon rate, which is double the six-month rate:[15]

Key In	Display Shows	Comments
96.5 [PV]	**96.50**	**Enters today's asking price**
9.26236 [PMT]	**9.26**	**Enters "new" annual coupon rate**
12.011999 [Enter]	**12.01**	**The settlement (purchase) date**
solving		
12.012002 [f] [YTM]	**10.66**	**The Yield to Call Date (IRR)**

Determining the Value of a Bond Using Call Price

Now that we own this quiet subterfuge, we can turn it around to determine the Price (PV) of a bond involving a call premium.
For example:

> A bond which carries a 7.75% coupon rate and a maturity date of Dec.1, 2005 also specifies a call date on Dec.1, 2002 at 105. What is the maximum price an investor should pay in order to realize a 10.65% yield if the settlement date will be Dec. 1, 1999?

Key In	Display Shows	Comments
9.26236 [PMT]	**9.26**	**Enters adjusted coupon rate**
10.66 [i]	**10.66**	**Enters desired yield (YTM)**
12.011999 [Enter]	**12.01**	**The settlement (purchase) date**[16]
solving 12.012002 [f] [Price]	**96.50**	**Value if called**

[15] The calculator automatically halves the annual coupon rate.

[16] Notice that the format for the date is M.DY (2,8). When this format is used, no date notation appears in the window. When the format D.MY is used (2,7), a notation of the format appears in the window. The entire date may be viewed by setting the decimal place to [f] 7

Unfortunately, this subterfuge works only when the settlement date falls on the 6-month or 12-month anniversary of a dividend, in order to avoid the problem of accrued interest. If they do not coincide in this way, the HP–12C will not deliver an accurate YTM or price.[17] The bond functions in Excel are very convenient and should be considered.

The HP-12C is *not* certified for bond calculations.

Using Excel's Bond Functions

For those whose calculator does not have a bond program, or for HP-12C owners working with a bond involving a call option, Microsoft's Excel® Spreadsheet program offers some excellent bond functions. In order to access these functions, the Analysis Tool Pack must be installed. To install, open an Excel page and from the Tools Menu select 'Add-Ins.' In the menu which appears, make sure that the box for the 'Analysis ToolPak' is checked. Click "O.K." Then pull down the Insert Menu and select the desired function. If the Analysis Tool Pack does not appear, it has not been installed with the original installation.

The great advantage to these functions is that they permit bond calculations involving something other than a $1,000 maturity value. Consider the function which delivers Price.

> In the formula bar, type ***=Price(*** and then hit *Control+Shift+A*. Excel will fill in all the names of the variables in the formula bar. All you need to do is to replace the variables with the appropriate values.
>
> **=Price(settlement, maturity, rate, yld, redemption, frequency, basis)**
> (Use the variables in the bond problem on page 6-8.)
>
> **=Price("12/1/99","12/1/02",0.0775,0.1065,105,2,0) = 96.37866**

The answer, 96.37866 is the price of the bond as a percent of $1,000. To convert to dollars, simply multiply by 10, = $963.79. This answer is quite close to the estimated price using an adjusted PMT. That answer was 96.50 or $965.00.

Note that the date in the formula is encapsulated in quotation marks and that the rate and yield are expressed as a decimal. The ***frequency*** denotes the number of coupon payment per year – which will almost always be 2. ***Basis*** refers to the method of counting the days. Using **0** for ***Basis*** means that the computer will use 30 days per month and 360 days per year.

The formula for the **yield** of a bond can be mounted in exactly the same way.

---

17 Some models of financial calculators do provide for the specification of a maturity price other than $1,000.

Does Yield Determine Price or Vice-Versa?

It is commonly assumed that the current yield (IRR) determines the current price when, in fact, it is the current price which determines the yield. It is valid to apply the current yield for one bond to determine the value of another of similar risk only when their maturity dates are the same and their coupon rates are the same.

If the current price must be fashioned before we can calculate the yield, then the question posed is "What determines the current price?" The answer is the **spot rate** for money. Using discount rates appropriate for each future period, a present value is calculated. Once the PV is known, the IRR can be expressed.

The **spot rate for money** is the market interest rate to be paid or received for the use of money for a definite period of time. The rate assumes no intervening PMTs.

Determining the Spot Rate for Money

The current market spot rate for money can be easily determined. For example, if it is desired to determine the spot rate for a money instrument with a maturity of 20 years, "purchase" two bonds of comparable risk which carry different coupon rates but which mature at the same time.

	Coupon Rate	Maturity	Current Price
Bond A	4%	20 years	$750
Bond B	8%	20 years	$1200

Buy two **Bonds A**, so that the total coupon payment is $80 per year (annual calculations have been used for simplicity), equal to the coupon payment for **Bond B**.

The cashflows from these bonds, A and B, can be depicted as :

$$\underline{\text{2 A-Bonds }}\text{PV}(\$1{,}500) = \frac{\$80}{(1+r)^1} + \frac{\$80}{(1+r)^2} + \frac{\$80}{(1+r)^3} + \ldots \frac{\$2080}{(1+r)^{20}} \quad \text{(see Note 18)}$$

$$\underline{\text{1 –Bond}}\ \text{PV}(\$1{,}200) = \frac{\$80}{(1+r)^1} + \frac{\$80}{(1+r)^2} + \frac{\$80}{(1+r)^3} + \ldots \frac{\$1080}{(1+r)^{20}}$$

Subtract the cashflow of Bond B from the cashflow of the two A-Bonds. The purpose of this is to eliminate the payments from consideration. As a result, we get:

[18] If necessary, buy fractional interests in the Bond A in order to make the PMT equal to Bond B.

$$\textbf{PV = (1500-1,200)} = \frac{2{,}080 - 1{,}080}{(1+r\}^{20}}$$

We will have invested \$300 more today in order to receive \$1,000 more in 20 years. Therefore:

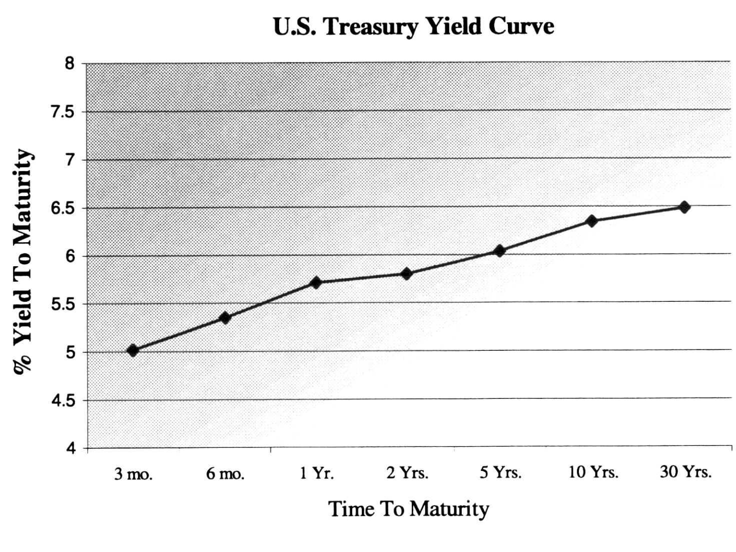
n	i	PV	PMT	FV
20	?	-300	0	1,000
Solving...	**6.205%**			

The spot rate for 20-year money (for this type instrument) is 6.205%. The yield on Bond A is 6.20%; for Bond B the yield is 6.24%.

Term Structure of Interest Rates, Yield Curves

The difficulty in considering the yield–to–maturity rate (the IRR) as *the* value for **r** for each future period to the maturity date of the bond is that it contradicts financial sense. We already know that as the receipt of cash gets pushed farther and farther into the future the spot rate for money usually rises, reflecting added risk. A single **r** value (IRR) does not show these differences, nor is it supposed to.. It is much more accurate to depict the PV of the bonds in question by the following amounts:

$$\textbf{PV} = \frac{PMT}{(1+r_1)^1} + \frac{PMT}{(1+r_2)^2} + \frac{PMT}{(1+r_3)^3} + \frac{PMT}{(1+r_4)^4} + \frac{PMT}{(1+r_5)^5} + \frac{PMT}{(1+r_6)^6}$$

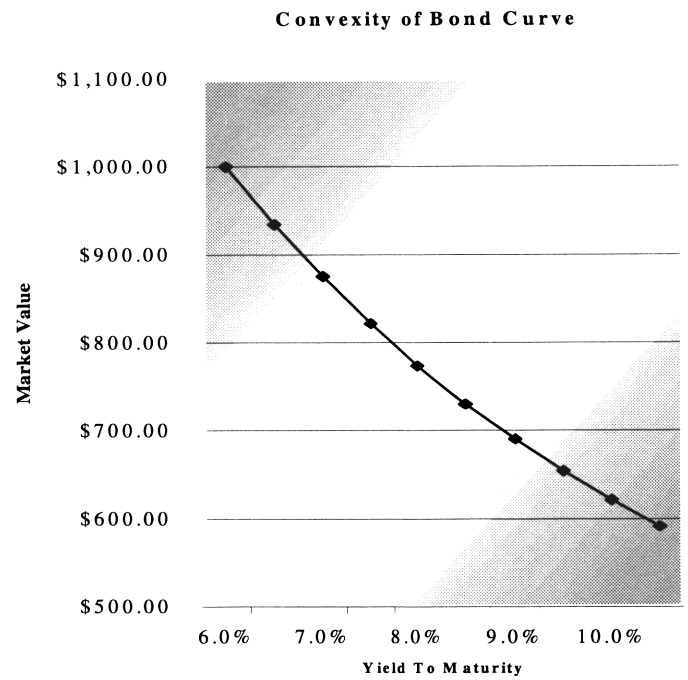


where the values $\mathbf{r_1}$ to $\mathbf{r_6}$ are the different spot rates for money, generally increasing to reflect the additional risk for time to receipt. These values are not set by computers, but rather by the **market.** They reflect investors' collective judgments regarding the risk of inflation, and the risk that the receipt of the full principal value may never be realized.[19] The phrase "term structure of rates" refers to the relationship between interest rates and the time to maturity.

19 In the case of "junk" bonds of low rating, the value **r** is quite high to reward the buyer for the inherent risk of default. High **r** values depress current PV.

It is important to recognize that there are many different yield curves, depending upon current market conditions and the class of security represented by the curve. Most financial media publish the yield curve for U.S. Treasury securities of various maturities, such as the example above.

While **not risk-free**, standard U.S. Treasuries represent the lowest risk, and therefore act as a base from which other rates, adjusted for risk, are frequently calculated.



Bond Duration

We already know that the (Present) Value of any investment is the sum total of all future financial benefits, each discounted at a rate commensurate with the perceived risk. We know, too, that as the receipt of a cash benefit is pushed farther and farther into the future, the present value of that benefit diminishes. Bonds are no exception.

Therefore the risk of recovering the full reversionary value of the bond increases with the time to maturity. But in the interim, a percent of the purchase price may be recovered as a function of the coupon rate of the bond. A bond with a higher coupon rate – say 10%, or $100 per year - will return a higher percentage of the bond's current market value over a given number of years when compared to a bond of similar maturity but with a lower coupon rate, say 7% per year.

The calculation of **Bond Duration** brings all these factors together in one number, allowing us to have a measurement of a bond's price sensitivity to changes in market interest rates.

Derivation of Macaulay's Duration Factor

We can represent the present value, or current market price, of the bond as:

$$\textbf{Value} = \sum_{t=1}^{n} \frac{CF_t}{(1+i)^t}$$

where **CF** = coupon payment per period **t**
i = current market yield rate per period of time (annual rate/2)
t = time (expressed in 6-month periods)
n = number of time periods to maturity

If we seek to measure the sensitivity of the Value (V) of the bond in response to changes in market yield rates (**i**), we need only take the first differentiation of **V** with respect to **i** :

$$\frac{dV}{di} = \sum_{t=1}^{n} \frac{-\mathbf{t} * CF_t}{(1+i)^{t+1}}$$

Note that Duration always carries a negative sign because of the sign of **t**.

When the value of **i** is small, as it will be when changes in market yield rates are small, the expression $\frac{1}{(1+i)}$ *substantially* equates to 1. To simplify the matter then, let's move $\frac{1}{(1+i)^t}$ outside the summation portion of the formula, leaving:

$$\frac{dV}{di} = \frac{1}{(1+i)} \sum_{t=1}^{n} \frac{-t * CF_t}{(1+i)^t}$$

Then, since it substantially equates to 1, we can ignore it altogether:

$$\frac{dV}{di} = \mathbf{1} * \sum_{t=1}^{n} \frac{-t * CF_t}{(1+i)^t} = \sum_{t=1}^{n} \frac{-t * CF_t}{(1+i)^t}$$

But the expression $\frac{CF_t}{(1+i)^t}$ represents the Present Value of the particular cashflow CF_t.

Therefore we can restate the equation as approximately $\frac{dV}{di} = \sum_{t=1}^{n} -t * (PV)CF_t$

In 1938, Frederick R. Macaulay defined Duration as the *total weighted average time for recovery of the payments and principal* *in relation to the current market price of the bond.* Bond Duration, therefore, is

$$\textbf{Duration} = \sum_{t=1}^{n} \frac{-t * (PV)CF_t}{\textbf{Market Price}}$$

where **Market Price** = $\sum_{t=1}^{n} \frac{CF_t}{(1+i)^t}$

and **(PV)CF**$_t$ = the Present Value of cashflow **t.**

How to Calculate Bond Duration

The calculation of a bond's Duration was a time–consuming task in Macaulay's day. Today the computer makes the measurement of bond value as a result in a change in market yield – or any other variable – a relatively minor chore. Yet, bond Duration is still a valuable tool in the hands of the bond trader, especially in assembling a portfolio of bonds.

Following Frederick Macaulay's formula, Bond Duration for a 3-year bond, bearing a 6% coupon and a market yield of 10%, is calculated as:

Year[20]	Pmt #	Coupon $	PV Factor	$PV	PV/Price	Duration[21]
-0.50	1.00	$30.00	0.952381	$28.57	0.0318	-0.0159
-1.00	2.00	$30.00	0.907029	$27.21	0.0303	-0.0303
-1.50	3.00	$30.00	0.863838	$25.92	0.0288	-0.0433
-2.00	4.00	$30.00	0.822702	$24.68	0.0275	-0.0549
-2.50	5.00	$30.00	0.783526	$23.51	0.0262	-0.0654
-3.00	6.00	$1030.00	0.746215	$768.60	0.8554	-2.5663
			Market Value =	**$898.49** [22]	1.0000	**-2.7761**

Therefore this bond, with a current value of $898.48, has a Duration of **-2.7761.**
The steps in calculating the Duration as it appears above are:

1. Determine the coupon rate. The coupon rate/2 * $1000 = PMT (Coupon $).
2. Determine the PV factor using the yield *per period*: $1/(1+ i)^t$ where **t** is the PMT # and **i** is the annual interest rate/2
3. Multiply the **PV Factor** * **Coupon$** to get the $**PV** of the Coupon payment.
4. Add the $**PVs** of all the cashflows to determine **Market Value** of the bond.
5. Divide each result of step #3 (**$PV**) by the current market value of the bond.
6. Multiply this factor by the **years** in column 1.

The **sum** of all final values in the right-hand column is the **Duration.**
Remember that Duration always carries a negative sign.

Determinants of Duration

As we can see from the equations above, coupon rate (which determines the size of the periodic cashflow), yield (which determines present value of the periodic cashflow), and time-to-maturity (which weights each cashflow) all contribute to the Duration factor.

Holding coupon rate and maturity constant –
Increases in market yield rates cause a decrease in the present value factors of each cashflow. Since Duration is a product of the present value of each cashflow and time, higher yield rates also lower Duration. Therefore Duration varies inversely with yield rates.

Holding yield rate and maturity constant –
Increases in coupon rates raise the present value of each periodic cashflow and therefore the market price. This higher market price lowers Duration. Therefore Duration varies inversely to coupon rate.

---

20 The time in years is negative to conform to Macaulay's formula.

21 Bond Duration is the product of **PV/Price** * the value under column **Year**. This is the reason that Duration is expressed in terms of years, but this is obviously not the capital pay-back period.

22 The Market Price is the summation of all the separate PVs in the cashflow.

Holding yield rate and coupon rate constant –
Increases in maturity increases Duration and cause the bond to be more sensitive to changes in market yields. Decreases in maturity decreases Duration and renders the bond less sensitive to changes in market yield. Therefore Duration varies directly with time-to-maturity (t).

Using Duration to Approximate Value Changes

The magnitude of the Duration is an index to the sensitivity of the bond to changes in market interest rates. Bonds with high Duration factors experience greater increases in value when rates decline, and greater losses in value when rates increase, compared to bonds with a lower Duration.

In order to more closely *approximate* the percent change in market value as the result of a percent change in yield, Macaulay derived **Modified Duration**, which is simply Duration times the factor which we removed from the formula for Duration above.

$$\textbf{Modified Duration } (\mathbf{D_M}) = \text{Duration} * \frac{\mathbf{1}}{\mathbf{(1+i)}}$$

In the example above, where Duration is –2.7761, the Modified Duration is:

$$\textbf{MDuration } (\mathbf{D_M}) = -2.7761 * \frac{1}{(1+\frac{0.10}{2})} = -2.6439$$

Note that the value of **i** (0.10) is the **annual** yield rate.

Macaulay used this Modified Duration, D_M, to *approximate* the percent change in bond value for a given percent change in yield, using the following formula:

Percent change in bond value = D_M * percent change in yield.

If yield rates rose from 10% to 10.5%, a 0.5% increase in rates, Macaulay's formula would predict a percent change in value as:

Percent change in bond value = D_M * percent change in yield.[23]
= – 2.6439 * (+ 0.5)
= – 1.3220%

The price change calculated by MDuration would be \$898.49 * – 1.322% = **–\$11.88** The new bond price would be approximately \$898.49 – \$11.88 = \$886.61. We can confirm the percent change and new price by entering these data into a spreadsheet: The change takes place in the **PV Factor** as a result of the change in market yield.

[23] Since Modified Duration is a negative value, a decrease in yield rate results in an increase in bond value. Multiplying the negative Duration times a decrease in yield results in an increase in bond value.

Year	Pmt #	Coupon	PV Factor	$PV	$PV÷ Price	Duration
-0.50	1	$30.00	0.95012	$28.5036	0.03215	-0.01608
-1.00	2	$30.00	0.90273	$27.0818	0.03054	0.03054
-1.50	3	$30.00	0.85770	$25.7309	0.02902	-0.04353
-2.00	4	$30.00	0.81491	$24.4474	0.02757	-0.05514
-2.50	5	$30.00	0.77426	$23.2279	0.02620	-0.06550
-3.00	6	$1030.00	0.73564	$757.7128	0.85453	-2.5636
			Price	**$886.70**	1.00000	**-2.77438**

As you can see, the computer indicates a decline in value from $898.49 to $886.70, a loss of $11.79 vs. $11.88 as predicted by Macaulay's approximation.

This difference in the answer we have obtained is caused by the **convexity** of the bond value curve. Macaulay's formula describes a straight line, but bond value in response to yield changes describes a convex curve. When yield changes are small (as in this example), the difference in value change is negligible, but when these differences are substantial (larger percent changes in market yield and higher Duration) then the differences in value increase.

If the Duration of our example bond were in the order -8 or -12, an increase of 1.0 % in interest rates would indicate a loss of approximately 8% ($71.88) and 12% ($107.82), respectively in bond price. But because of these large changes in yield, and the high Duration, the linearity of the Duration curve would result in larger pricing errors. Therefore the use of Duration to estimate change results in a reasonable *approximation*, especially when the changes in interest rates are not too large.

Significance of Duration

In the pre-computer days of Macaulay, Duration was conceived as a short-hand method of estimating price volatility as the result of changes in yield. Today, the value of Duration is somewhat less evident, since computer price programs are widely available which can indicate precisely the value of a bond with respect to all the important financial variables: coupon, yield and time. Still, Duration can be used by the bond investor to implement his investment strategy.

If the investor believes that market yields are going to decline, he may wish to alter his bond mix to include bonds carrying higher Durations in order to leverage the increase in bond value. If an increase in yields is expected, he may elect to change the mix to include bonds of lower Duration to minimize the negative effect on his portfolio..

Obviously bonds are subject to risk beyond changes in the coupon-yield-maturity variables, e.g. the risk of default, but Duration was not intended to reflect this kind of risk.

Duration and the Bond Portfolio

Perhaps the most prevalent use of Duration today is as a short-hand method of estimating the potential changes in the value of a portfolio of bonds.

Assume that a portfolio consists of (for simplicity) three bonds carrying the following current prices and Modified Durations:

Bond	Current Price	Mod. Duration
A	$845.57	4.12257
B	$625.95	7.3523
C	$884.17	4.04855

On a given day the market yield increases 20 basis points (+ 0.2%). What effect will this have on the value of this portfolio? Fortunately, the HP–12C has a set of statistical registers which will calculate a weighted mean. Here are the keystrokes (set decimal to [*f*] 5):

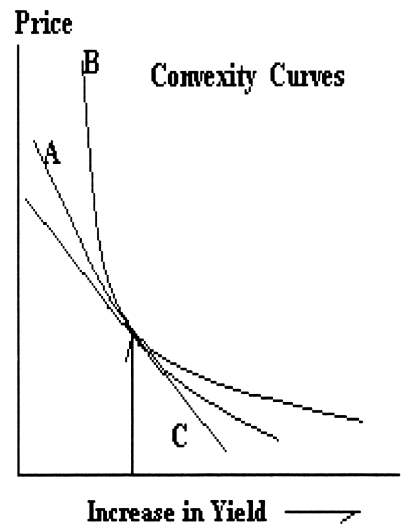
Key In	Display Shows	Comments
-4.12257 [Enter]	**-4.12257**	**Enters Mod. Duration**
0.2 [%]	**–0.00825**	**Mod. Duration x % change**
845.57 [Σ+] (4,9)	**1.00000**	**Puts price into statistical register**
–7.3523 [Enter]	**–7.3523**	**(same as above)**
0.2 [%]	**–0.01470**	
$625.95 [Σ+]	**2.00000**	
–4.04855 [Enter]	**–4.04855**	
0.2 [%]	**–0.00810**	
$884.17 [Σ+]	**3.00000**	

Now, by recalling R_2 (3,8), you will retrieve the total of all the original bond prices:

[RCL] 2	**2,355.69**	**Total of original prices.**
[RCL] 6	**–23.3354**	**The loss in value. (This is a loss since the rate increased).**
[+]	**2,332.3546**	**Adds the loss to show the new value of the portfolio.**
[f] 2	**$2,332.35**	**Re-sets price to 2 decimal places.**

Bond Convexity

The diagram at the right contains curves which depict the response of a bond's price to increasing market yield rates. (These curves are exaggerated to help illustrate the point.)



Curves **A** and **B** depict bonds with increased convexity. The "curve" labeled **C** represents Macaulay's formula, which suggests that the relationship between Price and Yield follows a nearly straight line.

If we were to rely on Macaulay's Modified Duration factor alone, we would calculate the change in bond price by multiplying the MDuration factor by *any* change in market yield, however small or large. The difficulty here is that the MDuration factor changes for each change in yield rates. Therefore if we were to stay with one MDuration factor when yield changes are more than minimal, we would incur significant errors in estimating a bond's value.

The idea behind the **Convexity** factor is to adjust Duration for the changing slope of the bond curve. Notice that as yield increases the *rate of loss* (slope of the curve) in bond value diminishes. While all three curves indicate lower prices as yield rates increase, the curve for Bond A shows that this bond is losing value at a rate lower than C, and bond B is losing value at a rate lower than A. The rate of loss is less with bonds of greater convexity and more with bonds of lesser convexity.

Notice also that Bond B will have consistently higher values, whether yields increase or decrease, than Bond A, and either bond A or B will have an actual value higher than the "curve" C would indicate. For this reason investors typically assign a higher market value (among bonds of similar maturity) to those bonds with higher convexity. This is especially so in anticipation of a strong upward move in interest rates and when dealing with bonds of longer maturity. In these cases, bonds of higher convexity lessen the amount of the loss. In the event of a decrease in market interest rates, the bond with greater convexity would acquire greater value.

The degree of a bond's convexity is determined by the interplay between the time to maturity and the coupon rate. But of much greater significance in estimating convexity as it influences value is the degree of change in interest rates.

Adjusting Duration Formula for Convexity

When significant changes in market yield are anticipated what we really want to know is the rate of change in the relationship between price and yield.

You will recall that the first differentiation of the bond valuation formula was:

$$\frac{dV}{di} = \sum_{t=1}^{n} \frac{-t * CF_t}{(1+i)^{t+1}}$$

The second differentiation of this formula will give us the *rate of change* we are looking for:

$$\frac{d^2P}{d\,i^2} = \sum_{t=1}^{n} \frac{t(t+1) * CF_t}{(1+i)^{t+2}}$$

Convexity is defined as $\frac{d^2P}{d\,i^2} * \frac{1}{P}$ where P is the Price of the bond, or $\frac{t(t+1) * CF_t}{(1+i)^{t+2} * P}$

The table on the opposite page configures the convexity for an 8% coupon rate, 5-year bond currently selling at par ($1,000) The result is the **Sum** (from the table below) divided by the **Price** ($1,000), or 80.7543

Duration is given in terms of **n** periods per year/**n**, but convexity is given in terms of **n** periods per year/ $\mathbf{n^2}$**.** Since n is two periods per year, the factor 80.7543 must be divided by 4, to deliver 20.1886. This is the Convexity factor in years.

The MDuration for this bond selling at par is –4.055448.
The following formula adjusts the current bond price for both MDuration and Convexity:

% Change in Bond Price = MDuration * % change in **i** +(0.5 * (Convexity) * (% change in **i**)^2.)
= –4.055448 * 0.02 + (0.5* 20.1886 * (0.02) $^{2)}$
= – 0.077071 = –7.7071%

Using this percentage decline, –7.7071%, the value of the bond will decline from $1,000 to $922.93. The computer generated value is $922.78 Had we used MDuration alone, the indicated value would be $918.89.
Notice that when market interest rates decline, (MDuration * **i**) will be positive, and so also will be the Convexity adjustment, since $\mathbf{i}^2$ will always be positive.

Convexity of an 8% Coupon, 5-Year Bond Selling At Par

Period	**CF**	**(1+i)^(t+2)**	**t(t+1)CF**	$\frac{\mathbf{t(t+1)CF}}{\mathbf{(1+i)^{(t+2)}}}$
1	$40.00	1.124864	80	71.119709
2	$40.00	1.169859	240	205.15301
3	$40.00	1.216653	480	394.52501
4	$40.00	1.265319	800	632.25162
5	$40.00	1.315932	1200	911.90138
6	$40.00	1.368569	1680	1227.5595
7	$40.00	1.423312	2240	1573.7943
8	$40.00	1.480244	2880	1945.6248
9	$40.00	1.539454	3600	2338.4914
10	$1040.00	1.601032	114400	71453.902
			Sum	80754.323

Zero Coupon Bonds

"Stripped Bonds" refers to bonds which are "stripped" of either 1) their coupons (dividend payments) or 2) their reversionary value. These strips can be bought to furnish a series of dividend payments but no reversionary value, or to deliver one lump sum at the time of maturity but no intervening payments.

The most popular of the "strips' are standard U.S. Treasury bonds.[24] Bonds are selected by security dealers, transferred to the Federal Reserve Bank in New York which creates the derivative instruments and returns the instruments to the bond dealer by Fedwire.[25] One can buy both the stripped interest coupons and/or the stripped principal. The Wall Street Journal denotes the former as **ci,** while the stripped principals are earmarked **bp.**

As *short*-term investments, Zero Coupon bonds (**bp**s) are very volatile derivative instruments popular with those who prefer to bet on changes in long-term interest rates.

For example, assume that current long-term (30-year) rates are at 9.5%. The Present Value (Ask) of a stripped-coupon Treasury maturing in 29 years is quoted at 7:22, meaning 7.6875, or $76.875 (per $1000 of reversionary value).

A trader foresees long–term interest rates declining to the 7.5% range within three years. Therefore the receipt of $1,000 in 26 years (29 – 3), discounted semi-annually at 7.5% will be:

n	i	PV	PMT	FV
26*2	7.5÷2	?	0	1,000
	Solving...	**–$147.44**		

This will represent an semi-annual **yield** in three years of:

n	i	PV	PMT	FV
3*2	?	-76.875	0	147.44
Solving..	**11.46**			

The annual yield will be 11.46 * 2 = 22.92%

A yield such as this is very tempting. But the door swings to and fro. Suppose that yield rates *increase* 2% in three years. The results demonstrate the reward and the risk of the high leverage implicit in Zeros:

---

24 Corporate bonds are also available as "strips."

25 The term STRIPS is an acronym for Separate Trading of Registered Interest and Principal of Securities, a Treasury program which allows bonds of maturities equal to 10 years or more to be transferred over Fedwire. This action has greatly reduced the cost of insurance customarily associated with transferring these derivatives.

n	i	PV	PMT	FV
52	**11.5** ÷ 2	?	0	1,000
	Solving..	**54.63**		

The yield now becomes:

n	i	PV	PMT	FV
3 *2	?	-76.875	0	54.63
Solving..	**-5.53**	**54.63**		

The annual yield will be -5.53* 2 = –11.06%

Duration Of Zero Coupon Bonds

This volatility of Zero Coupon bonds (stripped principal) can also be seen in their Duration. Since there are no coupon payments, the entire Duration is a function of the last payment, which is the principal. In the case of a 30-year Zero Coupon bond, discounted over 30 years (60 periods) @10% (5% per period), the Duration is:

Year	Pmt #	Coupon	PV Factor	$PV	$PV ÷ Price	Duration
-0.5	1	0	0.95238	0	0	0
-1.0	2	0	0.90703	0	0	0s
-30	60	1,000.00	0.05354	$53.53	1.00	-30.00
			Total $53.53			-30.00[26]

For all practical purposes then, the Duration of a Zero Coupon Bond, is approximately equal to its maturity (in years). *The longer the maturity, the higher the Duration,* and the more sensitive the instrument is to even small changes in market yield. A high Duration factor indicates that long-term Zero Coupon bonds are *very* sensitive to changes in market rates. An absolute change of only 0.5% in interest rates would result in approximately a 15% variation in value for a 30-year bond.

26 Modified Duration would be 28.57

Taxation of Zero Coupon Bonds

Although zero coupon bonds do not pay interest, they do result in *annual* taxable income because a zero coupon bond is treated as an Original Issue Discount bond (OID). The amount of the OID is the difference between the maturity value of the bond and the price at which the bond was acquired. The tax is levied on the annual increase in the value of the bond as it approaches maturity.

The amount which is currently recognized as taxable is determined by a ratio: the ratio is determined by dividing the number of days for which the bond has been held (in the tax year) by the number of days to maturity. This factor is applied to the excess of the maturity value of the bond over the acquisition cost. The result is that portion of the increased value of the bond which is currently subject to tax at the ordinary rate. (There is no long-term capital gains treatment available for these amounts. Gains are taxed as ordinary income)

For example, a bond acquired on July 1 for $500 will mature in exactly 10 years. The maturity value is $1,000. The ratio is 183 days / 3650 days, or 0.05014, or 5.014%. The amount recognized is ($1,000 - $500) x 5.0104% = $25.05 This amount is reported as interest earned..

Because these instruments produce no income during the holding period, and because they do result in taxable income, zero coupon bonds are frequently held in tax-deferred accounts.

Zeros As an Investment & Planning Tool

If U.S. Treasury Zero Coupon bonds are acquired for <u>long-term purposes</u> <u>without the need for intermediate liquidity</u>, they can be important <u>low-risk</u> instruments with which to fund future cashflow requirements. For example:

> An assessment of future income requirements for an education fund indicates that $250,000 will be required in 18 years. Zero Treasuries which will mature in 18 years are currently priced at 34:21, or $346.5625 at a time when long term interest rates are 6.06%. Two hundred fifty of these strips are acquired today at a cost of $86,640.
>
> What is the risk of loss of principal in year 18 if interest rates rise to 8.06% at the time of redemption ?

The answer is <u>zero risk</u>, since the bonds will be redeemed at $1,000 per bond. The risk inherent in this scenario is the calculation of the amount which will be required in year 18. This calculation must anticipate a reasonable inflation rate, since tuition fees in 18 years will not be at today's prices. But if held to maturity, there is virtually no risk in receiving th*e maturity value* of a U.S. Treasury bond. The *purchasing power* of the recovered principal, however, is another matter.

Inflation -Adjusted U.S. Treasury Securities

U.S. Treasury securities are regarded as the safest investments in the world. They are not, however, entirely risk-free. The risk associated with longer-term securities, especially, is the loss of purchasing power due to inflation. The reversionary value of a 30-year Treasury bond held to maturity in a 3.5% inflationary environment is only $353.13 in constant dollars..

In response to public requests for a Treasury instrument which will adjust with inflation, the federal government began issuing in January 1997 U.S. Treasury Inflation-Indexed Securities.[27] The bonds are similar to those issued in other countries, and quite similar to Canada's Real Return Bonds. U.S. Inflation-Indexed securities now range in maturity from one year Notes to 30-year Bonds.

On the dividend date the principal amount of the bond is adjusted for inflation using the non-seasonally adjusted *U.S. City Average All Items Consumer Price Index for All Urban Consumers* (CPI-U), which is published monthly by the Bureau of Labor Statistics. The CP-U Index used for the first day of each calendar month is the CP-U Index for the first day of the third preceding month. For example, the CP-U Index for April 1 would be the CP-U Index for the preceding January 1, which is published on February 1.

The original coupon *rate* for the bond does not change over the life of the instrument. Instead, the maturity amount of the bond is adjusted and the dividend paid each six months is determined using the adjusted principal amount. The CP-U for the current day is divided by the CP-U for the date of issue, and the result is multiplied by the reversionary value. Therefore in periods of disinflation the value of the bond will be reduced, as will the applicable dividend amount. The reversionary value of these bonds, however, will never be reduced below par. If the market value of the bond at maturity is less than the issue price, the government will bring the reversionary value to par.

Interest received on both paid dividends and the increased value of the reversion amount is taxable in the year received or credited. This tax treatment is similar to OID bonds. For this reason Inflation-Indexed bonds are well-adapted to tax-deferred accounts.

Inflation -adjusted bonds are eligible for the governments STRIPS program. Therefore stripped principal and stripped interest components for bonds of 10-year maturity or greater may become available.

Inflation-Indexed Bonds Indicate True Safe Rate

These bonds serve another important purpose. Since the effect of inflation is removed, these bonds are a valuable indicator of the current "safe rate" for money in the United States. Since inflation is not a factor, movement in the current price and current yield for these bonds is a direct result of basic market supply and demand forces. The "safe rate" of interest is the foundation rate used to construct capitalization and discount rates.

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[27] Sometimes referred to as TIPS: Treasury Inflation-Protected Securities.

A list of currently traded Inflation-Indexed U.S. Treasury bonds can be found in most financial newspapers, usually at the end of government bond section.

Chapter Summary

1. Bonds, as do all other investments, derive their Present Value from the discounted value of the cashflows realized over the holding period.
2. Most bonds are interest-bearing instruments whose price varies inversely to the current market rate interest (yield) for securities of equal risk and maturity.
3. The Yield-To-Maturity rate of a bond is its IRR, which tends to conceal the time-maturity rate of return inherent in the pricing of bonds.
4. The price of bonds is set by the spot rates for money. These rates typically increase with time to maturity resulting in an upwardly sloping yield curve.
5. Duration is a means of approximating the change in bond value as a result in a change in market interest rates.
6. Convexity adjustments overcome some of the error in using Duration and Modified Duration alone to calculate bond value.
7. Zero Coupon bonds are derivative components of bonds. They are highly sensitive to changes in market interest rates.
8. Bonds with "call" dates are valued to the call dates when the coupon rate exceeds the market rate, and to the maturity date when the coupon rate is below the market.
9. The YTM of a bond can be calculated by adding to the remaining interest payments an amount necessary to amortize the call premium.
10. Bond Duration is a sensitivity tool which can be used to approximate the change in bond price as the result of a change in market yield rates. It is particularly useful in estimating the impact of yield rate changes on a portfolio of bonds.
11. The Duration of a Zero Coupon Bond is approximately equal to its maturity, measured in years.
12. Zero Coupon Bonds are classified as Original Issue Discount Bonds. The accrued value of the bond is taxable to the holder even though no interest is received.
13. Inflation-Indexed U.S. Treasury securities protect the investor from the loss of bond value due to inflation, although coupon rates are correspondingly low.

Chapter 7 Risk & Reward

Risk is an element in every investment situation. Many corporations have now become sufficiently concerned about risk to appoint CROs – Corporate Risk Officers – whose job it is to identify, analyze and quantify risk..

While individual investors and financial planners may not be concerned about risk at the corporate level, they share similar concerns and objectives: the identification, analysis and quantification of risk.

We would be remiss, having discussed the valuation of both stocks and bonds in the previous chapters, not to consider the element of *risk* as it applies to these investments – indeed, to all investments. Quantifying risk will enable us to make appropriate adjustments to discount rates, to compare risk among different investments, and to choose investments which fit our risk comfort-zone. It is also important that we be rewarded for the investment risk we take.

In this chapter we will cover the very basic fundamentals concerning the quantification of risk. The goal is to help provide an insight to the degree of investment risk and how the investor may measure and mitigate the risk. The reader is also encouraged to explore further the quantification of risk by referring to more advanced texts dealing with statistics, probabilities and simulations.

Components of Discount Rates

So far we have arrived at the Present Value of various investments by discounting anticipated future cashflows by a selected discount rate. Discount rates, which are equivalent to desired return rates, are composed of:

1) a safe rate for money,
2) an allowance for inflation, and
3) **a premium for *risk*.**

For the most part, the safe rate for money and an allowance for inflation are matters of fact which can be objectively retrieved from existing data sources.

But assigning a risk-premium is a strictly a matter of judgment, or, if you will, a matter of assessing the risk/reward ratio inherent in every investment. The investor who perceives the risk of a certain investment to be low will add a low risk premium and use a lower discount rate to value the cashflows from the investment. The investor who perceives the risk to be greater will add a larger risk premium resulting in a larger discount rate. The first investor will pay more for the same investment cashflows than the second investor. It is this difference in perceived ***risk*** among different investors which creates differences in opinions of value, and therefore creates markets.

Let's first examine the *safe rate and the inflation rate.*

True Safe Rate and Inflation Rate

You will recall that in Chapter 6 (Bonds) we discussed the U.S. Inflation-Adjusted bonds which were introduced in 1997. These bonds are not only backed by the "full faith and credit" of the U.S. government, but also by the government's promise to eliminate any loss of future purchasing power due to inflation by adjusting the reversion value of the bond which, in turn, determines the periodic dividends. Therefore the "safest" investment in the United States is an investment in U.S. Treasury Inflation-Adjusted bonds, and we can truly refer to these investments as "risk-free." As such, their current market yield is also the current *true safe rate* for money in the United States.

Most financial writers refer to the standard U.S. Treasury obligations as the "safest" of all investments, and use the rates applicable to these securities as the "risk-free" rate. But the current yield on traditional U.S. Treasury securities always contains an allowance for inflation. If one chooses to use the current yield from standard Treasury bonds, then one must accept the judgment of the market as to the "built-in" inflation rate. Alternatively, one can add to the current safe rate for money, as determined by the yield on *Inflation-Adjusted* Treasury securities, one's own estimate of future inflation.

In either case, it is to this sub-total (safe rate + inflation rate) that a risk premium must be added. The total of these three rates will constitute the discount rate applicable to a particular investment.

Risk

There are a number of ways in which to define risk. One definition holds that risk is the probability of an *adverse* outcome multiplied by the severity of the outcome. Because most investors think of risk only in terms of an *adverse* outcome, risk becomes identified only with *loss*. But there is also the 'risk' of a positive outcome: the likelihood of making a gain. Both the risk of a loss, as well as the likelihood of a gain, are linked to the ***probabilities*** of all possible future outcomes.

What is Past *May* Be Prologue

If we want to make some predictions about how an investment - say, a stock- is likely to perform in the future, we can examine its past performance and extrapolate the results to the future. When we do this, we make an implicit assumption that the future will be somewhat like the past. But in this text we have consistently stressed that the value of any investment is the discounted value of all *future* returns which are likely to accrue to the investor over the expected holding period. We have also stressed that (to coin a phrase) "past performance is no reliable index of future performance."

A prospective approach to investing requires an estimate of *probable* future performance. Some investors, curiously, feel uncomfortable about any attempt to forecast future performance and prefer to rely on past performance. In contrast, professional investors and fund managers chiefly concern themselves with the *probabilities* of future performance and concentrate attention on the fundamental economic and market factors likely to influence and determine that performance.

Statistics and Probabilities

Estimates of future returns can be based either on the *statistics* of past returns or on the *probabilities* of future returns.

Statistics is that area of science concerned with the extraction of information from numerical data and its use in making inferences about the population[1] from which the data are obtained. In the next few pages we will be concerned with some rudimentary statistical and probability concepts and how they may be applied to identify and quantify future risk. After that, we will examine ways in which investment risk may be measured and mitigated.

In either approach, the objective is a forecast of future performance. Although there is a different method for each, and very likely a different outcome for each, the objective is the same: an **Expected Outcome, or Expected Rate of Return**.

Looking Backward

As an example, let's take the stock of PRQ corporation which has delivered the following annual rates of return over the past 5 years: 19%, 21%, 24%, 28% and 32%. If these data constitute the entire return history, we can determine the average (mean) rate of return, μ (mu), simply by averaging these past data.. In this case the average is:

$$\mu = \frac{\sum_{t=1}^{n} x_t}{n} = \mathbf{24.8} \text{ (see footnote }^{2}\text{)}$$

---

[1] The term ***population*** refers to all the data available to the sampler. A ***sample*** is a sub-set of the population.

[2] If the data collected represent the entire population, the average, $\boldsymbol{\mu}$, is determined by dividing the total by n; if the data collected represent a sample, the average, $\overline{x}$ (x bar), is determined by dividing by n–1.

where μ is the average return, x_t is the annual return for year **t** and **n** is the number of years, in this case 5. If we limit ourselves to these past 5 years, we can determine the degree to which the annual return rates for PRQ have varied, and – to the extent they vary – an estimate of how much future returns might deviate from this average.

Deviation, Variance and Standard Deviation of Historical Data

If we assume that the rates of return we have for PRQ are the entire population of data, we can proceed to determine the Variance and Standard Deviation for these past returns.

The mean rate of return of the entire population of data, is simply the average of the values given, or:

$$\mu = \frac{19\% + 21\% + 24\% + 28\% + 32\%}{5} = 24.8\%$$

The formula for the Variance of a population, σ^2, is defined as the sum of the squared deviation of the measurements, x_n, about their mean, μ, divided by n.

$$\sigma^2 = \frac{1}{n} * \sum_{t=1}^{n} (x_n - \mu)^2$$

% Rate of Return (x)	Mean μ	$(x - \mu)$	$(x - \mu)^2$	$(x - \mu)^2/n$
19	24.8	-5.80	33.64	6.728
21	24.8	-3.80	14.44	2.888
24	24.8	-0.80	0.64	0.128
28	24.8	3.20	10.24	2.048
32	24.8	7.20	51.84	10.368
			Variance = σ^2	22.160
			Standard Deviation = σ	4.707

If the values we have are but a sample of the entire population then the formula for the variance, s^2, becomes

$$s^2 = \frac{1}{n-1} * \sum_{t=1}^{n} (x_n - \overline{x})^2$$

Notice that the average of the sample is called $\overline{x}$ (x bar), and the numerator is divided by (n-1) instead of n. The standard deviation is the square root of the variance, or **s.**

[When dealing with an entire *population* the Variance is represented by the Greek (small) *sigma*, σ^2, while the Standard Deviation, being the square root of the Variance, is simply σ.
When dealing with a *sample* of the entire population, the Variance is represented as s^2, while the Standard Deviation is **s.**]

The meaning of the Standard Deviation is discussed after the following segment.

Standard Deviation on the Calculator

Determining Standard Deviation on the HP-12C is a relatively simple exercise. This calculator has a system of statistical functions all of which are accessed using the blue key, [g] .

Since we have only one variable in this simple situation, we can enter the rates of return as follows:

	Key In	Display Shows	Comment
	[f] [∑] (3,2)	0.00	Clears statistical registers
	19 [∑+] (4,9)	1.00	Enters first value
	21 [∑+]	2.00	Enters second value
	24 [∑+]	3.00	Enters third value
	28 [∑+]	4.00	Enters the fourth value
	32 [∑+]	5.00	Enters the fifth (value
Now press	[g] [$\bar{x}$] (4,7)	24.80	The average (mean) of the values

Notice that the calculator arrived at this value by dividing the sum of the values (124) by 5, which is equal to **n**.

But in the next step, <u>in the calculation of the standard deviation</u>, the calculator subtracts one from the counter, changing n to (n-1), or 4.. Therefore the calculator *assumes* that the data are derived from a s*ample* and not from an entire population.

If you require the standard deviation of a sample, enter the next line –

[g] [s] (4,8)	5.263	The standard deviation of the sample

But if you desire the standard deviation of an entire population (as in our example), do not execute the preceding line. Instead, continue by entering,

[g] [∑+] (4,9)	6.00	Adds 1 to n, and 24.80 to the values
[g] [s] (4,7)	4.707	The standard deviation of the population.

By adding one more data point which is equivalent to the average of the preceding 5, we do not disturb the average but we do change n from 4 to 5.

We will explore the significance of the Standard Deviation after considering the second possible approach to an obtaining an *expected return.*

Looking Forward

In looking forward, we choose to rely not on past performance but rather on estimates of future performance based on the ***probability*** of each possible future outcome. Assessing the probability of future performance, whether that performance is based on one or multiple factors, is always a matter of informed judgment. The formula for this approach sums up the probabilities, $\hat{x}$, (x-hat) for each possible Expected Return:

$$\hat{x} = \sum_{t=1}^{n} x_t * P_t$$

where $\hat{x}$ is the **Expected Return**, x_t is the possible outcome of event **t**, and $\mathbf{P_t}$ is the probability of the outcome **t .** Therefore the expected return is the sum of the weighted probabilities of each potential outcome.

As an example, let's assume that PRQ's current Rate of Return is 32%. Its management has entered its best estimate of the Rate of Growth for next year under each of 5 different economic scenarios. It estimates that PRQ could encounter any one of these scenarios, but with different probabilities for the occurrence of each.

If Economic Conditions are	PRQ's Change in Annual Return	Potential Return Based on Current 32%
Robust	+20%	38.4%
Active	+15%	36.8%
Average	+10%	35.2%
Slow	-10%	28.8%
Recession	-20%	25.6%

This table tells us how PRQ's *might* respond to the various economic scenarios, but it says nothing of the ***probability*** that each of these economic scenarios will occur. Therefore we need to enter our best informed estimate of the probability of each of these possible outcomes so that we can weight each potential outcome. The weights used are the probabilities assigned to each possible outcome. The **sum** of the weighted possible outcomes will be the **Expected Return**:

If Economic Conditions Are..	Rate of Return Might Be...	Probability of this is...	Weighted Outcome
Robust	38.4%	5%	1.92%
Active	36.8%	20%	7.36%
Average	35.2%	55%	19.36%
Slow	28.8%	15%	4.32%
Recession	25.6%	5%	1.28%
		Expected Return = Sum = $\hat{x}$ **=**	34.2% rounded

As you can see, there is a measurable difference between the Average Rate of Return as determined by the statistics of past performance, 24.8%, and the Expected Rate for next year, 34.2%, as determined by weighting future probabilities.

Variance and Standard Deviation of Expected Returns

Now that we know the Expected Rate we can revisit our possible outcomes to measure how far each outcome would deviate from the Expected Rate, and given that deviation, the Standard Deviation:

Economic Conditions	**Possible Rate** x_t	**Expected Rate** $\hat{x}$	**Difference** $(x_t - \hat{x})$
Robust	38.4%	34.2%	4.2%
Active	36.8%	34.2%	2.6%
Average	35.2%	34.2%	1.0%
Slow	28.8%	34.2%	–5.4%
Recession	25.6%	34.2%	–8.6%

The result is multiplied by the probability assigned to each outcome, and the results totaled to deliver the **Variance**. Again, the square root of the Variance yields the **Standard Deviation.**

Difference	**Difference Squared**[3]	**Probability**	**Product**
4.2%	17.64	5%	0.882
2.6%	6.76	20%	1.352
1.0%	1.0	55%	0.55
–5.4%	29.16	15%	4.374
–8.6%	73.96	5%	3.698
		Variance $= \sigma^2 =$	10.856%
	Standard Deviation $= \sigma =$		3.295%

Both the Variance and Standard Deviation are measurements of the degree to which the possible rates can vary from the Expected Rate.[4] In this case we could say that the expected rate of return is 34.2% ± 3.3%, the standard deviation. If a rate actually exceeds the Expected Rate, the investor has a pleasant outcome, but if the rate falls below the Expected Rate the investor has a less positive outcome – perhaps even a negative return, or loss.

In the case of PRQ, however, the chances of their having a negative rate of return are nil because the estimated worst case scenario, a Recession, still shows the firm would have a positive 25.6% rate of return (nice business). But not all businesses are so fortunate. Let's examine the returns

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3 Squaring the difference $(x_t - \hat{x})^2$ eliminates the problem of minus numbers.

4 If the data are historical data, we would be measuring the dispersion around the mean (average).

from two business with the same Expected Return but with greater potential variation (riskiness) in outcomes.

Company **CDF** is a manufacturer which enjoys low marginal[5] costs but competes in an industry very sensitive to economic conditions and therefore has returns which are quite volatile. **XYZ**, on the other hand, is a food distributor with normal marginal costs but competes in an industry where supply and demand vary little. Both firms have an Expected Return of 15%.

	CDF Manufacturing			XYZ Distributors		
Economy	**Probability**	**Possible Return**	**Weighted Return**	**Probability**	**Possible Return**	**Weighted Return**
Robust	25%	60%	15.00%	25%	20%	5.00%
Normal	50%	15%	7.50%	50%	15%	7.50%
Recession	25%	-30%	-7.50%	25%	10%	2.50%
		Expected Return	**15.00%**		**Expected Return**	**15.00%**

Despite the fact that the Expected Return for both stocks is the same, 15.0%, there is wide variation in the possible outcomes. The calculation of their Standard Deviations will show this:

CDF Corp.

Outcome	Expected Return	Deviation	Dev^2	x Prob. =	Product
x_t	$\hat{x}$	$(x_t - \hat{x})$	$(x_t - \hat{x})^2$		
60	15	45	2025	25%	506.25
15	15	0	0	50%	0
–30	15	–45	2025	25%	506.25
			Variance = σ^2 = Sum =		**1012.50**
			Standard Dev. = σ =		**31.82%**

XYZ Corp.

Outcome	Expected Return	Deviation	Dev^2	x Prob. =	Product
x_t	$\hat{x}$	$(x_t - \hat{x})$	$(x_t - \hat{x})^2$		
20	15	5	25	25.0%	6.25
15	15	0	0	50.0%	0
10	15	–5	25	25.0%	6.25
			Variance = σ^2 = Sum =		**12.5**
			Standard Dev. = σ =		**3.54%**

In the case of CDF Inc. we could say that the expected return is 15% ± 31.82%, or a range from + 46.82% to - 16.82%. CDF is a boom or bust stock with potentially high volatility.

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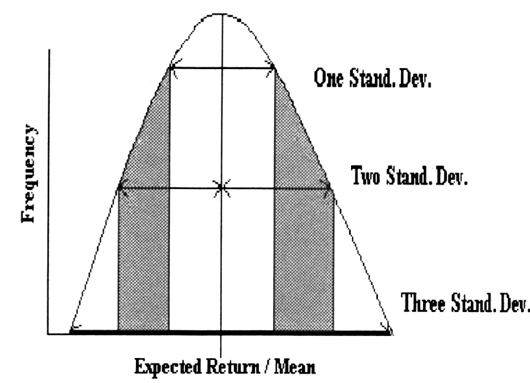
5 Costs, in excess of fixed costs, which vary according to units produced

XYZ, in contrast, could expect returns of 15% ± 3.54%, or a range of + 18.54% to +11.46%. It is clear which stock is potentially more volatile, and therefore more risky.

Meaning of Standard Deviation

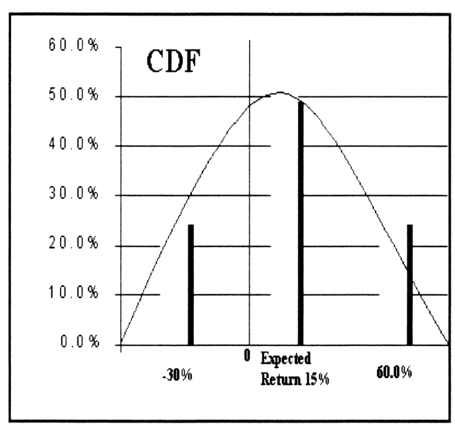
The significance of the *Standard Deviation* becomes much more meaningful after consideration of Tchebysheff's Theorem.



Tchebysheff, a Russian mathematician, demonstrated that when $\mathbf{x}$.......$\mathbf{x}_n$ is a series of measurements, and **k** is an integer equal to or greater than 1, then ***at least*** $(1-\frac{1}{k^2})$ percent of the data will fall within **k** Standard Deviations of the mean (or Expected Value). When **k** is equal to 1, the expression equates to zero, which doesn't help much; but when **k** = 2, then it predicts that ***at least*** $(1-\frac{1}{2^2})$, or 75% of the measurements will fall within 2 Standard Deviations of the mean (or Expected Return). When **k** is 3, then ***at least*** 89% of the data will fall within 3 Standard Deviations of the mean (or Expected Return).

The Empirical Rule

The histogram,[6] **CDF**, plots the frequency of the distribution points of the total return for CDF's stock against the probability of the return. Intervals, or bins, of the distribution points are plotted along the x-axis. The probability of the frequency of the distribution points per bin is plotted along the y-axis.



If we had chosen a continuous series of points between the pessimistic -30%and the optimistic +60%, their plots on this graph would follow the mound-shaped curve which is superimposed over the bar columns.[7] This curve resembles the "normal" bell curve, a pattern of distribution widely found in nature and in the collection of many types of "normal occurring" data.

Tychebysheff's Theorem applies to *any* distribution curve, but when the curve approaches the normal bell-shaped curve, the **Empirical Rule** applies. This rule

[6] A graph of a frequency distribution in which rectangles with bases on the horizontal (x) axis are given widths equal to the class interval (bin size) and heights on the vertical axis (y) equal to the frequency.

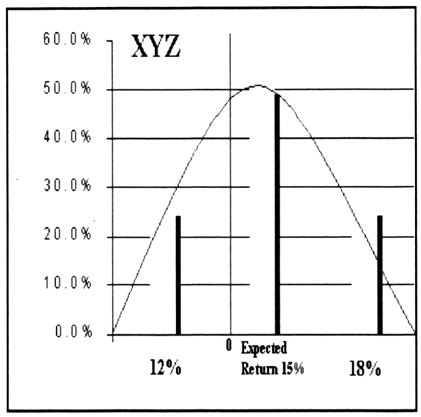
[7] Continuous distribution

states that **68.3%** of all the data will fall within ± 1 Standard Deviation of the mean, **95.5%** will fall within ± 2 Standard deviations, and **99.7%** will fall within ± 3 Standard Deviations.

In the case of CDF, one Standard Deviation would encompass values ranging from 15% ± 31.82%, or from -16.82% to +46.82% of the expected return.

The larger (wider) the Standard Deviation, the greater the potential variability; the greater the potential variability, the greater the risk.

In the case of **XYZ**, whose mound-shaped histogram appears nearby, the frequency of the distribution of the possible outcomes is much narrower. Its Standard Deviation is only 2.32%, indicating that 68.3% of the possible outcomes will fall between 15% ± 2.32% , or between +13.68% and +17.32% of the Expected Return. Ninety-five percent will fall within 2 Standard Deviations, or between +10.68% and +19.32%, while almost all possible outcomes (99.7%) occur between 8.04% and 21.96%.



Investors who are risk-averse most certainly will choose to invest in XYZ stock, since it requires more than 6 negative Standard Deviations in performance to produce a loss. In the case of CDF, less than one negative Standard Deviation will produce a negative return.

It is easy to see that the stock of CDF can be much more variable and carries much more risk than does XYZ. Therefore the market capitalization rate (discount rate) for CDF will carry a high risk premium component, and therefore indicate a lower price. XYZ's stock will carry a much lower risk premium and will therefore indicate a higher price. As we saw in Chapter 5, both these stocks will be sensitive to factors which contribute to growth in earnings. But the potential for extra-ordinary gain is with CDF, not with XYZ.

Semi-Variance and Semi- Standard Deviation

Since most investors define risk in terms of a potential for loss, one may ask..."why bother to measure standard deviation which includes positive variations. Doesn't it make more sense to focus only on negative variations and the potential for loss?"

The answer is that for the great majority of investments whose probability distribution is reasonably symmetric, standard deviation, taking into account both positive and negative deviations, is generally a good measure of risk - and one used by most analysts.

But some money managers and analysts choose instead to disregard positive deviations *above a certain targeted return rate.* Their purpose is to measure the potential for deviations below a targeted return. These deviations need not be losses, but they are defined as returns below targeted or desired yields.

Variances below the targeted return are known as **semi-variances**, and the standard deviation of these variances is the *semi*-standard deviation. These values can be calculated by eliminating from the calculations all deviations above the targeted yield, then calculating the mean, the variance and standard deviation of all the distribution points which remain. Although these results may be numerically positive, they are really to be interpreted as variances and deviations *below* the targeted return and therefore represent the likelihood not of a loss but of not achieving a pre-defined goal.

Risk Per Unit of Standard Deviation

Standard Deviation is a good index of risk when comparing stocks of equal, or nearly equal, returns, and when the data are reasonably symmetrical[8] around the Expected Returns. But a comparison is often sought between stocks which have quite different Expected Returns and different Standard Deviations.

For example, Stock A may have a Standard Deviation of 10% and an Expected Return of 30% while Stock B has a Standard Deviation of 5% and an Expected return of 10 %. In the case of Stocks A it would take 3+ negative Standard Deviations to create a loss, while Stock B would produce a loss with only 2+ negative Standard Deviations.

In these instances it makes more sense to measure the Standard Deviation (risk) per unit of Expected Return: "*How much risk must I assume to capture one unit of Expected Return*?"

This relation is expressed as the **Coefficient of Variation**, an index of risk which measures the Standard Deviation *per unit* of Expected Return

$$\textbf{Coefficient of Variation} = \textbf{CV} = \frac{\text{Standard Deviation}}{\text{Expected Return}}$$

If we apply this measurement to Stocks A and B, then

$$\mathbf{CV_A} = \frac{10.0\%}{30.0\%} = 0.33 \quad \text{while} \quad \mathbf{CV_B} = \frac{5.0\%}{10.0\%} = 0.50$$

Therefore, by this measure, Stock B is 50% riskier than Stock A, despite the fact that B has a lower Standard Deviation.

Sharpe's Index - Another Measure of Risk

A variant of the Coefficient of Variation is the Sharpe Index, named after the Nobel laureate in economics, Stanford's William Sharpe. The Sharpe Index measures the risk premium in relation to the Standard Deviation - *How much risk premium would I receive per unit of risk assumed?*

[8] When the curve approaches the normal bell-shaped curve.

The Sharpe Index, a ratio, is equal to $\frac{\overline{x} - x_{rf}}{\sigma}$, where $\overline{x}$ is the average return for one period and x_{rf} is the return on a risk-free investment (usually U.S. Treasuries). In the analysis of portfolios which extend over a number of periods, the geometric mean[9] of the investment is substituted for $\overline{x}$ and the geometric mean for Treasuries replaces x_{rf} .

Separating Risks

A stock held in isolation has two different sources of risks. One kind of risk, ***Unique Risk***, or ***unsystematic risk***, is the risk which attaches to one particular stock as the result of circumstances affecting that particular company, and perhaps its competitors. ***Market Risk***, or ***systematic risk***, is the risk to which all stocks are prey as the result of circumstances beyond their control or influence; e.g. inflation, increase in interest rates, economic recession, etc.

This is an important distinction because it has been shown that a large part of unique risk can be eliminated by a combination of certain stocks in a portfolio. Systematic risk, however, cannot be eliminated.

Correlation of Risk, Risk Minimization

It would be reasonable to assume that the risk of a two-stock portfolio would be the proportion of investment in stock A weighted by its Standard Deviation (its measurement of risk) *plus* the proportion of investment in stock B weighted by its Standard Deviation. But this is true only when the return of Stock A moves in lockstep with the return of Stock B, which is to say, when they are ***perfectly correlated***.

For example, if Stock A increases 10% and Stock B increases 10%, theses securities could be said to exhibit *perfect positive correlation* (+1). If Stock B declined 10% while Stock A increased 10%, these two would exhibit *perfect negative correlation* (-1). Combining two stocks which have perfect positive correlation will *not* defend against loss since they will go up and down together. Combining two stocks in equal proportion which have perfect negative correlation will completely eliminate the risk of loss since the gain in one will cancel the loss in the other. But it would also eliminate any gains.

But most stocks are not perfectly correlated. If any two stocks on the New York Stock Exchange are chosen at random, the correlation will be about +0.6, with a range of about +0.5 to +0.7. Therefore the choice of <u>two</u> stocks randomly chosen will not completely eliminate unique risk, <u>but it will reduce it.</u>

[9] See Chapter 4-10 for information about the geometric return.

Expected Return of Stocks Held in a Portfolio

The **Expected Return** of a portfolio, $\hat{x}_p$, is the **weighted average Expected Return** of all the stocks in the portfolio, where $\hat{x}_n$ is the Expected Return for Stock **n**. The applied *weight* (w_n) is the fraction of the total portfolio invested in each individual security. Therefore,

$$\boxed{\hat{x}_p = w_1\hat{x}_1 + w_2\hat{x}_2 + w_3\hat{x}_3 \ldots\ldots + w_n\hat{x}_n}$$

Assume the following Expected Returns for four stocks, S1 through S4:

	Proportion Invested	**Expected Return**
S1	15%	15%
S2	30%	16%
S3	35%	20%
S4	20%	22%

$$\hat{x}_p = .15*.15 + .30*.16 + .35*.20 + .20*.22 = .1845 = \mathbf{18.45\%}$$

Although the Expected Return of the portfolio, $\hat{x}_p$, is the weighted sum of the Expected Returns of each individual stock in the portfolio, it is <u>not</u> true that the *riskiness* of the portfolio (σ_p) is the weighted average of the Standard Deviation of each stock in the portfolio. This is due to the fact that, being imperfectly correlated, the price movement of one stock may reduce or completely eliminate the unique risk attached to another. As a result, the risk of a well-formed portfolio will be ***less than*** the average risk of all the stocks in the portfolio.

Determining Correlation Between Two Stocks

The degree to which two stocks move together is known as their **Correlation Coefficient**. A Correlation Coefficient of +1.0 indicates that the stocks move in positive lockstep; a Correlation Coefficient of –1 indicates that the stocks move exactly opposite to one another, while a Correlation Coefficient of zero, or near zero, indicates that the stocks move independently and with no relationship to one another.

The Correlation Coefficient = $r_{AB} = \dfrac{COV(AB)}{\sigma_A \sigma_B}$

The term COV(AB) represents the ***covariance*** of stocks A and B and is given by the formula

$$\textbf{Covariance} = COV_{AB} = \sum_{t=1}^{n}(x_{At} - \hat{x}_A)(x_{Bt} - \hat{x}_B)P_t$$

In the first expression following the summation sign, x_{At} is the **t** possible outcome of stock A; $\hat{x}_A$ is the Expected Return for stock A. The values in the second expression following the summation sign are correspondingly the same for stock B. P_t is the probability of outcome t. The degree of risk (as measured by Standard Deviation) for a portfolio (σ_p) composed of stocks A and B, is given by the following equation:

$$\textbf{Portfolio Standard Deviation} = \sigma_p = \sqrt{w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)r_{AB}\sigma_A\sigma_B}$$

(where r_{AB} is the Correlation Coefficient)

Risk of a Multi-Stock Portfolio, MPT

You can see that determining the risk of a portfolio of a larger number of stocks, such as in a Mutual Fund, would require the determination of the correlation coefficient for every stock in the portfolio and could become a complex undertaking. Yet diversification may be a very desirable strategy in reducing portfolio risk since it has been shown that assembling a portfolio of well-diversified stocks can materially reduce total unsystematic risk.[10]

Harry Markowitz, the father of **M**odern **P**ortfolio **T**heory (MPT), demonstrated that through the selection of a sufficient number of stocks with positive - though not perfect - correlation, unsystematic risk can virtually be eliminated.

Since the unique risk of a portfolio can be nearly eliminated by the proper choice of stocks, the remaining risk, – the systematic risk, which cannot be eliminated by diversification – is a function of the market risk of each stock in the portfolio in its relation to all other stocks in the portfolio.

Stock ßeta - Still Another Measurement of Risk

Rather than compare the risk of a stock to others in a particular portfolio, William Sharpe, together with Jack Treynor and John Lintner, devised the *Capital Asset Pricing Model* (CAPM) which relates the volatility of a particular stock to a broad index of all stocks.

Sharpe's group equated the overall volatility of the market to 1. A stock whose ßeta is greater than 1 is more volatile (and therefore supposedly riskier) than the broad market; a stock whose ßeta is less than 1 is less volatile (and therefore less risky) than the broad market. A stock whose beta is exactly 1 would rise and fall in lockstep with the market.

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[10] "Wide diversification is only required when investors do not understand what they are doing."
W. Buffett

Shape's CAPM formula states that:

(Expected Return Stock A – Risk-Free Rate) = **β***(Expected Return of Market - Risk Free Rate)

where **β** is the degree of volatility of the return on Stock A, as determined by Stock A's ***history,*** in comparison to the historical return on the broad market. The *Expected Return of Market* is the Expected (Average) Return of the S&P 500 Index of stocks. The Risk-free Rate is the rate delivered by U.S. Treasury Bonds.[11]

Squeezing the formula into a more compact form we have:

$$(\hat{x}_A - r_{rf}) = \beta\,(\hat{x}_M - r_{rf})$$

$$\text{Therefore, } \mathbf{Beta = (\beta)} = \frac{(\hat{x}_A - r_{rf})}{(\hat{x}_M - r_{rf})}$$

Note than the expression $(\hat{x}_A - r_{rf})$ is equivalent to the **risk premium** over the risk-free rate applied to Stock A, while $(\hat{x}_M - r_{rf})$ is the risk premium for the broad market. Therefore Beta measures the risk premium of a particular stock in relation to the risk premium of the broad market.

The utility of Beta relates to the fact that the expected riskiness of a portfolio, β_p , is a function of the weighted riskiness of each stock in the portfolio, as measured by its Beta, such that:

$$\text{Portfolio Risk} = \beta_p = \sum_{t=1}^{n} w_t \beta_t$$

If a number of stocks each with a Beta of 1.0 are contributed to a portfolio, the Beta of the portfolio will be 1.0. Replacing one of these stocks with a stock whose Beta is 0.8 will lower the riskiness of the portfolio. Adding a stock with a higher Beta will raise the riskiness of the portfolio.

The CAPM also enables one to set the expected (desired) return from a stock according to its Beta, the degree of risk in the broad market, and the current risk-free rate:

$$\hat{x}_A = \beta((\hat{x}_M - r_{rf}) + r_{rf}$$

where r_{rf} is the "risk-free" rate of return.

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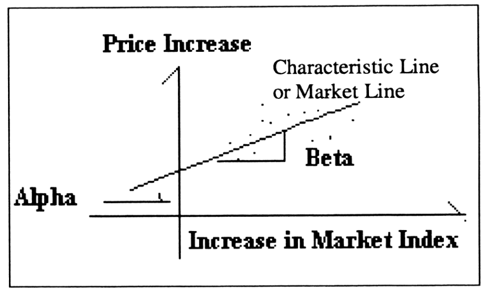
11 The "risk-free" rate may be determined either by the rate on U.S. Treasury Bills or Bonds, but more frequently on Bonds, and, as previously explained, carries an allowance for inflation.

Alpha



If we were to plot the price increase of a stock against the Market Index, we would expect that the regression line[12] drawn through the historical price points would pass through the origin of the graph. But in many cases it does not.

The point at which the regression line intercepts the y-axis (vertical axis), is known as the stock's Alpha. The distance from the x-axis is the difference between a stock's actual return and its expected return. The Alpha measurement is generally attributed to some non-market factor such as individual management. Therefore Sharpe's formula is usually modified to read:

$$(\hat{x}_A - r_{rf}) = \mathbf{A} + \beta\ (\hat{x}_M - r_{rf})$$

Finally, many brokerage houses which supply stock Betas add an error factor equal to 2 standard deviations of the confidence interval for the estimate of the Beta value. This error factor, **e,** is added to the formula:

$$(\hat{x}_A - r_{rf}) = A + \beta\ (\hat{x}_M - r_{rf}) + \mathbf{e}$$

R- Squared

The R-Squared value (0-100%) of a mutual fund measures the percentage of the fund's movement which can be attributed to changes in the Market Index. A high R-Squared factor indicates that most of the fund's movement can be traced to the changes in the Market Index, while a low factor indicates that very few of the changes can be attributed to the benchmark index.

The R-Square value is used in connection with a reading of the fund's Beta. A high R-Square suggests that the fund's Beta is more reliable than if the R-Square is low.

Limitations to the CAPM

While the theory of CAPM continues to be taught in most business schools, follow-up studies which have compared past Beta determinations with actual future stock performance suggest that in many cases there was little predictive value in published Betas.

The model for the CAPM is intended to be used with expected (ex-ante) data, but the values of Beta provided by investment service firms are calculated using historical (ex-post) data. Therefore in practice, the CAPM is predictive of future stock prices based on past performance, and not as the result of an analysis based on the probability of future performance. To the extent that a stock's Beta may be predictive of its future performance, the use of Betas to assemble a portfolio with reduced risk may or may not be helpful.

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[12] Characteristic Line

Arbitrage Pricing Theory (APT)

APT is one of the modifications to the CAPM meant to overcome its reliance on one factor alone, the measurement of a stock's risk in relation to market risk.. APT theory specifies that there may be a number of factors affecting this relationship: the manner in which a stock's price responds to interest rates, to inflation, to changes in economic activity, etc.. Two stocks carrying an equal potential for risk may respond to these myriad factors in different ways.

APT is a mathematical model which attempts to adjust the response to each of these factors. The derivation and construction of these mathematical models, however, is quite complex and is beyond the scope of this text, but it is interesting to note that many stock analysts and economists support their market predictions by reference to a model: "Our model indicates that ...."

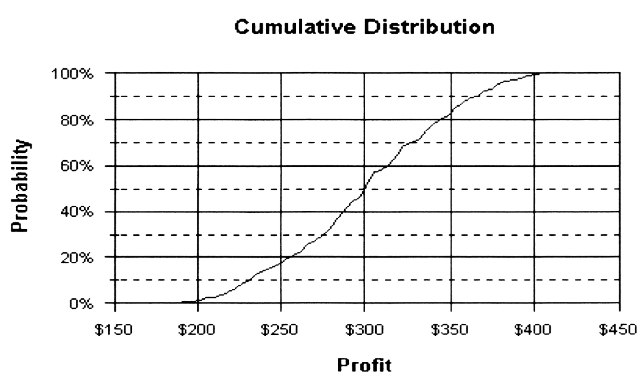
Monte Carlo Simulation

The development of computer spreadsheets has taken much of the drudgery out of reworking financial models. Spreadsheets enable us to determine in a few seconds the particular effect on a modeled outcome as the result of a discrete change in any variable, provided we are content to change one variable at a time. We can run any number of trials on the spreadsheet, changing a single variable each time, but when we are finished we have a large number of possible outcomes but no real sense of which outcome, or set of outcomes, is most probable.

When we encounter situations in which the variable can take on a large number of different values extending over a defined range of values, the spreadsheet approach proves very limiting. When the situation involves a large number of variables as well, the task becomes daunting.

The use of the computer comes to the rescue because it is capable of randomly generating a very large number of simulated outcomes for one or more variables expressed with limits, or *bounds*, set for each variable. Computer programs collect these results and organize them in a way which permits an estimate of probable outcomes.

These programs involve thousands of runs, the collection of the results for each run, and the organization of the results either in the form of a histogram or as a *Continuous Distribution* graph from which estimates of the most probable outcomes can be read.



In the accompanying illustration which depicts a Cumulative Distribution curve, one could say that the chance of obtaining a profit less than $250 is about 18%. The chance of receiving a return less than $350 is about 80%.

One such simulation which is proving attractive to financial professionals is **Monte Carlo Simulation.**

This simulation is a statistical technique[13] by which a quantity is calculated repeatedly using computer-generated random values within a specified range of probable outcomes.[14] Commercially available programs are capable of tasks such as pricing *derivatives*[15] or estimating the *value at risk*[16] of a portfolio, or the risk of attaining a capital sum at the time of retirement as the result of combining a number of different investment vehicles.

These programs involve thousands of runs, the collection of the results for each run, and the organization of the results either in the form of a histogram or as a *Continuous Distribution* graph from which estimates of the most probable outcomes can be read.

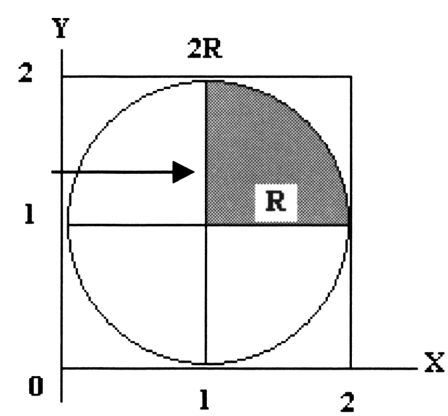
A Simple Monte Carlo Simulation

The following example illustrates how Monte Carlo Simulation might be used to calculate the numerical value of π (pi).[17]

Consider the area of a circle imbedded in a square. For simplicity's sake we will focus on the upper right quadrant. We have a *machine* which can throw numerical darts *randomly* at the upper right quadrant.

In order to be sure that all darts land within the upper right quadrant, we would program the computer to "throw darts" according to preset limits along both the X and Y axes; X values would be between 1 and 2, and Y values would also be between 1 and 2.

The machine is, of course, the computer which is programmed to generate random numbers of (x, y) points such that all darts land somewhere within the upper right quadrant. Some would land within the shaded area and some outside the shaded area.



It would be reasonable to assume that the number of darts landing within the shaded area, and the number landing within the quadrant but outside the shaded area, would be in the same proportion that the shaded area bears to the total area of the quadrant.

13 A mathematical technique for numerically solving differential equations
14 *Stochastic* analysis
15 A financial instrument whose value depends on the performance of some underlying instrument/
16 The upper bound on a confidence interval for the probability of the loss/profit that a portfolio will realize over a specific period of time.
17 Area of a circle = πr^2

Therefore,

$$\frac{\text{Number of darts in shaded area}}{\text{Total number of throws}} \approx \frac{\text{area of shaded area}}{\text{area of quadrant}}$$

$$\frac{\text{Number of darts in shaded area}}{\text{Total number of throws}} = \frac{\pi R^2/4}{R^2} = \frac{\pi R^2}{4 * R^2} = \frac{\pi}{4}$$

Therefore, $$\pi = \frac{\textbf{4 * Number of darts in shaded area}}{\textbf{Total number of throws}}.$$

It requires a very large number of "dart tosses" to produce a reasonably accurate estimation of the value of pi. The greater number of random tosses,[18] the more accurate the value of pi. An accurate value for the determination of pi might take 10-20,000 "tosses," or iterations of the computer for this simple determination. More complex simulations may require hundreds of thousands of iterations on a fast computer.

Because of the time and resources required, Monte Carlo simulations are most often reserved for those analysis problems for which no closed-form solution[19] exits, or is conveniently available. A compromise often sought is to reduce the total number of Monte Carlo iterations required to deliver a reliable answer. One means of reducing these computations is to reduce the required confidence interval or to tightened the limits of the possible outcomes (variance spread).

But one thing is certain: the more complex the model constructed to achieve a targeted goal (e.g. a retirement account of a stipulated capital value at some time in the future), the more frequently the simulation must be run in order to track the potential changes in a large number of variables.

Risk Associated with Bonds

In Chapter 6 we discussed bond risk as measured by Duration alone. But there are other factors influencing bond risk: liquidity, credit, currency and volatility. All these factors combine to determine the riskiness of an issue.

Bonds are graded either by Standard and Poor's, by Mode's Investment Services or by Fitch Inc. The first two services rate an issue according to "all the information provided to the public." The grade assigned to a bond gauges the issuer's ability to meet the promised dividend payments and the return of the principal, and therefore serves as an index to the bond's potential riskiness.
Both services use letter-coded grading systems which are similar, but not identical.

The following charts provide a brief explanation of the risk factors and list the grades assigned by each service.

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[18] The random numbers generated by a computed are quasi-random. Given a starting "seed' value, the computer will repeatedly generate the same set of "random" numbers. Commercially available programs, however, will pull a seed number from some transient value within the computer, and thus approach a true random series.

[19] A solution produced by a mathematical equation or computer algorithm

Liquidity...	refers to market demand for the bond and the ability of the bond holder to liquidate his position. Smaller issues may not be able to attract a sufficient number of potential buyers if the need to sell arises. Lowered liquidity increases risk and raises required yield.
Credit ...	refers to the creditworthiness of the issuer and applies directly to the likelihood of a default. Low credit (i.e. "junk") bonds require high yields to be marketable.
Duration...	refers to the bond's price sensitivity to changes in market yield. Duration is a function of market yield, coupon rate and time to maturity. Higher Durations increase risk.
Currency...	refers to changes in the exchange rate between the currency in which the bond was issued and the value of the holder's currency.
Volatility	Refers to the potential for the issue to respond to economic changes.

Rating	Standard & Poors	Moody's
Highest Quality	**AAA**	**Aaa**
High Quality	**AA**	**Aa**
Upper Medium Grade Obligations	**A**	**A**
Medium Grade Obligations	**BBB**	**Baa**
Somewhat Speculative	**BB**	**Ba**
Low Grade Speculative	**B**	**B**
Possibility of Default	**CCC**	**Caa**
Highly Speculative	**CC**	**Ca**
Lowest Graded - Extremely Poor Prospects	**C**	**C**

Bond Ratings Affect Discount Rates

Consistent with the Capital Asset Pricing Model, bond investors require higher yields for bonds of higher risk. The yield required is the discount rate which will be applied to all future cashflows arising from dividends and reversionary value. The higher the discount rate, the lower the Present Value of the bond.

In our discussion of Duration in Chapter 6, we discounted bonds according to the "market yield." If we were to construct a yield for a particular bond issue, it would be necessary to calculate an Expected Return incorporating all of the risk factors mentioned above. The rating services provide an index to the risk of the bond in their code-letter classifications, and therefore are a more convenient aid in establishing an appropriate discount rate.

In comparing bonds according to their Durations, it is important to compare bonds with similar risk ratings, since a risky bond with a high coupon rate may have a Duration equal to a less risky bond with a lower coupon rate. Duration measures the potential volatility of a bond to changes in market interest rates, but Duration alone does not measure changes in the ability of the issuer to repay interest and principal, the potential for currency rate changes, nor any of the other bond factors which are involved in risk.

In the Final Analysis...

When all is said and done, the evaluation of risk is a rather subjective exercise. *Statistics* can provide some guidelines as to historic performance and, by extension, to what the future may be like.

Probabilities involve predictive estimates of what the future may hold, irrespective of the past. But under both circumstances there is the likelihood of error: either that the future will not be very much like the past, or that our best estimates of the most probable outcomes will be well off the mark.

Nevertheless, an attempt to quantify risk does focus our attention on both past performance and the likelihood of future performance, and this attention is better than no attention at all. The degree of risk as seen through the eyes of one investor will be different from the degree of risk seen by another investor. As a result, each will add a different risk premium to the safe rate.

It is the combination of the safe rate, the inflation rate and the *risk premium* rate which adds up to the discount rate. And it is the discount rate which we use to convert expected future cashflows to an estimate of present value.

Chapter Summary

1. Risk is commonly understood as the likelihood of an adverse outcome.
2. Risk can be estimated by extrapolating past performance into the future using historic data.
3. Risk can also be estimated by identifying all possible outcomes and weighting the probability of each outcome. The sum of the weighted outcomes is the Expected Outcome.
4. The sum of the squares of the distance by which each individual value deviates from the Expected Outcome (as determined by probabilities) or Average Outcome (as determined by historical data) is the Variance. The Standard Deviation is the square root of the Variance.
5. The Standard Deviation is the degree to which the collected data deviate form the Mean or Expected Outcome.
6. It has been demonstrated that in a curve which follows the pattern of the normal bell-shaped curve, 68.35% of the data will fall within ± 1 standard deviation, 95.5% of the data will fall within ± 2 standard deviations, and practically all (99.7%) of the data will fall within ± 3 standard deviations of the mean (or expected return).
7. The Coefficient of Variation measures the amount of risk undertaken (as measured by the Standard Deviation) in relation to the Expected Return. This is a useful index of risk

which comparing investments of similar Expected Outcomes. It answers the question "*How much risk must I assume in order to achieve the Expected Outcome*?"

8. The Sharpe Index measures the risk premium obtained per unit of risk, as measured by Standard Deviation. Sharpe's Index asks *"How much premium do I receive for each unit of risk assumed?"*
9. All investments carry both systematic (unique) and unsystematic (market) risk. Unsystematic risk is the risk specific to a particular investment, while systematic risk is market risk common to all investments in the same category.
10. Diversification of a stock portfolio may be able to eliminate virtually all unsystematic 9r unique risk, but can eliminate none of the market or systematic risk.
11. The Correlation Coefficient measures the degree to which one stock varies with respect to another stock. Perfectly correlated stocks (+1) move in lockstep with one another; perfectly negative correlation (−1) indicates that two stocks move opposite to one another; 0 correlation indicates that the price movement of two stocks is completed unrelated.
12. The Capital Asset Pricing Model relates the risk associated with one stock to that of the broad market using stock Beta.
13. Stock Beta is a measurement of the risk premium of a single stock in relation to the risk premium of the broad market (as measured by the Standard & Poor's 500 Index).
14. Stock Alpha measures the difference between a stock's actual return and its expected return given a level of Beta risk.
15. Theoretically the systematic risk of a portfolio of stocks may be reduced, if not eliminated, by choosing a sufficient number of stocks which are positively correlated. In practice, the use of Beta to select stocks has been shown to be unreliable. A limitation is the reliance of Beta measurements on past performance.
16. Monte Carlo simulations offer a means of generating expected outcomes by the use of computer-generated random values in place of uncertainties.
17. Duration is a good index of volatility for bonds of similar risk but does not reflect all risk factors associated with bond investments.

Chapter 8
Promissory Notes & Mortgages

In this section we will evaluate cashflows arising from mortgages and promissory notes. Before we begin, it may be helpful to define some new terms and delineate a few important concepts related to promissory notes and mortgages.

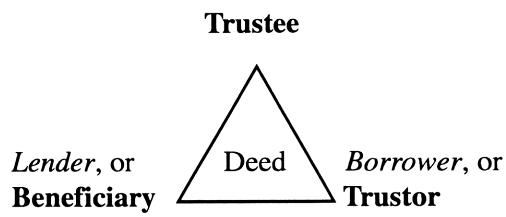
The most common financial method of acquiring real estate, as you know, is by combining owner funds (equity) with borrowed funds (debt). When stocks or bonds are acquired with borrowed funds, they are said to be bought "on margin." In either case, these two initial cashflows - equity and debt - taken together equal the acquisition price of the investment..

Distinguishing Mortgages & Trust Deed Loans

In eastern states, lenders commonly use an instrument called a **M**ortgage to evidence and secure their loans. The borrower executes a promissory note as *legal evidence* of the debt. The Mortgage is recorded against the owner's title and acts as a *security* device in the event of a failure of the borrower to repay the debt or otherwise fail to uphold the loan agreement. Most Mortgage forms combine these two instruments, the promissory note and security instrument, in a single document.

In western states, and some southern states, **M**ortgages are not used for reasons which reduce to the fact that they are legally cumbersome. In these states, the promissory note also acts as legal evidence of the debt, but the security device – which is always a separate, recordable instrument – is known as a *Trust Deed.* A Trust Deed *is* a recordable instrument, whereas a promissory note is not recordable.

In those states in which Mortgages are used, the relationship is a two-party arrangement directly between lender and borrower, with the borrower (mortgagor) retaining full title to the property. In those states in which Trust Deeds are used, the relationship is a three-party arrangement among 1) the borrower (trustor), 2) the lender (beneficiary) and 3) a trustee (a neutral third party).



In executing the Trust Deed, the Borrower-Trustor establishes a *Trust* and appoints the Trustee[1] as guardian of that which is to be placed in Trust – the deed to the Borrower's property. This deed, however, does not carry with it the usual rights to use and possess the property, only the right for the Trustee to sell the property under carefully proscribed legal conditions.[2]

If the Borrower-Trustor breaches his agreement with the Lender-Beneficiary, the Lender notifies the Trustee to initiate a *foreclosure* action to sell the property under a **Power of Sale** clause contained in the Trust Deed. To give public notice to whomever may be interested that the title to the property is encumbered by a *lien*,[3] the Lender-Beneficiary requires that the Trust Deed be officially recorded against the title to the Trustor's property.

Time of Recording Establishes Priority of Lien

First in Time
Is
First in Right

When more than one Trust Deed or mortgage is recorded against a Borrower's title, the order in which the liens are recorded establishes which is the "First Trust Deed," "Second Trust Deed." and so on.... The same is applicable to Mortgages. Each recorded lien is stamped with the exact date and time (hour and minute) of recording, and also assigned a sequential *Document Number*, by the recording clerk. If one wishes to determine whether a certain lien is a First, Second or Third Trust Deed or Mortgage, it is necessary to examine the actual document and note the time of recordation and the Document Number for each instrument. The Trust Deed or Mortgage document itself gives no other indication of its priority.

Trust Deeds which are recorded earlier in time than others are said to be "senior" notes, while those recorded later are "junior" notes. *Senior* and *junior* refer only to the time-priority of the Trust Deeds and not to the monetary value of the corresponding promissory notes which they secure.

These matters are relevant because they relate to investment risk.

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1 Usually, but not necessarily, a corporation created and owned by the title company which insures the title.

2 Which are listed in the Trust Deed document itself, and, in most states, in its Civil Code.

3 An encumbrance on property involving money.

If a lender forecloses under a Trust Deed which is senior to other Trust Deeds recorded on the same property, the lender takes full title to the property *in the condition in which it existed at the very hour and minute his Trust Deed was recorded.* All junior Trust Deeds are automatically expunged from the title.[4]

This fact creates **risk** for junior lien holders, and this risk is factored into the interest rates charged by lenders who make loans secured by junior Trust Deeds. This helps explain why second and third Trust Deed loans command higher interest rates.

If the reader seeks more information, he or she may wish to consult a text dealing with real estate principles or finance, since there is considerable misinformation about the workings of this important aspect of investment financing.

Is It a Trust Deed or a Note? Or is it a Mortgage?

One particular area of confusion derives from the fact that many people use the term "Trust Deed" when they are referring to the promissory note. The two are different; and it is more than just academic hair-splitting to want to maintain the distinction. If the Promissory Note can be separated from the Trust Deed, then some Promissory Notes can be securitized with collateral other than a Trust Deed on the subject property. And, by the same token, Trust Deeds may be used to collateralize something other than a note; e.g. a promise to do, or not to do, something.

Promissory notes are very commonly bought and sold in secondary financial markets. Trust Deeds never are.[5] It is the promissory note which not only *evidences* the debt, but also recites the term of the loan (**n**), the amount of the payment (**PMT**) and its due date, the interest rate (**i**), the principal amount (**PV**), and – in some cases – the amount due at maturity (**FV**), as well as other important conditions of the loan**.** While recorded Trust Deeds may cite some terms of the loan (**PV**, **PMT**, due date), they rarely recite all the terms of the Promissory Note. Our interest is not in the Trust Deed (nor in the Mortgage[6]) especially, but in the terms of the *Promissory Note*, because it is from the note that cashflows arise, not from the Trust Deed.

Lastly, be aware that many individuals use the term "*mortgage*" (lower case) to equal *loan,* or "*trust deed*" to equal *loan*. There's no way to discern what they mean except by patient inquiry.

In the following sections of this chapter we will discuss a number of common loan constructions and examine the cashflows which flow between borrower and lender. It is helpful to understand that lenders change loan programs change very frequently, adjusting the terms of loans in seemingly limitless ways, all designed to gain a marketing advantage, a financial advantage and

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4 Tax liens, however, are given automatic precedence over trust deeds.

5 When a promissory note is sold, any Trust Deed which acts as security for the note automatically transfers, by operation of the law, to the owner of the note. No action by the new owner is required.

6 Since most **M**ortgage documents usually contain the promissory note, a recorded Mortgage will provide the terms of the note. This is not the case with Trust Deed documents.

consumer appeal. Therefore, as you progress through these sections, be mindful of how the conventional elements of loans can be put together in unconventional ways.

A thorough understanding of the key elements of loan cashflows will help you evaluate a wide variety of existing loan programs, even loan programs yet to be concocted and, perhaps, enable you to solve many financing problems using innovative cashflow techniques.

A Word To the Wise

Investments in trust deed notes and Mortgages can offer excellent short-term yields, especially when they are acquired through an IRA or a 401(k) plan which enables the investor to defer taxes on the income and to roll-over the total proceeds into another investment. The overriding consideration, however, is safety of principal..

The first line of defense against loss is the credit worthiness of the trustor-borrower. A *current* credit report is an *absolute necessity.*

The second line of defense is a current and reputable appraisal of the property securing the debt. The property securing the debt need not be the property purchased with the loan proceeds. Any real property can be used as the security for any promissory note. The purpose of this appraisal is to ascertain the amount of net equity which the trustor has in the property and which is offered as security.

The third line of defense is a current report on the condition of title issued by a reputable title insurer which will confirm a legal description of the securing property, its current ownership, the existence and priority of senior liens, tax liens, recorded judgements against the owner's interest in the securing property and the existence of any non-standard easements or other encumbrances affecting the title. A junior lien holder should insist on being named "additionally insured" on the title and fire insurance policies.

Suffice to say, any financial planner or investor lacking the experience and know-how should obtain the professional assistance of an experienced real estate consultant or real estate attorney.

'Nuff said.

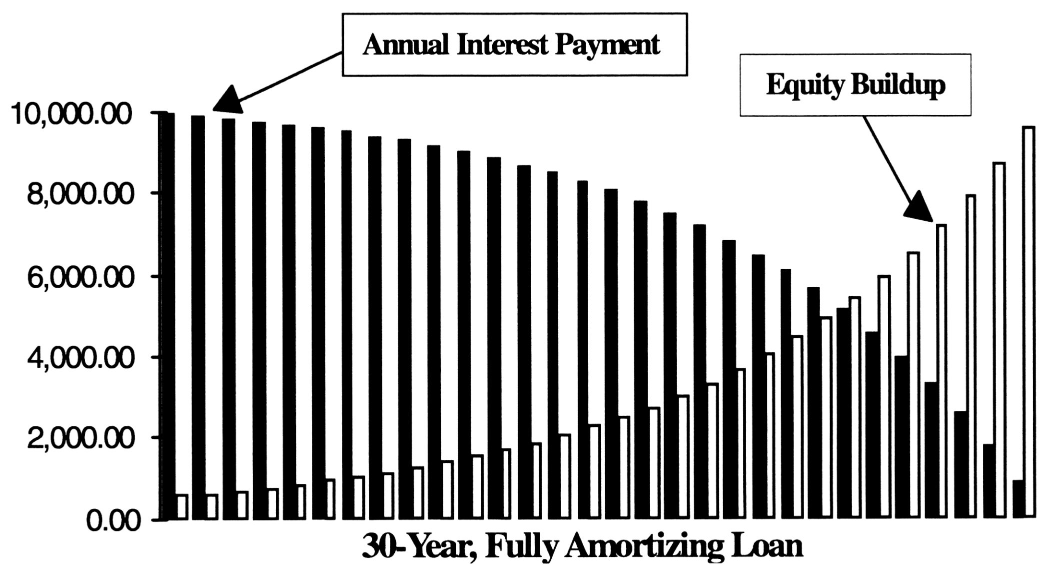
I - The Amortizing Loan

An amortizing loan is one which is set up on a payment schedule which would eventually eliminate the balance of the loan. The word *amortizing* derives from *a morte,* which means, literally, "to (the) death." This pertains to the fact that if the required loan payment is made each period, the balance of the loan will eventually be paid down to zero (i.e. "killed off"). In terms of cashflows, it means that the Future Value **(FV)** of *every* fully amortizing loan will eventually become zero. This is true even if only a few pennies from the regular **PMT** are available to reduce the principal.

Amortizing loans, therefore, are a type of *ordinary annuity* for a definite period of time whose PMTs can be calculated by formulas covered in Chapter 3.

The Amortization Schedule

The length of time required for the balance of an amortizing loan to reach zero is called the *amortization schedule* of the loan. The graph below illustrates the progress of a loan which will amortize over a 30-year period, and shows the cash from the payment devoted to interest vs. the cash devoted to reduction of the loan balance (the equity portion[7] of the payment). The hallmark of a *fixed-rate* amortizing loan is an *equal payment over the entire schedule of the loan* Therefore at any point in the progress of a fixed-rate amortizing loan, the addition of the interest and equity portions of the fixed payment will always total to the same number.



7 "Equity Portion" because it lowers the outstanding debt and increases the owner's cash position in his property.

The graph on the previous page should not be passed by lightly.

Notice that as an amortizing loan progresses toward a zero balance, the portion of each PMT devoted to the reduction of the loan increases. If one can assume (take over) an older loan in acquiring a property, each subsequent loan PMT by the new owner delivers more cash to reduce the principal. Less interest to the lender and more equity build-up to the owner increases the equity position of the owner.

Amortization Schedule vs. Due Date

Most commercial and residential amortizing loans are scheduled to be reduced to a zero balance over a period of from 15 to 40 years.[8] Since the payments needed to reduce the loan to zero are typically - but not always - made on a monthly basis, it would be more accurate to refer to these loans as loans which will amortize over 180-360 months.

The advantage to borrowers of loans which are constructed to amortize over long periods of time (30-40 years) is that the required payments are kept relatively low. But long amortization schedules place the lender of a fixed-rate loan at substantial risk to rising interest rates.[9] Therefore lenders will either 1) schedule the loan as though it were to last 25, 30 or even 40 years – but "call" the loan[10] at the end of the 5th, or 7th, 10th or ***x*** year in the schedule, or 2) charge substantially more in interest to compensate for the added risk.

This kind of loan is commonly used in connection with income-producing properties and less frequently, but not infrequently, with residential properties. Whenever a loan calls for a payment which is more than twice any of the preceding six regularly scheduled payments, that payment is called a **balloon payment.** Therefore all loans set up on fully amortizing schedules, but which require an early payoff by the borrower, contain **balloon** payments. Loans requiring a balloon payment must often meet certain legal obligations for notification to the borrower of the impending due date.

Only Fixed-Rate Loans Offer Constant Payments

Payments of the same amount, month-in and month-out, were once the hallmark of every amortizing loan. The amount of the payment (**PMT**) never changed over the life of the loan. Since the late 1970's, however, interest *rates* on certain kinds of amortizing loans periodically do change, and therefore the **PMT** necessary to "kill off" the balance of the loan over the amortization schedule originally selected also changes from time to time.

In order to distinguish between the two types of amortizing loans, we now refer to mortgages whose **PMT**s never change as **fixed-rate mortgages,** and to those whose **PMT**s change from time to time (as the result of a change in interest rates) as **Adjustable-Rate Mortgages**, or **ARM**s. Creative mortgage types have even created the hybrid - the adjustable rate mortgage

8 Although other schedules are frequently encountered.

9 Interest paid on deposits will exceed the interest rate obtained from outstanding loans.

10 Require that the loan be paid in full before the end of the amortization schedule.

which later converts to a fixed-rate mortgage, or the fixed-rate loan which later converts to an ARM. If you guessed that these loans are now called **convertible mortgages**, you are right.

The Fixed–Rate, Amortizing Loan

The elements of this loan are the same as for all cashflow problems:

n	**denotes** the number of *periods* over which a payment will be made. The total number of periods = the amortization schedule.
i	**denotes** the interest *rate* of the loan, expressed to correspond to the same time interval as **n** and **PMT.**
PV	**denotes** the **P**resent **V**alue of the loan at any point in its schedule.
PMT	**denotes** the payment per period **n** which is required to reduce the remaining balance of the loan to zero.
FV	**denotes** the **F**uture **V**alue, or remaining balance of the loan, at *any* point in its amortization schedule. For the *fully* amortizing loan, this amount, by definition, will always be zero at the end of the amortization schedule.

These five financial variables and their symbols are used by all financial calculators and by all financial (computer) spreadsheets. They are represented on the HP-12C by the same keys (1,1) through (1,5) which we used in solving cashflow problems in previous chapters.

n	i	PV	PMT	FV

On Microsoft Excel®, the order of the variables is changed: **n** becomes **nper,** and **i** becomes **rate.**

rate	nper	PMT	PV	FV	*type*

The ***type*** variable enables the computer user to set the PMT at the beginning of the period (BOP) or at the end (EOP). Set ***type*** to 1 for PMTs made BOP; set ***type*** to 0 for PMTs made EOP. The default value[11] is zero (EOP).

Determining the Payment Necessary to Fully Amortize a Loan

Problem:

> Calculate the monthly payment (**PMT**) necessary to amortize a loan of $100,000 (**PV**) over a period (**n**) of 30 years, if the lender charges an annual interest rate (**i**) of 10%, compounded monthly.

Remember that the time spans of **n, PMT** and **i** must all be the same. This problem, however, expresses **n** in years, **PMT** in months, and **i** in years. Since we are going to solve for the *monthly*

11 The value used by the computer when no value is specified.

PMT, we must first translate **n** into *months* and **i** into a *monthly* rate of interest to maintain time consistency among these cashflow variables.

The hard way to do this is to key-in 30 (years), multiply by 12 (months/year) and enter the result (360) into the **n** register (key (1,1). An easier way is to utilize the blue key [g] (4,3) .

Key–In	Display Shows
30 [g] [n]	**360.00**

This installs 360 (months) into the **n** register by automatically *multiplying* 30 x 12.

Similarly, we can conveniently convert the annual interest rate (10) into a monthly rate by using the blue key [g] as a prefix to the [i] entry:

Key–In	Display Shows
10 [g] [i]	**0.83**

In this case, using the blue key [g] before entering 10 into [i] automatically *divides* 10 by 12, = 0.83 We have now filled two of the Registers, [n] and [i]

n	i	PV	PMT	FV
360	0.83			

Now enter the **P**resent **V**alue (in this case the original loan amount, $100,000) into **PV:**

Key In	Display Shows	Comment
100000 CHS PV	**-100,000.00**	
	Pressing the CHS key (1,6) changes the sign of the number in the display	

We have now filled three of the Registers, [n] , [i] and [PV]

n	i	PV	PMT	FV
360	0.83	–100,000		

Notice that the figure 100,000 was entered as a negative number. We did this by first keying-in the number 100,000, then keying **CHS** (1,6) to change the sign, and then entering the result into **PV.** Had we not changed the sign, the payment (**PMT**) would be a negative number. There's nothing wrong with this, but you may wish to make it a general rule to enter all **PV** loan values – whenever appropriate – as a negative (from the lender's point of view) and read out all **PMTs** as positive. The sign convention requirement of the calculator requires that at least one of the cash entries must be a negative number.[12] This requirement is also true for computer spreadsheets.

Now we need to enter something into the **F**uture **V**alue register. Since this loan will be a completely amortizing loan, we know that in 360 months **FV** *must*, by definition, be zero. Therefore we enter **0** into **FV**.

n	i	PV	PMT	FV
360	**0.83…**	**–100,000**		**0**

Now all we need to do is to call for the answer by pressing **PMT** , the unknown:

n	i	PV	PMT	FV
360	**0.83…**	**–100,000**	**?**	**0**
			877.57…	

Therefore the **PMT** necessary to amortize a **$100,000** fixed-rate loan over **360 months,** with **monthly PMT**s including interest calculated at the rate of **10%** per annum, but applied monthly, is **$877.57… per month.**[13]

Solving this problem on the computer's spreadsheet is a matter of filling in the blanks. The formula is:

=PMT(rate, nper, pv, fv, type)
=PMT(0.10/12, 360, –100000, 0, 0)
= $877.57

Avoid using commas to format numbers inside Excel formulas

The ***type*** position enables specifying EOP or BOP PMTs. Notice that the interest rate is entered as a decimal, and not as a percent. Take care not to use commas except to separate formula entries.

Problem:

Suppose that you wish to determine the **PMT** for a loan of this amount, with an identical interest rate, but amortized over a 15-year schedule (180 months).

What **PMT** would be required?

---

12 Cash must flow out and in.

13 The ellipsis (…) indicates that there are additional values for the number which are not shown, 877.5715701

The only value in the registers which needs to be changed is the value in the [n] register. There is no need to clear the entire calculator and re-enter all the other values. (Don't worry about the value in **PMT**.) Simply *write-over* the value in [n] and re-solve for **PMT**:

	Key–In	Display Shows
	15 [g] [n]	**180.00**
solving		
	[PMT]	**1,074.61**

This solution says that a monthly payment of $1,074.61, which includes monthly interest at the annual rate of 10%, will completely amortize a $100,000 loan in 15 years (180 months).

Not all loans are institutional loans, as you well know. Many are made by private investors. Consider this situation:

Problem:

> Suppose that you are involved in the sale of a property whose owner has agreed to carry back a note in the amount of $15,000 as part of the purchase price. He requires, however, that he receive **PMT**s such that the note will be completely paid off (amortized) in 8 years, and he will settle for nothing less than 9% annual interest on the note. He also requires that the **PMT**s be made monthly.
> What **PMT** can he expect to receive?

Here's how to solve for **PMT** in this common situation:

n	will be 8 x 12	(expressed in months)
i	will be 9 ÷ 12	(as a percent per months)
PV	will be –15000	
PMT	**?**	(will be answered in PMT per month)
FV	will be 0	(the loan fully amortizes in 8 years)

The key-strokes are:

	Key–In	Display Shows
	f CLX	**0.00**
	8 g n	**96.00**
	9 g i	**0.75**
	15000 CHS PV	**-15,000.00**
	0 FV	**0.00**
solving		
	[PMT]	**219.75**

Solving for n, the Number of Periods

> Unfortunately you find that Buyer in the preceding problem cannot make this high a payment ($219.75). Following a discussion, the Buyer determines that he could make a PMT of not more than $200.00 per month.
>
> How long will it take for this PMT of $200 to amortize the loan?

n	will be ?
i	will be 9 ÷ 12
PV	will be –15000
PMT	200.00
FV	will be 0

	Key–In	Display Shows
	f CLX	**0.00**
	9 g i	**0.75**
	15000 CHS PV	**–15,000.00**
	200 PMT	**200.00**
	0 FV	**0.00**
solving		
	[n]	**111.00**

Therefore a **PMT** of $200 will completely amortize this debt *within* **111** months.

As previously discussed, one of the idiosyncrasies of the HP-12C calculator when it attempts to solve for **n** is that the answer is ***always*** given in integers (no decimal places). In almost all cases, the **PMT** made for **n** periods will overpay the loan. Therefore when solving for **n** always check for an overpayment by re-solving for **FV**. Enter these values:

	Key–In	Display Shows
	9 g i	**0.75**
	15000 CHS PV	**–15,000.00**
	200 PMT	**200.00**
	0 FV	**0.00**
	n	**111.00**
solving		
	[FV]	**-72.50**

Solving for a Balloon PMT

Continuing with the Problem:

> Unfortunately, the Seller refuses to carry this note that long. By way of compromise, the Buyer agrees to pay a balloon payment at the end of the 8th year, if the Seller will agree to the $200 per month payment and if the balloon payment is limited to not more than $2,500. We already know that a **PMT** less than $219.75 will not amortize this loan in 8 years at the given interest rate.
> But what will the balloon payment be?

n	will be 8 x 12
i	will be 9÷12 %
PV	will be -15000
PMT	200.00
FV	will be **?**

n	i	PV	PMT	FV
96	0.75	-15000	200	?
			Solving…	**2,762.59**

Here are the keystrokes:

	Key–In	Display Shows
	8 g n	**96.00**
	9 g i	**0.75**
	15000 CHS PV	**-15,000.00**
	200 PMT	**200.00**
solving	**FV**	**2,762.59**

Solving for the Interest Rate, i

Continuing with the Problem:

> Since the balloon payment of $2,762.59 would exceed the Buyer's limit of $2,500, the Buyer refuses to agree to these terms. As a compromise, you negotiate a modification whereby the Buyer will agree to a balloon payment of up to $2,500 *if* the Seller will lower the interest rate not more than one-half percent.
> What will the interest rate **i** be?

n	= 96
i	= **?** %
PV	= –15,000
PMT	= 200.00
FV	= 2,500

	Key–In	Display Shows
	8 g n	**96.00**
	15000 CHS PV	**-15,000.00**
	200 PMT	**200.00**
	FV	**2,500.00**
solving		
	$\boxed{\text{i}}$	**0.73 (Monthly interest rate)**
	12 $\boxed{\text{x}}$	**8.78**

The Seller agrees to this rate, 8.78%, as an acceptable alternative to not selling his property. Your negotiation is successfully concluded.

Payments When the Period is Not Monthly

Problem:

> You are negotiating the acquisition of a small retail center whose
> owner has agreed to carry back a note as part of the
> purchase price. She requires a 10% annual interest rate but
> has agreed to *quarterly* payments.
>
> If the amount of the note is $60,000 and the required annual
> interest rate is 10%, what quarterly **PMT** will be needed to
> amortize this note over a 7-year term?

The key to this kind of problem, involving something other than *monthly* **PMT**s, is to recognize that you must key-in both the number of periods, **n**, and the interest rate per period, **i,** in quarters of a year, and not in months. Therefore:

n	= 7 x 4 (quarters/year) = 28
i	= 10 ÷ 4 (quarters/year) = 2.5
PV	= –60,000
FV	= 0
PMT	= ?

Key–In	Display Shows
7 [Enter]	**7.00**
4 [x] [n]	**28.00**
10 [Enter]	**10.00**
4 [÷] [i]	**2.50**
60000 CHS PV	**-60,000.00**
0	**0.00**

solving

[PMT]	**3,005.28**

The *total annual* payments under this note would be 4 times the quarterly payments of $3.005.28, or $12.021.10.

Continuing,

4 [x] 12,021.10

Note: The blue key [g] is <u>not</u> used as a prefix to key-in either the interest rate, **i**, or the number of periods, **n**, since [g] [i] divides by 12, and [g] [n] multiplies by 12. It cannot be used when the period is <u>not</u> 12.

Let's turn this problem around so that you can see how to handle the determination of the annual interest rate when the loan is not payable monthly.

Problem:

> The owner of the retail center has agreed to carry back a note for $60,000 payable $$3,005.28 per quarter for 7 years. She agrees that these payments will be considered to fully amortize the promissory note.
> What interest will she realize on the note?

Solving for i

n	will be 7 x 4 = 28
i	= ?
PV	= –60000
FV	= 0
PMT	= 3,005.28

Key–In	Display Shows
7 [Enter]	7.00
4 [x] [n]	28.00
600000 [CHS] [PV]	–60,0000.00
3,005.28 [PMT]	3,005.28
0 [FV]	0.00

solving

[i]	2.50

But the interest rate, 2.50%, is the rate *per quarter.* This answer must be multiplied by 4 to express the annual rate.

Continuing,

4 [x]	10.00

Remember! Always keep the time-frame for **n, i,** and **PMT** the same – in this case quarterly. Failure to do so is a very common source of errors in cashflow analyses.

Generating an Amortization Schedule for a Fixed–Rate Loan

An amortization schedule is simply a numerical record of the progress of an amortizing loan as it proceeds toward a zero ending balance. The schedule typically shows periodic payment, the starting and ending balances, together with the interest paid and equity buildup (loan paydown) of the loan per period.

These schedules come in handy whenever it is necessary to calculate the interest paid over a number of consecutive periods, or the amount of loan reduction which takes place over an interval of time, or the remaining balance of a loan after a certain number of payments have been made.

For example, you may need to know how much tax-deductible interest you have paid on your home mortgage during the last tax year. Or you may need to know how much principal reduction will occur in a certain mortgage over the next three years. Or you may want to know what the balance of your mortgage will be in five years, when you intend to sell or refinance the loan.

Many financial professionals have access to “canned” computer programs which prompt the user for the required data and then produce a printed amortization schedule over the time period specified. But every financial professional should be able to generate an amortization schedule without a “canned” program. An understanding of amortization schedules makes it easy to construct one on a computer spreadsheet, which can be conveniently printed for third–party use.

Mortgage Interest is Paid in Arrears

One important fact about the typical amortizing mortgage is that the interest is paid in arrears. This means that the payment for the use of the money during the month is paid at the end of the month – not in advance at the beginning of the month.[14] Mortgages are Ordinary Annuities.

First–time home buyers are often confused about this because they are accustomed to paying apartment rent in advance at the beginning of the month. When they secure a purchase money mortgage through escrow, the escrow officer[15] is directed by the lender to collect interest on the mortgage money from the day it is credited to their account to the first of the next month. The first payment on the mortgage is usually due one month *proximo* (after that)

For example, suppose a sale escrow closes on the 16th of June. A purchase loan is funded through the same escrow at closing. The lender will typically instruct the escrow officer to collect interest on the total mortgage funds from the 16th of June through June 30th. These funds will be remitted by the escrow holder to the lender following the closing[16] and will appear on the buyer's closing statement as (prepaid) interest.[17] The first full, regularly-scheduled payment on the mortgage, however, will not be due until August 1, and will pay interest in arrears for the month of July, together with a small additional sum to begin to amortize the loan.

Let's consider the amortization schedule for the first two months of a loan. Let's also assume that the loan is in the original amount of $100,000, payable monthly over 30 years and bearing interest at the rate of 10% per annum.

Before creating an amortization schedule on the calculator, let's construct a **hand-made** amortization schedule for the first two months so that you can understand the process.

	Beginning Bal. of loan	**Payment**	**Interest**	**Paydown**	**Remaining Balance**
(1)	100,000.00	877.57	833.33	44.24	99,955.76
(2)	99,955.76	877,57	832.96	44.61	99,911.15

14 Contrast this to lease payments which are typically paid in advance.

15 Or administrating authority

16 These interest charges will appear on the buyer's closing statement and not on the lender's mortgage statement.

17 A tax deduction frequently overlooked.

Here's how these numbers were determined:

The **Interest** for the first period is equal to $100,000 x (.10 ÷ 12.) (The interest rate per month = 0.10 ÷ 12)

Key-in	Display Shows	Comment
[f] 4	**0.0000**	**Set decimal places to 4**
100000 [Enter]	**100,000.0000**	
0.10 [Enter]	**0.1000**	
12 [÷]	**0.0083**[18]	
[x]	**833.3333**	**Interest for month #1**

But the payment made was $877.57. Therefore the excess, $44.24 ($877.57 – 833.33), was available to reduce the **Beginning Balance** of the mortgage: $100,000 – 44.24 = $99,955.76, Continuing,

877.57 [x<>y] (3,4)	**833.3333**	
[-]	**44.2367Equity (paydown) portion**	

The amount by which the mortgage is reduced, $44.24, is referred to by some as the **loan paydown**, and by others as the owner's **equity buildup**. Both terms are common and either is acceptable.

The Beginning Balance of the mortgage on the first day of the *second* month is equal to the Remaining (ending) Balance from the previous month. The payment for the second period does not change. But the interest for the second month is less because the **P**resent **V**alue of the loan has been reduced.

Key-In	Display Shows
f 2	**Returns decimal setting to 2 places**
99,955.76 [Enter]	**99,955.76**
0.10 [Enter]	**0.10**
12 [÷]	**0.01**
[x]	**832.96**
877.57 [x<>y]	**832.96**
[-]	**44.61**

18 Decimal is set to 4 places to show interest rate to be 0.0083 rather than 0.01

Therefore the Remaining Balance after the second PMT = ($99,955.76 – 44.61) = $99,911.15. This sequence – which can easily be entered into a computer spreadsheet – continues in exactly the same fashion until the last **PMT** when the remaining balance should be zero.[19]

Amortization Schedules Using the HP-12C

The 12-C makes developing amortization schedules easy.

Problem:

> Recreate the two-period amortization schedule above using the calculator.

First, set up the loan in the calculator and *solve* for **PMT.** *Solving* for the **PMT** rather than keying-in the **PMT** improves the accuracy because the machine will calculate the **PMT** using 9 decimal places. You would probably key-in only 2.

n	i	PV	PMT	FV
360	**0.83**	**–100,000.00**	**?**	**0**
		solving	**877.57...**	

Now reset $\boxed{n}$ to **0** by simply "writing over" the value (360) which is now in the $\boxed{n}$ register. *Do not change any other register values.*

Page 1-1Key In	Display Shows
0 $\boxed{n}$	**0.00**

Your register should now look like this:

n	i	PV	PMT	FV
0	**0.83**	**–100,000.00**	**877.57**	**0**

To amortize the loan one (1) period, follow these steps:

19 Now that you understand how the loan amortization schedule is constructed, it should be a relatively simple matter for you to reproduce these calculations on a computer spreadsheet and have the capability to deliver a printed copy.

Key In	Display Shows	Comment
1 [f] [amort] (1,1)	833.33	The interest for the first period
[x<>y] (3,4)	44.24	The paydown for the first period
[RCL] [PV]	-99,955.76	The Remaining Bal. after one PMT

To amortize for five (5) *additional* periods:

5 [f] [amort]	4,161.07	The total interest for these five periods
[x<>y] [20]	226.78	The paydown for these five periods
[RCL] [PV]	-99,728.98	The Remaining Bal. after six PMTs

This means that if you want to determine the interest and paydown for the 120th **PMT** in a schedule, and the remaining balance after the 120th **PMT** in the schedule, you must first amortize 119 periods and then amortize one additional period. If you amortize 120 periods directly, the interest you obtain will be for the entire 120 periods, and the paydown will be the entire paydown for 120 periods. The remaining balance, however, would be correct.

Determining Remaining Balances

If you have no need to know the accrued interest amount, but simply want to know the remaining balance of a loan, there are two financial short-cuts you may want to use. These depend on the following two facts:

1. The remaining balance of an amortizing loan is equal to the **FV** of all the **made PMTs**, and,
2. The remaining balance of an amortizing loan is equal to the **PV** of all the **unmade PMTs**.

Problem:

> What is the remaining balance after the 36th **PMT** on a loan in the original amount of $100,000, amortized over 30 years, bearing interest at the annual rate of 10% and payable monthly?

Set up this loan in your calculator as before:

n	i	PV	PMT	FV
360	0.83	–100,000		0
			877.57…	

[20] This is key (3,4)

Since 36 **PMT**s will have been **made**, follow these simple steps:

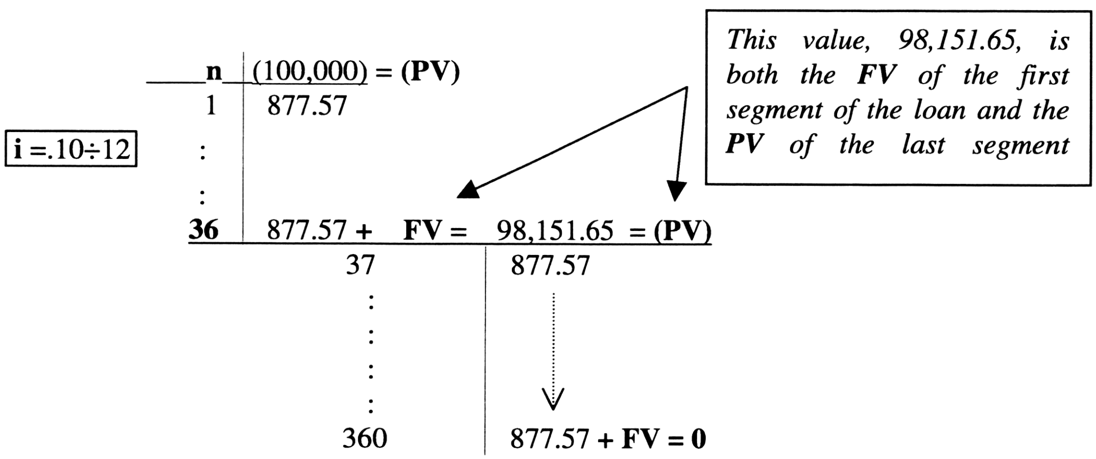
n	i	PV	PMT	FV
36	**0.83**	**-100,000**	**877.57…**	**?**
			Solving…	**-98,151.65**

Or, since 324 (360-36) **PMT**s remain **un**made, follow this solution *after resetting the calculator:*

n	i	PV	PMT	FV
324	**0.83**	**?**	**877.57…**	**0**
	Solving…	**-98,151.65**		

Perhaps the following T-Bars will help show you why both these methods are correct.

The first T-Bar represents a $100,000 loan on which PMTs have been made for three years (36 **PMT**s). The second segment of the T-Bar, represents the **PMT**s needed to be made over the remaining life of the loan (360 – 36 = 324).



As you can see, the Remaining Balance (FV) of the loan at the end of 36 months is $98,151.65. This number can also be considered to be the Beginning Balance (PV) of a 'new' loan, amortized over the remaining schedule (n = 324) at the same interest rate.

This structure will be very meaningful when we consider **Adjustable Rate Mortgages**, since we could easily change the interest rate at the end of the 36th period and calculate a new **PMT** to carry on the amortization to zero. This is, in fact, exactly what the lender does when it changes the interest rate on an Adjustable Rate Mortgage.

Section Summary

1. An **amortizing** loan is one in which the payment made will eventually reduce the balance of the loan to zero.
2. Some amortizing loans feature fixed payments for the life of the loan. The payments remain constant because the interest rate never changes. These loans are **fixed-rate loans**. Other amortizing loans vary the payments to accommodate changing interest rates. The latter are known as **ARM**s (Adjustable-Rate Mortgages).
3. The amortization period (schedule) represents the number of periods, **n**, over which a payment must be made to reduce the loan balance to zero.
4. Some loans call for at least one payment which is at least twice as large as the any of the preceding six payments in the schedule. These payments are known as **balloon** payments. Balloon payments introduce legal notification requirements upon the lender.
5. Amortization *schedules* chronicle the progress of the loan and depict the beginning balance, the payment made, amount of interest paid per period, equity buildup per period and remaining balance.
6. The **remaining balance** of an amortizing loan is equal to the p**resent value of all the unmade payments.**
7. The **remaining balance** of an amortizing loan is also equal to the **future value of all the made payments.**
8. Interest on amortizing loans is usually paid **in arrears.**

II-The Adjustable–Rate Mortgage

The adjustable–rate mortgage (**ARM**) is a financial device by which the lender transfers to the borrower most, but not all, of the risk of an increase in future interest rates.

It is the **interest** rate which is adjusted in an **ARM**. But in order to maintain the same amortization schedule, the **PMT** also changes to keep the loan on its course to a zero balance over the years specified in the original schedule of the loan.

The lender, in constructing this type mortgage, selects an interest rate base **Index**, then adds to the index a **margin**, or **spread**, to arrive at the total interest rate. The interest rate on an **ARM** mortgage is always equal to the index rate plus the spread rate.

Periodically, the lender adjusts the total interest rate applied to the loan to reflect changes in the underlying base index rate which, in turn, reflect changes in market rates of interest. In this way, the total **ARM** interest rate increases when the base index rate rises, and decreases when the base index rate falls. But in either case, the lender is always assured of a constant spread, or margin, regardless of the ups and downs in market interest rates. As the total interest rate changes, the monthly amount necessary to amortize the mortgage over the years remaining in the original schedule also changes. The actual change in the **PMT**, however, may lag the change in the interest rate in order to permit the borrower to follow some semblance of a budget.

This temporary discrepancy between the amount of the payment necessary to amortize the loan under the original schedule and the amount actually paid each month may create negative variations in the balance of the mortgage as compared to a fixed–rate mortgage. For example, when rates rise, the **PMT** made for a number of months may be insufficient to continue to amortize the remaining balance of the mortgage. Rather than decrease, the balance of the mortgage may actually (though not always) increase. This increase is sometimes referred to as a **negatively amortizing mortgage.**[1]

Because most of the risk of rising interest rates is transferred to the borrower, issuers of **ARM** mortgages are generally content with a lower total interest rate. If market interest rates rise, the **ARM** lender can pass along the increase to the borrower according to the terms of the **ARM** loan document, whereas in the case of a fixed-rate mortgage, the lender must absorb the added cost of increasing market interest rates and therefore settle for a lower profit, or even a loss, on his mortgage investment.

If a borrower opts for a fixed-rate mortgage, the lender will increase the total interest rate to provide room for the possibility of a future increase in market rates. If the borrower is willing to accept an **ARM** – which involves the lender's right to change the interest rate – the lender can afford to charge a lower rate because he is not exposed to the risk of higher rates.

[1] The term "negatively amortizing" is an oxymoron.

Selection of a Base Index

Lenders of Adjustable Rate Loans are required to identify in the loan document the Base Index which will determine the total interest charged. The lender is also restricted in the amount of the interest rate by the imposition of "caps" in the spread or margin he may apply. ARMs carry lifetime caps on the rate as well as annual caps.

Different lenders select different Indices to use as the **Base Index**. The following list depicts some of the Indices which are in current use by many **ARM** lenders:

1-year Treasury Bill	a U.S. Treasury security maturing in 1 year
3-year Treasury Note	a U.S. Treasury security maturing in 3 years
5-year Treasury Note	a U.S. Treasury security maturing in 5 years
10-year Treasury Bond	a U.S. Treasury security maturing in 10 years
National Average Mortgage Rates	Published by Freddie Mac[2]
LIBOR rate	the average of interbank offered rates for dollar deposits in the London market based on quotations at five major banks.[3]
Cost Of Funds Index (COFI)	the average cost of deposits to financial institutions affiliated with the Federal Home Loan Bank Board located within a designated District or region.
Index of U.S. Treasury Securities Adjusted to Constant Maturity of One Year	Average yield on a basketful of Treasury notes that will fall due over the coming year. Used in approx. 50% of ARMs.

Many other Indices are possible and a few others are used. In some cases, which are almost always restricted to commercial loans, the index used is the "prime rate."[4] But for the most part, commercial mortgage rates are tied to U.S. Treasuries of a comparable maturity.

The point is that each index establishes a base reference, or *link*, to current market interest rates. The lender's spread is added to the base to result in the effective rate for the **ARM** instrument. Since the indices vary quite a bit, the amount of the spread will also vary in order for the effective interest rate to reach competitive market levels. Therefore one cannot judge the acceptability of an **ARM** mortgage rate solely by the index used nor by the amount of the spread. The higher the index, the lower the spread; the lower the index, the higher the spread. The lender will add sufficient spread to any index it chooses to bring the total interest charged to the level of current, comparable mortgage rates.

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2 Freddie Mac = Federal Home Loan Mortgage Corporation

3 The banks are: Bank of America, Barclays, Swiss Bank, Deutsche Bank, Bank of Tokyo

4 The interest rate which lenders extend to major customers.

The choice of the Index should be made by other parameters. Some lenders prefer to index their **ARM** rates to short-term **T**(reasury)-**Bills**. The advantage here is that short term rates respond rapidly to changes in market interest rates. When interest rates are rising, short term T-Bill rates will rise quickly. A mortgage rate tied to these T-Bills will also rise rapidly, preventing any significant loss in market-rate interest to the lender.

But mortgage rates and payments which change rapidly, and sometimes sharply, wreak havoc with the average borrower's budgeting process. Surveys have shown that most informed consumers prefer to have their mortgage tied to an index which changes slowly – either up or down. Gradual changes help in budgeting.

The longer term Treasury **Notes** and the 10-year Treasury **Bond** tend to be somewhat less volatile, but the best index to select to achieve low volatility is the **Cost of Funds Index** (COFI). This Index is the average of the last four months of the cost of depository funds acquired by member institutions in the particular Federal Home Loan Bank District. In California, for example, the 11th District Federal Home Loan Bank's cost of funds is used; hence the acronym, 11th District **COFI**. The COFI is the moving average of the last four months of interest rates paid by institutions to attract funds. When a new rate becomes available, the oldest of the four monthly rates is dropped and the latest added. In this way, the curve is smoothed considerably. This explains why interest rates on **COFI-ARM**s do not decline as rapidly when general market rates are declining. But the converse is also true: rates tied to COFI also do not rise as rapidly when general rates are rising, and in some cases may even continue to decline for a short time.

It is difficult to construct a prospective amortization schedule for an **ARM** because the changes in interest rate cannot be predicted. The best one can do is to assume that the initial *stabilized* rate will be constant, and then to develop a schedule using that rate. This is not to say, however, that an ARM mortgage should not be examined for the payments which would become due under the most severe interest rate increases permissible under the terms of the loan. It should, and that technique is covered below.

Teaser Rates

We say adjustable mortgage *stabilized* rate, because some lenders, as an inducement to the borrower to take an **ARM** rather than a fixed-rate loan, will offer an introductory **teaser rate**. This rate is often substantially below the rate which currently applies to borrowers within the portfolio of the same lender. Teaser rates are marketing ploys and most often do not last more than a few (1-6) months. In order to curb abusive and misleading use of teaser rates, many states have now enacted laws prohibiting the use of teaser rates which understate the stabilized rate by more than a small margin (1-2 percentage points). Many lenders have abandoned teaser rates altogether.

It is possible, retrospectively, to reconstruct the progress of an ARM; but in order to do so you must have the initial terms of the loan (index used, spread, amortization schedule), and the months during which the interest rate changed and the months in which the **PMT** changed. No financial calculator can do this. It is even a daunting task on a computer.

ARMS Subject To Significant Lender Errors

As the result of studies by the General Accounting Office, Office of Thrift Supervision, National Credit Union Association and others, it has been estimated that mortgage lenders in the United States may have already overcharged borrowers as much as $10 billion in payments connected with **ARMs**. Most overpayments are associated with a lender's applying the wrong Index or the wrong spread rate to a mortgage. Those who have the skill to monitor the periodic changes in an adjustable rate mortgage are well advised to do so.

Tracking the Progress of an ARM

In order to demonstrate, as best we can, the amortization of an ARM, let's consider a brand new ARM offered under the following terms:

Principal Amount (**PV**)	$100,000	
Amortization Schedule (**n**)	30 years	
Beginning (teaser) Rate (**i**)	5.5%	Note that this is less than Index + spread rate
Index Rate	4.0%	
Spread	2.5%	(250 basis points[5])

A reading of the loan agreement indicates that the Beginning Rate will apply for the first 3 months of the loan. Beginning with the 4th month, and every month thereafter, the interest rate will adjust to the sum of the index rate + spread rate. The **PMT**, however, will remain constant for the first 6 months and then adjust in the seventh month and every six months thereafter to that amount necessary to amortize the then remaining balance of the loan over the time remaining in the *original* schedule.

This is a very typical ARM but you must be careful not to assume that all ARMs are the same. The variations are almost endless and *only a careful reading of the terms of the loan document can reveal the exact terms of any particular mortgage instrument.*

The initial PMT is determined as it would be for a fixed-rate loan:

n	i	PV	Pmt	FV
360	5.5÷12	-100,000	?	0
			567.79	

5 A basis point = 1/100 of one percent. Therefore there are 100 basis points in 1%.

The amortization schedule follows the typical format:

Period	Beginning Balance.	Payment	Interest	Paydown	Remaining Balance
1	100,000.00	$567.79	458.33	$109.46	$99,890.54
2	$99,890.54	$567.79	457.83	$109.96	$99,780.59
3	$99,780.59	$567.79	457.33	$110.46	$99,670.13

At this point in the progress of the loan, the interest rate will adjust upward to 6.5%, although the **PMT** will not yet change. Note the effect on the Interest paid, the Paydown and the progress of the Remaining Balance. To accomplish this on the calculator enter the new interest rate but do not resolve for PMT; then continue with the amortization procedure.

4	$99,670.13	$567.79	539.88	$27.91	$99,642.22
5	$99,642.22	$567.79	539.73	$28.06	$99,614.16
6	$99,614.16	$567.79	539.58	$28.21	$99,585.94

At this point the loan will be *recast* in order that it may continue to be amortized over the remaining 354 months in the schedule. Let's assume that the interest rate also changes in this month and increases from 6.5% to 6.625%. Again, note the effect on the Interest paid, the Paydown and the Remaining Balance.

n	i	PV	Pmt	FV
354	6.625÷12	-99,585.94	?	0
			641.10	

Amortize the loan one additional period to show:

7	$99,585.94	$641.10	549.80	$91.30	$99,494.64

Note: Amortizing the loan using Excel will result in very small differences in remaining balances.

You can see that determining the balance of an amortizing loan which has been amortizing for a number of years can be a daunting task. No doubt that the HP–12C could be programmed to handle an amortizing loan given entry stops to supply the various new rates and the interval of the PMTs. Except for some unique and unusual purpose, perhaps, such an exercise would be mental gymnastics at best, with little practical value. If the task seriously presents itself (and it occasionally does), it is much more easily accomplished on a computer spreadsheet which can be saved and printed..

Maximum Interest Rate Exposure - "Caps"

ARMs transfer most, but not all, of the of the interest rate risk. Most **ARM** lenders offer "caps" on the maximum interest rate which can ever be charged on the loan.[6] These "caps," known as "lifetime caps" are commonly in the order of 5-6% over the starting rate but can, and do, differ from lender to lender.

Many clients are interested to know at the outset the maximum amount of payment to which they could ever be exposed if interest rates increased dramatically. Since most ARMs limit the maximum annual increase in interest rates (usually to 2%), it is unlikely that the payor on the note would be faced with the need to pay the maximum interest rate increase represented by the lifetime "cap" in any one year.

For example, if the ARM given as an example above carried a lifetime (maximum) interest rate cap of 6% over the starting rate (5.5%) the payor could never be required to service the loan with an interest rate greater than 11.5% (6% + 5.5%). But even if the interest rates increased dramatically in any one year, the maximum *annual* increase would be limited to 7.5% (5.5% + 2.0%). The lender could, and would, apply increases the following year, and the year after that, in order to reach the 11.5% maximum limit. For this reason, it is possible than payments on an ARM may not decline even though the underlying Base Index has declined.

In order to determine the maximum exposure, in terms of monthly **PMT**s under the loan, we need only to substitute the maximum chargeable interest rate for the starting rate:

n	i	PV	Pmt	FV
360	11.5÷12	-100,000	?	0
			990.29	

Limit On Payment Increases Under ARMs

And additional safeguard available to the **ARM** borrower is a "cap" on the amount by which the annual **PMT** may increase, regardless of the increase in the interest rate. For example, in our example **ARM** the interest rate *could* increase as much as 2% in any one year.

Therefore the **PMT** based on this increase in interest rate would become:

n	i	PV	Pmt	FV
360	0.625	-100,000	?	0
			699.21	

6 In some states the lender is required by law to place a lifetime cap on the interest rate which may be charged on this type loan.

In this calculation we have added the annual cap in the interest rate, 2.0%, to the starting rate of 5.5%. Our monthly interest would be (5.5%+2.0%)/ 12 = 0.625%

But even this rate produces a **PMT** of $699.21 which is higher that the starting **PMT** of $567.79. How much higher?

Key–In	Display Shows
567.79 [Enter]	567.79
699.21 [Δ%] (2,4)	23.15%

Most **ARM**s specify that the **PMT** may not be increased more than 7.5% in any one year. Since 23.15% is far greater than the maximum allowable percentage increase in the **PMT**, the 7.5% limit , in this case, determines the maximum **PMT**:

Key–In	Display Shows
567.79 [Enter]	567.79
1.075 [x] (2,10)	610.37

Therefore the maximum allowable **PMT** for that year will be $610.37, which is 7.5% higher than the previous **PMT**.

Does that mean that the interest rate is also limited? No! The interest rate will remain at 7.5% (5.5% + 2.0%). The question is: "Is this higher **PMT** sufficient to amortize the loan over the remaining months in the schedule?"

Let's determine the effect on the loan if this dramatic increase occurred at the end of the second month (refer to amortization schedule above):

Period	Beginning Balance	Payment	Interest	Paydown	Remaining Balance
1	100,000.00	567.79	458.33	109.46	99,890.54
2	99,890.54	567.79	458.33	109.96	**99,780.58**
3	99,780.58	**610.37**	**623.63**	**-13.26**	**99,793.84**

In this situation, the maximum allowable **PMT** is insufficient to amortize the loan. The unpaid interest ($13.26) is added to the previous balance (of $99,780.58) resulting in a Remaining Balance at the end of the third period which is higher than the loan balance at the beginning of the same month. This is referred to as "negative amortization."

When this occurs, the borrower usually has a number of remedies: 1) he has the right to voluntarily increase the amount of the monthly **PMT**, or 2) to pay off the remaining balance of the loan without fees or penalties (to refinance), or 3) to convert to a fixed-rate mortgage, or 4) to increase the schedule of the loan not to exceed 40 years.

ARM PMTs May Not Decline In Step With Rates

ARM payments may not decline even when interest rates decline if an annual "cap" in the interest rate has restricted the lender's right to collect the full amount of an increased **PMT**, as the result of past increases in interest rates. The **PMT** may even *increase* under these circumstances until the lender has recovered all the interest which he was not able to collect due to the annual "cap." Information regarding this important point will be contained in the loan documents the borrower is asked to sign.

Recasting an ARM

In order to protect the lender against loans which could possibly grow to amounts which threaten the lender's security in the property, most ARMs specify an absolute limit beyond which the Remaining Balance may not grow, e.g. 100% to 125% of the original amount. If this limit is reached, the lender has the right to "recast" the loan and start all over again with a higher **PMT**.

In most cases, the ARM is recast over the months remaining in the original schedule of the loan, but some lenders may be amenable to recasting the loan over a period equal to the original amortization schedule. Should this occur, the remedies (above) available to the borrower apply.

When To Choose an ARM Over a Fixed-Rate Mortgage

If one could predict the future of interest rates, it would be a relatively simple matter to choose between an **ARM** and a fixed-rate mortgage.

Interest rates are affected by a myriad number of economic factors, but invariably begin to rise with recovery from economic slumps. As the economic recovery becomes more robust, demand for money increases and interest rates tend to rise even more. Eventually demand flattens under the weight of increasingly expensive capital. With the onset of recession, demand for capital usually wanes and interest rates typically recede, bottoming out six to nine months before the end of the recessionary period of the cycle.

Unfortunately, economic cycles, which used to run in approximately four year periods, have not been very predictable in the last 10-12 years. The recession that began in the United States in late 1990 did not followed a typical pattern of recovery. Recovery was slow and erratic, and interest rates remained quite low for a period of years. Since that time, aggressive actions by the Federal Reserve Bank have modulated both the downward and upward drift in interest rates.

It is probably a wise course to make a judgment regarding the selection of an **ARM** first by measuring your ability to sustain an increase in mortgage rates to the level represented at the maximum permissible interest rate and payment caps. If these increases could not be met, an **ARM** should not be seriously considered. If these caps can be met, then one may elect to gamble on the future direction of interest rates depending on where you believe you are in the present economic cycle. If you believe the economy is set for sustained growth, you may wish to "lock in" a low, fixed-rate mortgage. If you believe the economy is entering into a recessionary period, you may wish to gamble on a future interest rate decline by selecting an **ARM**.

For more specific information regarding ARMs, consult "Consumer Handbook on Adjustable Mortgages," available from any lending institution which is a member of the Federal Reserve Board or the Federal Home Loan Bank Board.

Segment Summary

1. ARMs are mortgages which are designed to transfer a portion of the risk of rising interest rates to the borrower rather than to the lender. By guaranteeing itself a "spread" over the underlying Index rate, the lender locks-in a dependable profit margin.
2. ARMs provide for periodic adjustments to the effective interest rate by linking the mortgage rate to an index reflective of general interest rates in the economy.
3. Lenders use a number of different indices as the base for their mortgage rate computations, but those indices which are slower to change afford more stability to the borrower and improve the ability to budget outflows.
4. Interest rates in ARMs contain "caps" which limit the annual and also the absolute increase the lender may charge over the entire life of the loan.
5. Payment changes, which are generally subject to change every six to twelve months, are also subject to "caps." Many ARMs limit the annual change in a payment to a maximum increase of 7.5% over the highest payment in the preceding 12 months.
6. If a limit on a payment results in a payment too low to pay the higher interest on the debt, the unpaid interest will be added to principal.
7. "Teaser" rates are low rates made by the lender to induce a borrower to select an ARM. These rates seldom last more than a few months.
8. It is difficult to check on the historical accuracy of a lender's adjustments to an existing ARM. Therefore most professionals should be able to confirm the accuracy of each adjustment to the payment and the interest rate as they occur. Keeping track of an ARM on a computer spread sheet would be an excellent method of double-checking for accuracy.
10. The selection of an ARM instead of a fixed-rate mortgage is an exercise in predicting the future of interest rates.

III Buydown Mortgages

Economic cycles in the United States, which once averaged four years, now appear to be in the process of setting new patterns. Nevertheless, as interest rates rise approaching the peak of the economic cycle, real estate (an interest-rate sensitive industry) begins to feel the pinch, and property sales typically decline. Buydown Mortgages are particularly useful during the part of the economic cycle when interest rates are high and when the buyer's ability to qualify for a loan may be temporarily impaired.

A Buydown Mortgage is a common marketing device originally conceived by developers who agreed to supplement a buyer's mortgage payments for a specified period of time. The amount contributed is funded from the seller's profits, so it is not uncommon to see prices raised slightly to make room for a Buydown. Buydowns are no longer the exclusive marketing tool of the developer; lenders now offer these mortgages to buyers seeking to reduce their monthly payments. Home sellers, too, have found that a buydown can often improve the marketability of their residential properties at a time when high interest rates make a sale difficult.

3-2-1s ... and other Arrangements

A "3-2-1" Buydown, for example, indicates that the seller will "Buydown" the rate of the mortgage the equivalent of 3 interest rate points[1] the first period, 2 interest rate points the second period, and 1 interest rate point the third period. The "period" typically is one year, but some builders arrange Buydowns in which the Buydown period is only six months. A "2-1" Buydown indicates that the seller will contribute the cash equivalent of 2 interest points the first period and 1 interest point the second period. Any combination or schedule is possible. There is no such thing as a "standard" Buydown.

A Builder's Buydown

> For example, NewStuds Inc. finishes a tract of $350,000 homes just as the economic cycle peaks and home mortgage interest rates reach 12% per year for a 30-year mortgage. Sales are depressed. The marketing department suggests a 2-1 Buydown to stimulate sales. A 20% down payment is required.
>
> What will be the buyer's payment , and how much will the builder contribute?

Under this arrangement, the lender will still receive payments reflecting a mortgage bearing 12% annual interest, payable monthly. However, the builder will contribute sufficient cash to reduce the monthly payment for the buyer *to the equivalent of* a 10% (12%-2%) mortgage the first year

[1] A "point" usually refers to one percent of the loan amount. But here we mean a reduction of 1% in the interest rate.

and an 11% (12%-1%) mortgage the second year. Beginning with the third year, the buyer will shoulder the **PMT**s alone.

The 3 T-Bars for the NewStuds Inc. mortgage Buydown look like this:

Months	Bank Receives		Buyer Pays		Builder Pays
	(280,000)		(280,000)		0
1	**2,880.12**	1	**2,457.20**	1	**422.92**
:	:	:	:	:	:
12	2,880.12	12	2,457.20	12	422.92
13	2,880.12	13	2,666.51	13	213.61
:	:	:	:	:	:
24	2,880.12+ 277,839	24	2,666.51+ 277,839	24	213.61
25	2,880.12	25	2,880.12	25	0

As you can see, during the first year (12 payments) the buyer makes payments on his mortgage as though the interest rate were 10% per annum. The builder supplies the balance of the monthly payment necessary to support the actual 12% interest-rate payment which the lender receives. At the beginning of the second year, the buyer's monthly payment increases to an amount necessary to support an 11% mortgage. The builder continues to supplement the payment to a 12% level. At the beginning of the third year (payment #25), the Buydown ends and the buyer assumes the full burden of the payments at the interest rate stated in the note.

During the Buydown period, the balance of the mortgage declines as though no Buydown were in place. The effective interest rate enjoyed by the Buyer is determined by the middle T-Bar: in this case, IRR = 10.61%. In order to calculate this IRR, calculate the remaining balance on the bank's loan at EOM_{24}. Using this loan balance, compute the IRR for the Buyer's cashflow series .

Some Buydowns are constructed using **ARM**s rather than fixed-rate mortgages in the anticipation that the economic cycle will have turned 180^{o} and interest rates will generally be lower when the Buydown expires.

Developer "Buys Out"

Most banks and all prudent buyers recognize that some developers/sellers may not be around for two years to continue to make these supplementary payments.[2] Therefore the developer pays an up-front, lump sum amount to the lender to implement the Buydown. This lump sum is the **P**resent **V**alue of the payments which the builder is obligated to make to the lender under the Buydown arrangement, discounted at a rate acceptable to the lender. If the lender, in the example above, discounted the builder's future **PMT**s using **i = 12%**, the mortgage rate, the lender may break even. If the lender discounts the builder's payments by something *less than* 12%, the

[2] Developers have a high mortality risk

Present Value (amount demanded of the builder) will be higher, and the lender may make an additional profit on the Buydown.

Determining Builder's Paydown

A T-Bar constructed to represent the builder's payments over the two year period looks like:

		(PV ?)
	1	422.92
	:	:
i = 12/12	12	422.92
	13	213.61
	:	:
	24	213.61

The keystrokes for the solution are as for any uneven cashflow:

	Key In	Display Shows
	f CLX	**0.00**
	0 g CFo	**0.00**
	422.92 g CFj	**422.92**
	12 g Nj	**12.00**
	213.61 g CFj	**213.61**
	12 g Nj	**12.00**
	12 g i	**1.00**
solving	**$\boxed{f}$ NPV**	**6,893.60**

Therefore the builder may be relieved of his future obligation to supplement the mortgage under the Buydown contract by making a lump sum payment to the lender in the amount of $6,893.60. If the lender agrees to discount the builder's obligation by 10% per annum, calculated for monthly payments, he would be required to pay the higher Present Value of all future obligations.

	Key In	Display Shows
	10 g i	**0.83**
solving		
	f NPV	**7,009.91**

Since Buydowns may be tied to **ARMs**, lenders may refuse to discount the future payments by the current market interest rate on the loan in order hedge a potential increase in market interest rates. Therefore the amount payable would be the sum total of all the future contributions.

Buydowns – A Small Price To Pay

As a percentage of the selling price, Buydown mortgages are often a small price to pay to make an otherwise difficult-to-sell property financially attractive. A reduction in price of less than 3 or

4% cannot be expected to have much impact on a potential buyer, and a price reduction always has a measurable impact on profits since all the reduction comes from the owner's equity. But the cost of a Buydown is often much less than the cost of an effective price reduction, and often more effective.

For example, in the case above, what percent of the selling price of $350,000 is represented by the Buydown amount for a 12% discount rate from the lender?

Key In	**Display Shows**
350000 **[Enter]**	**350,000.00**
6893.60 **[%T]** (1,3)	**1.97**

Buydown Available to Private Homesellers

While most Buydowns are used by developers and builders, there is no reason why they cannot also be used by the individual owner/seller to help make a property more marketable, especially during times of high interest rates. Many lenders now offer such programs for buyers of resale properties. As you can see, a Buydown such as the 2-1 Buydown depicted above represents only a 1.97% reduction in price, yet may be far more effective in obtaining a sale than an equivalent price reduction because lenders are willing to qualify the buyer at the beginning rate.

There is no reason why a Buydown could not also be used with seller-carry-back financing. But in those cases in which a homeseller wishes to do this, professional assistance in drawing the note and terms of the purchase agreement should be seriously considered.

Knowing how to analyze the lender's Buydown program enables the financial professional to help either the buyer or the seller to select the most favorable Buydown mortgage.

A Variation on the Buydown Theme

Certain lenders have created "Buydowns" which entail "no Buydown costs and no points." They also claim "no deferred interest added to your loan balance."

Consider a fixed-rate mortgage carrying a current interest rate of 7.75%, payable monthly over a 30–year term. In order to convert this fixed-payment, market-rate loan to a 2-1 Buydown in which the "lender pays the Buydown," the lender establishes a starter rate of 6.75% for the first year, then adds 1 interest point for second year. Beginning with the third year, and continuing for the remaining period of the original schedule, the interest rate is set at 2 points over the starting rate, or 8.75% (one full interest point over the current rate). The final payment schedule is based on these rates:

First year rate	6.75%
Second year rate	7.75%
Remaining years' rate	8.75%

Since there are no points and no fees, it is a relatively simple matter to calculate the Annual Percentage Rate (APR) or this loan.[3] Let's assume $100,000 of loan amount and determine what the payments will be for each year of the loan:

Keystrokes	Display	Comment
100000 CHS PV	**-100,000**	**Original amount of loan**
6.75 g i	**0.56**	**Enters interest rate, first yr.**
30 g n	**360.00**	**Amortization period**
0 FV	**0.00**	**Fully amortized loan**
PMT	**648.60**	**Payment/month, year 1**
12 n FV	**98,934.25**	**Remaining balance, year 1**
CHS PV	**-98,934.25**	**Beginning balance, year 2**
7.75 g i	**0.65**	**Interest rate, year 2**
348 n	**348.00**	**Amortization time remaining**
0 FV	**0.00**	**Fully amortized**
PMT	**715.05**	**Payment/month, year 2**
12 n FV	**97,987.87**	**Remaining balance, year 2**
CHS PV	**-97,987.87**	**Beginning balance, year 3**
8.75 g i	**0.73**	**Enters interest rate, yrs. 28-30**
28 g n	**336.00**	**Amortization time remaining**
0 FV	**0.00**	**Fully amortized**
PMT	**782.63**	**Payment/month, years 28-30**

Having identified what the payments will be for each year of the "Buydown," we can enter these PMTs into an uneven cashflow format and solve for the Internal Rate of Return.

	Key In	Display Shows	Comment
	100000 CHS g CFo	**-100,000.00**	**Starting balance of loan**
	648.60 g CFj	**648.60**	**Year 1 PMT**
	12 g Nj	**12.00**	**12 PMTs first year**
	715.05 g CFj	**715.05**	**Year 2 PMT**
	12 g Nj	**12.00**	**12 PMTs second year**
	782.63 g CFj	**782.63**	**PMT for Years 28-30**
	99 g Nj	**99.00**	
	782.63 g CFj	**782.63**	
	99 g Nj	**99.00**	**(Calculator can only store**
	782.63 g CFj	**782.63**	**99 entries in any one cell.)**
	99 g Nj	**99.00**	**Total 99+99+99+39 = 336**
	782.63 g CFj	**782.63**	
	39 g Nj	**39.00**	
	RCL n	**6.00**	**Checks for # of entries**
Solving	**f IRR**	**0.70**	**Monthly yield**
	12 x	**8.44**	**Annual yield = APR**

[3] The true cost of the loan to the borrower, reflecting all loan charges and points paid.

The lender can afford to advertise "no points and no costs" since his yield on this kind of loan is 69 basis points over the current market rate loan. The payments to be made by the borrower under this "Buydown" loan will total $21,419 greater than under the current loan. If the payments due under the "Buydown" loan are discounted at the market-rate interest, 7.75%, the **Net** Present Value of this loan to the lender is $6,980.05, or the approximate equivalent of a 7% loan fee. This is also the premium paid by the borrower, generally for the privilege of qualifying for a loan at a lower starting rate

This type "Buydown" loan.[4] is really a hybrid mortgage combining some features of an ARM (changing payment amounts and changing interest rates) with some features of a Graduated Payment loan (qualifying buyer at a lower starting rate). It differs from the conventional "Buydown" in that it raises interest rates later in the schedule in order to qualify the buyer at a rate slightly below market rates earlier in the schedule. The buyer, though initially qualified at a slightly lower rate for a short time, is committed to make payments at an interest rate above market rates for the remainder of the schedule. It differs from an ARM in that the buyer may take no advantage of a decline in rates.

Segment Summary

1. Buydowns are financial arrangements in which the seller assists in the mortgage payments for a specified period of time to make a property more affordable, especially during periods of high interest rates.
2. The effect to the buyer is to reduce the interest rate on the mortgage for the period of the Buydown. Thereafter, the interest rate in the note becomes the effective rate to the buyer.
3. Builders frequently employ Buydowns and cover some or all expense of the Buydown by small increases in the price of the property.
4. Lenders will require a lump sum payment from the builder/seller. This amount is the Present Value of the future payments the builder/seller is obligated to contribute, discounted at a rate acceptable to both.
5. If the lender discounts the Buydown payments by a low interest rate, the cost to the builder/seller increases, and vice versa.
6. Buydown are also available to individual sellers to attract buyers in a competitive market and during times of high interest rates.

4 These buydown terms were drawn from am actual offering by a major lender.

IV The Graduated Payment Mortgage

The Graduated Payment Mortgage is the most interesting of all commonly employed mortgage instruments, not because it is the most frequently used or the most complex, but because it affords the finance professional an excellent insight into cashflow concepts offering the greatest potential to craft innovative financial plans and other divers financial instruments to custom-fit problem situations. Understanding Graduate Payment Mortgages will also help one understand how to craft Graduated Payment Annuities as described in Chapter 3.

Why a Loan is Worth What It Is

By now the reader appreciates that a mortgage *is* an annuity, and the value of the annuity is the **PV** of all the **PMTs** due under the schedule of the promissory note, discounted at the rate **i.** The sum of the **PV** of each individual future **PMT** will always equal the **PV** of the loan.

To illustrate, calculate the **PV** of a **PMT** in the amount of $877.5715701 over a period of 360 months, discounted at the rate of 10% per year on a monthly basis:

n	i	PV	PMT	FV
360	0.833...	?	**877.5715701...**	0
	Solving...	**100,000**		

This loan has a principal value of $100,000 precisely because this amount is the total **P**resent **V**alue of all future payments, discounted at the indicated rate, **i,** over 360 periods.

In the fixed-rate amortizing loan, we held the **PMT**s constant; in the **ARM** we varied the **PMT**s according to periodic changes in the interest rate, such that the balance of the loan at any point in its schedule would be reduced to zero by a periodically adjusted **PMT**. The GPM loan, however, is different.

The GPM decides beforehand the amount and timing of the periodic changes in the **PMT**, and – given these changes – the amount of principal (**PV**) which the **PMT**s of the loan will support. Let's illustrate the GPM by a simple example.

> Suppose that you desire to create a loan instrument which will fully amortize a principal over 10 years with monthly **PMT**s discounted at an annual rate of 9% applied monthly. The borrower, however, will have a reduced ability to manage a fixed-rate **PMT** during the early years of the loan, and therefore requires a lower initial **PMT**. But the borrower can manage a 5% increase each year for years 2-5 (inclusive) of the loan on a reasonable, initial **PMT** amount.
> What initial and subsequent **PMTs** will fully amortize this loan, given five scheduled increases of 5% in the **PMT**s, over a 10-year term?

Unity (1) is a convenient assumption in many math problems, so let's use it here. Let's suppose that the first PMT is $1.00. If we increase the original PMT 5% each year for five years and then hold the PMT steady for the remainder of the ten–year schedule, we can determine the amount of PV which this $1 of PMT, periodically increased as stipulated, will support. Our T-Bar results in an uneven cashflow which would look like this:

		(PV) = ?
	Month # 1-12	1.00
	13-24	1.05
i = .09 ÷ 12	**25-36**	1.1025
	37-48	1.157625
	49-60	1.21550625
	61-72	1.276281563
	73-120	1.276281563

This series of cashflows begins with a $1.00 PMT. Each 12 periods (months), the PMT is increased 5%, for five successive increases. This is one of the formats used in widely available Graduated Payment Mortgages designed by FHA and VA lenders. Since you now know how to discount uneven cash flows these keystrokes should be easy to follow:

Key In	Display Shows	Comment
f CLX	**0.00**	**Clears all memory**
0 g CFo	**0.00**	**Set n counter to zero**
1 g CFj	**1.00**	**Enters first cashflow**
12 g Nj	**12.00**	**Enters number of times cashflow occurs**
1.05 g CFj	**1.05**	**Enters second cashflow**
12 g Nj	**12.00**	**Enters number of times cashflow occurs**
1.1025 g CFj	**1.10**[1]	**Enters third cashflow**
12 g Nj	**12.00**	**Enters number of times cashflow occurs**
1.157625 g CFj	**1.16**	**Enters third cashflow**
12 g Nj	**12.00**	**Enters number of times cashflow occurs**
1.2155065 g CFj	**1.22**	**Enters fourth cashflow**
12 g Nj	**12.00**	**Enters number of times cashflow occurs**
1.276281563 g CFj	**1.28**	**Enters last cashflow**
60 g Nj	**60.00**	**Enters number of times cashflow occurs**

[1] Assumes your calculator is set to show two places

	Key In	Display Shows	Comment
	9 g i	**0.75**	**Enters discount rate/mo.**
RCL	**n**	**6.00**	**Confirms six different cashflows entered**
solving	**f NPV (1,3)**	**92.04**	**(92.04369870, actually) the PV of this series of cashflows**

We have determined that $1.00 of initial monthly **PMT**, increased 5% per year for five increases, will support a **PV** of $92.04369870 amortized over ten years at a 9% annual interest/discount rate.

If we want to amortize $25,000 of principal under these same terms, the beginning **PMT,** *following the same pattern of increases*, would be exactly proportionate:

$$\frac{\$1.00}{\$92.04....} = \frac{X}{\$25{,}000}.$$

$$X = \frac{\$25{,}000}{\$92.04...} = \$271.61..\quad (\$271.6101195)$$

Determining the Payments

Now that we know the first year's **PMT**, $271.61, we can easily calculate the **PMT**s due under the entire schedule by increasing the initial **PMT** of $271.61 by 5% per year for *five increases*:

Months	Payment per Month
1 - 12	**$271.61**
13 - 24	**285.19**
25 - 36	**299.45**
37 - 48	**314.42**
49 - 60	**330.14**
61 - 120	**346.65**

GPM Loan Factors

Notice that the initial **PMT** of $271.61 is equal, in our example, to $25000 ÷ 92.04, which is the same as multiplying $25,000 by the reciprocal of 92.04.[2] The reciprocal of 92.04 is 1/92.04, or 0.010865.

[2] The reciprocal of a number is equal to 1 ÷ by that number. The reciprocal of 10 is 1/10, or .10

This "factor," 0.010865, is the *kind* supplied by FHA and VA to enable lenders to determine the amount of initial **PMT** for a Graduated Payment Mortgage of a particular amortization schedule and interest rate.[3] They need only multiply the principal amount of the new mortgage times the appropriate GPM factor for the term and interest rate, in order to determine the first payment under the GPM. These factors are called **GPM constants**.

Annual Percentage Increase Can Vary

In this case we increased the annual **PMT** amounts 5% per year, but we could just as easily have increased them 2.5% per year, or 6%, or 7.5% per year.... or any percentage we might have chosen. In fact, we could have suspended an increase in any year and reinstated it in some later year.

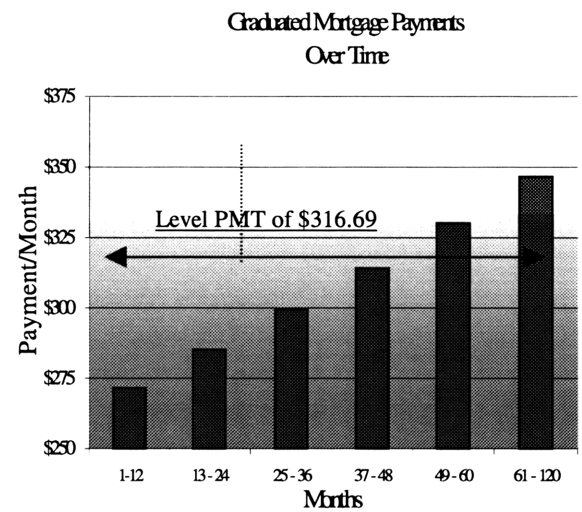
But since most residential loans are resold in the secondary market, managing an infinite variety of GPM loans, all with different percentage steps in the **PMT**s, would be an infinite nightmare for loan servicers. Therefore Federal Housing Authority (FHA)[4] and Veteran's Administration (VA) limit the annual percent increases in the **PMT**s to five basic choices: 2.5%, 5.0% and 7.5% annual increases for 5 years, or 2% and 3% increases over 10 years. Staring with the sixth year of the 5-year plans, and with the eleventh year of the 10-year plans, payments stay the same.

If the annual percent increase is low, the beginning **PMT** will be higher. If the annual percent increase is high, the initial **PMT** will be lower.

You can test the validity of our structured **GPM** loan by constructing a ten-year amortization schedule to determine whether this schedule of increasing **PMT**s will indeed amortize the loan to zero in ten years.

Some Practical Observations

Notice that the first **PMT** of $271.61 in our schedule is significantly below the fixed-rate $316.69 monthly **PMT** which would be needed to amortize $25,000 over 10 years at 9% annual interest. The final **PMT** amount is also measurably greater than what the fixed-rate, constant **PMT** amount would be.



3 In this case we used a very short amortization period for the convenience of the example. Therefore this factor will not apply to loans of a longer amortization period.

4 FHA Graduated payment Loans are known as Section 245 loans.

This means that **GPM** mortgages are best suited to those borrowers who can *reliably* anticipate the increases in the funds needed to service the debt.

GPM loans also allow the borrower to qualify for a larger loan since qualification of the buyer to make the **PMT**s is usually based on the ability to make the first year's **PMT**. As individual borrowers advance along in age, however, their increases in income generally begin to moderate, and income may eventually decline. Therefore it would be entirely inappropriate to place a couple about to retire in a residence utilizing a 7.5% increasing **GPM**. In just a few years, the increasing annual **PMT**s under the **GPM** schedule would likely outstrip their housing budget. So great care must be taken by the professional not to arrange or to approve a financial structure for those whose ability to meet future scheduled payments is problematical.

A second point worth noting is that longer scheduled (25-30 years) **GPM** loans fail to amortize in the early years because the lower payment is insufficient to cover the interest. Therefore the borrower accumulates for a number of years unpaid interest which is added to principal.

Consider, for example, that a $100,000 loan, with monthly **PMT**s including interest at 9% p.a. , and amortized over 30 years, would be fully amortized by a fixed monthly **PMT** of $804.62. The same loan, constructed as a **GPM** with payments increasing 7.5% in the second, third, fourth, fifth and sixth years, would require an initial monthly **PMT** of only $581.69. This loan, however, would fail to amortize in the early years –

Pmt #	**Beginning. Bal.**	**Payment**	**Equity Buildup**	**Ending Bal.**
1	100,000.00	$581.69	(168.31)	100,**168.31**
2	100,168.31	$581.69	(169.58)	100,**337.89**

This "negative amortization" builds more quickly with loans whose **PMT**s have steeper annual increments. For this reason, it is quite common to see GPM loans develop balances during the early life of the loan which are greater than the original balance. If a property subject to a 25-30 year **GPM** loan is to be sold during the first 5-6 years of the loan, the payoff required will be greater than the original loan amount, thereby reducing anticipated equity. This may come as a shock to some owners should they decide to sell.

Negative amortization is generally not a problem in the case of GPM loans amortized over short periods since these loans require **PMTs** large enough to easily exceed the interest due. Therefore there is no negative Equity Buildup.

GPM-ARMs

We have not covered the combination of an **ARM** and a **GPM**, but **GPM-ARM** loans are offered. They are arranged exactly as would be a **GPM** loan except that the interest rate adjusts periodically according to the underlying Index, forcing an adjustment in the remaining **PMT**s. These loans are difficult to program on the HP-12C because the entire loan must be "recast" with every change in the interest rate. Their calculation is best done on a computer, not a calculator, and therefore we will not attempt them here.

Versatility of Concept

The concept of the graduated payment mortgage can be applied to any personal financial plan designed to fund education expenses, retirement expenses or other future planning need. It is best applied to those situations in which the contributor to the plan has an improved earnings outlook in order to meet the increased payments required toward the back end of the schedule.

It is important to note that the ratios constructed involve Present Values and not Future Values. For example, consider the problem of constructing an education plan:

> Total costs of a 4-year private school education today are estimated to be $130,000, and are expected to rise at the rate of 5% per year for the indefinite future.
> A student's parents seek to establish an education fund over the next 10 years. They can increase their contribution to this fund by 5% for the first 5 years of the plan (five increases).
> If you can invest plan funds to yield 9.0% per annum, compounded monthly, what should be the amount of their first contribution beginning today (BOP).

We need to determine the expected cost of the 4-year program in 10 years. While it may be tempting to do so, this is not a situation calling for the use of the Inflation-Adjusted Rate since we are interested in knowing the actual nominal dollars which will be required in the future. The use of the IAR would deliver the purchasing power of the investment in 10 years, which has little to do with paying the bill in nominal dollars.

The cost of the education in 10 years would be:

EOP

n	i	PV	PMT	FV
120	5/12	–130,000	0	?
			Solving...	**214,111**

The required level BOP monthly PMT at the investment rate of 9/12% will be:

BOP

n	i	PV	PMT	FV
120	0.75	0	?	214,111
		Solving...	**-1,098.20**	

Required PV if invested in one lump sum:

n	i	PV	PMT	FV
120	0.75	?	0	214,111
	Solving...	**87,343.96**		

The GPM factor is calculated as before, except that we will remove the first $1.00 of PMT to accommodate a BOP plan. The GPM factor is 92.734..

$$\frac{\$1.00}{92.73....} = \frac{X}{\$87.343.96}$$

$$X = \frac{\$87,343.96}{89.1704...} = \mathbf{\$941.88}$$

This value, $941.88, is 14.2% lower that an even PMT of $1,098.20.
The entire PMT schedule would be:

Months	**PMT**
1-12	$ 941.98
13-24	$ 989.08
25-36	$1,038.53
37-48	$1,090.46
49-60	$1,144.98
61-72	$1,202.23
73-120	$1,262.34

Using a higher annual increase would result in lower PMTs during the early years, but higher PMTs in later years.

Using a 5% annual increase applied annually over the entire schedule reduces the initial PMT to $911.78, a 17% reduction. The 10th year PMT, however, would rise to $1,461.16.

Segment Summary

1. Construction of Graduated Payment financial plans depends on the fact that the Present Value of a cashflow is equal to the sum total of all the individual **PMT**s which can be expected under the schedule, discounted at a rate acceptable to the receiver of the cashflow. **PMTs need not be equal.**
2. Real state and financial planning professionals who make use of **Graduate Payment Annuities** may help qualify buyers for a home mortgage greater than that for which they could qualify under a standard **ARM** or fixed-payment mortgage.
3. Greater annual percentage increases in the **PMT** schedule enable buyers to qualify for a larger mortgage, but subject them to higher monthly payments later on. Smaller annual percentage rates ease the rate at which the initial payment will increase and are therefore less risky. Special care should be taken to ensure that the clients are confident that they will be able to meet the increasing monthly payments required to service the plan or the mortgage.

4. Long-term amortizing **GPM**ortgages result in a negatively progressing loan balance over the first 5 years of the loan.
5. The concept of the **GPM** need not be confined to mortgages but can be applied by financial advisors and planners to special circumstances such as education funds, retirement plans, and other investment programs requiring periodic investments over a number of years. The benefit of using this concept is that many more individuals and families will be able to participate in a graduated payment plan than in a fixed payment plan of comparable Present Value.

V The All-Inclusive Deed Of Trust Mortgage

The All-Inclusive Deed of Trust Mortgage (**AITD**), or "Wraparound Mortgage" as it is sometimes called, is a junior[1] lien whose principal (**PV**) includes (or "wraps around") the principal of one or more senior[2] notes.

The **PMT**s of an AITD also include the amount of the **PMT** or **PMT**s necessary to service the senior obligations. An AITD can wrap around one, some, or all *previous* notes, including other AITDs, secured by the property. An AITD should never be structured to "wrap" notes which are recorded in a position junior to its own.

The principal use of the **AITD** is to preserve the economic benefit of one or more favorable senior mortgages, most often for the benefit of the seller. Although **AITD**s can also be of value to the borrower, they are not commonly structured primarily for the buyer's benefit.

How They Come About

When a property is sold with an existing mortgage in place, the mortgage may either be "*assumed*" by the buyer or the property may be transferred to the buyer "*subject* to" the existing mortgage. Although these terms are sometimes carelessly interchanged, they ought not be, because there is a vast difference between the two. This difference has major legal and economic implications for both seller and buyer, but especially for the buyer.

Consider a property offered for sale for $400,000: The title is encumbered by a first trust deed lien securing a promissory note whose current balance is $150,635. This original note bears interest at the attractive rate of 6% per annum, payable $1,079.91 per month. This loan is all due and payable exactly ten years from now.

Let's further assume that the seller of this property is willing to carry-back part of the purchase price in the form of a note secured by a lien on the property. He requires a qualified buyer who is able to make a $100,000 down payment. Current mortgages for a property such as this one carry a 12% interest rate.

There are four ways in which the transaction might be structured for sale:

1. The buyer, with $100,000, in hand could find a new mortgage for $300,000 and ignore the seller's carry-back offer. He would, of course, pay the current 12% interest rate, points and loan costs on the new mortgage.
2. The property could be sold "subject to" the balance of the existing mortgage, $150,635. The seller would carry-back the difference as a second trust deed note ("mortgage") in the amount of $149,365.

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1 "Junior' means that the trust deed securing the promissory note is recorded as a lien on the property *after* one or more prior trust deeds securing one or more earlier promissory notes.

2 A note recorded against the title earlier in time than a succeeding note.

3. The buyer could "assume" the existing mortgage and the seller would carry back the difference as a second trust deed note in the amount of $149,365.
4. The seller could accept the buyer's down payment and carry back an All-Inclusive (second) Deed of Trust mortgage in the amount of $300,000.

Let's compare these capital structures:

	#1 New Mortgage	**#2 "Subject To" 1st T.D.**	**#3 "Assume" 1st T.D.**	**#4 AITD**
Sales Price	$400,000	$400,000	$400,0000	$400,000
New Mortgage	300,000	–	–	–
Existing Loan	–	150,635	150,635	–
New 2nd T.D.	–	149,365	149,365	–
New AITD	–-	–	–	300,000
Buyer's Equity (down payment)	100,000	100,000	100,000	100,000
Total Sales Price	$400,000	$400,000	$400,000	$400,000

As you can see, each arrangement results in the same equity position for the buyer.

There are important differences, however:

1. The New Mortgage option requires the buyer to apply for and secure a new $300,000 mortgage. At closing, the existing first trust deed will be expunged and the new first trust deed will be recorded against the title. The seller will have no further legal or financial interest in the property.

2. If the property is taken "subject to" the existing mortgage, title will transfer without *any* contact with the present lender. The buyer takes full advantage of the low 6% interest rate and the fact that the loan has been paid down for a number of years. As a result, a greater portion of the buyer's future payments will be devoted to equity buildup.[3] The buyer will enter into a contract with the seller to make the payments on the existing loan on behalf of the seller. *The seller will remain liable for the* $150,635 loan. In the event of a default by the buyer, the lender will foreclose against the seller,

3 A greater amount of the total payment will be used to reduce the balance of the loan.

since it is (presumably) unaware of any transfer of title.[4] The seller's second trust deed position may be wiped out unless he is willing to "step up" and cure the default on the first mortgage and then foreclose under the second which he holds.[5]

3. Under the third option, a formal "assumption," the buyer is contractually obligated to present himself to the existing lender and seek to be placed in the legal shoes of the seller. If the lender approves the assumption of the loan, the seller *may* be relieved of all liability for the loan.[6] Nevertheless, in the event of a default by the buyer on the first mortgage, the seller's second mortgage is still in jeopardy and must be defended, as above.

4. Under the AITD arrangement, the buyer signs a promissory note for an amount equal to the sales price less his down payment. *This amount includes the amount due on the first note.* The buyer pays the seller on the amount and at the interest rate specified in the AITD note. The seller removes from the buyer's **PMT** an amount necessary to service the underlying first mortgage, sends this amount to the holder of the first mortgage, and pockets the difference.

   At close of escrow two trust deeds encumber the property: a first trust deed securing a note whose balance is approximately $150,635 and a junior second deed of trust in the amount of $300,000. This second trust deed, however, legally must specify that it is an AITD and also stipulate the particulars regarding the underlying note which it "wraps around." These stipulations put any person interested in the property on notice that the property is encumbered by an AITD.

For the purposes of the example, let's assume that the seller offers an 11% interest rate to the buyer. This loan will be scheduled for 30 years, but all due and payable in 10 years.

Seller's Position	Principal	Mo. Payment	Seller's Annual Payments
AITD Face Value	$300,000	$2,856.97	$34,284 Received
1st Loan Bal.	150,635	1,079.91	12,959 Paid Out
Net Position	149,365	1,777.06	21,325 Retained

The seller appears to have lent $300,000, but $150,635 of this amount is the bank's funds. His equity in the AITD loan is the difference, $149,365, the same equity he would have had in a note if either the "subject to" or "assumption" option had been used to structure this sale.

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4 A default by the buyer will not necessarily affect the seller's credit rating since he may declare that he was not the owner of the property at the time of the foreclosure.

5 Trust deeds and mortgages typically give the beneficiary the right to foreclose on the loan if a default occurs in any senior note.

6 Some lenders will seek to hold the seller as "additionally liable" for the loan. Prudent sellers whose mortgages are assumed arrange to have their sales contract specify, as a condition of the sale, that the assumption of the loan provides for the full release of the seller's liability.

But the seller, using an AITD, is earning $21,325 on a net equity of $149,365, indicating a true annual interest rate of approximately **14.3%**.

This method of estimating the yield on an AITD is convenient and simple – but not entirely accurate because it ignores the actual payment amount on both loans and the fact that the senior note has a larger proportion of its payment devoted to equity buildup.

Calculating the Yield on an AITD Accurately

First, let's determine the payment amount (**PMT**) and then the remaining balance **(FV)** for the AITD note after ten years. We will do this in two steps: first calculating the **PMT** due under the note, and then the remaining balance, the **FV**, after 120 **PMT**s have been made.

n	i	PV	PMT	FV
360	0.92…	-300,000	?	0
		Solving..	2,856.97..	

Now we can determine the remaining balance after 120 **PMT**s (ten years) have been made. Do this by "writing over" the value in the **n** register. Then solve for FV:

n	i	PV	PMT	FV
120	0.92…	-300,000	2,856.97..	?
			Solving…	**276,787.67**

The Underlying First Trust Deed

We do not know the *original* amount of the underlying senior note, but we don't need to. We do know that its Present **Value** is now $150,635, that the loan **PMT** is $1,079.91 per month, and that the interest rate, **i**, is 6% per annum. We need to know, however, what the remaining balance of this note will be in ten years (120 months).[7] The keystrokes to accomplish this are:

n	i	PV	PMT	FV
120	6/12	-150.635	-1,079.91	?
			Solving…	**97,089.88**

Once again, constructing a T-Bar will help us visualize the AITD cashflows. Note that the payment and interest rate on the senior note are specified, not calculated.

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[7] **Take care** not to structure an AITD note to have a due date longer than the due date on the underlying loan(s). If you do, the holder of the AITD note would be required to pay off the balloon payment on the underlying senior note without the benefit of a balloon payment on the AITD. He may not have the cash. If the underlying senior note is completely amortized during the term of the AITD, however, then you need not be concerned about this.

	AITD Loan			**Senior Loan**			**Net Position**
	(300,000)			(150,635)			(149,365)
1	2,856.97		1	1,079.91		1	1,777.06
	2,856.97			1,079.91			1,777.06
	2,856.97			1,079.91			1,777.06
	2,856.97	This		1,079.91	equals this		1,777.06
	2,856.97	less this		1,079.91			1,777.06
	2,856.97			1,079.91			1,777.06
	2,856.97			1,079.91			1,777.06
	2,856.97			1,079.91			1,777.06
	2,856.97			1,079.91			1,777.06
120	2,856.97 + 276,788		120	1,079.91+ 97,090		120	1,777.06 +179698

The T-Bar represented by **Net Position** is the result of subtracting the **Senior Loan** T-Bar from The **AITD** T-Bar. Net Position provides us with the **PV**, the **Pmt**, and the **FV**. It fails only to specify **i.** But we can calculate the yield:

n	i	PV	PMT	FV
120	?	-149,365	1,777.06..	179,698
Solving...	**1.26284...**			

This yield, 1.26284%..., is the *yield per month.* The annual yield is 12 times this number, or 15.15% – measurably more than the 14.3% yield we more casually estimated above.

Note that the **FV** of the Net Position T-Bar is greater than the **PV** of the same T-Bar. It says that, in addition to the payments received by the seller over a ten year period of time, he will receive a net loan payoff greater than the amount he originally lent. The difference is accounted for by the extra cash earned by the seller in the form of the *equity buildup* in the maturing senior loan. Since the seller remained responsible for the underlying first trust deed loan, he benefited from its reduced balance, although the buyer's cash funded all the payments.

AITDs can be very profitable investments for the holder. But notice too, that the Buyer of the property benefits by obtaining an 11% loan from the seller at a time when the market rate is 12%. The "loser" in this transaction is the lender on the first mortgage since his loan is continued at a very low interest rate at a time when market rates are much higher. You can readily appreciate why lenders are usually resolute in calling below-market rate loans "due and payable" upon the transfer of the securing property.

Constructing an AITD to Produce a Specified Yield

Suppose that the seller in the above example desires to achieve an 16% *yield* on his net cash position over a ten–year period. What interest rate on the AITD loan would be required to produce this yield?

His net cash position of $149,365 is the **PV** of his loan. Let's amortize this loan over a 30 year schedule with interest at 16% p.a., payable monthly, and then determine the remaining balance after 10 years.

n	i	PV	PMT	FV
360	1.333...	-149,365	?	0
		Solving...	**2,008.60...**	

After ten years, the remaining balance on the Net Position T-Bar must be:

n	i	PV	PMT	FV
120	1.333...	-149,365	**2,008.60...**	?
			Solving...	**144,372.88**

Knowing the elements of the Net Position T-Bar, and the senior note T-Bar (a given), we can add these two T-Bars together to produce the AITD T-Bar:

	Net Position	*plus*		Senior Loan	*yields*		AITD
	(149,365)			(150,635)			(300,000)
1	2,008.60		1	1,079.91		1	3,088.51
	2,008.60			1,079.91			3,088.51
	2,008.60			1,079.91			3,088.51
	2,008.60 ← This			1,079.91	equals →		3,088.51
	2,008.60	**plus** this →		1,079.91			3,088.51
	2,008.60			1,079.91			3,088.51
	2,008.60			1,079.91			3,088.51
	2,008.60			1,079.91			3,088.51
	2,008.60			1,079.9			3,088.51
120	2,008.60 + 144,373		120	1,079.91 + 97,090		120	3,088.51+ 241,463

We know everything about the AITD except for **i,** but we can solve for that:

n	i	PV	PMT	FV
120	?	-300,000	3,088.51	241,463
Solving...	**0.941...**			

Again, this interest rate is a monthly rate. The annual rate for the AITD is 0.941..% x 12 = 11.29%. Since this rate is still below the market rate of 12%, the buyer may find this an attractive offer, especially in view of the fact that he will typically have neither points nor loan fees to pay.

AITDs Using Adjustable Interest Rates

There is no reason that the loan in this example could not also be set up as an **ARM**. For example, if the Cost of Funds Index were 9% at this time, the seller-lender could set the initial **AITD** interest rate equal to the COFI index (9%) plus a margin of 2.29% or 11.29%. If interest rates increased, the AITD rate might increase; if rates decline, the AITD rate would decline.[8]

If the underlying senior note is, indeed, an **ARM**, then it behooves the seller/lender to match the timing and term of the **AITD** with the timing and term of the underlying note. If he fails to take this precaution, he may be faced with increased interest rates and larger payments on the senior note which he cannot recover from the AITD borrower.

Multiple Notes Underlying an AITD

Most AITD notes wrap around only one senior note. But there are times, especially in the sale of certain types of commercial properties,[9] when the AITD wraps a number of prior loans. In these cases, simply insert the required T-Bars for each loan in our diagrams above and follow the same general netting procedures.

Escrow AITD Payments

There have been a number of instances in which a seller accepts a low down payment from the buyer and carries the balance of the purchase price in the form of an **AITD**, but then never makes the **PMT**s on the senior loans. *Therefore no **AITD PMTs** should ever be made directly to the seller-holder of the **AITD** note.* These **PMT**s should be made through an escrow account, or, preferably, the **AITD** note should be collected by a bank, title company, or other reputable financial institution which will extract from the **AITD PMT** the **PMT**s necessary to service all the underlying loans. The remaining balance may then be forwarded to the **AITD** note holder.

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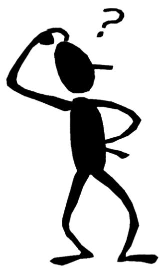
8 The alternative to an AITD loan for the seller is to demand cash, then invest his cash with the bank at, say, 9%. The bank would lend the buyer the required cash at 12%. AITDs can be an excellent investment for empty nesters.

9 Mobile Home Parks, motels and hotels, and small business sales are examples of real estate transactions which often involve AITDs which wrap more than one senior note.

What About the Deed?



In every discussion of AITDs someone always asks:

"What About the Deed?!!" "When do I get my deed?!!"

The buyer receives a deed to the property at close of the purchase escrow, in exactly the same way he would had he bought all–cash, or financed the acquisition with a new, conventional bank loan.

Confusion about this probably arises from transactions in which a buyer enters into a **Contract For Deed**, sometimes called a **Land Contract.** Under this arrangement, the buyer (vendee) does not receive legal title to the property until he has made a specified number of payments to the seller (vendor). During this time, the vendor retains legal title, conferring only "possessory" rights to the vendee. The vendee's interest is often referred to as an "equitable interest" in the property.

These contracts call for the vendee to obtain a loan at some time in the future, and to pay off the balance of the Contract held by the vendor. Only at that time does the vendee receive legal title to the property.

AITDs are not Land Contracts. They are kinds of Mortgages or Trust Deeds.[10] The buyer receives title to the property at the same time he would have had the financial structure not involved an All-Inclusive Mortgage.

(There is no rational reason why any buyer should arrange to acquire property using a Land Contract, and no rational reason why a seller should arrange to sell property using a Land Contract. It is an archaism fraught with problems and legal difficulties.)

CAVEAT

THERE ARE TIMES WHEN THE USE OF AN AITD CAN BE VERY BENEFICIAL TO BOTH BUYER AND SELLER IN A REAL ESTATE TRANSACTION. TODAY, HOWEVER, THE PRESENCE OF DUE-ON-SALE CLAUSES IN ALMOST ALL INSTITUTIONAL LOANS HAS GREATLY REDUCED THE OPPORTUNITY TO USE AITDS IN THE FINANCIAL STRUCTURE. EVEN WHEN THERE IS NO ACCELERATION CLAUSE, OR DUE-ON-SALE CLAUSE, IN A SENIOR NOTE, THE USE OF AN AITD IS NOT A SIMPLE REAL ESTATE LOAN SITUATION. IT IS STRONGLY RECOMMENDED THAT BOTH BUYER AND SELLER CONSULT AN EXPERIENCED REAL ESTATE ATTORNEY FAMILIAR WITH THE MANY PITFALLS THAT AWAIT THE UNWARY IN THE USE OF AITDS TODAY.

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10 A party cannot deliver a Trust Deed to a property unless he is the owner of the property.

Segment Summary

1. All-Inclusive Deed of Trust Mortgages (notes) may represent a useful financing tool under carefully selected circumstances. Both buyer and seller benefits may be found in these mortgage instruments.
2. The presence of a Due-On-Sale Clause in any senior note, however, makes the use an AITD very dangerous unless the approval of the senior lender is obtained.
3. The AITD operates principally for the benefit of the seller. If the prior note(s) contain no Due-On-Sale Clause, it is usually to the Buyer's benefit to take the property "subject to" the existing mortgages.
4. An AITD can "wraparound" one or more senior notes.
5. One of the principal sources of extra yield for the AITD holder is the paydown portion of the underlying loan which, as lender, he realizes.
6. AITDs are complex mortgage instruments for which the assistance of an experienced real estate attorney is essential.

VI The Reverse Mortgage

It is not possible, in keeping with the theme of this book, to calculate cashflows involved in reverse mortgages because the formulas involved in determining loan amounts involve regional loan limitations, the life-expectancy of the individual borrower and current interest rates. Nevertheless, the reverse mortgage is likely to become a more important device to aid some seniors in supplementing their retirement income, and as such, we want to include some information about it for the benefit of financial planners and other interested readers.

A Burgeoning Population Segment

In May 1999, Standard & Poor's estimated that the number of "elderly" (aged 65 and over) households would grow to more than 25.1 million by year 2010.The rating agency also estimated that the equity held in a home by these households now exceeds $1.8 trillion. For many of these homeowners this equity is the major component of their total financial wealth. For seniors who are relatively 'house-rich' but cash-poor, a reverse mortgage may not only enable a retirement but may even be the means to a comfortable one.

A Checkered History

Reverse mortgages have been around for more than 40 years, once drawing adverse publicity as the result of some unscrupulous private lenders who often defrauded senior homeowners by charging high interest rates and onerous loan fees.

In 1989 the federal Department of Housing and Development (HUD) was authorized to begin sponsorship of its own program, dubbed the *Home Equity Conversion Mortgage* (HECM) and insured by the Federal Housing Administration (FHA). In 1995 the program received a significant boost when the Federal National Mortgage Association (Fannie Mae) issued its own *Home Keeper* and *Home Keeper for Home Purchase* programs based on HECM, and began buying reverse mortgages from qualified lenders.[1] In 2000, Wall Street began issuing investment bonds collateralized by pools of reverse mortgages. As a result of these developments, the number of reverse mortgages issued is expected to grow apace the rapidly growing number of senior households.

Conventional vs. Reverse Mortgage

The conventional mortgage lender provides a lump sum to the borrower who then becomes obligated to repay the loan, with interest, over a specified amortization period. As time goes by, the amount of equity which the borrower has in the home increases until the loan is completely repaid, at which time the home is "free and clear." At that point, the owner's gross equity is equal to the market value of the home.

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[1] Fannie Mae is not a direct lender but sets the standards for mortgages which it will purchase from originating lenders.

The Reverse Mortgage creates just the opposite scenario: the owner who has substantial equity in the home[2] contracts with a lender who pays the owner a specified sum. The amount paid becomes the loan amount, which earns interest, drawing down the homeowner's equity and becoming a lien on the property. But from this point forward there are important differences between the conventional mortgage and the reverse mortgage, differences which should be of special interest to the financial planner.

Characteristics of Reverse Mortgage

Regardless of the plan selected, reverse mortgages do not require a monthly repayment. Therefore the borrowers are never at risk of losing their home because of a default in payments. They are required, however, to pay real estate taxes and maintain the property in good condition.

Funds received from the reverse mortgage are loan proceeds, and are not taxable as income. There is no requirement to repay the loan as long as the borrower(s) lives in the home as a principal residence. At the time they no longer live in the home, or when they sell the home, or permanently[3] move away, the balance of the mortgage becomes due and payable. In most cases this will require that the home be sold, but any excess of sale proceeds belongs to the owners, or to their heirs.

There is no restriction on how funds from a reverse mortgage may be used. If the funds are used to purchase an annuity, however, the proceeds from the annuity may be counted and taxed as ordinary income. This additional income may disqualify the owners for Medicaid and. Supplemental Security Income (SSI).[4]

All Reverse Mortgages are "non-recourse" loans, which means that if the loan balance grows to an amount which exceeds the value of the home at the time the owners no longer reside there, neither the owners nor their heirs will be responsible for any deficiency (amount of loan over proceeds from sale) when the home is sold. Ownership of the property is not affected by the reverse mortgage. The borrowers will remain the owners and their vesting will not be affected.. And there are no income, medical or credit requirements to obtain a reverse mortgage.

Qualifying for a Reverse Mortgage

The borrowers qualifications for a reverse mortgage are entirely different from those for a conventional mortgage.

1. Under these programs, both borrowers (if co-owners) must be 62 years of age or older.
2. If there is an existing mortgage on the property, it must be very small, or almost paid off.

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2 A very low mortgage or no mortgage at all.

3 "Permanently" means not having lived in the home for 12 months.

4 Proceeds from an annuity generally do not affect SSI and Social Security Benefits if spent in the same month in which proceeds are received.

3. The home must be the borrowers' principal residence. The borrowers must complete a cost-free, consumer education session offered by a HUD-approved counselor.

Loan Programs Available

Reverse mortgage loan programs available include:

1. Home Equity Conversion Mortgage (HECM insured by FHA)
2. Home Keeper Mortgage (Fannie Mae)
3. Home Keeper for Home Purchase (Fannie Mae)

Under any of the plans, the amount of the loan is a function of the age of the youngest co-owner, the current interest rate and the maximum "claim" amount. The claim amount for HECM mortgages is the lesser of the home's value or the maximum loan value for the geographic area in which the home is located.[5] Under Fannie Mae's *Home Keeper* program, the maximum amount is higher than under HUD's HECM program. Each of these plans provides a great deal of flexibility as to how the cash is to be received. There are three options:

Lump Sum: this option delivers the maximum cash in one payment. It is useful in the *Home Keeper for Home Purchase* plan since the lump sum amount can be added to other available cash and used to purchase a home. Unlike the conventional mortgage, the reverse mortgage obtained under the *Home Keeper for Home Purchase* program does not require repayment as long as the borrower remains in the home.

Payment Plans: cash may be received under five payment plans. These include,

<u>Term Option</u> – fixed, equal monthly payments for a fixed period of time as selected by the borrower

<u>Tenure Option</u> – equal monthly payments as long as the borrower occupies the home as the principal residence.

<u>Line of Credit Option</u> – draw up to a maximum line of credit as long as the borrower lives in the home.

<u>Modified Tenure Option</u> – set aside a portion of the maximum loan as a credit line and receive the rest as a series of equal monthly payments for as long as the borrower remains in the home.

<u>Modified Term Option</u> – set aside a portion of the maximum loan as a credit line and receive the rest as a series of equal monthly payments for a fixed period of time.

Creditline: under this option, the borrower may establish a line of credit and draw the maximum amount down as needed. Under HUD's program the creditline may be flat or growing, while Fannie Mae's creditline remains flat.

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5 FHA, a HUD subsidiary, publishes the amount of the maximum loan which they will insure. This schedule varies from region to region. This value caps the maximum reverse mortgage amount.

Costs of Reverse Mortgages

Reverse mortgages are more than 2-3 times more expensive than conventional loans. In addition to an origination fee and insurance premium, the lender may charge up to 2% of the maximum loan ("claim") amount, plus an on-going annual fee equal to 0.5% of the mortgage balance. There is also a monthly servicing fee ($15-30) which is charged for the administration costs of the loan. Total fees for a reverse mortgage can amount to 4-10% of the maximum ("claim") loan amount.

As is true with other consumer loans, the lender is required under federal law to inform the borrower of the estimated total annual loan costs (TALC). Since the bulk of TALC fees are paid up-front, the overall cost may be very high if , for any reason, the borrower lives in the home for a relatively short period after the loan is granted.

Furthermore, if the home has a high market value and the owners hold a very high amount of the value as equity, it may make more financial sense to extract cash from the home in the form of a conventional home equity loan, and to reinvest cash not immediately needed in other investments.

Counseling Required

As you may now suspect, the number of options available with a reverse mortgage can be confusing to many homeowners. For this reason, both HUD and Fannie Mae require that the consumer obtain counseling from a HUD-accredited counseling service.

But for some lower and moderate-income homeowners, a reverse mortgage may be the means to a more comfortable and secure retirement which may otherwise be unattainable.

An excellent source of information about reverse mortgages is the non-profit National Center for Home Equity Conversion (www.reservemortgage.org). HUD and Fannie Mae also maintain web sites which contain reliable and timely information.

VII - Loan Guidelines, LTV Ratios and APRs

Residential Loan Guidelines

In extending a loan to a residential borrower, the lender seeks to determine three things:

1. Can the borrower repay the loan?
2. Will the borrower repay the loan?
3. Is there adequate security for the loan?

Can the Borrower Repay?

In assessing the borrower's ability to repay the loan, the lender is primarily interested in the quantity and stability of the borrower's income. Lenders generally screen a prospective home buyer's ability to service the mortgage debt by the use of two ratios applied to monthly household cashflow: the **front-end** and the **back-end** ratios.

The front-end ratio is the ratio of total housing expenses to *gross* monthly household income before any deductions. Total housing expenses include the principal and interest payment on the loan, real estate taxes, property insurance and, in applicable cases, homeowner or other association fees which are required to be paid. If the borrower makes less than a 20% down payment, mortgage insurance is also generally required. All these costs are counted in determining the **front-end ratio.**

The range of front-end ratios used by residential lenders varies approximately from 28% to 34%. Therefore a household which enjoys $6,000 in gross monthly income (before deductions and income taxes) would generally be limited to 28%-34% of this amount, or $1,680 – $2,040. By subtracting from total housing costs all costs other than the principal and interest payment on the mortgage, the lender determines how many gross dollars remain to service the residential loan. Given a certain interest rate and amortization period, the amount of the residential loan is then easily calculated.

For example, using the data above, and estimating "all other housing costs" to be $300 per month, the family with gross household income of $6,000 could qualify for an 8%, 30-year loan as follows:

Gross Monthly Household Income	$6,000
30% Front-end ratio	0.30
Total housing costs not to exceed	1,800
Less all other housing costs	300
Amount available to service mortgage	1,500

n	i	PV	PMT	FV
360	8/12...	?	**1500**	0
	Solving...	**–204,425.24**		

Therefore this household income could support a mortgage of $204,425.

The **back-end ratio** is similar, except that it includes, in addition to total housing costs, all long-term consumer debt payments not likely to be repaid within 9-12 months. Back-end ratios range from 34% to 40%, and sometimes higher. Both front- and back-end ratios are subject to the individual lender's current underwriting standards, which frequently change.

Using the same data as before, and a back-end ratio of 38%, the borrower with a $6,000 gross monthly income would be limited to $2,280 ($6,000*38%) of total monthly debt. Since this total would include $1,800 in total housing debt, the difference ($480) would be available to service all other debt. "Other" debt includes debt such as credit cards, auto loans, installment debt etc. Borrowers whose total debt exceeds this limit may be denied the loan for which they applied, or may be eligible only for a reduced loan whose monthly costs would not exceed the monthly limit.

Will the Borrower Repay?...

In addition to the loan applicant's ability to repay the debt, the lender needs to know how likely it is that the applicant *will* repay the debt. This question speaks to the applicant's credit history which the lender will review using a current credit report from one of the three major credit information services.[1] In addition, 85% of mortgage originators use FICO scores [2] to measure potential credit risk. FICO scores are incorporated in and supplied by the credit bureaus themselves, and not by Fair, Isaac and Company which only supplies the software used by the bureaus.

FICO Score	Odds of Delinquency
595	2.25-1
600	4.5 -1
615	9-1
630	118-1
645	136-1
660	172-1
680	144-1
700	288-1
780	576-1

Each FICO report gathers information from the available credit sources and computes a number representing the risk that a borrower will have a delinquency greater than 90 days. The higher the score the less likely the risk. FICO scores below 600-620 are often the cause of loan denial.

FICO scores are not only used to forecast the risk of a delinquency, but are also used by most mortgage originators in setting the interest rate. Applicants with low FICO scores can expect to pay higher rates than those with high FICO

1 Equifax, Trans Union, and Experian (formerly TRW).

2 Fair, Isaac and Company. Other credit scorers provide a similar service.

scores. Therefore it is important that the potential borrower be aware of his or her credit standing and the FICO score before initiating loan inquiries. Submitting loan inquiries to a number of potential lenders has the effect of *lowering* FICO scores.

While it is not within the scope of this text to review FICO scores, it is virtually imperative that any applicant seeking a loan *first* research the voluminous information available on the world-wide web (WWW) dealing with FICO scores.

Loan-to-Value Ratios

Almost all lenders use Loan-to-Value ratios to further insulate themselves against the risk of loss in the event of a borrower's default. Despite the ability of the borrower to service a mortgage of a given size, the residential lender will generally restrict his loan to a percentage of the appraised value of the property. If a residential lender declares that it may not exceed an **LTV** ratio of 80%, it means that:

$$\text{Loan-to-Value Ratio} = \frac{\text{Amount of Loan}}{\text{Appraised Value of Property}} \leq 0.80$$

When the LTV ratio is above 80%, lenders will require that the borrower purchase mortgage insurance. Depending upon the amount of the mortgage in relation to the fair market value of the collateralizing real estate, this insurance will reimburse the lender in the event of a default by the borrower in amounts varying from 25-30% of the mortgage.

<u>Commercial Loan Guides</u>

Debt Service (Coverage) Ratios

Commercial and investment lenders frequently use a Loan-to-Value ratio as a secondary guideline. Their first preference is the Debt Service Ratio **(DSR)**, or Debt Coverage Ratio **(DCR)**, as it is sometimes called.

The preference is well justified. The primary source of repayment for a loan placed on income-producing property is its Net Operating Income (NOI), the income the property delivers after operating expenses.[3] This stands in contrast to the single-family residential property in which the primary source of repayment of the loan is the homeowner's income – not the property's income. Therefore commercial and investment (C&I) lenders have a vital interest in the **q**uantity, **q**uality and **d**uration[4] of the Net Operating Income.

The **NOI** is the income remaining after payment of operating expense but *before* loan payments. This is why every commercial lender requires an up-to-date operating statement from the prospective borrower together with copies of all current leases. If the lender doesn't care for the

3 Some lenders require that the borrower also have the capability of making the loan payments in the event the rent derived from the property proves insufficient. These are "recourse" or "credit" lenders.

4 "Quantity" refers to the size of the loan; "quality" refers to who is the source of the payments (the tenant); "duration" refers to how long this rent stream is likely to continue.

way the operating statement looks, he will construct his own and use it to determine a pro-forma **NOI**. He requests an examination of the leases not only to verify present income but also to assess, wherever possible, the financial strength of the tenant and the durability of the rental stream.

Once the **NOI** can be verified, the commercial lender will make a judgment regarding the **risk** associated with the income stream by examining the property's location, current leases and the credit worthiness of the tenants. If, for example, he judges that the risk associated with a residential apartment house of good quality in a healthy market is relatively low, he may require a Debt Service Ratio (DSR), or Debt Coverage Ratio (DCR) of 1.15. Properties which appear to be riskier – for instance, single-purpose, specialized buildings – command higher DSRs. In general, the riskier the property, the higher the DSR. A DSR of 1.15 means that:

$$\frac{\text{Net Operating Income}}{\text{Annual Debt Expense}} = 1.15$$

For example, if the **NOI** of a certain property were $250,000 per year and the lender held to a DSR or DCR of 1.15, we could determine the annual cost of servicing the mortgage as follows:

$$\frac{\$250000}{\text{Annual Debt Expense}} = 1.15$$

$$\text{Annual Debt Expense} = \$250{,}000 \div 1.15$$

$$\text{Annual Debt Expense} = \$217{,}391.30$$

Since the annual debt expense is known, we can compute the *monthly* debt expense and ask the lender what interest rate he requires, and how long the (amortization) schedule is on a loan to be paid monthly. He replies: "10% and 25 years, all due in 7 (years)."

Now we have all the information to determine the amount of loan this lender may lend on this property:

n	i	PV	PMT	FV
300	10/12	?	217,391.30/12	0
	Solving...	**–1,993,609.24**		

The lender, in considering both the LTV Ratio and the Debt Coverage Ratio will lend the *lesser* amount.

The APR – the Annual Percentage Rate

In Chapter 4 we devoted considerable attention to the Internal Rate of Return and its calculation using future cashflows. Chapter 4 also explained that *IRR*, *Yield* and *APR* are all one and the same measuring stick under different brand names. A banker's "yield" = automobile dealer's "APR" on car leases = investor's "IRR" on an apartment house.

Actually, bankers talk about "yield" when they discuss their return on a loan, but when they discuss the true cost of the loan to the residential borrower, they are required by federal law[5] to calculate and disclose the **A**nnual **P**ercentage **R**ate (**APR**), which is the true interest cost *when all loan related charges are factored in.* This federal law is not limited to real property loans (1-4 residential units), but applies to all kinds of consumer loans. It is not required to be disclosed to commercial borrowers, or to borrowers on residential properties greater than 4 units.

The **APR** is a very useful number because it enables the informed borrower to compare a myriad of different financial terms offered by different lenders, and to determine which loan offers the lowest total cost of borrowing.

We hasten to add that the least expensive loan is not always the best loan for the consumer because other terms of the loan agreement affect the value of the loan. For example, a loan may, by its terms, permit the borrower to transfer the loan to a second party for little or no assignment fee. This kind of loan may take on extra value on resale if rates take a turn to the upside and new money becomes more expensive.

Calculating APRs

Consider, for example, these two loans offered by different lenders:

> Lender A offers a loan of $100,000 at an annual interest rate of 9.375%, payable monthly, over an amortization schedule of 30 years, but all due and payable at the end of 5 years. Lender A will charge 2 points as loan fee.
> Lender B offers a loan of $100,000 at an annual interest rate of 9.50%, payable monthly, over an amortization schedule of 30 years, but all due and payable at the end of 5 years. Lender B will charge 1.75 points as loan fee.
>
> Which of these loans is less costly?

Follow these simple steps to determine the true cost of the loan:

1. Set the loan up in the calculator and solve for the monthly payment.
   This is the amount the borrower contracts to pay each month.
2. Calculate the remaining balance of the loan as the FV of all the made payments.

5 Regulation Z

3. Determine the charges [6] and points to be paid (Loan Amount x Points = Fee) and subtract this sum from the **PV**.
4. Enter this net amount into **PV** and re-solve for **i**, the true interest rate and **APR**.

Using this guide, let's calculate the true interest rate for Lender A's loan:

Key In	Display Shows	Comment
360 [n]	**360.00**	**Enters amort.period**
9.375 [g] [i]	**0.78**	**The monthly interest rate**
100000 [CHS] [PV]	**–100,000.00**	**Enters PV as a negative**
0 [FV]	**0.00**	**Inserts 0 into FV**
solving		
[PMT]	**831.75**	
60 [n] [FV]	**96,153.41**	**The balance at 60 months entered into FV**
[RCL] [PV] [CHS]	**100,000.00**	
[Enter]	**100,000.00**	**Puts 100,000 in Y register**
2 [%]	**2,000.00**	**The value of points paid[7]**
[–]	**98,000**	**The net amount after payment of loan points**
[CHS] [PV]	**-98,000.00**	**Replaces PV as a negative**
solving		
[i]	**0.82**	**Calculates true monthly interest rate**
12 [x]	**9.89**	**The true annual interest, and the APR for Loan A**

The APR for Loan B is calculated in exactly the same way, resulting in an APR of **9.95%**

As you can see, Loan A is the more economical loan even though it charges higher points than Loan B. This solution is specific for the terms quoted for these particular loans. If, for example, the due date on the loans were shorter or longer than 5 years, the APR would change.

6 Only those charges directly connected with the loan should be used.

7 If there are other loan costs, add them to this number.

How Long Will the Cash Be Used?

In determining the APR of a loan, it is absolutely essential to specify how long the borrower is likely to own the property. The APR number quoted in many newspapers which list competitive mortgage rates assumes that the loan will endure for its entire amortization period, generally 30 years. Therefore the costs of the loan, over and above the interest rate, are spread over a long period of time. But when these extra costs are spread over a much shorter period of time, they weigh heavily on the true cost of the financing and may reverse a seemingly obvious decision..

Junk Fees Make a Difference

Lenders augment their yields on loans by charging additional fees to the borrower at the time of loan closing. Some of these fees are legitimate costs incurred by the lender, while others can be of questionable parentage. The more specious charges have come to be known as "junk fees." Some lenders are adept at playing this variation of the "bait and switch" scam with borrowers. At first, the offered interest rate, loan points, and other terms of the loan appear very attractive; but at loan closing a whole new set of charges is billed to the buyer.

The practice has been, and still is, widespread. Therefore the informed borrower will also pay attention to the estimate of all loan costs and charges which the lender, under federal Regulation Z, is required to deliver to some borrowers[8] within three days of receiving the loan application.

All the loan fees charged to the borrower are rightfully counted in as "loan costs." These fees may include some of the following:

Appraisal fees	Preparation of Note	Recording Fees
Tax Notice fees	Loan escrow tie-in fees[9]	Misc. Loan fees

But the list of other fees charged by some lenders is often a fairy tale. Only the fees charged in connection with the loan are legally required to be included in a determination of the loan's APR. This technicality affords the less scrupulous lender the chance to add on fees and charges which it claims to be payable to third parties and not to the lender. Oftentimes, these "third parties" are owned by or directly controlled by the lender. If this is the case, the borrower should ignore the technicality and obtain a good-faith estimate of *all* charges, not to exceed a certain sum.

Before proceeding further with mortgages and loans, let's look at a few "hand tools" lenders use in formulating loans, computing loan amounts, and qualifying borrowers for a loan in relation to property values.

8 In connection with residential loans on 1 to 4 units
9 Fees charged by the escrow agent for services in connection with the loan.

Loan Constants

Prior to the advent of calculators and computers, loan amounts and loan payments were determined by using loan constants. Although "mortgage tables" are still sold in book stores, the availability of inexpensive financial calculators has made these reference books all but obsolete.

A mortgage loan constant (k) is nothing more than the PMT required to amortize $1.00 of loan principal (PV) over a defined period of time at a specified rate of interest..

We can determine the loan constant for a 30-year, fully amortizing loan, payable monthly including annual interest at the rate of 10%, by solving for the PMT necessary to amortize a PV of $1.00:

n	i	PV	PMT	FV
360	10/12	**-1.00**	?	0
		Solving...	**0.0087757**	

In the interest of brevity, mortgage table books express this PMT is terms of *cost per $1,000*, or $8.78.

If you own a financial calculator or computer you will rarely have a need to use loan constants since you can key-in the actual loan amount (PV) and obtain the precise PMT for the amount of the loan, or, knowing the desired PMT, you can determine the amount of loan that PMT will support. But for the sake of completeness, here are the math relationships:

$$(1)\quad \textbf{Loan Amt.} \times \textbf{Loan k} = \textbf{Loan Payment}$$

$$(2)\quad \frac{\textbf{Loan Payment}}{\textbf{Loan Amount}} = \textbf{Loan k}$$

$$(3)\quad \frac{\textbf{Loan Payment}}{\textbf{Loan k}} = \textbf{Loan Amount}$$

Segment Summary

1. A loan constant is that amount of **PMT** necessary to amortize $1.00 of loan principal over a certain amortization schedule and at a certain interest rate. It is also the quotient obtained by dividing the loan payment by the amount of the loan.
2. Financial calculators have made the use of loan constants all but obsolete.
3. A **Loan-to-Value** ratio refers to the amount of loan divided by the (appraised) Fair Market Value of the residential property.
4. An income-to-debt service cost is a variant of this ratio used in commercial and investment lending. This ratio, the **DSR** or **DCR**, substitutes the Net Operating

Income (NOI) as the numerator of the ratio, and the annual cost of servicing the debt as the denominator of the fraction.

5. Lenders manage **Loan-to-Value** and **Debt Service Ratios** to increase the safety of their loans.
6. High LTV ratio residential loans are likely to be risky loans. Commercial loans carrying high DCRs reflect the lender's risk concern.
7. The **APR** of a loan is the actual interest rate charged by the lender and paid by the borrower, after including all loan fees and charges. It is required to be given to the borrower within 3 days of loan application (1-4 residential units only).
8. Loans of different interest rates, including different "other loan fees," can be compared by determining their individual **APR**s.

Chapter 9 Valuing Cashflows from Real Estate

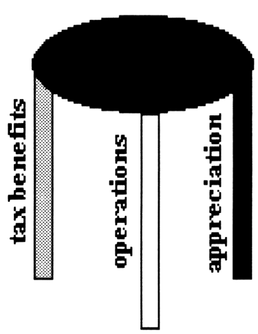
Real estate investments return cash to the investor somewhat differently from stocks and bonds.

Stocks and bonds have two major sources for the return of investment cash: in the case of stocks, dividends and (hopefully) price appreciation; in the case of bonds, interest payments and price appreciation.

Real estate, however, returns cash in three ways: 1) operating income, 2) price appreciation and 3) tax benefits. There are no tax benefits attached to owning stocks and bonds, only tax liabilities. But the non-cash deductions of depreciation and amortization available to the owner of income-producing real property are a significant source of cashflows. For this reason, investments in income real estate should always be measured on an after-tax basis.

A Three-Legged Stool

You can envision the ideal real estate investment as a stool sitting firmly on three cashflow legs.



Ideally, each leg, or source cashflow, would be equal, creating a stable investment platform in which risk would be spread evenly over all three cashflow sources.

Unfortunately, there is no such “ideal” investment. In most real estate investments one leg is longer than the others In at least one form of real estate investment, vacant land, there is but one leg, one source of cashflow return,[1] which makes this kind very risky.

Regardless of the proportion of cash benefits supplied by the three "legs", the percentages of return contributed by each cashflow leg

---

1 Appreciation

must add up to a market rate of return. If this were not so, real estate would fail to attract investment capital since, unlike water, investment money runs uphill to the highest rate of return. In any case, we will demonstrate a method of valuing the cashflows from real estate using skills and techniques already covered in prior chapters.

Quantifying and Valuing Income

The first step in quantifying the value of an income-producing property is to determine its current Net Operating Income. This ought to be a straightforward task, but it's not.

If a property is subject to multiple leases, one must obtain and read a complete and up-to-date copy of each lease in order to discover the property's current and future income potential. It is folly to read only one lease and assume that all other leases follow the same terms and conditions. Leases are individually negotiated and there can be wide variations affecting income in all the leases.

In reviewing leases, the prospective investor seeks to answer three important questions:

1. What is the *quantity* of current *and* future income?
2. What is the *quality* of the income source; i.e. who is the tenant and how credit-worthy is the tenant?
3. What will be the *duration* of the income?

The quantity of rent currently being collected, and other tenant information, should be verified whenever possible, by the use of *estoppel* certificates.[2]

These are certificates to be signed by the current tenant(s) indicating:

- that the lease is in full force and effect and that there are no defaults either on the part of the tenant or the landlord,
- the amount of the rent and that the rent is current (or not),
- the agreed-upon amount or method of determining future rent,
- that there are no offsets to the rent (such as a service performed by a tenant which would offset rent payable),
- the existence of any options remaining to extend the lease, and
- the terms of such options and methods by which the option is to be exercised.

It is good practice to furnish the tenant with a current copy of his lease and to request that he notify the prospective buyer if the lease copy is not current or has been altered.

2 A person is "estopped" from exercising a legal right if, when previously given an opportunity to declare that right, the person declined, failed to declare or misstated that right, which, if it were to be later exercised would operate to the serious detriment of the party requesting the declaration. Estopped = "legally barred."

Estoppel certificates are not employed when dealing with large multi-family residential properties usually managed by professional managers. They should be considered, however, if the property is owner-managed.

Attention to these details is very important since the prospective buyer is not just buying the property, but is buying the future income stream.

Vacancies and Credit Losses

There are few rental properties which do not experience a change of tenants and, therefore, vacancies. Allowances for vacancies and credit losses are deductions from Potential Income which fall directly to the bottom line (Net Operating Income). The best gauge of prospective vacancy losses depends on the type property owned.

Vacancies in smaller residential properties are very costly. A single vacancy for two months in a 4-plex represents a 4% loss of income (2/48); two such vacancies in a year represent an 8.3% loss in Potential Income. Regardless of the condition of the market, large apartment complexes always have vacancies due to normal move-in move-out tenant relocations. The minimum appears to be about 3% of Gross Scheduled Income per year.

Larger industrial properties are often leased for an extended number of years and a prospective buyer is frequently advised to "ignore vacancy because the lease has 10 years to run." Yet these are the very types of properties which can stand vacant for 6-12 months, or longer, following a tenant move-out. A loss of 6 months of income on a ten year lease is a 5% loss (6/120). A loss of 12 months of income is a 10% loss. Industrial properties are not immune to vacancy and a vacancy factor should always be used.

The more specialized a property, such as a bank building or theater, the more difficult it will be to find a new user, and the longer it will be vacant following a move-out.

Operating Expenses

It is not within the scope of this text to review all the considerations which must be made in estimating current operating expenses which vary considerably among different property types.

The purpose of gathering these data is to serve as a starting point for estimates of future income and expenses. Many operating expenses incurred by the new owner will be different from those incurred by the past owner. For example, some states reappraise real property for taxes only upon transfer of title. Thereafter, tax increases may be restricted by law. A new owner, acquiring a property long-held by the seller, may experience a significant increase in new real property taxes when the property is re-appraised. In other instances, the cost of insurance for a new owner

Annual Net Operating Income and Debt Service

Trade Name		**Date**		
Location		**Price**		
Type of Property		**Existing Loan(s)**		
Size of Property	(Sq. Ft./Units)	**Equity**		
Purpose				

Assessed Values		**% To Total**
Land		
Improvements		
Personal Property		
Total Assd. Value		
APR #		
Tax Area Code/Rate		

Existing	**Balance**	**Payment**	**Payable**	**Interest**	**Term**
1st					
2nd					
3rd					
Potential					
1st					
2nd					

	All Numbers Are Annual			**Comments**
1	**POTENTIAL RENTAL INCOME**			
2	Less: Vacancy & Credit Losses			
3	EFFECTIVE RENTAL INCOME			
4	Plus: Other Income			
5	**GROSS OPERATING INCOME**			
	OPERATING EXPENSES:			
6	Real Estate Taxes (new)			
7	Personal Property Taxes			
8	Property Insurance			
9	Off Site Management			
10	Payroll			
11	Expenses/Benefits			
12	Taxes/Worker's Compensation			
13	Repairs and Maintenance			
	Utilities:			
14	Gas			
15	Electric			
16	Water			
17	Trash			
18				
19	Accounting and Legal			
20	Advertising/Licenses/Permits			
21	Supplies			
22	Miscellaneous			
23	Reserves for Replacements			
24	Contract Services:			
25				
26				
27				
28	**TOTAL OPERATING EXPENSES**			
29	**NET OPERATING INCOME**			
30	Less: Annual Debt Service			
31	**CASH FLOW BEFORE TAXES**			

may be far greater than for the past owner who, perhaps, owned a number of properties insured under a blanket policy affording him substantial savings. In other cases, the amount of insurance carried by the seller underestimates the cost of replacement. A past history of income and operating expenses may be interesting and helpful, but the important objective is a sound estimate of future income and future expenses. As many of these expenses as possible such be obtained from entities other than the seller.

The form on the opposite page recapitulates the income and expense categories for the operation of an income-producing property. It also provides space for the calculation of debt service which, when subtracted from NOI, will yield the *Cashflow Before Tax* number or *Spendable Income.*

Estimating Cashflow After Taxes (CFAT)

Once an estimate of next year's Net Operating Income has been achieved, one can proceed to forecast the annual Cashflow After Tax. The following columns show how both the Cashflow and Taxable Income Statements are constructed:

Cashflow After Taxes	**Taxable Income & Tax**
Net Operating Income	Net Operating Income
Less Interest	Less Interest
Less Loan Paydown	--------------------
=Cashflow Before taxes	Less Depreciation
Less Taxes Due and Payable	Less Amortization
=Cashflow After Taxes (CFAT)	=Taxable Income
	x Tax rate
	=Taxes Due and Payable

Notice that the portion of the annual mortgage payment which is earmarked as Loan Paydown (on the left) is omitted from the calculation of the Taxable Income (on the right) since this paydown amount contributes to owner equity and is not tax-deductible.

Also be aware that this is a shortcut method of arriving at CFAT. In actual practice, the amount of Taxable Income would be carried over to the owner's regular tax return as Ordinary Income, where it would be added to other Ordinary Income. The total would determine the rate at which the owner would pay taxes. The only valid way to determine the tax attributable to the property is to calculate the tax due with and without the real estate. The difference is the tax attributable to the ownership of the property.

But here we approximate the tax value by multiplying the Taxable Income by the owner's incremental tax rate,[3] and then subtract the resulting estimate of tax from *Cashflow Before Taxes* to estimate CFAT.

3 The rate at which the taxpayer would pay taxes on the next dollar of ordinary income.

As you can see, the deductions for Depreciation and Amortization, non-cash items, reduce or "shelter" current Taxable Income from immediate tax recognition. Depending on the value of the depreciable improvement, many properties show a taxable loss for the first few years of normal operations. Then, as income rises, these deductions fail to cover all Taxable Income and the amount of tax payable increases. Still, as time goes on, these deduction continue to shelter some ordinary income from operations, and this sheltered income is not taxed in the year it is received..

Building a Pro-Forma

A Pro-forma is simply two or more Annual Operating Statements linked together over the contemplated holding period by a series of *Assumptions*. These assumptions constitute the basis for specific *forecasts* made with regard to future rents, expenses, debt costs, taxable income, and therefore the After-tax cashflows, or **PMTs**, which the owner may reasonable expect over the holding period. They are analogous to the assumptions which a stock analyst would make about the future performance of a particular issue or industry.

There is a great advantage to understanding how these cashflows are derived, because – once understood – the pro-forma can be re-configured to represent any situation which a particular investment situation presents. Commercially available pre-packaged, investment analysis computer programs do no more, but cost far more, and in the end are much less flexible, than your own self-generated spreadsheets.

A Model Pro-Forma

The model Pro-forma which follows on pages 9-7 and 9-8 depicts one possible operating scenario for a multi-family residential property over a 5-year holding period.

> Consider a 60-unit apartment house which is expected to generate $684,600 in Total Potential Income in the coming year. Three and one-half percent of this income is expected to be lost due to vacancies and credit problems. Its operating expenses are estimated to be $225,000 per year of which $57,600 will be property taxes.
>
> A $3,686,791 loan is available which will cost (first year) $321,548 in interest and $26,500 in loan reduction, for a total annual payment of $348,048. The fee to obtain this loan is 1.5 points, or 1.5% of the loan amount. Under these assumptions, what will be its first year Pre-tax Cashflow (CFBT) and After-tax Cashflow (CFAT)?
>
> What will be the 5-year IRR?

The variables for this analysis are stored in Column A from Line 2 to Line 19. When any of these variables are needed by a formula within the program, access is provided by reference, as opposed to placing the variable directly within the formula. By doing so, any one of these variables can be changed and the results of the change are immediately reflected in results throughout the spreadsheet.

	A	B	C	D	E	F	G	H
1	Real Estate Analysis 5 Years							
2	Property Ident.	100 Main St.		**This analysis assumes AGI <$100,000**				
3	Number of Units or Lseable Sq.Footage	60	80,000.00	per unit				
4	Acquisition Price	4,800,000						
5	Acquistion Costs @ 1.25%	60,000			**Loan Information**			
6	Total Acquisition Cost	4,860,000					**By LTV %**	
7	% To Improvement	70%					1st TD	3,840,000
8	Depreciable Basis	3,402,000					Int. Rate	8.75%
9	Estimated Expenses/Unit/Yr	$ 3,750					Amortization Schedule (Yrs.)	30
10	Rent Inflator /Yr	3.00%					Term (Yrs.)	10
11	Expense Inflator/ Yr	3.00%					Pmt/Mo.	$30,209.30
12	Tax Area Code Rate	1.20%					**By Debt Coverage**	
13	Tax Inflator /Yr	2.00%			Loan Points	1.5	NOI	435,060
14	Owner's Incr. Tax Bracket (Fed only)	33.0%			$ Points	55,302	DCR	1.25
15	Capitalization Rate In	9.06%			Capital Gains Rate	20%	Pmt/Mo.	29,004
16	Capitalization Rate Out	9.06%			Unrecaptured deprec. rate	25%	Max Loan	$3,686,791
17	Leverage (vs. Acq. Price)	80%					IRR	12.55%
18	Residential (Y/N)	Y		Program chooses lower loan amount				
19	Total Invested Equity	1,228,511						
20								
21		Year	1	2	3	4	5	6
22	Loan Progress	Beg. Bal.	3,686,791	3,660,291	3,631,377	3,599,830	3,565,408	3,527,851
23		Rate	8.75%	8.75%	8.75%	8.75%	8.75%	8.75%
24		Payment	348,048	348,048	348,048	348,048	348,048	348,048
25		Interest	321,548	319,134	316,500	313,626	310,491	307,069
26		Equity	26,500	28,914	31,548	34,422	37,557	40,979
27		Remain. Bal.	3,660,291	3,631,377	3,599,830	3,565,408	3,527,851	3,486,872
28		DCRatio	1.25	1.29	1.33	1.37	1.41	1.46
29								
30	Income							
31	Enter 1st Year's GSI per unit→		11,100					
32	Gross Scheduled Income		666,000	685,980	706,559	727,756	749,589	772,077
33	Other Occupancy Dependent Income		18,000	18,540	19,096	19,669	20,259	20,867
34	Total Potential Income		684,000	704,520	725,656	747,425	769,848	792,943
35	Vacancy/Credit Loss	3.5%	3.50%	3.50%	3.50%	3.50%	3.50%	3.50%
36	Vacancy $		23,940	24,658	25,398	26,160	26,945	27,753
37	Gross Operating Income		660,060	679,862	700,258	721,265	742,903	765,190
38								
39								
40	**Expenses**							
41	Operating Expenses ,< Taxes		167,400	172,422	177,595	182,922	188,410	194,062
42	RE Taxes		57,600	58,752	59,927	61,126	62,348	63,595
43	Total Operating Expenses		225,000	231,174	237,522	244,048	250,758	257,658
44	% Increase Expenses/Year		NA	2.7%	2.7%	2.7%	2.7%	2.8%
45	Percent Expenses to GOI		34.1%	34.0%	33.9%	33.8%	33.8%	33.7%
46	**Net Operating Income**		435,060	448,688	462,736	477,217	492,145	507,533
47								
48	**CashFlow Before Tax**							
49	Net Operating Income		435,060	448,688	462,736	477,217	492,145	
50	Annual Debt Service		348,048	348,048	348,048	348,048	348,048	
51	Cash Flow < Taxes		87,012	100,640	114,688	129,169	144,097	
52								
53	**Taxable Income, CFAT**							
54	Net Operating Income		435,060	448,688	462,736	477,217	492,145	
55	Less Interest		321,548	319,134	316,500	313,626	310,491	
56	Less Depreciation		118,555	123,709	123,709	123,709	118,555	
57	Less Amortized Loan Points		5,530	5,530	5,530	5,530	5,530	
58	Total Deductibles		445,633	448,373	445,740	442,866	434,575	
59	Total Taxable Income		(10,573)	314	16,996	34,352	57,570	
60	Incremental Tax Bracket		33.0%	33.0%	33.0%	33.0%	33.0%	
61	Tax		-3,489	104	5,609	11,336	18,998	
62	**Cash Flow After Tax**		90,501	100,536	109,079	117,833	125,099	

	A	B	C	D	E	F	G
64	**Adjusted Basis**						
65	Total Acquisition Cost		4,860,000	4,860,000	4,860,000	4,860,000	4,860,000
66	Less Depreciation Taken		118,555	242,264	365,973	489,682	608,236
67	Plus Costs of Disposition	5.50%	272,270	280,794	289,582	298,640	307,977
68	Total Adjusted Basis		5,013,715	4,898,530	4,783,609	4,668,958	4,559,741
69							
70	**Long Term Capital Gain**						
71	Gross Sales Price		4,950,355	5,105,348	5,265,120	5,429,817	5,599,591
72	Less Adjusted Basis		5,013,715	4,898,530	4,783,609	4,668,958	4,559,741
73	Indicated Gain		(63,360)	206,817	481,511	760,859	1,039,850
74							
75	**Tax Computation on Sale**						
76	Indicated Gain (Loss)		(63,360)	206,817	481,511	760,859	1,039,850
77	Depreciation Taken		118,555	242,264	365,973	489,682	608,236
78	Recapturable Depreciation		0	206,817	365,973	489,682	608,236
79	Depreciation Recapture @	25.0%	0	51,704	91,493	122,420	152,059
80	Remaining LTCG		0	0	115,538	271,177	431,613
81	LTCG Tax @	20.0%	0	0	23,108	54,235	86,323
82	Total LTCG+Recap Tax		0	51,704	114,601	176,656	238,382
83							
84	**Recovery Of Susp Losses & Points**						
85	Suspended Losses Per Year		0	0	0	0	0
86	Accum. Susp. Losses		0	0	0	0	0
87	Unamortized Loan Points		49,772	44,241	38,711	33,181	27,651
88	Total		49,772	44,241	38,711	33,181	27,651
89	Recovered @ Ordinary Tax rate		33.0%	33.0%	33.0%	33.0%	33.0%
90	Tax Credit		-16,425	-14,600	-12,775	-10,950	-9,125
91	**Total Tax on Sale**		-16,425	37,105	101,826	165,706	229,257
92							
93	**Net Proceeds**						
94	Sales Price		4,950,355	5,105,348	5,265,120	5,429,817	5,599,591
95	Less Costs of Sales		272,270	280,794	289,582	298,640	307,977
96	Less Mortgage Balance		3,660,291	3,631,377	3,599,830	3,565,408	3,527,851
97	Less Total Tax on Sale		-16,425	37,105	101,826	165,706	229,257
98	After Tax Proceeds		1,034,219	1,156,072	1,273,883	1,400,063	1,534,506
99							
100	**Performance Indices**						
101		Discount Rate	Value		Expenses/Unit	2,790	< Prop. Taxes
102	Capitalization Rate In	9.06%			Expenses/Unit	3,750	>Prop Taxes
103	Cash-on-Cash > Tax	7.7%					
104	Times Gross Multiplier	7.21			Cost/Unit	80,000	
105	After-tax Present Value	12.0%	$1,255,181		NOI/Unit	7,251	Pre-Tax
106	Net Present Value	12.0%	$26,670				
107	NPV Using IRR	12.55%	$0				
108							
109		**Internal Rate of Return Calculation**					
110	Initia	Investment.	**1**	**2**	**3**	**4**	**5**
111	1	(1,228,511)	1,124,720				
112	2	(1,228,511)	90,501	1,256,608			
113	3	(1,228,511)	90,501	100,536	1,382,962		
114	4	(1,228,511)	90,501	100,536	109,079	1,517,897	
115	5	(1,228,511)	90,501	100,536	109,079	117,833	1,659,605
116							
117		IRR =	-8.45%	4.89%	9.22%	11.36%	12.55%
118							
119							

The cell space D5:H16 contains a sub-module which calculates the available mortgage using both the Loan-to-Value approach and the DSR approaches which were covered in Chapter 8-63. The program compares the mortgage amounts and chooses the lesser mortgage. Calculation of the interest on Line 25 is accomplished by first determining the remaining balance of the loan at the end of each year. This balance is subtracted from the beginning balance, which delivers the loan paydown. The loan paydown is subtracted from the annual payment amount, leaving the interest portion of the total payment.

The calculation of the Net Operating Income begins on line 32 and concludes on line 46. The calculation of the Spendable Income or Cashflow Before Taxes (CFBT) is straightforward and presented on Lines 49-51. But in order to calculate the Taxable Income and CFAT we need to digress to cover Depreciation and Amortization deductions.

Determining Real Property Depreciation Allowance

The improvement on the land is a depreciating asset. Current tax code recognizes this and permits the owner to deduct from his annual operating income an allowance for the wasting of his asset. Changes in tax law have made the determination of deductible allowances for the wear and tear on the improvement a relatively easy task.

Since the value of the underlying land is non-depreciable, the real property investor must first make an **allocation** of the value of the property attributable to the land and, separately, to the improvements, or structures. *Only the value of the improvements is depreciable.* All surveying, engineering, inspection fees, escrow fees, legal fees, brokerage fees (if paid by the buyer), and any other legitimate cost of acquiring the property (excluding loan costs[4]), are includable in "acquisition cost" before the process of allocation.

Unfortunately there are no guidelines to assist the investor in selecting a percentage of the acquisition cost attributable to *existing* depreciable improvements. In the case of new construction, the value of the land and the cost of constructing the improvements are matters of recent record. But in the case of existing improvements, judgment must be the guide.

The safest procedure is to arrange for a formal appraisal of the property by a well-qualified appraiser. But this is an expensive solution not always followed by most private investors.[5] The local tax assessor will make his own judgment of the proportionate value of land and improvement but this may or may not reflect the true market value of the components.[6] Even so, if, on audit, the tax assessor's allocation is not acceptable to the IRS, it may require a formal appraisal by an accredited appraiser as the only way to defend the taxpayer's allocations.

4 Points on non-residential mortgages, and on mortgage for 5 or more residential units, must be capitalized and recovered ratably over the life of the loan.

5 The lender may require an appraisal to make a new loan, but this appraisal may not allocate land and improvement value. The lender's appraisal is made for loan purposes and not to establish market value.

6 If the property is re-appraised infrequently, the assessor's current allocation may be very outdated.

On the model Pro-forma an estimate of improvement value has been made at 70% of total Acquisition Value (property price plus acquisition costs), and entered in cell B-7. The first year's depreciation allowance (cost recovery allowance) will be based on the product of the Total Acquisition Cost (cell B-6) times the percent Allocation To Improvement (cell B-7). This value is the basis on which the appropriate depreciation allowance is entered in cell C-56.

Note that the cost of loan points is *not* included in the depreciable basis of the property (they are amortized separately) but it is included in the total of the investor's *initial investment*, B19, which will be used to measure return rates.

Choosing the Correct Depreciation Schedule

The theoretical purpose of the annual allowance for depreciation (cost recovery) of the improvement is to permit the owner to set aside a sum to compensate for the wearing out of the income-producing asset. The rate at which the investor could recover the cost of the improvement
was once open to a number of options. Tax law revisions, beginning in 1986, have now set the specific time periods over which the cost of real property improvements *must* be recovered:

- ✓ Residential real estate ....27.5 years
- ✓ Non-Residential real estate ....39 years

These "economic lives" are applicable regardless of the actual age of the property: a 60 year-old apartment house must be depreciated over not less than 27.5 years, even though its true remaining economic life may be only a few more years. A 50 year-old industrial building must be depreciated by a new owner over 39.0 years. These depreciable lives begin anew for each new owner beginning on the day he places the property in service.[7]

In the case of property composed of both residential dwelling units and non-residential units, the shorter period (27.5 years) may be used only if the total gross rental from the residential units is equal to or greater than 80% of the total gross rental income from the entire property in the taxable year. This becomes significant when dealing with new, mixed-use property which is composed of both residential and non-residential space. "Gross rental income" here means Collected, or Effective, or Gross Operating Income (Line 37). If Collected residential income is less than 80% of total collected income, the longer 39-year schedule must be used until the 80% level is reached.

Here's how the Annual Depreciation Amount of the model residential property was determined:

Total Acquisition Cost....................... $4,860,000
Allocation to Improvement.......................... 70%
Depreciable Basis 3,402,000
Depreciable Life ....27.5 years

7 Which is the date on which the property is offered for rent, not the date on which it is first rented.

$$\text{Allowance per year} = \frac{\$3{,}402{,}000}{27.5} = \$123{,}709$$

In those cases in which a property is acquired by means other than construction or cash purchase, the Acquisition **Basis** [8] of the property may be quite different from its acquisition cost. In the case of property acquired by gift, by exchange, by foreclosure or by inheritance, the Basis is likely to be much different and a competent tax advisor should be consulted.

Note: All rental real estate is subject to a "mid-month convention" which limits the depreciation deduction in the first month of operation to one-half month's depreciation allowance. If a real property is acquired at any time during the month of January, for example, it would be entitled to only one-half month's depreciation for the month of January, regardless of the day during the month the property were placed in service. Therefore, such a property would receive only 11.5/12 of the full year's allowance. The same mid-month convention applies, in exactly the same way, for the month of the sale or disposition of the property.

In the example above, the first year's depreciation allowance would be $118,555. This same depreciation allowance would be applicable in the year of the sale. The IRS provides a table[9] which indicates the allowable percentage of the depreciable basis which may be deducted for each year of the holding period, depending upon the month in which the property is first placed in service.

This explains why the depreciation deductions in Cell C56 and Cell G56 are less than the other years. (They are, in fact, 11.5/12ths of the allowances of the intervening years.)

How Loan Points are Handled

Points paid for a mortgage used to purchase a personal residence are deductible in the year paid. Points paid for commercial loans, or on residential income properties, must be capitalized and recovered ratably over the term of the mortgage. These points should be excluded from the Depreciable Basis since they must be deducted separately over the life of the mortgage and not over the depreciable life of the improvement.

8 *Basis* pertains to the value of a property for the purpose of determining taxable gains and losses. *Cost* refers to what one must pay for the property, and may have little or nothing to due with *Basis or Value*

9 Modified Accelerated Cost Recovery System (MACRS)

When a property is disposed of , the unrecovered loan points are deducted from that year's taxable income. Line 57 of the spreadsheet provides for an annual deduction from Taxable Income for the amortization of loan points. If the property is transferred, or the loan paid–off, before the entire cost of the points has been fully recovered, the unamortized portion of the points is treated as an ordinary expense, convertible to a tax savings at the owner's incremental tax rate (Lines 87– 90).

Determining the Original Basis

With respect to existing properties acquired by purchase, the Original Basis, Total Acquisition Cost, Line 66 , is something more than the contract price; it includes all the necessary and legitimate costs of placing title to the property in the investor's hands. Adjustments to this Original Basis are made during the holding period as the result of depreciation allowances taken, capital additions made to the property, and also as the result of any partial sales. Since these rules are complicated, professional advice is essential and should be obtained.

If personal property is included in the sale, these items should be excluded from the total and depreciated under more favorable MACRS rules allowing shorter recovery periods for personal property. These rules are contained in S. 167 and S.168 of the Internal Revenue Code and are available on the WWW.

The Adjusted Basis at Time of Disposition

As you can see on the Spreadsheet, the Adjusted Basis (Line 64) starts with the Original Basis, which in this case is equal to the total Acquisition Cost in cell B-6. To this value are *added* all the costs of final sale (cell Line 67) and the cost of any capital additions. Total depreciation (Cost Recovery) taken over the holding period (Line 66), and the Basis of partial sales, if any, are subtracted from this total to produce the **Total Adjusted Basis** (Line 68) at sale time .

If capital improvements are made during the holding period, this section of the Spreadsheet can be "opened up" and the depreciation allowance added in the appropriate year.[10] If a partial sale of the property had also occurred during the holding period, the proportionate Basis of property sold would be subtracted from the Original Basis,[11] lowering the Adjusted Basis. This will be offset by the lower sales price of the remaining property.

The Reversion Value of the Investment Property

The result of a sale of the property is the return, or **reversion**, to the owner of his **net** equity in the property. The net reversion value of the property is equivalent to the Future Value of the cashflow series, and occupies the same place on the analysis T-Bar as it did in Chapter 2. If we are to determine the reversion value of the property, it should be on an After-tax basis, since the computation of the annual cashflows (**PMT**s) was also done on an After-tax basis.

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10 If a capital improvement is added, the depreciation allowance on Line 57 would also be increased.

11 Alternately, partial sales can be *added* to Adjusted Basis

Estimating the Final Gross Sales Price and Net Proceeds

There are a number of ways in which the final gross sales price of the property can be estimated at the end of the holding period. Since income-producing property derives its value from the net income it produces, the most common method is to capitalize the net operating income which is projected for the property in the first full year *following* the sale date. Proponents of this method argue that this portrays the present value of the property in the hands of the new owner. Other analysts capitalize the net operating income at the *end of the last year* of the holding period, arguing that any increases in future income belong to the new buyer and not to the seller. Both arguments have merit, but if the property were to be appraised at the end of the holding period the appraiser would use the current NOI and not the NOI of the following year.

If the final sales price is determined using the NOI in the year following the sale,[12] it is necessary to extend the pro-forma for the 6th year through the NOI line, Line 46. Data below this line are irrelevant and therefore are omitted.

The owner's net equity at the end of Year 5 is the result of the following deductions from final sales price:

Final Sales Price (G71), say,	$5,599,591
Less Transaction fees @ 5.5%[13]	307,977
Equals Net Sales Price	$5,291,613
Less Capital Gains Tax	?
Less Mortgage Balance	?
Equals Owner's Net Equity & Proceeds	?

Let's estimate what the Capital Gains tax and remaining mortgage balance will be at the time of disposition.

Capital Gains and the Capital Gains Tax

A capital gain (or loss) is currently defined as a gain (or loss) arising from the disposition of a capital asset. Gains from real estate held for investment, or for use in a trade or business, are taxed as is a capital asset: gains from property held twelve months or less[14] are taxed as short term capital gains at *ordinary* income rates; gains from property held more than twelve months are taxed as *long-term* capital gains. Losses from the disposition of **real** property are fully deductible as *ordinary losses* regardless of the holding period.

12 If it is inappropriate to determine the reversion value by the capitalization method, then the reversion value must be determined by discounting the anticipated remaining cashflows which would accrue to the next owner. In some cases the value of the improvement will decline to zero, while the value of the land may either decline or increase in value.

13 Commissions, and some closing costs, are fully negotiable.

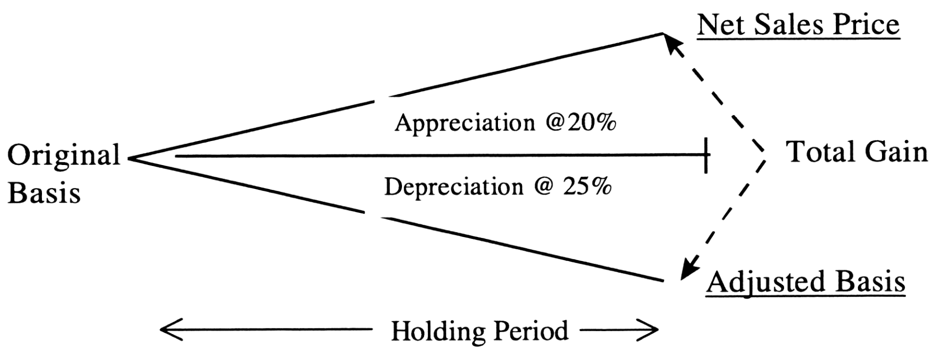
14 As of 2002. Congress constantly tinkers with the LTCG tax, so check current rules and regulations.

In July 1997,[15] Congress made substantial changes to the taxation of long term capital assets, but applied different rates to different classes of capital assets. Long-term gains from stocks and bonds, for example, enjoyed a reduction in rate from 28% to 20% With respect to real estate, however, Congress elected to impose a different rate on depreciation taken during the holding period. The rate applicable to *Depreciation Recapture*[16] will be 25%. Any gain remaining after the total depreciation is subtracted from the total long-term gain will be taxed at the 20% rate.

Beginning January 1, 2001, an asset held for a minimum of 5 years is taxed at the maximum LTCG rate of 18%. This rate requires that the property be held for a minimum period of 5 years *following* Dec.31,2000. For those in the 15% tax bracket, the maximum LTCG rate is to be reduced from 10% to 8% for any property which *has been held* for a minimum of five years. ("Collectibles," such as art, rugs, antiques, metals or gems, stamps or coins, continue to be taxed at the 28% rate.)

Estimating the Capital Gain at Sale Time

The accompanying diagram illustrates the sources of the capital gain from a depreciable real property investment. A portion of the total taxable gain stems from the appreciation of the property over the holding period. But a portion of total gain also arises from the total depreciation taken during the holding period.



In each year in which a depreciation allowance was taken as a deduction from ordinary income, the Original Basis was lowered by the amount taken. At the end of the holding period, the final *Adjusted* Basis in the property is equal to the Original Basis less the total accumulated depreciation deducted over the holding period. (Other additions and subtractions may apply.[17])

Note: Any depreciation taken prior to 1987 which was in excess of the amount which would have been allowed under the straight-line method is recaptured as *ordinary income* at the time of sale. These excess amounts were the result of using accelerated methods of Cost Recovery (e.g. declining balance methods) which were repealed beginning in 1987.

15 Made retroactive to May 7, 1997. Further reductions in the rate ar

16 S. 1250 Unrecaptured Depreciation

17 Capital additions and partial sales.

Graphically, the vertical distance between the Net Sales Price and the Adjusted Basis at the time of disposition represents the magnitude of the Indicated Long Term Capital Gain.[18]

Note that the Sales Price is net of (after) the costs of sale. The straight-line Depreciation allowances, taken in the past as deductions from taxable income, are now "recaptured" at the 25% rate. Gains attributable to appreciation in excess of the depreciation taken, will be taxed at the lower 20% LTCG rate[19] for individuals in a tax bracket greater than 15%.

The applicable rate is applied, therefore, to the total of the gains attributable to appreciation *plus* the total of the depreciation allowances taken over the holding period.

Since we know the amount of the depreciation taken over five years (allowing for the mid-month convention), we can calculate the Adjusted Basis of the property at the time of disposition. These calculations are represented on Lines 55-59, Column G, of the Spreadsheet

Original Basis	$4,860,000
Less Depreciation Taken (G-66)	–608,236
Plus Costs of Sale (G-67)	+307,977[20]
Less Basis of Partial Sale	0
Adjusted Basis at Sale Time(G-68)	$4,559,741

The total Gain is the difference between the Sales Price[21] and the Adjusted Basis:

Sales Price (G-71)	$5,599,591
Less Adjusted Basis (G-72)	4,559,741
Indicated (Realized) Gain (G-73)	1,039,850[22]

But of this total gain, the amount attributed to Depreciation Recapture will be taxed at the 25% rate, while the balance of the gain will be taxed at 20%.

Accrued Depreciation (G-77)		$608,236
Recapture Rate	25%	*$152,059*
Net LTCG (G-80)		$431,613
LTCG Rate	20%	*86,323*
Total Tax On Sale (G82)		$*238,382*

18 We assume the property has been held more than the required number of years.

19 Part of the appreciation is due to inflation, but Congress continues to tax gain due to inflation.

20 The Costs of Sale may be added to the Adjusted Basis or subtracted from the Gross Sales Price. We have chosen to add these costs to Basis.

21 We added the costs of sale to the Adjusted basis. We could have deducted them here instead.

22 Gains become *recognized* rather than simply *realized* when they become subject to current taxation.

Taxpayer's Recapture of Unamortized Loan Points

Of the total loan points paid, $55,302 ($3,686,791 * 1.5%), only $27,651 ($5,530.20 per year * 5 years) has been deducted. The amount of unamortized loan points, $27,651, is deductible in the year of the sale as an ordinary interest expense (G-88). The tax savings to the owner will be approximately equal to the unamortized points times the owner's incremental ordinary tax bracket (B-14 = 33%). This tax (savings) will be subtracted from any tax due, $238,382 – 9,125 = 229,257 (Lines G-82 and G-90).

Estimate of Net Proceeds After-tax

The balance of the mortgage is easily calculated following the procedures outlined in Chapter 8, Amortizing Mortgages. The final net cash proceeds can be estimated (Lines 94–98) on the Spreadsheet:

Final Sales Price, say, ...	$5,599,591
Less Transaction fees @ 5.0%	–307,977
Less Mortgage Balance	–3,527,851
Less Federal Capital Gains Taxes	–229,257
Equals Owner's Net Equity =	**$1,534,506 = Reversion = FV**

Finally, we have determined the projected **F**uture **V**alue of this investment to the investor after five years of ownership. The Spreadsheet provides the After-tax cashflows for each year of ownership enabling us to complete our 5-year T-Bar. The amount of the Initial Cash Investment (cell B-19) is equal to the down payment required plus the cost of the loan (points).

E-O-Y	1	90,501	
	2	100,536	(Line 63 of Spreadsheet)
	3	109,079	
	4	117,833	
	5	125,099 + 1,534,506	(Cell G-63 + G-98)

All this work has been done to complete this T-Bar and to answer the four most significant questions which an informed investor should ask:

1. *How much* cash must I invest ?
2. *When* must I advance this cash ?
3. *How much* cash can I expect to receive back ?
4. *When* am I to receive it ?

Since we now know the amount of the returning cashflows *and their timing*, the investor has the option of:

1) Asking the calculator (or computer), given a discount rate, to determine a **P**resent **V**alue for this cashflow series, or

2) Calculating the yield by supplying the amount of the initial investment and asking the calculator to solve for the **I**nternal **R**ate of **R**eturn, or
3) Determining the Net Present Value (**NPV**) of the investment by supplying the amount of the initial investment *and* a desired discount rate.[23]

Let's consider each of these solutions separately.

Option 1 requires that we provide a discount rate, which raises the question "How should a discount rate be chosen? "

Selecting an Appropriate Discount Rate

The discount rate should be equal to the constructed capitalization rate *plus* an estimated rate of inflation over the holding period. This assumes that the initial capitalization rate by which the property was acquired contains allowances for: 1) a satisfactory safe rate ON investment cash, 2) a satisfactory rate of return OF investment cash, and 3) a satisfactory risk premium. The overall capitalization rate does not contain an allowance for inflation.

To demonstrate that the discount rate is equal to the capitalization rate + an allowance for inflation, consider this situation.

> Your investment property returns $100 000 per year in NOI, which will increase an estimated 3% annually as the result of anticipated inflation. You hold the property 5 years and sell at the end of the fifth year, using a 9% capitalization rate applied to the sixth year's income to determine the sales price. The annual NOIs are:

Year	**1**	**2**	**3**	**4**	**5**	**6**
Income	$100,000	$103,000	$106,090	$109,273	$112,551	$115,927

If the present value of the property were determined by capitalizing the Year 1 income, the value would be:

$$\frac{100{,}000}{.09} = \$1{,}111{,}111$$

If the value of the property is to be determined by discounting future income, we need to determine the reversion amount of the sale at the end of year 5, by capitalizing the 6th year's estimated income:

[23] Note that The IRR and NPV always require an initial investment amount.

$$\text{Reversion} = \frac{115927}{.09} = \$1{,}288{,}078$$

The T– Bar would be:

		? →
EOY	1	100,000
	2	103,000
1 = 12%	3	106,090
	4	109,273
	5	112,551 + FV of **$1,288,078**

Applying the methods for discounting uneven cashflows (Chapter 2), the Present Value is determined to be **$1,111,111.**

Therefore an appropriate discount rate may be determined by adding to a constructed capitalization rate the inflation rate (3%) anticipated over the holding period.

Applying this discount rate (12%) to the T–Bar for the apartment house example, we can determine the Present Value of the Investment:

		(PV ?)
EOY	1	90,501
	2	100,536
i = 12%	3	109,079
	4	117,833
	5	125,099 + FV of $1,534,506

Using the same discount technique, we find the **Present Value** of the cashflows from this investment, discounted at 12% per year, is $1,255,181. This indicates that the Present Value of the future benefits, when discounted at 12% per year is greater than the initial cash investment of $1,228,511 (B-19).

In order to employ Option 2, (the IRR) above, we need to provide the total initial investment amount of $1,228,511 (Cell B–19). We are now asking the calculator (or computer) to determine the yield on the cash invested (the IRR):

		(1,228,511)
EOY	1	90,501
	2	100,536
IRR = ?	3	109,079
	4	117,833
	6	125,099 + FV of $1,534,506

The IRR, or yield, at the end of Year 5 is 12.55% (G-117)

In order to solve for the **Net** Present Value, Option 3, the investor would supply *both* the initial cash investment *and* the desired yield as a discount rate:

		NPV = $\sum$ PV – 1,228,511
EOY	1	90,501
	2	100,536
i = 12%	3	109,079
	4	117,833
	5	125,099 + 1,535,506

The **Net** Present Value, discounted @ 12%, is $26,670 (C-106).

The Net Present Value is $1,255,181–1,228,511 = $26,670. Since the **NPV** is positive, this investment would exceed the requirement of an investor seeking a 12% yield. (We already know that the actual yield is 12.55%.)

Calculating the IRR on a Spreadsheet

In order to calculate the IRR on a spreadsheet all the values of the T-Bar must be contiguous to one another, or in a continuous array, starting with the amount of the initial investment (cell B-19) and continuing through each year's operating income and the reversion value. This can be done by constructing a matrix which represents these values in an uninterrupted, lineal format. This matrix is depicted on Lines 111-115. The value of cell G-115 is the sum of the After-tax cashflow, cell G-62, and the reversionary amount, (cell G-98). The IRR function in each cell on Line 117 "reads" Lines (111-115) from left to right.

For example, the formula in Cell E-117 is " =IRR(B113:E113)." Note that the initial cash investment is a negative number in keeping with the sign convention which requires that the initial investment be a negative number.[24]

As you already know, this method of valuing a cashflow is known as Discounted Cashflow Analysis. While greatly to be preferred, there are other methods used to value real estate cashflows which deserve a brief mention.

Capitalization Rates

In Chapter 3 we discussed the method of determining the Present Value of an even cashflow which continues in perpetuity. You will recall that the Present Value of such an ordinary annuity is represented by the formula:

$$\textbf{Present Value} = \frac{\mathbf{C}}{\mathbf{i}}$$

where **C** is the even cashflow and **i** is the capitalization (discount) rate. (A capitalization rate *is* a discount rate used to convert a periodic PMT into a capital value.)

Understanding this, it is a small step to understand that the Capitalization Method of determining the fair market value of an income-producing real property is really a method of determining the Present Value of an investment by discounting in perpetuity the anticipated cashflow by a yield rate acceptable to the investor. Then, by changing our terminology a bit, we have:

24 Reversing all the signs will yield the same IRR.

$$\textbf{Fair Market Value (PV)} = \frac{\textbf{Net Operating Income}}{\textbf{Capitalization Rate}}$$

Most experienced investors understand the relationships of the capitalization rate, the Net Operating Income and the *Present* or *Fair Market* Value:

1. Fair Market Value	x	Capitalization Rate	=	Net Operating Income
2. Net Operating Income	÷	Capitalization Rate	=	Fair Market Value
3. Net Operating Income	÷	Fair Market Value	=	Capitalization Rate

Therefore a property having a Fair Market Value of $1,000,000 and producing Net Operating Income of $100,000 will be determined to have an overall[25] capitalization rate of:

$$\frac{\$100000}{\$1000000} = .10 = 10\% \quad \text{(Formula \#3 above)}$$

The second relationship, #2, is the relationship which real estate appraisers use to establish Fair Market Value using a method called the **Income Approach to Value**. The appraiser determines the Net Operating Income of the property by reviewing the rental income and operating expenses[26] and then *selecting* an appropriate **overall** capitalization rate.[27] He arrives at an estimate of Fair Market Value by dividing the NOI by the selected "cap" rate:

$$\frac{\text{Net Operating Income}}{\text{Capitalization Rate}} = \frac{\$100000}{0.10} = \$1{,}000{,}000 = \text{FMV}$$

Capitalization rates are most often derived from current and recent market data. It has already been shown that, given a capitalization rate, we can estimate the Fair Market Value of a property if we can determine its NOI.

For example:

$$\frac{\text{Net Operating Income}}{\text{Capitalization Rate}} = \text{Fair Market Value} = \frac{\$75000}{0.10} = \$750{,}000$$

The Fair Market Value varies inversely to the capitalization rate. Notice that as the cap rate increases, small changes in the rate result in large differences in Value:

$$\frac{\$75{,}000}{\mathbf{.07}} = \mathbf{\$1{,}074{,}285} \qquad \frac{\$75{,}000}{\mathbf{.08}} = \mathbf{\$937{,}599}$$

$$\frac{\$75{,}000}{\mathbf{.09}} = \mathbf{\$833{,}333} \qquad \frac{\$75{,}000}{\mathbf{.10}} = \mathbf{\$750{,}000}$$

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25 "Overall" because it measures the return on both the investor's equity and the lender's debt.

26 Effective (collected) income minus operating expenses = Net Operating Income (NOI).

27 Usually from comparative market data

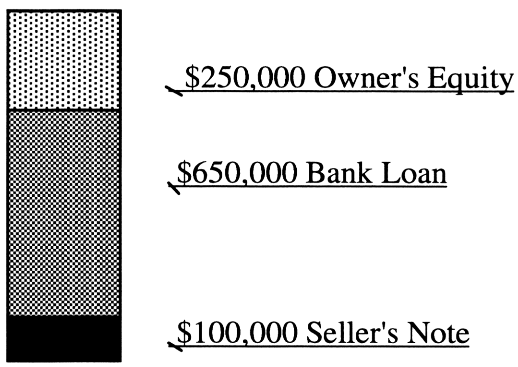
A few investors prefer to apply a *multiple* to the NOI. The multiple is nothing more than the reciprocal of the capitalization rate. For example, dividing the NOI by .07 is equivalent to multiplying the NOI by 14.3.

Band of Investments Approach to Cap Rates

In addition to the Market Data method of selecting Capitalization rates, a second method is an approach which relates the investor's risk to those incurred by other entities which have a concurrent financial interest in the property.

Consider the following *capital structure* which combines ownership (equity) and debt interests in a recently acquired office building

Total Value $1 Million



The question posed is:

*"As the result of this capital structure, which party bears the greatest **risk** in this investment?*

The Owner ?
The Bank ?
The Seller ?"

Most respondents judge that the Bank bears the greatest risk because it has the largest amount of cash invested; others say the recent Seller who holds a second trust deed, because a foreclosure by the Bank would expunge the Seller's position.

In truth, the greatest risk of loss is borne by the Owner since the property could decline in value as much as $250,000, eliminating all the Owner's equity, without reducing either the Bank's security interest or the second trust deed holder's security interest in the property.

A well established precept of investment is that *returns run with risk.* Those participants in an investment who bear greater risk are entitled to a greater reward. If the Bank charges an 8.5% interest rate on its share of the cash advanced, some observers would hold that the second trust deed holder is entitled to a 10% rate of return. If this is so, to what return is the Owner entitled if he carries the greatest risk of loss?

Many would opine that he is entitled, by comparison, to at least a 12%, or greater, return.

Therefore each investment position would require the following amounts of net operating cash to meet its particular yield requirements. The total of these sums is the amount of annual income, after expenses of operation, the property must produce, which also happens to be its **Net Operating Income:**

Risk Position	Cash at Risk	Required Rate	Cash Required
#1 Owner	$250,000	12%	$30,000
#2 Seller	$100,000	10%	$10,000
#3 Bank	$650,000	8.5%	$55,250
Total	**$1,000,000**	?	**$95,250**

But a Net Operating Income of $95,250 flowing from a property valued at $ 1 million indicates an overall capitalization rate of:

$$\frac{\$95{,}250}{\$1{,}000{,}000} = .09525 = \mathbf{9.525\%}$$

Overall Capitalization Rates, therefore, can be determined from Comparative Market Data and by the Band of Investment method. Of the two, the most commonly used method is the Comparative Market Data approach.

Strengths and Weaknesses of Cap Rates

For most investors, the capitalization method of determining market value is relatively quite simple. But many of these same investors do not appreciate that this method is based on two important assumptions: 1) the net operating income (cashflow) from the property will never change; and 2) the cashflow will go on forever (in perpetuity). Though convenient, neither of these implicit assumptions will stand the test of time.

A second objection to reliance on capitalization rates *gathered from historical market data* is that they are always *retro*spective in nature. They tell where the market has been, not where it may be headed. But income property (as does any investment) derives its value from the present worth of *future* benefits, not from the value of bygone benefits. By using last year's cap rates ("trailing rates"), an investor can overpay in a declining market or miss opportunities in a rising market. Buying real estate using historic cap rates is analogous to pricing a stock using a trailing P/E ratio, or to pricing a bond using last year's yield-to-maturity rates.

The last weakness pertains to the difficulty of finding truly comparable properties and then accurately uncovering their Net Operating Incomes. While it is common to see appraisals comparing income properties that are not in the same location, not of the same quality, not on the same side of town, and not of the same age, the Market Data approach to selecting capitalization rates to determine fair market value remains the most frequently used determinant of value for the average investor.

Cash-on-Cash & Gross Rent Multipliers

Two other single-period investment yardsticks are in popular use: **Cash-on-Cash**, and the **Gross Rent Multiplier.** The latter index, the Gross Rent Multiplier (**GRM**), is simply the quotient resulting from the division of a property's Fair Market Value by its Gross Scheduled Income:[28]

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28 Sometimes described as Gross Potential Income

$$\frac{\text{Fair Market Value}}{\text{Annual Gross Scheduled Income}} = \text{Gross Rent Multiplier}$$ [29]

If it can be ascertained that a property which recently sold for $1 million had a Gross Scheduled Income (**GSI**) of $125,000, then its "gross multiplier" would be:

$$\frac{\$1000000}{\$125000} = 8.0 \text{, the } \mathbf{GRM}$$

Therefore, a nearby property *of similar characteristics,* sporting a Gross Scheduled Income of $150,000, could be expected to have a market value of:

$$\$150{,}000 \times 8.0 = \$1{,}200{,}000$$

The GRM is very widely used in the marketing of smaller multi-family residential units, probably because operating expenses submitted by owners of these units are notoriously unreliable.[30] Any estimate of Net Operating Income based on understated or under-reported expenses consequently will be overstated, resulting in overstatement of market value.

The GRM is best used as a screening tool, since it measures nothing below the level of *Scheduled* rental income on the Operating Statement. It certainly provides no indication of a property's operating income due to vacancy or credit losses, no gauge of operating expenses, nor of future cashflows. But it is precisely future cashflows for which the informed investor is paying.

The **Cash-on-Cash** index is a useful, *single-period* measurement of value because it looks beyond vacancy, beyond operating expenses and beyond net operating income to income remaining after mortgage payments. It is the relationship of the *Spendable Income* (income after debt service, but before taxes)[31] to the Cash *initially* invested in the property, or:

$$\frac{\textbf{Cash After Debt Service}}{\textbf{Cash Initially Invested in the Property}}$$

It, too, may be faulted in that it does not encourage a look at the potential for future income. But it does give a rather good indication of the Return on Equity the investor can expect in relation to his cash investment *at the end of the first period of ownership*. Since it measures cash after operating expenses *and* after the costs of servicing the debt, it is also a reflection of the mortgageability of the property and its potential leverage.

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29 The Gross Rent Multiplier can be expressed either as an annual or monthly figure.

30 Frequently as the result of poor record keeping and poor management; sometimes as the result of dishonesty.

31 Cash-on-Cash after taxes is also, but less frequently, measured.

Investors who rely upon the Cash-on-Cash index operate on the assumption that if the property can perform to a reasonable standard in its first year of operation, it should, in the absence of any obvious or foreseeable problems, continue to perform well. Sometimes it does, sometimes it does not.

This useful and valuable index is often misused by promoters who forecast future (spendable) cashflows and then compare these future cashflows to the first year's cash investment. This is a serious mistake, similar to comparing $1 to be received in Year 50 to $1 invested today: the process ignores the time value of money. It has the same inborn-error-of-metabolism found in the example cited earlier in which a promoter is willing to pay $1,000,000 for a property worth only $250,000,[32] but neglects to tell the seller *when* the price is to be paid.

Distinguishing Cash-on-Cash, ROI, ROE & IRR

These four indices are often confused, and frequently mis·used.

The first, Cash-on-Cash, is a measure of the rate of pre-tax cash returned on *initially invested* capital and is commonly understood to be limited to the first year of ownership. For example, $100,000 in pre-tax spendable income from a property in which the investor placed $1,200,000 *equity* would indicate a Cash-on-Cash return of 8.3%.

The **R**eturn on **I**nvestment (ROI) is a measure of the cash return on *total* investment (equity + debt).

The **R**eturn on **E**quity (ROE) is a *single*-period measure of return on *currently invested* equity. It is sometimes referred to as the *Equity Capitalization Rate.* Note that the ROE is a single-period index of performance, whereas the Internal Rate of Return, which also measures return on invested equity, is a multi-period index involving *all* the cashflows from each year's operations *plus* the reversion value of the investment.

Why the Buildup of Equity Needs To Be Monitored

Investments in real estate differ from investments in mutual funds or stock/bond programs in which dividends can easily be re-invested in additional shares. Dividends from equities or payments from bond funds can be added to principal and earn additional cash dividends. This is an attractive advantage to mutual funds. The financial result is the *compounding* of the dividends at a rate equal to future dividend or distribution rates. All the distributed cash from the investment is always working.

Individual ownership of real estate, however, provides no practical opportunity for continued reinvestment (in real estate) of spendable income.[33] The annual cashflow from an individually

32 "If it sounds too good to be true, it usually is."

33 Real Estate Investment Trusts (REITs) often provide an opportunity for dividends to be periodically reinvested.

owned real estate investment is generally too small to serve as a down payment on a similar investment. Therefore the cash is typically withdrawn from the investment for other uses.

But as the net operating income of the property increases so does its market value (given a constant capitalization rate). The increase in market value is proportionate to the increase in Net Operating Income, but because of the multiplying effect of leverage, results in a disproportionate increase in owner equity.

For example, a $1 increase in NOI capitalized at 10% results in $10 added to market value, *all of which accrues to owner-equity*. This appreciation in value, however, tends to lie fallow in the property until it can be retrieved and reinvested either by a sale, exchange or re-finance of the property.

Annual paydown on mortgage debt also adds to owner's equity. As a result the owner's after-tax Return on Invested Equity (ROE) for any one year becomes:

$$\textbf{Return on Equity} = \frac{\textbf{Net Income After Taxes and Debt Service}}{\textbf{Initial Invesment + Appreciation + Loan Paydown}}$$

This ratio may be calculated using both pre-tax and after tax cashflows, and is a prime index of current investment performance. It does not measure, however, total return on investment since it omits income from other periods as well as the reversion value of the investment. We can use the IRR for that purpose.

Chapter Summary

1. The acquisition of income producing property can be considered title to the future cashflows which can reasonably be produced by the property.
2. Cashflows derived from improved property are comprised of: 1) annual proceeds from operations; 2) cash from Depreciation and Amortization deductions; and 3) the After-tax reversionary income derived from the disposition of the property at the end of the holding period.
3. Property may be valued using single-period indices such as the overall capitalization rate, gross rent multiplier and the cash–on–cash methods.
4. Multiple-period indices of performance, such as the Internal Rate of Return, Net Present Value and Present Value methods of valuation, require the investor to address in some detail the probable performance of the property and the resulting changes in cashflow for a number of years into the future.
5. The computer spreadsheet is the easiest way to perform Discounted Cashflow Analysis since values can be changed and new results obtained almost instantly.
6. An understanding of the underlying principles of investment analysis and DCF enables the investor or his advisor to devise spreadsheets capable of representing both the commonplace and unusual investment situation.

Chapter 10
Effect of Leverage on Investment Cashflows

Leverage is an important investment tool which can have a profound effect on yields from an investment.

Leverage can be defined simply as the use of borrowed money to control an investment.

When a great deal of other people's money is used to acquire the investment, we describe the investment as being "highly leveraged."
The advantage of financial leverage sources from two facts:

1. Money borrowed at a rate lower than the rate at which the investment earns income results in added cashflow to the owner.
2. Any increase in investment value belongs entirely to the owner of the investment.

Neutral, Positive and Negative Leverage

We know from previous discussion that the rate at which a real property of a given market value throws off Net Operating Income (NOI) is its capitalization rate. The property produces this NOI regardless of how much or how little of someone else's money may be invested in it. Therefore a property acquired at a capitalization rate of 10% will cast off $10.00 for each $100.00 invested in it, without regard to its capital structure.

But if $50.00 of invested capital is acquired through the use of lender's funds, and the lender charges 10% for the use of these funds, then the owner pays a rate for the use of money exactly equal to the rate at which the borrowed funds earn income. This is a situation of **neutral leverage**.

If, however, the lender requires 11% for the use of $50 ($5.50 interest), then the owner must take this 'extra' 50¢ from his share of the NOI. The owner *pays* a premium over the investment's yield rate equal to 1% of the total amount borrowed; hence, 1% of $50, or $0.50. He is left with $49.50. In this situation every borrowed dollar earns less than it costs. This is a position of **negative leverage.**

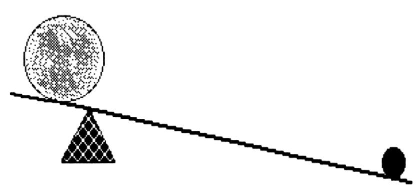
If the lender requires only 9% for the use of $50 ($4.50 interest) , the owner earns a *premium* on every dollar borrowed equal to 1% of the amount borrowed; hence 1% of $50 = $0.50. The borrower realizes $50.50. This is a position of **positive leverage.**

Therefore positive leverage occurs whenever the yield from the investment exceeds the interest rate on borrowed funds. Negative leverage occurs when the interest rate on borrowed funds exceeds the yield from the investment. Neutral leverage occurs when the yield is equal to the interest rate on borrowed funds, in which case the investor neither makes nor loses money on these borrowed funds.

Leveraging Stocks & Bonds

When equities and bonds are leveraged (bought on margin) the interest on borrowed funds typically exceeds dividends paid.

Therefore the owner who takes a maximum leveraged position must often "come up out of pocket" with additional funds in order to continue to maintain a leveraged position. This additional negative cashflow has a strongly negative impact on the IRR, and must be offset by gains in the sales price of the asset.



Give Me a Place To Stand and I Will Move the World

For this reason, in contrast to real estate, leveraged equities and bonds are not usually held for an extended period of time.

Leverage and Appreciation

If it were not for appreciation, it would make no financial sense at all to borrow any money when prevailing interest rates exceed the yield rate on an investment. But the market value of a leveraged asset is somewhat independent of its capital structure. In the case of real estate, the market value is tied to rents, and when these rents are increasing, as during periods of inflation, market value also increases. When it does, *the entire amount of the increase accrues to the equity owner*. This is the fact which justifies negative leverage during times of inflation.

When the stocks or bonds are accelerating in price, the rate of gain may exceed the interest rate on funds borrowed to acquire them. This makes both these markets sensitive to declining interest rates. The portion of the income which the owner may lose in additional cashflow to the lender because of negative leverage, he hopes to make up in cashflow due to substantial appreciation.

Risk of Leverage

The same high leverage which can be a wealth generator during inflationary periods can turn into a howling beast during times of deflation. These are times when property rents, and therefore NOIs, are declining. The property owner who is highly leveraged sends a greater and greater share of the NOI to his creditor, until – in terminal cases – the entire property, equity and all, is delivered to the lender via foreclosure.

In the case of a stock or bond, a declining market price or rising interest rates prompts a "margin call" since the value of the pledged asset is no longer adequate security for the broker's loan. Therefore the investor may be forced to sell exactly at the wrong time – when prices are declining. Increased margin sales can further depress stock and bond values prompting additional margin calls.[1]

Leverage is the original two-edged sword.

A particularly dangerous position is high leverage using loans that have short call (due) dates. If interest rates have risen substantially, the income from the asset may not be able to support a replacement loan. For this reason real estate lenders either limit the loan to value ratio or require that the borrower purchase mortgage insurance to safeguard the loan.

Stock brokers are limited by the Federal Reserve Board in the initial margin amount (50%) which an investor may use to acquire stock, but are not limited thereafter. Despite this limitation, some investors skirt the restriction by borrowing on the equity in their homes, while others invest in options and futures contracts. The crash of the NASDAQ market in the spring of 2001 erased $1.8 trillion of investor value leaving, in many cases, the investor with no value in his stock portfolio but with a high mortgage.

Effect on ROE

The effect of high leverage is easy to see.

Suppose that a property worth $100 is controlled by a small amount of equity, say $5.[2] During the year, the property's value increases 5% to $105. The gain due to appreciation, $5, belong entirely to the equity owner. Therefore the equity owner realizes a 100% return on invested capital ($5 return on $5 invested).

On the other hand, if the equity owner invests $95 in the property and borrows only $5 (5%), the property, unmindful of its capital structure, responds to market forces in the same way and rises

1 See 1929

2 Many homes are now purchased with a down payment as little as 3-5% of the purchase price.

the same $5 in value to $105. In this case, the equity owner realizes a 5.26% return on invested capital ($5 return on $95 invested).

On the other hand, if a 95% leveraged property declines 5% in value, the owner's equity is completely expunged.

This is the lure and risk of leverage.

Maximum Leverage

In Chapter 8, which reviews Debt Service Ratios, we discussed how real estate lenders insulate themselves from undue risk due to excessive leverage. Their main tools are, in the case of residential property, the loan-to-value ratio and mortgage insurance; in the case of commercial property they utilize the debt coverage (or service) ratio ,the loan-to-value ratio,[3] and personal guarantees.

The maximum leverage an income-producing real property can support depends on the cost of money. In terms of property operations, we can say that the maximum supportable debt occurs when:

$$\text{Net Operating Income} = \text{Cost of Debt Service}$$

$$\text{Value} * \text{Capitalization rate} = \text{Loan} * \text{Loan constant}$$

$$\frac{\text{Loan}}{\text{Value}} = \frac{\text{Capitalization Rate}}{\text{Loan constant}}$$

therefore,

$$\text{Leverage} = \frac{\text{Capitalization Rate}}{\text{Loan constant}}$$

Commercial lenders, however, are quite sensitive to the advantages and disadvantages of leverage and will not permit 100% of the NOI to be devoted to debt service. They require a cushion or margin of safety in the form of the Debt Coverage Ratio. Therefore:

$$\text{Maximum Leverage} = \frac{\text{Capitalization Rate}}{\text{Loan Constant x DCR}}$$

For example, a property with a capitalization rate of 10%, employing a loan whose annual constant is 0.1053, with a DCR of 1.25, could be leveraged:

$$\textbf{Maximum leverage} = \frac{0.10}{0.1053 * 1.25}$$

$$= 0.76 = 76\%$$

[3] Generally lending according to the mechanism which gives them the least risk exposure.

As you can see, both the loan constant (based partly on interest rates) and the debt coverage ratio vary inversely to maximum leverage. During periods of high interest rates, or when the lender perceives a property to be risky (and raises the DCR), the amount of loan leverage decreases.

Effect of Leverage on IRR

The IRR is extremely sensitive to a low initial investment by the investor. The lower the initial investment amount required to capture the cashflows, the higher will be the IRR because it requires a high discount rate to reduce large future cashflows to a small initial investment (Present Value).

But when more and more operating cash goes to service debt levels, the component of the IRR arising from annual operations declines steeply, and the component attributable to reversionary (sales) value rises steeply. This is simply another way of saying that if an asset is highly leveraged, most of the return will derive from the net sales proceeds at the end of the holding period and very little from cash returns *during* the holding period.

This "wait to see" effect of high leverage is the essence of the risk which almost always accompanies high IRRs.

Chapter Summary

1. Leverage involves the use of borrowed funds in the acquisition or refinance of an investment.
2. Positive leverage occurs when the yield from the investment exceeds the cost of borrowed funds.
3. Negative leverage occurs when borrowed funds cost more than the return rate.
4. Neutral leverage occurs when neither a premium nor a loss on borrowed funds occurs.
5. Low interest rates encourage the use of leverage since the capitalization rate on real property often exceeds the cost of borrowed funds.
6. Highly leveraged positions carry substantial risk from declining income, and, in the case of adjustable rate mortgages and stocks, rising interest rates.
7. Negatively leveraged assets have a negative impact on both ROI and IRR returns, which may be offset by the appreciation.

5. Equities acquired through margin accounts represent the same risk as does a highly leveraged real property: falling stock prices or rising interest rates may quickly erode owner equity forcing a sale when prices are declining.

Chapter 11 Problems... Problems... Problems...

Solving problems is the key to mastery of the mathematics of cashflow analysis. So here are some problems which you are invited to solve. *Answers appear at the end of the chapter.*

Consider all PMTs to be EOP unless specified or indicated otherwise.

Group I

1-1. What is the equivalent present value of a note which promises to pay you $1,000 in one year, if you could otherwise invest the cash at 12% p.a.(per annum)?

1-2. What would be the value of $2,000 deposited in a bank paying interest at the annual rate of 4.5% for two years if savings are compounded:
quarterly? _________
monthly? _________
daily (360 days/per year)? _________
continuously? _______________

1-3. How long would it take for an investment of $5,000 to double if the investment earns a return of 10% p.a., compounded quarterly, and the returns are similarly invested?

1-4. You plan to buy a new car which costs $16,000. What annual interest must your savings earn if you invest $5,000 initially and then $250 at the end of every month for three years? Assume earned interest to be credited to principal monthly. How much would you be required to invest monthly if you desired to buy the car at the end of the second year?

1-5. You decide to buy a new large-screen TV costing \$2,000. Dealer A offers to sell the TV for \$500 down and 36 monthly payments of \$54.23. Dealer B offers exactly the same machine with \$750 down and 36 monthly payments of \$49.04 over the same time period. What interest rate is charged by each dealer?

1-6. How much cash must you deposit in an account bearing 8.5% interest p.a., compounded monthly, if you wish to withdraw end-of-the-month payments of \$250 over the next 15 years?

1-7. Jones desires to establish a trust account which will provide \$1,500 per month in living expenses for 10 years, to commence 5 years from today. The payment will be made at the beginning of the month. He anticipates no contributions to the trust once payments begin. What sum must Jones deposit at the end of each month over the next 5 years if the trust invests the funds at 7% interest p.a., compounded monthly?

1-8. Mary Dee seeks to add to her assets by purchasing shares in a mutual fund which sells for \$52 per share and which pays annual dividends of \$3.50 per share. Disregarding share appreciation and a change in dividend policy, how many shares must she purchase at the beginning of each year if her goal is to accumulate \$10,000 at the end of 5 years? Assume that all dividends are reinvested.

1-9. Joseph Grange purchased a farm for \$275,000 with a down payment of \$100,000. The seller agreed to carry back a note for the remainder of the price for 5 years with monthly payments of \$1,500, including interest at the annual rate of 8%. What amount will farmer Grange need to refinance at the end of 5 years?

1-10. What end-of-month investment would be required in order to accumulate \$1 million at the end of 40 years if the cash could be invested at the rate of 10% p.a., compounded monthly?

1-11. What would be the future purchasing power in today's dollars of \$10,000 invested at the end of each year for 20 years at 8% interest, if the inflation rate over that time averaged 5% per year?

1-12. What sum today (in constant dollars) is financially equivalent to investing \$10,000 at the end of each year for 20 years if the investment earns 8% p.a. and inflation averages 5%?

1-13. If you desire to realize a 20% annual return on your investment, what price would you offer for a 3-year trust deed note in the amount of \$25,000 if the note provided interest-only, monthly payments of \$250.00 ?

Group II

2-1. What is the value today of a series of annual payments over three years which begins at $1000, payable at the end of the first year, and then increases 10% per year, if the opportunity cost of money is 15%?

2-2. You are scheduled to receive a guaranteed annuity of $2,000 per year for 5 years. The annuity will then increase to $3,000 for the next five years. How much could be borrowed from a banker who will lend 50% of the present value of the annuity and who discounts future cash receipts from this annuity at 12% per year?

2-3. What price would you pay for a lease which has a term of 15 years if the first 5 years of the lease requires rent of $3,000 per month, payable monthly in advance, and the remaining term requires rent at the rate of $3,750 per month in advance? You require a 12% return on invested capital.

2-4. If you were able to acquire the lease described in #3 for $275,000, what would the Net Present Value of this investment be?

2-5. You have been asked to invest $20,000 in a new venture which will not provide any returns for three years. Thereafter, the investment is expected to return $10,000 per year for 8 years? What is the Net Present Value of this investment if you require a 20% return on this type risk?

2-6. If the investment in problem #5 met forecasts, would be your yield on this investment?

2-7. If the start of the returns in problem #5 were delayed one year, what would the Net Present Value of the 8 annual payments be? Would you make this investment? Why?

2-8. The owner of an office building has secured a tenant for 10,000 s.f. The tenant has agreed to lease the property for 5 years, but requires very specialized space improvements that will cost an extra $120,000. The tenant can pay for half the estimated $12 / s.f. cost, but requests the owner to finance the balance and to add the amount to the agreed-upon base monthly rent of $1.90/s.f., payable in advance over the term of the lease. The owner requires a 15% return on invested capital. What will be the total monthly rent if the owner agrees to this proposal?

2-9. A fast food restaurant presently nets $180,000 per year, representing a growth in annual earnings of 20% per year over the past 5 years. If this growth rate slowed to 15% per year over the next 5 years, what would be the present value of the business to a new owner who requires a 25% p.a. return on capital? Ignore residual value.

Group III

3-1. A block of stock purchased on January 1 of a certain year for $100,000, and held for exactly 5 years, increased in value 8% each year. During this time, the stock returned a 5% dividend rate on its value at the end of the year. If all proceeds were reinvested in the stock, what is the investor's yield (IRR) on this investment if he liquidates the investment at the end of the 5th year.?

3-2. A small business owner acquired a business by discounting the anticipated income of $50,000 the first year at a 20% rate, $65,000 the second year by a 25% rate, and the third year's income of $85,000 by a 30% rate. If these incomes are realized at the end of each year, what will be his Internal Rate of Return on the price he paid? Disregard residual value.

3-3. An investor purchased a small residential investment property needing extensive rehabilitation. The property produced the following year-end net operating incomes over a five-year holding period: Year 1, (5,000); Year 2, $22,000; Year 3, $24,500; Year 4, $27,000; Year 5, $30,000. It was sold at the end of the 5th year and delivered $325,000 net after selling expenses. The owner paid $170,000 for the property, requires a return of 25% p.a., and has an opportunity cost of money equal to 10% per year. What is the Net Present Value of his investment? What was his yield on capital invested?

3-4. A utility stock purchased for $100 has produced consistent dividends of 6% of the original price each year for 7 years. To what value must the stock have appreciated for the owner to realize a 12% yield on invested capital if he chooses to sell at the end of the seventh year?

3-5. An investor with $20,000 cash is offered an attractive limited partner interest in a motion picture enterprise which will require $20,000 on signing. Each partner will also be responsible for additional payments of $5,000 the first year and $3,000 the second year. Thereafter, the investment is forecast to return $15,000 at the end of the third year, $18,000 at the end of the fourth year, and $25,000 at the end of the last year. The general partner will accept a note in lieu of the first and second year's payment, which note will earn interest at the rate of 12% p.a., payable monthly. If the investor buys this interest, and makes the monthly payments, what will be his IRR if this investment meets forecast?

3-6. An automobile dealer is willing to lease a car which costs him $23,000 for $399 per month for 36 months. The wholesale value of the car is estimated to be $14,000 at the end of the lease. What will be his yield on investment in this lease?
If the lessee pays an additional $2,500 in drive-off fees, what will be the dealer's yield?
What would be the dealer's yield, including drive-off fees of $2,500, if he can sell the car at a $4,000 margin over wholesale value at the end of the lease?

3-7. A beginning property investor purchased a condominium for $175,000 utilizing an 85% loan. Over a holding period of 7 years, she realized the following net annual amounts after taxes: $4,000, $4,200, $4,500, $4650, $4,875, $5,250, $5,400. What must be her net sales proceeds at the end of the 7th year if she is to achieve a yield of 12% on her original investment?

Group IV

4-1. What amount will retire a level-payment mortgage of $200,000 over a 15-year term if the monthly payments include interest at the rate of 8-7/8% per year?

4-2. How much total interest will be paid on the loan in problem #1?

4-3. What will the monthly payment be for the loan described in #1 above if the amortization schedule were 25 years? 30 years? 40 years?

4-4. What is the total interest to be paid for each option cited in problem #3?

4-5. At the end of which payment, for the loan described in problem #1 above, will the remaining balance of the loan be reduced below one-half the original amount borrowed?

4-6. Jones has completed 5 years of fixed, monthly payments of $1,345.60 on a 30-year, fully amortizing loan in the original amount of $175,000. How much interest will he pay in the coming year?

4-7. A home buyer originated an Adjustable Rate Mortgage (ARM) in the amount of $125,000 at a beginning interest rate of 5.5%, payable monthly over a 30-year term. At the end of the first 6 months, the interest rate is expected to adjust to 7.75%. What payment will be required to retire the loan over the time remaining?

4-8. If the ARM described in problem #7 contains a clause barring any changes in monthly payment until the end of the first year of the mortgage, what payment will be required to retire the loan over the time then remaining? Assume a constant 7.75% interest rate.

4-9. The interest rate of an adjustable rate mortgage, payable monthly, is programmed to be equal to the 1-year Treasury bill plus 250 basis points. The maximum rate of interest, however is 5 points over the starting rate of 6% p.a. What is the maximum payment for which a borrower could be liable on a $200,000 loan which was scheduled to be amortized over 30 years?

4-10 During a period of high interest rates, a builder suffers from slow sales on a new mid-price-range project. As his marketing consultant, you recommend the use of a 5-3-1 *buydown* to stimulate sales. Current interest rates for a fully amortized, fixed-rate loan, payable monthly over a 30-year term, are 11%. The average model retails for $450,000, and requires a 20% down payment. The builder's lender requires a lump sum payment from the builder in an amount equal to the builder's total 36-payment obligation. What would be the effective selling price to the builder if he accepts your recommendation and agrees to the lender's terms? What percent of the selling price does the cost of the buydown represent to the builder?

4-11 A self-storage property collects $225,000 after allowances for vacancy and bad debts. Its operating expenses are $62,000 per year. A lender for this type property requires a Debt Coverage Ratio of 1.4. What is the maximum loan which this lender will furnish if the current interest rate for a fixed-rate, 20-year mortgage, payable monthly, is 10.5%?
How much greater a mortgage would be available from a lender who requires a 1.3 DCR but charges the same interest rate for a loan amortized over 25 years?

4-12. If a mortgage in the amount of $125,000 requires a monthly payment of $1,096.95, what is its *annual* loan constant per $100 of loan?

4-13. A new home buyer is presented with two alternate loan offerings. The first loan, in the amount of $130,000 is offered at a fixed-interest rate of 8.875% per year, payable monthly over 30 years. Loan fees are 2 points plus estimated loan costs of an additional $1,750.
The second offering, in the same amount, proposes a fixed-rate, 25-year amortizing loan bearing interest at the rate of 9.375%, no points and no other fees.
Which loan is less expensive if the borrower intends to move in **ten** years?
Which loan is less expensive if the borrower intends to move in **five** years?
Which loan is less expensive if the borrower intends to move in **three** years?

4-14. A homeowner who originated a loan in the amount of $145,000, payable monthly over 29 years, including interest at the rate of 7.5% p.a., obtains the lender's agreement to recast the loan payments over the remaining schedule if the borrower reduces the loan balance by any one payment by at least $2,000. At the end of the second year of the loan, the borrower pays the remaining principal down by $3,000. What will his new payments be?

Group V

5-1. Calculate the overall capitalization rate of a property which collects $42,000 per month before expenses of $11,250 per month, and which recently sold for $4,100,000.

5-2. What would be the expected selling price for a property earning $126,000 net per year before taxes if comparable properties command a capitalization rate of 8.5%?

5-3. An office building recently sold for $3,750,000. Its next-door twin nets $350,000 before debt service. What is a likely capitalization rate for the property recently sold?

5-4. An investor is interested in acquiring a retail center. The negotiated sales price is expected not to exceed $6,000,000. A lender requiring a 10.5% interest rate loan will make a 65% loan on the property, amortized over 20 years. The seller has agreed to carry back 10% of the purchase price in an interest-only note 1 point higher than the interest rate on the first trust deed loan. What must the property earn in the first year of operation in order to return 14% cash-on-cash (pre-tax) to the buyer? What must be its minimum capitalization rate ?

5-5. An investor purchases a small strip-retail center for $650,000 with a down payment of 25%. The loan provides for monthly payments on a 25-year, fully amortizing schedule of payments, including interest at the rate of 9.75% p.a. If the center performs at a 10% capitalization rate in the first year of ownership, what will be the first year pre-tax Cash-on-Cash return to the owner?

5-6. A commercial loan of $2,000,000 with a ten-year due date was obtained at a cost of 1.5 points. What is the amount of the annual deduction from income attributable to the amortization of the points?

5-7. Given the mid-month convention used in calculating the depreciation allowance for non-residential property, what would be the first year's total depreciation deduction for the retail center in problem #5-5 if the value of the improvements is 75% of the acquisition cost?

5-8. If the owner of the property in problem #5 pays taxes at the marginal rate of 36%, what would be the net taxable income after deducting both his depreciation deduction and interest payments at the end of the first year of ownership?

5-9. A 6-unit residential income property was acquired for $450,000, all cash, plus closing costs of $4,500. The property was held for exactly 6 years when it was sold for $454,500 after deductions for all costs of sale. No capital improvements were made during the holding period and no part of the property was sold. What is the amount of the indicated gain if the improvements are 80% of total property value?

5-10. An industrial building was purchased for $1,750,000 utilizing a 75% loan-to-value, 20-year loan which cost 2 points. If the property were held for 5 years, what amount would be subtracted from the 5^{th} year's operating income to reflect unamortized loan points?

Group VI

6-1. Calculate the Duration of a zero coupon bond maturing in 29 years. Current yield is 5.5% p.a. What is its Modified Duration?

6-2. The owner of a property subject to a groundlease has come to you for advice. The lease on his property has only three more years to run, but he has an excellent investment opportunity which requires \$400,000 in up-front capital. His leasehold tenant is scheduled to pay \$12,500 per month, in advance, for the remaining three years of the groundlease. What yield could be achieved by an investor willing to advance the required \$400,000 with a personal guarantee of the rents?
What would be the return to the investor if your client were willing to accept \$350,000 for the remaining lease payments?

6-3. A Treasury bond carrying a coupon of 4.6% and maturing Aug. 15, 2006 is presently quoted at 89:18 (Ask). What would be the Yield-to-Maturity if the purchaser paid the Ask price on Dec. 10, 1997?

6-4. A corporate bond carries a coupon of 6.5% and provides for a call date of Jan. 2, 2000 at 105. The current rate for comparable issues is 7.2%. The bond matures on April 20, 2010. What is the most likely Ask price for this bond if purchased on Feb. 1, 1998?

6-5. A stock analyst projects that shares of USX (US Steel) will earn \$4.40 in the coming year and that the price will rise to \$35.20. USX pays out 25% in dividends and currently sells for \$32.00. What is the expected market capitalization for this stock?

6-6 A mutual fund has delivered the following annual return rates: 12.2%, 9.8%, 22.4%, –2.9%, 14.6%. What is the average return from this fund? What is the geometric mean?

Answers to Problems are indicated by bold faced type.

Group I

1-1. This problem asks for the PV of the future amount:

n	i	PV	PMT	FV
1	12	?	0	1,000
	Solving…	**-892.86**		

(Note the sign convention at work. The negative sign before 892.86 does not mean a negative value. It is a convention of the calculator that there must be at least 1 negative sign and at least 1 positive sign)

1-2. These four problems require you to correlate the time periods of $\boxed{n}$ and $\boxed{i}$:

n	i	PV	PMT	FV
8	4.5÷4	-2,000	0	?
			Solving…	**2.187.25**
24	4.5÷12	-2,000	0	**?**
				2,187.98
720	4.5÷360	-2,000	0	**?**
				2,188.34

<u>For continuous compounding</u>…

Multiply 0.045 * 2 = 0.090
Key-in g, e^x = 1.094171
Multiply the result by 2,000
FV = PV(e)$^{.045*2}$ = **2,188.35** (Notice how close this is to daily compounding)

1-3. The point will be reached when the FV = $10,000

n	i	PV	PMT	FV
?	10÷4	-5,000	0	10,000
29				

Ans. 29 quarters = 87 months . Resolve for FV. What does this tell you?

1-4. This problem emphasizes the importance of maintaining a singe vantage point, The $250 monthly payment is also a negative cashflow for the saver.

n	i	PV	PMT	FV
36	?	-5,000	-250	16,000
Solving...	**0.546**			

The PMT required at this rate for a 24 month plan would be:

n	i	PV	PMT	FV
24	0.546	-5,000	?	16,000
		Solving...	**-402.87..**	

1-5. Take the viewpoint of the dealer. Then construct your T-Bar.
Ans. Dealer A earns 18% per year; Dealer B earns 24% per year.

1-6. This problem depicts an initial negative cashflow of an unknown amount and a later positive cashflow of $250. The FV at the end of 15 years will be zero.

n	i	PV	PMT	FV
180	08.5÷12.	-?	250	0
	Solving...	**-25,387**		

1-7. First, determine how much the trust must have on hand when it begins its 10 year payout. Set calculator to BEGIN (1,7). Then determine how much Jones must deposit each year to reach this figure with end-of-the-month payments.
BEG

n	i	PV	PMT	FV
120	7÷12.	-?	-1,500	0
	Solving...	**-129,943.13**		

The trust must have accumulated $129,943.13. In order to accumulate this FV amount requires monthly PMTs (set calculator to END (1,8)) over 5 years of:

n	i	PV	PMT	FV
60	7÷12	0	?	129,943.13
		Solving...	-1,815.03	

1-8. If Mary invests $52 and has $55.50 at the end of one year, her investment is growing at a rate $= \frac{\$3.50}{\$52} = 6.73077\%$. Set calculator to $\boxed{\textbf{BEG}}$.

n	i	PV	PMT	FV
5	6.73077	0		10,000
		Solving…	**1,638.02**	

Ans. $1,638.02/year ÷ $52/ share = 31.50 shares per year.

To verify this answer, invest the single sum of $1,638.02 at 6.73077% for 5 years, then for 4 years, then for 3 years etc. etc... Then add all the future values together.

1-9. This is a problem in determining the remaining balance of a note. View this cashflow from the point of view of the lender who has advanced $175,000.

n	i	PV	PMT	FV
60	8÷12	-175,000	1,500	?
			Solving…	**-150,507.71**

1-10. This is a simple PMT problem:

n	i	PV	PMT	FV
480	10÷12	0	?	-1,000,000
		Solving…	**-158.13**	

1-11. Since you are working with constant dollars, the Inflation-Adjusted Rate can be used:

$$\text{Effective Rate} = \frac{1.08}{1.05} - 1 = .028571 * 100 = 2.8571\%$$

You must convert the decimal to a percentage because the HP-12C requires a percent in register **i.**

n	i	PV	PMT	FV
20	2.85714..	0	-10,000	?
			Solving…	**264,833.01**

1-12. Since you are seeking a **Present Value** *with PMTs made at the EOP*, the Inflation-Adjusted Rate will introduce a small error (1+g). See Appendix for additional information re IAR.

$$PV = \frac{\$10000}{.08-.05} * \left[1-\left(\frac{1.05}{1.08}\right)^{20}\right]$$

$$PV = \frac{\$10000 * 0.43074}{0.03}$$

$$PV = \mathbf{\$143,579.91}$$

Since the PMTs are equal you could also use the IAR and the horizontal register if you remember you will get a slightly inflated answer:

n	i	PV	PMT	FV
20	:.85714	?	–10,000	0
	Solving	150,758.91		

But since this is a PV problem using the IAR, the answer will be 1+g too high. Divide the result by 1.05: $150,758.91÷1.05 =**143,579.91**

1-13. If the note provides for interest-only PMTs, the FV will be equal to $25,000.

n	i	PV	PMT	FV
36	20÷12	?	250	25,000
	Solving	**:0,515.32**		

(Notice that both the PMT and FV are positive since they are cash in-flows to you.)

Group II

2-1. In this problem you are dealing with an *ordinary* annuity whose PMT increases at a steady 10% per period. Since this is a short cashflow, you can solve this by using the uneven cashflow registers: CFo, CFj, Nj. The 1st cashflow is $1,000, the 2nd is $1,100, the 3rd is $1,210. Discount @ 15%.
Ans. **$2,496.92**

2-2. Use the banker's 12% discount rate to calculate the PV of this uneven cashflow series. The loan value will be half this amount.
Ans. **$6,672.95**

2-3. This is a PV problem in uneven cashflows with PMTs in advance. *The BEG/END buttons have no effec*t when used with uneven cashflows. Store the first PMT, 3,000, in CFo, as a positive number. Then solve as though the remaining PMTs occur at the end of the periods. Remember that Nj s cannot exceed 99 per storage cell. Therefore, break up the 2nd CFj, $3,750, into 2 parts (e.g. 99 + 21) and enter normally.
Ans. **$281,527.36**

2-4. This NPV problem complicates the solution to #3 because you have already stored $3,000 in CFo. But if you pay $275,000 out and immediately receive $3,000 back at

the beginning of the lease, your net initial investment is –$272,000. Replace the $3,000 in memory cell 0 with –272,000 and re-solve. There is no need to re-enter the cashflows ***if*** you use STO -272,000 0
Ans. **NPV = $6,527.36**

2-5. This Net PV problem emphasizes the necessity of inserting zeros in a cashflow series to maintain the time relationship among the cashflows. The first cash entry, CFj, will be 0, occurring 3 times (Nj). The remaining cashflow, $10,000, then occurs 8 times.
Ans. **$2,205.79**

2-6. Without disturbing any other entries, simply solve for the IRR, f IRR
Ans. **21.94%**

2-7. Correct the Nj s for the first cashflow from 3 times to 4 times.
Ans. **-$1,495.18**. No, because it furnishes a negative Net Present Value, indicating hat your return would be less than 20%. Without disturbing the data entered, solve for the IRR to determine what your return would be. (18.78%)

2-8. The owner is being asked to amortize (finance) $60,000 in improvements over 60 months at a 15% interest rate. Set to BEG

n	i	PV	PMT	FV
60	15÷12	-60,000	?	0
		Solving…	1,409.77	

Ans. $1,409.77 per month over 10,000 s.f. = $0.14098/s.f. added to $1.90 = **$2.04098 /s.f. per month. ($2.04/s.f.)**

2-9. What will be next year's income: $180,000 or $207,000? And the years after that? This is an uneven cashflow problem. (The first cashflow is not $180,000 but $180,000* (1+.15) = $207,000.)
Ans. $705,701.24

Group III

3-1. This is a problem asking for the IRR of the investment. Since the percent increase each year is the same (1.08 * 1.05) and the IRR is equal to the percent increase per year, then the IRR is equal to (1.08)*(1.05). Things equal to the same thing are equal to each other.
Ans. = (1.08*1.05) - 1 = .134 = **13.4%**

3-2. This problem contrasts variable discount rates which can be applied to a cashflow with the IRR, which is a single discount rate. See Chapter 2.

Ans. IRR = 26.59% (only if the business has no residual value at the end of the 3rd year)

3-3. This is a problem in the use of the Modified Internal Rate of Return because of the negative cash flow at the EOY 1. How much more does the investor need to invest @10%, in addition to $170,000, to have the $5,000 available to cover the negative?
Ans. NPV = –$20,535.85.
MIRR = 21.6%

3-4. This problem, involving even PMTs, asks for the Future Value of the investment.

n	i	PV	PMT	FV
7	12	-100	6.00	?
			Solving	**160.53**

3-5. This problem must be solved on a monthly basis since the promissory note is paid monthly. In this case the limited partner would sign a note in the face amount of $8,000, payable $376.59 per month for 24 months, which PMTs will zero out the $5,000 and $3,000 future obligations. The limited partner will therefore have a monthly <u>negative</u> cashflow of $376.59 for 24 months. Thereafter he would have 11 monthly periods of no income, one month of $15,000, 11 more months of no income, one month of $18,000, 11 months of no income and one final cashflow of $25,000. Solve this problem as an uneven cashflow,
Ans. = 1.52214 per month, or 18.27% per year.

3-6. Since lease PMTs are paid in advance, the dealer makes an investment of –$23,000 but receives the first PMT, $399, at inception of the lease. Following this, the lessor will receive 35 additional PMTs of $399 and the return of the auto, worth $14,000 at the end of the 36th month. This is an uneven cashflow.
Ans. 0.81% per month, or 9.7% p.a.
Ans. 1.27% per month, or 15.25% p.a.
Ans. 1.73% per month, or 20.8% p.a.

3-7. The investor used $26,250 of her own funds. Calculate the NPV of this cashflow over 7 years, discounted at 12% per year. The result, -$5,303.48 is the NPV. Since this negative PV must be offset by the PV of the reversion amount, the reversion amount must be the FV of this sum when invested at 12% for 7 years.

n	i	PV	PMT	FV
7	12	-5,303.48	0	?
			Solving...	**11,724.31**

Group IV

4-1.

n	i	PV	PMT	FV
180	8.875÷12	–200,000	?	0
		Solving	**2,013.69..**	

4-2. The total interest over the life of the loan is the total amount of PMTs *less* the original loan amount.
Ans. = 180 x 2,013.69 – 200,000 = **$162,463.87**

4-3. **Ans. $1,661.31**
Ans. $1,591.29
Ans. $1,523.50

4-4. **Ans. $298,391.92**
Ans. $372,864.32
Ans. $531,282.02

4-5. This point will occur when the FV (Remaining Bal.) of the loan is equal to $100,000.

n	i	PV	PMT	FV
?	8.875÷12	–200,000	**-2,013.69..**	100,000
118 ←	**Solving**			

Ans. Before the 118th payment and after the 117th payment. (Resolve for FV)

4-6. You need first to calculate the interest on the loan. Since you know the PMT, this is easy.

n	i	PV	PMT	FV
360	?	–175,000	**-1,345.60..**	0
Solving..	.7083343			

Now amortize the loan 60 PMTs, and then 12 PMTs to find the answer.
Ans. $14,126.69

4-7. This is a common ARM problem. First, find the remaining balance at the end of the 6th month. Then amortize this balance over the time remaining at the higher interest rate.
Ans. $893.41

4-8. First, determine the initial monthly payment and then the balance of the loan at the end of the first 6 months.

n	i	PV	PMT	FV
360	5.5÷12	−125,000	?	0
		Solving..	**709.74..**	
6			**Solving…**	**124,169.62..**

Then, without changing the payment, change the interest to reflect a 7.75% rate payable monthly, and determine the remaining balance at the end of the next 6 months.

n	i	PV	PMT	FV
6	7.75÷12	−124,169.62	709.74	?
			Solving..	**124,731.78**

(Note that the balance grew because the fixed PMT was insufficient to cover the interest. The deficiency was added to the remaining balance.)

Now amortize this balance over the time *remaining* in the original schedule.

n	i	PV	PMT	FV
348	7.75÷12	−124,731.78	?	0
		Solving..	**901.51**	

4-9. Simply determine the highest payment due if the interest rate = (6+5%) = 11%.
Ans. $1,904.65

4-10. Calculate what monthly PMT the bank must receive at 11% interest rate. Calculate what the buyer must pay as though the interest rate were 5% less in the first year, and subtract this from the amount the bank must receive. This is the builder's obligation in the first year. Repeat this for each of the 2 succeeding years using 8% interest (11% - 5%) and then 10% (11%-3%). Enter these in your calculator as uneven cashflows. Each builder PMT occurs 12 times.
Put 0 into i. (No discount). Solve for NPV. Result is $27,910.81, the amount the builder must pay the bank. The builder's effective price is $450,000 - 27,910.81 **=$422,089.**
Ans. **6.2%** decrease in price.

4-11. Determine the NOI by subtracting the expenses from the Gross Operating Income of $225,000. Calculate the annual amount available for debt service by dividing by the DCR of 1.4. Divide by 12 to calculate the monthly payment. Then determine how large a loan this amounts will support, on a monthly basis, given the interest rate and amortization schedule.
Ans. 1.4 = $971,813
Ans. 1.3 =$1,106,643. Therefore, $134,830 larger.

4-12. The ***annual*** loan constant for a loan paid monthly is the ratio of one year's total monthly payments to the loan amount. Multiply by 100.
Ans. $10.530 per $100 of loan

4-13. This is a problem in calculating the APR of a loan. Set up the loan and solve for the PMT. Determine the remaining balance at 10 years. Deduct loan points and charges from PV and re-insert the remainder back into PV as a negative number . Change **n** to 120 months and recalculate **i**. Multiply the result by 12 o get annual APR.
Repeat for five and three years.
Ans. The APR for the loan requiring points and fees is as follows:
Move in **36** months, **APR = 10.1875%**
60 " APR = 9.738%
120 " APR = 9.413%

The APR for a loan requiring no points and no fees is always equal to the interest rate, 9.375% in this case, which is superior in all offerings.

4-14 Calculate the PMT and then the balance of the loan after 24 PMTs; deduct $3000 from the balance, use this as the PV of a new loan amortized over the remaining 27 years. Calculate the new payment
Ans. $1,001.67

Group V

5-1. Calculate the *annual* NOI by subtracting the expenses from the collected income and divide by the sales price.
Ans. 9.0%.

5-2. NOI ÷ Capitalization rate = Fair Market Value
Ans. $1,482,353

5-3. Net Operating Income ÷ Fair Market Value = Capitalization Rate
Ans. 9.3%

5-4. This is a problem in constructing a cap rate using the *Band of Investment* method.
Ans. 12.44% (See page 9-7)

5-5. Calculate the NOI ($650,000*10% = $65,000). Then calculate the annual debt service. The difference is the spendable income. Compare this to (divide by) the buyer's cash down payment.
Ans. 7.9%

5-6. The loan points ($2,000,000 * 1.5% = $30,000) must be amortized over 10 years.

Ans. $3,000

5-7. **Ans. $11,979.17** (11.5/12 of ($650,000*75% ÷ 39) (See Mid-month convention)

5-8. Ans.

NOI	$65,000.	(FMV * Capitalization rate)
Interest	-47,320	
Depreciation	-11,979	
Total Taxable	5,701	
Tax Rate	x 36%	
Tax	**2,052**	

5-9. The indicated gain would be equal net sales price less the Adjusted Basis, which is equal to the Original Basis of $454,500 less the depreciation amounts taken.

Original Basis	$454,500	
Allocation to Improvement	80.%	
Depreciable Basis	363,600	
Residential Schedule, in years	27.5	
Annual Deprec. Allowance	13,221.82	
Monthly Allowance	1,101.82	
Total 71-month depreciation	78,229	(Loss of 2 half-months)
Adjusted Basis	$376,271	

There is a loss of one-half month's depreciation at the beginning *and* end of the holding period. Therefore only 71 months of depreciation would be taken in 6 years. The indicated gain would be the net sales price of $454,500 less the Adjusted Basis of $376,271.
Ans. $78,229

5-10. The amount of the unamortized loan points would be the cost of the loan points ($26,250) less the amount amortized over the5-year holding period – $6,562.50
Ans. $19,687.50

Group VI

6-1. Since a zero coupon bond pays no dividends, the only cashflow contributing to the Duration factor will be the reversion value of $1,000. Nevertheless, the PV (and PV factor) must be discounted on a semi-annual schedule.

Yr.	Period	PMT	PV Factor	PV	PV/Price	Duration
29	58	1000	0.207326	207.33	1.00	**29**

Price = Total PVs = 207.33

The PV Factor is the PV of \$1.00 in FV to be received 58 periods in the future (n), discounted at 5.5%/2. The PV/Price is the total price (\$207.33) divided by the PV of \$1000 (\$207.33), or 1. The Duration is the PV/Price factor times the Yrs. (29)
The Modified Duration would be 29/(1+ 0.055/2) = **28.2238**

6-2. The Present Value of the remaining PMTs under the ground lease is an even cashflow situation, with PMTs occurring BOP. There is no reversion value. Therefore,

n	i	PV	PMT	FV
36	?	-400,000	12,500	0
Solving	0.6899			

Ans. 0.6899 * 12 = 8.28%
Ans. 18.2%

6-3. Convert (89+18 ticks) to (89+ 18/32). Use the bond program.
Ans. 6.17%

6-4. Ans. 94.36. Ignore the call date. The Bond will not be called since coupon rate is less than market rate.

6-5. $(1+r) = \dfrac{1.10 + 35.20}{32}$ $r = 1.13438 - 1 = .13438$

Ans. 13.4

6-6 The average return on this fund is the sum of the individual annual returns divided by the number of years; i.e. 56.10 ÷ 5 = 11.22%
The geometric mean assumes that the annual yield from the investment is left in the fund and bears a series of compounded returns. Assume you started with \$1.
Product = 1.122 * 1.098 * 1.224 * 0.971 * 1.146 = 1.67796
The value 0.971 (1–0.029) is used because the geometric mean calculation cannot include negative numbers. The geometric mean is the nth root of this product. Since n = 5 value, raise
1.67796 to the 1/5th power (0.2). Ans. = 1.10906 -1 = 10.91

Appendix

I - Derivations of Cashflow Formulas

(1) Simple Interest

In simple-interest situations, the capital invested earns interest at a constant rate (not compounded). Therefore:

$FV = PV + n * i * PV$, where **n** is the number of periods and **i** the rate.

$FV = PV(1+ (n * i))$

(2) Compound Interest

But when interest is compounded, the interest earned per period is added to the principle and the total then earns interest at the stated rate. At the end of the first period

$$FV = PV+ PV * i = PV(1+i)$$

At the end of the next period this total, PV(1+i), earns interest at rate i:

$$FV_2 = PV(1+i)(1+i) = PV(1+i)^2$$

Therefore $FV = PV(1+i)^n$ (This is the basic formula for compounded interest)

And, $PV=\dfrac{FV}{(1+i)^n}$

(3) PV of an infinite series of equal cashflows, payment at end of period.

This formula can be used to determine the amount necessary to fund a perpetual annuity whose payments begin one period following the date of funding. It is also the derivation of the **capitalization** method used to establish stock values using dividends, and real estate values using net operating income.

$$PV = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} + \frac{C}{(1+i)^4} \ldots\ldots\ldots\ldots\infty$$

Let $a = \dfrac{C}{(1+i)}$ and $x = \dfrac{1}{(1+i)}$ Then,

$$PV = a + ax + a x^2+ ax^3 + ax^4 \ldots\ldots \infty$$

Multiplying both sides by x,

$PVx = ax + ax^2 + ax^3 + a_x 4 \ldots\ldots \infty$

Subtracting last equation from first,

$PV - PVx = a$

$PV(1-x) = a$

$$PV = \frac{a}{(1-x)} = \frac{C}{(1+i)} \div 1 - \frac{1}{(1+i)} = \frac{C}{(1+i)} \div \frac{1+i-1}{(1+i)}$$

$$PV = \frac{\mathbf{C}}{\mathbf{i}}$$

(4) PV of an infinite series of equal cashflows, payment at beginning of period.

It is easy to see that the PV of an infinite series with the first payment made *in advance* is the same as the PV of an infinite series with payments at the end of the period, *plus* one payment, C.

Therefore, $PV = \frac{C}{i} + C$ *or* $PV = \frac{C}{i} * (1+i)$

(5) PV of a finite series of equal cashflows, n long, payment C at end of period (Ordinary Annuity)

$$PV = \frac{C}{(1+i)^1} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} \cdots \frac{C}{(1+i)^n}$$

Let $a = \frac{C}{(1+i)}$ and $x = \frac{1}{(1+i)}$

Then

$PV = a + ax + a x^2 + ax^3 + \ldots\ldots ax^{n-1}$

Multiplying both sides by x,

$PVx = ax + a x^2 + ax^3 + \cdots ax^n$

Subtracting the second equation from the first,

$PV - PVx = a - ax^n$

$$PV = \frac{C}{i} - \frac{C}{i(1+i)^n}$$

$$PV(1\text{-}x) = a(1\text{-}x^n)$$

$$PV = \frac{C}{i}\left(1 - \frac{1}{(1+i)^n}\right)$$

6) PV of a finite series of equal cashflows, n long, payment C at beginning of period (Annuity Due). Note that the series is foreshortened by one period, and that the first period is not discounted, but rather added at full value to the discounted total of those remaining.

$$PV = \underline{C} + \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \frac{C}{(1+i)^3} \cdots \frac{C}{(1+i)^{n-1}}$$

Let $a = C$ and $x = \frac{1}{(1+i)}$ Then,

$$PV = a + ax + a\,x^2 + ax^3 + \cdots\; ax^{n-1}$$

Multiplying both sides by x,

$$PVx = ax + ax^2 + ax^3 + \cdots\; ax^n$$

Subtracting the second equation from the first,

$$PV - PVx = a - ax^n$$

$$PV(1\text{-}x) = a(1\text{-}x^n)$$

$$PV = \frac{a(1\text{-}x^n)}{(1\text{-}x)} = C * \left(1 - \frac{1^n}{(1+i)^n}\right) \div (1\text{-}x)$$

$$PV = \frac{C\,(1+i)}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

(7) Present Value of an infinite series growing at constant rate g, payment C at end of period. Note that the first payment is not inflated by g.

$$PV = \frac{C}{(1+i)^1} + \frac{C(1+g)^1}{(1+i)^2} + \frac{C(1+g)^2}{(1+i)^3} + \frac{C(1+g)^3}{(1+i)^4} + \frac{C(1+g)^4}{(1+i)^5} \cdots\cdots\infty$$

Let $a = \frac{C}{(1+i)}$ and $x = \frac{(1+g)}{(1+i)}$ Then,

$PV = a + ax + a\,x^2 + ax^2 + ax^4 \ldots\ldots\ldots.\infty$

Multiplying both sides by x,

$PVx = ax + ax^2 + ax^2 + ax^4 \ldots\ldots.\infty$

Subtracting last equation from first,

$PV\text{-}PVx = a$

$PV(1\text{-}x) = a$

$$PV = \frac{a}{(1\text{-}x)} = \frac{C}{(1+i)} \div 1 - \frac{(1+g)}{(1+i)} = \frac{C}{(1+i)} \div \frac{1+i-1-g}{(1+i)} = \frac{C}{(1+i)} \div \frac{(i\text{-}g)}{(1+i)}$$

$$PV = \frac{C}{i\text{-}g}$$

(8) PV of a finite series growing at constant rate g, first payment at end of period. This involves a growth rate, g, and a discount rate, i, in a series n long.

$$PV = \frac{C}{(1+i)^1} + \frac{C(1+g)^1}{(1+i)^2} + \frac{C(1+g)^2}{(1+i)^3} + \frac{C(1+g)^3}{(1+i)^4} + \frac{C(1+g)^4}{(1+i)^5} \ldots.+ \frac{C(1+g)^{n-1}}{(1+i)^n}$$

Let $a = \dfrac{C}{(1+i)}$ and $x = \dfrac{(1+g)}{(1+i)}$ Then,

$PV = a + ax + ax^2 + ax^3 + ax^4 + ax^5 \ldots.\ ax^{n-1}$

Multiplying both sides by x

$PVx = ax + ax^2 + ax^3 + ax^4 + ax^5 \ldots\ ax^n$

Subtracting the second equation from the first,

$PV - PVx = a - ax^n$

$$PV = \frac{a*(1\text{-}x^n)}{(1\text{-}x)} = \frac{C}{(1+i)}\left(1 - \frac{(1+g)^n}{(1+i)^n}\right) * \frac{(1+i)}{(i\text{-}g)}$$

$$PV = \frac{C}{(i\text{-}g)}\left(1 - \frac{(1+g)^n}{(1+i)^n}\right)$$

(9) PV of a finite series growing at constant rate g, first payment at beginning of period.
This involves a growth rate, g, and a discount rate, i, in a series n long.

$$PV = C + \frac{C(1+g)^1}{(1+i)^1} + \frac{C(1+g)^2}{(1+i)^2} + \frac{C(1+g)^3}{(1+i)^3} + \frac{C(1+g)^4}{(1+i)^4} + \frac{C(1+g)^5}{(1+i)^5} \dots + \frac{C(1+g)^{n-1}}{(1+i)^{n-1}}$$

Let $a = C$ and $x = \frac{(1+g)}{(1+i)}$ Then,

$$PV = a + ax + ax^2 + ax^3 + ax^4 + ax^5 \dots ax^{n-1}$$

Multiplying both sides by x

$$PVx = ax + ax^2 + ax^3 + ax^4 + ax^5 \dots ax^n$$

Subtracting the second equation from the first,

$$PV - PVx = a - ax^n = a(1 - x^n)$$

$$PV = \frac{a*(1-x^n)}{(1-x)} = C\left(1 - \frac{(1+g)^n}{(1+i)^n}\right) * \frac{(1+i)}{(i-g)}$$

$$PV = \frac{C(1+i)}{(i-g)}\left(1 - \frac{(1+g)^n}{(1+i)^n}\right)$$

(10) Future Value of an even cashflow bearing interest for a finite number of periods.

When the PMT is to be received at the end of the period

$$FV = C + C(1+i)^1 + C(1+i)^2 + C(1+i)^3 + C(1+i)^4 \dots + C(1+i)^{n-1}$$

Let $a = C$ and $x = (1+i)$, then:

$$FV = a + ax + ax^2 + ax^3 + ax^4 + \dots ax^{n-1}$$

Multiplying both sides by x,

$$FVx = ax + ax^2 + ax^3 + ax^4 + \dots ax^n$$

Subtracting equation two from one,

$$FV - FVx = a - ax^n$$

$$FV = \frac{a\,(1-x^n)}{-i} = C\frac{(1-(1+i)^n)}{-i}$$

When the PMT is received at the beginning of the period:

$$FV = C(1+i)^1 + C(1+i)^2 + C(1+i)^3 + C(1+i)^4 \cdots\cdots + C(1+i)^n$$

Let $a = C$ and $x = (1+i)$, then:

$$FV = ax + ax^2 + ax^3 + ax^4 + \ldots.ax^n$$

Multiplying both sides by x,

$$FVx = ax^2 + ax^3 + ax^4 + \ldots.ax^{n+1}$$

Subtracting equation two from one,

$$FV - FVx = ax - ax^{n+1} = ax\,(1-x^n)$$

$$FV = \frac{ax(1-x^n)}{(1-x)} = C\frac{(1+i)(1-(1+i)n)}{1-(1+i)}$$

$$FV = C\,\frac{(1+i)(1-(1+i)^n)}{-i}$$

(11) Constant Payment Required to Retire Loan (PV) at i rate of interest, over n periods. where C = payment required per period

$$PV = \frac{C}{i} * \left[1 - \frac{1}{(1+i)^n}\right]$$ (the PV of a cashflow bearing i interest for a finite period, n)

Then,

$$C = PV * i \div \left[1 - \frac{1}{(1+i)^n}\right]$$

$$C = PV * i * \frac{(1+i)^n}{(1+i)^n - 1}$$

II- Using the Inflation-Adjusted Rate (IAR)

There are many instances when it is desirable to calculate returns on an investment in terms of today's dollars (constant dollars).
For example:
An investment of $100 today returns 9% per year for 15 years. Therefore its future value will be:

n	i	PV	PMT	FV
15	9	**-100**	0	?
			Solving...	**364.25**

If, however, the purchasing value of the dollar is reduced by an inflation rate of 4% per year, then the initial investment of $100 must increase by 4% per year just to maintain the same purchasing power.

n	i	PV	PMT	FV
15	4	**-100**	0	?
			Solving...	**180.09**

In this inflationary environment, an investment of $100 earning interest at the rate of 9% per year (compounded) will not earn 3.6425 times as much as it did 15 years ago, but only

$$\frac{364.25}{180.09} = 2.02254 \text{ times as much.}$$

Therefore $100, ***expressed in dollars of constant purchasing power***, grew to $202.254 The Inflation Adjusted Rate (IAR) at which it grew is only:

n	i	PV	PMT	FV
15	?	**-100**	0	202.254
Solving...	**4.807688**			

A simpler way to obtain this number is to use the formula for the IAR:

$$\frac{(1 + \text{nominal rate})}{(1 + \text{inflation rate})} - 1 = .04807688 \text{ or } 4.81\% \text{ (rounded)}$$

It may be helpful at times to know the **comparative** future purchasing power of a dollar which has been invested at a certain rate and discounted at another rate to reflect inflation. For example, if we wish to know the purchasing power, ***as measured in today's dollars***, of $100, invested @ 9% in a 4% inflationary environment, after 5 years, we can use the IAR for **i**:

n	i	PV	PMT	FV
5	4.807692	**-100**	0	?
			Solving...	**126.46**

This means that the dollars we will have in our hand at the end of 5 years will have the purchasing power of $126.46 *as measured in today's dollars.* But it does not mean that we will have $126.46 in our hand, but rather we will have:

n	i	PV	PMT	FV
5	<u>9</u>	**-100**	0	?
			Solving...	**153.86**

When *Not* To Use the IAR

Many problems in financial planning require a determination of a single amount to be put aside today (PV), or as a regular payment, in order to meet a future financial goal. For example

> A 4-year education program costs $100,000 today, and is expected to increase at the rate of 5% annually over the next ten years.
>
> What lump sum must be invested today, earning 9% per annum, in order to meet the future costs of the education program? In lieu of a lump sum, what end-of-the month (ordinary annuity) payment invested at the same rate would also accrue this amount?

<u>Do not use the IAR in this type problem</u> because it will give the FV of either the contributed PMTs or the PV *in terms of constant dollars*, which is not what is sought. The future obligation must be met with *nominal* dollars.

First we want to know what the future education cost will be in an inflationary environment increasing 5% per year over 10 years. This amount will be:

n	i	PV	PMT	FV
10	5	-100,000	0	?
			Solving...	**162,889.46**

Given this Future Value, we can determine either the lump sum amount (PV) or the monthly PMT necessary to reach this goal if the invested funds earn 9%/12 per period.

n	i	PV	PMT	FV
120	9÷12	**?**	0	162,889.46
	Solving...	**66,448.69**		

Or, in the case of periodic payment (EOP) per month...

n	i	PV	PMT	FV
120	9÷12	**0**	?	162,889.46
		Solving...	**841.74**	

If the PMT is to be made BOP, simply switch the calculator to the BEG mode. (**$835.48** per month).

The use of the Inflation Adjusted Rate is inappropriate in this type problem because we are not interested in a future amount *measured in today's dollars*, but in an actual amount measured in nominal dollars at that time. The use of the inflation rate (5%) simply tells us the estimated rate at which education costs will increase in the future.

Determining Amounts Needed To Provide an Annuity

One of the most important uses of the IAR is to calculate the PV dollar amount with which a plan, trust or endowment must be funded today in order to deliver future annuities of a constant purchasing power.

But when *ordinary annuities* are involved, the use of the IAR will lead to an incorrect amount. The following formulas may be used with ordinary annuity problems:

If no inflation is involved, then

$$PV = \frac{PMT}{i} \times \left(1 - \frac{1}{(1+i)^n} \right)$$

Where **PV** = the amount of the cash in the trust
PMT = the periodic cashflow (annuity)
i = the investment yield rate
n = number of periods

When the periodic annuity is to be increased by rate **g** to allow for growth or inflation, the formula is modified to:

$$PV = \frac{PMT}{i - g} \times \left(1 - \frac{(1+g)^n}{(1+i)^n} \right)$$

For example,

> With what lump sum amount must an account be funded **today** in order to provide for an annual ordinary annuity of \$3,000 for 15 years, if funds on deposit can earn 9% p.a.?

$$PV = \frac{3000}{.09} \times \left(1 - \frac{1}{(1+.09)^{15}} \right) = 24,182.07$$

If 4% annual inflation is involved, a larger sum must be funded, and this amount is calculated as:

$$PV = \frac{3000}{.09 - .04} \times \left(1 - \left(\frac{(1+.04)}{(1+.09)^{1}} \right)^{15} \right) = \$30,334.35$$

If the IAR is used in solving problems involving **ordinary annuities**, an error is introduced:

n	i	PV	PMT	FV
15	4.807692	?	3,000	0
	Solving...	**31,547.72**		

This value, 31,547.72 is exactly 4% <u>too high.</u>

The cause of this error can be seen from the following cashflow series which sums up the present value of a cashflow which begins with a first payment of \$3,000 made at the EOP, which payment
is then increased by 4% each subsequent period. Each payment is discounted by 9%, the investment yield rate, to produce the required PV.

$$PV = \frac{3,000}{(1+.09)^{1}} + \frac{3,000 * (1+.04)^{1}}{.(1+09)^{2}} \; \cdots\cdots\cdots \; \frac{3,000 * (1+.04)^{\mathbf{14}}}{.(1+09)^{15}}$$

Notice that in the case of an <u>ordinary annuity</u> (EOP), the first cashflow ,\$3,000, is *not* inflated by (1+g), or (1.04), since it is *stipulated* and occurs at the end of the first period. Notice, too, that the last expression of (1+.04) in the numerator is raised to the 14th power, not the 15th power.

But the IAR inflates *every* value in the numerator of the equation by (1+g) introducing the error.

Therefore the PV of the *ordinary annuity* cashflow using the IAR will always be inflated by (1+g). In our example, the answer found by using the IAR can be corrected by dividing by (1+g).

Or, $\frac{31547.72}{(1+.04)} = 30,334.35$, which is the correct answer.

Therefore the PV of an *ordinary* annuity, which is subject to inflation, and which is determined using the IAR will always be (1+g) *higher* than the correct amount.

Finding the PMT Which Can Be Supported by a PV

When the amount of the PV is stipulated[1], and the initial PMT (ordinary annuity) is to be determined, the use of the IAR will result in a number which is *lower* than the actual PMT which can be provided. Therefore the PMT obtained, using the IAR, must be *multiplied* by (1+g).

For example,

> A fund has been endowed with $30,334.35. What is the amount of an *ordinary* annuity which can be drawn from this fund for 15 years if funds remaining on the investment earn 9% and the annuity is increased 4% per year to offset inflation?

We can use the same formula as before: $PV = \frac{PMT}{(i - g)} \times \left(1 - \left(\frac{(1+g)}{(1+i)}\right)^n \right)$ but solve for PMT.

Let $V =$ the expression $1 - \left(\frac{(1+g)}{(1+i)}\right)^n$

Therefore, $PMT = \frac{PV(i - g)}{V} = \frac{30334.35 * (.09 - .04)}{0.550557} = 3{,}000$ (the annuity PMT)

The use of the IAR, however, returns:

n	i	PV	PMT	FV
15	4,807692	**30,334.35**	?	0
		Solving...	**2,884.62**	

This ordinary annuity amount, $2,884.62 is *too low*. When multiplied by (1+.04) the correct answer is obtained:

$2,884.62 x (1.04) = **$3,000.**

Neither of these problems involving *ordinary annuities* is encountered with *annuities-due* (BOP) because when the PMT is made at the beginning of the period the first payment is temporarily removed from the cashflow series and neither inflated nor discounted.

[1] As in the amount with which a trust, fund or endowment is funded.

The Use of the IAR in Future Value Problems

When the IAR is used to express a *future value* (FV), the answer obtained is in the form of today's (constant) dollars. For example:

> A prospective retiree estimates that her retirement income need will require a capital account of $500,000 , *as measured in today's dollars*, when she retires in 20 years. If the investment of a lump sum today will earn 9% p.a., and if inflation averages 4% p.a., what amount must she invest today to reach her goal?

This the future goal is expressed in terms of constant dollars, the IAR may be used.

n	i	PV	PMT	FV
20	4.807692	?	0	500,000
	Solving...	**-195,482.03**		

The balance of her account in 20 years will not be $500,000, but rather her initial investment will have grown at the rate of 9% p.a.:

n	i	PV	PMT	FV
20	9	**195,482.03**	0	?
			Solving...	**1,095,561.57**

This future sum of $1,095,561.57 will have the purchasing power of $500,000 *as measured in today's dollars*. If this future sum is to be accumulated as the result of an annual PMT, made at the *beginning* of each year, and invested @ 9% p.a., these PMT needs to be:

BOP

n	i	PV	PMT	FV
20	9	**0**	?	**1,095,561.57**
		Solving...	**- 19,646.21**	

If this future sum of $1,095,561.57 is to be acquired as the result of EOP PMTS, then:

n	i	PV	PMT	FV
20	9	**0**	?	**1,095,561.57**
		Solving...	**- 21,414.37**	

The following guidelines apply to the use of the IAR:

1. When calculating a PV sum from *annuity due* PMTs subject to inflation, the use of the IAR will deliver a correct PV.
2. When calculating the PMT of *annuity due* subject to inflation the use of the IAR will deliver a correct PMT.
3. When calculating a PV sum from *ordinary annuity* PMTs subject to inflation, the use of the IAR will deliver a PV which is (1+g) too high.
   Divide the PV by (1+g) to find the correct amount.
4. When calculating an ordinary annuity PMT subject to inflation which can be supported by a given PV, the use of the IAR will deliver a PMT (1+g) too low. Multiply the answer by (1+g) to find the correct PMT.
5. If presented with a future financial goal *which is stated in constant dollars*, use the IAR to determine the lump sum PV required.
   a. Carry the resulting PV forward at the investment yield rate to determine the future value of the investment in *nominal* dollars. Tomorrow's bills will be paid with tomorrow's dollars.
   b. If a PMT is required to meet these future nominal dollars, use the investment yield rate, not the IAR, to determine either a BOP or EOP PMT.
6. Do use the IAR to determine the PV required to meet a future goal (FV) *which is expressed in inflation-adjusted dollars.*
7. Remember that determining a future value using the IAR from either a PV or PMT series delivers a FV in *inflation-adjusted dollars*, not in nominal dollars.

Glossary of Some Common Financial Terms and Concepts

Adjustable-Rate Mortgage (ARM): A mortgage in which the interest rate is periodically determined by reference to the interest rate of a base index, such as the 1-year U.S. Treasury Bill, or some other index published by a neutral third party. The mortgage rate is constructed by adding an interest rate margin, or spread, to the base index rate. Purpose is to transfer to the borrower the major risk of an increase in market interest rates. In return, the borrower usually achieves a rate lower than a fixed-rate loan.

Adjusted Funds From Operations (AFFO): Funds From Operation (of a REIT) to which are added back certain necessary replacement costs which would ordinarily be covered by depreciation and amortization deductions. Funds From Operations do not include deductions for depreciation and amortization. See FFO.

AITD: An acronym for All-Inclusive Deed of Trust. A deed of trust securing a promissory note whose principal includes the principal of one or more senior promissory notes, and whose payment includes the payments due on these senior notes. Also called a **wrap-around mortgage.**

Alpha: The point at which a linear regression line crosses the y-axis on a graph which plots the *characteristic line* (rate of return on a stock) against the average market rate of return.

Amortizing Loan: A loan whose periodic repayments contain an amount designated for the reduction of the principal; therefore, a loan whose balance will eventually be reduced to zero. The **amortization schedule** is the time (number of periods) necessary for the balance to reach zero.

Annuity: A sum of money received or paid at regular intervals.

Annuity-due: An annuity (a financial instrument involving regularly occurring payments) made or received **at the beginning** of the payment period. Contrast to *Ordinary* Annuity.

APR: An acronym standing for Annual Percentage Rate. The APR of a loan is the yield, or Internal Rate of Return, which the lender realizes on a consumer loan over the term of the loan when all loan costs are added. Conversely, it is also the actual interest rate paid by the borrower when loan fees (points) and other loan charges are taken into account. Under Federal Reserve Regulation Z, the APR is required to be given to residential borrowers within three days of the loan application. Permits the consumer to compare loans with dissimilar interest rates and fees. Commercial loans (loans on non-residential property) and loans on 5 or more residential units are not required to specify the APR.

Auto Lease, "Money Factor:" A constant used by auto leasing agents to calculate the amount of the monthly lease payment on an automobile lease. The money factor is equal to the (monthly interest rate charged) divided by (the term of the lease in months x 100). To compare a money factor to an interest rate, multiply the factor x the term (in months) x 100. (Money factors are not equivalent to interest rates)

Beta: A measure of the correlation in the movement of a stock's price to that of an external index, most commonly the average price of the S&P 500 . It is based on a 60-month or 36-month historical regression of the return on the stock onto the market return: $\hat{x}_A = A + \beta(R_{rf}) + e$ where $\hat{x}_A$ is the monthly total return on the stock, A is the stock's Alpha, β is the stock's Beta, R_{rf} is the monthly total returns on the market (S&P 500), and e is the error term. A minimum of 12 monthly returns are required for this calculation. A beta of 1 means that the market and the stock move up or down together, at the same rate. That is, a 5% up or down move in the market should theoretically result in a 5% up or down move in the stock. A beta coefficient of 2 suggests that the stock is twice as volatile as the market. That is, if the market moves up 5%, then the stock should move up 10%. A beta coefficient of 0.5 indicates that the stock will move one-half as much as the market, either up or down. A negative beta indicates the stock tends to move in the opposite direction from the general market. That is, the stock price declines when the overall market is rising, or rises when the overall market is declining. The 60-Month Beta is a total return index while the 36-Month Beta is a price-only index. Negative Beta stocks are rare.

Bond Duration: A calculated factor used to approximate a bond's current price sensitivity to potential changes in market-rate yields. The Duration factor is the summation of the time-weighted ratios of the present value of the remaining payments and principal outstanding on a bond to the current market value of the bond. Bond Duration is always a negative number. The percent change in the bond is approximately equal to the percent change in market yield multiplied by the Duration factor. A more accurate estimate of the change is obtained by the use of the **Modified Duration**. The Modified Duration is equal to the Duration divided by (1 + (annual yield rate/2)).

Book Value: The reported net worth of a company. Generally, the original capital to start the firm, plus any capital raised by additional issues, less repurchased shares, plus retained earnings. Book value is often modified by accounting adjustments. Equivalent to "stockholders equity." Expressed as book value/share. Also total assets less total liabilities divided by the number of outstanding shares.

Boot: Unlike property received in an otherwise tax-deferred S.1031 exchange. For example, cash (Personal Property) received in an exchange of Real Property. The value of Boot is taxable to the receiver but not to the party furnishing the Boot.

Buydown Mortgage: A mortgage in which the seller reduces the nominal interest rate to the buyer by contributing specified (periodic) payments which cause the mortgage to resemble one with a lower interest rate. A 3-2-1 buydown would be one on which the buyer's loan behaves like a mortgage 3%, 2% and then 1% lower in interest rate during the first, second and third *periods* than the actual rate on the new mortgage. Payments would be funded in advance by the seller. The length of the *period* is usually, but not always, annual. Buydown mortgages are also available from the lender, in which case the borrower pays a cash lump sum up-front to obtain a lower interest rate.

Call Date: With reference to **mortgages**, a date on which a lender may require the remaining balance to become immediately due and payable. A "30 due in 7" mortgage, for example, would have payments structured as a 30-year loan, but would require a payoff by the borrower at the end of the 7th year.

With reference to **bonds**, a date on which the issuer of the bonds may call-in the bond for redemption. Bond call dates afford the issuer potential relief from a bond-obligation with a high interest rate during periods when market rates are substantially lower.

Capital Gain: A gain resulting from the sale of a capital asset. A *long-term capital gain* is the difference between the net sales price and the adjusted basis of an asset which has been held for the legally proscribed period of time. This time period changes from time to time. As of April 2002, the minimum holding time to qualify for Long Term Capital Gains treatment is one year and one day.

Capitalization rate: The rate at which a capital sum produces periodic payment. It is also the ratio which the payment of a constant payment, ordinary annuity bears to the total Present Value of the annuity. **In real estate**, the ratio of the annual Net Operating Income to the Fair Market Value of an income-producing property. (NOI/FMV = capitalization rate) More correctly, the overall capitalization rate when the NOI is compared to the total market value. The discount rate used to convert a perpetual annuity to a capital amount. The *overall rate* is the return rate on both equity invested plus debt. See Equity Return Rate.

Cashflow: Most simply, the flow of cash from one set of pockets into another. More precisely, Net Operating Income *plus* non-cash items which have already been deducted in determining the NOI, such as income depreciation and amortization allowances.

Characteristic line: A linear regression line drawn through a stock's historic price points on a graph comparing a stock's price to changes in a Market Index. Also **Market Line.**

Compound Interest: Interest earned on a principal sum to which prior earned interest has been added. Formula for compound interest forms the mathematical basis for the Time Value Of Money. (Cited by A. Einstein as the world's greatest invention.)

Conduit Loan (Lender): A real estate mortgage which is conveyed to a Real Estate Mortgage Investment Conduit (REMIC) which uses mortgages to securitize bonds sold to (institutional) investors.

Consumer Price Index (CPI): The cost of a basketful of goods purchased from time to time by the consumer. This index is maintained and published by the federal government's Bureau of Labor Statistics, and serves as a popular measure of inflation.

Convexity (of Bond Curve): Refers to the convex shape of the curve which results from plotting the price of a bond of a constant maturity and coupon rate against increasing market yields. Convexity is equal to the second differentiation of Macaulay's formula for Bond Duration and measures the rate of change in the slope of the bond curve.

Coupon Rate: The rate at which a bond pays annual dividends, expressed as a percent of $1,000. The annual dividend is customarily divided in two and paid every six months.

Debt Coverage Ratio (DCR): The ratio of the annual Net Operating Income of an income-producing property to the annual cost of servicing the loan. Also known as **Debt Service Ratio (DSR).**

Defeasance Clause: a pre-payment found in loans which are sold to conduit lenders. If the borrower seeks to pre-pay the loan at a time when the interest rates on U.S. Treasuries (of a maturity equal to the time to maturity for the loan) are below the loan's interest rate, the borrower must buy and deliver to the lender sufficient Treasury securities such that their interest and principal payments will make up the difference in lost interest to the lender.

Discount Rate: The opposite of an interest rate. A rate used to convert the future value of a single sum, or a series of payments, or a combination of the two, to an equivalent present value. A rate used to move the value of money *backward* in time. Interest rates move the value of money *forward* in time.

Discounted Cash Flow: A technique by which the values of future cashflows are converted into a financially equivalent, single Present Value using one or more **discount rates.**

DownREIT: A REIT which acts as a general partner in partnership with owners from whom it acquires real properties. The owners acquire interests in the partnership in exchange for the contribution of the property to the partnership. This is ordinarily a tax-free event. DownREITS differ from UpREITs in that the REIT owns other real properties previously acquired, while in the UPREIT the REIT owns no other properties. See UpREIT.

Duration: See **Bond Duration**

EBITDA: an acronym standing for Earnings Before Interest, Taxes, Depreciation and Amortization. Revenue, less cost of goods, selling and administrative costs, but before interest costs, income taxes, and non-cash deductions such as depreciation and amortization.

Effective Rate: The rate at which $1.00 will grow in one year as the result of compounding of the interest rate. To calculate the effective rate, determine the Future Value of $1.00 bearing interest at rate i/n over n periods per year. Subtract one from the result and multiply x 100 to convert to a percentage.

Equity Return Rate: The rate of return from an investment based on the equity portion of the investment only. This is in contrast to the overall capitalization rate which includes both equity and debt (borrowed funds).

Finite Annuity: A series of payments occurring on a regular schedule for a definite period of time.

Free Cash Flow: Cash available to debt and equity holders after investment required to maintain operating capacity. This investment money does not include cash for *new* investments. Alternately, FFO = cash available to finance expansion, reduce debt, pay dividends or repurchase equity. If historical depreciation allowances were equal to the cost of replacing capacity, then Net Income and Free Cash Flow would be identical.

Front-end, Back-end Ratios: Financial ratios used by mortgage underwriters to qualify a borrower for a residential loan. The Front-end ratio is the ratio of total monthly housing expenses (including taxes, ins. etc...) divided by total monthly gross income. The Back-end ratio is the total of all current periodic debt payments (including housing) divided by total monthly gross income.

Funds From Operations (FFO): Operating Income before deductions for depreciation and amortization. Does not include income from the one-time sale of assets. Used to express the income of a REIT. See AFFO

Future Value: The value of a Present Value sum, or a series of payments, or a combination of both, carried forward in time and increased by a periodic interest rate.

Geometric Mean: The nth root of a series of n annual returns of $(1+i)_n$ which are multiplied together, minus 1. Assumes that dividends are re-invested.

Graduated Payment Loan (GPM): A loan whose payments increase during the early years (often 1-6) and then remain level for the remainder of the term of the loan.

Gross Rent Multiplier: The ratio of the price of a real estate property to its gross scheduled rent.

Gross Scheduled Income: Rent which would be collected from a real property if every space were leased, and if there were no vacancies or credit losses. The maximum Potential Income.

Gross Operating Income: The amount of rent collected from the operation of a property after deductions for vacancy and credit losses, but before deductions for operating expenses. Sometimes called **Gross Effective Rent, or Gross Collected Rent.**

Hurdle Rate: The minimum rate of return which must be met on a project. Contrast to Opportunity Cost . The hurdle rate is often set equal to the cost of borrowed funds.

Inflation Adjusted Rate (IAR): The periodic rate of return on an investment after adjustment for inflation. Formula: I.A.R = (1+ nominal interest rate) / (1 + inflation rate) -1. (Multiply x 100 to convert to a percentage rate.) A rate used to express future sums in constant (non-inflated) dollars. Permits the measurement of the buying power of future dollars as measured in today's dollars.

Interest: Money paid or received for the use money. When the interest earned is added to the principal the interest earned on the total is known as compound interest. If the interest is not added to principal, the interest earned on principal is known as simple interest.

Internal Rate of Return (IRR): A single discount rate which, when applied to all future cashflows, causes the sum of their present values to equal the amount of the original investment. Or, that single discount rate which, when applied to a cashflow series, will result in a Net Present Value = zero. The rate of return on cash remaining in an investment over time. See Modified IRR

Leverage: The use of borrowed money in an investment. The ability to control an investment using borrowed funds. **Positive** leverage occurs when the rate of return on the investment exceeds the rate of interest paid on borrowed funds. **Negative** leverage is the opposite situation. **Neutral** leverage occurs when the rates are equal.

Loan Points : Fees paid over and above the nominal interest rate for the use of borrowed funds. One point is equal to one percentage point of the loan amount. Loan points are equivalent to pre-paid interest on a loan. In the case of residential real estate, points paid on a loan used to *acquire* a residence are fully deductible (on loans not in excess of $1Million) as interest paid in the year of acquisition. Points paid on loans used to refinance a principal residence, and on all non-residential property, must be amortized (deducted ratably) over the life of the loan.

Low-Floater Bond: A bond whose interest rate is adjusted periodically in order to maintain a constant maturity value.

Modified Internal Rate of Return: The Internal Rate of Return adjusted for negative cashflows using a "safe rate" to provide for future negative cashflows. Becomes more and more significant as IRR increases. Also known as the Financial Management Rate of Return (FMRR)

Net Asset Value (NAV): The underlying total value of assets owned by a REIT divided by the total number of outstanding shares of the REIT. A measurer of the price of the stock to the value of the assets owned.

Net Mortgage Relief: The amount by which the mortgage on a relinquished property exceeds the amount of mortgage on the replacement property in a S. 1031 exchange. Probably the most overlooked source of Boot in S. 1031 exchanges.

Net Operating Income: Operating revenue less operating expenses. Does not include the cost of servicing loans or non-cash items such as depreciation and amortization expense.

Net Present Value: The present value of a series of cashflows after having subtracted the amount of the initial investment. Widely used as a measurement stick for the acceptability of a project or venture when the required "hurdle" or discount rate is defined beforehand.

Opportunity Cost of Funds: The rate of return which could be realized on the *next best* available investment of similar risk.

Ordinary Annuity: A series of regular payments made or received **at the end** of the specified period.

Perpetual Annuity: A regularly occurring payment which continues forever.

Present Value: The present-day worth of a single sum, or a series of payments, or a combination of the two, when discounted at a specified rate. Synonymous with **Present Worth**. The present-day equivalent financial value of a future cashflow. The discounted value of future cashflows.

Residual Value: The Future Value of an investment at the end of the investment period. Also, **Reversion Value**.

R-Squared: The percent of total price movement of a mutual fund which can be attributed to the changes in the underlying Market Index.

Return on Equity: The rate of return **on the owner's cash** as opposed to the rate of return on all cash invested. The owner's Internal Rate of Return.

Return on Investment: Rate of Return on **all cash** invested in the project. This includes both then owner's equity and third-party loans.

REIT: An acronym for Real Estate Investment Trust, a corporation, trust or association which invests either in real properties, real property mortgages, or a combination of the two. REITs combine the limited liability advantage of the corporate form of ownership with the single-level taxation advantage of a partnership. In addition to other requirements for special tax treatment,, REITs must distribute (as of Jan. 1, 2000) 90% or more of annual earnings in the form of dividends. See UpREIT, DownREIT.

Reverse Mortgage: A mortgage on real property in which the lender pays the property owner a specified, periodic payment secured by the owner's equity in the property.

S. 1031 Exchange: A tax-deferred exchange of either real or personal property for like-kind property. "S. 1031" refers to the section of the tax code covering tax-deferred exchanges. Current law also permits the exchange of real property which is "taken" (by eminent domain, condemnation). This kind of exchange is covered by S. 1034 of the I.R.C., and is different from S. 1031 in many important details.

Safe Rate: The rate at which funds can be invested on a short-term basis with virtual certainty regarding the future return of capital and interest. Most likely, an investment in U.S. Treasury securities which limit the risk to inflation but promise the return of capital. US Inflation-Adjusted Treasury bonds are an excellent index of the current safe rate.

Senior Promissory Note and Trust Deed: a note whose securing trust deed (or mortgage) is recorded earlier in time than a later trust deed securing another promissory note. A senior trust deed has priority over a junior trust deed in the order in which they are to be satisfied in the event of a default.

Simple Interest: Interest earned on a capital sum to which no prior earned interest has been added. Contrast to *Compound Interest.*

Spendable Income: The amount of income remaining from an income property after deducting debt service but before taxes. Synonymous with Cashflow Before Taxes.

Stock Equilibrium Formula: The formula for an ordinary, infinite annuity. Used in the valuation of stocks which have earnings. Annual earnings are capitalized (divided) by the targeted yield rate. When the stock offers growth potential, the value of the growth rate is subtracted from the yield rate. Probably a misuse of the capitalization method which applies to periodic payments (dividends).

Standard Deviation: The degree of dispersion of data as measured from the Mean (average), or from the Expected Result determined by applying probabilities to the range of possible outcomes. In a series of measurements, x .... x_n, where **k** is an integer equal to or greater than 1, ***at least*** ($1 - 1/k^2$) percent of the measurements will fall within **k** Standard Deviations of the Expected Result. This is true for *any* distribution pattern, but in a normal "bell curve" shaped distribution pattern, 68% of measurements will fall within 1 Standard Deviation, 95.5% will fall within 2 Standard Deviations and 99.7% will fall within 3 Standard Deviations of the average or Expected Result. Standard Deviation is equal to the square root of the Variance.

Tick: In bond parlance, 1/32 of 1 point. One point = 1% of $1,000, or $10.00. Therefore one tick is equal to $0.3125. U.S. Treasury bonds are quoted using ticks, but Treasury bills are quoted in decimals.

Time Value of Money: A concept meant to convey the idea that the value of money varies according to the time for its receipt or payment.

TIPs: An acronym for Treasury Inflation-Protected Securities. Synonymous with Inflation-Adjusted Treasuries Securities.

Trust Deed: A recordable lien (money encumbrance) against real property used to secure a promise or a debt.

UpREIT: An acronym for Umbrella Partnership Real Estate Investment Trust. UpREITs are limited partnerships in which the REIT acts as the General Partner. The following definition is from the National Association of Real Estate Investment Trusts (NAREIT):

In the typical UPREIT, the partners of the Existing Partnerships and a newly-formed REIT become partners in a new partnership termed the Operating Partnership. For their respective interests in the Operating Partnership ("Units"), the partners contribute the properties from the Existing Partnership and the REIT contributes the cash proceeds from its public offering. The REIT typically

is the general partner and the majority owner of the Operating Partnership Units.
After a period of time (often one year), the partners may enjoy the same liquidity as the REIT shareholders by tendering their Units for either cash or REIT shares (at the option of the REIT or Operating Partnership). This conversion may result in the partners incurring the tax deferred at the UPREIT's formation. The Unit holders may tender their Units over a period of time, thereby spreading out such tax. In addition, when a partner holds the Units until death, the estate tax rules provide that the beneficiaries may tender the Units for cash or REIT shares without paying income taxes.

Variance: a measure of the dispersion of the possible outcomes of a series of historical or probable future measurements around an average or expected value. The sum of the squares of the distance of a number of points measured from a linear regression line drawn through them. Variance = $(\text{Standard Deviation})^2$ Both Standard Deviation and Variance are measurements of Risk.

Yield : A synonym for the Internal Rate of Return

Yield Curve: The curve which results from plotting the yield rate of a series of bonds of similar risk against their time-to-maturity. Most common yield curve is that of U.S. Treasury Bills, Notes and Bonds. It is published in most daily financial newspapers on the Bond page.

Yield-to-Maturity (YTM): The Internal Rate of Return on a bond investment held to the maturity *or* to the redemption (call) date of the bond.

Yield Maintenance Clause: A type pre-payment penalty found in loans issued by conduit lenders. (see Defeasance Clause)

Zero Coupon Bond: A bond which is separated from its dividend payments. Also called a "**strip**." The holder of this bond derivative receives only the redemption value of the bond and is not entitled to any of the dividend payments (coupons) carried by the original bond. Zeros are very volatile carrying a Duration factor approximately equal to their maturity in years.

Index

A

B

C

D

E

F

G

I

L

M

N

O

P

R

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